A stability/instability trichotomy for non-negative Lur’e systems

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Abstract—We identify a stability/instability trichotomy for a class of non-negative continuous-time Lur’e systems. Asymptotic as well as input-to-state stability concepts (ISS) are considered. The presented trichotomy rests on Perron-Frobenius theory, absolute stability theory and recent ISS results for Lur’e systems.

I. INTRODUCTION

Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) and \( b, c \in \mathbb{R}^n \) and consider the corresponding single-input single-output non-negative linear system

\[
\dot{x} = Ax + bu, \quad x(0) = \xi \in \mathbb{R}_+^n; \quad y = c^T x. \tag{1}
\]

We assume that

(A1) \( A \) is Metzler, \( b, c \in \mathbb{R}_+^n \) and \( b, c \neq 0 \) holds.

We recall that \( A = (a_{ij}) \) is Metzler if \( a_{ij} \geq 0 \) for \( i \neq j \) (all off-diagonal elements are non-negative).

System (1) is said to be non-negative if (A1) holds and \( u \geq 0 \). Non-negative systems of the form (1) occur naturally in biological, ecological and economic contexts.

We impose the following assumptions.

(A2) \( A \) is Hurwitz.

(A3) There exist non-negative numbers \( \alpha \) and \( \kappa \) such that \( \alpha I + A + \kappa bc^T \) is primitive.

Recall that (A3) means that the matrix \( (\alpha I + A + \kappa bc^T)^k \) is a positive matrix for some \( k \in \mathbb{N} \).

In the following, let \( G \) denote the transfer function of (1), that is, \( G(s) := c^T(sI - A)^{-1}b \).

Lemma 1.1: Assume that (A1)-(A3) hold. Then \( G(0) > 0 \) and \( \|G\|_{\infty} = G(0) \).

A proof of Lemma 1.1 can be found in [1].

Applying nonlinear non-negative feedback \( u = f(y) \) to (1), where \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) is locally Lipschitz, leads to the following non-negative Lur’e system

\[
\dot{x} = Ax + bf(c^T x), \quad x(0) = \xi \in \mathbb{R}_+^n. \tag{2}
\]

We assume that the following assumption holds.

(A4) \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) is locally Lipschitz and \( f(0) = 0 \).

Whilst absolute stability of Lur’e systems is a classical topic in control theory (see, for example, [2], [3], [8]), it seems that non-negative Lur’e systems have not received much attention (see however [7] which provides an analysis of the stability properties of a class of non-negative discrete-time Lur’e systems).

Assuming that (A1)-(A4) hold, we set

\[
p := \frac{1}{\|G(0)\|},
\]

and consider the following three cases.

Case 1. \( f(z)/z \leq p \) for all \( z > 0 \).

Case 2. \( \inf_{z > 0} f(z)/z > p \).

Case 3. There exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and

\[
\left| \frac{f(z) - f(y^*)}{z - y^*} \right| \leq p \text{ for all } z > 0, z \neq y^*.
\]

The condition in Case 3 means that the graph of \( f \) is “sandwiched” between the straight lines \( l_1 \) and \( l_2 \) given by \( l_1(z) = pz \) and \( l_2(z) = 2py^* - pz \), see Figure 1.

![Fig. 1. Case 3: graph of f “sandwiched” between the lines l₁ and l₂.](image)

II. LYAPUNOV STABILITY RESULTS

In this section, we present results which describe the stability/instability properties in each of three cases, where “stability” is interpreted in the sense of Lyapunov.

Let \( x(\cdot; \xi) \) denote the unique maximally defined forward solution of (2) with maximal interval of existence \([0, \omega_\xi]\), where \( 0 < \omega_\xi \leq \infty \).

The proposition below relates to Case 1. It follows from well known results in absolute stability theory, see, for example, [3].

Proposition 2.1: Assume that (A1)-(A4) hold.

(a) If \( f(z)/z \leq p \) for all \( z > 0 \), then the equilibrium 0 is stable in the large in the sense that there exists \( \Gamma \geq 1 \) such that, for every \( \xi \in \mathbb{R}_+^n \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi)\| \leq \Gamma \|\xi\| \quad \forall t \geq 0.
\]
(b) If \( f(z)/z < p \) for all \( z > 0 \), then the equilibrium 0 is globally asymptotically stable. In particular, for every \( \xi \in \mathbb{R}^n_+ \), \( \omega_\xi = \infty \) and \( x(t; \xi) \to 0 \) as \( t \to \infty \).

(c) If \( \sup_{z>0} f(z)/z < p \), then the equilibrium 0 is globally exponentially stable, that is, there exist \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in \mathbb{R}^n_+ \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi)\| \leq N e^{-\nu t}\|\xi\| \quad \forall t \geq 0.
\]

In Case 2, the solutions of (2) diverge to \( \infty \) for every non-zero initial condition. More precisely, we have the following result.

**Theorem 2.2:** Assume that (A1)-(A4) hold and let \( s > 0 \). Let \( \xi \in \mathbb{R}^n_+ \), \( \xi \neq 0 \), be such that the solution \( x(t; \xi) \) exists for all \( t \geq 0 \). Then

\[
\lim_{t \to \infty} x_i(t; \xi) = \infty \quad \forall i \in \{1, \ldots, n\},
\]

where \( x_i(\cdot; \xi) \) denotes the \( i \)-th component of \( x(\cdot; \xi) \).

We proceed to consider Case 3.

**Theorem 2.3:** Assume that (A1)-(A4) hold.

(a) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and

\[
|f(z) - f(y^*)| < p \quad \forall z > 0, \ z \neq y^*
\]

then \( x^* = -pA^{-1}by^* \) \( \in \mathbb{R}^n_+ \) is an equilibrium of (2) and \( x^* \) is stable in the large in the sense that there exists \( \Gamma \geq 1 \) such that, for every \( \xi \in \mathbb{R}^n_+ \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi) - x^*\| \leq \Gamma\|\xi - x^*\| \quad \forall t \geq 0.
\]

(b) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and

\[
|f(z) - f(y^*)| < p \quad \forall z > 0, \ z \neq y^*
\]

then \( 0, x^* = -pA^{-1}by^* \) \( \in \mathbb{R}^n_+ \) are the only equilibria of (2) and \( x^* \) is globally asymptotically stable in the sense that \( x^* \) is stable in the large (see statement (a) of this theorem) and, for every \( \xi \in \mathbb{R}^n_+ \) such that \( \xi \neq 0 \), \( \omega_\xi = \infty \) and \( x(t; \xi) \to x^* \) as \( t \to \infty \).

(c) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \),

\[
|f(z) - f(y^*)| < p \quad \forall z > 0, \ z \neq y^*
\]

and

\[
\lim sup_{y \to y^*} \frac{|f(z) - f(y^*)|}{y - y^*} < p,
\]

and if

\[
\lim inf_{z \to 0} \frac{f(z)}{z} > p,
\]

then \( 0, x^* = -pA^{-1}by^* \) \( \in \mathbb{R}^n_+ \) are the only equilibria of (2) and \( x^* \) is “semi-globally” exponentially stable in the sense that, for every compact set \( K \subset \mathbb{R}^n_+ \) with \( 0 \notin K \), there exists \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in K \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi) - x^*\| \leq Ne^{-\nu t}\|\xi - x^*\| \quad \forall t \geq 0.
\]
then the equilibrium $0$ of the unforced Lur’e system (2) is ISS in the sense that there exist $\psi \in K\mathcal{L}$ and $\varphi \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}_+^n$ and all non-negative $d \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, $x(\cdot;\xi,d)$ is defined on $\mathbb{R}_+$ and
\[
\|x(t;\xi,d)\| \leq \psi(||\xi||,t) + \varphi(||d||_{L^\infty(0,t)}) \quad \forall t \geq 0.
\]
The following theorem shows that, under suitable assumptions, the equilibrium $x^*$ has stability properties which are similar to ISS.

**Theorem 3.2:** Assume that (A1)-(A4) hold and that there exists $y^* > 0$ such that $f(y^*) = py^*$ and
\[
\left|\frac{f(z) - f(y^*)}{z - y^*}\right| < p \quad \forall z > 0, z \neq y^*, \quad (6)
\]
Furthermore, assume that (3) holds and
\[
pz - f(z) \to \infty \quad \text{as} \quad z \to \infty. \quad (7)
\]
Then $0$ and $x^* = -pA^{-1}by^* \in \mathbb{R}_+^n$ are the only equilibria of the unforced Lur’e system (2) and $x^*$ is "quasi ISS" in the sense that, for every $\delta > 0$, there exist $\psi \in K\mathcal{L}$ and $\varphi \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}_+^n$ with $||\xi|| \geq \delta$ and all non-negative $d \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, $x(\cdot;\xi,d)$ is defined on $\mathbb{R}_+$ and
\[
\|x(t;\xi,d) - x^*\| \leq \psi(||\xi - x^*||,t) + \varphi(||d||_{L^\infty(0,t)}) \quad \forall t \geq 0.
\]

To relate the conditions (6) and (7) to those in Proposition 3.1, we note that if (6) and (7) hold, then, for every $\varepsilon > 0$, there exists $\rho \in K_{\infty}$ such that
\[
|f(z) - f(y^*)| \leq p|z - y^*| - \rho(|z - y^*|) \quad \forall z \geq \varepsilon, \quad z \neq y^*.
\]
The proof of Theorem 3.2 is based on Proposition 3.1 and the following lemma.

**Lemma 3.3:** Assume that (A1)-(A4) hold. If (3) is satisfied and there exists $y^* > 0$ such that $f(y^*) = py^*$ and (6) holds, then, for every $\delta > 0$, there exist constants $\eta > 0$ and $\tau \geq 0$ such that for all $\xi \in \mathbb{R}_+^n$ with $||\xi|| \geq \delta$ and all non-negative $d \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, $x(\cdot;\xi,d)$ is defined on $\mathbb{R}_+$ and
\[
c^T x(t;\xi,d) \geq \eta \quad \forall t \geq \tau.
\]
This lemma also plays a key role in the proof of statements (b)-(d) of Theorem 2.3 (with disturbance $d = 0$). Detailed proofs of Proposition 3.1, Theorem 3.2 and Lemma 3.3 can be found in [1].

Finally, it follows from Proposition 2.4 that “global” ISS of $x^*$ (in the sense that there exist $\psi \in K\mathcal{L}$ and $\varphi \in \mathcal{K}$ such that (8) is satisfied for all $\xi \in \mathbb{R}_+^n$ with $\xi \neq 0$ and all non-negative $d \in L^\infty_{\text{loc}}(\mathbb{R}_+)$) does not hold.

**References**


