On a singular initial-value problem for the Navier-Stokes equations

L. E. Fraenkel and M. D. Preston

This paper presents a recent result for the problem introduced eleven years ago in [1], but described only briefly there. We shall prove the following, as far as space allows. The vorticity $\omega$ of a diffusing vortex circle in a viscous fluid has, for small values of a non-dimensional time, a second approximation $\omega_A + \omega_1$ that, although formulated for a fixed, finite Reynolds number $\lambda$ and exact for $\lambda = 0$ (then $\omega = \omega_A$), tends to a smooth limiting function as $\lambda \uparrow \infty$.

In §1 and §2 the necessary background and apparatus are described; §3 outlines the new result and its proof.

1 Introduction

In a certain weak sense, this paper is a continuation of [1]. However, no knowledge of [1] is required if the reader is willing to accept that a vorticity field in $\mathbb{R}^3$ (subject to mild restrictions, but not required to have any symmetry) has a centroid of vorticity moving with a velocity $U(t)$ that is given by an explicit formula when the vorticity $\omega(\cdot, t)$ throughout $\mathbb{R}^3$ is known. This result is essentially due to Saffman [2]; it was generalized a little (and perhaps clarified and sharpened) in [1].

We seek a solution of the Navier-Stokes equations with the initial condition illustrated in Figure 1: at time zero, vorticity $\omega$ is concentrated on, and is tangential to, a horizontal circle in $\mathbb{R}^3$. This initial vorticity induces an initial velocity field that has infinite kinetic energy. (The circle then diffuses and moves vertically, at first with infinite velocity; at all positive times $t > 0$ the kinetic energy is finite.)

More precisely, consider incompressible fluid occupying all of $\mathbb{R}^3$ and at rest at infinity there; let $x := (x_1, x_2, x_3)$ be such that the frame $(Ox_1, Ox_2, Ox_3)$ moves, relative to the motionless fluid at infinity, with the velocity $(0, 0, U(t))$ of the centroid of vorticity, the axes remaining parallel to their initial positions.

The fluid velocity relative to this moving frame is written $\mathbf{v}(x, t)$ and the vorticity is

$$\omega := \text{curl } \mathbf{v} = \nabla \times \mathbf{v}.$$ 

Our time variable is $t = \nu T$, where $T$ denotes physical time and $\nu$ is the kinematic viscosity (a given positive constant). This choice of $t$ simplifies the heat operator in (1.3) below and simplifies most subsequent equations.

In writing $U(t) = (0, 0, U(t))$, we have restricted attention to the cylindrical symmetry implied by the initial condition

$$\omega(x, 0) = \kappa \delta(z) \delta(r - a) e^{i \phi},$$ (1.1)
in which the circulation $\kappa$ and the radius $a$ are given positive constants, cylindrical co-ordinates $(z, r, \phi)$ are defined by $x =: (r \cos \phi, r \sin \phi, z)$, the unit vector $e^{\phi} := (-\sin \phi, \cos \phi, 0)$ and $\delta$ denotes the Dirac generalized function.

In terms of the vorticity $\omega$, the fluid velocity (relative to our moving frame) is

$$v(x, t) = -(0, 0, U(t)) + \nabla \times \int_{\mathbb{R}^3} \frac{1}{4\pi|x - x'|} \omega(x', t) dx'.$$  \hspace{1cm} (1.2)

With (1.1) and (1.2) understood, we seek $\omega(x, t)$ such that

$$\left( \frac{\partial}{\partial t} - \Delta \right) \omega = -\frac{1}{\nu} ((v \cdot \nabla)\omega - (\omega \cdot \nabla)v) \text{ in } \mathbb{R}^3 \times (0, t) \hspace{1cm} (1.3)$$

for some small $t > 0$.

Of course, it would be better to solve the problem (1.1) to (1.3) for all $t > 0$, but this is beyond us because we seek rather explicit answers. There are two excuses for considering only small $t$, or, rather, small $t/a^2$, which is non-dimensional. First, once a solution for $t > 0$ has been established, the general theory of the Navier-Stokes equations implies a continuation of the solution to all time, thanks to finite energy for $t > 0$, cylindrical symmetry and absence of a swirl velocity (of a velocity component in the direction $e^{\phi}$). Secondly, if the viscosity $\nu$ is small, which may be the case of primary interest, then the requirement that $\nu T/a^2$ be small does not demand that the physical time $T$ be small.

In view of (1.1), we write

$$\omega(x, t) =: \omega(z, r, t)e^{\phi},$$

and seek the solution of (1.1) to (1.3) in the scalar form $\omega = \omega_A + \omega_1 + \rho$, where $\omega_A$ is to be a first approximation for small $t/a^2$ and $\omega_A + \omega_1$ is to be a second (improved) approximation; the remainder $\rho$ is to make $\omega_A + \omega_1 + \rho$ an exact solution and is to be $o(\omega_1)$ as $t \downarrow 0$. Here are some details.
(i) The non-linear terms on the right-hand side of (1.3) are expected to be small for small $t/a^2$, because $\omega$ should be approximately constant and large on small circles in a meridional plane ($\phi = \text{constant}$) centred at $(z, r) = (0, a)$, so that $v$ is approximately tangential to such circles and approximately of constant magnitude on each of them. (If the initial vortex circle were a straight line, then these non-linear terms would vanish.) If the right-hand member of (1.3) is neglected, there results the formal approximation

$$\omega_A(z, r, t) = \frac{\kappa}{4\pi t} \exp \left( -\frac{s^2}{4t} \right) B \left( \frac{a r}{2t} \right),$$

(1.4)

where $s := \left( z^2 + (r-a)^2 \right)^{1/2}$ and $B$ is a known function such that $B(y) \to 1$ as $y \to \infty$; in fact,

$$B(y) := (2\pi y)^{1/2} e^{-y} I_1(y) \quad (0 \leq y < \infty),$$

(1.5)

where $I_1$ is the modified Bessel function of the first kind and of order 1 (as in [3], p.77).

(ii) The exponential in (1.4) prompts us to introduce inner variables

$$\sigma := \frac{s}{(4t)^{1/2}}, \quad \theta := \tan^{-1} \frac{r-a}{z};$$

(1.6)

then the amplitude $\kappa/4\pi t$ in (1.4) prompts us to pose

$$\omega_1(z, r, t) = (4t)^{-1/2} \tilde{\omega}_1(\sigma, \theta).$$

(1.7)

It suffices to consider $\omega_1$ in an inner region: $t \downarrow 0$ with $\sigma$ fixed, so that $s \downarrow 0$, because in an outer region: $t \downarrow 0$ with $s \geq \text{constant} > 0$, not only $\omega_A$, but also $\omega$, are exponentially small.

The rest of this paper is devoted mainly to description of $\tilde{\omega}_1$; the Reynolds number

$$\lambda := \frac{\kappa}{2\pi \nu}$$

(1.8)

will be an important parameter.

(iii) The problem for the remainder $\rho$ was sketched in [1]; the function $\rho(z, r, t)$ must be shown to exist and to be suitably small on the whole set $\mathbb{R} \times [0, \infty) \times (0, \bar{t}]$. Considerable progress has been made with this problem since [1] was written, but this analysis (which can only estimate $\rho$) is too long and too elaborate to be described here.

2 The perturbation $\omega_1$ for fixed Reynolds number $\lambda$

With $\omega_1$ as in (1.7), we adopt the notation

(a) $(\sigma, \theta) \in E := (0, \infty) \times (-\pi, \pi],$

(b) $\Delta_\sigma := \left( \frac{\partial}{\partial \sigma} \right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial \sigma} + \frac{1}{\sigma^2} \left( \frac{\partial}{\partial \theta} \right)^2,$

(c) $(A\tilde{\omega}_1)(\sigma_0, \theta_0) := \frac{1}{2\pi} \int \int_E \log \left| \frac{1}{\sigma e^{i\theta} - \sigma_0 e^{i\theta_0}} \right| \tilde{\omega}_1(\sigma, \theta) \sigma d\sigma d\theta,$

(d) $\omega_{A,0}(\sigma, t) := \frac{\kappa}{4\pi t} e^{-\sigma^2},$

(2.1)
in which \( A \tilde{\omega}_1 \) is a stream function describing the plane flow induced by vorticity \( \tilde{\omega}_1 \); the approximation \( \omega_{A,0} \) to \( \omega_A \) is that appropriate to \( t \downarrow 0 \) with \( \sigma \) fixed. We seek \( \tilde{\omega}_1(\sigma,\theta) \) by linearizing (1.2) and (1.3) about \( \omega_{A,0} \); the problem is then to solve the equation

\[
- \left( \Delta \sigma + 2\sigma \frac{\partial}{\partial \sigma} + 2 \right) \tilde{\omega}_1 + \frac{1}{\sigma^2} e^{-\sigma^2} \frac{\partial}{\partial \theta} \tilde{\omega}_1 - 4\lambda e^{-\sigma^2} \frac{\partial}{\partial \theta} (A \tilde{\omega}_1) = \frac{\kappa \lambda}{\pi a} g(\sigma) \cos \theta \quad \text{on } E,
\]

with side conditions

\[
\tilde{\omega}_1(\sigma,\theta) \to 0 \quad \text{as } \sigma \downarrow 0 \quad \text{and as } \sigma \uparrow \infty.
\]

The function \( g \) is a known, smooth function such that

(a) \( g(\sigma) = O(\sigma) \) as \( \sigma \downarrow 0 \);

(b) \( g(\sigma) = O(\sigma \log \sigma e^{-\sigma^2}) \) as \( \sigma \uparrow \infty \);

in fact,

(c) \( g(\sigma) := \sigma e^{-\sigma^2} \left( \frac{3}{2} \frac{1 - e^{-\sigma^2}}{\sigma^2} + \left( \log \frac{1}{\sigma} - \int_{\sigma}^{\infty} \frac{e^{-\rho^2}}{\rho^2} d\rho \right) - \frac{1}{2} (\gamma_E + 1 - \log 2) \right) \),

where \( \gamma_E = 0.5772... \) denotes Euler’s constant.

**Theorem 2.1.** For fixed \( \lambda \in [0, \infty) \), the problem (2.2) and (2.3) for \( \tilde{\omega}_1 \) has a pointwise, unique solution; in particular, \( \tilde{\omega}_1(\cdot,\theta) \in C^\infty[0, \infty) \), \( \tilde{\omega}_1(0,\theta) = 0 \) and \( \tilde{\omega}_1(\sigma,\theta) = o(e^{-\sigma^2/2}) \) as \( \sigma \uparrow \infty \).

Here we have space only to sketch the main steps of the proof.

(i) Under the transformation

\[
\tilde{\omega}_1(\sigma,\theta) = e^{-\sigma^2/2} q(\sigma,\theta),
\]

equation (2.2) becomes

\[
- (\Delta \sigma - \sigma^2) q + \frac{1}{\sigma^2} e^{-\sigma^2} \frac{\partial q}{\partial \theta} - 4\lambda e^{-\sigma^2} T \left( e^{-\sigma^2/2} q \right) = \frac{\lambda e^{\sigma^2/2}}{2} f(\sigma,\theta) \quad \text{on } E,
\]

where the operator \( T := (\partial/\partial \theta) A \) and \( f(\sigma,\theta) := (\kappa/\pi a) g(\sigma) \cos \theta \). Let

\[
(\xi, \eta) := (\sigma(\cos \theta, \sin \theta), \quad q_*(\xi, \eta) = q_*(\sigma \cos \theta, \sigma \sin \theta) := q(\sigma, \theta).
\]

The condition in (2.3) for \( \sigma \downarrow 0 \) will be implicit in what follows; it was imposed only to make \( q_* \) decent at the origin, because we shall find that \( q \) is of form \( q_*(\sigma) \cos \theta + q_*(\sigma) \sin \theta \). Henceforth the functions \( q_* \) and \( q \) will be identified wherever no confusion is possible. Similarly, the Cartesian-co-ordinate and polar-co-ordinate representations of other functions will be identified.

(ii) In the first instance we establish a weak solution of (2.6). Let the real Hilbert space \( Z \) be the completion of the set \( C^\infty_c(\mathbb{R}^2) \), of real-valued, infinitely
differentiable functions on $\mathbb{R}^2$ having compact support, in the norm implied by the inner product

$$\langle u, v \rangle_Z := \iint_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \sigma^2 uv) \, d\xi d\eta. \quad (2.8)$$

We shall say that $q$ is a weak solution of (2.6) if (and only if) $q \in Z$ and, for all test functions $u \in Z$,

$$B(u, q) := \iint_{\mathbb{R}^2} \left( \nabla u \cdot \nabla q + \sigma^2 u q + \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} u \frac{\partial q}{\partial \theta} - 4\lambda e^{-\sigma^2/2} u T \left( e^{-\sigma^2/2} q \right) \right) d\xi d\eta = \lambda \iint_{\mathbb{R}^2} e^{\sigma^2/2} f u \, d\xi d\eta. \quad (2.9)$$

(iii) Here is the key step of the proof.

**Lemma 2.2.** The bilinear form $B$ satisfies, for all $u$ and $v$ in $Z$,

$$B(u, u) = \|u\|^2, \quad (2.10)$$

$$|B(u, v)| \leq (1 + k_B \lambda) \|u\| \|v\|, \quad (2.11)$$

where $\| \cdot \| = \| \cdot \|_Z$ and $k_B$ is an absolute constant (independent of the variables, parameters and functions in question).

**Partial proof.** We shall prove only that (2.10) holds for all functions in $C_\infty^\infty (\mathbb{R}^2)$. The remainder of the proof is neither trivial nor immediate, but it is of a kind familiar in Sobolev-space theory and its application to partial differential equations.

In view of the definition of $B$ in (2.9), we wish to prove that, for all $\varphi \in C_\infty^\infty (\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2} \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial \varphi}{\partial \theta} \, d\xi d\eta = 0$$

and

$$\iint_{\mathbb{R}^2} e^{-\sigma^2/2} \varphi T(e^{-\sigma^2/2} \varphi) \, d\xi d\eta = 0.$$  

The first of these is immediate because $\int_{-\pi}^{\pi} \varphi \frac{\partial \varphi}{\partial \theta} \, d\theta = 0$. For the second, let $A(e^{-\sigma^2/2} \varphi) =: \psi$; then $e^{-\sigma^2/2} \varphi = -\Delta \psi$ and we wish to prove that

$$- \iint_{\mathbb{R}^2} (\Delta \psi) \frac{\partial \psi}{\partial \theta} \, d\xi d\eta = 0.$$  

Here it suffices to integrate over an open disc (or ball) $B(0, R)$ with centre the origin and radius $R$ so large that $B(0, R)$ contains the compact support of $\Delta \psi$. Thus the integral may be written

$$- \int_{\partial B(0, R)} \frac{\partial \psi}{\partial \sigma} \frac{\partial \psi}{\partial \theta} Rd\theta + \iint_{B(0, R)} \nabla \psi \cdot \frac{\partial}{\partial \theta} \nabla \psi \, d\xi d\eta.$$  

That this last integral over $B(0, R)$ vanishes is immediate as before. The boundary integral is now independent of $R$ and vanishes because $\partial \psi / \partial \sigma$ and
\(\partial \psi / \partial \theta\) are both \(O(R^{-1})\) as \(R \uparrow \infty\), by the definition (2.1)(c) of the operator \(A\).

(iv) Existence and uniqueness of a weak solution. The forcing function in (2.6) satisfies amply the condition
\[ \int \int_{\mathbb{R}^2} e^{\sigma^2/2} f(\sigma, \theta)^2 d\xi d\eta < \infty, \]  

because \(f(\sigma, \theta) = (\kappa/\pi a)g(\sigma)\cos \theta\) with \(g\) as in (2.4). This condition is sufficient to make the forcing integral in (2.9), namely,
\[ F(u) := \int \int_{\mathbb{R}^2} e^{\sigma^2/2} f u d\xi d\eta, \quad u \in Z, \]
a bounded linear functional evaluated at \(u\). In other words, \(F\) belongs to the dual space \(Z^*\) of \(Z\). The Lax-Milgram lemma now implies

**Lemma 2.3.** Equation (2.6) has a unique weak solution \(q\) and
\[ \frac{\lambda}{1 + k_B \lambda} \left\| F \right\|_{Z^*} \leq \| q \|_Z \leq \lambda \left\| F \right\|_{Z^*}. \]  

(v) Regularity theory: pointwise estimates. We separate the variables \(\sigma\) and \(\theta\). Let \(Y\) denote the real Hilbert space of functions \(y : [0, \infty) \to \mathbb{R}\) such that the functions having values \(y(\sigma)\cos \theta\) or \(y(\sigma)\sin \theta\) belong to \(Z\). It can be proved that, equivalently, \(Y\) is the completion of the set
\[ D := \{ \zeta \in C^\infty_c[0, \infty) \mid \zeta(0) = 0 \}, \]
where the compact support of \(\zeta\) may extend to the origin, in the norm implied by the inner product
\[ \langle v, w \rangle_Y := \int_0^\infty \left( v' w' + \left( \frac{1}{\sigma^2} + \sigma^2 \right) vw \right) \sigma d\sigma, \]
where the \((\cdot)'\) denotes differentiation.

It can then be proved that, if
(a) \(Q(\sigma) := q_c(\sigma) + iq_s(\sigma), \quad (q_c, q_s) \in Y^2; \) \[ Q(\sigma) := q_c(\sigma) + iq_s(\sigma), \quad (q_c, q_s) \in Y^2; \]  
(b) the operator \(T_1\) is defined by
\[ (T_1 y)(\sigma) := \frac{1}{2} \int_0^\infty \left( \frac{\rho \sigma}{\sigma^2} \right) y(\rho) \rho d\rho \quad \text{for all } y \in Y; \]
where \(a \wedge b\) denotes the lower envelope, or lesser, of \(a\) and \(b\);
(c) for all test functions \(v \in Y, \)
\[ \int_0^\infty \left( v' Q' + \left( \frac{1}{\sigma^2} + \sigma^2 \right) v Q - i \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} v Q + i 4 \lambda e^{-\sigma^2/2} v T_1 (e^{-\sigma^2/2} Q) \right) \sigma d\sigma \]
\[ = \lambda \int_0^\infty e^{\sigma^2/2} f_c v \sigma d\sigma, \]  
\[ (2.16) \]
where \( f_c(\sigma) := (\kappa/\pi a)g(\sigma) \);
(d) \( q(\sigma, \theta) := q_c(\sigma) \cos \theta + q_s(\sigma) \sin \theta; \) (2.17)
then \( q \) satisfies (2.9), so that the right-hand member of (2.17) is the unique weak solution of (2.6). Conversely, equations (2.9), (2.17) and (2.14) imply (2.16).

We now choose the test function in (2.16) to be a Green function of the operator
\[-\left(\frac{d}{d\sigma}\right)^2 - \frac{1}{\sigma} \frac{d}{d\sigma} + \left(\frac{1}{\sigma^2} + \sigma^2\right),\]
which results from insertion of (2.17) into (2.6). It is legitimate to choose \( v(\sigma) = K(\rho, \cdot) := \begin{cases} \frac{1}{\sigma} \sinh \frac{\sigma^2}{2} \cdot \frac{1}{\rho} \exp \left(-\frac{\rho^2}{2}\right) & \text{if } \sigma \leq \rho, \\ \frac{1}{\sigma} \exp \left(-\frac{\sigma^2}{2}\right) \cdot \frac{1}{\rho} \sinh \frac{\rho^2}{2} & \text{if } \sigma \geq \rho, \end{cases} \) (2.18)
because \( K(\rho, \cdot) \in Y \) for fixed \( \rho \in (0, \infty) \). Then (2.16) yields, after an integration by parts, the integral equation
\[Q(\rho) = \lambda \int_0^\infty K(\rho, \sigma)e^{\sigma^2/2}f_c(\sigma)\sigma d\sigma + i\lambda \int_0^\infty K(\rho, \sigma) \left( \frac{1 - e^{-\sigma^2}}{\sigma^2}Q(\sigma) - 4e^{-\sigma^2/2}T_1(e^{-\sigma^2/2}Q) \right)\sigma d\sigma.\] (2.19)

Since Lemma 2.3 provides bounds for \( \|q_c\|_Y \) and \( \|q_s\|_Y \), the regularity of \( Q \), and pointwise bounds, can be deduced from (2.19) and from Lemma 3.8 below without great difficulty.

**3 The perturbation \( \omega_1 \) as \( \lambda \uparrow \infty \)**

We return to equations (2.1) to (2.4) and define a stream function \( \tilde{\psi}_1 := A\tilde{\omega}_1 \). Then \( \tilde{\omega}_1 = -\Delta_\sigma \tilde{\psi}_1 \) and (2.2) becomes
\[
\left( \Delta_\sigma + 2\sigma \frac{\partial}{\partial \sigma} + 2 \right) \Delta_\sigma \tilde{\psi}_1 - \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial}{\partial \theta} \Delta_\sigma \tilde{\psi}_1 - 4\lambda e^{-\sigma^2} \frac{\partial \tilde{\psi}_1}{\partial \theta} = \frac{\kappa \lambda}{\pi a} g(\sigma) \cos \theta \quad \text{on } E.
\] (3.1)

In view of (2.5) and (2.17), the function \( \tilde{\psi}_1 \) has the form
\[\tilde{\psi}_1(\sigma, \theta) = \tilde{\psi}_{1c}(\sigma) \cos \theta + \tilde{\psi}_{1s}(\sigma) \sin \theta.\] (3.2)

We divide (3.1) by \( \lambda(1 - e^{-\sigma^2})/\sigma^2 \), write the \( \cos \theta \) and \( \sin \theta \) parts as separate equations and define, similarly to (2.14),
\[\Psi(\sigma) = \tilde{\psi}_{1c}(\sigma) + i\tilde{\psi}_{1s}(\sigma).\] (3.3)
With the notation
\[ \Delta_1 := \left( \frac{d}{d\sigma} \right)^2 + \frac{1}{\sigma} \frac{d}{d\sigma} - \frac{1}{\sigma^2}, \]

\[ \alpha(\sigma) := \frac{4\sigma^2}{e^{\sigma^2} - 1}, \]

\[ \beta(\sigma) := \frac{\sigma^2}{1 - e^{-\sigma^2}}, \]

\[ E := \Delta_1 + 2\sigma \frac{d}{d\sigma} + 2, \]

the problem for \( \tilde{\omega}_1 \) is to solve the equation

\[ -i\beta(\sigma) \frac{\lambda}{E} (\Delta_1 \Psi) + \left\{ \Delta_1 + \alpha(\sigma) \right\} \Psi = -i \frac{\kappa}{\pi a} \beta(\sigma) g(\sigma), \quad 0 < \sigma < \infty, \]

(3.5)

with the side conditions

as \( \sigma \downarrow 0 \), \( (\Delta_1 \Psi)(\sigma) \rightarrow 0 \), \( \Psi'(\sigma) = O(1) \) and \( \Psi(\sigma) = O(\sigma) \);

as \( \sigma \uparrow \infty \), \( (\Delta_1 \Psi)(\sigma) \rightarrow 0 \), \( \Psi'(\sigma) = O(\sigma^{-2}) \) and \( \Psi(\sigma) = O(\sigma^{-1}) \).

(3.6)

Here the conditions on \( \Delta_1 \Psi \) come from (2.3); the conditions on \( \Psi' \) and \( \Psi \) are implied by \( \Psi = -T_1(\Delta_1 \Psi) \), with \( T_1 \) as in (2.15), and by conditions on \( \Delta_1 \Psi \) much weaker than those in Theorem 2.1.

For \( \lambda \uparrow \infty \), equation (3.5) with (3.6) seems to form a singular perturbation problem, since a small parameter multiplies the highest derivatives. Surprisingly, this turns out not to be the case; nevertheless there is work to be done.

Apparently, if \( \Psi_0(\sigma) := \lim_{\lambda \uparrow \infty} \Psi(\sigma; \lambda) \) exists, then it must satisfy

\[ \left\{ \Delta_1 + \alpha(\sigma) \right\} \Psi_0 = -i \frac{\kappa}{\pi a} \beta(\sigma) g(\sigma), \quad 0 < \sigma < \infty, \]

(3.7)

and the six side conditions (3.6). We proceed to explore this problem.

**Lemma 3.1.** The equation

\[ \left\{ \Delta_1 + \alpha(\sigma) \right\} u = 0, \quad 0 < \sigma < \infty, \]

(3.8)

has solutions

\[ U(\sigma) := \frac{1}{\sigma} \left( 1 - e^{-\sigma^2} \right) \]

(3.9)

and

\[ V(\sigma) := \frac{1}{\sigma} - U(\sigma) \log \left( e^{\sigma^2} - 1 \right). \]

(3.10)

Here \( U \) is an eigensolution (with eigenvalue 0) in that it satisfies not only (3.8) but also all six side conditions (3.6).

**Proof.** This is a matter of direct calculation. \( \square \)

**Lemma 3.2.** The forcing function in (3.7) is orthogonal to the eigensolution \( U \) in the sense that

\[ \int_0^\infty U(\sigma) \beta(\sigma) g(\sigma) \sigma d\sigma = 0. \]

(3.11)
Hence the problem (3.7) and (3.6) has a (non-unique) solution

\[ \Psi_0(\sigma) = c_0 U(\sigma) + i \frac{K}{2\pi \alpha} \int_0^\sigma \{ U(\rho)V(\sigma) - U(\sigma)V(\rho) \} \beta(\rho) g(\rho) \rho \, d\rho \]  

(3.12)

for every \( c_0 \in \mathbb{C} \).

Proof. Again this is a matter of direct calculation, but the calculation is not short. With \( \beta \) defined by (3.4)(c) and \( g \) by (2.4)(c), the analytic proof of the orthogonality condition (3.11) seems to require a page. (However, with any machine capable of numerical integration, numerical verification of (3.11) is quick and easy.) We note that Liouville’s formula for Wronskians yields

\[ U(\sigma)V'(\sigma) - U'(\sigma)V(\sigma) = -\frac{2}{\sigma}, \quad 0 < \sigma < \infty. \]  

(3.13)

The following lemma is also relevant.

Lemma 3.3. Define, for suitable functions \( f \),

\[ (\mathcal{G}f)(\sigma) := \frac{1}{2} \int_0^\sigma \{ U(\rho)V(\sigma) - U(\sigma)V(\rho) \} f(\rho) \rho \, d\rho, \quad 0 < \sigma < \infty, \]  

(3.14)

and

\[ J(f) := \int_0^\infty V(\rho)f(\rho) \rho \, d\rho. \]  

(3.15)

Assume that \( f \in C[0, \infty) \), that \( \int_0^\infty U(\rho)f(\rho) \rho \, d\rho = 0 \), that \( f(\sigma) = O(\sigma) \) as \( \sigma \downarrow 0 \) and that \( f(\sigma) = O(\sigma^m e^{-\sigma^2}) \), with \( m \geq 1 \), as \( \sigma \uparrow \infty \). Then

\[ \{ \Delta_1 + \alpha(\sigma) \} (\mathcal{G}f)(\sigma) = -f(\sigma) \quad \text{in} \ (0, \infty); \]  

(3.16)

as \( \sigma \downarrow 0 \),

\[ (\mathcal{G}f)(\sigma) = O(\sigma^3) \quad \text{and} \quad (\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma); \]  

(3.17)

as \( \sigma \uparrow \infty \),

\[ (\mathcal{G}f)(\sigma) = -\frac{1}{2} J(f) \sigma^{-1} + O(\sigma^m e^{-\sigma^2}), \]  

(3.18)

\[ (\mathcal{G}f)'(\sigma) = \frac{1}{2} J(f) \sigma^{-2} + O(\sigma^{m-1} e^{-\sigma^2}), \]  

(3.19)

\[ (\mathcal{G}f)''(\sigma) = -J(f) \sigma^{-3} + O(\sigma^m e^{-\sigma^2}), \]  

(3.20)

and

\[ (\Delta_1 \mathcal{G}f)(\sigma) \quad \text{and} \quad (\mathcal{E}\mathcal{G}f)(\sigma) \quad \text{are} \quad O(\sigma^m e^{-\sigma^2}), \]  

(3.21)

Proof. Equation (3.16) follows from the definition of \( \mathcal{G}f \) and two differentiations. That \( (\mathcal{G}f)(\sigma) = O(\sigma^3) \) as \( \sigma \downarrow 0 \) also follows from the definition; then the differential equation (3.16) shows that \( (\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma) \) as \( \sigma \downarrow 0 \). In order to prove (3.18) and (3.19), we note that \( U(\sigma) \sim 1/\sigma \) and \( V(\sigma) \sim -\sigma \) as \( \sigma \uparrow \infty \), whence

\[ \int_0^\sigma U(\rho)f(\rho) \rho \, d\rho = \left( \int_0^\infty - \int_0^\sigma \right) U(\rho)f(\rho) \rho \, d\rho = 0 + O(\sigma^{m-1} e^{-\sigma^2}), \]

\[ \int_0^\sigma V(\rho)f(\rho) \rho \, d\rho = \left( \int_0^\infty - \int_0^\sigma \right) V(\rho)f(\rho) \rho \, d\rho = J(f) + O(\sigma^{m+1} e^{-\sigma^2}), \]
from which (3.18) and (3.19) follow.

The differential equation (3.16) and the estimate (3.18) of \( \mathcal{G}f \) imply that 
\((\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma^m e^{-\sigma^2})\) as \( \sigma \uparrow \infty \). What has been proved now implies the 
estimates of \((\mathcal{G}f)^{''}(\sigma)\) and \((\mathcal{E} \mathcal{G}f)(\sigma)\) for \( \sigma \uparrow \infty \).

The result (3.12) prompts two questions. How (if at all) is \( c_0 \) to be evaluated? 
How smooth is \( \Psi_0 \)? Analogues of both these questions will have to be answered 
more generally for each function \( \Psi_n \) in an identity

\[
\Psi(\sigma; \lambda) = \sum_{n=0}^{N} \lambda^{-n} \Psi_n(\sigma) + R_N(\sigma; \lambda).
\]

Here we anticipate later results and note that, with \( J \) as in (3.15),

\[
c_0 = i \frac{\kappa}{2\pi a} J(\beta g) = i \frac{\kappa}{a} (0.11527...).
\]

This follows from the equation governing \( \Psi_1 \), which requires an orthogonality 
condition involving \( \Psi_0 \).

Because of the function \( Q =: Q(\cdot; \lambda) \) defined by (2.14) and (2.16), we now define \( Q_0 = q_{0c} + iq_{0s} \) by

\[
Q_0(\sigma) := -e^{\sigma^2/2} (\Delta_1 \Psi_0)(\sigma)
\]

\[
= e^{\sigma^2/2} \left( 4c_0 \sigma e^{-\sigma^2} - \alpha(\sigma)(Gh_0)(\sigma) - h_0(\sigma) \right),
\]

where \( h_0 := -i(\kappa/\pi a)\beta g \). Evidently \( q_{0c} = 0 \).

It will emerge from Theorem 3.5 that \( Q_0 \) is the limit of \( Q(\cdot; \lambda) \) as \( \lambda \uparrow \infty \). In 
Figures 2 and 3, \( Q_0 \) is compared with \( Q(\cdot; \lambda) \) for large \( \lambda \); these values of \( Q(\cdot; \lambda) \) 
were obtained by numerical solution of the equation

\[
- (\Delta_1 - \sigma^2) Q - i\lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} Q + i4\lambda e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} Q) = \frac{\kappa \lambda}{\pi a} e^{\sigma^2/2} g(\sigma).
\]

This equation is equivalent to (2.6), because of (2.17); it is also the pointwise 
form of (2.16); with the condition that \( Q(\sigma) \rightarrow 0 \) as \( \sigma \downarrow 0 \) and as \( \sigma \uparrow \infty \), it has 
a pointwise, unique solution. Figures 2 and 3 are consistent with the result of 
Theorem 3.5 that, as \( \lambda \uparrow \infty \),

\[
q_c(\cdot; \lambda) = O(\lambda^{-1}) \quad \text{and} \quad q_s(\cdot; \lambda) - q_{0s} = O(\lambda^{-2}).
\]

**Definition.** We shall say that a function \( \varphi : [0, \infty) \rightarrow \mathbb{C} \) is **satisfactory on** \([0, k)\) 
if (and only if) there exist coefficients \( b_n \) and a number \( k > 0 \) such that

\[
\varphi(\sigma) := \sum_{n=0}^{\infty} b_n \sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < k.
\]

**Lemma 3.4.** (i) The function \( \beta g \) is satisfactory on \([0, (2\pi)^{1/2})\).

(ii) If \( f \) is satisfactory on \([0, (2\pi)^{1/2})\), then so is \( \mathcal{G}f \).

**Proof.** (i) We note that

\[
\beta(\sigma) g(\sigma) = \sigma e^{-\sigma^2} \left( \frac{3}{2} + \frac{\sigma^2}{1 - e^{-\sigma^2}} \left( \int_{\sigma}^{1} \frac{1 - e^{-\sigma^2}}{\rho} \, d\rho + C_0 + C_1 \right) \right),
\]

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Figure 2: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^{3/2\pi}$ and $\lambda = \infty$.

Figure 3: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^{4/2\pi}$ and $\lambda = \infty$.

where

$$C_0 := -\int_1^{\infty} \frac{e^{-\rho^2}}{\rho} \, d\rho, \quad C_1 := -\frac{1}{2} \left( \gamma_E + 1 - \log 2 \right),$$

and that the function $w$ defined by

$$w(z) = \frac{z}{1 - e^{-z}} \quad \text{if} \quad z \in \mathbb{C} \setminus \{0\} \setminus \{\text{poles}\} \quad \text{and} \quad w(0) = 1,$$

is holomorphic for $|z| < 2\pi$.

(ii) Let

$$W(\sigma) := \frac{1}{\sigma} - U(\sigma) \log \frac{e^{\sigma^2} - 1}{\sigma^2},$$

where the limiting value of $(e^{\sigma^2} - 1)/\sigma^2$ is taken at $\sigma = 0$. Then

$$V(\sigma) = W(\sigma) - 2U(\sigma) \log \sigma$$
Theorem 3.5. The perturbation $\tilde{\omega}_1$ has a representation

$$\tilde{\omega}_1(\sigma; \theta; \lambda) = \cos \theta \{ \lambda^{-1} \zeta_1(\sigma) + \lambda^{-2} \zeta_2(\sigma) + \zeta_3(\sigma; \lambda) \}$$

$$+ \sin \theta \{ \zeta_0(\sigma) + \lambda^{-2} \zeta_2(\sigma) + \zeta_4(\sigma; \lambda) \}$$

(3.25)

in which, for $m = 0, 1, 2, 3$ and $n = 4, 5$,
(a) the functions $\zeta_m$ and $\zeta_n(\cdot; \lambda)$ belong to $C^\infty[0, \infty)$ and are satisfactory on $[0, (2\pi)^{1/2})$;
(b) as $\sigma \uparrow \infty$, $\zeta_m(\sigma) = O(\sigma^{2m+1}e^{-\sigma^2})$ and $\zeta_n(\sigma; \lambda) = o(\sigma^{-\sigma^2})$ for fixed $\lambda$;
(c) as $\lambda \uparrow \infty$, $\zeta_n(\sigma; \lambda) = O(\lambda^{-n})$, uniformly over $\sigma \in [0, \infty)$.

The proof will be by means of further lemmas. Let $\Psi := \tilde{\psi}_1 + i\tilde{\psi}_1'$, as before, and let $\Omega := -\Delta_1 \Psi$, so that $\Omega = \tilde{\omega}_1 + i\tilde{\omega}_1'$. Our plan is to construct identities

$$\Psi(\sigma; \lambda) = \sum_{n=0}^N \lambda^{-n} \Psi_n(\sigma) + R_N(\sigma; \lambda),$$

$$\Omega(\sigma; \lambda) = \sum_{n=0}^N \lambda^{-n} \Omega_n(\sigma) + r_N(\sigma; \lambda),$$

(3.26)

(3.27)

in which estimates of the remainders $R_N$ and $r_N$ can be crude. In fact, we shall prove only that $R_N$ and $r_N$ are $O(\lambda^{-N})$, but this is sufficient for (3.25) if $N \geq 6$.

The terms of the expansion of $\Psi$ are to satisfy

$$\{ \Delta_1 + \alpha(\sigma) \} \Psi_n = h_n, \quad n = 0, 1, ..., N,$$

(3.28)

where

$$h_0(\sigma) = -i \frac{K}{\pi a} \beta(\sigma) g(\sigma),$$

(3.29)

$$h_n := i \beta \mathcal{E}(\Delta_1 \Psi_{n-1}) \quad \text{for} \quad n = 1, ..., N,$$

(3.30)

and

$$-i\lambda^{-1} \beta(\sigma) \mathcal{E}(\Delta_1 R_N) + \{ \Delta_1 + \alpha(\sigma) \} R_N = i\lambda^{-N-1} \beta(\sigma) \mathcal{E}(\Delta_1 \Psi_N);$$

(3.31)

then the right-hand member of (3.26) will satisfy the equation (3.5) governing $\Psi$.

Since $\Omega = -\Delta_1 \Psi$ and $\Psi = T_1 \Omega$, this scheme corresponds to

$$-\Omega_n + \alpha(\sigma) T_1 \Omega_n = h_n, \quad n = 0, 1, ..., N,$$

(3.32)

$$i\lambda^{-1} \beta(\sigma) \mathcal{E} \mathcal{R}_N - r_N + \alpha(\sigma) T_1 r_N = -i\lambda^{-N-1} \beta(\sigma) \mathcal{E} \Omega_N,$$

(3.33)

where $h_n = -i \beta(\sigma) \mathcal{E} \Omega_{n-1}$ for $n = 1, ..., N$. 

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Lemma 3.6. In order that equation (3.28), with the side conditions (3.6), have a solution, it is necessary that

$$\int_0^\infty U(\sigma)h_n(\sigma)\sigma d\sigma = 0, \quad n = 0, 1, \ldots, N;$$  \hspace{1cm} (3.34)

equivalently, that

$$\int_0^\infty \sigma^2 g(\sigma) d\sigma = 0 \quad \text{if} \quad n = 0,$$

and

$$\int_0^\infty \sigma^2 (\Omega_{n-1})(\sigma) d\sigma = 0 \quad \text{if} \quad n = 1, \ldots, N. \quad \hspace{1cm} (3.36)$$

Proof. Let $M := \Delta_1 + \alpha(\sigma)$. Assume that $u$ and $v$ are in $C^2(0, \infty)$, that $\sigma u(\sigma)v'(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$ and that $\sigma u'(\sigma)v(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$. Then integration by parts yields

$$\int_0^\infty u(Mv)\sigma d\sigma = \int_0^\infty (Mu)v\sigma d\sigma. \quad \hspace{1cm} (3.37)$$

Now let $u = U$ and $v = \Psi_n$. If $\Psi_n$ satisfies (3.6), then the foregoing hypotheses are satisfied. If also $M\Psi_n = h_n$, then

$$\int_0^\infty Uh_n\sigma d\sigma = \int_0^\infty (MU)\Psi_n\sigma d\sigma = 0. \quad \hspace{1cm} (3.38)$$

Equations (3.35) and (3.36) follow from the identity $U(\sigma)\beta(\sigma) = \sigma$ and from the definitions of $h_n$.

If $h_n$ satisfies not only the orthogonality condition (3.34), but also the other hypotheses on $f$ in Lemma 3.3 (and this will be the case), then the differential equation (3.28), with side conditions (3.6), has solutions

$$\Psi_n = c_nU - G h_n, \quad n = 0, 1, \ldots, N,$$  \hspace{1cm} (3.39)

whence

$$\Omega_n(\sigma) = - (\Delta_1 \Psi_n)(\sigma) = 4c_n\sigma e^{-\sigma} - \alpha(\sigma)(G h_n)(\sigma) - h_n(\sigma), \quad \hspace{1cm} (3.40)$$

for every $c_n \in \mathbb{C}$.

In order to evaluate $c_0, \ldots, c_N$ and in order to discuss $r_N$, we extend the definition (3.30) to $h_{N+1}$. Recall from Lemma 3.2 that for $n = 0$ the orthogonality condition (3.34) has already been established.

Lemma 3.7. For $n = 0, 1, \ldots, N$, the necessary condition $\int_0^\infty Uh_{n+1}\sigma d\sigma = 0$ implies that $c_n = -\frac{1}{2}J(h_n)$, where $J(\cdot)$ is defined by (3.15).

Proof. Extended to $\Omega_N$, the orthogonality condition (3.36) states that, for $n = 0, 1, \ldots, N$,

$$0 = \int_0^\infty \sigma^2 (\Omega_n)(\sigma) d\sigma = -4 \int_0^\infty \sigma^2 \Omega_n d\sigma,$$

by an integration by parts for which it suffices that $\Omega_n \in C^2(0, \infty)$, that $\Omega'_n(\sigma) = o(\sigma^{-2})$ both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, that $\Omega_n = o(\sigma^{-1})$ as $\sigma \downarrow 0$ and that $\Omega_n = o(\sigma^{-3})$ as $\sigma \uparrow \infty$. 

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Next, we observe that, if \( \Psi_n \in C^2(0, \infty) \) and if both \( \sigma^2 \Psi_n'(\sigma) \) and \( \sigma \Psi_n(\sigma) \) have limits both as \( \sigma \downarrow 0 \) and as \( \sigma \uparrow \infty \), then
\[
0 = \int_0^\infty \sigma^2 (\Delta_1 \Psi_n) \, d\sigma = \left[ \sigma^2 \Psi_n' - \sigma \Psi_n \right]_0^\infty,
\]
in which limiting values are implied on the right-hand side. In view of (3.39), the orthogonality condition is now
\[
c_n \left[ \sigma^2 U' - \sigma U \right]_0^\infty = \left[ \sigma^2 (G h_n)' - \sigma (G h_n) \right]_0^\infty.
\]
Referring to the definition of \( U \) in (3.9) and to the description of \( G f \) in Lemma 3.3, one is led to \( c_n = -\frac{1}{2} J(h_n) \). \( \square \)

It is time to relate the \( \zeta_n \) and \( \zeta_n(\cdot; \lambda) \) in Theorem 3.5 to the \( \Omega_n \) and \( r_N \) in (3.27). We noted after (3.23) that \( Q_0 \) is imaginary. Since \( \Omega(\sigma) = \exp(-\sigma^2/2)Q(\sigma) \), the function \( \Omega_0 \) is imaginary. Then, since \( h_1 = -i \beta \xi \Omega_0 \), the function \( h_1 \) and the coefficient \( c_1 \) are real. Equation (3.40) shows that \( \Omega_1 \) is real. An easy induction now shows that \( \Omega_n \) is imaginary if \( n \) is even and \( \Omega_n \) is real if \( n \) is odd.

Accordingly, if \( N \) is odd, then
\[
\zeta_1 = \Omega_1, \quad \zeta_3 = \Omega_3 \quad \text{and} \quad \zeta_5(\cdot; \lambda) = \lambda^{-5} \Omega_5 + \ldots + \lambda^{-N} \Omega_N + \text{Re} \, r_N(\cdot; \lambda),
\]
\[
\zeta_0 = -i \Omega_0, \quad \zeta_2 = -i \Omega_2 \quad \text{and} \quad \zeta_4(\cdot; \lambda) = -i \left( \lambda^{-4} \Omega_4 + \ldots + \lambda^{-N+1} \Omega_{N-1} \right) + \text{Im} \, r_N(\cdot; \lambda).
\]

If \( N \) is even, then there is a similar array.

Because of the explicit formula (3.40) for \( \Omega_n \) (with \( h_n = -i \beta \xi \Omega_{n-1} \), with \( c_n = -\frac{1}{2} J(h_n) \) and with the operator \( G \) described by Lemmas 3.3 and 3.4), enough may have been said about \( \Omega_n \) to justify the claims made for \( \zeta_0 \) to \( \zeta_3 \) in Theorem 3.5. For example, the result
\[
\zeta_m(\sigma) = O(\sigma^{2m+4} \sigma^{-2}) \quad \text{as} \quad \sigma \uparrow \infty
\]
follows for \( m = 0 \) from \( \beta(\sigma) \sim \sigma^2 \) and from the overestimate \( g(\sigma) = O(\sigma^2 \sigma^{-\sigma^2}) \), which imply that \( h_0 \) and \( \Omega_0 \) are \( O(\sigma^4 \sigma^{-\sigma^2}) \). Then repeated use of \( h_{n+1} = -i \beta \xi \Omega_n \) leads to (3.42).

On the other hand, the remainder \( r_N \) requires further discussion. Under the transformations
\[
r_N(\sigma) = e^{-\sigma^2/2} p_N(\sigma) = e^{-\sigma^2/2} \{ p_{N,\epsilon}(\sigma) + ip_{N,\xi}(\sigma) \},
\]
\[
p_N(\sigma, \theta) := p_{N,\epsilon}(\sigma) \cos \theta + p_{N,\xi}(\sigma) \sin \theta,
\]
equation (3.33) becomes
\[
-(\Delta_\sigma - \sigma^2) p_N + \frac{\lambda}{\beta(\sigma)} \frac{\partial}{\partial \theta} p_N - 4 \lambda e^{-\sigma^2/2} T(e^{-\sigma^2/2} p_N) = \lambda^{-N} e^{\sigma^2/2} f_N(\sigma, \theta),
\]
where
\[
f_N(\sigma, \theta) = \text{Re} \left\{ (E \Omega_N)(\sigma) e^{-i\theta} \right\}.
\]
The operator on the left-hand side of (3.45) is that in (2.6); as in §2, it follows that equation (3.45) has a unique weak solution bounded by

$$\|p_N\|_Z \leq \frac{\kappa}{a} A_N \lambda^{-N},$$

(3.47)

where $A_N$ depends only on $N$.

Choosing the test function in the definition of weak solution as in (2.18), we obtain the equation

$$P_N(\rho) = \lambda^{-N} \int_0^\infty K(\rho, \sigma) e^{\sigma^2/2} (\mathcal{E} \Omega_N)(\sigma) \sigma d\sigma + i\lambda \int_0^\infty K(\rho, \sigma) \left( \frac{P_N(\sigma)}{\beta(\sigma)} - 4e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} P_N) \right) \sigma d\sigma.$$  

(3.48)

This leads without difficulty to a pointwise solution $P_N \in C^2[0, \infty)$ such that $P_N(0) = 0$ and such that $P_N(\sigma) \to 0$ as $\sigma \uparrow \infty$. (Correspondingly, $r_N(\sigma)$ is $o(\exp(-\sigma^2/2))$ as $\sigma \uparrow \infty$.) Equation (3.48) and the bound (3.47) now imply that $P_N(\sigma; \lambda)$ is $O(\lambda^{-N+1})$ uniformly over $\sigma \in [0, \infty)$. Moreover, the equation

$$- (\Delta_1 - \sigma^2) P_N - \frac{i\lambda}{\beta(\sigma)} P_N + 4i\lambda e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} P_N) = \lambda - N e^{\sigma^2/2} \mathcal{E} \Omega_N,$$

may be written as

$$P_N'' = - \frac{1}{\sigma} P_N' + \left( \frac{1}{\sigma^2} + \sigma^2 \right) P_N - ... - \lambda^{-N} e^{\sigma^2/2} \mathcal{E} \Omega_N.$$

The right-hand member of this is in $C^1[k, \infty)$ for any $k > 0$, say in $C^1[1, \infty)$. Therefore $P_N' \in C^1[1, \infty)$. Repetition of this step shows that $P_N \in C^\infty[1, \infty)$.

It remains to prove that $P_N$ is better than $C^2$ at and near the origin. We return to equation (3.33) for $r_N$ and to the equation

$$\mathcal{E} \Omega + i\lambda \frac{\beta(\sigma)}{\beta'(\sigma)} \Omega - 4i\lambda e^{-\sigma^2} T_1 \Omega = -\frac{\kappa \lambda}{\pi a} g$$

for $\Omega = \tilde{\omega}_1 + i\tilde{\omega}_2$; our final lemma applies to both $r_N$ and $\Omega$.

**Lemma 3.8.** Assume that the equation

$$\mathcal{E} u + i\lambda \frac{\beta(\sigma)}{\beta'(\sigma)} u - 4i\lambda e^{-\sigma^2} T_1 u = \lambda f$$

(3.49)

has a solution $u \in C^2[0, \infty)$ such that $u(0) = 0$ and such that $u(\sigma)$ is $o(\exp(-\sigma^2/2))$ as $\sigma \uparrow \infty$. Assume also that $u$ is unique because it is the transformed version of a solution in the Hilbert space $Z$.

Then $u$ is satisfactory on $[0, (2\pi)^{1/2})$ whenever $f$ has this property.

**Proof.** We shall prove that there are coefficients $a_n$ such that

$$u(\sigma) = \sum_{n=0}^{\infty} a_n \sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < b$$

if $b \in (0, (2\pi)^{1/2})$. Here we are not constructing a series solution *ab initio* in the usual way; rather, we are establishing a regularity property of a known,
unique solution. Therefore we may regard $a_0 = u'(0)$ and $\int_0^\infty u d\sigma$ as known; we proceed to calculate the other coefficients in terms of these. The equation is satisfied, subject to convergence of the series, if for $n = 0, 1, 2, \ldots$

$$4(n+1)(n+2)a_{n+1} = -4(n+1)a_n - \sum_{j=0}^n (B_{n-j}a_j + A_{n-j}\tau_j) + \lambda f_n, \quad (3.50)$$

where

$$B_m = \frac{i\lambda(-1)^m}{(m+1)!}, \quad A_m = \frac{4i\lambda(-1)^{m+1}}{m!},$$

$$\tau_0 = \frac{1}{2} \int_0^\infty u d\sigma, \quad \tau_m = \frac{1}{4} \frac{a_{m-1}}{m(m+1)} \text{ for } m \geq 1,$$

and

$$f(\sigma) = \sum_{n=0}^{\infty} f_n \sigma^{2n+1} \text{ for } 0 \leq \sigma < (2\pi)^{1/2}.$$ 

Hence there is a constant $C = C(b)$ such that $|f_n| \leq Cb^{-2n}$.

Now, for every $p \in \{1, 2, 3, \ldots\}$ there is a number $\Gamma_p = \Gamma_p(b, \lambda)$ such that $|a_n| \leq \Gamma_p b^{-2n}$ for $n = 0, 1, \ldots, p$. We may suppose that $\Gamma_p \geq \kappa/\alpha$. Then (3.50) implies that

$$|a_{n+1}| \leq \Gamma_p b^{-2n-2}\phi(n, \lambda) \text{ for } n \leq p,$$

where

$$\phi(n, \lambda) := \frac{2\pi}{4(n+1)(n+2)} \left(4(n+1) + \lambda(1+\pi)e^{2\pi} + \frac{4\lambda\tau_0(2\pi)^n}{n!} + \lambda C\right).$$

For fixed $\lambda$, we choose $p$ so large that $\phi(p, \lambda) \leq 1$ and so large that $\phi(n, \lambda)$ decreases for $n \geq p$. Then $|a_m| \leq \Gamma_p b^{-2m}$ not merely for $m \leq p$, but also for $m \geq p + 1$. \hfill \Box

References

