PARTIAL REGULARITY FOR A LIOUVILLE SYSTEM

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ABSTRACT. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. We prove that the singular set of any extremal solution of the system

$$
\begin{aligned}
-\Delta u & = \mu e^v, \\ -\Delta v & = \lambda e^u
\end{aligned}
$$

in $\Omega$, with $u = v = 0$ on $\partial \Omega$, $\mu, \lambda \geq 0$, has Hausdorff dimension at most $n - 10$.

1. INTRODUCTION

In this article we consider the issue of partial regularity of extremal solutions to the Liouville system

$$
\begin{cases}
-\Delta u = \mu e^v & \text{in } \Omega, \\
-\Delta v = \lambda e^u & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1)

with $\Omega$ a bounded smooth open subset of $\mathbb{R}^n$, and $\lambda, \mu$ nonnegative parameters.

This system is a generalization of the equation

$$
\begin{cases}
-\Delta u = \lambda e^u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

(2)

where $\lambda$ denotes a positive parameter. It is well known that there is a maximal parameter $\lambda^* > 0$ for existence of solutions of (2) and for $0 < \lambda < \lambda^*$ there is a minimal solution $u_\lambda$. As $\lambda \to \lambda^*, \lambda < \lambda^*$ the solution $u_\lambda$ converges to the so-called extremal solution, which turns out to be smooth for $n \leq 9$, see [3, 11]. The interested reader may find in the book [7] the developments of the theory for the last six decades, with a particular focus on stable solutions.

Recently it was proved by K. Wang [13] that for $n \geq 10$ the extremal solution of (2) has a singular set of dimension at most $n - 10$. F. Da Lio [5] obtained partial regularity for any weak stationary solution in dimension 3 (not necessarily stable). See related results for the Lane-Emden equation in [14, 6].

Here we generalize the results of [13] to the system (1). For this system, M. Montenegro [12] proved the existence of a nonempty open set $U$ in the quarter plane $\lambda, \mu > 0$ such that for a couple of parameters $(\mu, \lambda)$ in $U$ there is a smooth minimal solution $(u, v)$ and no smooth solution exists if the couple is in the complement of

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Minimality means $u \leq \tilde{u}$ and $v \leq \tilde{v}$ in $\Omega$ for any other smooth solution $(\tilde{u}, \tilde{v})$ for the same $(\mu, \lambda)$.

For each slope $m > 0$, $U$ intersected with the line $\mu = m\lambda$ is a segment $\{(m\lambda, \lambda) : \lambda \in (0, \lambda^*(m))\}$ and at the extremal point $(m\lambda^*(m), \lambda^*(m)) \in \partial U$ there is a solution, called the extremal solution. It is defined as the limit as $\lambda \uparrow \lambda^*(m)$ of the minimal solution with parameters $(m\lambda, \lambda)$ and it may be singular. In a recent work [8], L. Dupaigne, A. Farina and B. Sirakov proved that the extremal solutions for the Liouville system (1) are smooth if $n \leq 9$. C. Cowan [1] had obtained the same conclusion under the restrictions $3 \leq n \leq 9$ and $\frac{n-2}{n} \leq \frac{\mu}{\lambda} \leq \frac{8}{n-2}$. In higher dimensions this fails at least in the radial case and for $\lambda = \mu$, where (1) reduces to (2).

Let us recall that an extremal solution $(u, v)$ satisfies (1) in the sense that $u, v \in L^1(\Omega)$, $e^{u \, \text{dist}(\cdot, \partial \Omega)}$, $e^{v \, \text{dist}(\cdot, \partial \Omega)} \in L^1(\Omega)$, and

$$
\int_{\Omega} u(-\Delta \varphi) = \int_{\Omega} \mu e^{v} \varphi, \quad \int_{\Omega} v(-\Delta \varphi) = \int_{\Omega} \lambda e^{u} \varphi,
$$

for all $\varphi \in C^2(\Omega)$ with $\varphi = 0$ on $\partial \Omega$.

We define the singular set $\Sigma$ of an extremal solution $(u, v)$ by $x \notin \Sigma$ if there is a neighborhood $W$ of $x$ such that $u$, $v$ are bounded in $W$. By elliptic regularity, $u$, $v$ are then smooth in this neighborhood.

**Theorem 1.1.** Assume $n \geq 10$ and let $(u, v)$ be an extremal solution of the Liouville system (1) and $\Sigma$ be its singular set. Then the Hausdorff dimension of $\Sigma$ is less or equal than $n - 10$.

The rest of the article is devoted to the proof of this theorem. We first recall a useful inequality which is valid for stable solutions of the system, obtained in C. Cowan, N. Ghoussoub [2] and L. Dupaigne, A. Farina, B. Sirakov [8]. We then state a comparison result between $u$ and $v$. Next, we perform a Moser iteration scheme to control the growth of some integrals of $e^u$ and $e^v$ on balls. The final step is an adaptation of an argument of K. Wang [13] using an $\varepsilon$-regularity result. The result in this paper is also closely related to the work of L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault [9] on stable solutions of $\Delta^2 u = e^u$ in a bounded domain or entire space.

### 2. Proof of Theorem 1.1

From [12] we know that for $(\mu, \lambda) \in U$, the associated minimal solution $(u, v)$ of (1), which is smooth, is stable in the sense that there exist $\varphi, \psi : \Omega \to \mathbb{R}$, smooth and positive in $\Omega$, satisfying

$$
\begin{cases}
-\Delta \varphi - \mu e^\psi \varphi = \eta \varphi & \text{in } \Omega, \\
-\Delta \psi - \lambda e^u \varphi = \eta \psi & \text{in } \Omega, \\
\varphi = \psi = 0 & \text{on } \partial \Omega,
\end{cases}
$$

for some $\eta > 0$. C. Cowan, N. Ghoussoub [2] and independently L. Dupaigne, A. Farina, B. Sirakov [8] have showed that this stability condition implies the following estimate.

**Lemma 2.1.** Let $(u, v)$ be a smooth stable solution of the system (1). For any $\varphi$ in $H^1_0(\Omega)$

$$
\sqrt{\lambda \mu} \int_{\Omega} \exp\left(\frac{u + v}{2}\right) \varphi^2 \leq \int_{\Omega} |\nabla \varphi|^2.
$$


2.1. **Comparison.** It will be useful later to have the following inequalities between the components of a solution of (1).

**Lemma 2.2.** Assume \( \lambda \geq \mu \). Then for any smooth solution to the Liouville system (1) we have:

\[
 u \leq v \leq u + \log \lambda - \log \mu.
\]

**Proof.** Introduce \( w = v - u - \log \lambda + \log \mu \). Then \( w \leq 0 \) on \( \partial \Omega \). We have \( -\Delta w = \lambda e^u - \mu e^v = -\lambda e^u (e^u - 1) \), and then

\[
 -\Delta w + \lambda e^u (\frac{e^u - 1}{w}) w = 0.
\]

Then due to the maximum principle \( w \leq 0 \) in \( \Omega \). For the first inequality in (4) introduce \( \tilde{w} = v - u \). Then \( -\Delta \tilde{w} = \lambda e^u - \mu e^v \geq \lambda (e^u - e^v) = -a(x) \tilde{w} \) where \( a(x) \geq 0 \). Then by the maximum principle \( \tilde{w} \geq 0 \) in \( \Omega \). \( \square \)

2.2. **Reverse Hölder inequality.** The following estimate is similar to the one obtained in [8] and [9], see also [4] for the scalar case. We assume that \((u, v)\) is a smooth stable solution of (1).

**Lemma 2.3.** For any \( 0 < \alpha < 4 \) there exists a constant \( C = C(n, \alpha, \lambda, \mu) \) such that for any \( \varphi \in C^\infty_c(\Omega) \) we have

\[
 ||\nabla(\exp(\frac{\alpha u}{2}) \varphi)||^2_{L^2(\Omega)} + ||\nabla(\exp(\frac{\alpha v}{2}) \varphi)||^2_{L^2(\Omega)} \
 \leq C \int_\Omega e^{\alpha u} (|\nabla \varphi|^2 + |\varphi \Delta \varphi|^2) + C \int_\Omega e^{\alpha v} (|\nabla \varphi|^2 + |\varphi \Delta \varphi|^2).
\]

**Remark 1.** Although the constant \( C \) depends on \( \mu, \lambda \) it remains bounded as \( (\mu, \lambda) \) approaches any extremal couple on \( \partial \Omega \).

**Proof.** Multiply \( -\Delta u = \mu e^v \) by \( e^{\alpha u} \varphi^2 \) and integrate by parts to obtain

\[
 \mu \int_\Omega e^{u+\alpha u} \varphi^2 = \int_\Omega \nabla u \nabla (e^{\alpha u} \varphi^2) = \frac{4}{\alpha} \int_\Omega \varphi^2 |\nabla (e^{\frac{\alpha u}{2}} \varphi)|^2 + \frac{1}{\alpha} \int_\Omega |\nabla (e^{\frac{\alpha u}{2}}) \nabla \varphi|^2.
\]

This reads also

\[
 \mu \int_\Omega e^{u+\alpha u} \varphi^2 = \frac{4}{\alpha} \int_\Omega |\nabla (e^{\frac{\alpha u}{2}} \varphi)|^2 + \frac{2}{\alpha} \int_\Omega e^{\alpha u} (|\nabla \varphi|^2 - \varphi \Delta \varphi).
\]

A similar equality is valid replacing respectively \( u \) by \( v \) and \( \mu \) by \( \lambda \). Introducing \( X = \int_\Omega |\nabla (e^{\frac{\alpha u}{2}} \varphi)|^2, \ Y = \int_\Omega |\nabla (e^{\frac{\alpha v}{2}} \varphi)|^2, \ A = \frac{2}{\alpha} \int_\Omega e^{\alpha u} (|\nabla \varphi|^2 - \varphi \Delta \varphi), \ B = \frac{2}{\alpha} \int_\Omega e^{\alpha v} (|\nabla \varphi|^2 - \varphi \Delta \varphi), \) we then have

\[
 \frac{4}{\alpha} X = \mu \int_\Omega e^{u+\alpha u} \varphi^2 + A,
\]

\[
 \frac{4}{\alpha} Y = \lambda \int_\Omega e^{v+\alpha v} \varphi^2 + B.
\]

We combine Hölder’s inequality and the stability estimate (3) to obtain

\[
 \mu \int_\Omega e^{u+\alpha u} \varphi^2 \leq \mu (\int_\Omega e^{\frac{\alpha u}{2}} e^{\alpha u} \varphi^2)^{1-\frac{\alpha}{2}} (\int_\Omega e^{\frac{\alpha u}{2}} e^{\alpha v} \varphi^2)^{\frac{\alpha}{2}} \leq (\frac{\mu}{\lambda})^{\frac{\alpha}{2}} X^{1-\frac{\alpha}{2}} Y^{\frac{\alpha}{2}}.
\]
Analogously, we have the same inequality replacing \( u \) by \( v \) and \( \mu \) by \( \lambda \). Hence we obtain
\[
\frac{4}{\alpha} X \leq \left( \frac{\mu}{\lambda} \right)^{\frac{1}{2}} X^{1-\frac{\alpha}{\lambda}} Y^{\frac{\alpha}{\lambda}} + A, \tag{6}
\]
\[
\frac{4}{\alpha} Y \leq \left( \frac{\lambda}{\mu} \right)^{\frac{1}{2}} Y^{1-\frac{\alpha}{\mu}} X^{\frac{\alpha}{\mu}} + B. \tag{7}
\]
Multiplying these inequalities leads to
\[
\left( \frac{16}{\alpha^2} \mu \right) XY \leq A \left( \frac{\lambda}{\mu} \right)^{\frac{1}{2}} Y^{1-\frac{\alpha}{\lambda}} + B \left( \frac{\mu}{\lambda} \right)^{\frac{1}{2}} X^{1-\frac{\alpha}{\mu}} + AB. \tag{10}
\]
Set \( \delta = \left( \frac{16}{\alpha^2} \mu \right) - 1 \). This implies that either
\[
\left( \frac{\mu}{\lambda} \right)^{\frac{1}{2}} X^{1-\frac{\alpha}{\lambda}} Y^{\frac{\alpha}{\lambda}} \leq \frac{A}{\delta} \left( 1 + \sqrt{1 + \delta} \right), \tag{8}
\]
or
\[
\left( \frac{\mu}{\lambda} \right)^{\frac{1}{2}} X^{1-\frac{\alpha}{\lambda}} Y^{\frac{\alpha}{\lambda}} \leq \frac{B}{\delta} \left( 1 + \sqrt{1 + \delta} \right) \tag{9}
\]
hold. Assuming that (8) is true and combining with (6) we get \( X \leq CA \). Using Young’s inequality in (7) we obtain \( Y \leq C (A + B) \) so that \( X + Y \leq C (A + B) \) holds, which is (5). Assuming the validity of (9) we obtain the same conclusion. \( \square \)

A consequence of the previous lemma is the following.

**Lemma 2.4.** Set \( 2^* = \frac{2n}{n-2} \). For any \( 0 < \alpha < \beta < 2(2^*) \), if \( B_{2^*}(x) \subset \Omega \) we have
\[
\left( r^{-n} \int_{B_r(x)} e^{\beta u} \right)^{\alpha/\beta} \leq C r^{-n} \int_{B_{2^*}(x)} e^{\alpha u} + e^{\alpha v}. \tag{10}
\]

**Proof.** Follows from repeated applications of Lemma 2.3, using Sobolev’s embedding and Hölder’s inequality. \( \square \)

**Remark 2.** Lemmas 2.3 and 2.4 are independent of the boundary conditions of \( u \) and \( v \), and do not use the comparison of \( u \) to \( v \) of Lemma 2.2.

### 2.3. Integrability of solutions.

**Lemma 2.5.** Assume \((u, v)\) is a stable smooth solution of (1) with parameter \((\mu, \lambda)\) of the form \( \mu = m\lambda \) for some fixed \( m > 0 \). For \( 1 \leq \alpha < 5 \) there is \( C \) independent of \( \lambda \) such that
\[
\int_{\Omega} e^{\alpha u} + e^{\alpha v} \leq C.
\]

We note that \( C \) in general depends on the slope \( m \). In this lemma we need the inequalities between \( u \) and \( v \) of Lemma 2.2. For the proof, we refer to [8] where the following was proved.

**Lemma 2.6.** Assume \( \lambda \geq \mu \). If \((u, v)\) is a stable smooth solution of (1) with parameter \((\mu, \lambda)\) of the form \( \mu = m\lambda \) for some fixed \( m > 0 \), then for \( 1 \leq \alpha < 5 \) there is \( C \) independent of \( \lambda \) such that
\[
\int_{\Omega} e^{\alpha u} \leq C.
\]

Lemma 2.5 follows from Lemmas 2.6 and 2.2 in the case \( \lambda \geq \mu \). By a symmetric argument we obtain the same conclusion if \( \lambda \leq \mu \).
2.4. \( \varepsilon \)-regularity. A crucial step is the following \( \varepsilon \)-regularity result, whose version for stable solutions in the scalar case is due to K. Wang [13], see also [9] for a biharmonic equation with exponential nonlinearity.

**Lemma 2.7.** Let \((u, v)\) be an extremal solution of (1). Then there is \( \varepsilon_2 > 0 \) such that if for some \( r_0 > 0 \) with \( B_{r_0}(x) \subset \Omega \) one has

\[
r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon_2
\]

then there is a neighborhood of \( x \) such that \( u, v \) are smooth in this neighborhood.

For the proof we need the following key step, which is adapted from [13] in the scalar case.

**Lemma 2.8.** There exists \( \varepsilon_0 > 0 \) and \( \theta > 0 \) depending only on \( n \) such that for any \( 0 < \varepsilon \leq \varepsilon_0 \), if \((u, v)\) is a stable smooth solution of (1), \( B_{r_0}(x) \subset \Omega \) and

\[
r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon
\]

then

\[
(\theta r_0)^{2-n} \int_{B_{\theta r_0}(x)} (e^u + e^v) \leq \varepsilon.
\]

**Proof.** Let us assume that \( x = 0 \) by shifting coordinates. We rescale the functions by setting

\[
\tilde{u}(x) = u(r_0 x) + 2 \log(r_0), \quad \tilde{v}(x) = v(r_0 x) + 2 \log(r_0),
\]

and note that the new functions (where the \( \tilde{\cdot} \) in the notation will be dropped) satisfy

\[-\Delta u = \mu e^{u}, \quad -\Delta v = \lambda e^{u}, \quad \text{in } B_1(0).\]

Let us decompose \( u = u_1 + u_2 \), \( v = v_1 + v_2 \) where

\[
\begin{align*}
\Delta u_1 &= 0 \quad \text{in } B_{1/2}(0), \quad u_1 = u \quad \text{on } \partial B_{1/2}(0), \\
-\Delta u_2 &= \mu e^{u} \quad \text{in } B_{1/2}(0), \quad u_2 = 0 \quad \text{on } \partial B_{1/2}(0), \\
\Delta v_1 &= 0 \quad \text{in } B_{1/2}(0), \quad v_1 = v \quad \text{on } \partial B_{1/2}(0), \\
-\Delta v_2 &= \lambda e^{u} \quad \text{in } B_{1/2}(0), \quad v_2 = 0 \quad \text{on } \partial B_{1/2}(0).
\end{align*}
\]

Let \( \gamma > 0 \), \( 0 < \theta < 1/4 \) to be fixed later on and \( \varepsilon > 0 \). Let us estimate

\[
\theta^{2-n} \int_{B_{\theta}(0)} e^u = \theta^{2-n} \int_{B_{\theta}(0) \cap \{u_2 \leq \varepsilon^\gamma\}} e^{u_1+u_2} + \theta^{2-n} \int_{B_{\theta}(0) \cap \{u_2 > \varepsilon^\gamma\}} e^u.
\]

For the first term we proceed by noting that \( e^{u_1} \) is subharmonic in \( B_{1/2}(0) \) and \( u_2 \geq 0 \), so

\[
\begin{align*}
\theta^{2-n} \int_{B_{\theta}(0) \cap \{u_2 \leq \varepsilon^\gamma\}} e^{u_1+u_2} &\leq \theta^{2-n} e^{\varepsilon^\gamma} \int_{B_{\theta}(0) \cap \{u_2 \leq \varepsilon^\gamma\}} e^{u_1} \\
&\leq \theta^{2-n} e^{\varepsilon^\gamma} \int_{B_{\theta}(0)} e^{u_1} \\
&\leq C \theta^2 e^{\gamma} \int_{B_{1/2}(0)} e^{u_1} \\
&\leq C \theta^2 e^{\gamma} \int_{B_{1/2}(0)} e^{u} \leq C \theta^2 e^{\gamma} \varepsilon,
\end{align*}
\]
where we have used (11). For the second term in (14) we have
\[\theta^{2-n} \int_{B_{\delta}(0) \cap \{u_2 > \varepsilon\}} e^u \leq \theta^{2-n} \varepsilon^{-\gamma} \int_{B_{\delta}(0) \cap \{u_2 > \varepsilon\}} u_2 e^u \]
\[\leq \theta^{2-n} \varepsilon^{-\gamma} \int_{B_{1/2}(0)} u_2 e^u \]
\[\leq \theta^{2-n} \varepsilon^{-\gamma} \|u_2\|_{L^2(B_{1/2}(0))} \|e^u\|_{L^2(B_{1/2}(0))}.\]
To estimate \(\|e^u\|_{L^2(B_{1/2}(0))}\) we apply (10) with \(\alpha = 1, \beta = 2\) to get
\[\|e^u\|_{L^2(B_{1/2}(0))} \leq C \varepsilon^{1/2}.\]
For \(\|u_2\|_{L^2(B_{1/2}(0))}\), first note that
\[\|e^v\|_{L^2(B_{1/2}(0))} \leq C \varepsilon^{1/2}.\]
Hence by \(L^2\) regularity theory
\[\|u_2\|_{W^{2,2}(B_{1/2}(0))} \leq C \varepsilon^{1/2}.\]
By using the Sobolev embedding \(W^{2,2} \subset L^{\frac{2n}{n-4}}\) we get
\[\|u_2\|_{L^{\frac{2n}{n-4}}(B_{1/2}(0))} \leq C \varepsilon^{1/2}.\]
By interpolation
\[\|u_2\|_{L^2(B_{1/2}(0))} \leq \|u_2\|_{L^1(B_{1/2}(0))} \|u_2\|_{L^{\frac{2n}{n-4}}(B_{1/2}(0))}^{1-m} \|u_2\|_{L^{\frac{2n}{n-4}}(B_{1/2}(0))}^m\]
where \(m = \frac{4}{n+4} \in (0, 1)\). But
\[\|u_2\|_{L^1(B_{1/2}(0))} \leq C \lambda \|e^u\|_{L^1(B_{1/2}(0))} \leq C \varepsilon,\]
so (19) combined with (18) and (20) yields
\[\|u_2\|_{L^2(B_{1/2}(0))} \leq C \varepsilon^m \varepsilon^{1-m/2} = C \varepsilon^{\frac{1+m}{2}}.\]
Therefore, using (16), (17) and (21) we find
\[\theta^{2-n} \int_{B_{\delta}(0) \cap \{u_2 > \varepsilon\}} e^u \leq C \theta^{2-n} \varepsilon^{1+m/2-\gamma}.\]
Combining this and (15) we obtain
\[\theta^{2-n} \int_{B_{\delta}(0)} e^u \leq C \theta^{2-\gamma} \varepsilon + C \theta^{2-\gamma} \varepsilon^{1+m/2-\gamma}.\]
Since \(m > 0\) we may choose \(0 < \gamma < m/2\). Then fix \(\theta > 0\) so that \(C \theta^2 \leq 1/2\) and
then choose \(\varepsilon_0 > 0\) sufficiently small so that \(C \theta^{2-n} \varepsilon_0^{m/2-\gamma} \leq 1/2\). It follows that
for any \(0 < \varepsilon \leq \varepsilon_0\)
\[\theta^{2-n} \int_{B_{\delta}(0)} e^u \leq \varepsilon.\]
A similar argument yields the corresponding estimate for \(e^v\). Rescaling back we obtain (12).

Applying the previous lemma we can prove
Lemma 2.9. There exists $\varepsilon_1 > 0$ and $\theta > 0$ depending only on $n$ such that for any $0 < \varepsilon \leq \varepsilon_1$, if $(u, v)$ is a stable smooth solution of (1), $B_{r_0}(x) \subseteq \Omega$ and
\[
\int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon
\]
then
\[
\int_{B_r(x)} (e^u + e^v) \leq 2^{n-2} \theta^{2-n} \varepsilon
\]
for any $y \in B_{r_0/2}(x)$ and any $0 < r \leq r_0/2$.

Proof. By shifting coordinates we can assume that $x = 0$ and by the scaling (13) that $r_0 = 1$. Let $\varepsilon_0$, $\theta$ be the constants of Lemma 2.8. We choose $\varepsilon_1$ so that $2^{n-2}\varepsilon_1 = \varepsilon_0$. Then, for any $y \in B_{1/2}(0)$ and $0 < \varepsilon \leq \varepsilon_1$ we have
\[
\left(\frac{1}{2}\right)^{2-n} \int_{B_{1/2}(y)} (e^u + e^v) \leq 2^{n-2} \int_{B_1(0)} (e^u + e^v) \leq 2^{n-2} \varepsilon \leq \varepsilon_0.
\]
Applying inductively Lemma 2.8, for any integer $k \geq 1$ we have
\[
(\theta^k)^{2-n} \int_{B_{\theta^k}(y)} (e^u + e^v) \leq 2^{n-2}\varepsilon.
\]
If $0 < r \leq 1/2$ is arbitrary we select $k \geq 1$ an integer such that $\theta^{k+1} \leq r \leq \theta^k$. Then
\[
\int_{B_r(y)} (e^u + e^v) \leq (\theta^{k+1})^{2-n} \int_{B_{\theta^k}(y)} (e^u + e^v) \leq 2^{n-2} \theta^{2-n} \varepsilon.
\]
\[\square\]

Proof of Lemma 2.7. The result of Lemma 2.9 holds also for any extremal solution. This can be proved by approximating an extremal solution $(u, v)$ of parameters $(m\lambda(m), \lambda^*(m)) \in \partial \mathcal{H}$ by minimal solutions with parameters $(m\lambda, \lambda)$ and $\lambda \uparrow \lambda^*(m)$. In this process, the constants appearing in the estimates remain bounded, see Remark 1.

Let $\varepsilon_1, \theta$ be the constants of Lemma 2.9. We take $0 < \varepsilon_2 < \varepsilon_1$ to be fixed later on. By the change of variables (13) we can assume that $x = 0$ and $r_0 = 1$, so now the hypothesis is
\[
\int_{B_1(0)} e^u + e^v \leq \varepsilon_2.
\]
Then by Lemma 2.9 we have
\[
\int_{B_{r}(y)} (e^u + e^v) \leq 2^{n-2} \theta^{2-n} \varepsilon_2
\]
for any $y \in B_{1/2}(0)$ and any $0 < r \leq 1/2$. This says that $e^u$, $e^v$ are in the Morrey space $M_{n/2}(B_{1/2}(0))$ and
\[
\|e^u\|_{M_{n/2}} + \|e^v\|_{M_{n/2}} \leq 2^{n-2} \theta^{2-n} \varepsilon_2.
\]
Let $\hat{u}$, $\hat{v}$ be the Newtonian potentials of $e^u\chi_{B_{1/2}}(0)$ and $e^v\chi_{B_{1/2}}(0)$ respectively. Then by [10] Lemma 7.20 we have
\[
\int_{B_1(0)} e^{\beta|\hat{u}|} + e^{\beta|\hat{v}|} \leq C_2
\]
for $\beta \leq \min(\frac{c_1}{\|\varepsilon_3\|_{Mn/2}}, \frac{c_2}{\|\varepsilon_3\|_{Mn/2}})$ where $c_1, C_2 > 0$ depend only on dimension.

By (22), choosing $\varepsilon_2 > 0$ small, we obtain that (23) holds for some $\beta > n/2$. Then $e^u, e^v \in L^\beta(B_{1/4}(0))$ for some $\beta > n/2$. By standard $L^p$ regularity $u, v \in L^\infty(B_{1/8}(0))$. Scaling back we have the conclusion. □

2.5. Proof of Theorem 1.1.

Proof. Let $1 \leq \alpha < 5$. We claim that

$$\Sigma \subset \left\{ x \in \Omega : \limsup_{r \to 0} r^{2\alpha-n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) > 0 \right\}.$$ 

Indeed, if $x \in \Omega$ and

$$\lim_{r \to 0} r^{2\alpha-n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) = 0$$

then by Hölder’s inequality also

$$\lim_{r \to 0} r^{2-n} \int_{B_r(x) \cap \Omega} (e^u + e^v) = 0.$$ 

Therefore for some $r_0 > 0$ so that $B_{r_0}(x) \subset \Omega$ we have

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon_2$$

where $\varepsilon_2 > 0$ is the constant from Lemma 2.7. Then by the same lemma $u, v$ are bounded in a neighborhood of $x$ and hence $x \notin \Sigma$.

Since $e^{\alpha u} + e^{\alpha v} \in L^1(\Omega)$ by Lemma 2.7, we obtain that $\mathcal{H}^{n-2\alpha}(\Sigma) = 0$, see e.g. [7, Theorem 5.3.4]. Letting $\alpha \uparrow 5$ we deduce that the Hausdorff dimension of $\Sigma$ is less or equal than $n-10$. □

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