Structure and rigidity of functions in $\mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ with gradients taking only 3 values

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Abstract

Consider a function $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that ∇u takes values in a fixed set of 3 vectors almost everywhere. This condition implies that u is piecewise affine away from a closed set of vanishing 1-dimensional Hausdorff measure. Furthermore, there is some rigidity in the sense that away from the exceptional set, small perturbations of u will result only in controllable changes of the structure.

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1 Introduction

The space $\mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ comprises all locally integrable functions $u \colon \mathbb{R}^2 \to \mathbb{R}$ that have a weak gradient ∇u and a distributional Hessian represented by an $\mathbb{R}^{2\times 2}$-valued Radon measure $D\nabla u$. The first main result of this paper may be formulated as follows.

Theorem. *Suppose that $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ is a function such that $\nabla u(x)$ belongs to a fixed set of 3 vectors for almost every $x \in \mathbb{R}^2$. Then there exists a closed set $\Sigma \subseteq \mathbb{R}^2$ of vanishing 1-dimensional Hausdorff measure such that u is piecewise affine in $\mathbb{R}^2 \setminus \Sigma$.*

We also have a rigidity result, which may informally be summarised as follows: if $\Omega \subseteq \mathbb{R}^2 \setminus \Sigma$ is an open set and v is another function satisfying the assumptions of the above theorem (for the same three vectors) and is a perturbation of u in the sense that $\|u - v\|_{L^1(\Omega)}$ and $\|D\nabla v\|(\Omega) - \|D\nabla u\|(\Omega)$ are both small enough, then the structure of v does not differ very much from the structure of u. Changes in structure are possible near specific points (in many situations, these are the local minima and maxima of u), but away from these points one can recover u from v, up to constants, by local translations in $\mathbb{R}^2$. The rigorous statement of this result is somewhat technical, and there are some subtleties to consider, owing to the different behaviour near certain points. Therefore, we will formulate it later, after introducing the necessary notation.

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The study of functions as described here is motivated by variational problems modelling the surface energy of nanocrystals and involving expressions such as

$$\int_\Omega \left(\epsilon |\Delta u|^2 + \frac{1}{\epsilon} W(\nabla u) \right) dx$$

for $u\colon \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^2$ is an open domain and $W\colon \mathbb{R}^2 \to \mathbb{R}$ is a three-well potential. If we take the limit $\epsilon \to 0$, this gives rise to a Modica-Mortola type theory and under suitable assumptions, we expect a limiting problem involving functions in $\mathrm{BV}^2_{\mathrm{loc}}(\Omega)$, the gradients of which take values in one of the wells of W almost everywhere. We give a further discussion at the end of the introduction.

When proving a statement such as the above theorem, we may assume without loss of generality that ∇u coincides almost everywhere with one of three *specific* vectors. It is convenient to choose

$$\alpha^1 = \left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}} \right), \quad \alpha^2 = \left(-\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}} \right), \quad \text{and} \quad \alpha^3 = (0, -\sqrt{2}),$$

all of length $\sqrt{2}$ with angle $2\pi/3$ between any two of them. If there is a different set $B = \{\beta^1, \beta^2, \beta^3\}$ such that the function $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ satisfies $\nabla u \in B$ almost everywhere, and if $\beta^1, \beta^2, \beta^3$ are affinely independent, then we may replace u by the function $v\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $v(x) = u(A^T x) + c \cdot x$, where $A \in \mathbb{R}^{2\times 2}$ and $c \in \mathbb{R}^2$ are determined by the equations

$$A(\beta^1 - \beta^3) = \alpha^1 - \alpha^3, \quad A(\beta^2 - \beta^3) = \alpha^2 - \alpha^3, \quad c = \alpha^3 - A\beta^3.$$

If $\beta^1, \beta^2, \beta^3$ are affinely dependent, then the statement of the theorem may be reduced to a question about functions in one variable and is easy to prove. We therefore do not discuss that situation here.

Thus we define the set $\mathcal{A}$, comprising all $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that $\nabla u(x) \in \{\alpha^1, \alpha^2, \alpha^3\}$ for almost all $x \in \mathbb{R}^2$. Henceforth, we discuss functions in $\mathcal{A}$ only.

There is a simple method to construct functions in $\mathcal{A}$, which gives a number of examples that are worth keeping in mind. For $j = 1, 2, 3$, let $\lambda_j\colon \mathbb{R}^2 \to \mathbb{R}$ denote the function with $\lambda_j(x) = \alpha^j \cdot x$ for $x \in \mathbb{R}^2$. (We will use this notation throughout the paper.) Let $\mathcal{L}$ denote the space of all functions of the form $\lambda_j + b$ for some $b \in \mathbb{R}$ and $j = 1, 2, 3$. Then clearly $\mathcal{L} \subseteq \mathcal{A}$. Combining functions from $\mathcal{L}$ with the operations $\wedge$ and $\vee$, where $(u \wedge v)(x) = \min\{u(x), v(x)\}$ and $(u \vee v)(x) = \max\{u(x), v(x)\}$ for any pair of functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ and any $x \in \mathbb{R}^2$, and allowing recursive (but for the moment finite) combinations, we obtain other functions in $\mathcal{A}$. These are easy to illustrate, as the sets $(\nabla u)^{-1}(\{\alpha^j\})$ will be (possibly infinitely extended) polygons in $\mathbb{R}^2$ for any such function u. An example is given in Fig. 1. When proving the above theorem, we answer the question to what extent this picture is generic. In order to formulate the statement more precisely, we use the following notation.

Definition 1. *Given $u \in \mathcal{A}$, define the following subsets of $\mathbb{R}^2$.*

1. *The set $\mathcal{F}(u)$ comprises all $x \in \mathbb{R}^2$ such that u coincides with a function from $\mathcal{L}$ in some neighbourhood of x.*
2. *The sets $\mathcal{E}^\wedge(u)$ and $\mathcal{E}^\vee(u)$ comprise all $x \in \mathbb{R}^2 \setminus \mathcal{F}(u)$ such that there exist $u_1, u_2 \in \mathcal{L}$ with $u = u_1 \wedge u_2$ or $u = u_1 \vee u_2$, respectively, in some neighbourhood of x. Furthermore, $\mathcal{E}(u) = \mathcal{E}^\wedge(u) \cup \mathcal{E}^\vee(u)$.*

Figure 1: A function $u \in \mathcal{A}$ represented in terms of the preimages of ∇u. Here α^1, α^2, and α^3 correspond to black, grey, and white regions, respectively.

3. *The sets* $\mathcal{V}^\wedge(u)$, $\mathcal{V}^\vee(u)$, $\mathcal{V}^{\wedge\vee}(u)$, *and* $\mathcal{V}^{\vee\wedge}(u)$ *comprise all* $x \in \mathbb{R}^2 \setminus (\mathcal{F}(u) \cup \mathcal{E}(u))$ *such that there exist* $u_1, u_2, u_3 \in \mathcal{L}$ *with* $u = u_1 \wedge u_2 \wedge u_3$ *or* $u = u_1 \vee u_2 \vee u_3$ *or* $u = (u_1 \wedge u_2) \vee u_3$ *or* $u = (u_1 \vee u_2) \wedge u_3$, *respectively, in some neighbourhood of* x. *Furthermore,* $\mathcal{V} = \mathcal{V}^\wedge(u) \cup \mathcal{V}^\vee(u) \cup \mathcal{V}^{\wedge\vee}(u) \cup \mathcal{V}^{\vee\wedge}(u)$.
4. *The set* $\mathcal{R}(u) = \mathcal{F}(u) \cup \mathcal{E}(u) \cup \mathcal{V}(u)$ *is called the* regular set *of* u.
5. *The set* $\mathcal{S}(u) = \mathbb{R}^2 \setminus \mathcal{R}(u)$ *is called the* singular set *of* u.

It is clear that $\mathcal{R}(u)$ is always open and $\mathcal{S}(u)$ is closed.

We use the symbol $\mathcal{H}^d$ to denote the d-dimensional Hausdorff measure, and dim will always denote the Hausdorff dimension. Then the above theorem is a consequence of the following.

Theorem 2. *If* $u \in \mathcal{A}$, *then* $\mathcal{H}^1(\mathcal{S}(u)) = 0$.

The result is sharp.

Proposition 3. *There exists* $v \in \mathcal{A}$ *such that* $\dim \mathcal{S}(v) = 1$.

The result breaks down if we allow *four* different values for the gradient.

Proposition 4. *For any* $\epsilon > 0$ *there exists a function* $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ *such that* $\nabla u(x) \in \{0, \alpha^1, \alpha^2, \alpha^3\}$ *for almost all* $x \in \mathbb{R}^2$ *and the set*

$$R = \{x \in \mathbb{R}^2 \colon u \text{ is piecewise affine in a neighbourhood of } x\}$$

satisfies $\mathcal{H}^2(R) < \epsilon$.

The construction in the proof of Proposition 4 relies on the fact that 0 belongs to the interior of the convex hull of $\{\alpha^1, \alpha^2, \alpha^2\}$. We give another example to show that Theorem 2 cannot be extended to the case of four gradient vectors, even if none of them belongs to the convex hull of the other three.

Proposition 5. *There exists a function* $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ *such that* $\nabla u(x) \in \{(1,0), (0,1), (-1,0), (0,-1)\}$ *for almost all* $x \in \mathbb{R}^2$ *and the set*

$$S = \{x \in \mathbb{R}^2 \colon u \text{ is not piecewise affine in any neighbourhood of } x\}$$

satisfies $\mathcal{H}^1(S) = \infty$.

We also study the rigidity of the structure implied by Theorem 2. Away from the singular set $\mathcal{S}(u)$, a function $u \in \mathcal{A}$ is given by a locally finite number of affine pieces. If we consider $v \in \mathcal{A}$ near u, to what extent are these affine pieces, and the way they are joined together, preserved? (Here we measure proximity in the local L^1-sense as well as in terms of $\|D\nabla u\|$; see Theorem 7 for details.) In part, this is a question about topological information. The interfaces where different affine pieces meet form a graph, and we are interested in possible changes of the topology of this graph.

It is easy to see that changes can occur, even away from the singular set. For example, if $u_0 = \lambda_1 \vee \lambda_2 \vee \lambda_3$ and $\tilde{u} = \lambda_1 \wedge \lambda_2 \wedge \lambda_3$, then the functions $u_t = u_0 \vee (\tilde{u} + t)$, for $t \geq 0$, give a continuous deformation of u_0 in a suitable topology, and the structure of u_0 is different from the structure of u_t for any $t > 0$ (see Fig. 2b below for an illustration). We will show, however, that this is the only way by which a change of topology is possible away from $\mathcal{S}(u)$, and moreover, it can only happen when the measure $\|D\nabla u\|$ increases nearby.

In order to determine how similar the structure of two functions is in a set $K \subseteq \mathbb{R}^2$, we fix $R > 0$ and introduce a pseudometric d_K^R as follows. (The fact that d_K^R is a pseudometric is not important, but we will prove it anyway in Proposition 32 below.) We use the notation $B_r(x)$ to denote the open ball in $\mathbb{R}^2$ of radius $r > 0$ centred at $x \in \mathbb{R}^2$.

Definition 6. *Let $K \subseteq \mathbb{R}^2$ and $R > 0$. For two functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ and for $\rho > 0$, let $\Delta_\rho^R(u, v)$ denote the set of all $x \in \mathbb{R}^2$ such that there exist $a \in B_\rho(0)$ and $b \in (-\rho, \rho)$ such that for all $y \in B_{R-\rho}(x)$,*

$$u(y) = v(y + a) + b \quad \text{and} \quad v(y) = u(y - a) - b.$$

Moreover, let

$$d_K^R(u, v) = \inf \left\{ \rho > 0\colon K \subseteq \Delta_\rho^R(u, v) \right\}.$$

Thus if $d_K^R(u, v)$ is small, then up to small constants, we may locally (but on quite large balls) obtain one of the functions from the other by small translations in the domain. If K is compact and u and v are piecewise affine near K such that $d_K^R(u, v)$ is sufficiently small, then this implies in particular that the two functions have the same structure near K.

Given $u \in \mathcal{A}$ and $\epsilon > 0$, and given an open set $\Omega \subseteq \mathbb{R}^2$, we also define

$$\mathcal{B}_\epsilon(u; \Omega) = \left\{ v \in \mathcal{A}\colon \|u - v\|_{L^1(\Omega)} \leq \epsilon,\ \|D\nabla v\|(\Omega) \leq \|D\nabla u\|(\Omega) + \epsilon \right\}.$$

We think of $\mathcal{B}_\epsilon(u; \Omega)$ as comprising small perturbations of u in Ω. This notion is related to the strict topology on the space of functions of bounded variation.

Theorem 7. *Let $u \in \mathcal{A}$ and suppose that $\Omega \subseteq \mathcal{R}(u)$ is an open set and $K \subseteq \Omega$ is compact. Furthermore, let $R_0 > 0$. Then there exists $R > 0$ such that for any $r > 0$ there exist $\epsilon > 0$ and a map $P\colon \mathcal{B}_\epsilon(u; \Omega) \to \mathcal{A}$ such that for all $v \in \mathcal{B}_\epsilon(u; \Omega)$,*

1. *$d_K^R(u, P(v)) \leq r$,*
2. *$P(v) = v$ on $\{x \in K\colon \operatorname{dist}(x, \mathcal{V}^\wedge(u) \cup \mathcal{V}^\vee(u)) \geq R_0\}$, and*
3. *$\|D\nabla P(v)\|(\Omega) < \|D\nabla v\|(\Omega)$ unless $P(v) = v$.*

In other words, while functions in $\mathcal{B}_\epsilon(u;\Omega)$ need not have the same structure as u in K, if ϵ is chosen sufficiently small, then changes in structure can occur only near $\mathcal{V}^\wedge(u)$ or $\mathcal{V}^\vee(u)$ and will necessarily increase the quantity $\|D\nabla v\|(\Omega)$. We may reverse these structural changes and obtain a map near u in the pseudometric d_K^R.

Of course, if $\mathcal{V}^\wedge(u)$ and $\mathcal{V}^\vee(u)$ do not intersect Ω, then the statement becomes simpler.

Corollary 8. *Suppose that $u \in \mathcal{A}$. Let $\Omega \subseteq \mathcal{R}(u) \setminus (\mathcal{V}^\wedge(u) \cup \mathcal{V}^\vee(u))$ be an open set and $K \subseteq \Omega$ compact. Then there exists $R > 0$ such that for any $r > 0$, there exists $\epsilon > 0$ such that any $v \in \mathcal{B}_\epsilon(u;\Omega)$ will satisfy the inequality $d_K^R(u,v) \leq r$.*

We conclude the introduction with a few further remarks about the motivation for the questions studied here. As mentioned previously, the set $\mathcal{A}$ arises in the context of a variational model for the surface energy of nanocrystals [21, 10, 19, 13, 9]. After a small-slope approximation, this model gives rise to a functional such as

$$\int_\Omega \left(\epsilon|\Delta u|^2 + \frac{W(\nabla u)}{\epsilon}\right) dx,$$

defined for functions u in the Sobolev space $W^{2,2}(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$ is an open domain and $W\colon \mathbb{R}^2 \to \mathbb{R}$ is a multi-well potential, for example [22]

$$W(p) = \frac{1}{6}(|p|^4 - |p|^2) + \frac{1}{9}(p_2^3 - 3p_1^2 p_2 + 1).$$

This specific function has zeros exactly at the points $\alpha^1/\sqrt{2}, \alpha^2/\sqrt{2}, \alpha^3/\sqrt{2}$ and is positive elsewhere. Depending on the boundary conditions, we may replace the Laplacian Δu by the Hessian $\nabla^2 u$ after some integrations by parts. If we study the behaviour of the resulting functionals as $\epsilon \to 0$, then we have a variant of the theory of Modica–Mortola [11, 12], but for gradients. It is easy to see that in the limit, we will have a function $u\colon \Omega \to \mathbb{R}$ satisfying conditions similar to those in the definition of $\mathcal{A}$.

There is a rich theory of similar problems for potentials with zeros on the unit sphere, initiated by Aviles–Giga [2], and problems for vector-valued functions (but potentials with isolated zeros) have also attracted some attention in the literature [7]. In both cases, however, the interesting phenomena are different in nature to what we study here.

Theorem 2 shows that away from a closed singular set of vanishing $\mathcal{H}^1$-measure (which may be negligible for certain questions), the function u is piecewise affine. Moreover, by Theorem 7, as long as we remain near u in the appropriate sense, and as long as we're prepared to apply the map P when necessary in order to reverse any structural changes, we essentially remain in a finite-dimensional space. This observation has the potential to simplify the analysis of the problem tremendously. It may also help for numerical computations; indeed, some algorithms for evolution problems have been formulated by Norris–Watson [16, 17] under the assumption that the underlying functions are piecewise affine and no uncontrolled changes of the structure occur.

Since the motivation for our theory comes from crystal surfaces, it is natural to ask whether there are similar results for surfaces, say in $\mathbb{R}^3$, which have a distributional second fundamental form given by a Radon measure and normal vectors in a fixed finite set almost everywhere. In order to obtain a genuine extension of the above, we need to allow at least four different normal vectors, for example with a tetrahedral symmetry. An easy modification of the example illustrated by Fig. 3 below, however, shows that no such results can be expected. (A more extreme example, with cubic symmetry but easily adapted to tetrahedral symmetry, is discussed in a previous paper [15].) This is in contrast to the one-dimensional situation, which allows a generalisation of the standard results for functions $\mathbb{R} \to \mathbb{R}$ to curves in $\mathbb{R}^2$ [3, 4, 5].

2 Counterexamples

We begin with the proofs of Propositions 3, 4, and 5, because the arguments used here are less technical than the proofs of Theorem 2 and Theorem 7.

Proof of Proposition 3. We first construct a function $u \in \mathcal{A}$, the singular set of which is not of dimension 1, but can be made arbitrarily close.

To this end, fix $\sigma \in (0, \frac{1}{4})$ and consider the functions $u_0 = \lambda_1 \vee \lambda_2 \vee \lambda_3$ and $\hat{u}_0 = \lambda_1 \wedge \lambda_2 \wedge \lambda_3$ in $\mathcal{A}$. Then u_0 is the function represented by Fig. 2a. Next we define $\hat{u}_1 = \hat{u}_0 + \frac{3\sigma}{\sqrt{2}}$ and $u_1 = u_0 \vee \hat{u}_1$. This is represented by Fig. 2b, and the parallelograms meeting in the centre of the picture have side length σ. In the next step, we define

$$\hat{u}_2'(x) = \hat{u}_0\left(x_1, x_2 - \sigma\right) + \frac{3\sigma^2 + \sigma}{\sqrt{2}},$$
$$\hat{u}_2''(x) = \hat{u}_0\left(x_1 + \frac{\sqrt{3}\sigma}{2}, x_2 + \frac{\sigma}{2}\right) + \frac{3\sigma^2 + \sigma}{\sqrt{2}},$$
$$\hat{u}_2'''(x) = \hat{u}_0\left(x_1 - \frac{\sqrt{3}\sigma}{2}, x_2 + \frac{\sigma}{2}\right) + \frac{3\sigma^2 + \sigma}{\sqrt{2}},$$

and $\hat{u}_2 = \hat{u}_2' \vee \hat{u}_2'' \vee \hat{u}_2'''$. Furthermore, set $\check{u}_2 = u_0 + \frac{3}{\sqrt{2}}(\sigma - \sigma^2)$. Finally, define

$$u_2 = (u_1 \vee \hat{u}_2) \wedge \check{u}_2.$$

This will give rise to a function as illustrated in Fig. 2c, with parallelograms of side length σ^2.

(a) Representiation of the initial function u_0 (b) The first step, leading to u_1 (c) The second step, leading to u_2

Figure 2: Construction of a function in $\mathcal{A}$ with self-similar singular set

Now we continue the construction recursively, with gives rise to a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$. It is easy to see that the sequence converges uniformly and we set $u = \lim_{n\to\infty} u_n$. Clearly u has a gradient in $\{\alpha^1, \alpha^2, \alpha^3\}$ almost everywhere. In order to verify that $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$, we note that for any open set $\Omega \subseteq \mathbb{R}^2$ with $B_1(0) \subseteq \Omega$, we can easily compute $\|D\nabla u_{n+1}\|(\Omega)$ in terms of $\|D\nabla u_n\|(\Omega)$. Simply counting and measuring the new jump lines for ∇u (and the disappearing jump lines), and observing that every jump is of size $\sqrt{6}$, we find that

$$\|D\nabla u_{n+1}\|(\Omega) = \|D\nabla u_n\|(\Omega) + 6\sqrt{6}\cdot 4^n\sigma^{n+1}.$$

Therefore,

$$\|D\nabla u_n\|(\Omega) = \|D\nabla u_0\|(\Omega) + 6\sqrt{6}\sigma\sum_{k=0}^{n-1}(4\sigma)^k.$$

As we have assumed that $\sigma < \frac{1}{4}$, the series on the right-hand side will converge when we let $n\to\infty$. (Incidentally, the assumption will also guarantee that there is no intersection between any two of the parallelograms introduced in a single step of the construction.) It follows that $u \in \mathcal{A}$.

The singular set $\mathcal{S}(u)$ is clearly self-similar, and so we can compute its dimension with the help of classical results. We claim that $\dim \mathcal{S}(u) = \frac{\log 4}{\log(1/\sigma)}$. In order to simplify the proof of this statement, we introduce the isometries $T_0, T_1, T_2, T_3\colon \mathbb{R}^2\to\mathbb{R}^2$, where

$$T_0(x_1,x_2) = (x_1,-x_2), \qquad T_1(x_1,x_2) = (x_1, x_2+1),$$
$$T_2(x_1,x_2) = \left(x_1 - \frac{\sqrt{3}}{2}, x_2 - \frac{1}{2}\right), \qquad T_3(x_1,x_2) = \left(x_1 - \frac{\sqrt{3}}{2}, x_2 + \frac{1}{2}\right).$$

Let $\Sigma_0 = \overline{B_{\sigma/(1-\sigma)}(0)}$ and

$$\Sigma_{n+1} = \bigcup_{j=0}^{3}\sigma T_j(\Sigma_n), \quad n = 0,1,\ldots.$$

Then outside of Σ_n, the functions $u_n, u_{n+1},\ldots$ in the construction of u will coincide for $n = 0,1,\ldots$. Therefore, the singular set $\mathcal{S}(u)$ will be contained in $\Sigma = \bigcap_{n=0}^{n}\Sigma_n$. On the other hand, near any point of Σ, the function u clearly cannot be represented by finitely many affine pieces. Hence $\mathcal{S}(u) = \Sigma$. It follows from classical results on self-similar sets [14, Theorem III] that $\dim\mathcal{S}(u) = \frac{\log 4}{\log(1/\sigma)}$.

This is not yet sufficient to prove Proposition 3. Therefore, we now combine the constructions for a sequence of numbers σ_n converging to $\frac{1}{4}$. More precisely, let $\sigma_0 = \frac{1}{8}$ and define $\sigma_{n+1} = 1/(6-8\sigma_n)$ for $n = 0,1,\ldots$. Then $\lim_{n\to\infty}\sigma_n = \frac{1}{4}$. (This is because the function $f(\sigma) = 1/(6-8\sigma)$ has a fixed point at $\frac{1}{4}$ and $\frac{2}{9} < f'(\sigma) < \frac{1}{2}$ for $0 < \sigma < \frac{1}{4}$.)

Now let $v_n \in \mathcal{A}$ denote the function constructed as above with scaling factor σ_n. Furthermore, let $v_n^{(0)} = v_n$ and recursively define

$$v_n^{(k+1)}(x) = \begin{cases} v_n^{(k)}(x) & \text{if } x \notin B_{\sigma_n/2}(0,\sigma_n), \\ \sigma_n v_{n+1}^{(k)}(x_1/\sigma_n, x_2/\sigma_n - 1) + \frac{\sigma_n}{\sqrt{2}} & \text{if } x \in B_{\sigma_n/2}(0,\sigma_n). \end{cases}$$

Finally, we define $v = \lim_{k\to\infty} v_0^{(k)}$. The effect of this definition is that v will coincide with v_0 outside of $B_{\sigma_0/2}(0,\sigma_0)$, with a scaled-down copy of v_1 (with scaling factor σ_0) in

$$B_{\sigma_0/2}(0,\sigma_0) \setminus B_{\sigma_0\sigma_1/2}(0,\sigma_0+\sigma_0\sigma_1),$$

with a scaled-down copy of v_2 (with scaling factor $\sigma_0\sigma_1$) in

$$B_{\sigma_0\sigma_1/2}(0,\sigma_0+\sigma_0\sigma_1) \setminus B_{\sigma_0\sigma_1\sigma_2/2}(0,\sigma_0+\sigma_0\sigma_1+\sigma_0\sigma_1\sigma_2),$$

and so on.

With the arguments used above, we see that any open set $\Omega \subseteq \mathbb{R}^2$ will satisfy

$$\|D\nabla v\|(\Omega) \leq \|D\nabla v_0\|(\Omega) + 6\sqrt{6}\left(\frac{\sigma_0}{1-4\sigma_0} + \frac{\sigma_0\sigma_1}{1-4\sigma_1} + \frac{6\sigma_0\sigma_1\sigma_2}{1-4\sigma_2} + \cdots\right).$$

But the choice of σ_n implies that

$$\frac{\sigma_{n+1}}{1-4\sigma_{n+1}} = \frac{1}{2(1-4\sigma_n)}, \quad n = 0,1,\ldots,$$

so this is a geometric series and in particular convergent. It follows that $v \in \mathcal{A}$. By the construction, the singular set $\mathcal{S}(v)$ contains a set congruent to $\mathcal{S}(v_n)$ for each $n \in \mathbb{N}$. Since $\dim \mathcal{S}(v_n) = \frac{\log 4}{\log(1/\sigma_n)} \to 1$ as $n \to \infty$, it follows that $\dim \mathcal{S}(v) \geq 1$. Theorem 2 then implies that $\dim \mathcal{S}(v) = 1$, which concludes the proof of Proposition 3. $\square$

Proof of Proposition 4. Recall that we study functions $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ satisfying $\nabla u \in \{0,\alpha^1,\alpha^2,\alpha^3\}$ almost everywhere here.

Define $u_0 = \lambda_1 \vee \lambda_2 \vee \lambda_3 - c_0$ for some number $c_0 > 0$ and set $u = 0 \wedge u_0$. This function coincides with 0 everywhere except in an equilateral triangle of side length $\sqrt{6}c_0$ centred at 0. Setting $u_1(x) = (\lambda_1 \vee \lambda_2 \vee \lambda_3)(x-a_1) - c_1$ for some $a_1 \in \mathbb{R}^2$ and $c_1 > 0$, we may generate another function $u = 0 \wedge u_0 \wedge u_1$ with gradient in $\{0,\alpha^1,\alpha^2,\alpha^3\}$ almost everywhere, which agrees with 0 except in the union of two equilateral triangles. (This is illustrated in Fig. 3.)

We now construct a sequence of functions of the form $0 \wedge u_0 \wedge \cdots \wedge u_n$, where each u_n is defined analogously to u_0 and u_1, using $a_n \in \mathbb{R}^2$ and $c_n > 0$ such that $\sum_{n=1}^\infty c_n < \infty$ and such that the corresponding triangles are pairwise disjoint and their union is dense in $\mathbb{R}^2$ and has an area less than ϵ. Then $0 \wedge u_0 \wedge \cdots \wedge u_n \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ is piecewise affine with gradient in $\{0,\alpha^1,\alpha^2,\alpha^3\}$ almost everywhere and the limit $u = \lim_{n\to\infty} 0 \wedge u_0 \wedge \cdots \wedge u_n$ exists almost everywhere. The above conditions guarantee that u has the required properties and the set R in the statement of the proposition is the union of the interiors of the above triangles. $\square$

Proof of Proposition 5. We first define $u_0(x) = \max\{x_2, -x_2\}$. Next, we choose $a_1 \in \mathbb{R}$ and $c_1 > 0$ and set $v_1(x) = \min\{x_1 - a_1, a_1 - x_1\} + c_1$. Then the function $u_1 = u_0 \vee v_1$ has the structure illustrated in Fig. 4a. Choose two other numbers $a_2 \in \mathbb{R}$ and $c_2 > 0$ such that the intervals (a_1-c_1, a_1+c_1) and (a_2-c_2, a_2+c_2) are disjoint and set $v_2 = \min\{x_1 - a_2, a_2 - x_1\} + c_2$ and $u_2 = u_1 \vee v_2$. Then continue the same process indefinitely with suitably chosen numbers a_n and c_n such that $\sum_{n=1}^\infty c_n < \infty$ and $\bigcup_{n=1}^\infty (a_n - c_n, a_n + c_n)$ is dense in $\mathbb{R}$, while the intervals are

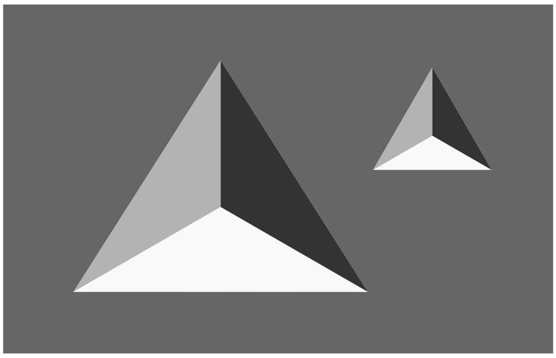
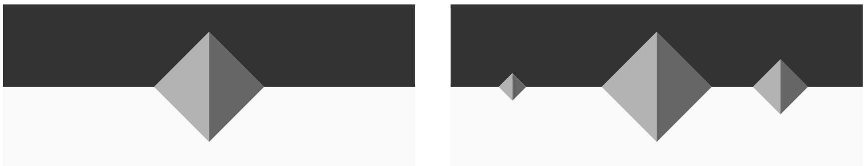


Figure 3: A function with gradient equal to 0 (dark grey), α^1 (black), α^2 (light grey), or α^3 (white) almost everywhere

pairwise disjoint. (The result after the third step is illustrated in Fig. 4b.) This gives rise to a sequence of functions u_n, and the limit $u = \lim_{n\to\infty} u_n$ exists and belongs to $\mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R})^2$. The construction guarantees that ∇u belongs to $\{(\pm 1, 0), (0, \pm 1)\}$ almost everywhere and it is easy to see that the singular set is

$$S = \left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (a_n - c_n, a_n + c_n)\right) \times \{0\},$$

which satisfies $\mathcal{H}^1(S) = \infty$. □



(a) The first step, leading to u_1

(b) The third step, leading to u_3

Figure 4: Construction of a function in $\mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R})^2$ with singular set of infinite 1-dimensional Hausdorff measure, the gradient of which takes one of the values $(1, 0)$ (light grey), $(0, 1)$ (black), $(-1, 0)$ (dark grey), or $(0, -1)$ (white) almost everywhere

3 Preliminaries

In the course of the proofs of Theorem 2 and Theorem 7, we will study the intersection of the graph of a given function $u \in \mathcal{A}$, denoted by $\mathrm{graph}(u)$, with certain planes in $\mathbb{R}^3$. We first introduce the tools to analyse the structure of the resulting curves in the plane. We begin with some notation and terminology.

Let $S^1_+ = \{a \in S^1 : a_1 > 0 \text{ and } a_2 > 0\}$. For $a \in S^1_+$ and $s \in \mathbb{R}$, define $L_s(a) = \{y \in \mathbb{R}^2 : a_1 y_2 - a_2 y_1 = s\}$. Thus fixing a and varying s, we obtain a foliation of $\mathbb{R}^2$ by lines parallel to a. We also write $L_s = L_s(1/\sqrt{2}, 1/\sqrt{2})$.

Definition 9. *We say that a set $S \subseteq \mathbb{R}^2$ is* descending *if it is closed and $S \cap L_s(a) \neq \emptyset$ for all $a \in S^1_+$ and all $s \in \mathbb{R}$. We say that S is* totally descending *if $S \cap L_s(a)$ contains exactly one point for all $a \in S^1_+$ and all $s \in \mathbb{R}$.*

We think of a descending set as stretching from the top left to the bottom right of the plane. For totally descending sets, we will give a rigorous statement related to this idea below (in Lemma 14).

We introduce a partial order on the set of all descending subsets of $\mathbb{R}^2$ as follows.

Definition 10. *Given two descending sets $S, T \subseteq \mathbb{R}^2$, we write $S \preceq T$ if for all $a \in S^1_+$ and $s \in \mathbb{R}$, the following holds true: if there exist $y \in L_s(a) \cap S$ and $z \in L_s(a) \cap T$ with $a \cdot z < a \cdot y$, then $S = T$.*

Proposition 11. *The relation $\preceq$ is a partial order on the set of all descending subsets of $\mathbb{R}^2$.*

Proof. Reflexivity is clear.

Consider two descending sets $S, T \subseteq \mathbb{R}^2$ with $S \preceq T$ and $T \preceq S$. Let $y \in S$. Choose $a \in S^1_+$ and $s \in \mathbb{R}$ such that $y \in L_s(a)$. Then, as T is descending, there exists a point $z \in T \cap L_s(a)$. If $z \neq y$, then it follows that $S = T$ (because $S \preceq T$ and $T \preceq S$). In any case, it follows that $y \in T$. This proves that $S \subseteq T$, and similarly we see that $T \subseteq S$. So $S = T$.

Finally, let S, T, U be three descending sets with $S \preceq T$ and $T \preceq U$. Let $a \in S^1_+$ and $s \in \mathbb{R}$, and suppose that $x \in S \cap L_s(a)$, $y \in T \cap L_s(a)$, and $z \in U \cap L_s(a)$. If $a \cdot z < a \cdot x$, then either $a \cdot z < a \cdot y$ or $a \cdot y < a \cdot x$; so either $S = T$ or $T = U$, and it follows that $S \preceq U$. □

In the following, we sometimes take limits of sequences of closed sets in $\mathbb{R}^2$. This is always in the sense of Hausdorff distance applied locally. That is, we have the convergence $S = \lim_{k\to\infty} S_k$ if $\operatorname{dist}((S \cap C) \cup \partial C, (S_k \cap C) \cup \partial C) \to 0$ for any non-empty, compact set $C \subseteq \mathbb{R}^2$ (where dist denotes the Hausdorff distance).

Lemma 12. *Suppose that $(S_k)_{k\in\mathbb{N}}$ and $(T_k)_{k\in\mathbb{N}}$ are two sequences of descending sets in $\mathbb{R}^2$ that converge to descending sets $S, T \subseteq \mathbb{R}^2$, repectively. If $S_k \preceq T_k$ for all $k \in \mathbb{N}$, then $S \preceq T$.*

Proof. For $a \in S^1_+$ and $s \in \mathbb{R}$, suppose that there exist $y \in L_s(a) \cap S$ and $z \in L_s(a) \cap T$ such that $a \cdot z < a \cdot y$. For every $k \in \mathbb{N}$, choose $v_k \in S_k$ and $w_k \in T_k$ such that $y = \lim_{k\to\infty} v_k$ and $z = \lim_{k\to\infty} w_k$. Let

$$a_k = \frac{v_k - w_k}{|v_k - w_k|},$$

which is well-defined and belongs to S^1_+ whenever k is sufficiently large. Moreover, it is clear that $a_k \to a$ as $k \to \infty$. Therefore, for k large enough, we have the inequality $a_k \cdot w_k < a_k \cdot v_k$. Since v_k and w_k both belong to $L_{s_k}(a_k)$ for some $s_k \in \mathbb{R}$, the hypothesis implies that $S_k = T_k$ for k large enough. Hence $S = T$. □

Lemma 13. *Suppose that $(S_k)_{k\in\mathbb{N}}$ is a sequence of descending sets in $\mathbb{R}^2$ such that $S_k \preceq S_{k+1}$ for all $k \in \mathbb{N}$. Further suppose that there exists a descending set $T \subseteq \mathbb{R}^2$ such that $S_k \preceq T$ for all $k \in \mathbb{N}$. Then $\lim_{k\to\infty} S_k$ exists and is descending.*

Proof. Since for any compact set $C \subseteq \mathbb{R}^2$, the metric space of all closed sets in C (with the Hausdorff distance) is compact, a diagonal sequence argument shows that $(S_k)_{k\in\mathbb{N}}$ has a convergent subsequence. We first prove that the limit of any subsequence must be descending. To this end, we may, without loss of generality, consider the whole sequence $(S_k)_{k\in\mathbb{N}}$ rather than a subsequence.

Suppose that $S = \lim_{k\to\infty} S_k$ exists. If $(S_k)_{k\in\mathbb{N}}$ is eventually constant, then there is nothing to prove. Otherwise, we may discard any duplicate elements and assume that $S_k \neq S_\ell$ whenever $k \neq \ell$. Now choose $a \in S^1_+$ and $s \in \mathbb{R}$. For each $k \in \mathbb{N}$, choose $y_k \in S_k \cap L_s(a)$ and set $h_k = a_k \cdot y_k$. Then $h_1 \leq h_2 \leq \dots$. If we also choose $z \in T \cap L_s(a)$ and set $m = a \cdot z$, then $h_k \leq m$ for all $k \in \mathbb{N}$. Therefore, the sequence $(h_k)_{k\in\mathbb{N}}$ is convergent, and this means that $(y_k)_{k\in\mathbb{N}}$ converges in $\mathbb{R}^2$. Let $y = \lim_{k\to\infty} y_k$. Then $y \in S \cap L_s(a)$. It follows that S is descending.

Now if S and S' are the limits of two subsequences of $(S_k)_{k\in\mathbb{N}}$, then Lemma 12 implies that $S' = S$. That is, we have only one possible limit, and convergence of the whole sequence $(S_k)_{k\in\mathbb{N}}$ follows. □

Lemma 14. *If $\Gamma \subseteq \mathbb{R}^2$ is totally descending, then there exists a Lipschitz continuous function $c\colon \mathbb{R} \to \mathbb{R}$, with Lipschitz constant 1 or less, such that*

$$\Gamma = \bigcup_{s\in\mathbb{R}} \left\{ y \in L_s \colon y_1 + y_2 = \sqrt{2}c(s) \right\}. \tag{1}$$

In other words, a totally descending set Γ is a Lipschitz graph over a line with slope -1. In particular, it is a curve in $\mathbb{R}^2$.

Proof. By the criterion from Definition 9, it is clear that there exists a function $c\colon \mathbb{R} \to \mathbb{R}$ such that Γ has the representation (1). It remains to show that c is Lipschitz continuous with Lipschitz constant 1 or less.

Let $s_0 \in \mathbb{R}$ and consider the unique point $y \in \Gamma \cap L_{s_0}$. Then $\Gamma \cap ((y_1, \infty) \times (y_2, \infty)) = \emptyset$ and $\Gamma \cap ((-\infty, y_1) \times (-\infty, y_2)) = \emptyset$. (Otherwise, the line connecting a point in one of these sets with y would violate the condition for the notion of 'totally descending' in Definition 9.) This, however, means that $|c(s) - c(s_0)| \leq |s - s_0|$ for all $s \in \mathbb{R}$. □

Finally, we need a tool of a different sort. This is an inequality involving the vectors $\alpha^1, \alpha^2, \alpha^3$.

Lemma 15. *Let $a_1, a_2, a_3 \geq 0$. If $x \in \mathbb{R}^2$ with $x \cdot \alpha^j \geq -1$ for $j = 1, 2, 3$, then*

$$\sum_{j=1}^{3} a_j \alpha_j \cdot x \geq 3 \min\{a_1, a_2, a_3\} - \sum_{j=1}^{3} a_j.$$

Proof. Consider the functions $f(x) = \sum_{j=1}^{3} a_j \alpha_j \cdot x$ and $g_j(x) = \alpha^j \cdot x$ for $j = 1, 2, 3$. Then by the linearity of f and g_j, the minimum of f subject to the constraints $g_j \geq -1$, $j = 1, 2, 3$, is attained at one of the points α^1, α^2, α^3 (the extremal points of the convex set determined by the constraints). We compute $f(\alpha^1) = 2a_1 - a_2 - a_3$, and we have similar identities for α^2 and α^3. □

4 Monotonicity

In the first few steps of the analysis of functions in $\mathcal{A}$, it is convenient to relax the condition on the gradient. We now assume that we have a function $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$\alpha^j \cdot \nabla u \geq -1 \tag{2}$$

almost everywhere for $j = 1, 2, 3$. (This means that $\nabla u(x)$ is in the triangle with corners $\alpha^1, \alpha^2, \alpha^3$ for almost every $x \in \mathbb{R}^2$.) Then u is automatically Lipschitz continuous. We assume furthermore that u is bounded. These are standing assumptions for the whole section and we will keep u with these properties fixed.

Our first observation is that (2) is equivalent to the condition that

$$u(x + s\alpha^j) \geq u(x) - s \tag{3}$$

for all $x \in \mathbb{R}^2$ and all $s \geq 0$. This is sometimes the more useful inequality, as it is satisfied everywhere, due to the continuity of u.

Since we often analyse points in $\mathbb{R}^3$ and their projections to certain planes simultaneously, the following convention is convenient.

Notation. *We use boldface symbols to denote points in $\mathbb{R}^3$, such as $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. The projection of $\boldsymbol{x}$ onto $\mathbb{R}^2 \times \{0\}$ is denoted by x. Thus $x = (x_1, x_2)$ and $\boldsymbol{x} = (x, x_3)$.*

Define the vectors

$$\boldsymbol{\nu}^1 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right), \quad \boldsymbol{\nu}^2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right), \quad \boldsymbol{\nu}^3 = \left(0, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right),$$

so that $\nu^j = -\alpha^j \nu_3^j$ for $j = 1, 2, 3$. Note that $\boldsymbol{\nu}^j$ is a normal vector to the graph of the affine function λ_j for $j = 1, 2, 3$ and $(\boldsymbol{\nu}^1, \boldsymbol{\nu}^2, \boldsymbol{\nu}^3)$ is an orthonormal basis of $\mathbb{R}^3$. (This is the reason for the specific choice of $\alpha^1, \alpha^2, \alpha^3$ above.) We often consider cyclic permutations of these three vectors. Therefore, given $j \in \mathbb{Z}$, we write $[j]$ for the corresponding equivalence class in $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ and we use the notation $\alpha^{[j]} = \alpha^j$ and $\boldsymbol{\nu}^{[j]} = \boldsymbol{\nu}^j$ for $j = 1, 2, 3$. For $i \in \mathbb{Z}_3$, we also define the rotations $\Phi_i \colon \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$\Phi_i(\boldsymbol{x}) = (\boldsymbol{x} \cdot \boldsymbol{\nu}^i, \boldsymbol{x} \cdot \boldsymbol{\nu}^{i+[1]}, \boldsymbol{x} \cdot \boldsymbol{\nu}^{i+[2]})$$

for $\boldsymbol{x} \in \mathbb{R}^3$. Thus Φ_i maps the vectors $\boldsymbol{\nu}^i, \boldsymbol{\nu}^{i+[1]}, \boldsymbol{\nu}^{i+[2]}$ to the standard basis vectors in $\mathbb{R}^3$.

Next, for every given $i \in \mathbb{Z}_3$, we define two functions $\overline{g}_i, \underline{g}_i \colon \mathbb{R}^2 \to \mathbb{R}$ that give some information about the intersection of graph(u) with lines parallel to $\boldsymbol{\nu}^i$. Namely, for $y \in \mathbb{R}^2$, we define

$$\overline{g}_i(y) = \inf\left\{t \in \mathbb{R} \colon u(t\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}) < t\nu_3^i + y_1\nu_3^{i+[1]} + y_2\nu_3^{i+[2]}\right\}$$

and

$$\underline{g}_i(y) = \sup\left\{t \in \mathbb{R} \colon u(t\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}) > t\nu_3^i + y_1\nu_3^{i+[1]} + y_2\nu_3^{i+[2]}\right\}.$$

In addition, for $t \in \mathbb{R}$ we define the sets

$$\Gamma_i(t) = \left\{ y \in \mathbb{R}^2 \colon \Phi_i^{-1}(t, y) \in \operatorname{graph}(u) \right\}.$$

These correspond to the intersections of graph(u) with planes normal to $\boldsymbol{\nu}^i$.

We first derive some basic properties of the functions $\underline{g}_i$ and $\overline{g}_i$. In particular, there is some monotonicity.

Lemma 16. *For any $i \in \mathbb{Z}_3$, the following statements hold true.*

1. *The function $\underline{g}_i$ is lower semicontinuous and $\overline{g}_i$ is upper semicontinuous.*
2. *The identity $\underline{g}_i = \overline{g}_i$ holds almost everywhere in $\mathbb{R}^2$.*
3. *For all $y \in \mathbb{R}^2$, the inequality $\underline{g}_i(y) \leq \overline{g}_i(y)$ holds true and the set*
$$\left\{ t\boldsymbol{\nu}^i + y_1\boldsymbol{\nu}^{i+[1]} + y_2\boldsymbol{\nu}^{i+[2]} \colon \underline{g}_i(y) \leq t \leq \overline{g}_i(y) \right\}$$
is contained in graph(u).
4. *Let $t \in \mathbb{R}$ and $y \in \mathbb{R}^2$. Then $y \in \Gamma_i(t)$ if, and only if, $\underline{g}_i(y) \leq t \leq \overline{g}_i(y)$.*
5. *For all $y \in \mathbb{R}^2$ and $a \in (0,\infty)^2$, the inequality $\overline{g}_i(y+a) \leq \underline{g}_i(y)$ is satisfied; and if equality holds, then $\underline{g}_i(y) = \underline{g}_i(y+sa) = \overline{g}_i(y+sa) = \overline{g}_i(y+a)$ for all $s \in (0,1)$.*
6. *For all $y \in \mathbb{R}^2$ and $a \in [0,\infty)^2$, the inequalities $\underline{g}_i(y) \geq \underline{g}_i(y+a)$ and $\overline{g}_i(y) \geq \overline{g}_i(y+a)$ are satisfied.*

Proof. 1. We only consider $\overline{g}_i$ here, as the proof for $\underline{g}_i$ is similar.

Let $y \in \mathbb{R}^2$ and $\epsilon > 0$. Fix $t \in \mathbb{R}$ with

$$u(t\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}) < t\nu_3^i + y_1\nu_3^{i+[1]} + y_2\nu_3^{i+[2]}$$

and $t \leq \overline{g}_i(y) + \epsilon$. Then by the continuity of u, there exists $\delta > 0$ such that

$$u(t\nu^i + z_1\nu^{i+[1]} + z_2\nu^{i+[2]}) < t\nu_3^i + z_1\nu_3^{i+[1]} + z_2\nu_3^{i+[2]}$$

for all $z \in B_\delta(y)$. Hence $\overline{g}_i(z) \leq t \leq \overline{g}_i(y) + \epsilon$ for all $z \in B_\delta(y)$. As $\epsilon > 0$ was chosen arbitrarily, this implies upper semicontinuity.

2.–4. Fix $y \in \mathbb{R}^2$ and consider the function

$$f(t) = u(t\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}) - t\nu_3^i - y_1\nu_3^{i+[1]} - y_2\nu_3^{i+[2]}.$$

Recall that u is bounded. Hence

$$\lim_{t \to \pm\infty} f(t) = \mp\infty,$$

and it follows that $\overline{g}_i(y)$ and $\underline{g}_i(y)$ are finite. Since u is Lipschitz continuous, so is f, and condition (3) implies that f is non-increasing. In particular, we have the inequality $\overline{g}_i(y) \geq \underline{g}_i(y)$, and if it happens that $\overline{g}_i(y) > \underline{g}_i(y)$, then f vanishes on the interval $[\underline{g}_i(y), \overline{g}_i(y)]$. In this case, it follows that the line segment

$$\left\{ t\boldsymbol{\nu}^i + y_1\boldsymbol{\nu}^{i+[1]} + y_2\boldsymbol{\nu}^{i+[2]} \colon \underline{g}_i(y) \leq t \leq \overline{g}_i(y) \right\}$$

is contained in graph(u) (as claimed in statement 3). As graph(u) has locally finite 2-dimensional Hausdorff measure, this can happen only for y in an $\mathcal{H}^2$-null set, which implies statement 2. Statement 4 follows as well.

5. Suppose that $a \in (0,\infty)^2$. We first note that

$$u\big(\underline{g}_i(y)\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}\big) = \underline{g}_i(y)\nu_3^i + y_1\nu_3^{i+[1]} + y_2\nu_3^{i+[2]} \tag{4}$$

and

$$\begin{aligned} u\big(\overline{g}_i(y+a)\nu^i + (y_1+a_1)\nu^{i+[1]} + (y_2+a_2)\nu^{i+[2]}\big)& \\ = \overline{g}_i(y+a)\nu_3^i + (y_1+a_1)\nu_3^{i+[1]} + (y_2+a_2)\nu_3^{i+[2]}&. \end{aligned} \tag{5}$$

Fix $\sigma, \tau \in \mathbb{R}$ with $\sigma < \tau$. For $t \in [0,1]$, define

$$\gamma(t) = ((1-t)\sigma + t\tau)\,\nu^i + (y_1 + ta_1)\nu^{i+[1]} + (y_2 + ta_2)\nu^{i+[2]}.$$

For almost every value of y, we then compute

$$\begin{aligned} &\frac{d}{dt}u(\gamma(t)) = (\tau-\sigma)\nu^i \cdot \nabla u(\gamma(t)) + a_1\nu^{i+[1]} \cdot \nabla u(\gamma(t)) + a_2\nu^{i+[2]} \cdot \nabla u(\gamma(t)) \\ &= -\frac{1}{\sqrt{3}}\Big((\tau-\sigma)\alpha^i \cdot \nabla u(\gamma(t)) + a_1\alpha^{i+[1]} \cdot \nabla u(\gamma(t)) + a_2\alpha^{i+[2]} \cdot \nabla u(\gamma(t))\Big) \end{aligned}$$

at almost every $t \in [0,1]$. Hence, by (2) and Lemma 15, we have the inequality

$$u\big(\tau\nu^i + (y_1+a_1)\nu^{i+[1]} + (y_2+a_2)\nu^{i+[2]}\big) - u\big(\sigma\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}\big) \le \frac{c}{\sqrt{3}}$$

for almost all $y \in \mathbb{R}^2$, where

$$c = \max\left\{\tau - \sigma + a_1 - 2a_2, \tau - \sigma + a_2 - 2a_1, a_1 + a_2 - 2(\tau - \sigma)\right\}.$$

By continuity, the inequality holds in fact everywhere. But since $c < \tau - \sigma + a_1 + a_2$, this means that

$$\begin{aligned} u\big(\tau\nu^i + (y_1+a_1)\nu^{i+[1]} + (y_2+a_2)\nu^{i+[2]}\big) - u\big(\sigma\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}\big)& \\ < \frac{1}{\sqrt{3}}(\tau - \sigma + a_1 + a_2) = (\tau-\sigma)\nu_3^i + a_1\nu_3^{i+[1]} + a_2\nu_3^{i+[2]}&. \end{aligned}$$

If we had the inequality $\overline{g}_i(y+a) > \underline{g}_i(y)$, this would contradict the identities (4) and (5). Hence $\overline{g}_i(y+a) \le \underline{g}_i(y)$.

If $\overline{g}_i(y+a) = \underline{g}_i(y)$, then we see that $\underline{g}_i(y) \ge \overline{g}_i(y+sa) \ge \underline{g}_i(y+sa) \ge \overline{g}_i(y+a) = \underline{g}_i(y)$ for any $s \in (0,1)$. Hence we have equality everywhere.

6. The upper semicontinuity entails that

$$\overline{g}_i(y) \ge \limsup_{b \searrow 0} \overline{g}_i(y_1 - b, y_2 - b).$$

Since statements 3 and 5 imply that

$$\overline{g}_i(y_1 - b, y_2 - b) \ge \overline{g}_i(y+a)$$

when $a_1 \ge 0$, $a_2 \ge 0$, and $b > 0$, the desired inequality follows. Similar arguments apply to $\underline{g}_i$. □

Next we analyse the sets $\Gamma_i(t)$. They satisfy a monotonicity property as well.

Lemma 17. *For any $i \in \mathbb{Z}_3$, the following statements hold true.*

1. *For all $t \in \mathbb{R}$, the set $\Gamma_i(t)$ is descending.*
2. *If $t_1 \geq t_2$, then $\Gamma_i(t_1) \preceq \Gamma_i(t_2)$.*
3. *For all $t \in \mathbb{R}$, either $\mathring{\Gamma}_i(t) \neq \emptyset$ or $\Gamma_i(t)$ is totally descending.*

Proof. 1. Let $a \in S^1_+$ and $y \in \mathbb{R}$. Then obviously the line

$$\left\{ t\boldsymbol{\nu}^i + (y_1 + \sigma a_1)\boldsymbol{\nu}^{i+[1]} + (y_2 + \sigma a_2)\boldsymbol{\nu}^{i+[2]} \colon \sigma \in \mathbb{R} \right\}$$

intersects the graph of u. Hence there exists σ_0 such that $y + \sigma_0 a \in \Gamma_i(t)$. It follows that $L_s(a) \cap \Gamma_i(t) \neq \emptyset$ for every $s \in \mathbb{R}$.

2. We know that $y \in \Gamma_i(t)$ if, and only if, the inequalities $\underline{g}_i(y) \leq t \leq \overline{g}_i(y)$ are satisfied by statement 4 of Lemma 16. Thus for $a \in S^1_+$ and $s \in \mathbb{R}$, if $y \in L_s(a) \cap \Gamma_i(t_1)$ and $z \in L_s(a) \cap \Gamma_i(t_2)$ with $a \cdot z < a \cdot y$, then statement 5 from Lemma 16 implies that

$$t_1 \leq \overline{g}_i(y) \leq \underline{g}_i(z) \leq t_2.$$

If $t_1 \geq t_2$, then we conclude that $t_1 = t_2$, and hence $\Gamma_i(t_1) = \Gamma_i(t_2)$. This proves that $\Gamma_i(t_1) \preceq \Gamma_i(t_2)$.

3. Suppose that $\mathring{\Gamma}_i(t) = \emptyset$. Consider $y \in \mathbb{R}^2$ and $a \in S^1_+$. We know that $y + \sigma a \in \Gamma_i(t)$ if, and only if, $\underline{g}_i(y + \sigma a) \leq t \leq \overline{g}_i(y + \sigma a)$. Thus, by statement 5 of Lemma 16, if there exist two numbers $\sigma_1, \sigma_2 \in \mathbb{R}$ with $y + \sigma_1 a, y + \sigma_2 a \in \Gamma_i(t)$, then the entire line segment between the two points belongs to $\Gamma_i(t)$ as well. That is, if $\sigma_1 \leq \sigma \leq \sigma_2$ or $\sigma_2 \leq \sigma \leq \sigma_1$, then $y + \sigma a \in \Gamma_i(t)$.

Define

$$\sigma_* = \inf \{\sigma \in \mathbb{R} \colon y + \sigma a \in \Gamma_i(t)\}$$

and

$$\sigma^* = \sup \{\sigma \in \mathbb{R} \colon y + \sigma a \in \Gamma_i(t)\}.$$

Set $L = \{y + \sigma a \colon \sigma_* \leq \sigma \leq \sigma^*\}$. Then $L \subseteq \Gamma_i(t)$. We claim that

$$\{z \in \Gamma_i(t) \colon z_1 > y_1 + \sigma_* a_1,\ z_2 > y_2 + \sigma_* a_2\} \subseteq L.$$

Indeed, if this were false, say if $z \in \Gamma_i(t) \setminus L$ with $z_1 > y_1 + \sigma_* a_1$ and $z_2 > y_2 + \sigma_* a_2$, then by the above arguments, the line segment L' connecting $y + \sigma_* a$ and z would be contained in $\Gamma_i(t)$. Moreover, for any point $w \in L' \setminus \{y + \sigma_* a\}$, there exists $\sigma' \in (\sigma_*, \sigma^*)$ such that

$$a'' = \frac{w - y - \sigma' a}{|w - y - \sigma' a|} \in S^1_+.$$

Then it would follow that the line segment connecting w with $y + \sigma' a$ is contained in $\Gamma_i(t)$. Since the line segments of this form fill a region with non-empty interior, this would contradict the choice of t.

Similarly,

$$\{z \in \Gamma_i(t) \colon z_1 < y_1 + \sigma^* a_1,\ z_2 < y_2 + \sigma^* a_2\} \subseteq L.$$

But if $\sigma^* > \sigma_*$, then this contradicts the fact that the line $L_{s'}(a)$ intersects $\Gamma_i(t)$ for every $s' \in \mathbb{R}$ (established in statement 1). Therefore, we have established that $\sigma_* = \sigma^*$, and $\Gamma_i(t)$ is totally descending. □

The sets $\Gamma_i(t)$ will play an important role in our analysis. Statement 3 in Lemma 17 indicates that there are two different cases to consider. If $\mathring{\Gamma}_i(t) \neq \emptyset$, then the following definitions are useful, too.

Given $i \in \mathbb{Z}_3$ and $t \in \mathbb{R}$, let

$$\hat{\Gamma}_i(t) = \lim_{t' \nearrow t} \Gamma_i(t') \quad \text{and} \quad \check{\Gamma}_i(t) = \lim_{t' \searrow t} \Gamma_i(t').$$

It follows from Lemma 13 and Lemma 17 that both of these limits exist and that $\hat{\Gamma}_i(t)$ and $\check{\Gamma}_i(t)$ are descending for all $t \in \mathbb{R}$. We can also prove the following.

Lemma 18. *Let $i \in \mathbb{Z}_3$. For every $t \in \mathbb{R}$, the following statements hold true.*

1. $\partial\Gamma_i(t) = \hat{\Gamma}_i(t) \cup \check{\Gamma}_i(t)$.
2. $\hat{\Gamma}_i(t)$ *and* $\check{\Gamma}_i(t)$ *are totally descending.*
3. *If* $\Gamma_i(t)$ *is totally descending, then* $\check{\Gamma}_i(t) = \Gamma_i(t) = \hat{\Gamma}_i(t)$.

Proof. 1. Consider $y \in \partial\Gamma_i(t)$. Then $y \in \Gamma_i(t)$, as this is a closed set, but for any $r > 0$, there exist points in $B_r(y)$ that do not belong to $\Gamma_i(t)$. It is clear, however, that $\mathbb{R}^2 = \bigcup_{\tau \in \mathbb{R}} \Gamma_i(\tau)$. Therefore, there exists a sequence $(t_k)_{k \in \mathbb{N}}$ in $\mathbb{R} \setminus \{t\}$ such that $\operatorname{dist}(y, \Gamma_i(t_k)) \to 0$ as $k \to \infty$.

For a fixed $k \in \mathbb{N}$, suppose that $t_k < t$. We claim that $\operatorname{dist}(y, \Gamma_i(t')) \leq \sqrt{2} \operatorname{dist}(y, \Gamma_i(t_k))$ for all $t' \in (t_k, t)$. In order to prove this, fix $y_k \in \Gamma_i(t_k)$ and let $a = (1/\sqrt{2}, 1/\sqrt{2}) \in S^1_+$. Then Lemma 17 implies that there exists $\sigma \geq 0$ such that $y_k - \sigma a \in \Gamma_i(t)$. Noting that $\overline{g}_i(z) \geq \overline{g}_i(y) \geq t$ for all $z \in (-\infty, y_1] \times (-\infty, y_2]$ by Lemma 16, we see that we can in fact choose σ such that $y_k - \sigma a \notin (-\infty, y_1) \times (-\infty, y_2)$. If $t_k < t' < t$, then there exists $\sigma' \in [0, \sigma]$ such that $y_k - \sigma' a \in \Gamma_i(t')$. But now it is easy to see that $|y - (y_k - \sigma' a)| \leq \sqrt{2}|y - y_k|$, and it follows that $\operatorname{dist}(y, \Gamma_i(t')) \leq \sqrt{2} \operatorname{dist}(y, \Gamma_i(t_k))$.

Now suppose that $t_k < t$ for infinitely many values of k. Then it follows that $\operatorname{dist}(y, \Gamma_i(t')) \to 0$ as $t' \nearrow t$. So $y \in \hat{\Gamma}_i(t)$. Similarly, if $t_k > t$ for infinitely many values of k, then $y \in \check{\Gamma}_i(t)$. This proves that $\partial\Gamma_i(t) \subseteq \hat{\Gamma}_i(t) \cup \check{\Gamma}_i(t)$.

Conversely, suppose that $y \in \hat{\Gamma}_i(t)$. Then there exist a sequence $(y_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^2$ and a sequence $t_k \nearrow t$ such that $y = \lim_{k \to \infty} y_k$ and $y_k \in \Gamma_i(t_k)$ for $k \in \mathbb{N}$. Thus $\Phi_i^{-1}(t_k, y_k) \in \operatorname{graph}(u)$. Since $\operatorname{graph}(u)$ is closed, it follows that $\Phi_i^{-1}(t, y) = \lim_{k \to \infty} \Phi_i^{-1}(t_k, y_k) \in \operatorname{graph}(u)$. That is, we conclude that $y \in \Gamma_i(t)$. But clearly it does not belong to the interior of $\Gamma_i(t)$; so $y \in \partial\Gamma_i(t)$. For $\check{\Gamma}_i(t)$, the arguments are similar.

2. It is clear that $\mathring{\Gamma}_i(t_1) \cap \mathring{\Gamma}_i(t_2) = \emptyset$ for $t_1 \neq t_2$. Therefore, we have only a countable number of values t with $\mathring{\Gamma}_i(t) \neq \emptyset$. Lemma 14 implies that the limit of totally descending sets is totally descending again, and so it follows from Lemma 17 that $\hat{\Gamma}_i(t)$ and $\check{\Gamma}_i(t)$ are totally descending for any $t \in \mathbb{R}$.

3. By statement 1, we have the inclusions $\hat{\Gamma}_i(t) \subseteq \Gamma_i(t)$ and $\check{\Gamma}_i(t) \subseteq \Gamma_i(t)$. Moreover, we know that $\hat{\Gamma}_i(t)$ and $\check{\Gamma}_i(t)$ are totally descending by statement 2. If $\Gamma_i(t)$ is totally descending as well, then Lemma 14 implies that all three sets are the same. ☐

The final lemma of this section gives some information about how $\Gamma_i(t)$ changes when the function u varies.

Lemma 19. *Suppose that* $(u_k)_{k\in\mathbb{N}}$ *is a sequence of uniformly bounded, continuous functions* $\mathbb{R}^2 \to \mathbb{R}$ *converging locally uniformly to* u*. For* $i \in \mathbb{Z}_3$ *and* $t \in \mathbb{R}$*, let*

$$G_k = \left\{ y \in \mathbb{R}^2 \colon u_k \left(t\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]} \right) = t\nu_3^i + y_1\nu_3^{i+[1]} + y_2\nu_3^{i+[2]} \right\}.$$

If $\Gamma_i(t)$ *is totally descending, then* $G_k \to \Gamma_i(t)$ *as* $k \to \infty$*.*

Proof. Clearly G_k is a closed set. We first claim that if the sequence $(G_k)_{k\in\mathbb{N}}$ converges at all, its limit must be $\Gamma_i(t)$, provided that $\Gamma_i(t)$ is totally descending.

In order to prove this claim, consider $G = \lim_{k\to\infty} G_k$. Then for every $y \in G$, there exist $y_k = (y_{1k}, y_{2k}) \in G_k$ such that $y = \lim_{k\to\infty} y_k$. Hence

$$\begin{aligned} u\left(t\nu^i + y_1\nu^{i+[1]} + y_2\nu^{i+[2]}\right) &= \lim_{k\to\infty} u_k\left(t\nu^i + y_{1k}\nu^{i+[1]} + y_{2k}\nu^{i+[2]}\right) \\ &= t\nu_3^i + y_1\nu_3^{i+[1]} + y_2\nu_3^{i+[2]} \end{aligned}$$

by the locally uniform convergence. Therefore, we have the inclusion $G \subseteq \Gamma_i(t)$. On the other hand, for every $s \in \mathbb{R}$, there exists a bounded sequence $(y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^2$ such that $y_k \in L_s \cap G_k$ for every $k \in \mathbb{N}$. (This is because the line

$$\left\{ t\boldsymbol{\nu}^i + y_1\boldsymbol{\nu}^{i+[1]} + y_2\boldsymbol{\nu}^{i+[2]} \colon y \in L_s \right\},$$

being parallel to a vector with non-vanishing third component, must intersect the graphs of each of the continuous, uniformly bounded functions u_k in some bounded set.) Hence $L_s \cap G$ cannot be empty. But since there exists exactly one point in $L_s \cap \Gamma_i(t)$, it follows that $G \cap L_s = \Gamma_i(t) \cap L_s$ for every $s \in \mathbb{R}$. Therefore $G = \Gamma_i(t)$.

By the compactness properties of the Hausdorff distance, every subsequence of $(G_k)_{k\in\mathbb{N}}$ has a convergent subsequence. Since the limit must necessarily be $\Gamma_i(t)$, the statement of the lemma follows. □

5 Curvature and its relation to $D\nabla u$

We still fix a bounded function $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that inequality (2) is satisfied almost everywhere for $j = 1, 2, 3$. We now consider the curvature of the sets $\Gamma_i(t)$ (in a generalised sense) and prove some inequalities relating it to the measure $\|D\nabla u\|$.

First we note that any totally descending set $\Gamma \subseteq \mathbb{R}^2$ allows a Lipschitz continuous parametrisation $\gamma \colon \mathbb{R} \to \Gamma$ by Lemma 14. Furthermore, we may always choose a parametrisation by arc length, i.e., such that $|\gamma'| = 1$ almost everywhere. The following is a classical notion related to ideas that have also been used for the one-dimensional counterpart of problems mentioned in the introduction [3, 4, 5].

Definition 20. *Consider a totally descending set* $\Gamma \subseteq \mathbb{R}^2$*, parametrised by arc length through* $\gamma \colon \mathbb{R} \to \mathbb{R}^2$*. Then for any open set* $\Omega \subseteq \mathbb{R}^2$*, the quantity*

$$\mathcal{C}(\Gamma; \Omega) = \sup \left\{ \int_{-\infty}^{\infty} \phi' \cdot \gamma' \, dt \colon \phi \in C_0^1(\gamma^{-1}(\Omega); \mathbb{R}^2) \text{ with } \sup_{\mathbb{R}} |\phi| \le 1 \right\}$$

is the total curvature *of* Γ *in* Ω*.*

Clearly, if $\mathcal{C}(\Gamma;\Omega)$ is finite, then $\gamma \in \mathrm{BV}^2(\gamma^{-1}(\Omega);\mathbb{R}^2)$. We note that the total curvature is lower semicontinuous.

Lemma 21. *Suppose that $(\Gamma_k)_{k\in\mathbb{N}}$ is a sequence of totally descending sets in $\mathbb{R}^2$ with $\Gamma_k \to \Gamma$ as $k \to \infty$. Then for any open set $\Omega \subseteq \mathbb{R}^2$,*

$$\mathcal{C}(\Gamma;\Omega) \leq \liminf_{k\to\infty} \mathcal{C}(\Gamma_k;\Omega).$$

Proof. We may assume that $\Gamma \cap \Omega$ is connected; otherwise we consider every connected component separately.

Choose parametrisations $\gamma_k : \mathbb{R} \to \mathbb{R}^2$ of Γ_k by arc length such that $\gamma_k(0) \to p$ for some $p \in \Gamma \cap \Omega$ as $k \to \infty$. Then, after extracting a subsequence, we have uniform convergence $\gamma_k \to \gamma$ for some Lipschitz continuous curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ by the theorem of Arzelà-Ascoli. Moreover, if $\liminf_{k\to\infty} \mathcal{C}(\Gamma_k;\Omega) < \infty$, then $\gamma_k'(s)$ converges to $\gamma'(s)$ for almost every $s \in \mathbb{R}$ with $\gamma(s) \in \Omega$ by standard results on BV-functions. Hence $\Gamma \cap \Omega$ is parametrised by arc length through γ. The inequality now follows from standard properties of the space $\mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2)$. □

It is clear that there is a relationship between $\|D\nabla u\|$ and the curvature of graph(u). Since the sets $\Gamma_i(t)$ correspond to curves on graph(u), it is not surprising that there is a relationship with the curvature of $\Gamma_i(t)$ as well. This takes the following form.

Lemma 22. *If $\Omega \subseteq \mathbb{R}^2$ is an open set and $a < b$, then*

$$\int_a^b \mathcal{C}(\Gamma_i(t);\Omega)\,dt \leq 2\|D\nabla u\|\left(\{x \in \mathbb{R}^2 : \Phi_i(x,u(x)) \in (a,b)\times\Omega\}\right).$$

Proof. Assume first that u is a smooth function with $\alpha^j \cdot \nabla u > -1$ for $j = 1,2,3$. Then graph(u) is a smooth surface and $\Gamma_i(t)$ is a smooth curve for every $t \in \mathbb{R}$. Let $\boldsymbol{A}$ denote the second fundamental form of graph(u). Then [6, (1.67)]

$$|\boldsymbol{A}(x,u(x))| \leq \frac{2|\nabla^2 u(x)|}{\sqrt{1+|\nabla u(x)|^2}}.$$

Hence

$$2\int_{\{x\in\mathbb{R}^2 :\, \Phi_i(x,u(x))\in(a,b)\times\Omega\}} |\nabla^2 u|\,dx \geq \int_{\{\boldsymbol{x}\in \mathrm{graph}(u) :\, \Phi_i(\boldsymbol{x})\in(a,b)\times\Omega\}} |\boldsymbol{A}|\,d\mathcal{H}^2.$$

On the other hand, the surface graph(u) is mapped onto $G_i = \bigcup_{t\in\mathbb{R}}\{t\}\times\Gamma_i(t)$ by Φ_i. If $\boldsymbol{A}_i$ denotes the second fundamental form of G_i, then obviously

$$\int_{\{\boldsymbol{x}\in \mathrm{graph}(u) :\, \Phi_i(\boldsymbol{x})\in(a,b)\times\Omega\}} |\boldsymbol{A}|\,d\mathcal{H}^2 = \int_{G_i\cap((a,b)\times\Omega)} |\boldsymbol{A}_i|\,d\mathcal{H}^2.$$

Let $\gamma_i(\cdot\,,t) : \mathbb{R} \to \Gamma_i(t)$ be a parametrisation of $\Gamma_i(t)$ by arc length. Then $\boldsymbol{\tau}_i = (0,\gamma_i')$ is a unit tangent vector to the curve $\{t\}\times\Gamma_i$. Furthermore, let θ_i denote the angle between the plane $\{0\}\times\mathbb{R}^2$ and the normal vector to G_i. Then we compute $|\boldsymbol{A}_i(\boldsymbol{\tau}_i,\boldsymbol{\tau}_i)| = |\gamma_i''|\cos\theta_i$ and $d\mathcal{H}^2 = \frac{ds\,dt}{\cos\theta_i}$. Hence

$$\int_a^b \int_{\{s\in\mathbb{R} :\, \gamma_i(s,t)\in\Omega\}} |\gamma_i''(s,t)|\,ds\,dt \leq \int_{G_i\cap((a,b)\times\Omega)} |\boldsymbol{A}_i|\,d\mathcal{H}^2.$$

Thus the desired inequality follows for this case.

In general, we approximate u by suitable smooth functions. Choose $\psi \in C_0^\infty(B_1(0))$ with $\int_{B_1(0)} \psi\, dx = 1$ and set $\psi_\epsilon(x) = \epsilon^{-2}\psi(x/\epsilon)$. Define $u_\epsilon = (1-\epsilon)\psi_\epsilon * u$. Then $\nabla u_\epsilon \cdot \alpha^j > -1$ for $j = 1, 2, 3$. We have the locally uniform convergence $u_\epsilon \to u$. For any $a', b' \in (a, b)$ and any open set $\Omega' \Subset \Omega$, we also know that

$$\limsup_{k\to\infty} \|D\nabla u_k\| \left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u_k(x)) \in (a', b') \times \Omega'\right\}\right) \\ \leq \|D\nabla u\| \left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (a, b) \times \Omega\right\}\right).$$

If $\Gamma_{i\epsilon}(t)$ is defined analogously to $\Gamma_i(t)$ (for u_ϵ instead of u), then using Lemma 19, we see that $\Gamma_{i\epsilon}(t) \to \Gamma_i(t)$ whenever $\Gamma_i(t)$ is totally descending (which is the case for almost every $t \in \mathbb{R}$ by statement 3 in Lemma 17). Therefore, using Lemma 21, we find that

$$\mathcal{C}(\Gamma_i(t); \Omega') \leq \liminf_{\epsilon\to 0} \mathcal{C}(\Gamma_{i\epsilon}(t); \Omega')$$

for almost every t. It now suffices to invoke Fatou's lemma to prove that

$$\int_{a'}^{b'} \mathcal{C}(\Gamma_i(t); \Omega')\, dt \leq 2\|D\nabla u\| \left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (a, b) \times \Omega\right\}\right)$$

for any $a', b' \in (a, b)$ and any open set $\Omega' \Subset \Omega$. Now we apply this inequality to, say, $a_k = a + 1/k$, $b_k = b - 1/k$, and $\Omega_k = \{x \in \Omega \colon |x| < k,\ \operatorname{dist}(x, \partial\Omega) > 1/k\}$ for $k \in \mathbb{N}$ and use Beppo Levi's theorem. The desired inequality follows. □

6 The structure of $\partial\Gamma_i(u)$

Again we fix u in this section, but now we assume that it belongs to $\mathcal{A}$. That is, we assume that $u \in \mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $\nabla u(x) \in \{\alpha^1, \alpha^2, \alpha^3\}$ almost everywhere. This is stronger than condition (2) used in the last two sections.

We still assume that u is bounded. This is no loss of generality for the proofs of Theorem 2 or Theorem 7, because both give *local* statements. We may modify any function in $\mathcal{A}$ outside a given compact set $K \subseteq \mathbb{R}^2$ such that it remains in $\mathcal{A}$ but becomes bounded. (For example, we may choose an equilateral triangle $T \subseteq \mathbb{R}^2$ with $K \subseteq T$ and with sides parallel to $\alpha^1, \alpha^2, \alpha^3$. Then we may define a function $\tilde{u} \colon \mathbb{R}^2 \to \mathbb{R}$ that agrees with u in T as follows. First, for $x \in T$, we set $\tilde{u}(x) = u(x)$. Next, we extend $\tilde{u}$ by even reflection across each side of T. Then $\tilde{u}$ is defined in another equilateral triangle with twice the side length of T. We continue the construction indefinitely, thus defining $\tilde{u}$ on all of $\mathbb{R}^2$. It is then easy to see that $\tilde{u} \in \mathcal{A}$.)

Unsurprisingly, under the stronger assumptions, we can prove further properties of the sets $\Gamma_i(t)$. It turns out that almost all of them will be of the following form.

Definition 23. *A set Γ is called a* staircase *if it is totally descending and there exist locally finite sets $P, Q \subseteq \mathbb{R}$ such that $\Gamma \subseteq (P \times \mathbb{R}) \cup (\mathbb{R} \times Q)$.*

In other words, a staircase is a locally finite union of horizontal and vertical line segments, arranged to form a Lipschitz graph over a line of slope -1 (according to Lemma 14).

Lemma 24. *Let $i \in \mathbb{Z}_3$. Then $\Gamma_i(t)$ is a staircase for almost every $t \in \mathbb{R}$.*

Proof. Consider the function $w_i(x) = x \cdot \nu^i + u(x)\nu_3^i$, which belongs to $\mathrm{BV}^2_{\mathrm{loc}}(\mathbb{R}^2)$ and is Lipschitz continuous. Note that

$$\begin{aligned}\{t\} \times \Gamma_i(t) &= \Phi_i\left(\{\boldsymbol{x} \in \mathrm{graph}(u) \colon \boldsymbol{x} \cdot \boldsymbol{\nu}^i = t\}\right) \\ &= \Phi_i\left(\{(x, u(x)) \colon x \in \mathbb{R}^2 \text{ with } w_i(x) = t\}\right).\end{aligned}$$

By the results of Dorronsoro [8, Theorem 1] and Pavlica–Zajíček [18, Theorem 3.3], there exists a countably 1-rectifiable set $E \subseteq \mathbb{R}^2$ such that the Fréchet derivative of w_i exists and ∇w_i is approximately continuous at every point $x \in \mathbb{R}^2 \setminus E$. Being countably 1-rectifiable, the set E intersects $w_i^{-1}(\{t\})$ in an $\mathcal{H}^1$-null set for all but countably many values of t. In conjunction with a verson of Sard's theorem of Pavlica–Zajíček [18, Theorem 4.4], this implies that for almost every $t \in \mathbb{R}$ there exists an $\mathcal{H}^1$-null set $N_t \subseteq w_i^{-1}(\{t\})$ such that $\nabla w_i(x)$ exists and is approximately continuous at x with $\nabla w_i(x) \neq 0$ for every $x \in w_i^{-1}(\{t\}) \setminus N_t$.

If ∇w_i is approximately continuous at x, then so is ∇u, and it follows that $\nabla u(x) \in \{\alpha^1, \alpha^2, \alpha^3\}$. But if $\nabla w_i(x) \neq 0$, then $\nabla u(x) \neq \alpha^i$. Hence $\nabla u(x) = \alpha^{i+[1]}$ or $\nabla u(x) = \alpha^{i+[2]}$ at every $x \in w_i^{-1}(\{t\}) \setminus N_t$. The set $\{(x, u(x)) \colon x \in N_t\}$ is still an $\mathcal{H}^1$-null set, as is $\Phi_i(\{(x, u(x)) \colon x \in N_t\})$.

For almost every t, the set $\Gamma_i(t)$ is totally descending by Lemma 17. Hence Lemma 14 implies that there exists a parametrisation $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ by arc length such that the derivative $\gamma'(s)$ exists at almost every $s \in \mathbb{R}$. We have the identity

$$u\big(t\nu^i + \gamma_1(s)\nu^{i+[1]} + \gamma_2(s)\nu^{i+[2]}\big) = t\nu_3^i + \gamma_1(s)\nu_3^{i+[1]} + \gamma_2(s)\nu_3^{i+[2]}$$

for every $s \in \mathbb{R}$. Differentiating, we obtain

$$\begin{aligned}\big(\gamma_1'(s)\nu^{i+[1]} + \gamma_2'(s)\nu^{i+[2]}\big) \cdot \nabla u\big(t\nu^i + \gamma_1(s)\nu^{i+[1]} + \gamma_2(s)\nu^{i+[2]}\big)& \\ = \gamma_1'(s)\nu_3^{i+[1]} + \gamma_2'(s)\nu_3^{i+[2]}&.\end{aligned}$$

By the above observations, the vector

$$\nabla u\big(t\nu^i + \gamma_1(s)\nu^{i+[1]} + \gamma_2(s)\nu^{i+[2]}\big)$$

is either $\alpha^{i+[1]}$ or $\alpha^{i+[2]}$ at almost every s. If it is the former, then it follows that $\gamma_1'(s) = 0$. If it is the latter, then $\gamma_2'(s) = 0$. Therefore, for almost every $t \in \mathbb{R}$, away from an $\mathcal{H}^1$-null set, the tangent vectors of $\Gamma_i(t)$ are horizontal or vertical.

Moreover, by Lemma 22, almost every $\Gamma_i(t)$ has locally finite total curvature. That is, we find that $\gamma' \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^2)$ with only four possible values. We infer that γ' has a locally finite number of jumps. The claim of the lemma then follows. ☐

As a consequence, we can estimate not just the curvature, but also the length of $\Gamma_i(t)$ in terms of $\|D\nabla u\|$.

Lemma 25. *Let $i \in \mathbb{Z}_3$. For any open set $\Omega \subseteq \mathbb{R}^2$ and almost every $t \in \mathbb{R}$,*

$$\mathcal{H}^1(\Gamma_i(t) \cap \Omega) \leq \sqrt{2}\|D\nabla u\|\left(\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in \mathbb{R} \times (\Gamma_i(t) \cap \Omega)\}\right).$$

In particular, there exists a null set $N \subseteq \mathbb{R}$ such that for any countable set $\Theta \subseteq \mathbb{R}$,

$$\mathcal{H}^1\left(\Omega \cap \bigcup_{t\in\Theta\setminus N} \Gamma_i(t)\right) \leq \sqrt{2}\|D\nabla u\|\left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in \mathbb{R}\times\Omega\right\}\right).$$

Proof. Let $t \in \mathbb{R}$ such that $\Gamma_i(t)$ is a staircase. Let $y \in \Gamma_i(t)$, and assume for simplicity that y is on a vertical piece of $\Gamma_i(t)$, say $y_2 \in (a,b)$ and $\{y_1\}\times(a,b) \subseteq \Gamma_i(t)$. (The arguments are similar for a horizontal piece, and since the corners form an $\mathcal{H}^1$-null set, we can ignore them.)

For every $z \in (-\infty, y_1)\times(a,b)$, we know that $\underline{g}_i(z) > t$ by Lemma 16, whereas for every $z \in (y_1,\infty)\times(a,b)$, we know that $\overline{g}_i(z) < t$. Choose $\delta > 0$. For $s \in (a,b)$, define $\tau_s^- = \underline{g}_i(y_1-\delta, s)$ and $\tau_s^+ = \overline{g}_i(y_1+\delta, s)$. Then $\Phi_i^{-1}(\tau_s^-, y_1-\delta, s) \in \operatorname{graph}(u)$ and $\Phi_i^{-1}(\tau_s^+, y_1+\delta, s) \in \operatorname{graph}(u)$. Noting that the map $\Phi_{i+[2]}\circ\Phi_i^{-1}$ just permutes the coordinates cyclically, we infer that $\Phi_{i+[2]}^{-1}(s, \tau_s^\pm, y_1\pm\delta) \in \operatorname{graph}(u)$, and therefore $(\tau_s^\pm, y_1\pm\delta) \in \Gamma_{i+[2]}(s)$ for all $s\in(a,b)$. It is also clear that $(t,y_1)\in\Gamma_{i+[2]}(s)$. As $\tau_s^+ < t < \tau_s^-$, it follows that $\Gamma_{i+[2]}(s)\cap(\mathbb{R}\times(y_1-2\delta, y_2+2\delta))$ cannot be contained in a single horizontal or vertical line for any $s\in(a,b)$. But $\Gamma_{i+[2]}(s)$ is a staircase for almost every $s\in\mathbb{R}$ by Lemma 24. Hence there must be a corner in $\mathbb{R}\times(y_1-2\delta, y_1+2\delta)$ for almost every $s\in(a,b)$, which means that

$$\mathcal{C}\left(\Gamma_{i+[2]}(s); \mathbb{R}\times(y_1-2\delta, y_1+2\delta)\right) \geq \sqrt{2}.$$

Now Lemma 22 implies that

$$\begin{aligned}\frac{b-a}{\sqrt{2}} &\leq \|D\nabla u\|\left(\left\{x\in\mathbb{R}^2\colon \Phi_{i+[2]}(x,u(x)) \in (a,b)\times\mathbb{R}\times(y_1-2\delta, y_1+2\delta)\right\}\right)\\ &= \|D\nabla u\|\left(\left\{x\in\mathbb{R}^2\colon \Phi_i(x,u(x)) \in \mathbb{R}\times(y_1-2\delta, y_1+2\delta)\times(a,b)\right\}\right).\end{aligned}$$

Letting $\delta\to 0$, we obtain

$$\frac{b-a}{\sqrt{2}} \leq \|D\nabla u\|\left(\left\{x\in\mathbb{R}^2\colon \Phi_i(x,u(x)) \in \mathbb{R}\times\{y_1\}\times(a,b)\right\}\right).$$

Finally, we obtain the first inequality of the lemma by taking the sum over all pieces of $\Gamma_i(t)$ in Ω.

The second inequality is an obvious consequence. □

We already know that the sets $\Gamma_i(t)$ have a nice structure for almost all $t\in\mathbb{R}$ by Lemma 24. For the exceptional values of t, we consider the sets $\hat{\Gamma}_i(t)$ and $\check{\Gamma}_i(t)$ instead, defined on page 16. They are not staircases in general, but behave similarly away from a set of vanishing one-dimensional Hausdorff measure.

Proposition 26. *Let $i\in\mathbb{Z}_3$. For every $t\in\mathbb{R}$, there exists a closed set $F\subseteq\hat{\Gamma}_i(t)$ with $\mathcal{H}^1(F)=0$, such that for every $y\in\hat{\Gamma}_i(t)\setminus F$, there exists $r>0$ such that $\hat{\Gamma}_i(t)\cap B_r(y)$ is a horizontal or vertical line segment. The same statement is true for $\check{\Gamma}_i(t)$.*

Proof. By Lemma 24, there exists a sequence $t_k \nearrow t$ such that $\Gamma_i(t_k)$ is a staircase for every $k\in\mathbb{N}$. Moreover, Lemma 25 implies that

$$\mathcal{H}^1\left(\Omega\cap\bigcup_{k=1}^{\infty}\Gamma_i(t_k)\right) < \infty \tag{6}$$

for any bounded, open set $\Omega \subseteq \mathbb{R}^2$. Every $\Gamma_i(t_k)$, being a staircase, consists of a locally finite union of horizontal and vertical line segments.

For $s \in \mathbb{R}$, recall that $L_s = \{z \in \mathbb{R}^2 \colon z_2 - z_1 = s\sqrt{2}\}$. Let $\Sigma \subseteq \mathbb{R}$ be the set of all $s \in \mathbb{R}$ such that the following two conditions are satisfied:

- the intersection
$$L_s \cap \bigcup_{k=1}^{\infty} \Gamma_i(t_k)$$
is finite and
- there exists $K \in \mathbb{N}$ such that for all $k \geq K$, the single element of $L_s \cap \Gamma_i(t_k)$ is *not* a corner of $\Gamma_i(t_k)$.

Then $\mathcal{H}^1(\mathbb{R}\backslash\Sigma) = 0$ because of (6) and because the set of all corners of the curves $\Gamma_i(t_k)$, $k \in \mathbb{N}$, is countable.

Now for every $k \in \mathbb{N}$, let V_k be the set of all $s \in \Sigma$ such that $L_s \cap \Gamma_i(t_k) = L_s \cap \hat{\Gamma}_i(t)$ and this set (of one element) is contained in the interior of a vertical line segment of $\Gamma_i(t_k)$. Let $H_k \subseteq \Sigma$ be defined similarly for horizontal line segments.

Fix $k \in \mathbb{N}$. Consider $s \in V_k$ and let $L_s \cap \Gamma_i(t_k) = \{y\}$. Let $\{y_1\} \times [a, b]$ be the maximal vertical line segment containing y and contained in $\Gamma_i(t_k)$. Note that $y \in \hat{\Gamma}_i(t)$ by the definition of V_k, and therefore $y \in \Gamma_i(t)$ by Lemma 18. Hence the the monotonicity of Lemma 17 implies that $y \in \Gamma_i(t')$ for every $t' \in [t_k, t]$.

It is clear that any totally descending set $G \subseteq \mathbb{R}^2$ with $y \in G$ and $G \preceq \Gamma_i(t_k)$ will satisfy $\{y_1\} \times [a, y_2] \subseteq G$. Hence Lemma 17 also implies that $\{y_1\} \times [a, y_2] \subseteq \Gamma_i(t')$ for every $t' \in [t_k, t]$. Therefore, we also see that $\{y_1\} \times [a, y_2] \subseteq \hat{\Gamma}_i(t)$. In particular, if we have another $s' \in \mathbb{R}$ such that L_s intersects $\{y_1\} \times (a, y_2]$, then $s' \in V_k$. Given that the number of vertical line segments in $\Gamma_i(t_k)$ is locally finite, this means that V_k is the union of a locally finite set of intervals. Moreover, if s is in the interior of V_k, then it follows that there exists $r > 0$ such that $\hat{\Gamma}_i(t) \cap B_r(y)$ is a vertical line segment. The same reasoning applies to H_k.

So if we set $\Sigma' = \bigcup_{k\in\mathbb{N}}(\mathring{V}_k \cup \mathring{H}_k)$, then $\bigcup_{k\in\mathbb{N}}(V_k \cup H_k) \setminus \Sigma'$ is a countable set (comprising at most all the end points of the intervals constituting V_k and H_k). But $\Sigma \subseteq \bigcup_{k\in\mathbb{N}}(V_k \cup H_k)$ by construction. Therefore, we conclude that $\mathcal{H}^1(\mathbb{R} \setminus \Sigma') = 0$. For the closed set

$$F = \hat{\Gamma}_i(t) \setminus \bigcup_{s\in\Sigma'} L_s,$$

this means that $\mathcal{H}^1(F) = 0$ by Lemma 14. We have seen that every point of $\hat{\Gamma}_i(t) \setminus F$ has the required property, so this concludes the proof for $\hat{\Gamma}_i(t)$.

The same arguments apply to $\check{\Gamma}_i(t)$. □

7 The structure of u

Again we fix $u \in \mathcal{A}$ and assume that it is bounded. Recall the definitions of the functions $\underline{g}_i$ and $\overline{g}_i$ for $i \in \mathbb{Z}_3$ on page 12. For any $y \in \mathbb{R}^2$, we have seen that

$$\{t \in \mathbb{R} \colon y \in \Gamma_i(t)\} = [\underline{g}_i(y), \overline{g}_i(t)].$$

Now let

$$\hat{\rho}(t,y) = \sup\left\{r > 0 \colon \Gamma_i(t) \cap B_r(y) \subseteq (-\infty, y_1] \times (-\infty, y_2]\right\},$$

and

$$\check{\rho}_i(t,y) = \sup\left\{r > 0 \colon \Gamma_i(t) \cap B_r(y) \subseteq [y_1, \infty) \times [y_2, \infty)\right\}.$$

We now define

$$\hat{g}_i(y) = \inf\left\{t \geq \underline{g}_i(y) \colon \hat{\rho}(t,y) > 0\right\}$$

and

$$\check{g}_i(y) = \sup\left\{t \leq \overline{g}_i(t) \colon \check{\rho}_i(t,y) > 0\right\}.$$

These functions give us some information about the structure of u.

Lemma 27. *Let $i \in \mathbb{Z}_3$. Then for any $y \in \mathbb{R}^2$,*

$$\underline{g}_i(y) \leq \check{g}_i(y) \leq \hat{g}_i(y) \leq \overline{g}_i(y).$$

Let $t_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. Then $\boldsymbol{x}_0 = \Phi_i^{-1}(t_0, y_0)$ satisfies the following statements.

1. *If $\underline{g}_i(y_0) < t_0 < \check{g}_i(y_0)$, then u coincides with the function*

$$x \mapsto u(x_0) + \max\{(x - x_0) \cdot \alpha^{i+[1]}, (x - x_0) \cdot \alpha^{i+[2]}\}$$

in a neighbourhood of x_0.

2. *If $\check{g}_i(y_0) < t_0 < \hat{g}_i(y_0)$, then u coincides with one of the functions*

$$x \mapsto u(x_0) + (x - x_0) \cdot \alpha^{i+[1]} \quad \text{or} \quad x \mapsto u(x_0) + (x - x_0) \cdot \alpha^{i+[2]}$$

in a neighbourhood of x_0.

3. *If $\hat{g}_i(y_0) < t_0 < \overline{g}_i(y_0)$, then u coincides with the function*

$$x \mapsto u(x_0) + \min\{(x - x_0) \cdot \alpha^{i+[1]}, (x - x_0) \cdot \alpha^{i+[2]}\}$$

in a neighbourhood of x_0.

Proof. Because $\Gamma_i(t)$ is closed and therefore $\hat{\rho}(t,y) > 0$ and $\check{\rho}_i(t,y) > 0$ when $y \notin \Gamma_i(t)$, we immediately obtain the inequalities $\underline{g}_i(y) \leq \hat{g}_i(y) \leq \overline{g}_i(y)$ and $\underline{g}_i(y) \leq \check{g}_i(y) \leq \overline{g}_i(y)$ for all $y \in \mathbb{R}^2$.

If $t \in [\underline{g}_i(y), \overline{g}_i(y)]$ with $\hat{\rho}(t,y) > 0$, then for any $t' \in (t, \overline{g}_i(y)]$, it is clear that $\hat{\rho}(y,t') \geq \hat{\rho}(t,y)$ by the monotonicity of Lemma 17. A similar statement holds for $\check{\rho}_i$. As almost every $\Gamma_i(t)$ is totally descending, Lemma 14 implies that we cannot have the inqualities $\hat{\rho}(t,y) > 0$ and $\check{\rho}_i(t,y) > 0$ at the same time for almost every $t \in [\underline{g}_i(y), \overline{g}_i(y)]$. It follows that $\check{g}_i(y) \leq \hat{g}_i(y)$ for every $y \in \mathbb{R}^2$.

Remember that for almost all $t \in \mathbb{R}$, the set $\Gamma_i(t)$ comprises a locally finite union of horizontal and vertical line segments by Lemma 24. If this is the case, then $t \geq \hat{g}_i(y)$ or $t \leq \check{g}_i(y)$ if y happens to be a corner of $\Gamma_i(t)$. (We can tell which of the two by the type of corner.) If $y \in \Gamma_i(t)$, but y is not a corner, then $\check{g}_i(y) \leq t \leq \hat{g}_i(y)$. Moreover, if $\check{g}_i(y) < \hat{g}_i(y)$, then y is in the interior of a horizontal piece of $\Gamma_i(t)$ for almost all $t \in (\check{g}_i(y), \hat{g}_i(y))$ or in the interior of a vertical piece of $\Gamma_i(t)$ for almost all $t \in (\check{g}_i(y), \hat{g}_i(y))$; but we can never have both situations simultaneously by the monotonicity of Lemma 17.

Now fix $y_0 \in \mathbb{R}^2$. Statements 1 and 3 are proved by the same arguments, so we give the details only for statement 3.

Suppose that $t_0 \in (\hat{g}_i(y_0), \overline{g}_i(y_0))$. Choose $t_1, t_2 \in (\hat{g}_i(y_0), \overline{g}_i(y_0))$ with $t_1 < t_0 < t_2$ and such that both $\Gamma_i(t_1)$ and $\Gamma_i(t_2)$ are staircases. Set $r = \hat{\rho}(t_1, y_0)$ and $\Gamma_0 = y_0 + ((-r, 0] \times \{0\}) \cup (\{0\} \times (-r, 0])$. Then $\Gamma_i(t) \cap B_r(y_0) = \Gamma_0$ for all $t \in (t_1, t_2)$. Now we note that in a neighbourhood of the point $\boldsymbol{x}_0 = \Phi_i^{-1}(t_0, y_0)$, the set $\Phi_i^{-1}((t_1, t_2) \times \Gamma_0)$ coincides with the graph of

$$x \mapsto u(x_0) + \min\{(x - x_0) \cdot \alpha^{i+[1]}, (x - x_0) \cdot \alpha^{i+[2]}\}.$$

This concludes the proof of statement 3.

Finally, in order to prove statement 2, assume that $t_0 \in (\check{g}_i(y_0), \hat{g}_i(y_0))$. Then the above observations imply that there exists $r > 0$ and there exist $t_1 < t_0$ and $t_2 > t_0$ such that $\Gamma_i(t) \cap B_r(y_0)$ is the same horizontal or vertical line segment for all $t \in (t_1, t_2)$. The desired statement then follows. □

The following is an almost immediate consequence.

Proposition 28. *Let $i \in \mathbb{Z}_3$. Suppose that $\boldsymbol{x} \in \mathbb{R}^3$ and $I \subseteq \mathbb{R}$ is an interval such that $\boldsymbol{x} + I\boldsymbol{\nu}^i \subseteq \operatorname{graph}(u)$. Then there exists a set $S \subseteq I$ with at most four points such that $x + (I \setminus S)\nu^i \subseteq \mathcal{F}(u) \cup \mathcal{E}(u)$.*

Proof. Let $L = \boldsymbol{x} + I\boldsymbol{\nu}^i$. Then $\Phi_i(L) \subseteq [\underline{g}_i(y), \overline{g}_i(y)] \times \{y\}$ for some $y \in \mathbb{R}^2$; that is, $[\underline{g}_i(y), \overline{g}_i(y)] = \boldsymbol{\nu}^i \cdot \boldsymbol{x} + I$. It then follows from Lemma 27 that $\Phi_i^{-1}(t, y) \in (\mathcal{F}(u) \cup \mathcal{E}(u)) \times \mathbb{R}$ for all $t \in (\underline{g}_i(y), \overline{g}_i(y))$ unless $t = \check{g}_i(y)$ or $t = \hat{g}_i(y)$. Thus $S = \{\underline{g}_i(y), \check{g}_i(y), \hat{g}_i(y), \overline{g}_i(y)\} - \boldsymbol{\nu}^i \cdot \boldsymbol{x}$ has the required properties. □

Lemma 29. *Let $i \in \mathbb{Z}_3$ and $t \in \mathbb{R}$. Then there exist a closed set $F \subseteq \mathbb{R}^2$ with $\mathcal{H}^1(F) = 0$ such that $\{t\} \times (\partial\mathring{\Gamma}_i(t) \setminus F) \subseteq \Phi_i(\mathcal{E}(u) \times \mathbb{R})$.*

Proof. Write $G = \mathring{\Gamma}_i(t)$. Note that $\partial G \subseteq \partial\Gamma_i(t) = \hat{\Gamma}_i(t) \cup \check{\Gamma}_i(t)$ according to Lemma 18. Therefore, we first consider the sets $\hat{\Gamma}_i(t)$ and $\check{\Gamma}_i(t)$.

Let $\hat{E}$ be the set of all $y \in \hat{\Gamma}_i(t)$ with the property that there exists a radius $r > 0$ such that $B_r(y) \cap \hat{\Gamma}_i(t)$ is a horizontal or vertical line segment. Then $\mathcal{H}^1(\hat{\Gamma}_i(t) \setminus \hat{E}) = 0$ by Proposition 26. Clearly $\hat{E}$ is open relative to $\hat{\Gamma}_i$ and consists of a countable union of horizontal and vertical line segments, the orthogonal projections of which onto the line $\{y \in \mathbb{R}^2 \colon y_1 + y_2 = 0\}$ are pairwise disjoint by Lemma 18 and Lemma 14.

If $L \subseteq \hat{E}$ is one of these line segments, then $\Phi_i^{-1}(\{t\} \times L)$ is a line segment parallel to $\boldsymbol{\nu}^{i+[1]}$ or $\boldsymbol{\nu}^{i+[2]}$ and contained in $\operatorname{graph}(u)$. Therefore, by Proposition 28, there exists a finite set $A \subseteq L$ such that $\{t\} \times (L \setminus A) \subseteq \Phi_i((\mathcal{F}(u) \cup \mathcal{E}(u)) \times \mathbb{R})$. Removing the finite exceptional sets from all of the line segments in $\hat{E}$, we still obtain a relatively open set $\hat{E}' \subseteq \hat{\Gamma}_i(t)$ such that $\hat{E}' \subseteq \Phi_i((\mathcal{F}(u) \cup \mathcal{E}(u)) \times \mathbb{R})$ and $\mathcal{H}^1(\hat{\Gamma}_i(t) \setminus \hat{E}') = 0$.

Similar arguments apply to $\check{\Gamma}_i(t)$, giving rise to a set $\check{E}'$ with the corresponding properties. Set $F = \partial G \setminus (\hat{E}' \cup \check{E}')$. Then F is closed and $\mathcal{H}^1(F) = 0$. Since any point of $\{t\} \times \partial G$ clearly cannot belong to $\Phi_i(\mathcal{F}(u) \times \mathbb{R})$, it follows that $\{t\} \times (\partial G \setminus F) \subseteq \Phi_i(\mathcal{E}(u) \times \mathbb{R})$, as required. □

Proposition 30. *Let $i \in \mathbb{Z}_3$, $R > 0$, and $\boldsymbol{x}_0 \in \operatorname{graph}(u)$. Suppose that*

$$\Lambda = \left\{\boldsymbol{x}_0 + r\boldsymbol{\nu}^i : r \in [0, R]\right\} \subseteq \operatorname{graph}(u)$$

is a maximal line segment in $\operatorname{graph}(u)$ *(in the sense that if Λ' is a line segment with $\Lambda \subseteq \Lambda' \subseteq \operatorname{graph}(u)$, then $\Lambda' = \Lambda$). Then for every $\epsilon > 0$ there exists a curve $\Gamma \subseteq \operatorname{graph}(u)$ with $\boldsymbol{x}_0 \in \Gamma$ and $\mathcal{H}^1(\Gamma \setminus (\mathcal{E}(u) \times \mathbb{R})) \leq \epsilon$, such that either $\boldsymbol{x}_0 + R\boldsymbol{\nu}^i \in \Gamma$ or Γ is unbounded.*

Proof. By Proposition 28, there exists a set $S \subseteq \Lambda$ with at most four points, such that $\Lambda \setminus S \subseteq (\mathcal{F}(u) \cup \mathcal{E}(u)) \times \mathbb{R}$. Let $\Lambda_0 = \Lambda \cap (\mathcal{E}(u) \times \mathbb{R})$.

Let $t_1, t_2 \in \mathbb{R}$ be the numbers such that $\Phi_{i+[1]}(\Lambda) \subseteq \{t_1\} \times \Gamma_{i+[1]}(t_1)$ and $\Phi_{i+[2]}(\Lambda) \subseteq \{t_2\} \times \Gamma_{i+[2]}(t_2)$. For any $\boldsymbol{x} \in \Lambda$, if $x \in \mathcal{F}(u)$, then either $\Phi_{i+[1]}(\boldsymbol{x}) \in \{t_1\} \times \mathring{\Gamma}_{i+[1]}(t_1)$ or $\Phi_{i+[2]}(\boldsymbol{x}) \in \{t_2\} \times \mathring{\Gamma}_{i+[2]}(t_2)$, but never both. Let Λ_1 and Λ_2, respectively, denote the sets of points in $\Lambda \setminus (S \cup \Lambda_0)$ where one or the other of these conditions holds true. Then we have the disjoint decomposition $\Lambda = S \cup \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$. Moreover, we can decompose Λ_1 and Λ_2 disjointly into their connected components, say

$$\Lambda_1 = \bigcup_{k \in K_1} \Lambda_{1k} \quad \text{and} \quad \Lambda_2 = \bigcup_{k \in K_2} \Lambda_{2k}$$

for two countable index sets $K_1, K_2 \subseteq \mathbb{N}$ (as Λ_1 and Λ_2 are open relative to Λ).

We will construct Γ by modifying Λ. The set S is negligible for the desired statement, whereas Λ_0 is already contained in $\mathcal{E}(u) \times \mathbb{R}$. We now modify Λ_1 and Λ_2 step by step as follows.

Consider a connected component Λ_{1k} of Λ_1. Then $\Phi_{i+[1]}(\Lambda_{1k})$ corresponds to a line segment in $\mathring{\Gamma}_{i+[1]}(t_1)$, and its end points belong to $\partial\mathring{\Gamma}_{i+[1]}(t_1) \subseteq \hat{\Gamma}_{i+[1]}(t_1) \cup \check{\Gamma}_{i+[1]}(t_1)$ by Lemma 18. It also follows from Lemma 18 that $\hat{\Gamma}_{i+[1]}(t_1)$ and $\check{\Gamma}_{i+[1]}(t_1)$ are totally descending. Thus Lemma 14 implies that there exist two Lipschitz functions $\hat{c}, \check{c} \colon \mathbb{R} \to \mathbb{R}$ such that

$$\hat{\Gamma}(t_1) = \bigcup_{s \in \mathbb{R}} \left\{y \in L_s : y_1 + y_2 = \sqrt{2}\hat{c}(s)\right\}$$

and

$$\check{\Gamma}(t_1) = \bigcup_{s \in \mathbb{R}} \left\{y \in L_s : y_1 + y_2 = \sqrt{2}\check{c}(s)\right\}.$$

Then clearly

$$\mathring{\Gamma}_{i+[1]}(t_1) \subseteq \bigcup_{s \in \mathbb{R}} \left\{y \in L_s : \sqrt{2}\check{c}(s) < y_1 + y_2 < \sqrt{2}\hat{c}(s)\right\}.$$

We distinguish two cases here. If there exists $s \in \mathbb{R}$ such that $\check{c}(s) = \hat{c}(s)$, then every connected component of $\mathring{\Gamma}_{i+[1]}(t_1)$ has a path connected boundary. In particular, there exists a curve G_{1k} within $\partial\mathring{\Gamma}_{i+[1]}(t_1)$ that connects the two end points of $\Phi_{i+[1]}(\Lambda_{1k})$. We then replace Λ_{1k} by $\Phi_{i+[1]}^{-1}(\{t_1\} \times G_{1k})$. If $\check{c}(s) < \hat{c}(s)$ for all $s \in \mathbb{R}$, then we consider the end point of Λ_{1k} closer to $\boldsymbol{x}_0$; say if $\Lambda_{1k} = \boldsymbol{x}_0 + (a, b)\boldsymbol{\nu}^i$, then we consider the point $\boldsymbol{x}_0 + a\boldsymbol{\nu}^i$. Suppose that $\Phi_i(\boldsymbol{x}_0 + a\boldsymbol{\nu}^i) = (t_1, y) \in \{t_1\} \times \partial\mathring{\Gamma}_{i+[1]}(t_1)$. Then there exists an unbounded curve G_{1k} within $\partial\mathring{\Gamma}_{i+[1]}(t_1)$ beginning at y. In this case, we replace $\boldsymbol{x}_0+(a,R)\boldsymbol{\nu}^i$ by $\Phi_i^{-1}(\{t_1\}\times G_{1k})$.

The same construction works for the connected components of Λ_2. We replace in succession $\Lambda_{11},\Lambda_{21},\Lambda_{12},\Lambda_{22},\dots$, skipping any pieces that have already been replaced. Then after finitely many steps, we have replaced all but a subset of $\Lambda_1\cup\Lambda_2$ of measure ϵ or less. The resulting curve has the desired properties by Lemma 29. □

In the following we consider balls in $\mathbb{R}^3$ as well as in $\mathbb{R}^2$. We use the notation $\boldsymbol{B}_r(\boldsymbol{x})$ to denote the open ball in $\mathbb{R}^3$ of radius $r>0$ centred at $\boldsymbol{x}\in\mathbb{R}^3$.

Lemma 31. *Suppose that $x\in\mathcal{S}(u)$. Then for every $r>0$ there exists a curve $\Gamma\subseteq\operatorname{graph}(u)$ with $\boldsymbol{B}_r(x,u(x))\cap\Gamma\neq\emptyset$ and $\Gamma\setminus\boldsymbol{B}_{2r}(x,u(x))\neq\emptyset$ and*

$$\mathcal{H}^1\left(\Gamma\cap\boldsymbol{B}_{2r}(x,u(x))\cap(\mathcal{E}(u)\times\mathbb{R})\right)\geq\frac{r}{2}.$$

Proof. Choose $i\in\mathbb{Z}_3$ and let $(t,y)=\Phi_i(x,u(x))$. Then, since $(t,y)\in\Phi_i(\mathcal{S}(u)\times\mathbb{R})$, we see from Lemma 27 that $t\in\{\underline{g}_i(y),\check{g}_i(y),\hat{g}_i(y),\overline{g}_i(y)\}$. We distinguish several possibilities.

Case 1: $\underline{g}_i(y)=t=\overline{g}_i(y)$. Since $x\in\mathcal{S}(u)$, it is clear that $y\notin\mathring{\Gamma}_i(t)$. So Lemma 18 implies that $y\in\partial\Gamma_i(t)=\hat{\Gamma}_i(t)\cup\check{\Gamma}_i(t)$. If $y\in\hat{\Gamma}_i(t)$, then it follows from Lemma 17 and the definition of $\hat{\Gamma}_i(t)$ that for any $r>0$, there exists $t'<t$ such that $\Gamma_i(t')$ is totally descending and $B_{r/2}(y)\cap\Gamma_i(t')\neq\emptyset$. Lemma 17 then also implies that $B_{r/2}(y)\cap\Gamma_i(t'')\neq\emptyset$ for all $t''\in(t',t)$. If $y\in\check{\Gamma}_i(t)$, then we conclude similarly that for any $r>0$ there exists $t'>t$ such that $B_{r/2}(y)\cap\Gamma_i(t'')\neq\emptyset$ for all $t''\in(t,t')$. In particular, in both cases, we may choose t'' such that $|t-t''|<\frac{r}{2}$ and $\Gamma_i(t'')$ is a staircase. Because of Lemma 25, we may furthermore choose t'' such that $\Gamma_i(t'')\cap B_{r/2}(y)$ is *not* a single line segment. (Otherwise we would have infinitely many parallel line segments approaching y, and their total length would be infinite.) Thus $\Gamma_i(t'')$ is a locally finite union of horizontal or vertical line segments, each of which corresponds to a maximal line segment parallel to $\boldsymbol{\nu}^{i+[1]}$ or $\boldsymbol{\nu}^{i+[2]}$ in $\operatorname{graph}(u)$. Moreover, at least one of the end points is contained in $\Phi_i^{-1}(B_{r/2}(y)\times(t-r/2,t+r/2))\subseteq\boldsymbol{B}_r(x,u(x))$. Applying Proposition 30 to each of the line segments, we can now easily construct the desired path.

Case 2: $\underline{g}_i(y)=t=\check{g}_i(y)<\hat{g}_i(y)$. There are two possibilities here. Either there exists $t'\in(t-r/2,t)$ such that $\Gamma_i(t')\cap B_{r/2}(y)\neq\emptyset$, or there exists $t'\in(t,t+r/2)$ with $t'<\hat{g}_i(y)$, such that $\Gamma_i(t')\cap B_{r/2}(y)$ is *not* a single line segment. (If neither were the case, then it would follow that $\hat{\Gamma}(t)\cap B_{r/2}(y)=\emptyset$ and $\check{\Gamma}_i(t)\cap B_{r/2}(y)=(y_1-r/2,y_1+r/2)\times\{y_2\}$ or $\check{\Gamma}_i(t)\cap B_{r/2}(y)=\{y_1\}\times(y_2-r/2,y_2+r/2)$. Lemma 18 would then imply that $\Gamma_i(t)\cap B_{r/2}(y)=\left\{z\in B_{r/2}(y)\colon z_1\geq y_1\right\}$ or $\Gamma_i(t)\cap B_{r/2}(y)=\left\{z\in B_{r/2}(y)\colon z_2\geq y_2\right\}$, and we would conclude that $x\in\mathcal{E}(u)$, in contradiction to the assumption that $x\in\mathcal{S}(u)$.) In both cases, we can argue as in Case 1.

Case 3: $\underline{g}_i(y)=t<\check{g}_i(y)$. Then we can choose $t'\in(t,t+r/2)$ with $t'<\check{g}_i(y)$, such that $\Gamma_i(t')$ is a staircase. As we automatically have a corner at y, we can now argue as above.

Case 4: $\underline{g}_i(y)<t=\check{g}_i(y)$. This case is similar to Case 3, but we choose $t'\in(t-r/2,t)$.

Case 5: $\hat{g}_i(y) = t < \overline{g}_i(y)$. This case is similar to Case 4.

Case 6: $\hat{g}_i(y) < t = \overline{g}_i(y)$. This case is similar to Case 3.

Case 7: $\breve{g}_i(y) < \hat{g}_i = t = \overline{g}_i(t)$. This case is similar to Case 2.

Now all possible cases are covered. □

We are finally in a position to prove the first main result of this paper.

Proof of Theorem 2. Consider a compact set $C \subseteq \mathbb{R}^2$. It suffices to prove that $\mathcal{H}^1(\mathcal{S}(u) \cap C) = 0$. By the arguments at the beginning of Sect. 6, we may assume without loss of generality that u is bounded, so that all of the above results apply.

Let $\rho > 0$ and define $U_\rho = \left\{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \mathcal{S}(u) \cap C) < \rho\right\}$. In view of the compactness and by Vitali's covering lemma, there exist finitely many points $x_1, \ldots, x_K \in \mathcal{S} \cap C$ such that $B_\rho(x_k) \cap B_\rho(x_\ell) = \emptyset$ for $1 \leq k, \ell \leq K$ with $k \neq \ell$ and

$$\mathcal{S}(u) \cap C \subseteq \bigcup_{k=1}^{K} B_{5\rho}(x_k). \tag{7}$$

By Lemma 31, for every $k = 1, \ldots, K$ there exists a curve $\Gamma_k \subseteq \operatorname{graph}(u)$ that connects a point in $\boldsymbol{B}_{\rho/2}(x_k, u(x_k))$ with a point outside of $\boldsymbol{B}_\rho(x_k, u(x_k))$ and such that

$$\mathcal{H}^1\left(\Gamma_k \cap \boldsymbol{B}_\rho(x_k, u(x_k)) \cap (\mathcal{E}(u) \times \mathbb{R})\right) \geq \frac{\rho}{4}.$$

This means in particular that

$$\mathcal{H}^1\left(\boldsymbol{B}_\rho(x_k, u(x_k)) \cap (\mathcal{E}(u) \times \mathbb{R})\right) \geq \frac{\rho}{4}.$$

But it is clear that every point in $\mathcal{E}(u)$ belongs to the jump set of ∇u and it is easy to compute the contribution of $\|D\nabla u\|$. Indeed, we find that

$$\|D\nabla u\|\left(B_\rho(x_k) \cap \mathcal{R}(u)\right) \geq \frac{\rho}{2}.$$

It follows that

$$K\rho \leq 2\|D\nabla u\|(U_\rho \cap \mathcal{R}(u)).$$

Using (7) and letting $\rho \to 0$, we conclude that

$$\mathcal{H}^1(\mathcal{S}(u) \cap C) \leq 20 \liminf_{\rho \to \infty} \|D\nabla u\|(U_\rho \cap \mathcal{R}(u)).$$

But since $\bigcap_{\rho > 0} U_\rho \cap \mathcal{R}(u) = \emptyset$, it follows that $\mathcal{H}^1(\mathcal{S}(u) \cap C) = 0$. □

8 The pseudometric $d_K^R(u, v)$

Let $R > 0$ and $K \subseteq \mathbb{R}^2$. Recall the functions d_K^R from Definition 6, defined on pairs of functions $u, v \colon \mathbb{R}^2 \to \mathbb{R}$. In this section we prove that d_K^R is a pseudometric (and in fact a metric if restricted to continuous functions and if functions that agree in the R-neighbourhood of K are identified). This is not essential for the proof of Theorem 7, but the observation is interesting for the interpretation of the result, as it may be thought of as 'almost continuity' of the identity map on $\mathcal{A}$ with respect to two seemingly rather different topologies.

Proposition 32. *The following statements hold true.*

1. *For any $R > 0$ and any set $K \subseteq \mathbb{R}^2$, the function d_K^R is a pseudometric on the space of all functions $\mathbb{R}^2 \to \mathbb{R}$.*

2. *Two continuous functions u, v agree on the set $\{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, K) \le R\}$ if, and only if, $d_K^R(u, v) = 0$.*

Proof. First we note that statement 2 follows immediately from the definition.

It is also clear that $d_K^R(u, v)$ is finite for any two functions u and v; indeed, it satisfies $d_K^R(u, v) \le R$, as $\Delta_\rho^R(u, v) = \mathbb{R}^2$ trivially when $\rho > R$. Symmetry of d_K^R is clear as well. Thus it suffices to prove the triangle inequality.

Suppose that $u, v, w \colon \mathbb{R}^2 \to \mathbb{R}$ are three functions and $\rho > d_K^R(u, v)$ and $\rho' > d_K^R(v, w)$. Fix $x \in K$ and choose $a \in B_\rho(0)$, $a' \in B_{\rho'}(0)$, $b \in (-\rho, \rho)$, and $b' \in (-\rho', \rho')$ such that

$$u(y) = v(y + a) + b \quad \text{and} \quad v(y) = u(y - a) - b$$

for all $y \in B_{R-\rho}(x)$ and

$$v(y) = w(y + a') + b' \quad \text{and} \quad w(y) = v(y - a') - b'$$

for all $y \in B_{R-\rho'}(x)$. Now let $y \in B_{R-\rho-\rho'}(x)$. Since $y \in B_{R-\rho}(x)$ and $y + a \in B_{R-\rho'}(x)$, it follows that

$$u(y) = v(y + a) + b = w(y + a + a') + b + b'.$$

Similarly, we conclude that

$$w(y) = v(y - a') - b' = u(y - a - a') - b - b'.$$

As obviously $a + a' \in B_{\rho+\rho'}(0)$ and $-\rho - \rho' < b + b' < \rho + \rho'$, it follows that $K \subseteq \Delta_{\rho+\rho'}^R(u, v)$ and $d_K^R(u, v) \le \rho + \rho'$. □

9 Local rigidity

The purpose of this section is to provide some tools for the proof of Theorem 7. As in the previous sections, we choose $u \in \mathcal{A}$ and this will stay fixed throughout. Again we assume that u is bounded, recalling that this does not entail any loss of generality by the arguments in Sect. 6.

Recall that in Lemma 22, we estimate the total variation of ∇u in terms of the total curvature of the curves $\Gamma_i(t)$. In view of Theorem 2, we can now improve this as follows.

Proposition 33. *Suppose that $\Omega \subseteq \mathbb{R}^2$ is an open set. Then*

$$\|D\nabla u\|(\Omega) = \sqrt{2} \sum_{i \in \mathbb{Z}_3} \int_{-\infty}^{\infty} \mathcal{C}\left(\Gamma_1(t); \left\{y \in \mathbb{R}^2 \colon (t, y) \in \Phi_i(\Omega \times \mathbb{R})\right\}\right) dt.$$

Proof. Since Theorem 2 tells us that $\mathcal{H}^1(\mathcal{S}(u)) = 0$, we infer that

$$\Phi_i\left(\{(x, u(x)) \colon x \in \mathcal{S}(u)\}\right) \cap \left(\{t\} \times \Gamma_i(t)\right) = \emptyset$$

for almost every $t \in \mathbb{R}$ and for all $i \in \mathbb{Z}_3$. Furthermore, as $\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, standard results on BV-functions [1, Lemma 3.76] imply that $\|D\nabla u\|(\mathcal{S}(u)) = 0$. Therefore, replacing Ω by $\Omega \cap \mathcal{R}(u)$ will not change the desired identity, and we may assume without loss of generality that $\Omega \cap \mathcal{S}(u) = \emptyset$. But then the quantity $\|D\nabla u\|(\Omega)$ is determined by the jumps of ∇u, and every jump point corresponds to a corner of $\Gamma_i(t)$ for exactly one $i \in \mathbb{Z}_3$ and exactly one $t \in \mathbb{R}$.

It remains to verify that the weight $\sqrt{2}$ in the desired formula is correct. But this is now an exercise in elementary Euclidean geometry and is left to the reader. □

In the following, we will examine the sets $\Gamma_i(t)$ again. We need a measure of distance between any two of them, which is given by the following notion, related to the flat metric on spaces of integer multiplicity currents [20, §31].

Suppose that $\Gamma, \Gamma' \subseteq \mathbb{R}^2$ are two totally descending sets. Then there is an open set $G \subseteq \mathbb{R}^2$ between then; that is, G comprises all $y \in \mathbb{R}^2$ such that there exist $r, r' > 0$ with $(y_1 - r, y_2 - r) \in \Gamma$ and $(y_1 + r', y_2 + r') \in \Gamma'$ or vice versa. For an open set $\Omega \subseteq \mathbb{R}^2$, we then define

$$A(\Gamma, \Gamma'; \Omega) = \mathcal{H}^2(G \cap \Omega).$$

Thus $A(\Gamma, \Gamma'; \Omega)$ is the area of the region in Ω between Γ and Γ'. We similarly define $A(\Gamma, \infty; \Omega)$ and $A(\Gamma, -\infty; \Omega)$ as the area of the region in Ω above and below Γ, respectively (where 'above' and 'below' are defined in a manner consistent with the relations from Definition 10).

We furthermore define $\Gamma_+ = ([0, \infty) \times \{0\}) \cup (\{0\} \times [0, \infty))$ and $\Gamma_- = ((-\infty, 0] \times \{0\}) \cup (\{0\} \times (-\infty, 0])$. Given $y \in \mathbb{R}^2$ and $r > 0$, we write $Q_r(y) = (y_1 - r, y_1 + r) \times (y_2 - r, y_2 + r)$.

The proof of Theorem 7 is based on the following lemmas. Here we should think of u as a perturbation of another, fixed function in $\mathcal{A}$. We study how u differs from the other function locally by placing a small box around a point on a face, edge, or vertex of the graph of the latter and analysing the behaviour in that box (after translation) in terms of the sets $\Gamma_i(t)$.

Lemma 34. *Fix $i \in \mathbb{Z}_3$ and $\ell, h \in [\frac{1}{2}, 2]$. Let $\Gamma_0 = \mathbb{R} \times \{0\}$ or $\Gamma_0 = \{0\} \times \mathbb{R}$. Then for any $\delta \in (0, \frac{1}{4}]$ there exists $\epsilon > 0$ such that the following holds true for any $R > 0$. Suppose that*

$$\int_{-hR}^{hR} A(\Gamma_i(t), \Gamma_0; (-R, R) \times (-\ell R \ell R)) \leq \epsilon R^2 \tag{8}$$

and

$$\|D\nabla u\| \left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (-hR, hR) \times (-R, R) \times (-\ell R, \ell R)\right\}\right) \leq \epsilon R. \tag{9}$$

Then there exists $y \in Q_{\delta R}(0)$ such that $\Gamma_i(t) \cap ((-R, R) \times (-\ell R, \ell R)) = (y + \Gamma_0) \cap ((-R, R) \times (-\ell R, \ell R))$ for all $t \in ((\delta - h)R, (h - \delta)R)$.

Proof. We may assume without loss of generality that $R = 1$, and the more general statement then follows by scaling. Moreover, we assume for simplicity that $\Gamma_0 = \mathbb{R} \times \{0\}$ and that $\ell = h = 1$. The arguments are essentially the same in general.

Recall that $\Gamma_i(t)$ is a staircase for almost every t by Lemma 24. If (9) is satisfied and ϵ is sufficiently small, then Proposition 33 implies that for every $t \in (-1, 1)$ with the exception of a set of measure $\delta/4$ or less, the curve $\Gamma_i(t)$ does not have any corners in $Q_1(0)$. If we have inequality (8) as well for a sufficiently small ϵ, then for every $t \in (-1, 1)$ with the exception of a set of measure $\delta/2$ or less, the set $\Gamma_i(t) \cap Q_1(0)$ is a horizontal line segment of distance less than δ from Γ_0.

Fix $t_1 \in (-1, \delta - 1)$ and $t_2 \in (1 - \delta, 1)$ with this property. Then there exist unique points $y' \in \Gamma_i(t_1)$ and $y'' \in \Gamma_i(t_2)$ with $y'_1 = y''_1 = 0$ and $y'_2, y''_2 \in (-\delta, \delta)$. By Lemma 17, they satisfy $y'_2 \geq y''_2$. We claim that in fact $y'_2 = y''_2$.

Indeed, if this were false, then for all $r \in (-1, 1)$, we would have the inequality $\underline{g}_i(r, s) \geq t_2$ when $s \leq y''_2$ and $\overline{g}_i(r, s) \leq t_1$ when $s \geq y'_2$, while $t_1 < \underline{g}_i(r, s) \leq \overline{g}_i(r, s) < t_2$ when $s \in (y''_2, y'_2)$. Therefore, almost each of the curves $\Gamma_{i+[1]}(r)$ would have at least two corners in $Q_1(0)$. We would then find that

$$\int_{-1}^{1} \mathcal{C}(\Gamma_{i+[1]}(r); Q_1(0))\, dr \geq 4\sqrt{2},$$

which, by Proposition 33, contradicts (9). Therefore, we conclude that $y'_2 = y''_2$.

By the monotonicity of Lemma 17, it follows that $\Gamma_i(t) \cap Q_1(0) = (y' + \Gamma_0) \cap Q_1(0)$ for all $t \in (t_1, t_2)$. This concludes the proof. $\square$

Lemma 35. *Fix $i \in \mathbb{Z}_3$ and $h \in [\frac{1}{2}, 2]$. Let $\Gamma_0 = \Gamma_+$ or $\Gamma_0 = \Gamma_-$. Then for any $\delta \in (0, \frac{1}{4}]$ there exists $\epsilon > 0$ such that the following holds true for any $R > 0$. Suppose that*

$$\int_{-hR}^{hR} A(\Gamma_i(t), \Gamma_0; Q_R(0)) \leq \epsilon R^2$$

and

$$\|D\nabla u\| \left(\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (-hR, hR) \times Q_R(0)\}\right) \leq 4hR + \epsilon R. \tag{10}$$

Then there exists $y \in Q_{\delta R}(0)$ such that $\Gamma_i(t) \cap Q_R(0) = (y + \Gamma_0) \cap Q_R(0)$ for all $t \in ((\delta - h)R, (h - \delta))R$. In particular,

$$\|D\nabla u\| \left(\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (-hR, hR) \times Q_R(0)\}\right) \geq 4(h - \delta)R.$$

Proof. For simplicity we assume that $\Gamma_0 = \Gamma_+$ and $R = h = 1$.

For any horizontal or vertical line L, it is clear that $A(L, \Gamma_+; Q_1(0)) \geq 1$. Therefore, if ϵ is sufficiently small, then for all $t \in (-1, 1)$ with the exception of a set of measure $\delta/8$ or less, the curve $\Gamma_i(t)$ is a staircase with at least one corner in $Q_1(0)$. Furthermore, comparing (10) with Proposition 33, we see that for ϵ small enough, there is a single corner, which belongs to $Q_\delta(0)$, for all $t \in (-1, 1)$ with the exception of a set of measure $\delta/2$ or less.

Choose $t_1 \in (-1, \delta - 1)$ and $t_2 \in (1 - \delta, 1)$ such that $\Gamma_i(t_1)$ and $\Gamma_i(t_2)$ have this property. Note that by the above observations,

$$\int_{-1}^{1} \mathcal{C}(\Gamma_i(t); Q_1(0))\, dt \geq 2\sqrt{2}(1 - \delta). \tag{11}$$

With arguments similar to those used in the proof of Lemma 34, we show that $\Gamma_i(t_1) = \Gamma_i(t_2)$. (Otherwise, we could derive the inequalities

$$\int_{\delta}^{1} \mathcal{C}(\Gamma_{i+[1]}(t); Q_1(0))\, dt \geq 2\sqrt{2}(1 - \delta)$$

or

$$\int_\delta^1 \mathcal{C}(\Gamma_{i+[2]}(t); Q_1(0))\,dt \geq 2\sqrt{2}(1-\delta).$$

But together with (11), this contradicts (10).)

Again as in the proof of Lemma 34, it follows that $\Gamma_i(t) = \Gamma_i(t_1)$ for all $t \in (t_1, t_2)$. The inequality follows from Proposition 33. □

Lemma 36. *Fix $i \in \mathbb{Z}_3$. Then for any $\delta \in (0, \frac{1}{4}]$ there exists $\epsilon > 0$ such that for any $R > 0$ with*

$$\|D\nabla u\| \left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (-R, R) \times Q_R(0)\right\}\right) \leq (6+\epsilon)R, \tag{12}$$

the following holds true.

1. *If*

$$\int_{-R}^0 A(\Gamma_i(t), \infty; Q_R(0))\,dt + \int_0^R A(\Gamma_i(t), \Gamma_-; Q_R(0)) \leq \epsilon R^2, \tag{13}$$

   then there exists $y \in Q_{\delta R}(0)$ and there exists $T \in (-\delta R, \delta R)$ such that $\Gamma_i(t) \cap Q_{(1-\delta)R}(0) = \emptyset$ for all $t \in (-R, T)$ and $\Gamma_i(t) \cap Q_{(1-\delta)R}(0) = (y + \Gamma_-) \cap Q_{(1-\delta)R}(0)$ for all $t \in (T, (1-\delta)R)$.

2. *If*

$$\int_0^R A(\Gamma_i(t), -\infty; Q_R(0))\,dt + \int_{-R}^0 A(\Gamma_i(t), \Gamma_+; Q_R(0)) \leq \epsilon R^2, \tag{14}$$

   then there exists $y \in Q_{\delta R}(0)$ and there exists $T \in (-\delta R, \delta R)$ such that $\Gamma_i(t) \cap Q_{(1-\delta)R}(0) = \emptyset$ for all $t \in (T, R)$ and $\Gamma_i(t) \cap Q_{(1-\delta)R}(0) = (y + \Gamma_+) \cap Q_{(1-\delta)R}(0)$ for all $t \in ((\delta - 1)R, T)$.

In both cases,

$$\|D\nabla u\| \left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (-R, R) \times Q_R(0)\right\}\right) \geq 6(1-2\delta)R.$$

Proof. We may assume without loss of generality that $R = 1$. For simplicity, we only consider case 1. The proof for case 2 is similar.

If ϵ is small enough, then clearly there exists $t_1 \in (-\delta/4, 0]$ such that $\Gamma_i(t_1)$ is a staircase and $A(\Gamma_i(t_1), \infty; Q_1(0)) \leq \frac{\delta^2}{256}$. Then the monotonicity of Lemma 17 implies that $\Gamma_i(t) \cap Q_{1-\delta/16}(0) = \emptyset$ for every $t \leq t_1$.

For $m \in \mathbb{N}$, let Θ_m denote the set of all $t \in (0,1)$ such that $\Gamma_i(t)$ is a staircase with at least m corners in $Q_1(0)$. Then (13) implies that $\mathcal{H}^1((0,1) \setminus \Theta_1) \leq \frac{\delta}{4}$ whenever ϵ is small enough. Therefore,

$$\int_0^1 \mathcal{C}\left(\Gamma_i(t); Q_1(0)\right)\,dt \geq \sqrt{2}\left(1 - \frac{\delta}{4}\right). \tag{15}$$

Moreover, Proposition 33 and (12) imply that $\mathcal{H}^1(\Theta_4) \leq \frac{6+\epsilon}{8}$. Thus, given $\eta > 0$, there exists $t_2 \in (0,1)$ such that $\Gamma_i(t_2)$ is a staircase with at least one and at most three corners in $Q_1(0)$ and such that $A(\Gamma_i(t_2), \Gamma_-; Q_1(0)) \leq \eta$ (provided that ϵ is small enough). This can only happen if one of these corners is near 0 and the other two are either close to one another or near the boundary of $Q_1(0)$. In particular, if η is chosen sufficiently small, it follows that

$$\Gamma_i(t_2) \cap Q_{1-\delta/16}(0) \subseteq \left\{y \in Q_{1-\delta/16}(0)\colon\ \mathrm{dist}(y, \Gamma_-) \leq \delta/16\right\}.$$

Therefore, for every $r \in (\delta/16 - 1, -\delta/16)$, we know that $\underline{g}_i(r,s) \geq t_2$ when $s < -\delta/16$ and $t_2 > \overline{g}_i(r,s) \geq \underline{g}_i(r,s) \geq t_1$ when $\delta/16 < s < 1 - \delta/16$. It follows that $\Gamma_{i+[1]}(r)$, if it is a staircase, must have at least one corner in $(-\frac{1}{2}, \frac{1}{2}) \times (-1, 1)$. Therefore,

$$\int_{-1}^{0} \mathcal{C}\left(\Gamma_{i+[1]}(r); (-1/2, 1/2) \times (-1, 1)\right) \, dr \geq \sqrt{2}\left(1 - \frac{\delta}{8}\right). \tag{16}$$

Similarly, we see that

$$\int_{-1}^{0} \mathcal{C}\left(\Gamma_{i+[2]}(s); (-1, 1) \times (-1/2, 1/2)\right) \, ds \geq \sqrt{2}\left(1 - \frac{\delta}{8}\right). \tag{17}$$

Therefore, by Proposition 33 and (12), (16), (17), we have the inequality

$$\int_{0}^{1} \mathcal{C}\left(\Gamma_i(t); Q_1(0)\right) \, dt \leq \sqrt{2}\left(1 + \frac{\delta}{4} + \frac{\epsilon}{2}\right). \tag{18}$$

Now we conclude that $\mathcal{H}^1(\Theta_2) \leq \frac{\delta}{2} + \frac{\epsilon}{2}$, for otherwise we would find a contradiction to (18), bearing in mind that $\mathcal{H}^1((0,1) \setminus \Theta_1) \leq \frac{\delta}{4}$. If ϵ is small enough, we can therefore find $t_3 \in (1 - \delta, 1)$ such that $\Gamma_i(t_3)$ has *exactly* one corner $y \in Q_\delta(0)$ in $Q_1(0)$; more precisely, such that $\Gamma_i(t_3) \cap Q_1(0) = (y + \Gamma_-) \cap Q_1(0)$. Define $T = \underline{g}_i(y)$. We claim that $\Gamma_i(t) \cap Q_{1-\delta}(0) = \emptyset$ for all $t \in (-1, T)$.

If this were false, then there would be a number $t \in (t_1, T)$ such that $\Gamma_i(t)$ is a staircase and intersects $Q_{1-\delta}(0)$. We distinguish two cases.

Case 1. If $\Gamma_i(t)$ intersects $[\delta - 1, 1 - \delta] \times \{1 - \delta\}$ or $\{1 - \delta\} \times [\delta - 1, 1 - \delta]$, then the length of $\Gamma_i(t) \cap (Q_{1-\delta/16}(0) \setminus (-1, 1/2)^2)$ is at least $30\delta/16$. Hence with the same arguments as before, we then find that

$$\begin{aligned}\int_{-1}^{\frac{1}{2}} \mathcal{C}\left(\Gamma_{i+[1]}(r); (1/2, 1) \times (-1, 1)\right) \, dr + \int_{\frac{1}{2}}^{1} \mathcal{C}\left(\Gamma_{i+[1]}(r); Q_1(0)\right) \, dr \\ + \int_{-1}^{\frac{1}{2}} \mathcal{C}\left(\Gamma_{i+[2]}(s); (-1, 1) \times (1/2, 1)\right) \, ds + \int_{\frac{1}{2}}^{1} \mathcal{C}\left(\Gamma_{i+[2]}(s); Q_1(0)\right) \, ds > \delta\sqrt{2}.\end{aligned}$$

Together with (15), (16), and (17), this then contradicts (12).

Case 2. If $\Gamma_i(t)$ intersects neither $[\delta - 1, 1 - \delta] \times \{1 - \delta\}$ nor $\{1 - \delta\} \times [\delta - 1, 1 - \delta]$, then $\Gamma_i(t) \cap Q_{1-\delta}(0)$ is a curve connecting a point of $\{\delta - 1\} \times (y_2, 1 - \delta)$ with a point of $(y_1, 1 - \delta) \times \{\delta - 1\}$ within the set $Q_{1-\delta}(0) \setminus ((-1, y_1] \times (-1, y_2])$. This is because $\Gamma_i(t)$ is totally descending and $\underline{g}_i(y) = T > t$. Therefore, any line segment of the form $\{r\} \times (y_2, 1 - \delta)$ with $r \in (\delta - 1, -\delta)$ will intersect $\Gamma_i(t)$. We then conclude that for almost every $r \in (\delta - 1, -\delta)$, the curve $\Gamma_{i+[1]}(r)$ has at least three corners in $Q_1(0)$. Therefore, we can improve (16) as follows:

$$\int_{-1}^{0} \mathcal{C}\left(\Gamma_{i+[1]}(r); Q_1(0)\right) \, dr \geq 3\sqrt{2}(1 - 2\delta).$$

Similarly, we obtain

$$\int_{-1}^{0} \mathcal{C}\left(\Gamma_{i+[2]}(s); Q_1(0)\right)\, ds \geq 3\sqrt{2}(1-2\delta).$$

Combining these inequalities with (15), we obtain a contradiction to (12) again.

To summarise, we have shown that $\Gamma_i(t) \cap Q_{1-\delta}(0) = \emptyset$ for all $t \in (-1, T)$. Next we claim that $\Gamma_i(t) \cap Q_{1-\delta}(0) = (y + \Gamma_-) \cap Q_{1-\delta}(0)$ for all $t \in (T, 1-\delta)$.

Suppose this were false. Then there exists $t \in (T, t_3)$ such that $\Gamma_i(t)$ is a staircase and $\Gamma_i(t) \cap Q_{1-\delta}(0) \neq (y + \Gamma_-) \cap Q_{1-\delta}(0)$. We distinguish two cases again.

Case 3. If $\Gamma_i(t)$ intersects $(-1, 1] \times \{1\}$ or $\{1\} \times (-1, 1]$, then, as it is a totally descending curve containing y by the choice of t, the length of $\Gamma_i(t) \cap (Q_1(0) \setminus (-1, 1/2)^2)$ must be at least $1/2$. Thus we can argue exactly as in Case 1 above.

Case 4. If $\Gamma_i(1)$ intersects neither $(-1, 1] \times \{1\}$ nor $\{1\} \times (-1, 1]$, then we can argue similarly to Case 2 above. In this case, since $\Gamma_i(t) \cap Q_{1-\delta}(0) \neq (y + \Gamma_-) \cap Q_{1-\delta}(0)$, there exists $z \in \Gamma_i(t) \cap Q_{1-\delta}(0)$ with either $z_1 > y_1$ or $z_2 > y_2$. In the second case, because $\Gamma_i(t)$ is totally descending, any line of the form $\{r\} \times (-1, 1)$ with $r \in (-1, \delta - 1)$ will intersect $\Gamma_i(t)$ and $\Gamma_i(t_3)$ at two different points. Hence $\Gamma_{i+[1]}(r)$ will have at least 3 corners for $r \in (-1, \delta - 1)$, while it still has at least one corner for $\delta - 1 < r < -\delta$. Thus we find that

$$\int_{-1}^{1} \mathcal{C}\left(\Gamma_{i+[1]}(r); Q_1(0)\right)\, dr \geq \sqrt{2}\,(1+\delta)\,.$$

If $z_1 > y_1$, then

$$\int_{-1}^{1} \mathcal{C}\left(\Gamma_{i+[2]}(s); Q_1(0)\right)\, ds \geq \sqrt{2}\,(1+\delta)\,.$$

Together with (15) and (16) or (17), this still gives a contradiction to (12), provided that ϵ is sufficiently small.

So $\Gamma_i(t) \cap Q_{1-\delta}(0) = (y + \Gamma_-) \cap Q_{1-\delta}(0)$ for all $t \in (T, 1-\delta)$ and $\Gamma_i(t) \cap Q_{1-\delta}(0) = \emptyset$ for all $t \in (-1, T)$. This means that we have the desired structure. The desired inequality then follows from Proposition 33 again. □

The preceding lemma is useful to analyse the structure near $\mathcal{V}^{\vee\wedge}(u)$ and $\mathcal{V}^{\wedge\vee}(u)$. As discussed in the introduction, we cannot expect a similar rigidity result near corners that correspond to $\mathcal{V}^{\wedge}(u)$ or $\mathcal{V}^{\vee}(u)$. Nevertheless, we can still derive an inequality.

Lemma 37. *Fix $i \in \mathbb{Z}_3$. Then for any $\delta \in (0, \frac{1}{4}]$ there exists $\epsilon > 0$ such that the following holds true for any $R > 0$. If either*

$$\int_{-R}^{0} A(\Gamma_i(t), \infty; Q_R(0))\, dt + \int_{0}^{R} A(\Gamma_i(t), \Gamma_+; Q_R(0)) \leq \epsilon R^2$$

or

$$\int_{0}^{R} A(\Gamma_i(t), -\infty; Q_R(0))\, dt + \int_{-R}^{0} A(\Gamma_i(t), \Gamma_-; Q_R(0)) \leq \epsilon R^2,$$

then

$$\|D\nabla u\|\left(\left\{x \in \mathbb{R}^2 \colon \Phi_i(x, u(x)) \in (-R, R) \times Q_R(0)\right\}\right) \geq 6(1-2\delta)R.$$

Proof. Again we may assume without loss of generality that $R = 1$. For simplicity, we only consider the case where the first assumption (involving Γ_+) holds.

If ϵ is small enough, then there exist $t_0 \in (-1, 0)$, $t_1 \in (0, \delta)$, and $t_2 \in (1-\delta, \delta)$ such that $\Gamma_i(t_0)$, $\Gamma_i(t_1)$, and $\Gamma_i(t_2)$ are staircases satisfying

$$A(\Gamma_i(t_0), \infty; Q_1(0)) + A(\Gamma_i(t_1), \Gamma_+; Q_1(0)) + A(\Gamma_i(t_2), \Gamma_+; Q_1(0)) \leq \delta^2.$$

Then clearly $\Gamma_i(t_0) \cap Q_{1-\delta}(0) = \emptyset$. Furthermore, since $\Gamma_i(t_1)$ and $\Gamma_i(t_2)$ are totally descending, it is easy to see that $\Gamma_i(t_1) \cap Q_{1-\delta}(0)$ and $\Gamma_i(t_2) \cap Q_{1-\delta}(0)$ are contained in $([-\delta, 1-\delta] \times [-\delta, \delta]) \cup ([-\delta, \delta] \times [-\delta, 1-\delta])$. Lemma 17 then implies that

$$\Gamma_i(t) \subseteq ([-\delta, 1-\delta] \times [-\delta, \delta]) \cup ([-\delta, \delta] \times [-\delta, 1-\delta])$$

for all $t \in [t_1, t_2]$. In particular, if $t \in [t_1, t_2]$ is such that $\Gamma_i(t)$ is a staircase, then there must be a corner in $Q_{1-\delta}(0)$. It follows that

$$\int_\delta^{1-\delta} \mathcal{C}(\Gamma_i(t); Q_1(0)) \geq \sqrt{2}(1 - 2\delta).$$

For any $r \in (\delta, 1-\delta)$, we conclude furthermore that $t_0 \leq \underline{g}_i(r, s) \leq t_1$ when $s \geq \delta$, while $\overline{g}_i(r, s) \geq t_2$ when $s \leq -\delta$. Therefore, with the arguments used in the proof of Lemma 36, we see that

$$\int_\delta^{1-\delta} \mathcal{C}(\Gamma_{i+[1]}(t); Q_1(0)) \geq \sqrt{2}(1 - 2\delta).$$

Similarly, we obtain the estimate

$$\int_\delta^{1-\delta} \mathcal{C}(\Gamma_{i+[2]}(t); Q_1(0)) \geq \sqrt{2}(1 - 2\delta).$$

It now suffices to combine these inequalities with Proposition 33. □

10 Rigidity

We need one more lemma before we prove our second main result. This result will help to reverse the changes in structure that may arise near points of $\mathcal{V}^\wedge(u)$ or $\mathcal{V}^\vee(u)$, as discussed in the introduction.

Lemma 38. *Let $u_0 = \lambda_1 \vee \lambda_2 \vee \lambda_3$ or $u_0 = \lambda_1 \wedge \lambda_2 \wedge \lambda_3$ and let H be the closed hexagon in $\mathbb{R}^2$ with corners*

$$(\cos((2k-1)\pi/6), \sin((2k-1)\pi/6)), \quad k = 1, \ldots, 6.$$

Suppose that $u \in \mathcal{A}$ and $R > 1$. If $u(x) = u_0(x)$ for all $x \in RH \setminus H$, then $\|D\nabla u\|(H) \geq 3\sqrt{6}$. Furthermore, equality holds if, and only if, $u = u_0$ in H.

Proof. We assume for simplicity that $u_0 = \lambda_1 \vee \lambda_2 \vee \lambda_3$. Write $p_k = \cos((2k-1)\pi/6), \sin((2k-1)\pi/6))$, $k = 1, \ldots, 6$, and $\mu = \sqrt{3/2}$. Then $u(p_k) = \sqrt{2}$ if k is odd and $u(p_k) = 1/\sqrt{2}$ if k is even. Therefore,

$$\Phi_1(p_1, u(p_1)) = (0, \mu, \mu), \quad \Phi_1(p_2, u(p_2)) = (0, 0, \mu), \quad \Phi_1(p_3, u(p_3)) = (\mu, 0, \mu),$$
$$\Phi_1(p_4, u(p_4)) = (\mu, 0, 0), \quad \Phi_1(p_5, u(p_5)) = (\mu, \mu, 0), \quad \Phi_1(p_6, u(p_6)) = (0, \mu, 0).$$

Thus when we apply Φ_1, the restriction of graph(u) to ∂H is mapped to the union of line segments between consecutive pairs of these points. This means in particular that

$$\Phi_1(\mathrm{graph}(u|_{\partial H})) \supseteq ([0,\mu] \times \{(\mu,0),(0,\mu)\}).$$

Hence $\Gamma_1(t)$ contains the points $(0,\mu)$ and $(\mu,0)$ for every $t \in (0,\mu)$. The same reasoning applies to $u|_{\partial(rH)}$ for any $r \in (1,R]$, and it follows that $\Gamma_1(t)$ contains the line segments $\{0\} \times [\mu, R\mu]$ and $[\mu, R\mu] \times \{0\}$.

If $\Gamma_1(t)$ is a staircase, then it necessarily has a corner in $[0,\mu]^2$. If there is exactly one corner, then $\Gamma_1(t) \cap [0,\mu]^2 = (\{0\} \times [0,\mu]) \cup ([0,\mu] \times \{0\})$. Hence for any open set $\Omega' \subseteq \mathbb{R}^2$ with $[0,\mu]^2 \subseteq \Omega'$,

$$\int_0^\mu \mathcal{C}(\Gamma_1(t);\Omega')\,dt \geq \sqrt{3}.$$

The same reasoning applies to $\Gamma_2(t)$ and $\Gamma_3(t)$. If $\Omega \subseteq \mathbb{R}^2$ is an open set with $H \subseteq \Omega$, then Proposition 33 implies that

$$\|D\nabla u\|(\Omega) \geq 3\sqrt{6}.$$

The desired inequality follows. Moreover, if we have equality, then it follows that $\Gamma_i(t) \cap [0,\mu]^2 = (\{0\} \times [0,\mu]) \cup ([0,\mu] \times \{0\})$ for all $i \in \mathbb{Z}_3$ and almost all $t \in (0,\mu)$, from which we infer that $u(x) = u_0(x)$ for all $x \in H$. $\square$

We now have all the ingredients for the proof of our second main result.

Proof of Theorem 7. By the arguments in Sect. 6, we may assume without loss of generality that $u \in \mathcal{A}$ is bounded. We fix $\eta > 0$. We may enlarge K if necessary, so that $\|D\nabla u\|(\Omega \setminus K) < \eta$. Choose a precompact, open set $\Omega' \Subset \Omega$ with $K \subseteq \Omega'$. Since $\Omega \subseteq \mathcal{R}(u)$, the restriction of u to Ω' is piecewise affine with finitely many pieces. Let $G = \mathrm{graph}(u) \cap (\Omega' \times \mathbb{R})$.

Let $R > 0$ such that $10R < R_0$ and $100R \leq \mathrm{dist}(K, \partial\Omega')$, as well as

$$100R \leq \min_{x_1,x_2 \in \Omega' \cap \mathcal{V}(u)} |x_1 - x_2|.$$

Note that there are finitely many points in $\Omega' \cap \mathcal{V}(u)$. Consider an open cube in $\mathbb{R}^3$ of side length $2R$ and edges parallel to $\boldsymbol{\nu}^1$, $\boldsymbol{\nu}^2$, and $\boldsymbol{\nu}^3$, centred at $(x, u(x))$ for each $x \in \Omega \cap \mathcal{V}(u)$. We discard any such cubes that do not intersect $K \times \mathbb{R}$ and denote the remaining, finitely many cubes by $C_1, \ldots, C_M$.

Now for any edge $E \subsetneq \mathrm{graph}(u)$ connecting two points of $(\Omega' \cap \mathcal{V}(u)) \times \mathbb{R}$, say $(x_1, u(x_1))$ and $(x_2, u(x_2))$, there exists a number $h \in [\frac{1}{2}, 2]$ such that $E \setminus (\boldsymbol{B}_R(x_1, u(x_1)) \cup \boldsymbol{B}_R(x_2, u(x_2)))$ is covered disjointly, up to finitely many points, by open cylinders of height $2hR$, with axes contained in E, and with square cross-sections of side length $2R$ and sides parallel to $\boldsymbol{\nu}^1$, $\boldsymbol{\nu}^2$, or $\boldsymbol{\nu}^3$. For any edge in graph(u) that intersects $K \times \mathbb{R}$ but does not connect two points of $(\Omega' \cap \mathcal{V}(u)) \times \mathbb{R}$, we may use cubes of side length R instead to cover (up to finitely many points) any parts in $K \times \mathbb{R}$. We discard anything not intersecting $K \times \mathbb{R}$ and label the remaining cylinders $C_1', \ldots, C_{M'}'$.

Finally, we may cover the faces of graph(u) that intersect $K \times \mathbb{R}$ (up to a one-dimensional set) by finitely many boxes of side lengths $2R$, $2hR$, and $2\ell R$, with $h, \ell \in [\frac{1}{2}, 2]$, and denote them $C_1'', \ldots, C_{M''}''$. We may achieve that the union of all the sets C_m ($1 \leq m \leq M$), $C'_{m'}$ ($1 \leq m' \leq M'$), and $C''_{m''}$ ($1 \leq m'' \leq M''$) disjointly cover $G_R \cap (K \times \mathbb{R})$, where

$$G_R = \bigcup_{\boldsymbol{x} \in \mathrm{graph}(u)} \left\{ \boldsymbol{x} + r\boldsymbol{\nu}^1 + s\boldsymbol{\nu}^2 + t\boldsymbol{\nu}^3 \colon -R < r, s, t < R \right\},$$

up to a 2-dimensional set.

Now let $\epsilon > 0$ and suppose that $v \in \mathcal{B}_\epsilon(u; \Omega)$. Since u and v are Lipschitz continuous with Lipschitz constant $\sqrt{2}$, the inequality $\|u - v\|_{L^1(\Omega)} \leq \epsilon$ implies that $\|u - v\|_{L^\infty(\Omega')} \leq 2(3\epsilon/\pi)^{1/3}$, provided that ϵ is sufficiently small. In particular, we may assume that $\mathrm{graph}(v) \subseteq G_R$.

Define $U_m = \left\{x \in \mathbb{R}^2 \colon (x, u(x)) \in C_m\right\}$ and $V_m = \left\{x \in \mathbb{R}^2 \colon (x, v(x)) \in C_m\right\}$ for $m = 1, \ldots, M$, and define $U'_{m'}$, $V'_{m'}$ ($m' = 1, \ldots, M'$) and $U''_{m''}$, $V''_{m''}$ ($m'' = 1, \ldots, M''$) analogously. Then the sets U_m ($1 \leq m \leq M$), $U'_{m'}$ ($1 \leq m' \leq M'$), and $U''_{m''}$ ($1 \leq m'' \leq M''$) are pairwise disjoint and cover almost all of K. The sets V_m, $V'_{m'}$, and $V''_{m''}$, on the other hand, do not necessarily cover almost all of K, but they are pairwise disjoint. Moreover, the construction and Proposition 33 imply that

$$\|D\nabla u\|(U_m) = 6R, \quad m = 1, \ldots, M.$$

For $m' = 1, \ldots, M'$ such that $C_{m'}$ has height $2hR$, we find that

$$\|D\nabla u\|(U'_{m'}) = 4hR,$$

and for $m'' = 1, \ldots, M''$,

$$\|D\nabla u\|(U''_{m''}) = 0.$$

Consider $x_0 \in \mathcal{E}(u)$ such that $(x_0, u(x_0))$ is the centre of one of the cylinders $C'_{m'}$ (for $1 \leq m' \leq M'$). Then we may apply Lemma 35 to the function $x \mapsto v(x + x_0) - u(x_0)$ and conclude that

$$\|D\nabla v\|(V'_{m'}) \geq 4hR - \frac{\eta}{M + M'},$$

provided that $2hR$ is the height of $C'_{m'}$ and ϵ is sufficiently small. Similar arguments apply if $x_0 \in \mathcal{V}(u)$ such that $(x_0, u(x_0))$ is the centre of one of the cubes C_m (for $1 \leq m \leq M$). In this case, we use Lemma 36 or Lemma 37, which gives the estimate

$$\|D\nabla v\|(V_m) \geq 6R - \frac{\eta}{M + M'},$$

provided that ϵ is sufficiently small.

Recalling that $\|D\nabla v\|(\Omega) \leq \|D\nabla u\|(\Omega) + \epsilon$ and $\|D\nabla u\|(\Omega \setminus K) < \eta$, and comparing the above inequalities for u and v, we find that we necessarily have the inequalities

$$\|D\nabla v\|(V_m) \leq 6R + 3\eta, \quad m = 1, \ldots, M,$$

provided that ϵ is small enough. For $m' = 1, \ldots, M'$ such that $C_{m'}$ has height $2hR$, we find that

$$\|D\nabla v\|(V'_{m'}) \leq 4hR + 3\eta,$$

and for $m'' = 1, \ldots, M''$,

$$\|D\nabla v\|(V''_{m''}) \leq 3\eta.$$

Now that we have these inequalities, we can extract more information out of Lemmas 34, 35, and 36. It follows that away from the boundaries, the structure of u in U_m or $U'_{m'}$ or $U''_{m''}$ is the same as the structure of v in V_m or $V'_{m'}$ or $V''_{m''}$, up to translations and additive constants. In particular, given $\delta > 0$, the following is true: if $x \in U'_{m'}$ with $\operatorname{dist}(x, \partial U'_{m'}) \geq R/4$ or $x \in U''_{m''}$ with $\operatorname{dist}(x, \partial U''_{m''}) \geq R/4$, then $x \in \Delta_{\delta R}^{R/10}(u,v)$. The analogous statement for U_m is true, provided that m corresponds to a point in $\mathcal{V}^{\wedge\vee}(u)$ or $\mathcal{V}^{\vee\wedge}(u)$ (not $\mathcal{V}^{\wedge}(u)$ or $\mathcal{V}^{\vee}(u)$).

Using similar constructions, but varying the sizes of the boxes involved, we can make similar statements about points not covered so far. We omit the details, because they would be tedious, but it is not difficult to see that

$$\{x \in \Omega' \colon \operatorname{dist}(x, \mathcal{V}^{\wedge}(u) \cup \mathcal{V}^{\vee}(u)) > 10R\} \subseteq \Delta_{\delta R}^{R/10}(u,v).$$

For any $x_0 \in \mathcal{V}^{\vee}(u) \cap K$, it then follows that v coincides with a function of the form

$$x \mapsto (\lambda_1 \vee \lambda_2 \vee \lambda_3)(x - x_0 + a) + u(x_0) + b$$

on $B_{20R}(x_0) \setminus B_{10R}(x_0)$, where $a \in B_{\delta R}(0)$ and $-\delta R \leq b \leq \delta R$. We simply replace v by this function in $B_{10R}(x_0)$. With the analogous construction near every point of $\mathcal{V}^{\wedge}(u) \cap K$, we obtain a function $P(v)$ such that $d_K^{R/10}(u, P(v)) \leq \delta R$. Lemma 38 shows that $\|D\nabla P(v)\|(\Omega) \leq \|D\nabla v\|(\Omega)$, with equality only if $P(v) = v$. It is clear that the statement of Theorem 7 follows. ☐

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