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BUBBLE TOWER SOLUTIONS FOR SUPERCRITICAL ELLIPTIC PROBLEM IN \mathbb{R}^N

WENJING CHEN, JUAN DÁVILA, AND IGNACIO GUERRA

ABSTRACT. We consider the following problem

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, & u > 0 \quad \text{in } \mathbb{R}^N; \\ u(z) \rightarrow 0 \quad \text{as } |z| \rightarrow \infty, \end{cases}$$

where $p = p^* + \varepsilon$, with $p^* = \frac{N+2}{N-2}$, $1 < q < \frac{N+2}{N-2}$ if $N \geq 4$, $3 < q < 5$ if $N = 3$, $\lambda > 0$, and ε is a positive parameter. We prove that for $\varepsilon > 0$ small enough, it has a solution with the shape of a tower of bubbles.

Keywords: elliptic equation, non-uniqueness, bubble-tower solutions

1. INTRODUCTION

We are interested in the elliptic equation

$$(1.1) \quad \begin{cases} -\Delta u + u = u^p + \lambda u^q, & u > 0 \quad \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \end{cases}$$

where $N \geq 3$, $\lambda > 0$ and $1 < q < p$. This problem arises in the study of standing waves of a nonlinear Schrödinger equation with two power type nonlinearities, see for example Tao, Visan and Zhang [28].

If $p = q$, equation (1.1) reduces to

$$(1.2) \quad \begin{cases} -\Delta u + u = u^p, & u > 0 \quad \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \end{cases}$$

after a suitable scaling.

Thanks to the classical result of Gidas, Ni and Nirenberg [15], solutions of (1.1) and (1.2) are radially symmetric about some point, which we will assume is always the origin.

It is well known that problem (1.2) has a solution if and only if $1 < p < \frac{N+2}{N-2}$. Existence was proved by Berestycki and Lions [2], while non-existence follows from the Pohozaev identity [26]. Uniqueness also holds and was fully settled by Kwong [16], after a series of contributions [4, 17, 23, 24, 22, 21]. See also Felmer, Quaas, Tang and Yu [10] for further properties.

Concerning (1.1), the work of Berestycki and Lions [2] is still applicable if $1 < q < p < \frac{N+2}{N-2}$, and one obtains existence of a solution. If $p, q \geq \frac{N+2}{N-2}$ there is no solution, again from the Pohozaev identity.

Recently, Dávila, del Pino and Guerra [5] proved that uniqueness does not hold in general for (1.1), if $1 < q < p < \frac{N+2}{N-2}$. More precisely if $N = 3$, the authors

obtained at least three solutions to problem (1.1) if $1 < q < 3$, $\lambda > 0$ is sufficiently large and fixed, and $p < 5$ is close enough to 5.

Let us mention some contributions to the question of existence for (1.1) when one exponent is subcritical and other is critical or supercritical. If $1 < q < p = \frac{N+2}{N-2}$ in (1.1), Alves, de Morais Filho and Souto [1] proved:

- when $N \geq 4$, there exists a nontrivial classical solution for all $\lambda > 0$ and $1 < q < \frac{N+2}{N-2}$;
- when $N = 3$, there exists a nontrivial classical solution for all $\lambda > 0$ and $3 < q < 5$;
- when $N = 3$, there exists a nontrivial classical solution for $\lambda > 0$ large enough and $1 < q \leq 3$.

Moreover, Ferrero and Gazzola [11] proved that for $q < \frac{N+2}{N-2} \leq p$, there exists $\bar{\lambda} > 0$, such that if $\lambda > \bar{\lambda}$, then (1.1) has at least one solution, while for $q < \frac{N+2}{N-2} < p$, there exists $0 < \underline{\lambda} < \bar{\lambda}$ such that if $\lambda < \underline{\lambda}$, then there is no solution.

In this paper, we are interested in multiplicity of solutions of (1.1), and for this we take an asymptotic approach, that is, we consider

$$(1.3) \quad \begin{cases} -\Delta u + u = u^p + \lambda u^q, & u > 0 \quad \text{in } \mathbb{R}^N, \\ u(z) \rightarrow 0 \quad \text{as } |z| \rightarrow \infty, \end{cases}$$

where $p = p^* + \varepsilon$, with $p^* = \frac{N+2}{N-2}$, $\lambda > 0$ and $\varepsilon > 0$ are parameters, and q satisfies

$$(1.4) \quad 1 < q < \frac{N+2}{N-2} \quad \text{if } N \geq 4; \quad 3 < q < 5 \quad \text{if } N = 3.$$

Our result can be stated as follows.

Theorem 1.1. *Let $\lambda > 0$ and let q satisfy (1.4). Given an integer $k \geq 1$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there is a solution $u_\varepsilon(z)$ of problem (1.3) of the form*

$$(1.5) \quad u_\varepsilon(z) = (N(N-2))^{\frac{N-2}{4}} \sum_{j=1}^k \frac{\varepsilon^{-[(j-1) + \frac{1}{p^*-q}]} (\Lambda_j^*)^{-\frac{N-2}{2}}}{\left(1 + \varepsilon^{-\frac{4}{N-2}[(j-1) + \frac{1}{p^*-q}]} (\Lambda_j^*)^{-2} |z|^2\right)^{\frac{N-2}{2}}} (1 + o(1)),$$

where the constants $\Lambda_j^* > 0$, $j = 1, 2, \dots, k$, can be computed explicitly and depend on k, N, q .

The expansion (1.5) is valid if $\frac{1}{C} \varepsilon^{\frac{2}{N-2}[(i-1) + \frac{1}{p^*-q}]} \leq |z| \leq C \varepsilon^{\frac{2}{N-2}[(i-1) + \frac{1}{p^*-q}]}$, with some $i \in \{1, 2, \dots, k\}$, and $o(1) \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$ in this region.

The solutions described in this result behave like a superposition of “bubbles” of different blow-up orders centered at the origin, and hence have been called bubble-tower solutions. By bubbles we mean the functions

$$(1.6) \quad w_\mu(z) = \alpha_N \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |z|^2)^{\frac{N-2}{2}}}, \quad \text{with } \alpha_N = (N(N-2))^{\frac{N-2}{4}},$$

where $\mu > 0$, which are the unique positive solutions (except translations) of

$$-\Delta w = w^{p^*} \quad \text{in } \mathbb{R}^N.$$

Based on numerical simulations we present bifurcation diagrams for solutions of (1.3) where q satisfies (1.4). In Figure 1 (left) we show the bifurcation diagram as

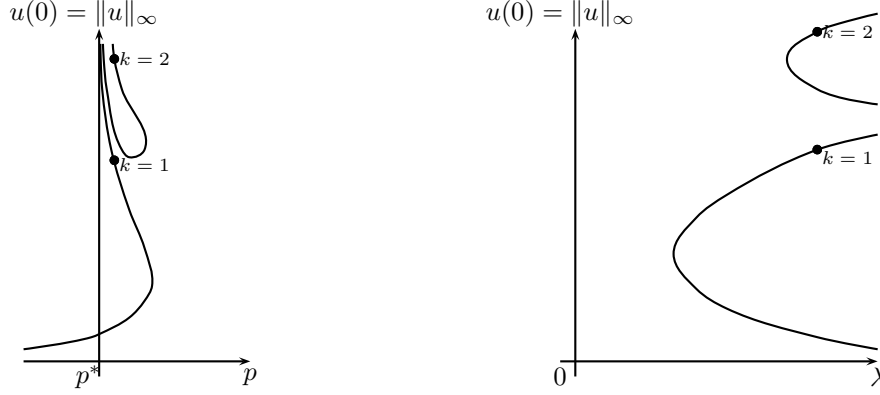


FIGURE 1. Left: $u(0)$ vs. p for λ large and fixed. Right: $u(0)$ vs. λ for $p = p^* + \varepsilon$, $\varepsilon > 0$ small and fixed.

a function of p for a fixed large λ , and in Figure 1 (right) we show the diagram as a function of λ for $p = p^* + \varepsilon$, $\varepsilon > 0$ small and fixed. In both diagrams we observe branches of solutions, with the upper part having unbounded solutions as $\varepsilon \rightarrow 0$ or $\lambda \rightarrow \infty$. We believe that the solutions constructed in Theorem 1.1 are located on these upper branches, and are shown in the diagrams for the cases of 1 and 2 bubbles.

Bubble-tower solutions were found by del Pino, Dolbeault and Musso [6] for a slightly supercritical Brezis-Nirenberg problem in a ball, and after that have been studied intensively [3, 7, 8, 9, 13, 14, 18, 19, 20, 25]. In particular we mention the work of Campos [3] who considered the existence of bubble-tower solutions to a problem related to ours:

$$\begin{cases} -\Delta u = u^{p^* \pm \varepsilon} + u^q, & u > 0 \quad \text{in } \mathbb{R}^N; \\ u(z) \rightarrow 0 \quad \text{as } |z| \rightarrow \infty, \end{cases}$$

with $\frac{N}{N-2} < q < p^* = \frac{N+2}{N-2}$, $N \geq 3$.

For the proof of Theorem 1.1, we consider a variation of the so-called Emden-Fowler transformation:

$$v(x) = \left(\frac{p^* - 1}{2} \right)^{\frac{2}{p^*-1}} r^{\frac{2}{p^*-1}} u(r),$$

with

$$r = |z| = e^{-\frac{p^*-1}{2}x}, \quad x \in (-\infty, +\infty).$$

Then finding a radial solution $u(r)$ to (1.3) corresponds to solving the problem

$$(1.7) \quad \begin{cases} \mathcal{L}_0(v) = \alpha_\varepsilon e^{\varepsilon x} v^{p^* + \varepsilon} + \lambda \beta_N e^{-(p^*-q)x} v^q & \text{in } (-\infty, +\infty); \\ v(x) > 0 & \text{for } x \in (-\infty, +\infty); \\ v(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

where

$$(1.8) \quad \mathcal{L}_0(v) = -v'' + v + \left(\frac{2}{N-2} \right)^2 e^{-\frac{4}{N-2}x} v,$$

is the transformed operator associated to $-\Delta + I$, and $\alpha_\varepsilon, \beta_N$ are constants, see (2.5).

Under the Emden-Fowler transformation the bubbles w_μ take the form

$$(1.9) \quad W(x - \xi) = \left(\frac{4N}{N-2} \right)^{\frac{N-2}{4}} e^{-(x-\xi)} \left(1 + e^{-\frac{4}{N-2}(x-\xi)} \right)^{-\frac{N-2}{2}}$$

with $\mu = e^{-\frac{2}{N-2}\xi}$, and solve

$$\begin{cases} W'' - W + W^{p^*} = 0, & \text{in } (-\infty, +\infty); \\ W'(0) = 0; \\ W(x) > 0, \quad W(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

In Section 2, we build an approximate solution to (1.7) as a sum of suitable projections of the transformed bubbles W centered at $0 < \xi_1 < \dots < \xi_k$ with $\xi_1 \rightarrow \infty$. After the study of the linearized problem at the approximate solution in Section 3, and solvability of a nonlinear projected problem in Section 4, we perform a Lyapunov-Schmidt reduction procedure as in [12, 18, 3]. Then the problem becomes to find a critical point of some functional depending on $0 < \xi_1 < \dots < \xi_k$. This is done in Section 5 where Theorem 1.1 is proved.

From the technical point of view, one difficulty is due to the form of the linearized operator. As $r \rightarrow \infty$ dominates $-\Delta + I$ (or \mathcal{L}_0 as $x \rightarrow -\infty$ after the change of variables) while near the regions of concentration the important part of the linearization is $\Delta + p^*w_\mu^{p^*-1}$. This is taken into account in the norm we use for the solutions of linearized problem, and it is more naturally written for the functions after the Emden-Fowler transformation. This is different from many previous works, but is already contained in [5].

2. THE FIRST APPROXIMATE SOLUTION

In this section, we build the first approximate solution to (1.3). In order to do this, we introduce U_μ as the unique solution of the following problem

$$(2.1) \quad \begin{cases} -\Delta U_\mu + U_\mu = w_\mu^{p^*} & \text{in } \mathbb{R}^N, \\ U_\mu(z) \rightarrow 0 & \text{as } |z| \rightarrow \infty, \end{cases}$$

where w_μ are the bubbles (1.6). We write

$$U_\mu(z) = w_\mu(z) + R_\mu(z).$$

Then $R_\mu(z)$ satisfies

$$-\Delta R_\mu(z) + R_\mu(z) = -w_\mu(z) \quad \text{in } \mathbb{R}^N, \quad R_\mu(z) \rightarrow 0 \quad \text{as } |z| \rightarrow \infty.$$

We have the following result, whose proof is postponed to the Appendix.

Lemma 2.1. *Assume $0 < \mu \leq 1$, we have*

- (a) $0 < U_\mu(z) \leq w_\mu(z)$, for $z \in \mathbb{R}^N$.
- (b) One has

$$U_\mu(z) \leq C\mu^{\frac{N-2}{2}}|z|^{-(N+2)}, \quad \text{for } |z| \geq R,$$

where R is a large positive number but fixed.

(c) Given any $\mu > 0$ small, we have

$$(2.2) \quad |R_\mu(z)| \leq C \frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}} \quad \text{for } N \geq 3, \quad |z| \geq 1.$$

$$(2.3) \quad |R_\mu(z)| \leq C \begin{cases} \mu^{-\frac{N-6}{2}} & \text{for } N \geq 5; \\ \mu \log \frac{1}{\mu} & \text{for } N = 4; \\ \mu^{\frac{1}{2}} & \text{for } N = 3. \end{cases} \quad |z| \leq \frac{\mu}{2}.$$

$$(2.4) \quad |R_\mu(z)| \leq C \begin{cases} \mu^{-\frac{N-6}{2}} \frac{1}{(1+|\frac{z}{\mu}|^2)^{\frac{N-4}{2}}} & \text{for } N \geq 5; \\ \mu \log \frac{1}{|z|} & \text{for } N = 4; \\ \mu^{\frac{1}{2}} & \text{for } N = 3. \end{cases} \quad \frac{\mu}{2} \leq |z| \leq 1.$$

We define the following Emden-Fowler transformation

$$v(x) = \mathcal{T}(u(r)) = \left(\frac{p^* - 1}{2} \right)^{\frac{2}{p^* - 1}} r^{\frac{2}{p^* - 1}} u(r), \quad r = |z| = e^{-\frac{p^* - 1}{2}x}$$

with $x \in (-\infty, +\infty)$. Using this transformation, finding a radial solution $u(r)$ to problem (1.3) corresponds to that of solving problem (1.7), where

$$(2.5) \quad \alpha_\varepsilon = \left(\frac{p^* - 1}{2} \right)^{-\frac{2\varepsilon}{p^* - 1}}, \quad \beta_N = \left(\frac{p^* - 1}{2} \right)^{\frac{2(p^* - q)}{p^* - 1}}.$$

Define

$$V_\xi(x) = \mathcal{T}(U_\mu)(r), \quad \text{with } r = e^{-\frac{p^* - 1}{2}x}, \quad \mu = e^{-\frac{2}{N-2}\xi}.$$

Then $V_\xi(x)$ is the solution of the problem

$$\begin{cases} \mathcal{L}_0 V_\xi(x) = W(x - \xi)^{p^*} & \text{in } (-\infty, +\infty); \\ V_\xi(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

Note that \mathcal{L}_0 is the transformed operator associated to $-\Delta + Id$ and given in (1.8).

We write

$$V_\xi(x) = W(x - \xi) + R_\xi(x),$$

where W is given in (1.9) and $R_\xi(x) = \mathcal{T}(R_\mu)(r)$. By the Emden-Fowler transformation and as a consequence of Lemma 2.1, we have the following estimates.

Lemma 2.2. For $\xi > 0$, we have

- (a) $0 < V_\xi(x) \leq W(x - \xi) = O(e^{-|x - \xi|})$, for $x \in \mathbb{R}$.
(b)

$$(2.6) \quad V_\xi(x) \leq C e^{\frac{N+6}{N-2}x} e^{-\xi}, \quad \text{for } -\infty < x \leq -\frac{N-2}{2} \log R,$$

for $R > 0$ is a fixed large number as Lemma 2.1.

(c) For $N \geq 3$, there is a positive constant C , such that

$$|R_\xi(x)| \leq C \begin{cases} e^{-|x - \xi|} & \text{if } x \leq 0; \\ e^{-|x - \xi|} e^{-\frac{2}{N-2} \min\{x, \xi\}} & \text{if } x \geq 0. \end{cases}$$

Define

$$Z_\xi(x) := \partial_\xi V_\xi(x) = \partial_\xi W(x - \xi) + \partial_\xi R_\xi(x).$$

Note that $\partial_\xi W(x - \xi) = O(e^{-|x-\xi|})$ and

$$\partial_\xi W(x - \xi) = -\frac{2}{N-2} \mu \mathcal{T}(\partial_\mu w_\mu(r)),$$

$$(2.7) \quad Z_\xi(x) = -\frac{2}{N-2} \mu \mathcal{T}(\tilde{Z}_\mu(r)) \quad \text{with} \quad \tilde{Z}_\mu(z) = \partial_\mu U_\mu(z),$$

$$(2.8) \quad \partial_\xi R_\xi(x) = -\frac{2}{N-2} \mu \mathcal{T}(\partial_\mu R_\mu(r)).$$

Then from (6.1), (2.8) and Lemma 2.2 (c), we have for $N \geq 3$,

$$|\partial_\xi R_\xi(x)| \leq C \begin{cases} e^{-|x-\xi|} & \text{if } x \leq 0; \\ e^{-|x-\xi|} e^{-\frac{2}{N-2} \min\{x, \xi\}} & \text{if } x \geq 0. \end{cases}$$

Therefore

$$Z_\xi(x) = O(e^{-|x-\xi|}) \quad \text{for } \forall x \in \mathbb{R}.$$

Moreover, from (6.2) and (2.7), we find

$$|Z_\xi(x)| \leq C e^{\frac{N+6}{N-2}x} e^{-\xi}, \quad \text{for } -\infty < x \leq -\frac{N-2}{2} \log R,$$

for a fixed large $R > 0$.

Let $\eta > 0$ be a small but fixed number. Given an integer number k , let Λ_j , for $j = 1, \dots, k$, be positive numbers and satisfy

$$(2.9) \quad \eta < \Lambda_j < \frac{1}{\eta}.$$

Set

$$(2.10) \quad \mu_1 = \varepsilon^{\frac{2}{(N+2)-(N-2)q}} \Lambda_1 \quad \text{and} \quad \mu_j = \varepsilon^{\frac{2}{N-2}(j-1) + \frac{2}{(N+2)-(N-2)q}} \Lambda_j$$

for $j = 2, \dots, k$. We observe that

$$\frac{\mu_{j+1}}{\mu_j} = \varepsilon^{\frac{2}{N-2}} \frac{\Lambda_{j+1}}{\Lambda_j}, \quad j = 1, \dots, k-1.$$

Define k points in \mathbb{R} as

$$\mu_j = e^{-\frac{2}{N-2}\xi_j}, \quad j = 1, \dots, k.$$

Then we have that

$$0 < \xi_1 < \xi_2 < \dots < \xi_k,$$

and

$$(2.11) \quad \begin{cases} \xi_1 = -\frac{1}{p^*-q} \log \varepsilon - \frac{N-2}{2} \log \Lambda_1, \\ \xi_j - \xi_{j-1} = -\log \varepsilon - \frac{N-2}{2} \log \frac{\Lambda_j}{\Lambda_{j-1}}, \quad j = 2, \dots, k. \end{cases}$$

Set

$$(2.12) \quad W_j = W(x - \xi_j), \quad R_j = R_{\xi_j}(x), \quad V_j = W_j + R_j, \quad V = \sum_{j=1}^k V_j.$$

We look for a solution of (1.3) of the form $u = \sum_{j=1}^k U_{\mu_j} + \psi$ corresponds to find a solution of (1.7) of the form $v = V + \phi$, where V is given by (2.12) and $\phi = \mathcal{T}(\psi)$ is a small term. Thus problem (1.7) becomes

$$(2.13) \quad \begin{cases} \mathcal{L}_\varepsilon(\phi) = N(\phi) + E & \text{in } (-\infty, +\infty); \\ \phi(x) > 0 & \text{for } x \in (-\infty, +\infty); \\ \phi(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

where

$$\begin{aligned} \mathcal{L}_\varepsilon(\phi) &= \mathcal{L}_0(\phi) - \alpha_\varepsilon(p^* + \varepsilon)e^{\varepsilon x}V^{p^*+\varepsilon-1}\phi - \lambda q\beta_N e^{-(p^*-q)x}V^{q-1}\phi, \\ N(\phi) &= \alpha_\varepsilon e^{\varepsilon x} \left[(V + \phi)^{p^*+\varepsilon} - V^{p^*+\varepsilon} - (p^* + \varepsilon)V^{p^*+\varepsilon-1}\phi \right] \\ &\quad + \lambda\beta_N e^{-(p^*-q)x} \left[(V + \phi)^q - V^q - qV^{q-1}\phi \right] \end{aligned}$$

and

$$\begin{aligned} E &= \alpha_\varepsilon e^{\varepsilon x}V^{p^*+\varepsilon} - \mathcal{L}_0(V) + \lambda\beta_N e^{-(p^*-q)x}V^q \\ &= \alpha_\varepsilon e^{\varepsilon x}V^{p^*+\varepsilon} - \sum_{j=1}^k W_j^{p^*} + \lambda\beta_N e^{-(p^*-q)x}V^q. \end{aligned}$$

where \mathcal{L}_0 is defined by (1.8).

3. THE LINEAR PROBLEM

In order to solve problem (2.13), we consider first the following problem: given points $\xi = (\xi_1, \dots, \xi_k)$, finding a function ϕ such that for certain constants c_1, c_2, \dots, c_k ,

$$(3.1) \quad \begin{cases} \mathcal{L}_\varepsilon(\phi) = N(\phi) + E + \sum_{j=1}^k c_j Z_j & \text{in } (-\infty, +\infty); \\ \lim_{|x| \rightarrow \infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall j = 1, \dots, k, \end{cases}$$

where $Z_j(x) = Z_{\xi_j}(x) = \partial_{\xi_j} V_{\xi_j}(x)$ for $j = 1, 2, \dots, k$.

To solve (3.1), it is important to understand its linear part, thus we consider the following problem: given a function h , finding ϕ such that

$$(3.2) \quad \begin{cases} \mathcal{L}_\varepsilon(\phi) = h + \sum_{j=1}^k c_j Z_j & \text{in } (-\infty, +\infty); \\ \lim_{|x| \rightarrow \infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall j = 1, \dots, k, \end{cases}$$

for certain constants c_j .

Now we analyze invertibility properties of the operator \mathcal{L}_ε under the orthogonality conditions. Let σ satisfy

$$(3.3) \quad 0 < \sigma < \min \left\{ q - 1, 1, \frac{(N + 2)(2q - 1)}{N + 6}, \frac{3q - p^*}{2} \right\}.$$

We define the real number M as follows

$$(3.4) \quad M = \begin{cases} 0, & \text{if } 1 \geq \frac{4}{N-2} + \sigma; \\ \max\{0, \gamma\}, & \text{if } 1 \leq \frac{4}{N-2} + \sigma, \end{cases}$$

where γ satisfies

$$\left(1 - \left(\frac{4}{N-2} + \sigma\right)^2\right) e^{-\frac{4}{N-2}\gamma} = -\frac{1}{2} \left(\frac{2}{N-2}\right)^2.$$

We define the following norms for functions ϕ, h defined on \mathbb{R} ,

$$(3.5) \quad \|\phi\|_* = \sup_{x \leq -M} e^{-(\frac{4}{N-2} + \sigma)x} e^{\sigma\xi_1} |\phi(x)| + \sup_{x \in \mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|} \right)^{-1} |\phi(x)|,$$

and

$$\|h\|_{**} = \sup_{x \in \mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|} \right)^{-1} |h(x)|.$$

The choice of norm here is motivated by the presence of 2 regimes in the solution of the linearized problem. Near the concentration points ξ_j we have a right hand side of the form $|h(x)| \leq C e^{-\sigma|x-\xi_j|}$ and near these points the dominant terms in the linear operator \mathcal{L}_ε are

$$-\phi'' + \phi - \alpha_\varepsilon(p^* + \varepsilon) e^{\varepsilon x} V^{p^* + \varepsilon - 1} \phi,$$

so we can expect the solution ϕ to be controlled by $|\phi(x)| \leq C e^{-\sigma|x-\xi_j|}$. For $x \leq 0$ the dominant part of the linear operator is

$$\left(\frac{2}{N-2}\right)^2 e^{-\frac{4}{N-2}x} \phi.$$

Since the right hand side is controlled by $e^{-\sigma|x-\xi_1|}$, we can control ϕ using as supersolution $e^{(\frac{4}{N-2} + \sigma)x} e^{-\sigma\xi_1}$. Actually this will be a super solution for the whole linear operator for $x \leq -M$, where M is defined in (3.4).

The main result in this section is solvability of problem (3.2).

Proposition 3.1. *There exist positive numbers ε_0 , and $C > 0$ such that if the points $0 < \xi_1 < \xi_2 < \dots < \xi_k$ satisfy (2.11), then for all $0 < \varepsilon < \varepsilon_0$ and all functions $h \in C(\mathbb{R}; \mathbb{R})$ with $\|h\|_{**} < +\infty$, problem (3.2) has a unique solution $\phi =: T_\varepsilon(h)$ with $\|\phi\|_* < +\infty$. Moreover,*

$$(3.6) \quad \|\phi\|_* \leq C \|h\|_{**} \quad \text{and} \quad |c_j| \leq C \|h\|_{**}.$$

We first consider a simpler problem

$$(3.7) \quad \begin{cases} \mathcal{L}_0(\phi) - \alpha_\varepsilon(p^* + \varepsilon) e^{\varepsilon x} V^{p^* + \varepsilon - 1} \phi = h + \sum_{j=1}^k c_j Z_j & \text{in } (-\infty, +\infty); \\ \lim_{|x| \rightarrow \infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall j = 1, \dots, k, \end{cases}$$

for certain constants c_j , here \mathcal{L}_0 is defined by (1.8).

Lemma 3.2. *Under the assumptions of Proposition 3.1, then for all $0 < \varepsilon < \varepsilon_0$ and any h, ϕ solution of (3.7), we have*

$$(3.8) \quad \|\phi\|_* \leq C \|h\|_{**},$$

and

$$(3.9) \quad |c_j| \leq C \|h\|_{**}.$$

Proof. To prove (3.8), by contradiction, we suppose that there exist sequences ϕ_n , h_n , ε_n and c_j^n that satisfy (3.7), with

$$\|\phi_n\|_* = 1, \quad \|h_n\|_{**} \rightarrow 0, \quad \varepsilon_n \rightarrow 0.$$

We get a contradiction by the following steps.

Step 1: $c_j^n \rightarrow 0$ as $n \rightarrow +\infty$.

Multiplying (3.7) by Z_i^n and integrating by parts twice, we get that

$$(3.10) \quad \sum_{j=1}^k c_j^n \int_{\mathbb{R}} Z_j^n Z_i^n = - \int_{\mathbb{R}} h_n Z_i^n + \int_{\mathbb{R}} \left[\mathcal{L}_0(Z_i^n) - \alpha_{\varepsilon_n} (p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^* + \varepsilon_n - 1} Z_i^n \right] \phi_n.$$

Note that

$$\int_{\mathbb{R}} Z_j^n Z_i^n = C \delta_{ij} + o(1),$$

where δ_{ij} is Kronecker's delta. Then (3.10) defines a linear system in the c_j^n 's which is almost diagonal as $n \rightarrow \infty$.

Since $Z_i^n(x) = \partial_{\xi_i^n} V_{\xi_i^n}(x) = O(e^{-|x - \xi_i^n|})$, we then have

$$(3.11) \quad \begin{aligned} \left| \int_{\mathbb{R}} h_n Z_i^n \right| &\leq C \|h_n\|_{**} \int_{\mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x - \xi_j^n|} \right) e^{-|x - \xi_i^n|} dx \\ &\leq Ck \|h_n\|_{**} \int_{\mathbb{R}} e^{-|y|} dy \leq C \|h_n\|_{**}. \end{aligned}$$

Moreover, Z_i^n satisfy

$$\mathcal{L}_0(Z_i^n) = p^* W^{p^* - 1}(x - \xi_i^n) \partial_{\xi_i^n} W(x - \xi_i^n),$$

so we get

$$(3.12) \quad \left| \int_{\mathbb{R}} \left[\mathcal{L}_0(Z_i^n) - \alpha_{\varepsilon_n} (p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^* + \varepsilon_n - 1} Z_i^n \right] \phi_n \right| = o(1) \|\phi_n\|_*.$$

From (3.10)-(3.12), we obtain

$$(3.13) \quad |c_j^n| \leq C \|h_n\|_{**} + o(1) \|\phi_n\|_*.$$

Thus $\lim_{n \rightarrow \infty} c_j^n = 0$.

Step 2: For any $L > 0$, any $l \in \{1, 2, \dots, k\}$, we have

$$(3.14) \quad \sup_{x \in [\xi_l^n - L, \xi_l^n + L]} |\phi_n(x)| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Indeed, suppose not, we assume that there exist $L > 0$ and some $l \in \{1, 2, \dots, k\}$ such that

$$|\phi_n(x_{n,l})| \geq c > 0, \quad \text{for some } x_{n,l} \in [\xi_l^n - L, \xi_l^n + L].$$

By elliptic estimates, there is a subsequence of ϕ_n converging uniformly on compact sets to a nontrivial bounded solution $\tilde{\phi}$ of

$$\mathcal{L}_0(\tilde{\phi}) = p^* W^{p^* - 1}(x - \xi_l) \tilde{\phi},$$

where $\xi_l = \lim_{n \rightarrow \infty} \xi_l^n$. By nondegeneracy [27], it is well known that $\tilde{\phi} = cZ_l$ for some constant $c \neq 0$. But taking the limit in the orthogonality condition $\int_{\mathbb{R}} Z_l^n \phi_n = 0$, we obtain $\tilde{\phi} = 0$, which is a contradiction. Thus (3.14) holds.

Step 3: We prove that $\|\phi_n\|_ \rightarrow 0$ as $n \rightarrow \infty$.*

Claim: For any $L > 0$ and $j \in \{1, 2, \dots, k\}$, we have

$$(3.15) \quad \sup_{\mathbb{R} \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]} \left(\sum_{j=1}^k e^{-\sigma|x - \xi_j^n|} \right)^{-1} |\phi_n(x)| \rightarrow 0,$$

and

$$(3.16) \quad \sup_{x \leq -M} e^{-(\frac{4}{N-2} + \sigma)x} e^{\sigma \xi_1^n} |\phi_n(x)| \rightarrow 0,$$

as $n \rightarrow +\infty$.

By the definition of $\|\cdot\|_*$ in (3.5), using (3.14), (3.15) and (3.16), we get that $\|\phi_n\|_* \rightarrow 0$ as $n \rightarrow \infty$.

Now we prove the above claim. We note that

$$h_n + \sum_{j=1}^k c_j^n Z_j^n \leq (C_0 \|h_n\|_{**} + o(\|\phi_n\|_*)) \sum_{j=1}^k e^{-\sigma|x - \xi_j^n|}, \quad \text{with } C_0 > 0.$$

For $x \in \mathbb{R} \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$, let us define

$$\begin{aligned} \tilde{\psi}_n(x) &= \left(C_0 \|h_n\|_{**} + e^{\sigma L} \sup_{\cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]} |\phi_n(x)| + o(\|\phi_n\|_*) \right) \sum_{j=1}^k e^{-\sigma|x - \xi_j^n|} \\ &\quad + \varrho \sum_{j=1}^k e^{-\bar{\sigma}|x - \xi_j^n|} \end{aligned}$$

with $\varrho > 0$ small but fixed and $0 < \bar{\sigma} < \sigma$. Then by choosing suitable large $L > 0$, we get

$$\begin{aligned} \mathcal{L}_0(\tilde{\psi}_n(x)) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^* + \varepsilon_n - 1} \tilde{\psi}_n(x) \\ \geq \mathcal{L}_0(\phi_n(x)) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^* + \varepsilon_n - 1} \phi_n(x). \end{aligned}$$

On the other hand, we have that for any $L > 0$ and $j \in \{1, 2, \dots, k\}$,

$$\tilde{\psi}_n(\xi_j^n - L) \geq \phi_n(\xi_j^n - L) \quad \text{and} \quad \tilde{\psi}_n(\xi_j^n + L) \geq \phi_n(\xi_j^n + L).$$

Moreover, there exists $R > 0$ large enough, such that

$$\tilde{\psi}_n(R) \geq \phi_n(R),$$

and

$$\tilde{\psi}_n(-R) \geq \phi_n(-R).$$

By the maximum principle, we get

$$\phi_n(x) \leq \tilde{\psi}_n(x) \quad \text{for } x \in [-R, R] \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Similarly, we obtain $\phi_n(x) \geq -\tilde{\psi}_n(x)$ for $x \in [-R, R] \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$. Thus

$$|\phi_n(x)| \leq \tilde{\psi}_n(x) \quad \text{for } x \in [-R, R] \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Letting $R \rightarrow +\infty$, we get

$$|\phi_n(x)| \leq \tilde{\psi}_n(x) \quad \text{for } x \in \mathbb{R} \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Letting $\rho \rightarrow 0$, for $x \in \mathbb{R} \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$, we have that

$$|\phi_n(x)| \leq \left(C_0 \|h_n\|_{**} + e^{\sigma L} \sup_{\cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]} |\phi_n(x)| + o(\|\phi_n\|_*) \right) \sum_{j=1}^k e^{-\sigma|x-\xi_j^n|}.$$

So (3.15) holds.

For $x \leq -M$, let $\rho > 0$ small and $C_1 > 0$ be chosen later, we define

$$\psi_n(x) = C_1 (C_0 \|h_n\|_{**} + o(\|\phi_n\|_*)) e^{(\frac{4}{N-2} + \sigma)x} e^{-\sigma\xi_1^n} + \rho e^{\frac{4}{N-2}x}.$$

By the maximum principle, we get

$$\phi_n(x) \leq \psi_n(x) \quad \text{for } x \in [-R, -M],$$

if $R > 0$ is large enough. By a similar argument, we obtain $\phi_n(x) \geq -\psi_n(x)$ for $x \in [-R, -M]$. Thus

$$|\phi_n(x)| \leq \psi_n(x) \quad \text{for } x \in [-R, -M].$$

Let $R \rightarrow +\infty$, we get

$$|\phi_n(x)| \leq \psi_n(x) \quad \text{for } x \in [-\infty, -M].$$

Let $\rho \rightarrow 0$, we have

$$|\phi_n(x)| \leq C_1 (C_0 \|h_n\|_{**} + o(\|\phi_n\|_*)) e^{(\frac{4}{N-2} + \sigma)x} e^{-\sigma\xi_1^n} \quad \text{for } x \in [-\infty, -M].$$

So we obtain that (3.16) holds.

Moreover, estimate (3.9) follows from (3.13) and (3.8). \square

Proof of Proposition 3.1. From Lemma 3.2, for ϕ and h satisfying (3.2), we then have

$$\|\phi\|_* \leq C \left(\|h\|_{**} + \|e^{-(p^*-q)x} V^{q-1} \phi\|_{**} \right),$$

and

$$|c_j| \leq C \left(\|h\|_{**} + \|e^{-(p^*-q)x} V^{q-1} \phi\|_{**} \right).$$

In order to establish (3.6), it is sufficient to show that

$$(3.17) \quad \|e^{-(p^*-q)x} V^{q-1} \phi\|_{**} \leq o(1) \|\phi\|_*.$$

Indeed,

$$(3.18) \quad \begin{aligned} \|e^{-(p^*-q)x} V^{q-1} \phi\|_{**} &\leq \sup_{x \leq -M} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j^n|} \right)^{-1} \left| e^{-(p^*-q)x} V^{q-1} \phi \right| \\ &\quad + \sup_{x \geq -M} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j^n|} \right)^{-1} \left| e^{-(p^*-q)x} V^{q-1} \phi \right| := Q_1 + Q_2. \end{aligned}$$

Now we estimate Q_1 and Q_2 respectively, we first have

$$(3.19) \quad \begin{aligned} Q_1 &\leq C \sup_{x \leq -M} e^{\sigma|x-\xi_1^n|} |\phi(x)| e^{-(p^*-q)x} V^{q-1} \\ &\leq C e^{-(q-1)\xi_1^n} \sup_{x \leq -M} e^{-(\frac{4}{N-2} + \sigma)x} e^{\sigma\xi_1^n} |\phi(x)|. \end{aligned}$$

For Q_2 , if $-M \leq x \leq \xi_1$, then we have

$$\begin{aligned} e^{-(p^*-q)x} V^{q-1} &\leq \sum_{j=1}^k e^{-(p^*-q)x} e^{-(q-1)|x-\xi_j|} \leq C e^{(2q-p^*-1)x} e^{-(q-1)\xi_1} \\ &\leq C \max \left\{ e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1} \right\}. \end{aligned}$$

If $x \geq \xi_1$, then we have

$$e^{-(p^*-q)x} V^{q-1} \leq \sum_{j=1}^k e^{-(p^*-q)x} e^{-(q-1)|x-\xi_j|} \leq C e^{-(p^*-q)x} \leq C e^{-(p^*-q)\xi_1}.$$

Thus we find

$$(3.20) \quad Q_2 \leq C \max \left\{ e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1} \right\} \sup_{x \geq -M} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|} \right)^{-1} |\phi(x)|.$$

From (3.18), (3.19) and (3.20), we get

$$\|e^{-(p^*-q)x} V^{q-1} \phi\|_{**} \leq C \max \left\{ e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1} \right\} \|\phi\|_* = o(1) \|\phi\|_*.$$

So estimate (3.17) holds.

We now prove the existence and uniqueness of solution to (3.2). Consider the Hilbert space

$$H = \left\{ \phi \in H^1(\mathbb{R}) : \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall j = 1, 2, \dots, k. \right\}$$

with inner product

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}} (\phi' \psi' + \phi \psi) dx.$$

Then problem (3.7) is equivalent to find $\phi \in H$ such that

$$(3.21) \quad \begin{aligned} \langle \phi, \psi \rangle &= \int_{\mathbb{R}} \left[\alpha_\varepsilon (p^* + \varepsilon) V^{p^* + \varepsilon - 1} \phi + \lambda q \beta_N e^{-(p^*-q)x} V^{q-1} \phi \right. \\ &\quad \left. + \left(\frac{2}{N-2} \right)^2 e^{-\frac{4}{N-2}x} \phi + h \right] \psi dx, \end{aligned}$$

for all $\psi \in H$. By the Riesz representation theorem, (3.21) is equivalent to solve

$$(3.22) \quad \phi = K(\phi) + \tilde{h}$$

with $\tilde{h} \in H$ depending linearly on h and $K : H \rightarrow H$ being a compact operator. Fredholm's alternative yields there is a unique solution to problem (3.22) for any h provided that

$$(3.23) \quad \phi = K(\phi)$$

has only the zero solution in H . (3.23) is equivalent to problem (3.2) with $h = 0$. If $h = 0$, estimate (3.6) implies that $\phi = 0$. This ends the proof of Proposition 3.1.

Now we study the differentiability of the operator T_ε with respect to $\xi = (\xi_1, \dots, \xi_k)$. Consider the Banach space

$$C_* = \{f \in C(\mathbb{R}) : \|f\|_{**} < \infty\}$$

endowed with the $\|\cdot\|_{**}$ norm. The following result holds.

Proposition 3.3. *Under the assumptions of Proposition 3.1, the map $\xi \mapsto T_\varepsilon$ is of class C^1 . Moreover,*

$$\|D_\xi T_\varepsilon(h)\|_* \leq C\|h\|_{**}$$

uniformly on the vectors ξ which satisfy (2.11).

Proof. Fix $h \in \mathcal{C}_*$ and let $\phi = T_\varepsilon(h)$ for $\varepsilon < \varepsilon_0$. Let us recall that ϕ satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = h + \sum_{j=1}^k c_j Z_j & \text{in } (-\infty, +\infty); \\ \lim_{|x| \rightarrow \infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall j = 1, \dots, k, \end{cases}$$

for certain constants c_j . Differentiating above equation with respect to ξ_l , $l \in \{1, \dots, k\}$. Set $Y = \partial_{\xi_l} \phi$ and $d_j = \partial_{\xi_l} c_j$, we have

$$\begin{cases} \mathcal{L}_\varepsilon(Y) = \bar{h} + \sum_{j=1}^k d_j Z_j & \text{in } (-\infty, +\infty); \\ \lim_{|x| \rightarrow \infty} Y(x) = 0; \\ \int_{\mathbb{R}} Y Z_j + \phi \partial_{\xi_l} Z_j = 0, \quad \forall j = 1, \dots, k, \end{cases}$$

where

$$\bar{h} = \alpha_\varepsilon(p^* + \varepsilon)(p^* + \varepsilon - 1)e^{\varepsilon x} V^{p^* + \varepsilon - 2} Z_l \phi + \lambda q(q-1)\beta_N e^{-(p^* - q)x} V^{q-2} Z_l \phi + c_l \partial_{\xi_l} Z_l.$$

Let $\eta = Y - \sum_{i=1}^k b_i Z_i$, where $b_i \in \mathbb{R}$ is chosen such that

$$\int_{\mathbb{R}} \eta Z_j = 0,$$

that is,

$$(3.24) \quad \sum_{i=1}^k b_i \int_{\mathbb{R}} Z_i Z_j = \int_{\mathbb{R}} Y Z_j = \int_{\mathbb{R}} \partial_{\xi_l} \phi Z_j = - \int_{\mathbb{R}} \phi \partial_{\xi_l} Z_j.$$

This is an almost diagonal system, it has a unique solution and we have

$$(3.25) \quad |b_i| \leq C\|\phi\|_*.$$

Moreover, η satisfies

$$(3.26) \quad \begin{cases} \mathcal{L}_\varepsilon(\eta) = g + \sum_{j=1}^k d_j Z_j & \text{in } (-\infty, +\infty); \\ \lim_{|x| \rightarrow \infty} \eta(x) = 0; \\ \int_{\mathbb{R}} \eta Z_j = 0, \quad \forall j = 1, \dots, k, \end{cases}$$

with

$$g = \bar{h} - \sum_{i=1}^k b_i \mathcal{L}_\varepsilon(Z_i).$$

From Proposition 3.1, there is a unique solution $\eta = T_\varepsilon(g)$ to (3.26) and

$$(3.27) \quad \|\eta\|_* \leq C\|g\|_{**}.$$

Moreover, we have

$$\begin{aligned}
\|g\|_{**} &\leq C\|e^{\varepsilon x}V^{p^*+\varepsilon-2}Z_l\phi\|_{**} + C\|e^{-(p^*-q)x}V^{q-2}Z_l\phi\|_{**} \\
&\quad + \|c_l\partial_{\xi_l}Z_l\|_{**} + \sum_{i=1}^k |b_i|\|\mathcal{L}_\varepsilon(Z_i)\|_{**} \\
(3.28) \quad &\leq C(\|\phi\|_* + |c_l| + |b_i|) \leq C\|h\|_{**},
\end{aligned}$$

because $|b_i| \leq C\|\phi\|_*$, $\|\phi\|_* \leq C\|h\|_{**}$ and $|c_l| \leq C\|h\|_{**}$.

By (3.25), (3.27), (3.28) and $\|Z_i\|_* \leq C$, we obtain that

$$\|\partial_{\xi_l}\phi\|_* \leq \|\eta\|_* + \sum_{i=1}^k |b_i|\|Z_i\|_* \leq C\|h\|_{**}.$$

Besides $\partial_{\xi_l}\phi$ depends continuously on ξ in the considered region for this norm. \square

4. NONLINEAR PROBLEM

In this section, our purpose is to study nonlinear problem. We first have the validity of the following result.

Lemma 4.1. *We have*

$$(4.1) \quad \|N(\phi)\|_{**} \leq C \left(\|\phi\|_*^{\min\{p^*, 2\}} + \|\phi\|_*^{\min\{q, 2\}} \right),$$

and

$$(4.2) \quad \|\partial_\phi N(\phi)\|_{**} \leq C \left(\|\phi\|_*^{\min\{p^*-1, 1\}} + \|\phi\|_*^{\min\{q-1, 1\}} \right),$$

for $\|\phi\|_* \leq 1$.

Proof. By the fundamental theorem of calculus and the definition of $\|\cdot\|_{**}$, we have

$$\begin{aligned}
&\|N(\phi)\|_{**} \\
&\leq \alpha_\varepsilon(p^* + \varepsilon) \sup_{x \in \mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|} \right)^{-1} e^{\varepsilon x} \left| \int_0^1 [(V+t\phi)^{p^*+\varepsilon-1} - V^{p^*+\varepsilon-1}] \phi dt \right| \\
&\quad + \lambda q \beta_N \sup_{x \in \mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|} \right)^{-1} e^{-(p^*-q)x} \left| \int_0^1 [(V+t\phi)^{q-1} - V^{q-1}] \phi dt \right| \\
&:= N_1 + N_2.
\end{aligned}$$

Using

$$\|a+b\|^q - \|a\|^q \leq C \begin{cases} |a|^{q-1}|b| + |b|^q & \text{if } q \geq 1; \\ \min\{|a|^{q-1}|b|, |b|^q\} & \text{if } 0 < q < 1, \end{cases}$$

if $p^* \geq 2$ and for $\|\phi\|_* \leq 1$, we have

$$\begin{aligned}
N_1 &\leq C \sup_{x \in \mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|} \right)^{-1} e^{\varepsilon x} V^{p^*+\varepsilon-2} |\phi|^2 + C \sup_{x \in \mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|} \right)^{-1} e^{\varepsilon x} |\phi|^{p^*+\varepsilon} \\
&\leq C\|\phi\|_*^2 + C\|\phi\|_*^{p^*+\varepsilon} \leq C\|\phi\|_*^2.
\end{aligned}$$

Similarly, if $1 < p^* < 2$, we find that $N_1 \leq C\|\phi\|_*^{p^*}$. Thus we get

$$N_1 \leq C\|\phi\|_*^{\min\{p^*, 2\}}.$$

Moreover, by similar computations as N_1 , we can conclude that

$$N_2 \leq C\|\phi\|_*^{\min\{q, 2\}}.$$

Thus we get (4.1).

We differentiate $N(\phi)$ with respect to ϕ , we have

$$\partial_\phi N(\phi) = \alpha_\varepsilon(p^* + \varepsilon)e^{\varepsilon x} \left[(V + \phi)^{p^* + \varepsilon - 1} - V^{p^* + \varepsilon - 1} \right] + \lambda\beta_N q e^{-(p^* - q)x} \left[(V + \phi)^{q-1} - V^{q-1} \right].$$

By a similar argument as $\|N(\phi)\|_{**}$, (4.2) holds. \square

Lemma 4.2. *Let $\sigma > 0$ satisfy (3.3) and $0 < \xi_1 < \xi_2 < \dots < \xi_k$ satisfy (2.11). If q satisfies (1.4), then there exist $\tau \in (\frac{1}{2}, 1)$ and a constant $C > 0$, such that*

$$\|E\|_{**} \leq C\varepsilon^\tau, \quad \|\partial_\xi E\|_{**} \leq C\varepsilon^\tau.$$

Proof. We have

$$\begin{aligned} E &= \alpha_\varepsilon e^{\varepsilon x} (V^{p^* + \varepsilon} - V^{p^*}) + (\alpha_\varepsilon e^{\varepsilon x} - 1)V^{p^*} + \left(V^{p^*} - \left(\sum_{j=1}^k W_j \right)^{p^*} \right) \\ &\quad + \left(\left(\sum_{j=1}^k W_j \right)^{p^*} - \sum_{j=1}^k W_j^{p^*} \right) + \lambda\beta_N e^{-(p^* - q)x} V^q \\ (4.3) \quad &:= E_1 + E_2 + E_3 + E_4 + E_5. \end{aligned}$$

Estimate of E_1 :

$$|E_1| = \left| \varepsilon \alpha_\varepsilon e^{\varepsilon x} \int_0^1 V^{p^* + t\varepsilon} \log V dt \right| \leq C\varepsilon \sum_{j=1}^k e^{-\sigma|x - \xi_j|}.$$

Estimate of E_2 : By the Taylor expansion, we have

$$\begin{aligned} |E_2| &= \left| \left(\left(\frac{p^* - 1}{2} \right)^{-\frac{2\varepsilon}{p^* - 1}} e^{\varepsilon x} - 1 \right) V^{p^*} \right| \\ &= \left(\varepsilon x \int_0^1 e^{t\varepsilon x} dt + O(\varepsilon) e^{\varepsilon x} \right) V^{p^*} \leq C\varepsilon |\log \varepsilon| \sum_{j=1}^k e^{-\sigma|x - \xi_j|}. \end{aligned}$$

Estimate of E_3 : Since

$$|E_3| = \left| V^{p^*} - \left(\sum_{j=1}^k W_j \right)^{p^*} \right| \leq C V^{p^* - 1} \sum_{j=1}^k |R_{\xi_j}(x)|.$$

Thanks to Lemma 2.2, for $x \leq 0$, we have

$$|E_3| \leq C V^{p^* - 1} \sum_{j=1}^k e^{-|x - \xi_j|} \leq C V^{p^* - 1} e^{-\xi_1} \leq C \varepsilon^{\frac{1}{p^* - q}} \sum_{j=1}^k e^{-\sigma|x - \xi_j|}.$$

For $0 \leq x \leq \xi_1$,

$$\begin{aligned} |E_3| &\leq CV^{p^*-1} \sum_{j=1}^k e^{-|x-\xi_j|} e^{-\frac{2}{N-2} \min\{x, \xi_j\}} \\ &\leq C \sum_{j=1}^k e^{-\sigma|x-\xi_j|} \begin{cases} \varepsilon^{\frac{2}{N+2-(N-2)q}} & \text{if } N \geq 4; \\ \varepsilon^{\frac{1}{5-q}} & \text{if } N = 3. \end{cases} \end{aligned}$$

If $x \geq \xi_1$, for $0 < \sigma < p^* - 1$, we have

$$\begin{aligned} |E_3| &\leq CV^{p^*-1} \sum_{j=1}^k e^{-|x-\xi_j|} e^{-\frac{2}{N-2} \min\{x, \xi_j\}} \\ &\leq CV^{p^*-1} e^{-\frac{2}{N-2} \xi_1} \leq C \varepsilon^{\frac{2}{N+2-(N-2)q}} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}. \end{aligned}$$

Therefore we get for $x \in \mathbb{R}$,

$$|E_3| \leq C \sum_{j=1}^k e^{-\sigma|x-\xi_j|} \begin{cases} \varepsilon^{\frac{2}{N+2-(N-2)q}} & \text{if } N \geq 4; \\ \varepsilon^{\frac{1}{5-q}} & \text{if } N = 3. \end{cases}$$

Estimate of E_4 : If $-\infty < x \leq \frac{\xi_1 + \xi_2}{2}$, we have

$$\begin{aligned} |E_4| &\leq \left| \left(\sum_{j=1}^k W(x - \xi_j) \right)^{p^*} - W(x - \xi_1)^{p^*} \right| + \left| \sum_{j=2}^k W(x - \xi_j)^{p^*} \right| \\ &\leq p^* \left(\sum_{j=1}^k W(x - \xi_j) \right)^{p^*-1} \sum_{j=2}^k W(x - \xi_j) + \sum_{j=2}^k W(x - \xi_j)^{p^*} \\ &= p^* \left(\sum_{j=1}^k W(x - \xi_j) \right)^{p^*-1-\theta} \left(\sum_{j=1}^k W(x - \xi_j) \right)^{\theta} \sum_{j=2}^k W(x - \xi_j) + \sum_{j=2}^k W(x - \xi_j)^{p^*} \end{aligned}$$

with a positive number θ , satisfying $0 < \theta < p^* - 1 - \sigma$. Note that

$$\left(\sum_{j=1}^k W(x - \xi_j) \right)^{\theta} \sum_{j=2}^k W(x - \xi_j) \leq C \varepsilon^{\frac{1+\theta}{2}}.$$

Moreover,

$$\sum_{j=2}^k W(x - \xi_j)^{p^*} \leq C \varepsilon^{\frac{p^*-\sigma}{2}} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

Thus

$$|E_4| \leq C \varepsilon^{\frac{1+\theta}{2}} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}, \quad \text{for } -\infty < x \leq \frac{\xi_1 + \xi_2}{2},$$

with $0 < \theta < p^* - 1 - \sigma$. Similarly, for $\frac{\xi_{l-1} + \xi_l}{2} \leq x \leq \frac{\xi_l + \xi_{l+1}}{2}$ with $l = 2, \dots, k-1$, and $x \geq \frac{\xi_{k-1} + \xi_k}{2}$, we get

$$|E_4| \leq C\varepsilon^{\frac{1+\theta}{2}} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

Therefore for $x \in \mathbb{R}$, we have

$$|E_4| \leq C\varepsilon^{\frac{1+\theta}{2}} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}, \quad \text{where } 0 < \theta < p^* - 1 - \sigma.$$

The estimate of E_5 is similar as the previous ones and we get

$$|E_5| \leq C \max\{\varepsilon, \varepsilon^{\frac{q-\sigma}{p^*-q}}\} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

From (4.3) and the previous estimates, for $0 < \theta < p^* - 1 - \sigma$ with σ satisfying (3.3), we have

$$\|E\|_{**} \leq C \begin{cases} \max\left\{\varepsilon|\log \varepsilon|, \varepsilon^{\frac{2}{N+2-(N-2)q}}, \varepsilon^{\frac{1+\theta}{2}}, \varepsilon^{\frac{q-\sigma}{p^*-q}}\right\} & \text{if } N \geq 4; \\ \max\left\{\varepsilon|\log \varepsilon|, \varepsilon^{\frac{1}{5-q}}, \varepsilon^{\frac{1+\theta}{2}}, \varepsilon^{\frac{q-\sigma}{p^*-q}}\right\} & \text{if } N = 3. \end{cases}$$

Therefore if q satisfies (1.4), we find that there exists $\tau \in (\frac{1}{2}, 1)$ such that

$$\|E\|_{**} \leq C\varepsilon^\tau.$$

Differentiating E with respect to ξ_i ($i = 1, 2, \dots, k$), we have

$$\begin{aligned} \partial_{\xi_i} E &= \alpha_\varepsilon(p^* + \varepsilon)e^{\varepsilon x} V p^* + \varepsilon^{-1} \partial_{\xi_i} V - p^* \sum_{j=1}^k W(x - \xi_j)^{p^*-1} \partial_{\xi_i} W(x - \xi_j) \\ &\quad + \lambda \beta_N q e^{-(p^*-q)x} V q^{-1} \partial_{\xi_i} V. \end{aligned}$$

The proof of estimate for $\|\partial_{\xi_i} E\|_{**}$ is similar to that of $\|E\|_{**}$. \square

Proposition 4.3. *Assume that $0 < \xi_1 < \xi_2 < \dots < \xi_k$ satisfy (2.11). Then there exists $C > 0$ such that for $\varepsilon > 0$ small enough, there exists a unique solution $\phi = \phi(\xi)$ to problem (3.1) with*

$$\|\phi\|_* \leq C\varepsilon^\tau,$$

for some $\tau \in (\frac{1}{2}, 1)$ satisfying Lemma 4.2. Moreover, the map $\xi \mapsto \phi(\xi)$ is of class C^1 for the $\|\cdot\|_*$ norm, and

$$\|\partial_{\xi_i} \phi\|_* \leq C\varepsilon^\tau.$$

Proof. Problem (3.1) is equivalent to solve a fixed point problem

$$\phi = T_\varepsilon(N(\phi) + E) := A_\varepsilon(\phi).$$

We will show that the operator A_ε is a contraction map in a proper region. Set

$$\mathcal{F}_\gamma = \{\phi \in C(\mathbb{R}) : \|\phi\|_* \leq \gamma\varepsilon^\tau\},$$

where $\gamma > 0$ will be chosen later.

For $\phi \in \mathcal{F}_\gamma$, by Lemmas 4.1 and 4.2, we get

$$\|A_\varepsilon(\phi)\|_* = \|T_\varepsilon(N(\phi) + E)\|_* \leq C\|N(\phi)\|_{**} + \|E\|_{**}$$

$$\leq C \left(\gamma^{\min\{p^*, 2\}} \varepsilon^{\min\{p^* - 1, 1\}\tau} + \gamma^{\min\{q, 2\}} \varepsilon^{\min\{q - 1, 1\}\tau} + 1 \right) \varepsilon^\tau.$$

Then we have $A_\varepsilon(\phi) \in \mathcal{F}_\gamma$ for $\phi \in \mathcal{F}_\gamma$ by choosing γ large enough but fixed.

Moreover, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$, by writing

$$N(\phi_1) - N(\phi_2) = \int_0^1 N'(\phi_2 + t(\phi_1 - \phi_2)) dt (\phi_1 - \phi_2).$$

By Proposition 3.1 and using (4.2), we find

$$\begin{aligned} & \|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \leq C \|N(\phi_1) - N(\phi_2)\|_{**} \\ & \leq C \left(\left(\max_{i=1,2} \|\phi_i\|_* \right)^{\min\{p^* - 1, 1\}} + \left(\max_{i=1,2} \|\phi_i\|_* \right)^{\min\{q - 1, 1\}} \right) \|\phi_1 - \phi_2\|_* \\ & \leq C \varepsilon^\kappa \|\phi_1 - \phi_2\|_* \end{aligned}$$

with some $\kappa > 0$. This implies that A_ε is a contraction map from \mathcal{F}_γ to \mathcal{F}_γ . Thus A_ε has a unique fixed point in \mathcal{F}_γ .

Now we consider the differentiability of $\xi \mapsto \phi(\xi)$. We write

$$B(\xi, \phi) := \phi - T_\varepsilon(N(\phi) + E).$$

First we observe that $B(\xi, \phi) = 0$. Moreover,

$$\partial_\phi B(\xi, \phi)[\theta] = \theta - T_\varepsilon(\theta(\partial_\phi(N(\phi)))) \equiv \theta + M(\theta),$$

where

$$M(\theta) = -T_\varepsilon(\theta(\partial_\phi(N(\phi)))).$$

By a direct calculation, we get

$$\|M(\theta)\|_* \leq C \|\theta(\partial_\phi(N(\phi)))\|_{**} \leq C \varepsilon^\kappa \|\theta\|_*.$$

So for $\varepsilon > 0$ small enough, the operator $\partial_\phi B(\xi, \phi)$ is invertible with uniformly bounded inverse in $\|\cdot\|_*$. It also depends continuously on its parameters. Let us differentiate with respect to ξ , we have

$$\partial_\xi B(\xi, \phi) = -(\partial_\xi T_\varepsilon)(N(\phi) + E) - T_\varepsilon((\partial_\xi N)(\xi, \phi) + \partial_\xi E),$$

where all these expressions depend continuously on their parameters. The implicit function theorem yields that $\phi(\xi)$ is of class C^1 and

$$\partial_\xi \phi = -(\partial_\phi B(\xi, \phi))^{-1} [\partial_\xi B(\xi, \phi)]$$

so that

$$\|\partial_\xi \phi\|_* \leq C (\|N(\phi)\|_{**} + \|E\|_{**} + \|(\partial_\xi N)(\xi, \phi)\|_{**} + \|\partial_\xi E\|_{**}) \leq C \varepsilon^\tau.$$

□

5. THE FINITE-DIMENSIONAL VARIATIONAL REDUCTION

According to the results of the previous section, our problem has been reduced to find points $\xi = (\xi_1, \xi_2, \dots, \xi_k)$, such that

$$(5.1) \quad c_j(\xi) = 0 \quad \text{for all } j = 1, \dots, k.$$

If (5.1) holds, then $v = V + \phi$ is a solution to (1.7), and $u = \sum_{j=1}^k U_{\mu_j} + \psi$ is the solution to problem (1.3) with $\psi = \mathcal{T}^{-1}(\phi)$.

Define the function $\mathcal{I}_\varepsilon : (\mathbb{R}^+)^k \rightarrow \mathbb{R}$ as

$$\mathcal{I}_\varepsilon(\xi) := I_\varepsilon(V + \phi),$$

where V is defined by (2.12) and I_ε is the energy functional of (1.7) defined by

$$\begin{aligned} I_\varepsilon(v) &= \frac{1}{2} \int_{-\infty}^{+\infty} (|v'(x)|^2 + |v|^2) dx + \frac{1}{2} \left(\frac{2}{N-2} \right)^2 \int_{-\infty}^{+\infty} e^{-\frac{4}{N-2}x} v^2 dx \\ &\quad - \frac{1}{p^* + \varepsilon + 1} \alpha_\varepsilon \int_{-\infty}^{+\infty} e^{\varepsilon x} |v|^{p^* + \varepsilon + 1} dx - \frac{1}{q+1} \lambda \beta_N \int_{-\infty}^{+\infty} e^{-(p^* - q)x} |v|^{q+1} dx. \end{aligned}$$

We have the following fact.

Lemma 5.1. *The function $V + \phi$ is a solution to (1.7) if and only if $\xi = (\xi_1, \dots, \xi_k)$ is a critical point of $\mathcal{I}_\varepsilon(\xi)$, where $\phi = \phi(\xi)$ is given by Proposition 4.3.*

Proof. For $s \in \{1, 2, \dots, k\}$, we have

$$\begin{aligned} \partial_{\xi_s} \mathcal{I}_\varepsilon(\xi) &= \partial_{\xi_s} (I_\varepsilon(V + \phi)) = DI_\varepsilon(V + \phi) [\partial_{\xi_s} V + \partial_{\xi_s} \phi] \\ &= \sum_{j=1}^k c_j \int_{\mathbb{R}} Z_j [\partial_{\xi_s} V + \partial_{\xi_s} \phi] = \sum_{j=1}^k c_j \left(\int_{\mathbb{R}} Z_j Z_s dx + o(1) \right), \end{aligned}$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly for the norm $\|\cdot\|_*$. This implies that the above relations define an almost diagonal homogeneous linear equation system for the c_j . Thus ξ is the critical point of I_ε if and only if $c_j = 0$ for all $j = 1, 2, \dots, k$. \square

Lemma 5.2. *The following expansion holds*

$$\mathcal{I}_\varepsilon(\xi) = I_\varepsilon(V) + o(\varepsilon)$$

as $\varepsilon \rightarrow 0$, where $o(\varepsilon)$ is uniform in the C^1 -sense on the vectors ξ satisfying (2.11).

Proof. By the fact that $DI_\varepsilon(V + \phi)[\phi] = 0$ and using the Taylor expansion, we have

$$\begin{aligned} \mathcal{I}_\varepsilon(\xi) - I_\varepsilon(V) &= I_\varepsilon(V + \phi) - I_\varepsilon(V) = \int_0^1 D^2 I_\varepsilon(V + t\phi) [\phi^2] t dt \\ &= \int_0^1 t dt \int_{-\infty}^{+\infty} (N(\phi) + E) \phi dx \\ &\quad + (p^* + \varepsilon) \alpha_\varepsilon \int_0^1 t dt \int_{-\infty}^{+\infty} e^{\varepsilon x} [V^{p^* + \varepsilon - 1} - (V + t\phi)^{p^* + \varepsilon - 1}] \phi^2 dx \\ &\quad + \lambda \beta_N q \int_0^1 t dt \int_{-\infty}^{+\infty} e^{-(p^* - q)x} [V^{q-1} - (V + t\phi)^{q-1}] \phi^2 dx. \end{aligned}$$

Since $\|\phi\|_* \leq C\varepsilon^\tau$ and $\|E\|_{**} \leq C\varepsilon^\tau$ with $\tau > \frac{1}{2}$, we get

$$\mathcal{I}_\varepsilon(\xi) - I_\varepsilon(V) = O(\varepsilon^{2\tau}) = o(\varepsilon)$$

uniformly on the points ξ which satisfy (2.11).

Moreover, differentiating with respect to ξ_s , we have

$$\begin{aligned} \partial_{\xi_s} (\mathcal{I}_\varepsilon(\xi) - I_\varepsilon(V)) &= \int_0^1 \int_{-\infty}^{+\infty} \partial_{\xi_s} [(N(\phi) + E) \phi] t dx dt \\ &\quad + \alpha_\varepsilon (p^* + \varepsilon) \int_0^1 t dt \int_{-\infty}^{+\infty} e^{\varepsilon x} \partial_{\xi_s} \left([V^{p^* + \varepsilon - 1} - (V + t\phi)^{p^* + \varepsilon - 1}] \phi^2 \right) dx \\ &\quad + \lambda \beta_N q \int_0^1 t dt \int_{-\infty}^{+\infty} e^{-(p^* - q)x} \partial_{\xi_s} \left([V^{q-1} - (V + t\phi)^{q-1}] \phi^2 \right) dx. \end{aligned}$$

By the fact that $\|\partial_\xi \phi\|_* \leq C\varepsilon^\tau$ and $\|\partial_\xi E\|_{**} \leq C\varepsilon^\tau$ with $\tau > \frac{1}{2}$, we deduce that

$$\partial_{\xi_s} (\mathcal{I}_\varepsilon(\xi) - I_\varepsilon(V)) = O(\varepsilon^{2\tau}) = o(\varepsilon).$$

□

Now we consider the energy functional of problem (1.3), which is defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) - \frac{1}{p^* + 1 + \varepsilon} \int_{\mathbb{R}^N} |u|^{p^* + 1 + \varepsilon} - \frac{\lambda}{q + 1} \int_{\mathbb{R}^N} |u|^{q+1}.$$

By a direct calculation, we have that

$$(5.2) \quad I_\varepsilon(V) = \left(\frac{2}{N-1} \right)^{N-1} \frac{1}{\omega_{N-1}} J(U),$$

where V is defined by (2.12), ω_{N-1} is the volume of the unit sphere in \mathbb{R}^N and $U(z) = \sum_{j=1}^k U_{\mu_j}(z)$ with U_{μ_j} satisfying problem (2.1).

We give the following expansion of $J(U)$, whose proof is in the Appendix.

Lemma 5.3. *Assume that (2.9) and (2.10) hold, then we have the following expansion:*

$$(5.3) \quad J(U) = a_1 + a_2\varepsilon - \varphi(\Lambda_1, \dots, \Lambda_k)\varepsilon + a_3\varepsilon \log \varepsilon + o(\varepsilon),$$

where

$$(5.4) \quad \varphi(\Lambda_1, \dots, \Lambda_k) = a_4 \Lambda_1^{\frac{N+2-(N-2)q}{2}} - a_5 \sum_{i=1}^k \log \Lambda_i + a_6 \sum_{l=1}^{k-1} \left(\frac{\Lambda_{l+1}}{\Lambda_l} \right)^{\frac{N-2}{2}},$$

and as $\varepsilon \rightarrow 0$, $o(\varepsilon)$ is uniform in the C^1 -sense on the Λ_i 's satisfying (2.9), and

$$\begin{aligned} a_1 &= \frac{k}{N} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz, \\ a_2 &= \frac{k}{(p^*+1)^2} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \\ &\quad - \frac{k}{p^*+1} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} \log \frac{\alpha_N}{(1+|z|^2)^{\frac{N-2}{2}}} dz, \\ a_3 &= \frac{(N-2)^2}{4N} \left(\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \right) \\ &\quad \times \sum_{i=1}^k \left(\frac{2(i-1)}{N-2} + \frac{2}{N+2-(N-2)q} \right), \\ a_4 &= \frac{\lambda}{q+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz, \\ a_5 &= \frac{(N-2)^2}{4N} \left(\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \right), \\ a_6 &= \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \frac{1}{|z|^{N-2}} dz. \end{aligned}$$

Now we are ready to prove our main result.

Proof of Theorem 1.1. Thanks to Lemma 5.1, we know that

$$u = \sum_{j=1}^k U_{\mu_j} + \psi \quad \text{with } \psi = \mathcal{T}^{-1}(\phi)$$

is a solution to problem (1.3) if and only if ξ is a critical point of $\mathcal{I}_\varepsilon(\xi)$, where the existence of ϕ is guaranteed by Proposition 4.3.

Finding a critical point of $\mathcal{I}_\varepsilon(\xi)$ is equivalent to find that of $\tilde{\mathcal{I}}_\varepsilon(\xi)$, which is defined as

$$\tilde{\mathcal{I}}_\varepsilon(\xi) = - \left(\frac{N-1}{2} \right)^{N-1} \frac{\omega_{N-1}}{\varepsilon} \mathcal{I}_\varepsilon(\xi) + \frac{a_1}{\varepsilon} + a_2 + a_3 \log \varepsilon.$$

On the other hand, from Lemmas 5.2 and 5.3, using (5.2), we have

$$\begin{aligned} \mathcal{I}_\varepsilon(\xi) &= I_\varepsilon(V) + o(\varepsilon) = \left(\frac{2}{N-1} \right)^{N-1} \frac{1}{\omega_{N-1}} J(U) + o(\varepsilon) \\ &= \left(\frac{2}{N-1} \right)^{N-1} \frac{1}{\omega_{N-1}} [a_1 + a_2 \varepsilon - \varphi(\Lambda_1, \dots, \Lambda_k) \varepsilon + a_3 \varepsilon \log \varepsilon] + o(\varepsilon), \end{aligned}$$

as $\varepsilon \rightarrow 0$, where $\varphi(\Lambda)$ is defined by (5.4) and $o(\varepsilon)$ is uniform in the C^1 -sense. Then we have

$$(5.5) \quad \tilde{\mathcal{I}}_\varepsilon(\xi) = \varphi(\Lambda) + o(1),$$

where $o(1)$ is uniform in the C^1 -sense as $\varepsilon \rightarrow 0$.

We set $s_1 = \Lambda_1$, $s_j = \frac{\Lambda_j}{\Lambda_{j-1}}$, then we can write $\varphi(\Lambda_1, \dots, \Lambda_k)$ as

$$\begin{aligned} \varphi(s_1, \dots, s_k) &= a_4 s_1^{\frac{N+2-(N-2)q}{2}} - a_5 k \log s_1 - \sum_{j=2}^k \left[a_5 (k-j+1) \log s_j - a_6 s_j^{\frac{N-2}{2}} \right] \\ &:= \tilde{\varphi}_1 - \sum_{j=2}^k \tilde{\varphi}_j, \end{aligned}$$

with

$$\tilde{\varphi}_1 = a_4 s_1^{\frac{N+2-(N-2)q}{2}} - a_5 k \log s_1$$

and

$$\tilde{\varphi}_j = a_5 (k-j+1) \log s_j - a_6 s_j^{\frac{N-2}{2}}, \quad j = 2, \dots, k.$$

We note that

$$(5.6) \quad \bar{s}_1 = \left(\frac{2a_5 k}{a_4 (N+2-(N-2)q)} \right)^{\frac{2}{N+2-(N-2)q}}$$

is the critical point of $\tilde{\varphi}_1$, and

$$(5.7) \quad \bar{s}_j = \left(\frac{2a_5 (k-j+1)}{(N-2)a_6} \right)^{\frac{2}{N-2}}, \quad j = 2, \dots, k,$$

is the critical point of $\tilde{\varphi}_j$. Moreover

$$\tilde{\varphi}_1''(\bar{s}_1) < 0, \quad \tilde{\varphi}_j''(\bar{s}_j) < 0, \quad j = 2, \dots, k.$$

So $(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_k)$ is a nondegenerate critical point of $\varphi(s_1, \dots, s_k)$. Thus

$$\Lambda^* := (\bar{s}_1, \bar{s}_2 \bar{s}_1, \bar{s}_3 \bar{s}_2 \bar{s}_1, \dots, \bar{s}_k \times \dots \times \bar{s}_2 \bar{s}_1)$$

is a nondegenerate critical point of $\varphi(\Lambda)$. It follows that the local degree $\deg(\nabla\varphi(\Lambda), \mathcal{O}, 0)$ is well defined and is nonzero, here \mathcal{O} is an arbitrarily small neighborhood of Λ^* . Hence from (5.5), for $\varepsilon > 0$ small enough, we have that $\deg(\nabla_\xi \tilde{\mathcal{I}}_\varepsilon(\xi), \bar{\mathcal{O}}, 0) \neq 0$, where $\bar{\mathcal{O}}$ is a small neighborhood of $\xi^* = (\xi_1^*, \dots, \xi_k^*)$ and

$$\xi_j^* = \left[(j-1) + \frac{1}{p^* - q} \right] \log \frac{1}{\varepsilon} - \frac{N-2}{2} \log(\bar{s}_j \bar{s}_{j-1} \cdots \bar{s}_1), \text{ for } \forall j = 1, \dots, k.$$

So ξ^* is a critical point of $\tilde{\mathcal{I}}_\varepsilon(\xi)$, which implies there is a critical point of \mathcal{I}_ε .

Furthermore, if for some i , $|x - \xi_i| \leq C_0$ with some $C_0 > 0$, then we have $|\phi| = o(W(x - \xi_i))$. Thus $\psi(|z|) = \mathcal{T}^{-1}(\phi(x)) = o(w_{\mu_i})$ for $\frac{1}{\mathcal{O}}\mu_i \leq |z| \leq C\mu_i$. Moreover, from (c) of Lemma 2.1, we get that $R_{\mu_i} = o(w_{\mu_i})$ for $\frac{1}{\mathcal{O}}\mu_i \leq |z| \leq C\mu_i$. Therefore we obtain (1.5) holds with

$$\Lambda_j^* = \bar{s}_j \bar{s}_{j-1} \cdots \bar{s}_1, \quad j = 1, \dots, k,$$

where \bar{s}_j are given by (5.6) and (5.7). This finishes the proof. \square

6. APPENDIX

6.1. Proof of Lemma 2.1. In order to prove Lemma 2.1, we introduce the Green function. For a fixed $z \in \mathbb{R}^N$, let $G(z, y)$ be the Green function of $-\Delta + I$, which satisfies

$$\begin{aligned} -\Delta G(z, y) + G(z, y) &= \delta_z(y) \quad \text{in } \mathbb{R}^N, \\ G(z, y) &\rightarrow 0 \quad |y| \rightarrow \infty. \end{aligned}$$

We have the following result.

Lemma 6.1. *We have*

$$|G(z, y)| \leq \frac{C}{|y - z|^{N-2}} \quad \text{for } 0 < |y - z| \leq 1,$$

and

$$|G(z, y)| \leq C|y - z|^{\frac{1-N}{2}} e^{-|y-z|} \quad \text{for } |y - z| \geq 1.$$

Proof. By radial symmetry, we can write $G(z, y) = G(r)$ with $r = |y - z|$. Since $G(r)$ is singular at zero and tends to zero at infinity, we can verify that G is given by

$$G(r) = \frac{N-2}{(2\pi)^{\frac{N}{2}} \Gamma(\frac{N}{2})^2} r^{\frac{2-N}{2}} K_{\frac{N-2}{2}}(r),$$

where $K_{\frac{N-2}{2}}(r)$ is a Modified Bessel Function of the Second Kind, see [15]. For $N = 3$, the function G has the explicit form $G(r) = \frac{e^{-r}}{4\pi r}$. In general, we have that $K_{\frac{N-2}{2}}(r) \sim \frac{\Gamma(\frac{N-2}{2})}{2} (\frac{2}{r})^{\frac{N-2}{2}}$ for r close to 0, and $K_{\frac{N-2}{2}}(r) \sim \sqrt{\frac{\pi}{2r}} e^{-r}$ for r large. Using these estimates, we obtain the result. \square

Proof of Lemma 2.1. (a) It is a direct consequence of the maximum principle.

(b) Define the barrier function $Q(z) = \mu^{\frac{N-2}{2}} |z|^{-(N+2)}$. It satisfies $-\Delta Q(z) + Q(z) \geq c\mu^{\frac{N-2}{2}} |z|^{-(N+2)}$ for all $|z| \geq R$ with $R > 0$ a large constant, here c is positive constant. Since $Q(z) = \mu^{\frac{N-2}{2}} R^{-(N+2)}$ for $|z| = R$ and $U_\mu(z) \leq w_\mu(z) \leq \alpha_N \mu^{\frac{N-2}{2}} |z|^{-(N-2)}$ for all $|z| \geq 0$. Set $\varphi(z) = AQ(z) - U_\mu(z)$ for some constant $A > 0$, we then have $-\Delta\varphi(z) + \varphi(z) \geq 0$ for $|z| \geq R$, and $\varphi(z) \geq 0$ for $|z| = R$ by

choosing suitable constant A . By the maximum principle we get $U_\mu(z) \leq AQ(z) = A\mu^{\frac{N-2}{2}}|z|^{-(N+2)}$ for $|z| \geq R$.

(c) Using the representation

$$R_\mu(z) = \int_{\mathbb{R}^N} G(y-z)w_\mu(y)dy$$

and standard convolution estimates we can obtain the stated bounds for R_μ . \square

Set

$$\tilde{Z}_\mu(z) = \partial_\mu U_\mu(z), \quad \bar{Z}_\mu(z) = \partial_\mu w_\mu(z),$$

then $\tilde{Z}_\mu(z)$ satisfies

$$\begin{cases} -\Delta \tilde{Z}_\mu + \tilde{Z}_\mu = \frac{N+2}{N-2} w_\mu^{\frac{4}{N-2}} \bar{Z}_\mu & \text{in } \mathbb{R}^N, \\ \tilde{Z}_\mu(z) \rightarrow 0 & \text{as } |z| \rightarrow \infty. \end{cases}$$

We can write

$$\tilde{Z}_\mu(z) = \bar{Z}_\mu(z) + \partial_\mu R_\mu(z),$$

then $\partial_\mu R_\mu(z)$ satisfies

$$\begin{cases} -\Delta(\partial_\mu R_\mu(z)) + \partial_\mu R_\mu(z) = -\partial_\mu w_\mu(z) & \text{in } \mathbb{R}^N, \\ \partial_\mu R_\mu(z) \rightarrow 0 & \text{as } |z| \rightarrow \infty. \end{cases}$$

We observe that $|\partial_\mu w_\mu(z)| \leq C\mu^{-1}w_\mu$, then we have

Corollary 6.2. *One has*

$$(6.1) \quad |\partial_\mu R_\mu(z)| \leq C\mu^{-1}|R_\mu(z)| \quad \text{for } \forall z \in \mathbb{R}^N.$$

Moreover, by the maximum principle, we have that

$$(6.2) \quad |\tilde{Z}_\mu(z)| \leq C\mu^{\frac{N-4}{2}}|z|^{-(N+2)} \quad \text{for } |z| \geq R,$$

where R is a large positive number but fixed in Lemma 2.1.

6.2. Expansion of energy.

Proof of Lemma 5.3. The proof is very similar to the one in [20]. The difference is that we have more terms in the energy and the initial approximation is also somewhat different. We have

$$\begin{aligned} J(U) &= \left[\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2) - \frac{1}{p^*+1} \int_{\mathbb{R}^N} U^{p^*+1} \right] \\ &\quad + \left[\frac{1}{p^*+1} \int_{\mathbb{R}^N} U^{p^*+1} - \frac{1}{p^*+1+\varepsilon} \int_{\mathbb{R}^N} U^{p^*+1+\varepsilon} \right] - \frac{\lambda}{q+1} \int_{\mathbb{R}^N} U^{q+1} \\ (6.3) \quad &:= J_1 + J_2 + J_3, \end{aligned}$$

where $U = \sum_{j=1}^k U_{\mu_j}$ with $U_{\mu_j} = w_{\mu_j} + R_{\mu_j}$.

As in [20] but using the estimates of R_μ in Lemma 2.1 we can get

$$\begin{aligned} J_1 &= \frac{k}{N} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \\ (6.4) \quad &- \varepsilon \sum_{l=1}^{k-1} \left(\frac{\Lambda_{l+1}}{\Lambda_l} \right)^{\frac{N-2}{2}} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \frac{1}{|z|^{N-2}} dz + o(\varepsilon). \end{aligned}$$

Also as in [20] we obtain

$$\begin{aligned}
J_2 &= \varepsilon \frac{k}{(p^* + 1)^2} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |z|^2)^N} dz \\
&\quad - \varepsilon \frac{k}{p^* + 1} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |z|^2)^N} \log \frac{\alpha_N}{(1 + |z|^2)^{\frac{N-2}{2}}} dz \\
&\quad + \varepsilon \frac{(N-2)^2}{4N} \left(\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |z|^2)^N} dz \right) \sum_{i=1}^k \log \Lambda_i \\
&\quad + \frac{(N-2)^2}{4N} \left(\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |z|^2)^N} dz \right) \\
(6.5) \quad &\quad \times \sum_{i=1}^k \left(\frac{2(i-1)}{N-2} + \frac{2}{N+2-(N-2)q} \right) \varepsilon \log \varepsilon + o(\varepsilon).
\end{aligned}$$

We will do with detail the estimate of the term J_3 .

Given $\delta > 0$ small but fixed. Let μ_1, \dots, μ_k be given by (2.10), set $\mu_0 = \frac{\delta^2}{\mu_1}$ and $\mu_{k+1} = 0$. Define the following annulus

$$A_i := B(0, \sqrt{\mu_i \mu_{i-1}}) \setminus B(0, \sqrt{\mu_i \mu_{i+1}}), \quad \text{for } i = 1, \dots, k.$$

We observe that $B(0, \delta) = \bigcup_{i=1}^k A_i$. On each A_i , the leading term in $\sum_{j=1}^k U_{\mu_j}$ is U_{μ_i} .

Then we have

$$\begin{aligned}
-(q+1)J_3 &= \lambda \sum_{l=1}^k \int_{A_l} \left[\left(U_{\mu_l} + \sum_{j=1, j \neq l}^k U_{\mu_j} \right)^{q+1} - U_{\mu_l}^{q+1} - (q+1)U_{\mu_l}^q \sum_{j=1, j \neq l}^k U_{\mu_j} \right] \\
&\quad + \lambda \sum_{l=1}^k \int_{A_l} U_{\mu_l}^{q+1} + \lambda(q+1) \sum_{l=1}^k \int_{A_l} \sum_{j=1, j \neq l}^k U_{\mu_l}^q U_{\mu_j} + \lambda \int_{\mathbb{R}^N \setminus B(0, \delta)} \left(\sum_{j=1}^k U_{\mu_j} \right)^{q+1} \\
&:= J_{3,1} + J_{3,2} + J_{3,3} + J_{3,4}.
\end{aligned}$$

By the mean value theorem, for some $t \in [0, 1]$, we have

$$\begin{aligned}
J_{3,1} &= \lambda \frac{q(q+1)}{2} \sum_{l=1}^k \int_{A_l} \left(U_{\mu_l} + t \sum_{j=1, j \neq l}^k U_{\mu_j} \right)^{q-1} \left(\sum_{j=1, j \neq l}^k U_{\mu_j} \right)^2 \\
&\leq C\lambda \sum_{j,l=1, j \neq l}^k \int_{A_l} w_{\mu_l}^{q-1} w_{\mu_j}^2 + C\lambda \sum_{i,j,l=1, i,j \neq l}^k \int_{A_l} w_{\mu_i}^{q-1} w_{\mu_j}^2.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{j,l=1, j \neq l}^k \int_{A_l} w_{\mu_l}^{q-1} w_{\mu_j}^2 &= \sum_{j,l=1, j \neq l}^k \int_{A_l} (w_{\mu_l}^{q-1} w_{\mu_j}^{\frac{q-1}{q}}) w_{\mu_j}^{\frac{q+1}{q}} \\
(6.6) \quad &\leq \sum_{j,l=1, j \neq l}^k \left(\int_{A_l} w_{\mu_l}^q w_{\mu_j} \right)^{\frac{q-1}{q}} \left(\int_{A_l} w_{\mu_j}^{q+1} \right)^{\frac{1}{q}},
\end{aligned}$$

and

$$(6.7) \quad \sum_{i,j,l=1, i,j \neq l}^k \int_{A_l} w_{\mu_i}^{q-1} w_{\mu_j}^2 \leq \sum_{i,j,l=1, i,j \neq l}^k \left(\int_{A_l} w_{\mu_i}^{q+1} \right)^{\frac{q-1}{q+1}} \left(\int_{A_l} w_{\mu_j}^{q+1} \right)^{\frac{2}{q+1}}.$$

If $j > l$, then

$$\begin{aligned} \int_{A_l} w_{\mu_i}^q w_{\mu_j} dz &= \alpha_N^{q+1} \int_{\sqrt{\mu_l \mu_{l+1}} \leq |z| \leq \sqrt{\mu_l \mu_{l-1}}} \frac{\mu_l^{\frac{N-2}{2}q}}{(\mu_l^2 + |z|^2)^{\frac{N-2}{2}q}} \frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2 + |z|^2)^{\frac{N-2}{2}}} dz \\ &= \left(\frac{\mu_j}{\mu_l} \right)^{\frac{N-2}{2}} \mu_l^{-\frac{N-2}{2}q + \frac{N+2}{2}} \left[\alpha_N^{q+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |z|^2)^{\frac{N-2}{2}q}} \frac{1}{|z|^{N-2}} dz + o(1) \right]. \end{aligned} \quad (6.8)$$

If $j < l$, then

$$\begin{aligned} \int_{A_l} w_{\mu_i}^q w_{\mu_j} dx &= \alpha_N^{q+1} \int_{\sqrt{\mu_l \mu_{l+1}} \leq |z| \leq \sqrt{\mu_l \mu_{l-1}}} \frac{\mu_l^{\frac{N-2}{2}q}}{(\mu_l^2 + |z|^2)^{\frac{N-2}{2}q}} \frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2 + |z|^2)^{\frac{N-2}{2}}} dz \\ &= \left(\frac{\mu_l}{\mu_j} \right)^{\frac{N-2}{2}} \mu_l^{-\frac{N-2}{2}q + \frac{N+2}{2}} \alpha_N^{q+1} \int_{\sqrt{\frac{\mu_{l+1}}{\mu_l}} \leq |z| \leq \sqrt{\frac{\mu_{l-1}}{\mu_l}}} \frac{1}{(1 + |z|^2)^{\frac{N-2}{2}q}} \frac{1}{(1 + (\frac{\mu_l}{\mu_j})^2 |z|^2)^{\frac{N-2}{2}}} dz \\ &\leq \left(\frac{\mu_l}{\mu_j} \right)^{\frac{N-2}{2}} \mu_l^{-\frac{N-2}{2}q + \frac{N+2}{2}} \alpha_N^{q+1} \int_{\sqrt{\frac{\mu_{l+1}}{\mu_l}} \leq |z| \leq \sqrt{\frac{\mu_{l-1}}{\mu_l}}} \frac{1}{(1 + |z|^2)^{\frac{N-2}{2}q}} dz. \end{aligned} \quad (6.9)$$

For $i \neq l$, we have

$$(6.10) \quad \int_{A_l} w_{\mu_i}^{q+1} \leq C \mu_i^{-\frac{N-2}{2}q + \frac{N+2}{2}} \begin{cases} \left(\frac{\mu_l}{\mu_i} \right)^{\frac{N}{2}} & \text{if } i \leq l-1 < l; \\ \left(\frac{\mu_i^2}{\mu_l \mu_{l-1}} \right)^{\frac{N-2}{2}q-1} & \text{if } i \geq l+1 > l. \end{cases}$$

From (6.6)-(6.10), (1.4) and (2.10), we get $J_{3,1} = o(\varepsilon)$.

Moreover,

$$\begin{aligned} J_{3,2} &= \lambda \sum_{l=1}^k \int_{A_l} w_{\mu_l}^{q+1} + \lambda \sum_{l=1}^k \int_{A_l} (U_{\mu_l}^{q+1} - w_{\mu_l}^{q+1}) \\ &= \varepsilon \Lambda_1^{\frac{N+2-(N-2)q}{2}} \lambda \int_{\mathbb{R}^N} \frac{1}{(1 + |z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(\varepsilon). \end{aligned}$$

From (6.8) and (6.9), we have

$$J_{3,3} \leq C \lambda \sum_{l=1}^k \int_{A_l} \sum_{j=1, j \neq l}^k U_{\mu_l}^q U_{\mu_j} \leq C \lambda \sum_{l=1}^k \int_{A_l} \sum_{j=1, j \neq l}^k w_{\mu_l}^q w_{\mu_j} = o(\varepsilon).$$

Finally,

$$J_{3,4} = \lambda \int_{\mathbb{R}^N \setminus B(0,\delta)} \left(\sum_{j=1}^k U_{\mu_j} \right)^{q+1} \leq C \sum_{j=1}^k \int_{\mathbb{R}^N \setminus B(0,\delta)} w_{\mu_j}^{q+1} dz = o(\varepsilon).$$

Thus we get

$$(6.11) \quad J_3 = -\varepsilon \Lambda_1^{\frac{N+2-(N-2)q}{2}} \frac{\lambda}{q+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(\varepsilon).$$

From (6.3), (6.4), (6.5) and (6.11), we obtain (5.3). \square

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