MULTIPOLE VORTEX PATCH EQUILIBRIA FOR ACTIVE SCALAR EQUATIONS

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Abstract. We study how a general steady configuration of finitely-many point vortices, with Newtonian interaction or generalized surface quasi-geostrophic interactions, can be desingularized into a steady configuration of vortex patches. The configurations can be uniformly rotating, uniformly translating, or completely stationary. Using a technique first introduced by Hmidi and Mateu [36] for vortex pairs, we reformulate the problem for the patch boundaries so that it no longer appears singular in the point-vortex limit. Provided the point vortex equilibrium is non-degenerate in a natural sense, solutions can then be constructed directly using the implicit function theorem, yielding asymptotics for the shape of the patch boundaries. As an application, we construct new families of asymmetric translating and rotating pairs, as well as stationary tripole pairs. We also show how the techniques can be adapted for highly symmetric configurations such as regular polygons, body-centered polygons and nested regular polygons by integrating the appropriate symmetries into the formulation of the problem.

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1. Introduction

1.1. Historical discussion. In this note we consider the generalized surface quasi-geostrophic (gSQG) equations, which describe the evolution of the potential temperature $\omega$ through the transport equation

$$\begin{cases}
\partial_t \omega + v \cdot \nabla \omega = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
v = \nabla^\perp \psi, \\
\psi = -(-\Delta)^{-1+\frac{\alpha}{2}} \omega, \\
\omega|_{t=0} = \omega_0.
\end{cases} \quad (1.1)$$

Here $\nabla^\perp = (-\partial_2, \partial_1)$ while $\alpha \in [0, 1)$ is a real parameter. The vector field $v$ is the flow velocity and the fractional Laplacian operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is of convolution type, defined by

$$-(-\Delta)^{-1+\frac{\alpha}{2}} \omega(x) = \int_{\mathbb{R}^2} K_\alpha(x - y) \omega(y) dy$$

with

$$K_\alpha(x) := \begin{cases}
\frac{1}{2\pi} \ln|x| & \text{if } \alpha = 0, \\
-\frac{C_\alpha}{2\pi} \frac{1}{|x|^\alpha} & \text{if } \alpha \in (0, 1), \quad \text{with} \quad C_\alpha = \frac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})},
\end{cases} \quad (1.2)$$

where $\Gamma$ is the gamma function. This model was proposed by Córdoba et al. [12] as an interpolation between Euler equations and the surface quasi-geostrophic model, which correspond to $\alpha = 0$ and $\alpha = 1$, respectively.

The main purpose of this note is to show the existence of new families of periodic global solutions of (1.1) in the vortex patch setting, namely when $\omega(\cdot, t)$ is the characteristic function of finite collection of bounded domains. Such patterns are a special class of Yudovich solutions where $\omega(\cdot, t)$ is merely bounded and integrable. Yudovich solutions are known to be unique and to exist globally in time in the case of Euler equations [54], but for $\alpha > 0$ the situation is more delicate because the velocity field $v$ is in general not Lipschitz. Nonetheless, when the initial datum has a patch structure, one can locally construct a unique solution which remains a patch. The motion of the boundary of the patch is governed by so-called contour dynamics equations; see [20, 47]. It is worth mentioning that, while the boundary’s regularity is globally preserved for $\alpha = 0$ [11, 3], for $\alpha > 0$ numerical evidence [12] suggests singularity formation in finite time.

There are very few explicit solutions to the gSQG and Euler equations. The only known explicit simply-connected vortex patch solutions are the Rankine vortex, which is stationary, and the Kirchhoff ellipses [29] for the Euler equation, which are rotating. Nevertheless, a family of uniformly rotating patches with $m$-fold symmetry, called V-states, was numerically computed by Deem and Zabusky [17]. Later, Burbea [4] gave an analytical proof of their existence, based on a conformal mapping parametrization and local bifurcation theory. Recently, Burbea’s branches of solutions were extended to global ones [32]. The regularity and the convexity of the V-states have been investigated in [37, 32, 8]. Similar research has been carried out for the gSQG equations: The construction of simply connected V-states was established in [30, 17], and their boundary regularity was discussed in [8].

We point out that there is a large literature dealing with rotating vortex patches and related problems. For instance, we mention the existence results of rotating patches close to Kirchhoff’s ellipses [34, 8], multiply-connected patches [35, 16, 35, 14, 46, 26], patches in bounded domains [15], non-trivial rotating smooth solutions [9] and rotating vortices with non-uniform densities [24]. The radial symmetry properties of stationary and uniformly-rotating solutions was studied in a series of works [19, 33, 28].
All of the above analytical results treat connected patches. The first numerical works revealing the existence of translating symmetric pairs of simply connected patches for Euler equation are due to Deem and Zabusky [17] and Pierrehumbert [45]. Similar studies were performed by Saffman and Szeto in [50] for the symmetric co-rotating vortex pairs and by Dritschel [18] for asymmetric pairs. Later, Turkington gave in [51] an analytical proof using variational arguments, where he considered an unbounded fluid domain with \( N \) symmetrically arranged vortex patches rotating about the origin. Implementing the same approach, Keady [39] proved the existence of translating pairs of symmetric patches and Wan [52] studied the existence and stability of desingularizations of a general system of rotating point vortices. Very recently, Godard-Cadillac, Gravejat and Smets [25] extended Turkington’s result to the gSQG equations, while Ao, Dávila, Del Pino, Musso and Wei [1] have obtained related families of smooth solutions via gluing techniques. See [41, 42, 44, 49, 53] for additional references on multiply connected patches.

The variational arguments [51, 39, 25] do not give much information about the shape of the vortex patches, or about the uniqueness of solutions. In [36], however, Hmidi and Mateu gave a direct proof showing the existence of co-rotating and counter-rotating vortex pairs, using an elegant desingularization of the contour dynamics equations and an application of the implicit function theorem. The same technique was implemented for the desingularization of the asymmetric pairs [31], Kármán street [21] and the vortex polygon [22]. See [6] for related results where point vortices are instead desingularized into doubly-connected patches. We mention that, using more sophisticated Nash–Moser techniques, Gómez-Serrano, Park and Shi [27] have very recently constructed stationary configurations of multi-layered patches with finite kinetic energy. Also, García and Haziot [23] have combined ideas from [36] and [32] to prove a global bifurcation result for co-rotating and counter-rotating pairs.

In this note we show how the technique in [36] can be extended to arbitrary configurations of finitely-many point vortices. For a general configuration, the problem reduces via Lyapunov–Schmidt to a finite-dimensional nonlinear equation. Under an natural non-degeneracy assumption on the point vortex configuration alone, one can instead simply apply a modified version of the implicit function theorem. For highly symmetric configurations, we may also simplify the problem by reformulating it in spaces which take these symmetries into account.

1.2. Statement of the general result. Recall that the gSQG point vortex model for \( N \) interacting vortices in the complex plane \( \mathbb{C} \) is given by the Hamiltonian system

\[
\frac{d}{dt}z_j(t) = i\tilde{C}_\alpha \frac{\beta}{2} \sum_{k=1,k\neq j}^{N} \gamma_k \frac{z_j(t) - z_k(t)}{|z_j(t) - z_k(t)|^{\alpha+2}}, \quad j = 1, \ldots, N, \tag{1.3}
\]

where \( z_1(t), \ldots, z_N(t) \) are the point vortex locations, \( \pi \gamma_1, \ldots, \pi \gamma_N \in \mathbb{R} \setminus \{0\} \) are the circulations and

\[
\tilde{C}_\alpha := \alpha C_\alpha = \frac{2^\alpha \Gamma(1+\alpha/2)}{\Gamma(1-\alpha/2)}. \tag{1.4}
\]

The case \( \alpha = 0 \) corresponds to the classical point vortex Eulerian interaction. A general review about the \( N \)-vortex problem and vortex statics can be found in [2] for the Newtonian interaction and [48] for gSQG interactions. We are concerned with periodic solutions for which the configuration of vortices is instantaneously moving as a rigid body, so that

\[
\frac{d}{dt}z_j(t) = iU + i\Omega z_j(t), \tag{1.5}
\]

where here \( U \in \mathbb{R} \) is the constant linear velocity and \( \Omega \in \mathbb{R} \) is the constant angular velocity. Such solutions are known as relative equilibria or vortex crystals. Explicitly solving (1.5), we see that whenever \( \Omega \neq 0 \) we can shift coordinates so that \( U = 0 \). Thus there is no loss of generality in restricting our attention to equilibria which are either rotating with \( \Omega \neq 0 \) and \( V = 0 \), translating with \( U \neq 0 \) and \( \Omega = 0 \), or stationary with \( U = \Omega = 0 \).
Figure 1. (a) A solution of the point vortex system \((1.6)\). The vortices are located at the points \(z_j(0) := z_j\) and have circulations \(\pi\gamma_j\). (b) A desingularization into vortex patches. The vortex at \(z_j\) has become a small nearly-circular patch \(D_{\varepsilon_j}\). To leading order in the small parameter \(\varepsilon\), the radius of the patch is \(\varepsilon b_j\), and the net circulation is \(\pi\gamma_j\).

Setting \(z_j = x_j + iy_j := z_j(0)\), \((1.3)\) reduces to the algebraic system

\[
P_\alpha^\lambda(\lambda) = \Omega z_j + U - \frac{C_\alpha}{2} \sum_{k=1,k\neq j}^N \gamma_k \frac{z_j - z_k}{|z_j - z_k|^\alpha + 2} = 0, \quad j = 1, \ldots, N, \tag{1.6}
\]

where here \(\lambda = (x_1, \ldots, x_N, y_1, \ldots, y_N, \gamma_1, \ldots, \gamma_N, \Omega, U)\). Taking real and imaginary parts, this defines a mapping \(P^\alpha(\lambda)\) with values in \(\mathbb{R}^{2N}\).

**Definition 1.1.** We call a rigidly rotating or translating solution \(\lambda^*\) of \((1.6)\) non-degenerate if, after a reordering of the entries of \(\lambda\), we can write

\[
\lambda = (\lambda_1, \lambda_2) \quad \text{with} \quad \lambda_1 \in \mathbb{R}^{2N-1} \quad \text{and} \quad \text{codim ran } D_{\lambda_1} P^\alpha(\lambda^*) = 1. \tag{1.7}
\]

We call a stationary solution \(\lambda^*\) of \((1.6)\) (with \(\Omega = U = 0\)) non-degenerate if, after a reordering of the entries of \(\lambda\), we can write

\[
\lambda = (\lambda_1, \lambda_2) \quad \text{with} \quad \lambda_1 \in \mathbb{R}^{2N-3} \quad \text{and} \quad \text{codim ran } D_{\lambda_1} P^\alpha(\lambda^*) = 3. \tag{1.8}
\]

Informally stated, our first result is the following; see Theorems 2.7 and 2.8 for a precise version.

**Theorem 1.2.** Let \(\alpha \in [0,1)\). Then any non-degenerate solution \(\lambda\) of \((1.6)\) can be desingularized into a family of vortex patch equilibria depending smoothly on a small parameter \(\varepsilon > 0\) measuring the size of the patches.

See Figure 1 for an illustration.

**Remark 1.3.** As we shall see later in Theorems 2.7 and 2.8, the vortex patches in Theorem 1.2 are small \(C^{1+\beta}\) perturbations of the unit disk, whose boundaries are given by conformal parametrizations which can be explicitly expanded to any order in the small parameter \(\varepsilon\). In particular, the conformal parametrizations of the boundaries \(\phi_j: \mathbb{T} \to \partial \Omega^\varepsilon_j\) have the explicit Fourier asymptotic expansions

\[
\phi_j(\varepsilon, w) = w + (\varepsilon b_j)^{\alpha+2} \Xi_\alpha \sum_{k=1,k\neq j}^N \gamma_k \frac{(\bar{z}_k - \bar{z}_j)^2}{|z_k - z_j|^\alpha + 4w} + o(\varepsilon^{2+\alpha}), \quad \Xi_\alpha := \frac{(\alpha + 2)\Gamma(1 - \frac{\alpha}{2})\Gamma(3 - \frac{\alpha}{2})}{4\Gamma(2 - \alpha)}.
\]

Furthermore, with slight modifications in the proof we can show that the boundary of each vortex patch belongs to \(C^{n+\beta}\) for any fixed \(n \in \mathbb{N}\). The range of \(\varepsilon\) would be not uniform with respect to \(n\) but would shrink to zero as \(n\) goes to infinity. However, using the approach developed in [8], we expect that the conformal mappings possess holomorphic extensions outside of a small disc and...
thus that the boundaries are analytic. In the Euler case, the analyticity of sufficiently smooth patch boundaries could alternatively be proved using the same technique as in \[32\].

**Remark 1.4.** The leading-order ratios between the sizes of the patches can be specified a priori. Moreover, the range of \( \varepsilon \) is uniform as some of these ratios are sent to zero, allowing us to recover solutions involving a combination of point vortices and vortex patches. See Remark 2.9 for more details.

**Remark 1.5.** While the proof is valid for \( \alpha \in [0, 1) \), we expect that a similar result can be proved for \( \alpha \in [1, 2) \) using the spaces introduced in [7]; see [5].

### 1.3. Applications

There are many point vortex equilibria satisfying the non-degeneracy assumption in Theorem 1.2. We shall give several examples where this assumption can be easily checked and the resulting vortex patch solutions are, to the best of our knowledge, new.

The most elementary solutions to (1.6) are co-rotating and counter-rotating vortex pairs. A family of asymmetric co-rotating pairs is given by

\[
\lambda^* := (x_1^*, x_2^*, y_1^*, y_2^*; \gamma_1^*, \gamma_2^*; \Omega^*, U^*) = \left(d, -cd, 0, 0; c\gamma, \gamma; \frac{\gamma C_{\alpha}}{2d^{\alpha+2}(1+c)^{\alpha+1}}, 0\right),
\]

where \( \gamma \in \mathbb{R} \setminus \{0\} \), \( d > 0 \) and \( 0 < |c| < 1 \); see Figure 2(a). In the time-dependent problem, the two vortices steadily rotate about the origin with angular velocity \( \Omega^* \). Counter-rotating pairs given by

\[
\lambda^* := (x_1^*, x_2^*, y_1^*, y_2^*; \gamma_1^*, \gamma_2^*; \Omega^*, U^*) = \left(d, -d, 0, 0; -\gamma, \gamma; 0, \frac{\gamma C_{\alpha}}{2^{\alpha+2}d^{\alpha+1}}\right),
\]

instead steadily translate along the \( y \)-axis; see Figure 2(b). We also consider asymmetric stationary tripole of the form

\[
\lambda^* := (x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*; \gamma_1^*, \gamma_2^*, \gamma_3^*; \Omega^*, U^*)
\]

\[
= \left(1, 0, -a, 0, 0, 0; \gamma, -\gamma\left(\frac{a}{a+1}\right)^{\alpha+1}, \gamma a^{\alpha+1}; 0, 0\right),
\]

where \( a \in (0, 1) \); see Figure 2(c). Note that all of these above configurations are invariant under reflections about the \( x \)-axis. Furthermore, horizontal translations of (1.10) and rotations of (1.9) are also solutions.

As we shall see in Section 3, these configurations are non-degenerate in the sense of Definition 1.1. Using Theorem 1.2, they can therefore be desingularized into steady vortex patch equilibria. As no two vortices in (1.9) or (1.11) can be identified with one another, the same is true of the corresponding vortex patches. While the two vortices in (1.10) can be identified, this symmetry is broken if we require the leading-order ratio between the sizes of the patches to be different from 1. For the pairs, this extends the desingularization result of [31], obtained in the Eulerian case \( \alpha = 0 \), to gSQG equations (1.1) with \( \alpha \in (0, 1) \). To the best of our knowledge, the asymmetric tripole patch solutions are new both for the Euler and gSQG equations. Furthermore, this appears to be the first existence proof for stationary solutions to the gSQG equations involving multiple patches; see [27] for stationary solutions to the Euler equations with multiple multi-layered patches and [26] for stationary doubly connected solutions to the gSQG equations.

**Figure 2.** (a) An asymmetric co-rotating vortex pair given by (1.9). (b) A symmetric counter-rotating vortex pair given by (1.10). (c) A asymmetric point vortex tripole given by (1.11).
Theorem 1.6. Let $\alpha \in [0, 1)$ and $b_1, b_2, b_3 \in (0, \infty)$, and let $\gamma, d, c, a$ be as above. Then, the following results hold true.

(i) For any $\varepsilon > 0$ sufficiently small, there are two strictly convex domains $O_1^\varepsilon, O_2^\varepsilon$, 1-fold symmetric, $C^{1+\beta}$ perturbations of the unit disc, and real numbers $x_1(\varepsilon) = \delta + o(\varepsilon)$, $x_2(\varepsilon) = -cd + o(\varepsilon)$ such that

$$\omega_0^\varepsilon = \frac{\gamma}{\varepsilon^2 b_1^2} \chi D_1^\varepsilon + \frac{c \gamma}{\varepsilon^2 b_2^2} \chi D_2^\varepsilon \quad \text{with} \quad D_1^\varepsilon := \varepsilon b_1 O_1^\varepsilon + x_1(\varepsilon), \quad D_2^\varepsilon := \varepsilon b_2 O_2^\varepsilon + x_2(\varepsilon),$$

generates a co-rotating vortex pair for (1.1) with angular velocity $\Omega^* = \frac{1}{2} \gamma \hat{C}_\alpha d^{-\frac{1}{2} - (1 + \varepsilon) - \alpha - 1}$. 

(ii) For any $\varepsilon > 0$ sufficiently small, there are two strictly convex domains $O_1^\varepsilon, O_2^\varepsilon$, 1-fold symmetric, $C^{1+\beta}$ perturbations of the unit disc, and real numbers $x_2(\varepsilon) = -\delta + o(\varepsilon)$, $\gamma_1(\varepsilon) = -\gamma + o(\varepsilon)$ such that

$$\omega_0^\varepsilon = \frac{\gamma_1(\varepsilon)}{\varepsilon^2 b_1^2} \chi D_1^\varepsilon + \frac{\gamma}{\varepsilon^2 b_2^2} \chi D_2^\varepsilon \quad \text{with} \quad D_1^\varepsilon := \varepsilon b_1 O_1^\varepsilon + d, \quad D_2^\varepsilon := \varepsilon b_2 O_2^\varepsilon + x_2(\varepsilon),$$

generates a counter-rotating vortex pair for (1.1) with speed $U^* = \gamma \hat{C}_\alpha 2^{-\frac{1}{2} - d - \alpha - 1}$.

(iii) For any $\varepsilon > 0$ sufficiently small, there are three strictly convex domains $O_1^\varepsilon, O_2^\varepsilon, O_3^\varepsilon$, 1-fold symmetric, $C^{1+\beta}$ perturbations of the unit disc, and two real numbers $x_3(\varepsilon) = -a + o(\varepsilon)$, $\gamma_2(\varepsilon) = -\gamma \left( \alpha \frac{1}{a+1} \right)^{\alpha + 1} + o(\varepsilon)$ such that

$$\omega_0^\varepsilon = \frac{\gamma}{\varepsilon^2 b_1^2} \chi D_1^\varepsilon + \frac{\gamma_2(\varepsilon)}{\varepsilon^2 b_2^2} \chi D_2^\varepsilon + \frac{\gamma a^{\alpha + 1}}{\varepsilon^2 b_3^2} \chi D_3^\varepsilon$$

with $D_1^\varepsilon := \varepsilon b_1 O_1^\varepsilon + 1, \quad D_2^\varepsilon := \varepsilon b_2 O_2^\varepsilon, \quad D_3^\varepsilon := \varepsilon b_3 O_3^\varepsilon + x_3(\varepsilon),$

generates a stationary vortex triple for (1.1).

Remark 1.7. By sending some of $b_1, b_2, b_3$ to zero, we can recover configurations involving a mixture of vortex patches and point vortices; see Remark 2.9.

Remark 1.8. The reflection symmetry property can be checked using the boundary equations, the uniqueness of the constructed curve of solutions and invariance under reflections about the $x$-axis of the point vortex configuration; see Section 3.

While the above examples are asymmetric and have only two or three vortices, there are other well-known point vortex equilibria which are highly symmetric and have many vortices. When seeking similarly symmetric desingularizations of such equilibria, it convenient to integrate these additional symmetries into the statement of the problem. In particular, the relevant non-degeneracy conditions on the point vortex equilibria can be much simpler to verify than Definition 1.1.

As a concrete example, we consider two concentric regular $m$-gons with a vortex at each vertex, and assume that the vortices of a same polygon have the same vorticity $\gamma_1 \in \mathbb{R} \setminus \{0\}$ or $\gamma_2 \in \mathbb{R}$. We place, in addition, a point vortex at the center of the regular $m$-gons with intensity $\gamma_0 \in \mathbb{R}$. More specifically, we are concerned with the system of point vortices

$$\omega_0^j(z) = \pi \gamma_0 \delta_0^j(z) + \sum_{j=1}^{2} \sum_{j=0}^{m-1} \pi \gamma_j \delta_{j,k}(0)^j(z), \quad z_{jk}(0) := \begin{cases} d_1 e^{2k\varpi/m} & \text{if } j = 1, \\ d_2 e^{(2k+\varpi)\vartheta/m} & \text{if } j = 2, \quad \vartheta \in \{0, 1\}. \end{cases} \quad (1.12)$$

The value $\vartheta = 0$ corresponds to the configuration where the vertices of the polygons are radially aligned with each other and $\vartheta = 1$ refers to the case where the vertices are out-of-phase by an angle $\pi/m$. If we assume that (1.12) performs a perpetual uniform rotation, then we may easily check that the system of $2m+1$ equations in (1.6) can be reduced to a system of two real equations:

$$\Omega d_j - \frac{\hat{C}_\alpha}{2d_j^2 + \alpha} \left( \gamma_0 + \gamma_j \sum_{k=1}^{m-1} \frac{1 - e^{2k\varpi/m}}{1 - e^{2k\varpi/m}} - 2 + \gamma_3 - \sum_{k=0}^{m-1} \left| 1 - e^{-2k\varpi/m} \frac{d_3 - j}{d_j} \right|^{2+\alpha} \right) = 0, \quad j = 1, 2. \quad (1.13)$$
Figure 3. Sketch of the vortex patch solutions in Theorem 1.9 with \( m = 3 \). The point vortex of circulation \( \pi \gamma_0 \) at the origin has been desingularized into a patch with \( m \)-fold symmetry, while the vortices with circulations \( \pi \gamma_1, \pi \gamma_2 \) at the vertices of two regular \( m \)-gons have been desingularized into vortex patches with 1-fold symmetry. (a) shows the case aligned case \( \vartheta = 0 \), while (b) shows the staggered case \( \vartheta = 1 \).

Observe that the last system is linear in \( \Omega \) and \( \gamma_2 \), and, thus, under a simple non-degeneracy condition we can explicitly solve the system (1.13) for \( \Omega \) and \( \gamma_2 \neq 0 \).

For the sake of clarity we shall give an elementary statement about the desingularization of these configurations; for a complete statement see Theorem 4.3.

**Theorem 1.9.** Let \( \alpha \in [0, 1) \), \( \vartheta \in \{0, 1\} \), \( b_1, b_2 \in (0, 1) \), \( \gamma_0, \gamma_1 \in \mathbb{R} \setminus \{0\} \), \( d_1, d_2 \in (0, \infty) \) and let \( (\Omega^*, \gamma_2^*) \in (\mathbb{R} \setminus \{0\})^2 \) be a solution of (1.13) satisfying the non-degeneracy conditions (4.7) and (4.9). Then, for any \( \varepsilon > 0 \) sufficiently small, there are three strictly convex domains \( O_{\varepsilon}^0, O_{\varepsilon}^1, O_{\varepsilon}^2 \), \( C^{1+\beta} \) perturbations of the unit disc, and a real number \( \gamma_2 = \gamma_2(\varepsilon) \) such that

\[
\omega_{0, \varepsilon} = \frac{\gamma_0}{\varepsilon^2 b_0^2} \chi_{\varepsilon b_0 O_0} + \sum_{j=1}^{2} \frac{\gamma_j}{\varepsilon^2 b_j^2} \sum_{k=0}^{m-1} \chi_{\varepsilon_{jk}} \quad \text{with} \quad D_{jk}^\varepsilon := \begin{cases} e^{\frac{2k\pi i}{m}} (\varepsilon b_1 O_1^\varepsilon + d_1) & \text{if } j = 1, \\ e^{\frac{(2k+\vartheta)\pi i}{m}} (\varepsilon b_2 O_2^\varepsilon + d_2) & \text{if } j = 2, \end{cases}
\]

(1.14) generates a rotating solution for (1.1) with some constant angular velocity \( \Omega(\varepsilon) \). Moreover \( O_0^\varepsilon \) is \( m \)-fold symmetric and \( O_1^\varepsilon, O_2^\varepsilon \) are 1-fold symmetric.

See Figure 3 for an illustration.

**Remark 1.10.** The complete statement of Theorem 1.9 given in Theorem 4.3 explicitly computes the asymptotic behavior of the conformal parametrizations \( \phi_j^\varepsilon : T \to \partial O_j^\varepsilon \).

**Remark 1.11.** The parameter \( \varepsilon > 0 \) can be chosen uniformly as any of the parameters \( b_0, b_1 \) and \( b_2 \) tend to 0, and therefore we may recover the point vortex-vortex patch configurations discussed in [13, 53]; see Remark 2.9.

**Remark 1.12.** The proof can be easily adapted to the rotating vortex polygon (with \( \gamma_0 = \gamma_2 = 0 \) in (1.14)) and which has been studied in [11, 25, 51, 22]. This remains equally true for the body-centered polygonal configurations (\( \gamma_2 = 0 \)), treated in [52], as well as for the nested polygons without a central patch (\( \gamma_0 = 0 \)). The latter solutions were first observed numerically in [53].

1.4. **Idea of the proof.** We shall briefly explain the basic ideas behind Theorem 1.2 for the Euler equations in the rotating case; a similar strategy is followed for the gSQG equations and for traveling
or stationary patches. We seek simply connected bounded domains \( O_j^\varepsilon \) such that the initial datum

\[
\omega_0^\varepsilon(z) = \sum_{j=1}^N \frac{\gamma_j}{\varepsilon^2 b_j^2} \varphi_j^\varepsilon(z) \quad \text{with} \quad \varphi_j^\varepsilon := \varepsilon b_j O_j^\varepsilon + z_j
\]

performs a uniform rotation around the center of mass of the system, taken to be the origin, with an angular velocity \( \Omega \). Here the parameters \( b_j \) will allow us to specify the relative sizes of the patches; see Remark 1.4. After moving to the rotating frame, the boundaries of the system are subject to the following stationary system, see for instance [16 Page 1896],

\[
\Re \{ \gamma_j (v^\varepsilon(z) + i \Omega \overline{z} \overline{n}) \} = 0 \quad \text{for all} \quad z \in \partial O_j^\varepsilon, \quad (1.15)
\]

where \( \overline{n} \) is the exterior unit normal vector to the boundary at the point \( z \). In virtue of the Biot–Savart law and Green’s theorem, we may write

\[
\overline{v^\varepsilon(z)} = \frac{1}{4\pi} \sum_{k=1}^N \frac{\gamma_k}{\varepsilon^2 b_k^2} \int_{\partial O_k^\varepsilon} \frac{\overline{\xi - z}}{\xi - z} d\xi
\]

for all \( z \in \mathbb{C} \).

Following Hmidi and Mateu [36], we reformulate (1.15) in terms of the conformal parametrizations of the boundaries \( \phi_j : \mathbb{T} \to \partial O_j^\varepsilon \), which we assume have the form

\[
\phi_j(w) = w + \varepsilon b_j f_j(w).
\]

In other words, we shall look for domains \( O_j^\varepsilon \) which are small perturbations of the unit disc with an amplitude of order \( \varepsilon b_j \). While the resulting problem initially appears to have terms which are singular in \( \varepsilon \), as in [36] there is a cancellation — essentially due to the symmetry of the disk — which eliminates these terms. This leads to a nonlinear system

\[
\mathcal{G}_j(\varepsilon, f; \lambda) = 0, \quad j = 1, \ldots, N
\]

for the perturbations \( f := (f_1, \ldots, f_N) \) of the patch boundaries, where the nonlinear operator \( \mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_N) : (-\varepsilon_0, \varepsilon_0) \times B_0 \times \Lambda \to W_0 \) is well-defined and of class \( C^1 \). Here \( \Lambda \) in a small neighborhood of \( \lambda^* \), which is the solution to the point vortex system (1.6) with \( \alpha = 0 \), \( B_0 \) is the unit ball in \( \mathbb{C} \) and

\[
\begin{align*}
\mathcal{V}_0 &:= \bigotimes_{N \text{ times}} V_0, \\
\mathcal{W}_0 &:= \bigotimes_{N \text{ times}} W_0, \\
V_0 &:= \left\{ f \in C^{1+\beta}(\mathbb{T}) : f(w) = \sum_{n \geq 1} a_n \overline{w}^n, \ a_n \in \mathbb{C} \right\}, \\
W_0 &:= \left\{ g \in C^\beta(\mathbb{T}) : g(w) = \sum_{n \neq 0} c_n w^n, \ c_{-n} = \overline{c_n} \in \mathbb{C} \right\}
\end{align*}
\]

for some fixed Hölder exponent \( \beta \in (0, 1) \). Moreover, for \( (\varepsilon, f; \lambda) = (0, 0; \lambda) \) we find

\[
\mathcal{G}_j(0, 0; \lambda)(w) = \Im \{ \mathcal{P}_j^0(\lambda \overline{w}) \}, \quad (1.16)
\]

so that \( \mathcal{G}(0, 0; \lambda) = 0 \) is equivalent to (1.6).

Intending to apply the implicit function theorem, we next linearize \( \mathcal{G} \) about the point vortex solution \((0, 0; \lambda^*)\). The linearized operator with respect to the patch boundaries \( f \)

\[
D_j \mathcal{G}_j(0, 0; \lambda^*) h(w) = \gamma_j \Im \{ h'_j(w) \},
\]

which has a trivial kernel and a range with finite codimension \( 2N \). To deal with this deficiency, we also linearize with respect to the point vortex parameters \( \lambda \) using (1.16). Assuming that \( \lambda^* \) is non-degenerate in the sense of Definition 1.1, we deduce that the linearized operator \( D_{(f, \lambda_0)} \mathcal{G}(0, 0; \lambda^*) \)
has trivial kernel and a range with codimension 1. As this final deficiency is caused by a nonlinear identity
\[
\sum_{j=1}^{N} \frac{\gamma_j}{\pi} \int_{T} G_j^2 (\varepsilon, f; \lambda) (w) (\varepsilon b_j |w + \varepsilon b_j f_j(w)|^2 + \varepsilon b_j \Re \left[ z_j f_j(w) \right] + \Re [\overline{z_j} w]) \overline{w} dw = 0
\]
satisfied by the functional \(G\), we can conclude by applying a modified version of the implicit function theorem (Lemma 2.6).

1.5. Notation. Let us end this part by summarizing some notation to be used in the paper. We will denote the unit disc by \(D\) and its boundary by \(T\). For continuous functions \(f: T \to \mathbb{C}\) we introduce the notation
\[
\int_{T} f(\tau) d\tau := \frac{1}{2\pi i} \int_{T} f(\tau) d\tau,
\]
where \(d\tau\) stands for complex integration. For any \(x \in \mathbb{R}\) and \(n \in \mathbb{N}\), we use the notation \((x)_n\) to denote the Pochhammer symbol defined by
\[
(x)_n := \begin{cases} 
1 & \text{if } n = 0, \\
x(x+1) \cdots (x+n-1) & \text{if } n \geq 1.
\end{cases}
\]
Finally, we use the notation \(\delta_{ij}\) to denote the Kronecker delta defined by
\[
\delta_{ij} := \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

1.6. Outline of the paper. In Section 2 we consider completely general vortex equilibria, without any symmetry assumptions, show how the problem is desingularized and prove Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.6. Finally, in Section 4 we prove Theorem 1.9 by imposing suitable rotation and reflection symmetries.

2. Desingularization of general vortex equilibria

In this section we consider a general configuration of finitely many point vortices in uniform rotation or translation, and show how these vortices can be desingularized into small vortex patches. Using the approach developed in [36], we first write down the contour dynamics equations governing the \(N\) steady vortex patches, and then find the suitable function spaces where the problem is well-posed. Finally, we prove Theorem 1.2 using a simple extension of the implicit function theorem.

Consider \(N\) bounded simply connected domains \(O_j^\varepsilon, j = 1, \ldots, N\), containing the origin and contained in the ball \(B(0,2)\). Given \(b_j \in (0, \infty), z_j \in \mathbb{C}\) and \(\varepsilon \in (0, \varepsilon_0)\), we define the domains
\[
D_j^\varepsilon := \varepsilon b_j O_j^\varepsilon + z_j,
\]
where \(\varepsilon_0 > 0\) is chosen small enough that the sets \(D_j^\varepsilon\) are pairwise disjoint,
\[
D_j^\varepsilon \cap D_k^\varepsilon = \emptyset, \quad j \neq k.
\]
Consider the initial vorticity
\[
\omega_0^\varepsilon(z) = \frac{1}{\varepsilon^2} \sum_{j=1}^{N} \frac{\gamma_j}{b_j^2} \chi_{D_j^\varepsilon}(z).
\]
Note that, if \(\varepsilon \to 0\) and \(|O_j^\varepsilon| \to |D|\) in (2.3), we find the point vortex distribution
\[
\omega_0^0(z) = \pi \sum_{j=1}^{N} \gamma_j \delta_{z_j}(z),
\]
whose evolution is described by (1.3).
2.1. Integral identities for the stream function. We shall give in this subsection some identities related to the (non-relative) stream function,

$$\forall z \in \mathbb{C} \mapsto \psi^\varepsilon(z) = \frac{1}{\varepsilon^2} \sum_{k=1}^{N} \frac{\gamma_k}{b^2_k} \int_{D^\varepsilon_k} K_\alpha(z - \xi) dA(\xi), \quad (2.4)$$

associated to the vortex patch \([2,3]\), where \(K_\alpha\) is defined in \([1,2]\). As we shall see in the following subsections, these identities will be useful to explain the degeneracy of the functional defining the V-states. While they can be derived using the symmetries and variational structure of the problem, we shall give here a simple proof using the structure of the kernel \(K_\alpha\).

**Lemma 2.1.** For all \(\varepsilon \in (0, \varepsilon_0)\), the stream function \(\psi^\varepsilon(z)\) satisfies the following identities:

(i) \(\sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b^2_j} \int_{\partial D^\varepsilon_j} \psi^\varepsilon(z) d\xi = 0\),

(ii) \(\mathfrak{Re} \left[ \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b^2_j} \int_{\partial D^\varepsilon_j} \bar{z} \psi^\varepsilon(z) d\xi \right] = 0\).

**Proof.** Applying the complex version of Green’s theorem,

$$2i \int_D \partial_\xi f(\xi, \bar{\xi}) dA(\xi) = \int_{\partial D} f(\xi, \bar{\xi}) d\xi, \quad (2.5)$$

to \((2.4)\) we find

$$\psi^\varepsilon(z) = \frac{1}{2i} \sum_{k=1}^{N} \frac{\gamma_k}{\varepsilon^2 b^2_k} \int_{\partial D^\varepsilon_k} \hat{K}_\alpha(z - \xi) d\xi \quad \text{with} \quad \hat{K}_\alpha(\xi) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \ln |\xi| - \frac{1}{2} & \text{if } \alpha = 0, \\
- \frac{C_\alpha}{2\pi(1 - \frac{\alpha}{2})} & \text{if } \alpha \in (0,1).
\end{array} \right.$$  

It follows that

$$\sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b^2_j} \int_{\partial D^\varepsilon_j} \psi^\varepsilon(z) d\xi = \frac{1}{2i} \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b^2_j} \int_{\partial D^\varepsilon_k} \hat{K}_\alpha(z - \xi) d\xi d\xi.$$

The integrand \(\hat{K}_\alpha(z - \xi)\) changes sign when the roles of \(z\) and \(\xi\) are reversed, and so (i) follows.

Next we apply formula \((2.5)\) to \(\bar{z} \psi^\varepsilon\), which yields

$$\int_{\partial D^\varepsilon_j} \bar{z} \psi^\varepsilon(z) d\xi = 2i \int_{D^\varepsilon_j} \psi^\varepsilon(z) dA(z) + 2i \int_{D^\varepsilon_j} \bar{z} \partial_\xi \psi^\varepsilon(z) dA(z).$$

Since \(\psi^\varepsilon(z)\) is real, the last identity implies

$$\mathfrak{Re} \left[ \int_{\partial D^\varepsilon_j} \bar{z} \psi^\varepsilon(z) d\xi \right] = 2 \mathfrak{Im} \left[ \int_{D^\varepsilon_j} \bar{z} \partial_\xi \psi^\varepsilon(z) dA(z) \right]. \quad (2.6)$$

Differentiating \((2.4)\) with respect to \(z\) we get

$$\partial_z \psi^\varepsilon(z) = \sum_{k=1}^{N} \frac{\gamma_k}{\varepsilon^2 b^2_k} \int_{D^\varepsilon_k} \hat{K}_\alpha(z - \xi) dA(z) \quad \text{with} \quad \hat{K}_\alpha(\xi) = \left\{ \begin{array}{ll}
\frac{1}{4\pi} \frac{1}{|\xi|} & \text{if } \alpha = 0, \\
\frac{\alpha C_\alpha}{4\pi} \frac{\xi}{|\xi|^{\alpha+2}} & \text{if } \alpha \in (0,1).
\end{array} \right.$$
The desired identity (ii) now follows from the fact that the integrand is purely real, and the proof is complete.

2.2. Boundary equations. First suppose that \( \omega^\varepsilon \) gives rise to \( N \) rotating patches for the model (1.1) about the centroid of the system, assumed to be the origin, with an angular velocity \( \Omega \). More precisely, we are looking for a solution \( \omega^\varepsilon(t) \) of (1.1) of the form

\[
\omega^\varepsilon(t, z) = \omega^\varepsilon_0(e^{-i\Omega t} z).
\]

Inserting this expression into (1.1) we obtain

\[
(v^\varepsilon(z) - i\Omega z) \cdot \nabla \omega^\varepsilon_0(z) = 0,
\]

with \( v^\varepsilon \) is the velocity field associated to \( \omega^\varepsilon_0 \). Using the patch structure, we conclude that (16 Page 1896)

\[
\text{Re} \left\{ \gamma_j (v^\varepsilon(z) + i\Omega z) \bar{n} \right\} = 0 \quad \text{for all} \quad z \in \partial D_j^\varepsilon, \quad j = 1, \ldots, N, \quad (2.7)
\]

where \( \bar{n} \) is the exterior unit normal vector to the boundary at the point \( z \).

If instead \( \omega^\varepsilon_0 \) translates vertically with uniform velocity \( U \), that is

\[
\omega^\varepsilon(t, z) = \omega^\varepsilon_0(z - iTU),
\]

the analogue of (2.7) is

\[
\text{Re} \left\{ \gamma_j (v^\varepsilon(z) + iTU) \bar{n} \right\} = 0 \quad \text{for all} \quad z \in \partial D_j^\varepsilon, \quad j = 1, \ldots, N, \quad (2.8)
\]

For the sake of abbreviation and simplicity we shall unify (2.7) and (2.8) as follows

\[
\text{Re} \left\{ \gamma_j (v^\varepsilon(z) + iTU + i\Omega z) \bar{n} \right\} = 0 \quad \text{for all} \quad z \in \partial D_j^\varepsilon, \quad j = 1, \ldots, N, \quad (2.9)
\]

and assume that either \( \Omega \) or \( U \) vanishes.

2.2.1. Euler equation. In view of the Biot–Savart law one has

\[
\overline{v^\varepsilon(z)} = -2i \partial_z \psi^\varepsilon(z) = -\frac{i}{2\pi} \sum_{k=1}^{N} \frac{\gamma_k}{\varepsilon^2 b_k^2} \int_{D_k^\varepsilon} \frac{dA(\zeta)}{z - \zeta}
\]

for all \( z \in \mathbb{C} \). By the complex form of Green’s theorem (2.5), we may replace the integral over \( D_k \) in (2.10) with an integral along \( \partial D_k \):

\[
\overline{v^\varepsilon(z)} = \frac{i}{2} \sum_{k=1}^{N} \frac{\gamma_k}{\varepsilon^2 b_k^2} \int_{\partial D_k^\varepsilon} \frac{\xi - \bar{z}}{\xi - z} d\xi.
\]

Inserting the last identity into (2.9) leads to

\[
\gamma_j \text{Re} \left\{ (\Omega z + U + V^\varepsilon(z)) z' \right\} = 0, \quad \forall z \in \partial D_j^\varepsilon, \quad j = 1, \ldots, N, \quad (2.11)
\]
where \( z' \) denotes a tangent vector to the boundary at the point \( z \) and

\[
V^\varepsilon(z) := \sum_{k=1}^{N} \frac{\gamma_k}{2 \varepsilon b_k} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{\bar{\xi} - z}{\bar{b}_k \bar{\xi} + z_k - z} d\xi.
\]

In view of (2.1), a suitable change of variables gives

\[
V^\varepsilon(z) = \sum_{k=1}^{N} \frac{\gamma_k}{2 \varepsilon b_k} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{\varepsilon b_k \bar{\xi} - z_k - \varepsilon b_j \bar{\xi} - z_j}{\varepsilon b_k \xi + z_k - \varepsilon b_j z - z_j} d\xi.
\]

Observe, from (2.2), that for any \( j \neq k \) and \( z \in \partial \mathcal{O}_j^\varepsilon \) one has

\[
\varepsilon b_j z + z_j + z_k \notin \varepsilon b_k \mathcal{O}_k^\varepsilon.
\]

Thus, by the residue theorem, for every \( z \in \partial \mathcal{O}_j^\varepsilon \), we may write

\[
V^\varepsilon(\varepsilon b_j z + z_j) = \sum_{k=1}^{N} \frac{\gamma_k}{2 \varepsilon b_k} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{\varepsilon b_k \bar{\xi} - z_k - \varepsilon b_j \bar{\xi} - z_j}{\varepsilon b_k \xi + z_k - \varepsilon b_j z - z_j} d\xi
\]

\[= \frac{\gamma_j}{2 \varepsilon b_j} \int_{\partial \mathcal{O}_j^\varepsilon} \frac{\bar{\xi} - z}{\xi - z} d\xi + \sum_{k=1}^{N} \frac{\gamma_k}{2} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{\bar{\xi}}{\varepsilon b_k \xi + z_k - \varepsilon b_j z - z_j} d\xi.
\]

Replacing \( z \) by \( \varepsilon b_j z + z_j \) in (2.11) and using the last identity we get

\[
\gamma_j \text{ Re} \left\{ \left( \Omega(\varepsilon b_j \bar{\xi} + z_j) + U + \frac{\gamma_j}{2 \varepsilon b_j} \int_{\partial \mathcal{O}_j^\varepsilon} \frac{\bar{\xi} - z}{\xi - z} d\xi \right. \right.
\]

\[+ \left. \sum_{k=1}^{N} \frac{\gamma_k}{2} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{\bar{\xi}}{\varepsilon b_k \xi + z_k - \varepsilon b_j z - z_j} d\xi \right\} = 0, \quad \forall z \in \partial \mathcal{O}_j^\varepsilon. \tag{2.12}
\]

We shall look for domains \( \mathcal{O}_j^\varepsilon \), which are perturbations of the unit disc with an amplitude of order \( \varepsilon b_j \). More precisely, we shall consider \( \phi_j : \mathbb{C} \setminus \bar{D} \rightarrow \mathbb{C} \setminus \partial \mathcal{O}_j^\varepsilon \) the unique conformal map with the expansion

\[
\phi_j(w) = w + \varepsilon b_j f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^{\infty} \frac{a_j^m}{w^m}, \quad a_j^m \in \mathbb{C}. \tag{2.13}
\]

By the Kellogg–Warschawski theorem [45, Theorem 3.6], since the boundary \( \partial \mathcal{O}_j^\varepsilon \) is assumed to be a smooth Jordan Curve, \( \phi_j \) extends to a smooth mapping \( \mathbb{C} \setminus \bar{D} \rightarrow \mathbb{C} \setminus \partial \mathcal{O}_j^\varepsilon \), and its trace, that we shall also denote by \( \phi_j \), is a smooth parametrization of \( \partial \mathcal{O}_j^\varepsilon \). Thus, making the change of variable \( z = \phi_j(w) \), \( z' = iw\phi_j'(w) \) in (2.12), we obtain

\[
\text{Im} \left\{ \gamma_j \left( \Omega(\varepsilon b_j \phi_j(w) + z_j) + U + \frac{\gamma_j}{2 \varepsilon b_j} \int_{\mathcal{T}} \frac{\phi_j(\tau) - \phi_j(w)}{\phi_j(\tau) - \phi_j(w)} \phi_j'(\tau) d\tau \right. \right.
\]

\[+ \left. \sum_{k=1}^{N} \frac{\gamma_k}{2} \int_{\mathcal{T}} \varepsilon b_k \phi_k(\tau) + z_k - \varepsilon b_j \phi_j(w) - z_j \right\} w\phi_j'(w) = 0 \tag{2.14}
\]

for any \( w \in \mathbb{T} \) and \( j = 1, \ldots, N \).
To desingularize this system in $\varepsilon$, we follow the ideas of [35] and write, by virtue of (2.13),

$$
\frac{1}{2\varepsilon b_j} \int_T \frac{\phi_j(\tau) - \phi_j(w)}{w - \tau} \phi'_j(\tau) d\tau = \frac{1}{2\varepsilon b_j} \int_T \frac{\overline{w - \tau} + \varepsilon b_j (f_j(\tau) - f_j(w))}{w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w))} \left[ 1 + \varepsilon b_j f'_j(\tau) \right] d\tau
$$

$$
= \frac{1}{2} \int_T \frac{\overline{w - \tau} + \varepsilon b_j (f_j(\tau) - f_j(w))}{w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w))} f'_j(\tau) d\tau - \frac{1}{2\varepsilon b_j} \int_T \frac{\overline{w - \tau}}{w - \tau} d\tau
$$

$$
+ \frac{1}{2} \int_T \frac{(w - \tau)(f_j(\tau) - f_j(w)) - (\overline{w - \tau})(f_j(\tau) - f_j(w))}{(w - \tau)(w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w)))} d\tau
$$

$$
:= \mathcal{I}[\varepsilon, f_j](w) - \frac{1}{2\varepsilon b_j} \int_T \frac{\overline{w - \tau}}{w - \tau} d\tau.
$$

From the obvious identity

$$
\int_T \frac{\overline{w - \tau}}{w - \tau} d\tau = w,
$$

the expression of $\phi_j$ in (2.13), and the symmetry of the disk we can get rid of the singular term from the full nonlinearity,

$$
\text{Im} \left\{ \frac{1}{2\varepsilon b_j} \left( \int_T \frac{\phi_j(\tau) - \phi_j(w)}{w - \tau} \phi'_j(\tau) d\tau \right) w \phi'_j(w) \right\} = \text{Im} \left\{ \mathcal{I}[\varepsilon, f_j](w) \left( 1 + \varepsilon f'_j(w) \right) - \frac{1}{2} f'_j(w) \right\}.
$$

Inserting the last equation into (2.14), we conclude that

$$
\gamma_j G_j^0(\varepsilon, f; \lambda)(w) := -\gamma_j \text{Im} \left\{ \left( \Omega(\varepsilon b_j \overline{w} + \varepsilon^2 b_j^2 \overline{f_j(w)} - z_j) + U + \gamma_j \mathcal{I}[\varepsilon, f_j](w) \right) \right. 
$$

$$
+ \sum_{k=1, k\neq j}^N \gamma_k \mathcal{J}_{kj}[\varepsilon, f_k, f_j](w) w \left( 1 + \varepsilon b_j f'_j(w) \right) - \frac{\gamma_j}{2} f'_j(w) \right\} = 0 \tag{2.15}
$$

for all $w \in \mathbb{T}$ and $j = 1, \ldots, N$, where $f = (f_1, \ldots, f_N)$,

$$
\lambda = (x_1, \ldots, x_N, y_1, \ldots, y_N, \gamma_1, \ldots, \gamma_N, \Omega, U) \tag{2.16}
$$

are the point vortex parameters and

$$
\mathcal{I}[\varepsilon, f_j](w) := \frac{1}{2} \int_T \frac{\overline{w - \tau} + \varepsilon b_j (f_j(\tau) - f_j(w))}{w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w))} f'_j(\tau) d\tau
$$

$$
+ \frac{1}{2} \int_T \frac{2i \text{Im} \left\{ (w - \tau)(f_j(\tau) - f_j(w)) \right\}}{(w - \tau)(w - \tau + \varepsilon b_j f_j(\tau) - \varepsilon b_j f_j(w))} d\tau, \tag{2.17}
$$

$$
\mathcal{J}_{kj}[\varepsilon, f_k, f_j](w) := \frac{1}{2} \int_T \frac{\left( \overline{\tau + \varepsilon b_k f_k(\tau)} (1 + \varepsilon b_k f'_k(\tau)) \right)}{\varepsilon b_k \tau + \varepsilon^2 b_k^2 f_k(\tau) + z_k - \varepsilon b_j (w + \varepsilon b_j f_j(w)) - z_j} d\tau. \tag{2.18}
$$

Here we have used the notation $\mathcal{I}$ and $\mathcal{J}_{kj}$ with a complex conjugate in order to unify the notation with the functions $\mathcal{I}^\alpha$, $\mathcal{J}_{kj}^\alpha$, and $G_j^\alpha$ that we shall introduce in the next subsection for the gSQG equations. Furthermore, both sides of (2.15) are multiplied by $\gamma_j$ to ensure that the system is valid even if we set some of the $\gamma_j$ equal to zero.

### 2.2.2. gSQG equations

The velocity can be recovered from the boundary as follows

$$
v^e(z) = \frac{C_\alpha}{2\pi} \sum_{k=1}^N \gamma_k \varepsilon^{2b_k^2} \int_{\partial \mathcal{D}_k} \frac{d\xi}{|z - \xi|^\alpha} \tag{2.19}
$$
for all \( z \in \mathbb{C} \), see for instance [30]. In view of (2.11), suitable change of variables gives

\[
v^\varepsilon(z) = i \sum_{k=1}^{N} \frac{\gamma_k \alpha \varepsilon \int_{\partial \mathcal{O}^\varepsilon_k} \frac{1}{|\varepsilon b_k \xi + z_k - z|^{\alpha}} d\xi.}
\]

Inserting the last identity into (2.9) and taking a complex conjugate inside the real part leads to

\[
\text{Re} \left\{ \gamma_j \left( \Omega z + U - \sum_{k=1}^{N} \frac{\gamma_k \alpha}{\varepsilon b_k} \int_{\partial \mathcal{O}^\varepsilon_k} \frac{1}{|\varepsilon b_k \xi + z_k - z|^{\alpha}} d\xi \right) \varepsilon b_j \right\} = 0 \quad \forall z \in \partial \mathcal{O}^\varepsilon_j, \quad j = 1, \ldots, N,
\]

where \( z' \) denotes a tangent vector to the boundary at the point \( z \). Replacing \( z \) by \( \varepsilon b_j z + z_j \) in the last system gives

\[
\text{Re} \left\{ \gamma_j \left( \Omega \varepsilon b_j z + z_j + U - \frac{\gamma_j \alpha \varepsilon}{|\varepsilon b_j^{1+\alpha}} \int_{\partial \mathcal{O}^\varepsilon_j} \frac{1}{|\xi - z|^{\alpha}} d\xi \right) \right\} = 0 \quad \forall z \in \partial \mathcal{O}^\varepsilon_j.
\]

We shall look for conformal parametrizations \( \phi_j : \mathbb{T} \to \partial \mathcal{O}^\varepsilon_j \) having the expansions

\[
\phi_j(w) = w + \varepsilon |\alpha| b_j^{1+\alpha} f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^{\infty} \frac{a_{m}^{j}}{w^m}, \quad a_{m}^{j} \in \mathbb{C}, \quad j = 1, \ldots, N. \quad (2.21)
\]

Here, the coefficient \( |\varepsilon|^{\alpha} \) in the definition of the conformal mapping \( \phi_j \) comes from the singularity of the gSOG kernel. For every \( w \in \mathbb{T} \), the tangent vector is given by \( z' = i w \phi_j'(w) \) and therefore (2.20) becomes

\[
\text{Im} \left\{ \gamma_j \left( \Omega \varepsilon b_j \phi_j(w) + z_j + U - \frac{\gamma_j \alpha \varepsilon}{|\varepsilon b_j^{1+\alpha}} \int_{\mathbb{T}} \frac{\phi_j'(\tau)}{|\phi_j(\tau) - \phi_j(w)|^{\alpha}} d\tau \right) \right\} = 0.
\]

In order to desingularize the system (2.22) we shall use the following Taylor formula,

\[
\frac{1}{|A + B|^{\alpha}} = \frac{1}{|A|^{\alpha}} - \alpha \int_{0}^{1} \frac{1}{|A + tB|^{2+\alpha}} dt = \frac{1}{|A|^{\alpha}} \left( \frac{1}{|A + B|^{\alpha}} - \alpha \int_{0}^{1} \frac{1}{|A + tB|^{2+\alpha}} dt \right) = \frac{1}{|A + B|^{\alpha}}.
\]

which is true for any complex numbers \( A, B \) such that \(|B| < |A|\). Taking \( A = z_k - z_j \) and \( B = b_k \phi_k(\tau) - b_j \phi_j(w) \), one may write

\[
\frac{1}{|\varepsilon b_k \phi_k(\tau) + z_k - \varepsilon b_j \phi_j(w) - z_j|^{\alpha}} = \frac{1}{|z_k - z_j|^{\alpha}} - \alpha \varepsilon \int_{0}^{1} \frac{1}{|\varepsilon b_k \phi_k(\tau) + z_k - \varepsilon b_j \phi_j(w) - z_j|^{\alpha+2}} dt.
\]

It follows that

\[
\frac{1}{\varepsilon} \int_{\mathbb{T}} \frac{\phi_k'(\tau)}{|\varepsilon b_k \phi_k(\tau) + z_k - \varepsilon b_j \phi_j(w) - z_j|^{\alpha}} d\tau = -\alpha \int_{0}^{1} \frac{1}{\varepsilon} \int_{\mathbb{T}} \frac{1}{|\varepsilon b_k \phi_k(\tau) + z_k - \varepsilon b_j \phi_j(w) - z_j|^{\alpha+2}} \phi_k'(\tau) d\tau d\tau
\]

\[
-\alpha \int_{0}^{1} \frac{1}{\varepsilon} \int_{\mathbb{T}} \frac{1}{|\varepsilon b_k \phi_k(\tau) + z_k - \varepsilon b_j \phi_j(w) - z_j|^{\alpha+2}} \phi_k'(\tau) d\tau d\tau.
\]
On the other hand, from (2.21), one has
\[
\frac{C_\alpha}{\varepsilon |\phi_j(\tau) - \phi_j(w)|} \int_T \frac{d\tau}{|\phi_j(\tau) - \phi_j(w)|^{\alpha}} = \frac{C_\alpha}{\varepsilon |\phi_j(\tau) - \phi_j(w)|} \int_T \frac{d\tau}{|\tau - w|^{\alpha}} + \frac{C_\alpha}{\varepsilon |\phi_j(\tau) - \phi_j(w)|} \int_T \frac{b_j^{1+\alpha} f_j'(\tau) d\tau}{|\tau - w|^{\alpha}}
\]

Using the identity [30] Page 337,
\[
\frac{C_\alpha}{\varepsilon |\phi_j(\tau) - \phi_j(w)|} \int_T \frac{d\tau}{|\tau - w|^{\alpha}} = \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{(2 - \alpha)\Gamma^2(1 - \frac{\alpha}{2})} w =: \mu_\alpha w,
\]

and applying the formula (2.23) with \(A = \tau - w\) and \(B = \varepsilon |\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))\), we find
\[
\frac{C_\alpha}{\varepsilon |\phi_j(\tau) - \phi_j(w)|} \int_T \frac{d\tau}{|\phi_j(\tau) - \phi_j(w)|^{\alpha}} = \frac{\mu_\alpha w + C_\alpha b_j^{1+\alpha} f_j'(\tau)}{\varepsilon |\phi_j(\tau) - \phi_j(w)|^{\alpha}} d\tau
\]

As in the Euler case, by (2.21) and the symmetry of the disc the singular term disappears from the nonlinearity,
\[
\text{Im} \left\{ \frac{C_\alpha}{\varepsilon |\alpha b_j^{1+\alpha} f_j'(\tau)} \int_T \frac{d\tau}{|\phi_j(\tau) - \phi_j(w)|^{\alpha}} \right\} = 0
\]

Inserting (2.24) and (2.26) into (2.22) we get
\[
\gamma_j \mathcal{G}_j^\alpha(\varepsilon, f; \lambda)(w) := \gamma_j \text{Im} \left\{ \left( \Omega(z b_j w + \varepsilon^2 |\alpha b_j^{2+\alpha} f_j(w) + z_j) + U + \gamma_j \mathcal{T}_\alpha^\alpha(\varepsilon, f_j)(w) + \sum_{k=1, k \neq j}^N \gamma_k \mathcal{J}_k^\alpha(\varepsilon, f_k, f_j)(w) \right) \overline{w(1 + \varepsilon |\alpha b_j^{1+\alpha} f_j'(w))} - \mu_\alpha \gamma_j f_j'(w) \right\} = 0
\]

for all \(w \in \mathbb{T}\) and \(j = 1, \ldots, N\), where \(f = (f_1, \ldots, f_N)\), \(\lambda\) denotes the point vortex parameters (2.16) and
\[
\mathcal{T}_\alpha^\alpha(\varepsilon, f_j)(w) := -C_\alpha \int_T \frac{f_j'(\tau)}{|\tau - w + \varepsilon |\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))|^{\alpha}} d\tau
\]

\[
+ \alpha C_\alpha \int_T \int_0^1 \frac{\text{Re} \left[ (f_j(\tau) - f_j(w))(\tau - w) \right] + \varepsilon |\alpha t f_j(\tau) - f_j(w)|^2 dt d\tau}{|\tau - w + \varepsilon |\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))|^{\alpha}}
\]

\[
\mathcal{J}_k^\alpha(\varepsilon, f_k, f_j; \lambda)(w) := \frac{\alpha C_\lambda}{b_k} \int_T \int_0^1 \frac{\text{Re} \left[ (z_k - z_j)(b_k \phi_k(\tau) - b_j \phi_j(w)) \right]}{|z_k - z_j|^{\alpha+2} b_k(d\tau d\tau)} + \int_T \int_0^1 \frac{\varepsilon |b_k \phi_k(\tau) - b_j \phi_j(w)|^2 \phi_k(\tau)}{|z_k - z_j|^{\alpha+2} d\tau d\tau}.
\]
2.3. Regularity and linearization of the functional $G^\alpha$. For any $\beta \in (0,1)$, we denote by $C^\beta(\mathbb{T})$ the space of continuous functions $f: \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_{C^\beta(\mathbb{T})} := \|f\|_{L^\infty(\mathbb{T})} + \sup_{x \neq y \in \mathbb{T}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$ 

For any integer $n$, the space $C^{n+\beta}(\mathbb{T})$ stands for the set of functions $f$ of class $C^n$ whose $n$-th order derivatives are Hölder continuous with exponent $\beta$. It is equipped with the usual norm,

$$\|f\|_{C^{n+\beta}(\mathbb{T})} := \|f\|_{L^\infty(\mathbb{T})} + \left\| \frac{d^n f}{dx^n} \right\|_{C^\beta(\mathbb{T})}.$$ 

We henceforth fix some $\beta \in (0,1)$, and for $\alpha \in [0,1]$ define

$$\eta_\alpha := \begin{cases} 1 - \alpha & \text{if } 0 < \alpha < 1, \\ \beta & \text{if } \alpha = 0. \end{cases}$$

Consider the Banach spaces

$$V_0^\alpha := V_0^\alpha \times \cdots \times V_0^\alpha_{N \text{ times}}, \quad W_0^\alpha := W_0^\alpha \times \cdots \times W_0^\alpha_{N \text{ times}}, \quad \text{and } \tilde{W}_0^\alpha := \tilde{W}_0^\alpha \times \cdots \times \tilde{W}_0^\alpha_{N \text{ times}}$$

with

$$V_0^\alpha := \left\{ f \in C^{1+\eta_\alpha}(\mathbb{T}) : f(w) = \sum_{n \geq 1} a_n w^n, \ a_n \in \mathbb{C} \right\},$$

$$W_0^\alpha := \left\{ g \in C^\alpha(\mathbb{T}) : g(w) = \sum_{n \neq 0} c_n w^n, \ c_{-n} = \overline{c_n} \in \mathbb{C} \right\},$$

$$\tilde{W}_0^\alpha := \left\{ g \in W^\alpha : c_1 = 0 \right\}.$$

Here the subscript 0 refers to the fact that no a priori symmetry assumption is made on the patch boundaries. The restriction on the Fourier coefficients of functions in $V_0^\alpha$ guarantees that these functions can be extended to holomorphic functions on $\mathbb{C} \setminus \mathbb{D}$. The restriction on the Fourier coefficients of functions in $W_0^\alpha$, on the other hand, simply says that these functions are real-valued.

We denote by $B_0^\alpha$ the unit ball in $V_0^\alpha$,

$$B_0^\alpha := \left\{ f \in V^\alpha : \|f\|_{C^{\alpha+1}(\mathbb{T})} < 1 \right\}$$

and by $B_0^\beta$ the unit ball in $W_0^\alpha$,

$$B_0^\beta := \left\{ f \in W^\alpha : \|f\|_{C^\alpha(\mathbb{T})} < 1 \right\}.$$

We shall unify the expression of the function $G^\alpha_j$ in (2.27) and the function $G^0_j$ in (2.15) as follows: For every $f \in V_0^\alpha$ and $w \in \mathbb{T}$,

$$G^\alpha_j(\varepsilon, f; \lambda)(w) := \text{Im} \left\{ \left( \Omega(\varepsilon b_j w + \varepsilon^2 |\varepsilon|^\alpha b_j^{2+\alpha} f_j(w) + z_j) + U \right) \overline{w} \left( 1 + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} f_j^\prime(w) \right) \\
- \mu_\alpha \gamma_j f_j^\prime(w) + \left( \gamma_j \mathcal{T}[\varepsilon, f_j](w) + \sum_{k=1, k \neq j}^N \gamma_k \mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w) \right) \overline{w} \left( 1 + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} f_j^\prime(w) \right) \right\},$$

(2.31)

where $\mu_\alpha$ is defined in (2.25), and $\mathcal{T}[\varepsilon, f_j](w)$ and $\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w)$ are given by (2.28)–(2.29) if $\alpha \in (0,1)$ and by (2.17)–(2.18) if $\alpha = 0$. We then define the nonlinear operator

$$G^\alpha(\varepsilon, f; \lambda) := (G^\alpha_1(\varepsilon, f; \lambda), \ldots, G^\alpha_N(\varepsilon, f; \lambda)).$$

Proposition 2.2. Let $\alpha \in [0,1)$ and let $\lambda^*$ solve (1.6).

(i) There exists $\varepsilon_0 > 0$ and a small neighborhood $\Lambda$ of $\lambda^*$ such that $G^\alpha$ can be extended to a $C^1$ mapping $(-\varepsilon_0, \varepsilon_0) \times B_0^\alpha \times \Lambda \to W_0^\alpha$. 

Proof. Note that the term

gives (ii). Differentiating (2.33) with respect to \( \alpha \) implies (iii) in the case \( \alpha = 0 \).

For all \( \lambda \in \Lambda \) one has

\[
\mathcal{G}^\alpha_j(0, 0; \lambda)(w) = \text{Im} \left\{ \mathcal{P}^\alpha_j(\lambda)w \right\}
\]

for \( w \in \mathbb{T} \), where \( \mathcal{P}^\alpha_j(\lambda) \) is given by (1.6).

(iii) The Fréchet derivative of \( \mathcal{G}^\alpha \) with respect to \( f \) at \( (0, 0; \lambda) \) is given by

\[
D_f \mathcal{G}^\alpha(0, 0; \lambda)h(w) = \sum_{n \geq 1} M_n^\alpha \left( \begin{array}{c} \gamma_1 \text{Im} \left\{ a_n^1 w^{n+1} \right\} \\ \vdots \\ \gamma_N \text{Im} \left\{ a_n^N w^{n+1} \right\} \end{array} \right),
\]

where here \( h = (h_1, \ldots, h_N) \in \mathcal{V}_0^\alpha \) with \( h_j(w) := \sum_{n \geq 1} a_n^j w^n \), and

\[
M_n^\alpha := \frac{\Gamma(1 + \frac{n}{2}) \Gamma(1 - \alpha)}{2^{2-\alpha} \Gamma^{\alpha/2} (1 - \frac{n}{2})} \left( 2(n + 1) \frac{1}{1 - \frac{n}{2}} - (1 + \frac{n}{2})_n - (1 + \frac{n}{2})_{n+1} \right).
\]

(iv) For any \( \lambda \in \Lambda \), the linear operator \( D_f \mathcal{G}^\alpha(0, 0; \lambda): \mathcal{V}_0^\alpha \to \mathcal{V}_0^\alpha \) is an isomorphism.

Next, we shall prove (ii) and (iii) in the case \( \alpha = 0 \). Substituting \( \varepsilon = 0 \) in (2.17) and (2.18) gives

\[
\overline{\mathcal{J}}^0_{kj}[0, f_k, f_j; \lambda](w) = -\frac{1}{2} \frac{1}{z_j - z_k},
\]

\[
\overline{\mathcal{T}}^0[0, f_j](w) = \frac{1}{2} \int_{\mathbb{T}} \frac{\overline{w} - \tau}{w - \tau} f_j'(\tau) d\tau + \int_{\mathbb{T}} \frac{i \text{Im} \left\{ (w - \tau)(f_j(\tau) - f_j(w)) \right\}}{(w - \tau)^2} d\tau = 0,
\]

where we have used the residue theorem in the last identity. Then, from (2.15), we get

\[
\mathcal{G}^0_j(0, f; \lambda)(w) = -\text{Im} \left\{ \left( -\Omega \overline{z}_j + U - \frac{1}{2} \sum_{k=1, k \neq j}^N - \frac{\gamma_k}{z_j - z_k} \right) w - \frac{\gamma_j}{2} f_j'(w) \right\}
\]

and hence

\[
\mathcal{G}^0_j(0, 0; \lambda)(w) = -\text{Im} \left\{ \left( -\Omega \overline{z}_j + U - \frac{1}{2} \sum_{k=1, k \neq j}^N - \frac{\gamma_k}{z_j - z_k} \right) w \right\}
\]

which shows (ii). Differentiating (2.33) with respect to \( f \) gives

\[
\partial_{f_k} \mathcal{G}^0_j(0, 0; \lambda)h_j(w) = \delta_{jk} \frac{\gamma_j}{2} \text{Im} \left\{ h_j'(w) \right\},
\]

implying (iii) in the case \( \alpha = 0 \).
To prove (ii) and (iii) in the case $\alpha \in (0, 1)$, we substitute $\varepsilon = 0$ in (2.28) and (2.29) and obtain
\[ T^\alpha[0, f_j](w) = -C_\alpha \int_\mathbb{T} f_j'(\tau) - U \alpha C_\alpha \int_0^1 \frac{\Re \left[ (f_j(\tau) - f_j(w))(\tau - w) \right]}{|w + \tau|^{2+\alpha}} \, d\tau dt, \]
\[ J^\alpha_k[0, f_k, f_j; \lambda](w) = \frac{\alpha C_\alpha}{2} \frac{z_k - z_j}{|z_k - z_j|^{\alpha+2}}. \]

Thus, by (2.27), one has
\[ G_j^\alpha(0, f; \lambda)(w) = \operatorname{Im} \left\{ \left( \Omega z_j + U + \frac{\alpha C_\alpha}{2} \sum_{k=1, k \neq j}^N \gamma_k \frac{z_k - z_j}{|z_k - z_j|^{\alpha+2}} \right) \bar{w} \right. \]
\[ + \left. \gamma_j \bar{\lambda} T^\alpha[0, f_j](w) - \mu_\alpha \gamma_j f_j(w) \right\}. \] (2.34)

It follows that
\[ G_j^\alpha(0, 0; \lambda)(w) = \operatorname{Im} \left\{ \left( \Omega z_j + U - \frac{\alpha C_\alpha}{2} \sum_{k=1, k \neq j}^N \gamma_k \frac{z_j - z_k}{|z_j - z_k|^{\alpha+2}} \right) \bar{w} \right\}. \]

Comparing the last expression with (1.6) concludes (ii). Next, differentiating (2.34) with respect to $f$ gives
\[ \partial_f G_j^\alpha(0, 0; \lambda) h_j(w) = \delta_{jk} \gamma_j \operatorname{Im} \left\{ T^\alpha[0, h_j](w) \bar{w} - \mu_\alpha h_j(w) \right\}. \] (2.35)

The last expression was explicitly computed in [36] Pages 726–728 and takes the form
\[ \partial_f G_j^\alpha(0, 0; \lambda) h_j(w) = \gamma_j \sum_{n \geq 1} \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{4 \Gamma^2(1 - \frac{n}{2})} \left( \frac{2(n + 1)}{1 - \frac{n}{2}} - \frac{(1 + \frac{n}{2})_n}{(1 - \frac{n}{2})_n} \right) \operatorname{Im} \left\{ a_j^n w^{n+1} \right\}, \]

getting the announced result.

The proof of (iv) is elementary in the case $\alpha = 0$, since
\[ D_f G^\alpha(0, 0; \lambda) h(w) = \sum_{n \geq 1} \frac{1}{2} \begin{pmatrix} \gamma_1 \operatorname{Im} \left\{ k_1'(w) \right\} \\ \vdots \\ \gamma_N \operatorname{Im} \left\{ k_N'(w) \right\} \end{pmatrix}. \]

For the case $\alpha \in (0, 1)$, we reproduce similar arguments to [36]. In particular, we observe from (2.35) that $D_f G_j^\alpha(0, 0; \lambda): \mathcal{V}_0^\alpha \rightarrow \mathcal{W}_0^\alpha$ is a compact perturbation of a Fredholm operator of index zero, since $T^\alpha: \mathcal{V}_0^\alpha \rightarrow \mathcal{W}_0^\alpha$ is smoothing. Therefore $\partial_f G_j^\alpha(0, 0; \lambda): \mathcal{V}_0^\alpha \rightarrow \mathcal{W}_0^\alpha$ is Fredholm with index zero. To check that is has a trivial kernel we can argue as in [36] page 728. This concludes the proof of the proposition. \hfill \square

Remark 2.3. Since $\mathcal{W}_0^\alpha \subset \mathcal{W}_0^\alpha$ has codimension $2N$, Proposition 2.2(iv) implies that $D_f G^\alpha(0, 0; \lambda^*)$ is Fredholm with finite index $-2N$. As we will see, even for non-generate equilibria the full linearized operator $D_{(f, \lambda)} G^\alpha(0, 0; \lambda^*)$ will have a range with nonzero codimension. While this rules out an immediate application of the implicit function theorem, one can still, in a small neighborhood of $(0, 0; \lambda^*)$, the nonlinear problem $G^\alpha(\varepsilon, f; \lambda) = 0$ will always admit a Lyapunov–Schmidt reduction to an equation in finite dimensions; see for instance [40]. Of course, studying this reduced equation may still be quite challenging, and may in particular involve evaluating further Fréchet derivatives of $G^\alpha$.
2.4. Integral identities for the functional $G^\alpha$. In this short section we prove several integral identities for the functional $G^\alpha_j$ which follow from Lemma 2.1.

First, we show how that functional $G^\alpha_j$ defined in (2.27) can be written in terms of the relative stream function

$$\Psi^\varepsilon(z) := -\frac{1}{2} \Omega|z|^2 - \frac{1}{2} Uz + \psi^\varepsilon(z)$$

restricted to points on the boundary $\varepsilon bj\phi_j(w) + z_j \in \partial D^j_\varepsilon$. Writing $w \in \mathbb{T}$ as $w = e^{i\theta}$, we claim that

$$\partial_\theta \Psi^\varepsilon(\varepsilon bj\phi_j(w) + z_j) = -\varepsilon b_j G^\alpha_j(\varepsilon, f; \lambda)(w).$$

To see this, we use $\partial_\theta \phi_j(w) = i\varepsilon b_j \phi_j(w)$ and $\nu^\varepsilon(z) = 2i\partial_\varepsilon \psi^\varepsilon(z)$ to rewrite

$$\partial_\theta \Psi^\varepsilon(\varepsilon bj\phi_j(w) + z_j) = 2\varepsilon b_j \Re \left\{ \partial_\varepsilon \Psi^\varepsilon(\varepsilon bj\phi_j(w) + z_j) \partial_\theta b_j(w) \right\}$$

$$= -\varepsilon b_j \Re \left\{ 2i\partial_\varepsilon \Psi^\varepsilon(\varepsilon bj\phi_j(w) + z_j) \overline{\phi_j(w)} \right\}$$

$$= -\varepsilon b_j \Re \left\{ \left( -i\Omega(\varepsilon bj\phi_j(w) + z_j) - iU + \nu^\varepsilon(\varepsilon bj\phi_j(w) + z_j) \right) \overline{\phi_j(w)} \right\}$$

$$= -\varepsilon b_j \Im \left\{ \left( \Omega(\varepsilon bj\phi_j(w) + z_j) + U + i\nu^\varepsilon(\varepsilon bj\phi_j(w) + z_j) \right) \overline{\phi_j(w)} \right\}$$

$$= -\varepsilon b_j G^\alpha_j(\varepsilon, f; \lambda)(w).$$

We can now prove the following lemma.

**Lemma 2.4.** Let $(\varepsilon, f, \lambda) \in (-\varepsilon_0, \varepsilon_0) \times B_0^\alpha \times \Lambda$. Then, the following identities hold:

(i) If $\Omega = 0$ then

$$\Re \left[ \sum_{j=1}^N \frac{\gamma_j}{\pi} \int_{\mathbb{T}} G^\alpha_j(\varepsilon, f; \lambda)(w)(1 + \varepsilon bj\phi_j(w)\overline{\phi_j})dw \right] = U \sum_{j=1}^N \frac{\gamma_j}{\pi} \int_{\mathbb{T}} G^\alpha_j(\varepsilon, f; \lambda)(w)(1 + \varepsilon bj\phi_j(w)\overline{\phi_j})dw,$$

(ii) If $U = 0$ then

$$\sum_{j=1}^N \frac{\gamma_j}{i\pi} \int_{\mathbb{T}} G^\alpha_j(\varepsilon, f; \lambda)(w)(\varepsilon bj|w + \varepsilon bj\phi_j(w)|^2 + \varepsilon b_j \Re \left[ \overline{\phi_j} f_j(w) \right] + \Re \left[ \overline{\phi_j} w \right])dw = 0.$$

**Proof.** By continuity, it is enough to consider $\varepsilon \neq 0$, and thanks to the $\varepsilon \mapsto -\varepsilon$ symmetry in (2.68), we can further restrict to $\varepsilon > 0$.

We shall first prove (i). Averaging (2.36) over the boundaries, with $\Omega = 0$, then summing and using Lemma 2.1 yields

$$\sum_{j=1}^N \frac{\gamma_j}{\varepsilon^2 b_j^2} \int_{\partial D^j_\varepsilon} \Psi^\varepsilon(z)dz = -\frac{U}{2} \sum_{j=1}^N \frac{\gamma_j}{\varepsilon^2 b_j^2} \int_{\partial D^j_\varepsilon} zdz.$$

In view of (2.1) and (2.13), by making simple changes of variables we get

$$\sum_{j=1}^N \frac{\gamma_j}{\varepsilon b_j} \int_{\mathbb{T}} \Psi^\varepsilon(\varepsilon bj\overline{\phi_j}(w) + z_j)\phi_j'(w)dw = -\frac{U}{2} \sum_{j=1}^N \frac{\gamma_j}{\varepsilon b_j} \int_{\mathbb{T}} (\varepsilon bj\overline{\phi_j}(w) + z_j)\phi_j'(w)dw.$$
Writing $w \in \mathbb{T}$ as $w = e^{i\theta}$ then integrating by parts and using (2.37), the left hand side of (2.40) becomes
\[
\sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon b_j} \int_{\mathbb{T}} \Psi^e (\varepsilon b_j \phi_j (w) + \overline{z}_j) \phi'_j (w) dw = \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon b_j} \int_{0}^{2\pi} \Psi^e (\varepsilon b_j \phi_j (w) + z_j) \partial_{\theta} \phi_j (w) d\theta
\]
\[
= -\sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon b_j} \int_{0}^{2\pi} \partial_{\theta} \Psi^e (\varepsilon b_j \phi_j (w) + z_j) \phi_j (w) d\theta
\]
\[
= -i \sum_{j=1}^{N} \gamma_j \int_{\mathbb{T}} G^\alpha_j (\varepsilon, f; \lambda) (w) \phi_j (w) dw.
\] (2.41)

Combining (2.40)–(2.41) and using the simple identity
\[
\frac{1}{\varepsilon b_j} \int_{\mathbb{T}} (\varepsilon b_j \phi_j (w) + \overline{z}_j) \phi'_j (w) dw = 2\pi i + \varepsilon^2 b^2 \int_{\mathbb{T}} f_j (w) f'_j (w) dw
\]
we conclude that
\[
\sum_{j=1}^{N} \frac{\gamma_j}{\pi} \int_{\mathbb{T}} G^\alpha_j (\varepsilon, f; \lambda) (w) (1 + \varepsilon b_j f_j (w) \overline{w}) dw = U \sum_{j=1}^{N} \gamma_j \left( 1 + \varepsilon^2 b^2 \int_{\mathbb{T}} f_j (w) f'_j (w) dw \right).
\]

Then (i) follows from the identity
\[
\int_{\mathbb{T}} f_j (w) f'_j (w) dw - \int_{\mathbb{T}} f_j (w) f'_j (w) dw = \frac{1}{2\pi i} \int_{0}^{2\pi} \partial_{\theta} |f_j (w)|^2 d\theta = 0.
\]

As for (ii), integrating (2.36) with $U = 0$ over the boundaries then summing and taking the real part we obtain, by virtue of Lemma 2.1
\[
\text{Re} \left[ \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b_j^2} \int_{\partial D_j^i} \tau \Psi^e(z) dz \right] = -\frac{\Omega}{2} \text{Re} \left[ \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b_j^2} \int_{\partial D_j^i} |z|^2 dA(z) \right].
\]

According to (2.5), the quantity
\[
\sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b_j^2} \int_{\partial D_j^i} |z|^2 dA(z) = 4i \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b_j^2} \int_{D_j} |z|^2 dA(z)
\]
is purely imaginary. It follows that
\[
\text{Re} \left[ \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon^2 b_j^2} \int_{\partial D_j^i} \tau \Psi^e(z) dz \right] = 0.
\]

Then, in view of (2.1) and (2.13), by making simple changes of variables we get
\[
\sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon b_j} A_j := \sum_{j=1}^{N} \frac{\gamma_j}{\varepsilon b_j} \text{Re} \left[ \int_{\mathbb{T}} (\varepsilon b_j \phi_j (w) + \overline{z}_j) \Psi^e (\varepsilon b_j \phi_j (w) + z_j) \phi'_j (w) dw \right] = 0.
\] (2.42)
Writing \( w \in \mathbb{T} \) as \( w = e^{i\theta} \) then integrating by parts and using (2.37) we find
\[
A_j = \int_0^{2\pi} \Re \left[ (\epsilon b_j \phi_j(w) + \overline{z}_j) \partial_\theta \phi_j(w) \right] \Psi^\epsilon (\epsilon b_j \phi_j(w) + z_j) d\theta
\]
\[
= \int_0^{2\pi} \partial_\theta \left( \epsilon b_j |\phi_j(w)|^2 + \Re \left[ \overline{\phi}_j(w) \right] \right) \Psi^\epsilon (\epsilon b_j \phi_j(w) + z_j) d\theta
\]
\[
= - \int_0^{2\pi} \left( \epsilon b_j |\phi_j(w)|^2 + \Re \left[ \overline{\phi}_j(w) \right] \right) \partial_\theta \left[ \Psi^\epsilon (\epsilon b_j \phi_j(w) + z_j) \right] d\theta
\]
\[
= -i\epsilon b_j \int_\mathbb{T} \left( \epsilon b_j |\phi_j(w)|^2 + \Re \left\{ \overline{\phi}_j(w) \right\} \right) \mathcal{G}_j^\alpha(\epsilon, f; \lambda)(w) \overline{w} dw. \tag{2.43}
\]
Combining (2.42) and (2.43) gives the desired result. \( \square \)

**Remark 2.5.** Substituting \((\epsilon, f) = (0, 0)\) in the identities of Lemma 2.4 and using Proposition 2.2(ii) we obtain well known identities for point vortices: If \( \Omega = 0 \) we have
\[
\sum_{j=1}^N \gamma_j \mathcal{P}_j^\alpha(\lambda) = U \sum_{j=1}^N \gamma_j
\]
and if \( U = 0 \) we get
\[
\Im \left\{ \sum_{j=1}^N \gamma_j \mathcal{P}_j^\alpha(\lambda) \overline{\gamma}_j \right\} = 0.
\]
See, for instance, [43]. These identities can be seen as coming from the symmetries of the Hamiltonian system (1.3) under translations and rotations.

### 2.5. An abstract lemma

From Proposition 2.2, we know that the operator \( D_f \mathcal{G}_j^\alpha(0, 0; \lambda^*) \) is not onto. As discussed in Remark 2.3, the linearized operator \( D_{(f, \lambda)} \mathcal{G}_j^\alpha(0, 0; \lambda^*) \) is also not onto. Thankfully, we will be able to “explain” this latter degeneracy using the nonlinear identities in Lemma 2.4. More precisely, for non-degenerate equilibria, we will be able to apply the following mild generalization of the usual implicit function theorem.

**Lemma 2.6.** Let \( G: U \times V \to Z \) and \( F: Z \times U \times V \to \mathbb{R}^n \) be \( C^1 \) mappings satisfying
\[
G(0, 0) = 0, \tag{2.44}
\]
\[
F(0, x, y) = 0, \tag{2.45}
\]
\[
F(G(x, y), x, y) = 0, \tag{2.46}
\]
for all \((x, y) \in U \times V\), where \( X, Y, Z \) are Banach spaces and \( U \subset X \) and \( V \subset Y \) are open sets containing the origin. If the linearizations of these mappings at the origin satisfy
\[
\ker D_y G(0, 0) = \{0\}, \tag{2.47}
\]
\[
\text{codim ran } D_y G(0, 0) = n, \tag{2.48}
\]
\[
\text{ran } D_z F(0, 0, 0) = \mathbb{R}^n, \tag{2.49}
\]
then there exists a neighborhood \( \tilde{U} \times \tilde{V} \) of the origin in \( U \times V \) and a \( C^1 \) mapping \( g: \tilde{U} \to \tilde{V} \) such that
\[
g(0) = 0,
\]
\[
G(x, g(x)) = 0 \quad \text{for all } x \in \tilde{U}. \tag{2.50}
\]
Moreover, every solution of \( G(x, y) = 0 \) in \( \tilde{U} \times \tilde{V} \) is of the form \( (x, g(x)) \), and the operator \( D_x g(0) \) is uniquely determined by the equation
\[
D_x G(0,0) + D_y G(0,0) D_x g(0) = 0
\]
 obtained by implicitly differentiating \((2.50)\).

**Proof.** We first claim that the linearizations of \( G \) and \( F \) at the origins in \( U \times V \) and \( Z \times U \times V \) satisfy
\[
\text{ran } D_y G(0) = \text{ker } D_x F(0).
\]
To see this, we differentiate \((2.45)\) and \((2.46)\) with respect to \( x \) and \( y \) to find
\[
D_x F(0) = 0, \quad D_y F(0) = 0, \quad D_x F(0) D_y G(0) + D_x F(0) = 0, \quad D_x F(0) D_y G(0) + D_y F(0) = 0.
\]
This in particular implies that \( D_x F(0) D_y G(0) = 0 \), and hence \( \text{ran } D_y G(0) \subseteq \ker D_x F(0) \). As \((2.49)\) and \((2.48)\) force \( \text{codim } \ker D_x F(0) = \text{codim } \text{ran } D_y F(0) = n \), the only possibility is that the two spaces are equal and \((2.51)\) holds.

Again using \((2.49)\), we know that there is an \( n \)-dimensional subspace \( Z_1 \subseteq Z \) such that \( D_x F(0) \) restricts to an invertible map \( Z_1 \to \mathbb{R}^n \). Moreover, by \((2.51)\) we have
\[
Z = Z_1 \oplus \text{ker } D_x F(0) = Z_1 \oplus \text{ran } G_y(0).
\]
Consider the augmented mapping
\[
H : Z_1 \times U \times V \to Z, \quad H(z_1, x, y) = G(x, y) - z_1.
\]
The linearized operator \( D_{(z_1, y)} H(0) : Z_1 \times Y \to Z \), given by
\[
D_{(z_1, y)} H(0) \left( \begin{array}{c} z_1 \\ y \end{array} \right) = D_y G(0) y - z_1,
\]
is invertible. Indeed, it is onto by \((2.52)\). If \( (z_1, y) \) lies in its kernel, applying \( D_x F(0) \) to \((2.53)\) and using the definition of \( Z_1 \) \((2.51)\) yields \( z_1 = 0 \), at which point \( y = 0 \) follows from \((2.47)\). Thus, by the implicit function theorem there is a neighborhood \( \tilde{Z}_1 \times \tilde{U} \times \tilde{V} \) of the origin in \( Z_1 \times U \times V \) and \( C^1 \) mappings \((h, g) : \tilde{U} \to \tilde{Z}_1 \times \tilde{V} \) satisfying
\[
(h(0), g(0)) = 0,
\]
\[
H(h(x), x, g(x)) = G(x, g(x)) - h(x) = 0 \quad \text{for all } x \in \tilde{U}.
\]
Moreover, all solutions of \( H(z, x, y) = 0 \) in \( \tilde{Z}_1 \times \tilde{U} \times \tilde{V} \) are of the form \( (h(x), x, g(x)) \), and the linearizations of \((h, g)\) at the origin are uniquely determined by
\[
0 = D_z H(0) D_x h(0) + D_x H(0) + D_y h(0) D_x g(0)
\]
\[
= -D_x h(0) + D_x G(0) + D_y G(0) D_x g(0).
\]
The proof will therefore be complete if we can show that \( h \equiv 0 \), for which will need the full force of the nonlinear conditions \((2.45)\) and \((2.46)\) and not just their linearizations at the origin. Consider the restriction
\[
F_1 = F \big|_{Z_1 \times U \times V} : Z_1 \times U \times V \to \mathbb{R}^n.
\]
By our choice of \( Z_1 \), the linearized operator \( D_{z_1} F_1(0) = D_{z_1} F(0) |_{Z_1} \) is invertible. Applying the implicit function theorem and using \((2.45)\) we conclude that, possibly after shrinking \( \tilde{Z}_1 \times \tilde{U} \times \tilde{V} \), all solutions of \( F_1(z_1, x, y) = 0 \) in \( \tilde{Z}_1 \times \tilde{U} \times \tilde{V} \) are of the form \( (0, x, y) \). By \((2.46)\) and \((2.54)\) we have
\[
0 = F(G(x, g(x)), x, g(x)) = F(h(x), x, g(x)) = F_1(h(x), x, g(x)),
\]
for all \( x \in \tilde{U} \), and so this forces \( h \equiv 0 \) as desired. \( \square \)
2.6. Existence of vortex patch equilibria. We shall give in this subsection a detailed statement of Theorem 1.2, as well as a proof based on Lemma 2.6. The stationary case is more degenerate than the rigid motion case due to the additional symmetries in the problem, and so we will treat it separately.

For rigidly rotating or translating vortex patch solutions, our main result is the following.

**Theorem 2.7.** Let $\alpha \in [0,1)$ and let $\lambda^*$ be a non-degenerate solution, in the sense of Definition 1.1, to the $N$-vortex problem (1.6), with one of $\Omega, U$ nonzero. Then the following hold true.

(i) There exists $\varepsilon_1 > 0$ and a unique $C^1$ function $(f, \lambda_1) : (-\varepsilon_1, \varepsilon_1) \rightarrow \mathcal{B}^\alpha \times \mathbb{R}^{2N-1}$ satisfying
\[
G^\alpha(\varepsilon, f(\varepsilon); \lambda_1(\varepsilon), \lambda_2^*) = 0, \tag{2.55}
\]
with $\lambda_1(\varepsilon) = \lambda_1^* + o(\varepsilon)$ and
\[
f_j(\varepsilon, w) = \varepsilon b_j \Xi_\alpha \sum_{k=1,k\neq j}^N \frac{\gamma_k}{\gamma_j} \frac{(z_k - z_j)^2}{|z_k - z_j|^{\alpha+1}} w + o(\varepsilon), \quad \Xi_\alpha := \frac{(\alpha + 2)\Gamma(1 - \frac{\alpha}{2})\Gamma(3 - \frac{\alpha}{2})}{4\Gamma(2 - \alpha)}.
\]

(ii) These solutions enjoy the symmetries
\[
f(-\varepsilon)(w) = f(\varepsilon)(-w), \quad \lambda_1(-\varepsilon) = \lambda_1(\varepsilon).
\]

(iii) For all $\varepsilon \in (-\varepsilon_1, \varepsilon_1) \setminus \{0\}$ the domains $\mathcal{O}_j^\varepsilon$, whose boundaries are given by the conformal parametrizations $\phi_j^\varepsilon = \text{Id} + \varepsilon\varepsilon|^{\alpha} b_j^{1+\alpha} f_j : T \rightarrow \partial \mathcal{O}_j^\varepsilon$, are strictly convex.

**Proof.** In view of Proposition 2.2, for any $h \in \mathcal{V}_0^\alpha$ and $\lambda_1 \in \mathbb{R}^{2N-1}$, we have
\[
D_{(f; \lambda_1)} G^\alpha(0,0; \lambda^*) \left( \frac{h}{\lambda_1} \right)(w) = D_f G^\alpha(0,0; \lambda^*) h(w) + \text{Im} \left\{ D_{\lambda_1} \mathcal{P}_j^\alpha(\lambda^*) \lambda_1 \right\}, \tag{2.56}
\]
where $D_f G^\alpha(0,0; \lambda^*)$ is an isomorphism from $\mathcal{V}_0^\alpha$ to $\mathcal{W}_0^\alpha$. From the hypothesis on the matrix $D_{\lambda_1} \mathcal{P}_j^\alpha(\lambda^*)$, the second linear operator $D_{\lambda_1} G^\alpha(0,0; \lambda^*)$ on the right hand side has a trivial kernel and
\[
\text{ran}[D_{\lambda_1} G^\alpha(0,0; \lambda^*)] \subset \mathbb{W} := \{ w \mapsto \text{Im}[c_1 w] : c_1 \in \mathbb{C}^N \}
\]
is codimension 1. Moreover, it is easy to see that
\[
\mathcal{W}_0^\alpha = \mathcal{W}_0^\alpha \oplus \mathbb{W}. \tag{2.57}
\]
Thus, one has
\[
\text{codim ran} D_{(f; \lambda_1)} G^\alpha(0,0; \lambda^*) = 1 \quad \text{and} \quad \ker D_{(f; \lambda_1)} G^\alpha(0,0; \lambda^*) = \{0\}. \tag{2.58}
\]

In the case of pure translation ($\Omega = 0$ and $U \neq 0$) we set
\[
\Phi(\varepsilon, f, g; \lambda) := \text{Im} \left\{ \sum_{j=1}^N \frac{\gamma_j}{\pi} \int_T g_j(w) \left( 1 + \varepsilon b_j \overline{w} f_j(w) \right) dw \right\} \tag{2.59}
\]
while in the case of pure rotation ($\Omega \neq 0$ and $U = 0$) we instead set
\[
\Phi(\varepsilon, f, g; \lambda) := \text{Im} \left\{ \sum_{j=1}^N \frac{\gamma_j}{\pi} \int_T g_j(w) \left( \varepsilon b_j |w| + \varepsilon b_j f_j(w) |w| + \varepsilon b_j \text{Re}[\overline{z}_j f_j(w)] + \text{Re}[\overline{z}_j w] \right) w dw \right\} \tag{2.60}
\]
with $g = (g_1, \ldots, g_N) \in \mathcal{W}_0^\alpha$. It is clear that the mapping $\Phi : (-\varepsilon_0, \varepsilon_0) \times \mathcal{B}_0^\alpha \times \Lambda \rightarrow \mathbb{R}$ is $C^1$ and that for all $(\varepsilon, f, \lambda) \in (-\varepsilon_0, \varepsilon_0) \times \mathcal{B}_0^\alpha \times \Lambda$ one has
\[
\Phi(\varepsilon, f, 0; \lambda) = 0. \tag{2.61}
\]
Moreover, by (2.59)–(2.60) and Lemma 2.4(i), we have
\[
\Phi(\varepsilon, f, G^\alpha(\varepsilon, f; \lambda); \lambda) = 0. \tag{2.62}
\]
By differentiating (2.59)–(2.60) with respect to \( g \) in the direction \( \tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_N) \in \mathcal{W}_0^\alpha \),

\[
\tilde{g}_j(w) = \sum_{n \neq 0} c_{n,j} w^n, \quad c_{-n,j} = \overline{c}_{n,j} \in \mathbb{C}, \quad j = 1, \ldots, N,
\]  

(2.63)

we get, for \( \Omega = 0 \) and \( U \neq 0 \),

\[
D_g \Phi(0, 0, 0; \lambda) \tilde{g} = \text{Im} \left\{ \sum_{j=1}^{N} \frac{\gamma_j}{\pi} \int_{\mathbb{T}} \tilde{g}_j(w) dw \right\} = -2 \sum_{j=1}^{N} \gamma_j \text{Re}[c_{1,j}]
\]

and, for \( \Omega \neq 0 \) and \( U = 0 \),

\[
D_g \Phi(0, 0, 0; \lambda) \tilde{g} = \text{Im} \left\{ \sum_{j=1}^{N} \frac{\gamma_j}{\pi} \int_{\mathbb{T}} \tilde{g}_j(w) \text{Re} \left[\overline{z_j} w \right] dw \right\} = 2 \sum_{j=1}^{N} \gamma_j \text{Re}[z_j c_{1,j}].
\]

In either case, we easily check that

\[
\text{ran} \ [D_g \Phi(0, 0, 0; \lambda^*)] = \mathbb{R}.
\]

(2.64)

Consequently, the existence and uniqueness in (i) follow from (2.58)–(2.64) and Lemma 2.6.

Next, differentiating (2.55) with respect to \( \varepsilon \) at the point \((0; 0; \lambda^*)\) we get

\[
D_{(f, A_1)} \mathcal{G}^\alpha(0, 0; \lambda^*) \partial \varepsilon (f(\varepsilon); \lambda(\varepsilon)) \bigg|_{\varepsilon=0} = -\partial \varepsilon \mathcal{G}^\alpha(0, 0; \lambda^*).
\]

(2.65)

In view of (2.27), for all \( \alpha \in (0, 1) \) we have

\[
\partial \varepsilon \mathcal{G}_j^\alpha(0, 0; \lambda^*)(w) = \frac{\alpha}{2} \left( \frac{\alpha}{2} + 1 \right) C_\alpha \sum_{k=1, k \neq j}^{N} \gamma_k b_j \text{Im} \left\{ \frac{(z_k - z_j)^2}{|z_k - z_j|^{\alpha + 4}} \right\}
\]

(2.66)

and, by (2.15), for \( \alpha = 0 \) we get

\[
\partial \varepsilon \mathcal{G}_j^\alpha(0, 0; \lambda^*)(w) = -\frac{1}{2} \sum_{k=1, k \neq j}^{N} \gamma_k b_j \text{Im} \left\{ \frac{w^2}{(z_k - z_j)^2} \right\}.
\]

Thus, for all \( \alpha \in [0, 1) \) we have

\[
\partial \varepsilon \mathcal{G}_j^\alpha(0, 0; \lambda^*) \in \mathcal{W}_0^{\alpha}. \]

Since the linear operator \( D_j \mathcal{G}^\alpha(0, 0; \lambda^*): \mathcal{W}_0^\alpha \rightarrow \mathcal{W}_0^{\alpha} \) is an isomorphism and, by hypothesis, the kernel of the operator \( D_\lambda \mathcal{P}_{j}^\alpha(\lambda^*) \) is trivial, combining (2.56), (2.65), (2.66) and Proposition 2.2(iii) we conclude that

\[
\partial \varepsilon \lambda(\varepsilon) \bigg|_{\varepsilon=0} = 0 \quad \text{and} \quad \partial \varepsilon f_j(\varepsilon) \bigg|_{\varepsilon=0}(w) = \begin{cases} 
\left( \frac{\alpha}{2} + 1 \right) \frac{\alpha C_\alpha}{2M_1^\alpha} \sum_{k=1, k \neq j}^{N} \gamma_k b_j \frac{(z_k - z_j)^2}{|z_k - z_j|^\alpha} & \text{if } \alpha \in (0, 1), \\
\sum_{k=1, k \neq j}^{N} \gamma_k b_j \frac{1}{(z_k - z_j)^2} & \text{if } \alpha = 0.
\end{cases}
\]

Finally, straightforward computations yield

\[
\frac{\alpha C_\alpha}{M_1^\alpha} = \frac{\Gamma(1 - \frac{\alpha}{2})\Gamma(3 - \frac{\alpha}{2})}{\Gamma(2 - \alpha)},
\]

(2.67)

completing the proof of (i).

By the uniqueness in (i), in order to prove (ii) it suffices to show that

\[
\mathcal{G}^\alpha_j(\varepsilon, f; \lambda)(-w) = \mathcal{G}^\alpha_j(-\varepsilon, \hat{f}; \lambda)(w),
\]

(2.68)
where $\tilde{f}(w) = f(-w)$. From (2.31) one has

$$G_j^\alpha(-\varepsilon, \tilde{f}; \lambda)(w) = \text{Im} \left\{ \left( \Omega \left( -\varepsilon b_j w + \varepsilon^2 |\varepsilon|^{\alpha} b_j^2 + \tilde{f}_j(w) + z_j \right) + U \right. \right.$$  

$$+ \gamma_j \mathcal{I}^\alpha[-\varepsilon, \tilde{f}_j](w) + \sum_{k=1, k \neq j}^N \gamma_k \mathcal{J}^\alpha_k[\varepsilon, f_k, f_j](w) \right| w \left( 1 - \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} \tilde{f}_j(w) \right) - \mu_\alpha \gamma_j \tilde{f}_j(w) \right\}. \tag{2.69}$$

Since $\tilde{f}_j(w) = -f_j'(w)$, by (2.28) we have

$$\mathcal{I}^\alpha[-\varepsilon, \tilde{f}_j](w) := C_\alpha \int_T \frac{f_j'(-\tau)}{\tau - w - \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(-\tau) - f_j(-w))} d\tau$$

$$+ \alpha C_\alpha \int_0^1 \left[ \frac{((f_j(-\tau) - f_j(-w))(\tau - w) - \varepsilon |\varepsilon|^{\alpha} t (f_j(-\tau) - f_j(-w))^2}{|\tau - w - t\varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(-\tau) + f_j(-w))} \right] dt d\tau.$$  

Making the change of variable $\tau \mapsto -\tau$ we get

$$\mathcal{I}^\alpha[-\varepsilon, \tilde{f}_j](w) = -C_\alpha \int_T \frac{f_j'(\tau)}{\tau + w + \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(\tau) - f_j(-w))} d\tau$$

$$+ \alpha C_\alpha \int_0^1 \left[ \frac{((f_j(\tau) - f_j(-w))(\tau + w) + \varepsilon |\varepsilon|^{\alpha} t (f_j(\tau) - f_j(-w))^2}{|\tau + w + t\varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(\tau) + f_j(-w))} \right] dt d\tau$$

$$= \mathcal{I}^\alpha[\varepsilon, f_j](w).$$

In a similar way we can check that

$$\mathcal{J}^\alpha_k[-\varepsilon, \tilde{f}_j, f_j; \lambda](w) = \mathcal{J}^\alpha_k[\varepsilon, f_k, f_j; \lambda](-w).$$

Inserting the two last identities into (2.69) and using the fact that $\tilde{f}_j(w) = f_j(-w)$ yields (2.68) as desired.

As mentioned in Remark 1.3, with only minor modifications the above proof still holds when $V_0^\alpha$ and $W_0^\alpha$ in (2.30) are replaced by spaces with higher Hölder regularity $C^{\alpha+1+\eta_0}$ and $C^{\alpha+\eta_0}$ for any fixed $n \in \mathbb{N}$, at the cost of possibly shrinking $\varepsilon_1$. In particular, by uniqueness we may assume that $f_j$ and hence $\phi_j$ are $C^2$, which allows us to prove the convexity of the domains $\mathcal{O}_j^\varepsilon$ by following the same argument as in [36].

Recall that the curvature can be expressed, in terms of the conformal mapping, by the formula

$$\kappa(w) = \frac{1}{|\phi_j''(w)|} \text{Re} \left( 1 + w \frac{\phi_j''(w)}{\phi_j'(w)} \right).$$

As $\phi_j(w) = w + \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} f_j(w)$, we easily verify that

$$1 + \frac{\phi_j''(w)}{\phi_j'(w)} = 1 + O(\varepsilon |\varepsilon|^{\alpha}),$$

uniformly in $w$. Thus the curvature is strictly positive and therefore the domain $\mathcal{O}_j^\varepsilon$ is strictly convex.

Now, we treat the stationary case, where $\Omega = U = 0$. We have the following result.

**Theorem 2.8.** Let $\alpha \in [0, 1)$ and let $\lambda^*$ be a non-degenerate solution, in the sense of Definition 1.1(ii), to the $N$-vortex problem (1.6) with $\Omega = U = 0$. Then the conclusions of Theorem 2.7 hold, except that now $\lambda_1(\varepsilon)$ takes values in $\mathbb{R}^{2N-3}$ rather than $\mathbb{R}^{2N-1}$. 
Proof. We shall only give the proof of the existence and uniqueness of (i). The proof of the asymptotic expansion and (ii)–(iii) follow the same lines of Theorem \ref{thm:2.7}.

From Proposition \ref{prop:2.2} the hypothesis on the matrix \(D_{\lambda_1} P^\alpha(\lambda^*)\), \eqref{eq:2.56} and \eqref{eq:2.57} we conclude that

\[
\text{codim ran } D_{(f;\lambda_1)} \mathcal{G}^\alpha(0; 0; \lambda^*) = 3 \quad \text{and} \quad \ker D_{(f;\lambda_1)} \mathcal{G}^\alpha(0; 0; \lambda^*) = \{0\}. \tag{2.70}
\]

For all \((\varepsilon, f, \lambda) \in (-\varepsilon_0, \varepsilon_0) \times B_0^\alpha \times \Lambda\), we set

\[
\tilde{\Phi}(\varepsilon, f, g; \lambda) := \sum_{j=1}^N \gamma_j \frac{\pi}{\varepsilon} \begin{pmatrix}
\text{Re}\left\{ \int_T g_j(w)(1 + \varepsilon b_j \bar{w} f_j(w))dw \right\} \\
\text{Im}\left\{ \int_T g_j(w)(1 + \varepsilon b_j \bar{w} f_j(w))dw \right\}
\end{pmatrix}.
\]

The mapping \(\tilde{\Phi}: (-\varepsilon_0, \varepsilon_0) \times B_0^\alpha \times B_0^\alpha \times \Lambda \to \mathbb{R}^3\) is \(C^1\) and satisfies

\[
\tilde{\Phi}(\varepsilon, f, 0; \lambda) = 0. \tag{2.71}
\]

Moreover, by Lemma \ref{lem:2.4} we have

\[
\tilde{\Phi}(\varepsilon, f, \mathcal{G}^\alpha(\varepsilon, f; \lambda); \lambda) = 0. \tag{2.72}
\]

Differentiating \(\tilde{\Phi}\) with respect to \(g\) in the direction \(\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_N) \in W_0^\alpha\) in \eqref{eq:2.63} gives

\[
D_{\tilde{g}} \tilde{\Phi}(0, 0, 0; \gamma) \tilde{g} = \text{Im}\left\{ \sum_{j=1}^N \frac{\gamma_j}{\pi} \int_T \tilde{g}_j(w)dw \right\} = -2 \sum_{j=1}^N \left( \begin{array}{c}
\text{Im}[c_{1, j}] \\
\text{Re}[c_{1, j}]
\end{array} \right) \left( \begin{array}{c}
y_j \\
x_j \text{Im}[c_{1, j}] - x_j \text{Re}[c_{1, j}]
\end{array} \right).
\]

We can easily check that

\[
\text{ran } D_{\tilde{g}} \tilde{\Phi}(0, 0, 0; \lambda^*) = \mathbb{R}^3. \tag{2.73}
\]

Thus, from \eqref{eq:2.70}–\eqref{eq:2.73} and using Lemma \ref{lem:2.6} we conclude the desired result. \(\square\)

Remark 2.9. In this section we have suppressed the dependence of \(\mathcal{G}^\alpha\) on the parameters \(b_j \in (0, \infty)\).

Just as with \(\varepsilon\), one can check that \(\mathcal{G}^\alpha\) is in fact \(C^1\) in \(b_j\). This is true even for \(b_j = 0\), corresponding to the case where \(j\)-th point vortex is not desingularized into a vortex patch but instead remains a point vortex. Applying Lemma \ref{lem:2.6} as in the proof of Theorem \ref{thm:2.7} one obtains families of solutions made up of a combination of point vortices and small vortex patches. The same can be done in the examples of Sections 3 and 4 below.

3. Examples of asymmetric vortex equilibria

In this section we shall give some explicit examples of point vortex solutions to the \(N\)-vortex problem \eqref{eq:1.6} satisfying the non-degeneracy condition in Definition \ref{def:1.1}. In particular, we prove Theorem \ref{thm:1.6} by simply applying Theorems \ref{thm:2.7} and \ref{thm:2.8}.

Asymmetric co-rotating pairs. Set \(N = 2\) and consider the rotating solution \(\lambda^*\) given by \eqref{eq:1.9} to the \(N\)-vortex problem \eqref{eq:1.6}. The differential of the mapping

\[
P^\alpha := (\text{Re}[P_1^\alpha], \text{Im}[P_1^\alpha], \text{Re}[P_2^\alpha], \text{Im}[P_2^\alpha])
\]

with respect to \(\lambda_1 = (x_1, x_2, y_2)\) at the point \(\lambda^*\) is

\[
D_{\lambda_1} P^\alpha(\lambda^*) \begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_2
\end{pmatrix} = \frac{\gamma \hat{C}_\alpha}{2d_\alpha + 2(1 + c)^{\alpha + 2}} \begin{pmatrix}
2 + c + \alpha & -(\alpha + 1) & 0 \\
0 & 0 & 1 \\
-c(\alpha + 1) & 1 + c(2 + \alpha) & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_2
\end{pmatrix}.
\]

Eliminating the second row we get a matrix with Jacobian determinant \((\alpha + 2)(1 + c)^2\), which is nonzero if \(c \neq -1\). Thus, this matrix has rank 3, which implies that the kernel is trivial and the
image has codimension 1. Therefore, Theorem 2.7 applies yielding the existence of $\varepsilon > 0$ and a unique $C^1$ function $(f; \lambda_1) = (f_1, f_2; x_1, x_2, y_2): (-\varepsilon, \varepsilon) \to B_1^0 \times \mathbb{R}^3$ satisfying

$$G^\alpha(\varepsilon, f(\varepsilon); \lambda_1(\varepsilon), \lambda_2^*) = 0. \quad (3.1)$$

It remains only to check the reflection symmetry property. From (2.31), one has

$$G_j^\alpha(\varepsilon, f; \lambda_1, \lambda_2)(\overline{w}) := -\Im \left\{ \left( \Omega(\varepsilon b_j w + \varepsilon^2 |\varepsilon|^{\alpha+1} f_j(\overline{w}) + \varepsilon j) \right) w(1 + \varepsilon |\varepsilon|^{\alpha+1} f_j(\overline{w})) - \mu_\alpha \gamma_j f_j(\overline{w}) + \left( \gamma_j J^\alpha \varepsilon, f_j(\overline{w}) + \gamma_{3-j} J_{3-j}^\alpha \varepsilon, f_k, f_j(\overline{w}) \right) w(1 + \varepsilon |\varepsilon|^{\alpha+1} f_j(\overline{w})) \right\}.$$ 

Set

$$f_j(w) := f_j(\overline{w}) \quad \text{and} \quad \lambda_1 = (x_1, x_2, -y_2). \quad (3.2)$$

Since $\tilde{f}_j(w) = \overline{f}_j(\overline{w})$, then we can easily check, from (2.17)–(2.18) and (2.28)–(2.29), that

$$\overline{T}^\alpha \varepsilon, f_j(\overline{w}) = T^\alpha \varepsilon, f_j(w) \quad \text{and} \quad \overline{J}_{3-j}^\alpha \varepsilon, f_{3-j}, f_j; \lambda_1, \lambda_2(\overline{w}) = J_{3-j}^\alpha \varepsilon, f_{3-j}, f_j; \lambda_1, \lambda_2(w).$$

It follows that

$$G_j^\alpha(\varepsilon, f; \lambda_1, \lambda_2)(\overline{w}) = -G_j^\alpha(\varepsilon, \bar{f}, \bar{\lambda}_1, \lambda_2)(w). \quad (3.3)$$

Since $\lambda_1^*(0) = \bar{\lambda}_1^*(0)$, by uniqueness of the solution of (3.1) we conclude that

$$\overline{f}_j(w) = f_j(\overline{w}) \quad \text{and} \quad y_2 = 0,$$

which implies that the Fourier coefficients of $f_j \in V_0^\alpha$ are real and the domain associated to the conformal mapping $\phi_j = \Id + \varepsilon |\varepsilon|^{\alpha+1} f_j$ is symmetric with respect to the real axis.

**Asymmetric counter-rotating pairs.** Set $N = 2$ and consider the translating solution $\lambda^*$ given by (1.10) to the $N$-vortex problem (1.6). The differential of the mapping

$$\mathcal{P}^\alpha := \left( \text{Re}[\mathcal{P}_1^\alpha], \text{Im}[\mathcal{P}_1^\alpha], \text{Re}[\mathcal{P}_2^\alpha], \text{Im}[\mathcal{P}_2^\alpha] \right)$$

with respect to $\lambda_1 = (x_2, y_2, \gamma_1)$ is

$$D_{\lambda_1} \mathcal{P}^\alpha(\lambda^*) \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{\gamma}_1 \end{pmatrix} = \frac{1}{2^{\alpha+3} d^{\alpha+2}} \begin{pmatrix} -\gamma(\alpha + 1) & 0 & 0 \\ 0 & \gamma & 0 \\ -\gamma(\alpha + 1) & 2d & 0 \\ 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{\gamma}_1 \end{pmatrix}.$$ 

By eliminating the last line we get a matrix with Jacobian determinant $2\gamma d(\alpha+1)$, which is nonzero. Therefore, this matrix has rank 3, which implies that the kernel is trivial and the image has codimension one. Hence, the existence of counter-rotating vortex patch pair follows from Theorem 2.7.

The reflection symmetry property can be checked similarly to the co-rotating case.

**Stationary tripole.** Set $N = 3$ in (1.6) and consider the stationary tripole $\lambda^*$ given by (1.11). The differential of the mapping

$$\mathcal{P}^\alpha := \left( \text{Re}[\mathcal{P}_1^\alpha], \text{Im}[\mathcal{P}_1^\alpha], \text{Re}[\mathcal{P}_2^\alpha], \text{Im}[\mathcal{P}_2^\alpha], \text{Re}[\mathcal{P}_3^\alpha], \text{Im}[\mathcal{P}_3^\alpha] \right)$$

with respect to $\lambda_1 = (x_3, y_3, \gamma_2)$ at the point $\lambda^*$ is

$$D_{\lambda_1} \mathcal{P}^\alpha(\lambda^*) \begin{pmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{\gamma}_2 \end{pmatrix} = \frac{\gamma \tilde{C}_\alpha}{2} \begin{pmatrix} 0 & 0 & -1/\gamma \\ (\alpha+1)a^{\alpha+1}(a+1)^{-\alpha-2} & a^{\alpha+1}(a+1)^{-\alpha-2} & 0 \\ 0 & a^{-1} & 0 \\ -(\alpha+1)a^{-1} & 0 & 0 \\ -\gamma(\alpha + 1) & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{\gamma}_2 \end{pmatrix},$$

which has rank 3. Thus, Theorem 2.8 guarantees the existence of stationary vortex patch tripole. The reflection symmetry property with respect to the real axis can be checked similarly to the co-rotating pairs.
4. Nested Polygonal Vortex Patch Equilibria

In this section we shall construct 2m + 1 multipolar vortex equilibria in which a central patch is surrounded by 2m satellite patches centered at the vertices of two nested regular m-gons. The vertices of the polygons are either radially aligned with each other or out of phase by an angle \( \pi/m \), and the patches on each polygon are identical with the same strength; see Figure 3.

More precisely, we shall desingularize the following system of point vortices

\[
\omega_0^0(z) = \pi \left( \gamma_0 \delta_{z_0(0)}(z) + \gamma_1 \sum_{k=0}^{m-1} \delta_{z_{1k}(0)}(z) + \gamma_2 \sum_{k=0}^{m-1} \delta_{z_{2k}(0)}(z) \right)
\]

with

\[
z_0(0) = 0 \quad \text{and} \quad z_{jk}(0) := \left\{ \begin{array}{ll}
d_1 e^{\frac{2k\pi i}{m}} & \text{if } j = 1 \text{ and } 0 \leq k \leq m - 1, \\
d_2 e^{\frac{(2k+\vartheta)\pi i}{m}} & \text{if } j = 2 \text{ and } 0 \leq k \leq m - 1,
\end{array} \right.
\]

where \( \gamma_0, \gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\} \), \( d_2 > d_1 > 0 \) and \( \vartheta = 0 \) corresponds to the aligned configuration while \( \vartheta = 1 \) refers to the staggered configuration. Assuming that \( z_0(t) = z_0(0) \) and \( z_{jk}(t) = e^{it}z_{jk}(0) \), one may easily check that the system of 2m + 1 equations in (1.6) can be reduced to

\[
\gamma_j F_j^0(\lambda) = 0, \quad j = 0, 1, 2,
\]

where \( \lambda = (\Omega, \vartheta) \) and

\[
F_0^0(\lambda) := \hat{C}_\alpha \left( \frac{\gamma_1}{d_1^{1+\alpha}} + \frac{\gamma_2}{d_2^{1+\alpha}} e^\frac{\vartheta \pi i}{m} \right) \sum_{k=0}^{m-1} e^{\frac{2k\pi i}{m}},
\]

\[
F_1^0(\lambda) := \Omega d_1 \left( \frac{\gamma_0 + \gamma_1}{d_1^{1+\alpha}} \sum_{k=1}^{m-1} \frac{1-e^{\frac{2k\pi i}{m}}}{1-e^{\frac{2k\pi i}{m}}} + \frac{1-e^{\frac{(2k+\vartheta)\pi i}{m}}}{1-e^{\frac{(2k+\vartheta)\pi i}{m}}} \right),
\]

\[
F_2^0(\lambda) := \Omega d_2 \left( \frac{\gamma_0 + \gamma_1}{d_2^{1+\alpha}} \sum_{k=1}^{m-1} \frac{1-e^{\frac{2k\pi i}{m}}}{1-e^{\frac{2k\pi i}{m}}} \right),
\]

with \( d := d_2/d_1 \). By symmetry arguments, one may easily check that

\[
\sum_{k=0}^{m-1} e^{\frac{2k\pi i}{m}} = 0, \quad \sum_{k=1}^{m-1} \frac{1-e^{\frac{2k\pi i}{m}}}{1-e^{\frac{2k\pi i}{m}}} = 1 \sum_{k=1}^{m-1} \left( \frac{\sin \left( \frac{k\pi}{m} \right)}{\sin \left( \frac{\pi}{m} \right)} \right)^\alpha =: S_\alpha,
\]

\[
\sum_{k=0}^{m-1} \frac{1-d^{\pm1} e^{\frac{(2k\pm\vartheta)\pi i}{m}}}{1-d^{\pm1} e^{\frac{(2k\pm\vartheta)\pi i}{m}}} \left( \alpha + 2 \right) = 1 \sum_{k=0}^{m-1} \frac{1-d^{\pm1} \cos \left( \frac{(2k\pm\vartheta)\pi i}{m} \right)}{1-d^{\pm1} \cos \left( \frac{(2k\pm\vartheta)\pi i}{m} \right)} \frac{\alpha + 2}{2} =: T_\alpha^\pm(d, \vartheta).
\]

Thus, the identities in (4.4) become

\[
F_0^0(\lambda) = 0,
\]

\[
F_1^0(\lambda) = \Omega d_1 \left( \frac{\gamma_0 + \gamma_1}{d_1^{1+\alpha}} + \gamma_2 T_\alpha^+(d, \vartheta) \right),
\]

\[
F_2^0(\lambda) = \Omega d_2 \left( \gamma_0 + \gamma_1 T_\alpha^-(d, \vartheta) + \frac{\gamma_2}{2} S_\alpha \right).
\]

Moreover, the differential of the mapping \( Q^0 := (Q_\alpha^0, Q_\alpha^2) \) with respect to \( \lambda = (\Omega, \vartheta) \) is given by

\[
D_\lambda F^\alpha(\Omega) \left( \frac{\dot{\Omega}}{\dot{\vartheta}} \right) = \begin{pmatrix}
\frac{d_1}{2} & -\frac{\hat{C}_\alpha}{2} \frac{T_\alpha^+(d, \vartheta)}{S_\alpha} \\
\frac{d_2}{2} & -\frac{\hat{C}_\alpha}{2} \frac{S_\alpha}{S_\alpha}
\end{pmatrix}
\]
If the Jacobian determinant is non-trivial,
\[
\det \left( D_\lambda F^\alpha(\lambda) \right) = -\frac{\hat{C}_\alpha d_1}{2d_2^{\alpha+1}} \left( \frac{S_\alpha}{2} - d^{\alpha+2} T^+_\alpha(d, \vartheta) \right) \neq 0,
\]
(4.7) then the system (4.5) has a unique solution \( \lambda^* = (\Omega^*, \gamma^2) \) given by
\[
\gamma_2^2 := \frac{(d^{\alpha+2} - 1) \gamma_0 + \frac{1}{2} S_\alpha d^{\alpha+2} - T^-_\alpha(d, \vartheta)}{\frac{1}{2} S_\alpha - T^+_\alpha(d, \vartheta)} \gamma_1,
\]
\[
\Omega^* := \frac{\hat{C}_\alpha}{2(d_1^{\alpha+2} + d_2^{\alpha+2})} \left( \gamma_0 + \gamma_1 \left( \frac{1}{2} S_\alpha + T^-_\alpha(d, \vartheta) \right) \gamma_2^2 \left( T^+_\alpha(d, \vartheta) + \frac{1}{2} S_\alpha \right) \right). \tag{4.8}
\]
In order to ensure that \( \gamma_2 \) is non-vanishing, one has to assume that \( \gamma_0 \) and \( \gamma_1 \) verify the condition
\[
(d^{\alpha+2} - 1) \gamma_0 + \frac{1}{2} S_\alpha d^{\alpha+2} - T^-_\alpha(d, \vartheta) \gamma_1 \neq 0. \tag{4.9}
\]

Remark 4.1. For the Eulerian interaction \( \alpha = 0 \), one may easily check that
\[
S_\alpha = m - 1 \quad \text{and} \quad T^\pm_\alpha(d, \vartheta) = \frac{m}{1 - (-1)^d d^{\pm m}}. \]

It follows that (4.7) and (4.9) can be written as
\[
\frac{m-1}{2} + \frac{md^2}{1 - (-1)^d d^{\pm m}} \neq 0, \quad (d^2 - 1) \gamma_0 + \left( \frac{m - 1}{2} d^2 - \frac{md^m}{d^m - (-1)^d} \right) \gamma_1 \neq 0.
\]
This amounts to the study of the roots of two polynomials of order \( m \). A detailed analysis is given in [10, Pages 10–13] and [2, Pages 18–22].

Remark 4.2. While for general non-degenerate equilibria the differential \( D_\lambda P^\alpha(\lambda^*) \) was never onto, in this more symmetric setting the differential \( D_\lambda F^\alpha(\lambda^*) \) is onto whenever (4.7) holds. This will enable us to directly apply the implicit function theorem to the vortex patch equations, avoiding Lemma 2.6 and the use of integral identities.

4.1. Boundary equations. Let \( m \) be a positive integer and \( \mathcal{O}_0^\varepsilon, \mathcal{O}_1^\varepsilon, \mathcal{O}_2^\varepsilon \) be three bounded simply connected domains containing the origin and contained in the ball \( B(0, 2) \). Assume in addition that \( \mathcal{O}_0^\varepsilon \) is \( m \)-fold symmetric, that is
\[
eq \frac{2\pi}{m} \mathcal{O}_0^\varepsilon = \mathcal{O}_0^\varepsilon, \tag{4.10}
\]
and \( \mathcal{O}_0^\varepsilon, \mathcal{O}_1^\varepsilon, \) and \( \mathcal{O}_2^\varepsilon \) are symmetric about the real axis. Given \( b_0, b_1, b_2 \in \mathbb{R}_+ \) and \( d_1, d_2 \in \mathbb{R}_+ \) and \( \varepsilon \in (0, \varepsilon_0) \), with \( \varepsilon_0 \ll 1 \), we define the domains
\[
\mathcal{D}_0^\varepsilon := \varepsilon b_0 \mathcal{O}_0^\varepsilon, \tag{4.11}
\]
\[
\mathcal{D}_1^\varepsilon := \varepsilon \frac{2\pi}{m} (\varepsilon b_1 \mathcal{O}_1^\varepsilon + d_1), \quad j = 0, \ldots, m - 1, \tag{4.12}
\]
\[
\mathcal{D}_2^\varepsilon := \varepsilon \frac{2\pi}{m} (\varepsilon b_2 \mathcal{O}_2^\varepsilon + d_2), \quad j = 0, \ldots, m - 1, \tag{4.13}
\]
where we recall that \( \vartheta = 0 \) corresponds to the aligned configuration and \( \vartheta = 1 \) refers to the staggered configuration. Let \( \gamma_0, \gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\} \) and consider the initial vorticity
\[
\omega_0^\varepsilon = \frac{\gamma_0}{\varepsilon^2 b_0^2} \chi_{\mathcal{D}_0^\varepsilon} + \frac{\gamma_1}{\varepsilon^2 b_1^2} \sum_{j=0}^{m-1} \chi_{\mathcal{D}_1^\varepsilon} + \frac{\gamma_2}{\varepsilon^2 b_2^2} \sum_{j=0}^{m-1} \chi_{\mathcal{D}_2^\varepsilon}. \tag{4.14}
\]
Now assume that the evolution of \( \omega_0^\varepsilon \) is prescribed by the (2.9) with \( U = 0 \). While this initially gives \( N = 2m + 1 \) equations, one for each patch, using the fact that
\[
\mathcal{D}_1^\varepsilon = e^{\frac{2\pi}{m}} \mathcal{D}_0^\varepsilon, \quad \text{and} \quad \mathcal{D}_2^\varepsilon = e^{\frac{2\pi}{m}} \mathcal{D}_{20},
\]
we shall show that this system can be reduced to a system of three equations, on the boundaries of \( \mathcal{O}_0^\varepsilon, \mathcal{O}_1^\varepsilon \) and \( \mathcal{O}_2^\varepsilon \).
4.1.1. Euler equation. From (2.11) one has

\[
\text{Re} \left\{ \gamma_0 (\Omega \bar{z} + \mathcal{V}^\varepsilon(z)) z' \right\} = 0, \quad \forall z \in \partial \mathcal{D}_{00},
\]
\[
\text{Re} \left\{ \gamma_1 (\Omega \bar{z} + \mathcal{V}^\varepsilon(z)) z' \right\} = 0, \quad \forall z \in \partial \mathcal{D}_{1n}, \quad n = 0, \ldots, m - 1,
\]
\[
\text{Re} \left\{ \gamma_2 (\Omega \bar{z} + \mathcal{V}^\varepsilon(z)) z' \right\} = 0, \quad \forall z \in \partial \mathcal{D}_{2n}, \quad n = 0, \ldots, m - 1,
\]

where \(z'\) denotes a tangent vector to the boundary at the point \(z\) and

\[
V^\varepsilon(z) = \frac{\gamma_0}{2 \varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi + \sum_{\ell=1}^{2} \frac{\gamma_\ell}{2 \varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \int_{\partial \mathcal{D}_{ik}} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi.
\]

In view of (4.12) and (4.13), the change of variables \(\xi \mapsto e^{\frac{2\pi i k}{m}} \xi\) leads to

\[
V^\varepsilon(z) = \frac{\gamma_0}{2 \varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}} \frac{\bar{\xi} - \bar{z} - \frac{2\pi i k}{m} \bar{z}}{\xi - \frac{2\pi i k}{m} z} d\xi + \sum_{\ell=1}^{2} \frac{\gamma_\ell}{2 \varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \int_{\partial \mathcal{D}_{i0}} \frac{\bar{\xi} - \frac{2\pi i k}{m} \bar{z}}{\xi - \frac{2\pi i k}{m} z} d\xi.
\]

For any \(n \in \{1, \ldots, m - 1\}\) one has

\[
V^\varepsilon(e^{\frac{2\pi i k}{m} z}) = \frac{\gamma_0}{2 \varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}} \frac{\bar{\xi} - \frac{2\pi i k}{m} \bar{z}}{\xi - e^{\frac{2\pi i k}{m} z}} d\xi + \sum_{\ell=1}^{2} \frac{\gamma_\ell}{2 \varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \int_{\partial \mathcal{D}_{i0}} \frac{\bar{\xi} - \frac{2\pi i k}{m} \bar{z}}{\xi - \frac{2\pi i k}{m} z} d\xi.
\]

In view of (4.10), making the change of variables \(\xi \mapsto e^{\frac{2\pi i k}{m} \xi}\) in the first integral gives

\[
V^\varepsilon(e^{\frac{2\pi i k}{m} z}) = e^{\frac{2\pi i k}{m} V^\varepsilon(z)}.
\]

From (4.12), (4.13) and (4.17) we conclude that if (4.15) is satisfied for \(n = 0\), then it also satisfied for all \(n \in \{1, \ldots, m - 1\}\). Thus, the system (4.15) is reduced to

\[
\text{Re} \left\{ \gamma_j (\Omega \bar{z} + V^\varepsilon(z)) z' \right\} = 0, \quad \forall z \in \partial \mathcal{D}_{j0}, \quad j = 0, 1, 2.
\]

We assume that the boundaries of the domains \(\mathcal{O}_j^\varepsilon, j = 0, 1, 2\) in (4.12), (4.13) are parametrized by conformal mappings \(\phi_j : T \to \partial \mathcal{O}_j^\varepsilon\) satisfying

\[
\phi_j(w) = w + \varepsilon b_j f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^{\infty} \frac{a^j_m}{w^m}, \quad a^j_m \in \mathbb{R}.
\]

Following the steps established in Section 2.2.1, more precisely (2.15), we may conclude that the dynamics the three bounders is governed by the system

\[
\gamma_0 G_0^\varepsilon(\varepsilon, f; \lambda)(w) = -\gamma_0 \text{Im} \left\{ \left( \Omega (\varepsilon b_0 \bar{w} + \varepsilon^2 b_0^2 f_0(\bar{w})) + \gamma_0 \mathcal{V}^\varepsilon(\varepsilon, f_0)(w) \right) \right. \\
+ \sum_{\ell=1}^{2} \frac{\gamma_\ell}{2 \varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \mathcal{K}_\ell^\varepsilon(\varepsilon, f_\ell, f_0)(w) \left. \right\} w(1 + \varepsilon b_0 f_0'(w)) - \frac{\gamma_0}{2} f_0'(w) = 0,
\]

\[
\gamma_j G_j^\varepsilon(\varepsilon, f; \lambda)(w) := -\gamma_j \text{Im} \left\{ \left( \Omega (\varepsilon b_j \bar{w} + \varepsilon^2 b_j^2 f_j(\bar{w}) + d_j) + \gamma_j \mathcal{V}^\varepsilon(\varepsilon, f_j)(w) + \gamma_0 \mathcal{K}_j^\varepsilon(\varepsilon, f_0, f_j)(w) \right) \right. \\
+ \sum_{\ell=1}^{2} \frac{\gamma_\ell}{2 \varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \mathcal{K}_\ell^\varepsilon(\varepsilon, f_\ell, f_j)(w) \left. \right\} w(1 + \varepsilon b_j f_j'(w)) - \frac{\gamma_j}{2} f_j'(w) = 0.
\]
for all $w \in T$ and $j = 1, 2$, where

$$
\overline{T}_k[\varepsilon, f_j](w) = \frac{1}{2} \int_T \frac{w - \tau + \varepsilon b_j(f_j(\tau) - f_j(\overline{\tau}))}{w - \tau + \varepsilon b_j(f_j(\tau) - f_j(w))} f_j'(\tau) d\tau \\
+ \frac{i}{2} \int_T (w - \tau)(w - \tau + \varepsilon b_j f_j'(\tau)) d\tau,
$$

(4.21)

$$
\overline{K}_k[\varepsilon, f, f_n](w) := \frac{1}{2} \int_T \nu_{k\ell}(\varepsilon b_\ell \varepsilon + 2b_\ell^2 f_\ell(\tau) + d_\ell) - \varepsilon b_n(w + \varepsilon b_n f_n(w)) - d_j d\tau,
$$

(4.22)

with the convention $d_0 = 0$ and $\nu_{k\ell} := \exp(2k\pi i/m + (\delta_{2\ell} - \delta_{2j})\vartheta\pi i/m)$.

4.1.2. gSQG equations. From (2.9) and (2.19) one has

$$
\text{Re} \left\{ \gamma_0 (\Omega z + iv^\varepsilon(z)) \overline{z} \right\} = 0 \quad \forall z \in \partial D_{00}^c,
$$

$$
\text{Re} \left\{ \gamma_1 (\Omega z + iv^\varepsilon(z)) \overline{z} \right\} = 0 \quad \forall z \in \partial D_{1n}^c, \quad n = 0, \ldots, m - 1,
$$

(4.23)

$$
\text{Re} \left\{ \gamma_2 (\Omega z + iv^\varepsilon(z)) \overline{z} \right\} = 0 \quad \forall z \in \partial D_{mn}^c, \quad n = 0, \ldots, m - 1,
$$

where $z'$ denotes a tangent vector to the boundary at the point $z$ and

$$
v^\varepsilon(z) = \frac{C_\alpha}{2\pi} \frac{\gamma_0}{\varepsilon^2 b_0^2} \int_{\partial D_{00}^c} \frac{d\xi}{|z - \xi|^\alpha} + \sum_{m=1}^2 \sum_{k=0}^{m-1} \frac{C_\alpha}{2\pi} \int_{\partial D_{\ell k}^c} \frac{d\xi}{|z - \xi|^\alpha},
$$

for all $z \in \mathbb{C}$. In view of (4.12) and (4.13), a suitable change of variables gives

$$
v^\varepsilon(z) = \frac{C_\alpha}{2\pi} \frac{\gamma_0}{\varepsilon^2 b_0^2} \int_{\partial D_{00}^c} \frac{d\xi}{|z - \xi|^\alpha} + \sum_{\ell=1}^2 \sum_{k=0}^{m-1} \frac{C_\alpha}{2\pi} \int_{\partial D_{\ell k}^c} \frac{b_\ell^2 d\xi}{|z - \xi|^\alpha}.
$$

Observe that for any $n \in \{1, \ldots, m - 1\}$ one has

$$
v^\varepsilon(e^{2\pi i/n} z) = \frac{C_\alpha}{2\pi} \frac{\gamma_0}{\varepsilon^2 b_0^2} \int_{\partial D_{00}^c} \frac{d\xi}{|e^{2\pi i/n} z - \xi|^\alpha} + \sum_{\ell=1}^2 \sum_{k=0}^{m-1} \frac{C_\alpha}{2\pi} \int_{\partial D_{\ell k}^c} \frac{b_\ell^2 e^{2\pi i/n} d\xi}{|e^{2\pi i/n} z - \xi|^\alpha}.
$$

From (4.10), the change of variable $\xi \mapsto e^{2\pi i/n} \xi$ in the first integral leads to

$$
v^\varepsilon(e^{2\pi i/n} z) = e^{2\pi i/n} v^\varepsilon(z).
$$

From the last identity and by (4.12) and (4.13), we conclude that the system (4.23) of $2m + 1$ equations can be reduced to a system of three equations,

$$
\gamma_j \text{Re} \left\{ (\Omega z + iv^\varepsilon(z)) \overline{z} \right\} = 0 \quad \text{for all} \quad z \in \partial D_{j0}^c, \quad j = 0, 1, 2.
$$

Assume that the boundaries of the domains $O_j^c$, $j = 0, 1, 2$ in (4.12), (4.13) are parametrized by the conformal mappings $\phi_j : T \rightarrow \partial O_j^c$ satisfying

$$
\phi_j(w) = w + \varepsilon |\varepsilon|^\alpha b_j^1 + \alpha f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^\infty \frac{a_j^m}{y_m^m}, \quad a_j^m \in \mathbb{R}.
$$
Then, from (2.27) one may conclude that the dynamics of three boundaries is described by
\[
\gamma_0 \mathcal{G}_0^\alpha(\varepsilon, f; \lambda)(w) := \gamma_0 \Im \left\{ \left( \Omega(\varepsilon b_0 w + b_0^{\alpha+2} \varepsilon^2 |f|^\alpha f_0(w)) + \gamma_0 \mathcal{I}^\alpha[\varepsilon, f_0](w) \right) \right\} = 0,
\]
for all \( w \in \mathbb{T} \) and \( j = 1, 2 \), where \( \mu_\alpha \) is defined in (2.25) and
\[
\mathcal{I}^\alpha[\varepsilon, f_0](w) = -C_\alpha \int_\mathbb{T} \frac{f_j^\prime(\tau)}{|\phi_j(\tau) - \phi_j(w)|^\alpha} d\tau + \alpha C_\alpha \int_\mathbb{T} \int_0^1 \Re \left\{ (f_j^\prime(\tau) - f_j(w))(\tau - \overline{w}) \right\} \frac{1}{|w - \tau + \varepsilon \tau|^{\alpha}(f_j(\tau) - f_j(w))^2 + \varepsilon \tau} d\tau d\tau,
\]
for all \( \lambda \neq 0 \) and \( \nu_{k\ell} = \texttt{exp}(2k\pi i/m + (\delta_{k\ell} - \delta_{k\ell})/\pi i/m) \).

4.2. Existence of the nested polygonal patch equilibria. For any \( m \geq 2 \) we define the Banach spaces
\[
\mathcal{V}_m^\alpha := V_m^\alpha \times V_1^\alpha \times V_1^\alpha \quad \text{and} \quad \mathcal{W}_m^\alpha := W_m^\alpha \times W_1^\alpha \times W_1^\alpha,
\]
with \( V_m^\alpha := \{ f \in V_1^\alpha : f(e^{2\pi i} w) = e^{2\pi i} f(w) \} \) and \( W_m^\alpha := \{ g \in W_1^\alpha : g(e^{2\pi i} z) = g(z) \} \),
\[
\begin{align*}
\phi_0(w) &= w + \varepsilon |f|^\alpha b_0^{1+\alpha} f_0(w) = w \left( 1 + \varepsilon |\phi|^\alpha b_0^{1+\alpha} \sum_{n=1}^\infty \frac{a_{nm-1}}{w^{nm}} \right),
\end{align*}
\]
which provides the \( m \)-fold symmetry of the associated patch. We denote by \( B^\alpha \) the open unit ball in \( \mathcal{V}_m^\alpha \). Define the mapping
\[
\mathcal{G}_m^\alpha(\varepsilon, f; \lambda) := (\mathcal{G}_0^\alpha(\varepsilon, f; \lambda), \mathcal{G}_1^\alpha(\varepsilon, f; \lambda), \mathcal{G}_2^\alpha(\varepsilon, f; \lambda)),
\]
where \( f = (f_0, f_1, f_2) \), \( \lambda = (\Omega, \gamma_2), \mathcal{G}_j^\alpha \) is given by (4.24)–(4.25) for \( \alpha \in (0, 1) \) and by (4.19)–(4.20) for \( \alpha = 0 \).

The proof of the existence of the co-rotating nested polygons follows from the next theorem, which gives the full statement of Theorem 1.9.

**Theorem 4.3.** Let \( \alpha \in [0, 1) \), \( b_1, b_2, d_1, d_2 \in (0, \infty) \) such that \( d = d_2/d_1 > 0 \) satisfies (4.7), and let \( \gamma_0, \gamma_1 \in \mathbb{R} \setminus \{0\} \) such that (4.9) holds. Then

(i) There exists \( \varepsilon_0 > 0 \) and a neighborhood \( \Lambda \) of \( \lambda^* \) in \( \mathbb{R}^2 \) such that \( \mathcal{G}_m^\alpha \) can be extended to a \( C^1 \)

mapping \((-\varepsilon_0, \varepsilon_0) \times B^\alpha \times \Lambda \rightarrow \mathcal{W}_m^\alpha \).
(ii) $G^\alpha(0,0;\lambda^*) = 0$, where $\lambda^* = (\Omega^*, \gamma_0^*)$ is given by (4.8).

(iii) The linear operator $D(f, \lambda)G^\alpha(0,0;\lambda^*): \mathcal{V}^\alpha \times \mathbb{R}^2 \rightarrow \mathcal{W}^\alpha$ is an isomorphism.

(iv) There exists $\varepsilon_1 > 0$ and a unique $C^1$ function $(f, \lambda): (-\varepsilon_1, \varepsilon_1) \rightarrow \mathcal{B}^\alpha \times \mathbb{R}^2$ such that

$$G^\alpha(\varepsilon, f(\varepsilon); \lambda(\varepsilon)) = 0,$$

with $\lambda(\varepsilon) = \lambda^* + o(\varepsilon)$ and

$$f(\varepsilon) = \Xi_\alpha \left(0, \frac{\varepsilon b_1 Q_1^\alpha}{\gamma_1 d_1^2 + 2}, \frac{\varepsilon b_2 Q_2^\alpha}{\gamma_2 d_2^2 + 2} \right) + o(\varepsilon),$$

$$Q^\alpha_j := \gamma_0 + \sum_{\ell=1}^2 \sum_{k=\delta_j}^{\delta_j+1} \left(\frac{2k\pi i + (\delta_k - \delta_j)}{\alpha} \frac{d\ell}{d\ell - 1}\right)^2 - 4\Gamma(2 - \alpha)/4\Gamma(3 - \frac{\alpha}{2}). \quad (4.30)$$

(v) For all $\varepsilon \in (-\varepsilon_1, \varepsilon_1) \setminus \{0\}$ the domains $O^\varepsilon_j$, whose boundaries are given by the conformal parametrizations $\phi_j^\varepsilon = \text{Id} + \varepsilon |\alpha| b_j^{1+\alpha} f_j: \mathbb{T} \rightarrow \partial O^\varepsilon_j$, are strictly convex.

Proof. The regularity of the nonlinear operator $G^\alpha$ follows from Proposition 2.2. In order to prove the reflection symmetry property we shall assume that the Fourier coefficients of $f_1, f_2$ are real, that is

$$\begin{align*}
\bar{f}_j(w) &= f_j(w),
\end{align*} \quad (4.31)$$

and prove that

$$G^\alpha_j(\varepsilon, f; \lambda)(\bar{w}) = -G^\alpha_j(\varepsilon, f)(w). \quad (4.32)$$

It is obvious that if $f_1, f_2$ satisfy (4.31), then

$$w \mapsto \text{Im} \left\{ \Omega(\varepsilon b_j w + \varepsilon^2 |\alpha| b_j^{1+\alpha} f_j(w) + d_j) \bar{w} (1 + \varepsilon |\alpha| b_j^{1+\alpha} f_j(w)) - \mu \alpha \gamma_j f_j(w) \right\}$$

satisfies (4.32). Moreover, using (4.26) and (4.27), we check that the Fourier coefficients of $\mathcal{I}^\alpha[\varepsilon, f_j](w)$ and $\mathcal{J}^\alpha_k[\varepsilon, f_k, f_j](w)$ are also real for every $f$ satisfying (3.2), namely,

$$\mathcal{I}^\alpha[\varepsilon, f_j](w) = \mathcal{I}^\alpha[\varepsilon, f_j](\bar{w}), \quad \text{and} \quad \mathcal{J}^\alpha_k[\varepsilon, f_k, f_j; \lambda](w) = \mathcal{J}^\alpha_k[\varepsilon, f_k, f_j; \lambda](\bar{w}).$$

Then using (4.24) we conclude (4.32). Thus, it remains to check the $m$-fold symmetry property of $G^\alpha_0$, namely, that if

$$f_0(e^{\frac{2\pi i}{m}} w) = e^{\frac{2\pi i}{m}} f_0(w) \quad \forall w \in \mathbb{T}, \quad (4.33)$$

then

$$G^\alpha_0(\varepsilon, f; \lambda)(e^{\frac{2\pi i}{m}} w) = G^\alpha_0(\varepsilon, f; \lambda)(w) \quad \forall w \in \mathbb{T}. \quad (4.34)$$

We shall give the details of the proof in the case $\alpha \in (0,1)$; the case $\alpha = 0$ can be checked in a similar way. From (4.21) one has

$$\begin{align*}
\mathcal{I}^\alpha[\varepsilon, f_0](e^{\frac{2\pi i}{m}} w) &= \frac{1}{2} \int_{\mathbb{T}} e^{\frac{2\pi i}{m} w - \tau} + \varepsilon b_0(f_0(\tau) - f_0(e^{\frac{2\pi i}{m} w})) f_0'(\tau) d\tau
\end{align*}$$

$$+ \int_{\mathbb{T}} \frac{i \text{Im} \left\{ (e^{\frac{2\pi i}{m} w - \tau})(f_0(\tau) - f_0(e^{\frac{2\pi i}{m} w})) \right\} }{(e^{\frac{2\pi i}{m} w - \tau})(e^{\frac{2\pi i}{m} w - \tau} + \varepsilon b_0 f_0(\tau) - \varepsilon b_0 f_0(e^{\frac{2\pi i}{m} w}))} d\tau.$$
Then, by (4.33), we deduce that

\[ \mathcal{T}[\varepsilon, f_0](e^{\pi i} w) = e^{-\frac{2\pi i}{m}} \mathcal{T}[\varepsilon, f_0](w). \]

In view of (4.22) and (4.33) we have

\[ \mathcal{K}_k^0[\varepsilon, f_\ell, f_0](e^{\pi i} w) = e^{-\frac{2\pi i}{m}} \frac{1}{2} \int_\mathcal{T} e^{(k-1)\frac{2\pi i}{m} + \delta_2\frac{2\pi i}{m}} \left( \tau + \varepsilon \beta_\ell \beta_\ell(\tau) \right) \left( 1 + \varepsilon \beta_\ell \beta_\ell'(\tau) \right) d\tau. \]

Summing over \( k \) then gives

\[ \sum_{k=0}^{m-1} \mathcal{K}_k^0[\varepsilon, f_\ell, f_0](e^{\pi i} w) = e^{-\frac{2\pi i}{m}} \sum_{k=0}^{m-1} \mathcal{K}_k^0[\varepsilon, f_\ell, f_0](w), \]

concluding the proof of (i). The proof of (ii) follows immediately from Proposition 2.2(ii), (4.5) and (4.8). In order to show (iii) we use Proposition 2.2(ii) and (4.6) to get, for all \( h = (h_0, h_1, h_2) \in \mathcal{V}^\alpha \) and \( (\Omega, \gamma_2) \in \mathbb{R}^2 \),

\[ D_{(f, \lambda)} \mathcal{G}^\alpha(0, 0; \lambda^*) \begin{pmatrix} \Omega \\ \gamma_2 \\ h \end{pmatrix}(w) = - \begin{pmatrix} d_1 & 0 \\ \frac{\hat{C}_n}{2} T_0^\circ(d, \theta) d_2 & \frac{\hat{C}_n}{2} S_0 d_2 \end{pmatrix} \begin{pmatrix} \hat{\Omega} \\ \gamma_2 \end{pmatrix} \operatorname{Im}\{w\} + \sum_{n \geq 1} M_n^\alpha \begin{pmatrix} \gamma_0 a_0^\alpha \\ \gamma_1 a_1^\alpha \end{pmatrix} \operatorname{Im}\{w^{n+1}\}, \]

where \( M_n^\alpha \) is given by (2.32). Proposition 2.2(iv) and the assumption (4.7) then imply (iii).

The existence and uniqueness in (iv) follow form the implicit function theorem. In order to compute the asymptotic of the solution, we shall use the formula

\[ \partial_\varepsilon (f(\varepsilon), \lambda(\varepsilon))|_{\varepsilon=0} = -D_{(f, \lambda)} \mathcal{G}^\alpha(0, 0; \lambda^*)^{-1} \partial_\varepsilon \mathcal{G}^\alpha(0, 0; \lambda^*). \]

(4.35)

For any \( H \in \mathcal{W}^\alpha \) with the expansion

\[ H(w) = \sum_{n \geq 0} \begin{pmatrix} A_0 \\ A_1^n \end{pmatrix} \operatorname{Im}\{w^{n+1}\}, \]

with \( A_n^0 = 0 \) if \( n \) is not a multiple of \( m \), we have

\[ D_{(f, \lambda)} \mathcal{G}^\alpha(0, 0; \lambda^*)^{-1} H(w) = \]

\[ \begin{pmatrix} \frac{1}{\gamma_0} \sum_{n \geq 1} \frac{A_n^0}{M_n^\alpha} \overline{w}^n \\ \frac{1}{\gamma_1} \sum_{n \geq 1} \frac{A_n^1}{M_n^\alpha} \overline{w}^n \\ \frac{1}{\gamma_2} \sum_{n \geq 1} \frac{A_n^2}{M_n^\alpha} \overline{w}^n \end{pmatrix} \frac{\alpha^{n+2} F_1^+(d, \theta) A_0^2 - \frac{1}{2} S_n A_n^0}{\det(D_\lambda \mathcal{P}^\alpha(\lambda^*))} \frac{A_0^2 d_1 - A_1^0 d_2}{\det(D_\lambda \mathcal{P}^\alpha(\lambda^*))}, \]

where \( \det(D_\lambda \mathcal{P}^\alpha(\lambda^*)) \) was calculated in (4.7). On the other hand, from (4.24)–(4.25) and (4.19)–(4.20) we have

\[ \mathcal{G}_0^\alpha(\varepsilon, 0; \lambda)(w) = \operatorname{Im}\left\{ \left( \sum_{\ell=1}^{2} \gamma_\ell \sum_{k=0}^{m-1} \mathcal{K}_k^0[\varepsilon, f_\ell, f_0](w) \right) \overline{w} \right\}, \]

\[ \mathcal{G}_j^\alpha(\varepsilon, 0; \lambda)(w) = \operatorname{Im}\left\{ \Omega_{d_j} w + \left( \gamma_0 \mathcal{K}_0^0[\varepsilon, f_0, f_j](w) + \sum_{\ell=1}^{2} \gamma_\ell \sum_{k=b_\ell}^{m-1} \mathcal{K}_k^0[\varepsilon, f_\ell, f_j](w) \right) \overline{w} \right\}, \]

with \( j = 1, 2 \).
Case $\alpha = 0$. Differentiating (4.22) with respect to $\varepsilon$ gives

$$
\partial_\varepsilon \mathcal{K}_k^{0}[\varepsilon, 0, 0](w)|_{\varepsilon=0} = \frac{1}{2(\nu_{k\ell j} d_\ell - d_j)^2} \int_\mathbb{T} \tau (\nu_{k\ell j} b_\ell \tau - b_n w) d\tau
$$

$$
= \frac{2(\nu_{k\ell j} d_\ell - d_j)^2}{2},
$$

with $\nu_{k\ell j} = \exp(2k\pi i/m + (\delta_2 - \delta_2)\vartheta i/m)$. It follows from (4.37) that

$$
\partial_\varepsilon \mathcal{G}_0^0(\varepsilon, 0; \lambda)|_{\varepsilon=0}(w) = \frac{b_0}{2} \text{Im} \left\{ \sum_{\ell=1}^{2} \gamma_\ell \sum_{k=0}^{m-1} \frac{\gamma_\ell}{\nu_{k\ell j} d_\ell - d_j} \right\} = 0,
$$

$$
\partial_\varepsilon \mathcal{G}_j^0(\varepsilon, 0; \lambda)|_{\varepsilon=0}(w) = \frac{b_j}{2d_j^2} \left( \gamma_0 + \sum_{\ell=1}^{2} \sum_{k=0}^{m-1} \frac{\gamma_\ell}{\nu_{k\ell j} d_\ell - d_j} \right) \text{Im\{w\}} = \frac{b_j}{2d_j^2} Q_j^0 \text{Im\{w\}}.
$$

Combining the two last identities with (4.35) and (4.36) yields

$$
\partial_\varepsilon (f(\varepsilon), \lambda(\varepsilon))|_{\varepsilon=0} = \left( 0, \frac{b_1 Q_1^0}{\gamma_1 d_1^{\alpha+2}} w, \frac{b_2 Q_2^0}{\gamma_2 d_2^{\alpha+2}} w ; 0, 0 \right).
$$

Case $\alpha \in (0, 1)$. From (4.27) we have

$$
\mathcal{K}^\alpha_k[\varepsilon, 0, 0](w) = \alpha C_\alpha \nu_{k\ell j} \left[ \int_\mathbb{T} \int_0^1 \text{Re} \left\{ \frac{(\nu_{k\ell j} d_\ell - d_j)(\nu_{k\ell j} b_\ell \tau - b_j w)}{\nu_{k\ell j} (t\varepsilon b_\ell \tau + d_\ell) - (t\varepsilon b_j w + d_j)^{\alpha+2}} dt d\tau \right\} \right.
$$

$$
\left. + \int_\mathbb{T} \int_0^1 \frac{t\varepsilon \nu_{k\ell j} b_\ell \tau - b_j w}{\nu_{k\ell j} (t\varepsilon b_\ell \tau + d_\ell) - (t\varepsilon b_j w + d_j)^{\alpha+2}} dt d\tau \right].
$$

with $\nu_{k\ell j}$ defined as for $\alpha = 0$. Applying formula (2.23) gives

$$
\mathcal{K}^\alpha_k[\varepsilon, f_\ell, f_j](w) = \frac{\alpha C_\alpha}{4} \left[ 2(\nu_{k\ell j} d_\ell - d_j) + \varepsilon b_j w \right] + (\alpha + 2) \left( \frac{(\nu_{k\ell j} d_\ell - d_j)^2}{\nu_{k\ell j} d_\ell - d_j} \right) + o(\varepsilon).
$$

Inserting the last identity into (4.37) and then differentiating with respect $\varepsilon$, we obtain

$$
\partial_\varepsilon \mathcal{G}_0^\alpha(0, 0; \lambda)(w) = \frac{(\alpha + 2)\alpha C_\alpha}{4} \text{Im} \left\{ b_0 w^2 \sum_{\ell=1}^{2} \frac{\gamma_\ell}{d_\ell^{\alpha+2}} \sum_{k=0}^{m-1} \frac{\gamma_\ell}{\nu_{k\ell j} d_\ell - d_j} \right\} = 0,
$$

$$
\partial_\varepsilon \mathcal{G}_j^\alpha(0, 0; \lambda)(w) = \frac{(\alpha + 2)\alpha C_\alpha}{4d_j^{\alpha+2}} b_j \left( \gamma_0 + \sum_{\ell=1}^{2} \sum_{k=0}^{m-1} \frac{\gamma_\ell}{\nu_{k\ell j} d_\ell - d_j} \right) \text{Im\{w\}}.
$$

Combining the two last identities with (4.35), (4.36) and (2.67), we get

$$
\partial_\varepsilon (f(\varepsilon), \lambda(\varepsilon))|_{\varepsilon=0} = \Xi_\alpha \left( 0, \frac{b_1 Q_1^\alpha}{\gamma_1 d_1^{\alpha+2}} w, \frac{b_2 Q_2^\alpha}{\gamma_2 d_2^{\alpha+2}} w ; 0, 0 \right),
$$

where $Q_j^\alpha$ and $\Xi_\alpha$ are given by (4.30).

The convexity in (v) is established in exactly the same way as in the proof of Theorem 2.7. This ends the proof of the theorem. \qed

Remark 4.4. Note that, by setting $\gamma_2 = 0$ and $\lambda = \Omega$ in (2.3), (4.24)–(4.25) and (4.19)–(4.20), we can recover the existence and uniqueness result of the body-centered polygonal configuration through Theorem 4.3. This remains equally true for the rotating vortex polygon by setting $\gamma_0 = \gamma_2 = 0$ and $\lambda = \Omega$. 

Acknowledgments

Miles H. Wheeler was partially supported by NSF-DMS grant 1400926. The work of Z. Hassainia is supported by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE.

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