Adaptive sampled-data integral control of stable infinite-dimensional linear systems *

Zhenqing Ke †, Hartmut Logemann ‡
Department of Mathematical Sciences
University of Bath, Bath, BA2 7AY, UK

Stuart Townley §
School of Mathematical Sciences
University of Exeter, Exeter, EX4 4QE, UK

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Abstract. In this paper, adaptive discrete-time low-gain integral control strategies are presented for tracking constant reference signals for infinite-dimensional discrete-time power-stable linear systems. The discrete-time results are applied in the development of adaptive sampled-data low-gain integral control of well-posed infinite-dimensional exponentially stable linear systems. Our results considerably extend, improve and simplify previous work by two of the authors [*IEEE Trans. Automat. Contr.* 42 (1997), pp. 22-37].

Keywords. Adaptive control, adaptive tracking, discrete-time systems, integral control, low-gain control, sampled-data control, infinite-dimensional systems.

1 Introduction

There has been much interest in low-gain integral control over the last thirty years. The following principle has become well established (see Davison [3] and Morari [10]): an application of the integrator $(\varepsilon/s)I$ to an asymptotically stable, finite-dimensional continuous-time plant, with transfer function matrix $\mathbf{G}(s)$, leads to an asymptotically stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that the gain parameter $\varepsilon > 0$ is sufficiently small and the eigenvalues of the steady-state matrix $\mathbf{G}(0)$ have positive real parts. This principle has been extended to various classes of infinite-dimensional systems (see, for example, the pioneering contribution by Pohjolainen [11] and the paper by Logemann and Townley [6]).

If the plant uncertainty is large, then it is natural to tune the parameter ε adaptively. For continuous-time plants, low-gain universal adaptive controllers which achieve asymptotic tracking of constant reference signals have been presented by Cook [1] and Miller and Davison [8], [9] in the finite-dimensional case and by Logemann and Townley [6, 7] in the infinite-dimensional case. By "universal" we mean that the controllers are not based on system identification or plant parameter estimation algorithms.

*This work was supported in part by UK EPSRC under Grant GR/S94582/01.
†Email: kezhenqing@hotmail.com
‡Email: hl@maths.bath.ac.uk
§Email: townley@maths.ex.ac.uk

In this paper, we first consider the problem of adaptive low-gain integral control of discrete-time power-stable infinite-dimensional systems. These discrete-time results are then applied to the main objective of this paper, namely, adaptive sampled-data set-point control of well-posed systems. We remark that the class of well-posed linear systems is the largest class of infinite-dimensional systems for which a well-developed state-space and frequency-domain theory exists. Well-posed systems are rather general in the sense that they capture most distributed parameter systems and all time-delay systems which are of interest in applications.

In Section 2, we improve a result in [5] on adaptive low-gain control of discrete-time systems. Theorem 3.2 in [5] shows that the adaptive controller

$$u(k+1) = u(k) + \gamma^{-q}(k)(r - y(k)), \quad \gamma(k+1) = \gamma(k) + \|r - y(k)\|^2, \tag{1.1}$$

achieves asymptotic tracking of arbitrary constant reference signals r, provided that the following three assumptions are satisfied:

(1) the plant is power stable;

(2) the steady-state gain matrix $\mathbf{P}(1)$ is symmetric and positive definite, where $\mathbf{P}(z)$ is the transfer function of the discrete-time plant;

(3) the parameter q in (1.1) satisfies $q \in (0, 1/2)$.

The symmetry assumption in (2) is restrictive and highly nonrobust, essentially limiting the applications of the above result to single-input single-output systems. The main result of Section 2 (Theorem 2.1) shows that assumption (2) can be replaced by the considerably weaker (and essentially necessary) assumption that all the eigenvalues of $\mathbf{P}(1)$ have positive real parts, and (3) can be replaced by $q \in (0, 1]$. Furthermore, in comparing the analysis presented here to that in [5], we use a change of coordinates technique which is the discrete-time counterpart to that used in [7], leading to a dramatic simplification of the proofs.

In Section 3, we study adaptive low-gain sampled-data control for well-posed systems. Our results are extensions and improvements of those in [5] with respect to the following aspects.

• The plant is assumed to belong to the class of exponentially stable well-posed systems, which is more general than the class of exponentially stable regular systems considered in [5]. We emphasize that it is often considerably easier to verify well-posedness than to show regularity.

• In [5], it is assumed that $\mathbf{G}(0)$ is symmetric and positive definite, where $\mathbf{G}$ denotes the transfer of the continuous-time plant. As discussed above, this assumption is restrictive and highly nonrobust. In the present paper, we only assume that the eigenvalues of $\mathbf{G}(0)$ have positive real parts.

• The sampled-data controller used in Section 3 is based on an adaptive control law similar to (1.1): it processes a sampled version of the plant output obtained by the application of a generalized sampling operation, a special case of which is the simple averaging prototype used in [5].

• The range of the parameter q is $(0, 1]$ instead of $(0, 1/2)$ in [5].

• The analysis of the behaviour of the tracking error has been considerably improved, see statements (4) and (5) of Theorem 3.2 and part (3) of Remark 3.3.

We illustrate the main result by a heat equation example with two point controls and two point observations.

Notation. Let Z be a Banach space. The space of all Z-valued sequences defined on $\mathbb{Z}_+$ is denoted by $F(\mathbb{Z}_+, Z)$ and $PC(\mathbb{R}_+, Z)$ denotes the set of piecewise continuous functions defined on $\mathbb{R}_+$ with values in Z. For $\alpha \in \mathbb{R}$, define $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ and define the exponentially weighted L^2-space $L^2_\alpha(\mathbb{R}_+, Z)$ by

$$L^2_\alpha(\mathbb{R}_+, Z) := \{f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Z) \,|\, f(\cdot)e^{-\alpha\cdot} \in L^2(\mathbb{R}_+, Z)\}.$$

We define

$$
\begin{aligned}
H^2(\mathbb{C}_\alpha, Z) &:= \{f : \mathbb{C}_\alpha \to Z\} \,|\, f \text{ is holomorphic and } \sup_{x>\alpha} \int_{-\infty}^{\infty} \|f(x+i\sigma)\|^2 d\sigma < \infty\}\,, \\
H^\infty(\mathbb{C}_\alpha, Z) &:= \{f : \mathbb{C}_\alpha \to Z \,|\, f \text{ is holomorphic and bounded}\}\,.
\end{aligned}
$$

The set of all bounded linear operators from a Banach space Z_1 to a Banach space Z_2 is denoted by $\mathcal{B}(Z_1, Z_2)$; if $Z_1 = Z_2 = Z$, then we write $\mathcal{B}(Z)$ for $\mathcal{B}(Z_1, Z_2)$. For $T \in \mathcal{B}(Z_1, Z_2)$, let $\sigma(T)$ denote the spectrum of T. The *forward difference operator* $\triangle : F(\mathbb{Z}_+, Z) \to F(\mathbb{Z}_+, Z)$ is defined by $(\triangle u)(k) := u(k+1) - u(k)$. The Laplace transform is denoted by $\mathscr{L}$.

2 Adaptive discrete-time low-gain control

Let X, U and Y be Hilbert spaces. Consider the discrete-time system

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k)\,; \quad x(0) = x^0 \in X\,, && \text{(2.1a)} \\
y(k) &= Cx(k) + Du(k)\,, && \text{(2.1b)}
\end{aligned}
$$

where $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(U, X)$, $C \in \mathcal{B}(X, Y)$ and $D \in \mathcal{B}(U, Y)$. The transfer function of (2.1) is

$$
\mathbf{P}(z) := C(zI - A)^{-1}B + D\,.
$$

System (2.1) is called *power stable* if A is power stable, i.e., there exist $M \geq 1$ and $\rho \in (0, 1)$ such that $\|A^k\| \leq M\rho^k$ for all $k \in \mathbb{Z}_+$. It is well-known that (2.1) is power stable if and only if the spectral radius of A is smaller than 1.

The aim is to find an adaptive controller which achieves setpoint tracking. Following [5], consider the adaptive controller given by

$$
\begin{aligned}
u(k) &= Kw(k)\,, && \text{(2.2a)} \\
w(k+1) &= w(k) + \gamma^{-q}(k)(r - y(k))\,; \; w(0) = w^0, && \text{(2.2b)} \\
\gamma(k+1) &= \gamma(k) + \|r - y(k)\|^2\,; \; \gamma(0) = \gamma^0 > 0\,, && \text{(2.2c)}
\end{aligned}
$$

where $r \in Y$ is the reference vector, $K \in \mathcal{B}(Y, U)$ and $q \in (0, 1]$. Note that the scalar adaptive variable $\gamma(k)$ is increasing (with rate of change given by $\|r - y(k)\|^2$), so that the gain $\gamma^{-q}(k)$ is decreasing, hence the terminology "low-gain" controller.

The following theorem is the main result of this section. It forms the discrete-time counterpart of the continuous-time result in [7].

Theorem 2.1. *Assume that* (2.1) *is power stable, there exists* $K \in \mathcal{B}(Y, U)$ *such that*

$$
\sigma(\mathbf{P}(1)K) \subset \mathbb{C}_0\,,
$$

and $q \in (0, 1]$. *Then, for all* $(x^0, w^0) \in X \times Y$, *all* $\gamma^0 > 0$ *and all* $r \in Y$, *the closed-loop system given by* (2.1) *and* (2.2) *has the following properties:*

(1) $r - y \in \ell^2(\mathbb{Z}_+, Y)$, *so in particular* $\lim_{k\to\infty} y(k) = r$;

(2) $\lim_{k\to\infty} \gamma(k) = \gamma^\infty < \infty$;

(3) $u - u^\infty, \triangle u \in \ell^2(\mathbb{Z}_+, U)$, *where* $u^\infty := K(\mathbf{P}(1)K)^{-1}r$;

(4) $x - x^\infty \in \ell^2(\mathbb{Z}_+, X)$, *where* $x^\infty := (I-A)^{-1}Bu^\infty$.

Proof. We proceed in several steps.

Step 1: A change of coordinates. We note first that, if the limit $\lim_{k\to\infty} w(k) =: w^\infty$ exists, then $\lim_{k\to\infty} x(k) = (I-A)^{-1}BKw^\infty$ and $\lim_{k\to\infty} y(k) = \mathbf{P}(1)Kw^\infty$. In particular, if $w^\infty = (\mathbf{P}(1)K)^{-1}r$, then $y(k)$ will converge to r as $k \to \infty$. This motivates the following change of coordinates:

$$z(k) := x(k) - (I-A)^{-1}BKw(k)\,, \quad \forall k \in \mathbb{Z}_+\,, \tag{2.3a}$$

$$v(k) := w(k) - (\mathbf{P}(1)K)^{-1}r\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{2.3b}$$

Invoking the identity $A(I-A)^{-1} + I = (I-A)^{-1}$ together with (2.1)-(2.3), a routine calculation gives

$$\begin{aligned} z(k+1) &= x(k+1) - (I-A)^{-1}BKw(k+1) \\ &= Ax(k) + Bu(k) - (I-A)^{-1}BKw(k+1) \\ &= Az(k) + [A(I-A)^{-1} + I]BKw(k) - (I-A)^{-1}BKw(k+1) \\ &= Az(k) + (I-A)^{-1}BK[w(k) - w(k+1)] \\ &= Az(k) - \gamma^{-q}(k)\Gamma e(k)\,, \quad \forall k \in \mathbb{Z}_+\,, \end{aligned} \tag{2.4}$$

where $\Gamma := (I-A)^{-1}BK$ and $e := r - y$, and

$$\begin{aligned} v(k+1) &= w(k+1) - (\mathbf{P}(1)K)^{-1}r \\ &= w(k) - (\mathbf{P}(1)K)^{-1}r + \gamma^{-q}(k)e(k) \\ &= v(k) + \gamma^{-q}(k)e(k)\,, \quad \forall k \in \mathbb{Z}_+\,. \end{aligned} \tag{2.5}$$

Moreover,

$$\begin{aligned} e = r - y &= r - Cx - Du \\ &= r - Cz - C(I-A)^{-1}BKw - DKw \\ &= -Cz - \mathbf{P}(1)K[w - (\mathbf{P}(1)K)^{-1}r] \\ &= -[Cz + \mathbf{P}(1)Kv]\,. \end{aligned} \tag{2.6}$$

Step 2: A Lyapunov-type argument. Since A is power stable and $\sigma(\mathbf{P}(1)K) \subset \mathbb{C}_0$, there exist $P \in \mathcal{B}(X)$, $P = P^*$, $P > 0$ and $Q \in \mathcal{B}(Y)$, $Q = Q^*$, $Q > 0$ such that

$$A^*PA - P = -I\,, \quad (\mathbf{P}(1)K)^*Q + Q(\mathbf{P}(1)K) = I\,, \tag{2.7}$$

(see [12], Proposition 5 and [14], p. 231, Theorem 18), where P^* and Q^* are the adjoint operators of P and Q, respectively. Set

$$V(k) := \langle z(k), Pz(k)\rangle + \langle v(k), Qv(k)\rangle\,, \quad \forall k \in \mathbb{Z}_+\,.$$

By the positivity of P and Q, $V(k) \geq 0$ for all $k \in \mathbb{Z}_+$. In the following, we will estimate $V(k+1) - V(k)$ in terms of $z(k)$, $v(k)$, $e(k)$ and $\gamma(k)$. To this end, note that there exists $M_1 \geq 0$ such that

$$\langle \Gamma d, P\Gamma d\rangle \leq M_1\|d\|^2\,, \quad \langle d, Qd\rangle \leq M_1\|d\|^2\,, \qquad \forall d \in Y\,.$$

It follows from (2.4)-(2.7) that there exists $M_2 \geq 0$ such that

$$
\begin{aligned}
&\langle z(k+1), Pz(k+1)\rangle - \langle z(k), Pz(k)\rangle \\
&= \langle Az(k) - \gamma^{-q}(k)\Gamma e(k), P[Az(k) - \gamma^{-q}(k)\Gamma e(k)]\rangle - \langle z(k), Pz(k)\rangle \\
&\leq \langle z(k), (A^*PA - P)z(k)\rangle + 2\gamma^{-q}(k)|\langle Az(k), P\Gamma e(k)\rangle| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\
&\leq -\|z(k)\|^2 + 2\gamma^{-q}(k)|\langle Az(k), P\Gamma Cz(k)\rangle| + 2\gamma^{-q}(k)|\langle Az(k), P\Gamma \mathbf{P}(1)Kv(k)\rangle| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\
&\leq -\|z(k)\|^2 + M_2\gamma^{-q}(k)\|z(k)\|^2 + M_2\gamma^{-q}(k)\|z(k)\|\|v(k)\| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\
&\leq -\|z(k)\|^2 + M_2\gamma^{-q}(k)(1+\frac{\mu}{2})\|z(k)\|^2 + \frac{M_2\gamma^{-q}(k)}{2\mu}\|v(k)\|^2 + M_1\gamma^{-2q}(k)\|e(k)\|^2\,, \quad \forall k \in \mathbb{Z}_+\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle v(k+1), Qv(k+1)\rangle - \langle v(k), Qv(k)\rangle \\
&= \langle v(k) + \gamma^{-q}(k)e(k), Q[v(k) + \gamma^{-q}(k)e(k)]\rangle - \langle v(k), Qv(k)\rangle \\
&\leq -\gamma^{-q}(k)\langle [Q(\mathbf{P}(1)K) + (\mathbf{P}(1)K)^*Q]v(k), v(k)\rangle + 2\gamma^{-q}(k)|\langle v(k), QCz(k)\rangle| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\
&\leq -\gamma^{-q}(k)\|v(k)\|^2 + M_2\gamma^{-q}(k)\|z(k)\|\|v(k)\| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\
&\leq -\gamma^{-q}(k)\|v(k)\|^2 + \frac{M_2\gamma^{-q}(k)\mu}{2}\|z(k)\|^2 + \frac{M_2\gamma^{-q}(k)}{2\mu}\|v(k)\|^2 + M_1\gamma^{-2q}(k)\|e(k)\|^2\,, \quad \forall k \in \mathbb{Z}_+\,,
\end{aligned}
$$

where $\mu > 0$ is arbitrary. Hence

$$
\begin{aligned}
V(k+1) - V(k) \quad &\leq \quad -[1 - M_2(1+\mu)\gamma^{-q}(k)]\|z(k)\|^2 + (-1 + \frac{M_2}{\mu})\gamma^{-q}(k)\|v(k)\|^2 \\
&\qquad + 2M_1\gamma^{-2q}(k)\|e(k)\|^2\,, \quad \forall k \in \mathbb{Z}_+\,.
\end{aligned}
\tag{2.8}
$$

Step 3: Proof of statement (2). By (2.2c), γ is non-decreasing, so that statement (2) will follow if we show that γ is bounded. To this end, seeking a contradiction, suppose that γ is not bounded. Then, since $q > 0$, $k \mapsto \gamma^{-q}(k)$ is monotonically decreasing and converging to 0. Hence, there exists $N_1 \in \mathbb{Z}_+$ such that

$$\gamma^{-q}(k) \leq \frac{1}{2M_2(1+2M_2)}\,, \quad \forall k \geq N_1\,.$$

Choosing $\mu = 2M_2$, it follows from (2.8) that

$$V(k+1) - V(k) \leq -\frac{1}{2}(\|z(k)\|^2 + \gamma^{-q}(k)\|v(k)\|^2) + 2M_1\gamma^{-2q}(k)\|e(k)\|^2\,, \quad \forall k \geq N_1\,.$$

Note from (2.6) that

$$\|e\|^2 = \|Cz + \mathbf{P}(1)Kv\|^2 \leq 2(\|Cz\|^2 + \|\mathbf{P}(1)Kv\|^2)\,.$$

Consequently, there exists $M_3 > 0$ such that

$$
\begin{aligned}
V(k+1) - V(k) \quad &\leq \quad -4M_3\gamma^{-q}(k)(\|Cz(k)\|^2 + \|\mathbf{P}(1)Kv(k)\|^2) + 2M_1\gamma^{-2q}(k)\|e(k)\|^2 \\
&\leq \quad [-2M_3 + 2M_1\gamma^{-q}(k)]\gamma^{-q}(k)\|e(k)\|^2\,, \quad \forall k \geq N_1\,.
\end{aligned}
$$

By the monotonicity of $k \mapsto \gamma^{-q}(k)$ and (2.2c), there exists $N_2 \geq N_1$ such that

$$V(k+1) - V(k) \leq -M_3\gamma^{-q}(k)\|e(k)\|^2 = -M_3\gamma^{-q}(k)[\gamma(k+1) - \gamma(k)]\,, \quad \forall k \geq N_2\,.$$

Summing up over k, we obtain

$$V(k) - V(N_2) \le -M_3 \sum_{j=N_2}^{k-1} \gamma^{-q}(j)[\gamma(j+1) - \gamma(j)]\,, \quad \forall k \ge N_2 + 1\,.$$

Since $k \mapsto \gamma^{-q}(k)$ is monotonically decreasing and the fact that V is non-negative, it follows that

$$\int_{\gamma(N_2)}^{\gamma(k)} s^{-q} ds = \sum_{j=N_2}^{k-1} \int_{\gamma(j)}^{\gamma(j+1)} s^{-q} ds \le \sum_{j=N_2}^{k-1} \gamma^{-q}(j)[\gamma(j+1) - \gamma(j)] \le \frac{V(N_2) - V(k)}{M_3} \le \frac{V(N_2)}{M_3}\,. \tag{2.9}$$

However, since $q \in (0, 1]$ and $\gamma(k) \to \infty$ as $k \to \infty$, we have that $\int_{\gamma(N_2)}^{\gamma(k)} s^{-q} ds \to \infty$ as $k \to \infty$, contradicting (2.9). Consequently, γ is bounded, completing the proof of statement (2).

Step 4: Proof of statements (1), (3) and (4). It follows immediately from (2.2c) that

$$r - y = e \in \ell^2(\mathbb{Z}_+, Y)\,, \tag{2.10}$$

so that, in particular, $\lim_{k\to\infty} y(k) = r$, showing that statement (1) is true. Hence, by (2.2a) and (2.2b), $\triangle u = K \triangle w \in \ell^2(\mathbb{Z}_+, U)$. Since A is power stable, statement (2) together with (2.4) and (2.10) imply

$$z \in \ell^2(\mathbb{Z}_+, X)\,, \tag{2.11}$$

so that $Cz \in \ell^2(\mathbb{Z}_+, Y)$. It follows from (2.6), (2.10) and the invertibility of $\mathbf{P}(1)K$ that $v \in \ell^2(\mathbb{Z}_+, Y)$. Invoking (2.2a) and (2.3b) completes the proof of statement (3). Since

$$x - (I - A)^{-1} B u^\infty = z + (I - A)^{-1} B (u - u^\infty)\,,$$

statement (4) follows from statement (3) and (2.11). □

Remark 2.2. Assume that (2.2a) is replaced by $u(k) = Kw(k) + d$, where $d \in U$ is a constant input disturbance. Then the conclusions of Theorem 2.1 remain valid with $u^\infty = K(\mathbf{P}(1)K)^{-1}(r - \mathbf{P}(1)d) + d$. To see this, we observe that the proof of Theorem 2.1 still applies, provided that the change of coordinates (2.3) is replaced by $z(k) := x(k) - (I - A)^{-1} B(Kw(k) + d)$ and $v(k) := w(k) - (\mathbf{P}(1)K)^{-1}(r - \mathbf{P}(1)d)$. ◇

3 Adaptive sampled-data low-gain control

We first recall briefly some facts about well-posed continuous-time systems (see, for example, [13], [15], [16], [17] for details). The class of well-posed systems captures the systems theoretic properties of linearity, time-invariance, and causality together with natural continuity properties. Every well-posed system has a well-defined transfer function. Throughout this section, we shall consider a well-posed system Σ with state-space X, input space U, and output space Y (all Hilbert spaces), generating operators (A, B, C), input-output operator G and transfer function $\mathbf{G}$. Here A is the generator of a strongly continuous semigroup (C_0-semigroup) $\mathbf{T}$ on X, $B \in \mathcal{B}(U, X_{-1})$, and $C \in \mathcal{B}(X_1, Y)$, where X_1 denotes the domain of A, as an operator defined on X, endowed with the graph norm $\|x\|_1 := \|x\| + \|Ax\|$, and X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} := \|(\beta I - A)^{-1} x\|$. Here β is in the resolvent set of A. It can be verified that different choices of β leads to equivalent norms. We have $X_1 \hookrightarrow X \hookrightarrow X_{-1}$. It is known that $\mathbf{T}$ restricts to a C_0-semigroup on X_1 and extends to a C_0-semigroup on X_{-1} with the exponential growth constant being the same on all three spaces X_1, X and X_{-1}. The generator of the restricted (extended) semigroup is a restriction (extension) of A. The restricted/extended semigroups and their generators will be denoted by the same symbols $\mathbf{T}$ and A, respectively.

The control operator B is *admissible*, that is, for every $t \geq 0$, there exists $b_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}(t-s)Bv(s) \right\| \leq b_t \|v\|_{L^2}, \quad \forall v \in L^2([0,t], U),$$

and the observation operator C is also *admissible*, that is, for every $t \geq 0$, there exists $c_t \geq 0$ such that

$$\int_0^t \|C\mathbf{T}(t)z\|^2 dt \leq c_t \|z\|^2, \quad \forall z \in X_1.$$

The so-called Λ-*extension* of C is defined by

$$C_\Lambda z := \lim_{\lambda \to \infty,\, \lambda \in \mathbb{R}} C\lambda(\lambda I - A)^{-1} z, \quad \forall z \in \operatorname{dom}(C_\Lambda),$$

where $\operatorname{dom}(C_\Lambda)$ is the set of all $z \in X$ for which the above limit exists. Clearly, $X_1 \subset \operatorname{dom}(C_\Lambda)$. For each $z \in X$, $\mathbf{T}(t)z \in \operatorname{dom}(C_\Lambda)$ for almost all $t \geq 0$, and if $\alpha > \omega(\mathbf{T})$, then $C_\Lambda \mathbf{T} z \in L^2_\alpha(\mathbb{R}_+, Y)$, where

$$\omega(\mathbf{T}) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbf{T}(t)\|$$

denotes the exponential growth constant of $\mathbf{T}$. The transfer function $\mathbf{G}$ satisfies

$$\frac{\mathbf{G}(s) - \mathbf{G}(\eta)}{s - \eta} = -C(sI - A)^{-1}(\eta I - A)^{-1} B, \quad \forall s, \eta \in \mathbb{C}_{\omega(\mathbf{T})},\ s \neq \eta, \tag{3.1}$$

and $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha, \mathcal{B}(U,Y))$ for every $\alpha > \omega(\mathbf{T})$. Moreover, the input-output operator $G : L^2_{\text{loc}}(\mathbb{R}_+, U) \to L^2_{\text{loc}}(\mathbb{R}_+, Y)$ is continuous and shift-invariant; for every $\alpha > \omega(\mathbf{T})$, $G \in \mathcal{B}(L^2_\alpha(\mathbb{R}_+, U), L^2_\alpha(\mathbb{R}_+, Y))$ and

$$(\mathscr{L}(Gv))(s) = \mathbf{G}(s)(\mathscr{L}(v))(s), \quad \forall s \in \mathbb{C}_\alpha,\ \forall v \in L^2_\alpha(\mathbb{R}_+, U).$$

For $x^0 \in X$ and $v \in L^2_{\text{loc}}(\mathbb{R}_+, U)$, let x and y denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function v. Then

$$x(t) = \mathbf{T}(t)x^0 + \int_0^t \mathbf{T}(t-s)Bv(s)ds, \quad \forall t \geq 0, \tag{3.2}$$

$x(t) - (\eta I - A)^{-1}Bv(t) \in \operatorname{dom}(C_\Lambda)$ for almost all $t \geq 0$, and

$$\dot{x}(t) = Ax(t) + Bv(t); \quad x(0) = x^0 \in X, \quad \text{for a.a. } t \geq 0, \tag{3.3a}$$

$$y(t) = C_\Lambda[x(t) - (\eta I - A)^{-1}Bv(t)] + \mathbf{G}(\eta)v(t), \tag{3.3b}$$

where $\eta \in \mathbb{C}_{\omega(\mathbf{T})}$ is arbitrary. The differential equation (3.3a) has to be interpreted in X_{-1}. In the following, we identify Σ and (3.3) and refer to (3.3) as a well-posed system. We say that (3.3) is *exponentially stable* if $\mathbf{T}$ is exponentially stable, i.e., $\omega(\mathbf{T}) < 0$.

Let $\tau > 0$ be the sampling period and let $u = (u(k))_{k \in \mathbb{Z}_+} \in F(\mathbb{Z}_+, U)$ be an arbitrary sequence. Define the the (zero-order) *hold operator* $\mathcal{H}$ by

$$(\mathcal{H}u)(t) := u(k), \quad \forall t \in [k\tau, (k+1)\tau).$$

Let $a \in L^2([0,\tau],\mathbb{R})$ be such that

$$\text{(i)} \quad \int_0^\tau a(t)dt = 1\,, \qquad \text{(ii)} \quad \int_0^\tau a(t)\mathbf{T}(t)z\,dt \in X_1\,, \quad \forall z \in X\,.$$

Whilst the above condition (ii) is difficult to check for general a, it is easy to show (using integration by parts) that (ii) holds if there exists a partition $0 = t_0 < t_1 < \cdots < t_m = \tau$ such that $a|_{(t_{j-1},\,t_j)} \in W^{1,1}((t_{j-1},t_j),\mathbb{R})$ for $j = 1,2,\ldots,m$. An example for a is $a(t) \equiv 1/\tau$.

We define a *generalized sampling operator* $\mathcal{S} : L^2_{\mathrm{loc}}(\mathbb{R}_+,Y) \to F(\mathbb{Z}_+,Y)$ by

$$y_k := (\mathcal{S}y)(k) := \int_0^\tau a(t)y(k\tau+t)dt\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{3.4}$$

Define $L : X \to X_1$ by

$$Lz := \int_0^\tau a(t)\mathbf{T}(t)z dt\,.$$

By the closed-graph theorem, we know that $L \in \mathcal{B}(X,X_1)$. Define

$$\begin{pmatrix} A_\tau & B_\tau \\ C_\tau & D_\tau \end{pmatrix} := \begin{pmatrix} \mathbf{T}(\tau) & \displaystyle\int_0^\tau \mathbf{T}(s)dsB \\ CL & CLA^{-1}B + \mathbf{G}(0) \end{pmatrix} \tag{3.5}$$

Trivially, $A_\tau \in \mathcal{B}(X)$. Moreover, $B_\tau \in \mathcal{B}(U,X)$, $C_\tau \in \mathcal{B}(X,Y)$ and $D_\tau \in \mathcal{B}(U,Y)$.

Proposition 3.1. *Assume that* (3.3) *is exponentially stable and consider* (3.3) *with* $v = \mathcal{H}u$ *where* $u \in F(\mathbb{Z}_+,U)$. *Set* $x_k := x(k\tau)$ *for all* $k \in \mathbb{Z}_+$, *where* x *is the solution of* (3.3a) *given by* (3.2), *and define* y_k *by* (3.4). *Then*

$$x_{k+1} = A_\tau x_k + B_\tau u(k)\,, \tag{3.6a}$$

$$y_k = C_\tau x_k + D_\tau u(k)\,. \tag{3.6b}$$

Moreover, A_τ *is power stable and*

$$\mathbf{G}_\tau(1) = C_\tau(I - A_\tau)^{-1}B_\tau + D_\tau = \mathbf{G}(0)\,,$$

where $\mathbf{G}_\tau$ *denotes the transfer function of the discrete-time system* (3.6).

Proof. The equation (3.6a) follows easily from (3.2). To prove (3.6b), it is useful to note that

$$\int_0^\tau a(t)C_\Lambda\mathbf{T}(t)zdt = CLz = C_\tau z\,, \quad \forall z \in X\,. \tag{3.7}$$

Without loss of generality, we may choose $\eta = 0$ in (3.3b) to obtain that

$$\begin{aligned} y(k\tau+t) &= C_\Lambda\left[\mathbf{T}(t)x_k + \int_0^t \mathbf{T}(s)Bu(k)ds + A^{-1}Bu(k)\right] + \mathbf{G}(0)u(k) \\ &= C_\Lambda[\mathbf{T}(t)x_k + \mathbf{T}(t)A^{-1}Bu(k)] + \mathbf{G}(0)u(k)\,, \quad \forall k \in \mathbb{Z}_+\,,\ \forall t \in [0,\tau)\,. \end{aligned}$$

Hence, by (3.7),

$$\begin{aligned} y_k &= \int_0^\tau a(t)y(k\tau+t)dt \\ &= \int_0^\tau a(t)C_\Lambda\mathbf{T}(t)(x_k + A^{-1}Bu(k))dt + \mathbf{G}(0)u(k) \\ &= CLx_k + CLA^{-1}Bu(k) + \mathbf{G}(0)u(k) \\ &= C_\tau x_k + D_\tau u(k)\,, \quad \forall k \in \mathbb{Z}_+\,. \end{aligned}$$

Moreover, A_τ is power stable since $\mathbf{T}(t)$ is exponentially stable. Finally, since $B_\tau = (\mathbf{T}(\tau)-I)A^{-1}B$, it follows that

$$\mathbf{G}_\tau(1) = C_\tau(I - A_\tau)^{-1}B_\tau + D_\tau = -CLA^{-1}B + CLA^{-1}B + \mathbf{G}(0) = \mathbf{G}(0)\,. \qquad \square$$

We seek an adaptive controller which achieves setpoint tracking. To this end, consider the adaptive control law given by

$$\begin{aligned} v(t) &= (\mathcal{H}u)(t)\,, && (3.8a)\\ u(k) &= Kw(k)\,, && (3.8b)\\ w(k+1) &= w(k) + \gamma^{-q}(k)(r - (\mathcal{S}y)(k))\,;\ w(0) = w^0, && (3.8c)\\ \gamma(k+1) &= \gamma(k) + \|r - (\mathcal{S}y)(k)\|^2\,;\ \gamma(0) = \gamma^0\,, && (3.8d)\end{aligned}$$

where $(\mathcal{S}y)(k)$ is defined in (3.4), $r \in Y$ is the reference vector, $K \in \mathcal{B}(Y,U)$ and $q \in (0,1]$.

Theorem 3.2. *Assume that* (3.3) *is exponentially stable, there exists $K \in \mathcal{B}(Y,U)$ such that*

$$\sigma(\mathbf{G}(0)K) \subset \mathbb{C}_0\,, \tag{3.9}$$

and $q \in (0,1]$. Then, for all $(x^0, w^0) \in X \times Y$, $\gamma^0 > 0$ and all $r \in Y$, the closed-loop sampled-data system given by (3.3) *and* (3.8) *has the following properties.*

(1) $\lim_{k\to\infty} \gamma(k) = \gamma^\infty < \infty$.

(2) $\lim_{t\to\infty} v(t) = v^\infty$ *and* $v - v^\infty \in L^2(\mathbb{R}_+, U)$, *where* $v^\infty := K(\mathbf{G}(0)K)^{-1}r$.

(3) $\lim_{t\to\infty} x(t) = x^\infty := -A^{-1}Bv^\infty$ *and* $x - x^\infty \in L^2(\mathbb{R}_+, X)$.

(4) *The error $e := r - y$ can be decomposed as $e = e_1 + e_2$, where $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2(\mathbb{R}_+, Y)$.*

(5) *Under the additional assumption that*

$$\lim_{t\to\infty} (Gf)(t) = 0\,, \quad \forall f \in PC(\mathbb{R}_+, U) \cap L^2(\mathbb{R}_+, U) \text{ with } \lim_{t\to\infty} f(t) = 0\,, \tag{3.10}$$

the error signal $e = r - y$ can be decomposed as $e = e_1 + e_2$, where $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2_\alpha(\mathbb{R}_+, Y)$ for every $\alpha > \omega(\mathbf{T})$; furthermore, if (3.10) *holds and, for some $t_0 \geq 0$, $\mathbf{T}(t_0)(Ax^0 + BKw^0) \in X$, then $\lim_{t\to 0} e(t) = 0$.*

Proof. Let $(x^0, w^0) \in X \times Y$ and $\gamma^0 > 0$. We obtain x, y, $(u(k))_{k\in\mathbb{Z}_+}$ and $(\gamma(k))_{k\in\mathbb{Z}_+}$ by applying (3.8) to (3.3). Set $x_k := x(k\tau)$ for all $k \in \mathbb{Z}_+$ and define y_k by (3.4), that is $y_k = (\mathcal{S}y)(k)$. It follows from Proposition 3.1 that x_k, $u(k)$ and y_k satisfy (3.6), with $(A_\tau, B_\tau, C_\tau, D_\tau)$ given by (3.5). By exponential stability of (3.3), Proposition 3.1 guarantees that A_τ is power stable and, by (3.9),

$$\sigma(\mathbf{G}_\tau(1)K) = \sigma(\mathbf{G}(0)K) \subset \mathbb{C}_0\,,$$

where $\mathbf{G}_\tau$ denotes the transfer function of the discrete-time system (3.6). Therefore, applying Theorem 2.1 to the discrete-time system (3.6) and the discrete-time controller given by (3.8b)-(3.8d), we see that $\lim_{k\to\infty} \gamma(k) = \gamma^\infty$, showing that statement (1) is true. Moreover,

$$u - v^\infty \in \ell^2(\mathbb{Z}_+, U)\,, \quad \triangle u \in \ell^2(\mathbb{Z}_+, U)\,. \tag{3.11}$$

Hence, it is easy to see that $v - v^\infty = \mathcal{H}(u - v^\infty) \in L^2(\mathbb{R}_+, U)$ and

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} (\mathcal{H}u)(t) = v^\infty\,,$$

so that statement (2) follows. To prove statement (3), note that, for each $k \in \mathbb{N}$ and $t \in [k\tau, (k+1)\tau)$,

$$\begin{aligned} x(t) &= \mathbf{T}(t)x^0 + \mathbf{T}(t-k\tau)\sum_{j=0}^{k-1}\int_{j\tau}^{(j+1)\tau} \mathbf{T}(k\tau - s)Bu(j)ds + \int_{k\tau}^{t} \mathbf{T}(t-s)Bu(k)ds \\ &= \mathbf{T}(t)x^0 + \mathbf{T}(t-k\tau)[\mathbf{T}(\tau) - I]\sum_{j=0}^{k-1} \mathbf{T}((k-j-1)\tau)A^{-1}B(u(j) - v^\infty) \\ &\quad + [\mathbf{T}(t-k\tau) - I]A^{-1}B(u(k) - v^\infty) + [\mathbf{T}(t) - I]A^{-1}Bv^\infty\,. \end{aligned} \tag{3.12}$$

Consequently, for each $k \in \mathbb{N}$ and $t \in [k\tau, (k+1)\tau)$,

$$\begin{aligned} \|x(t) - x^\infty\| &\le \|\mathbf{T}(t)\|\|x^0\| + M\|A^{-1}B\|\|\mathbf{T}(\tau) - I\|\|\sum_{j=0}^{k-1} \mathbf{T}(k-1-j)(u(j) - v^\infty)\| \\ &\quad + (M+1)\|A^{-1}B\|\|u(k) - v^\infty\| + \|\mathbf{T}(t)\|\|x^\infty\|\,, \end{aligned}$$

where $M := \max_{t\in[0,\tau]} \|\mathbf{T}(t)\|$. Therefore statement (3) follows from the exponential stability of $\mathbf{T}$ and the fact that $u - v^\infty \in \ell^2(\mathbb{Z}_+, U)$.

To prove statement (4), define the integral operator J by

$$(Jv)(t) := \int_0^t v(s)ds\,, \quad \forall v \in L^1_{\mathrm{loc}}(\mathbb{R}_+, U)\,,\ \forall t \in \mathbb{R}_+\,,$$

and define the function $\theta : \mathbb{R}_+ \to \mathbb{R}$ by $\theta(t) := 1$ for all $t \in \mathbb{R}_+$. For every $t \in \mathbb{R}_+$, let $k_t \in \mathbb{Z}_+$ be such that $t \in [k_t\tau, (k_t+1)\tau)$. Then,

$$\begin{aligned} (J\mathcal{H}(\triangle u))(t) &= \sum_{j=0}^{k_t-1}\int_{j\tau}^{(j+1)\tau} (\mathcal{H}(\triangle u))(s)ds + \int_{k_t\tau}^{t} (\mathcal{H}(\triangle u))(s)ds \\ &= \tau\sum_{j=0}^{k_t-1}[u(j+1) - u(j)] + (t - k_t\tau)(\mathcal{H}(\triangle u))(t) \\ &= \tau(\mathcal{H}u)(t) - \tau\theta(t)u(0) + h(t)\,, \quad \forall t \ge 0\,, \end{aligned} \tag{3.13}$$

where $h(t) := (t - k_t\tau)(\mathcal{H}(\triangle u))(t)$ for all $t \ge 0$. It follows from (3.13) that

$$\begin{aligned} GJ\mathcal{H}(\triangle u) - \mathbf{G}(0)J\mathcal{H}(\triangle u) &= \tau G(\mathcal{H}u) - \tau\mathbf{G}(0)\mathcal{H}u - \tau G(\theta u(0)) \\ &\quad +\ \tau\mathbf{G}(0)\theta u(0) + Gh - \mathbf{G}(0)h\,. \end{aligned}$$

Consequently, we conclude that

$$e = r - y = r - C_\Lambda\mathbf{T}(t)x^0 - G(\mathcal{H}u) = e_1 + e_2\,,$$

where

$$e_1 := -\frac{1}{\tau}(GJ - \mathbf{G}(0)J)\mathcal{H}(\triangle u) - \frac{1}{\tau}\mathbf{G}(0)h + r - \mathbf{G}(0)\mathcal{H}u\,, \tag{3.14}$$

and

$$e_2 := -C_\Lambda \mathbf{T}(t)x^0 - [G(\theta u(0)) - \mathbf{G}(0)\theta u(0)] + \frac{1}{\tau} Gh\,. \tag{3.15}$$

We first prove that $\lim_{t\to\infty} e_1(t) = 0$. By (3.11), it is clear that $\mathcal{H}(\triangle u) \in L^2(\mathbb{R}_+, U)$. Noting that the function $s \mapsto [\mathscr{L}(GJ - \mathbf{G}(0)J)](s) = s \mapsto (1/s)(\mathbf{G}(s) - \mathbf{G}(0))$ is in $H^\infty(\mathbb{C}_0, \mathcal{B}(U,Y))$, it follows that $GJ - \mathbf{G}(0)J \in \mathcal{B}(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y))$. Hence

$$(GJ - \mathbf{G}(0)J)\mathcal{H}(\triangle u) \in L^2(\mathbb{R}_+, Y)\,. \tag{3.16}$$

Moreover, since, by shift-invariance, G and J commute,

$$[(GJ - \mathbf{G}(0)J)\mathcal{H}(\triangle u)]' = (G - \mathbf{G}(0))\mathcal{H}(\triangle u) \in L^2(\mathbb{R}_+, Y)\,. \tag{3.17}$$

As a consequence of (3.16) and (3.17), we obtain

$$\lim_{t\to\infty} [(GJ - \mathbf{G}(0)J)\mathcal{H}(\triangle u)](t) = 0\,. \tag{3.18}$$

Moreover, (3.11) implies that

$$\lim_{t\to\infty} h(t) = 0\,, \quad h \in L^2(\mathbb{R}_+, U) \cap PC(\mathbb{R}_+, U)\,, \tag{3.19}$$

and

$$\lim_{t\to\infty} \mathbf{G}(0)(\mathcal{H}u)(t) = \mathbf{G}(0)v^\infty = r\,. \tag{3.20}$$

Combining (3.14), (3.18)-(3.20) gives $\lim_{t\to\infty} e_1(t) = 0$. We proceed to prove that $e_2 \in L^2(\mathbb{R}_+, Y)$. Obviously,

$$C_\Lambda \mathbf{T}x^0 \in L^2_\alpha(\mathbb{R}_+, Y)\,, \quad \forall \alpha > \omega(\mathbf{T})\,,\ \forall x^0 \in X\,. \tag{3.21}$$

Now

$$[\mathscr{L}(G(\theta u(0)) - \mathbf{G}(0)\theta u(0))](s) = \frac{1}{s}[\mathbf{G}(s) - \mathbf{G}(0)]u(0)\,,$$

and we see that $\mathscr{L}(G(\theta u(0)) - \mathbf{G}(0)\theta u(0)) \in H^2(\mathbb{C}_\alpha, U)$ for all $\alpha > \omega(\mathbf{T})$. Hence, by the Paley-Wiener theorem,

$$G(\theta u(0)) - \mathbf{G}(0)\theta u(0) \in L^2_\alpha(\mathbb{R}_+, U)\,, \quad \forall \alpha > \omega(\mathbf{T})\,. \tag{3.22}$$

Using $G \in \mathcal{B}(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y))$ and $h \in L^2(\mathbb{R}_+, U)$, we see that $Gh \in L^2(\mathbb{R}_+, Y)$. Combining this with (3.15), (3.21), (3.22) and the exponential stability of $\mathbf{T}$, yields that $e_2 \in L^2(\mathbb{R}_+, Y)$. This proves statement (4).

To prove statement (5), we assume that $(Gf)(t) \to 0$ as $t \to 0$ for all $f \in PC(\mathbb{R}_+, U) \cap L^2(\mathbb{R}_+, U)$ with $\lim_{t\to\infty} f(t) = 0$. Then by (3.19), we have

$$\lim_{t\to\infty} (Gh)(t) = 0\,. \tag{3.23}$$

Writing $e = \tilde{e}_1 + \tilde{e}_2$, where

$$\tilde{e}_1 := \frac{1}{\tau} Gh - \frac{1}{\tau}(GJ - \mathbf{G}(0)J)\mathcal{H}(\triangle u) - \frac{1}{\tau}\mathbf{G}(0)h + r - \mathbf{G}(0)\mathcal{H}u\,,$$

and

$$\tilde{e}_2 := -C_\Lambda \mathbf{T}(t)x^0 - [G(\theta u(0)) - \mathbf{G}(0)\theta u(0)]\,, \tag{3.24}$$

it follows from (3.18)-(3.20) and (3.23) that $\lim_{t\to\infty} \tilde{e}_1(t) = 0$, and from (3.21) and (3.22) that $\tilde{e}_2 \in L^2_\alpha(\mathbb{R}_+, Y)$ for all $\alpha > \omega(\mathbf{T})$. This proves the first claim of statement (5). Moreover, assume that there exists $t_0 \geq 0$ such that $\mathbf{T}(t_0)(Ax^0 + BKw^0) \in X$. To prove the second claim of statement (5), it suffices to show that $\lim_{t\to\infty} \tilde{e}_2(t) = 0$. Laplace transform of (3.24) gives

$$(\mathscr{L}(\tilde{e}_2))(s) = -C(sI - A)^{-1}x^0 - \frac{1}{s}[\mathbf{G}(s) - \mathbf{G}(0)]u(0)\,.$$

It follows from (3.1) with $\eta = 0$ that

$$\frac{1}{s}[\mathbf{G}(s) - \mathbf{G}(0)] = C(sI - A)^{-1}A^{-1}B\,,$$

so that

$$(\mathscr{L}(\tilde{e}_2))(s) = -C(sI - A)^{-1}A^{-1}(Ax^0 + BKw^0)\,.$$

Thus, for $t \geq t_0$,

$$\tilde{e}_2(t) = -C_\Lambda \mathbf{T}(t)A^{-1}(Ax^0 + BKw^0) = -CA^{-1}\mathbf{T}(t - t_0)\mathbf{T}(t_0)(Ax^0 + BKw^0)\,.$$

Since $\mathbf{T}(t_0)(Ax^0 + BKw^0) \in X$ and $\mathbf{T}$ is exponentially stable, $\lim_{t\to\infty} \tilde{e}_2(t) = 0$. □

Remark 3.3. (1) Denoting the Lebesgue measure on $\mathbb{R}_+$ by μ_L, statement (4) of Theorem 3.2 implies that, for every $\varepsilon > 0$,

$$\lim_{T\to\infty} \mu_L\big(\{t \geq T : \|e(t)\| \geq \varepsilon\}\big) = 0\,,$$

showing that the error $e(t)$ "converges to 0 in measure" as $t \to \infty$.

(2) If U and Y are finite-dimensional and the impulse response of G is a (matrix-valued) Borel measure on $\mathbb{R}_+$, then G satisfies (3.10). Furthermore, in this case, if $\mathbf{T}(t_0)x^0 \in X_1$ for some $t_0 \geq 0$, then it can be shown that $\lim_{t\to 0} e(t) = 0$.

(3) Assume that (3.8a) is replaced by $u(t) = (\mathcal{H}u)(t) + d$, where $d \in U$ is a constant input disturbance. It follows from Remark 2.2 that the conclusions of Theorem 3.2 remain valid, provided that v^∞ is re-defined by $v^\infty := K(\mathbf{G}(0)K)^{-1}(r - \mathbf{G}(0)d) + d$ and, in statement (5), the condition $\mathbf{T}(t_0)(Ax^0 + BKw^0) \in X$ is replaced by $\mathbf{T}(t_0)(Ax^0 + B(Kw^0 + d)) \in X$.

(4) The proof of statement (4) of Theorem 3.2 is inspired by the proof of Proposition 7.3.4 in [2]. ◇

Example 3.4. For purpose of illustration, we consider the problem of heating a bar of length 1. We keep both endpoints at temperature 0 and inject heat of magnitude $v_j(t)$ at the point $\xi_j \in (0,1)$, $j = 1, 2$. Temperature measurements are taken at the points $\eta_1, \eta_2 \in (0,1)$. The system to be controlled can be formulated as follows

$$\begin{aligned}
z_t(\xi,t) &= \kappa z_{\xi\xi}(\xi,t) + \delta(\xi - \xi_1)v_1(t) + \delta(\xi - \xi_2)v_2(t)\,, \quad \forall \xi \in (0,1)\,,\ \forall t > 0\,, && (3.25a)\\
y_1(t) &= z(\eta_1,t)\,, \quad y_2(t) = z(\eta_2,t)\,; \quad \forall t > 0\,, && (3.25b)\\
z(0,t) &= z(1,t) = 0\,, \quad \forall t \geq 0\,; \quad z(\xi,0) = z^0(\xi)\,, \quad \forall \xi \in (0,1)\,. && (3.25c)
\end{aligned}$$

Here κ is a positive constant. Non-adaptive continuous-time low-gain integral control of this system was studied in [4].

System (3.25) can be formulated as a well-posed system with state space $X = L^2(0,1)$. In particular, the semigroup $\mathbf{T}(t)$, given by

$$(\mathbf{T}(t)z^0)(\xi) = \sum_{n=1}^{\infty} 2\exp(-\kappa n^2\pi^2 t)\sin(n\pi\xi)\int_0^1 \sin(n\pi\lambda)z^0(\lambda)d\lambda\,,$$

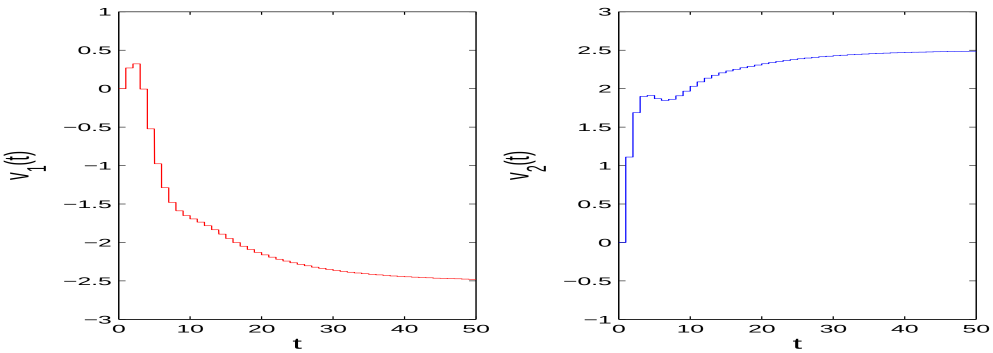

Figure 3.1: Input signals v_1, v_2.

is exponentially stable. Assuming that

$$0 < \xi_1 \le \eta_1 \le \xi_2 \le \eta_2 < 1 \,,$$

the transfer function $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \begin{pmatrix} \dfrac{\sinh((1-\eta_1)\sqrt{s/\kappa})\sinh(\xi_1\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} & \dfrac{\sinh((1-\xi_2)\sqrt{s/\kappa})\sinh(\eta_1\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} \\ \dfrac{\sinh((1-\eta_2)\sqrt{s/\kappa})\sinh(\xi_1\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} & \dfrac{\sinh((1-\eta_2)\sqrt{s/\kappa})\sinh(\xi_2\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} \end{pmatrix} .$$

It is then easy to see that

$$\mathbf{G}(0) = \frac{1}{\kappa}\begin{pmatrix} (1-\eta_1)\xi_1 & (1-\xi_2)\eta_1 \\ (1-\eta_2)\xi_1 & (1-\eta_2)\xi_2 \end{pmatrix} .$$

As a consequence, the characteristic polynomial of $\mathbf{G}(0)$ is given by

$$\det(\lambda I - \mathbf{G}(0)) = \lambda^2 - \kappa^{-1}[(1-\eta_1)\xi_1 + (1-\eta_2)\xi_2]\lambda + \kappa^{-2}\xi_1(1-\eta_2)(\xi_2-\eta_1) \,.$$

Since $\xi_1, \xi_2, \eta_1, \eta_2 \in (0,1)$, it follows that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$ if and only if $\xi_2 > \eta_1$. We sample the output using the simple averaging sampling operation defined by

$$(\mathcal{S}y)(k) = \frac{1}{\tau}\int_0^\tau y(k\tau + t)dt \,, \quad (\text{i.e., } a(t) \equiv 1/\tau) \,.$$

To be specific, we set

$$\xi_1 = 0.2 \,,\ \xi_2 = 0.6 \,,\ \eta_1 = 0.4 \,,\ \eta_2 = 0.8 \,,\ \tau = 1 \,,\ K = I \,,\ \kappa = 0.1 \,,$$

$$z^0(\xi) = \sin(\pi\xi) \,,\ r = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,,\ q = 0.55 \,,\ \gamma^0 = 2 \,,\ w^0 = 0 \,.$$

Matlab simulations of the closed-loop system given by (3.25) and (3.8) (with $v = (v_1, v_2)^T$ and $y = (y_1, y_2)^T$) are shown in Figures 3.1-3.3. By Theorem 3.2, we know that

$$\lim_{t\to\infty} v(t) = (\mathbf{G}(0))^{-1} r = \begin{pmatrix} -2.5 \\ 2.5 \end{pmatrix} ,$$

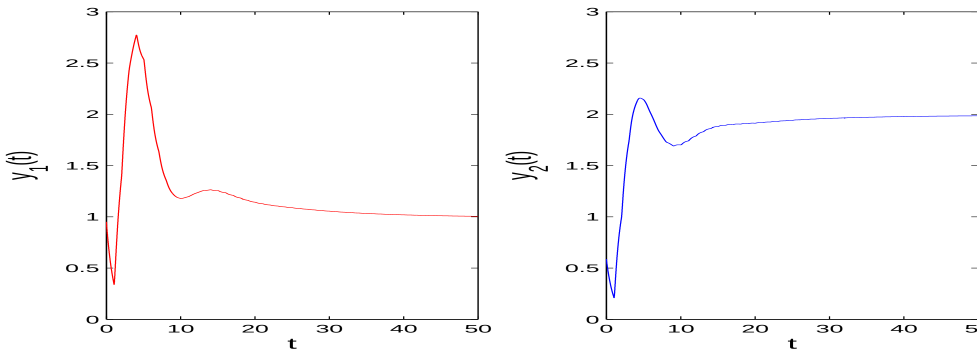

Figure 3.2: Temperature measurements y_1, y_2.

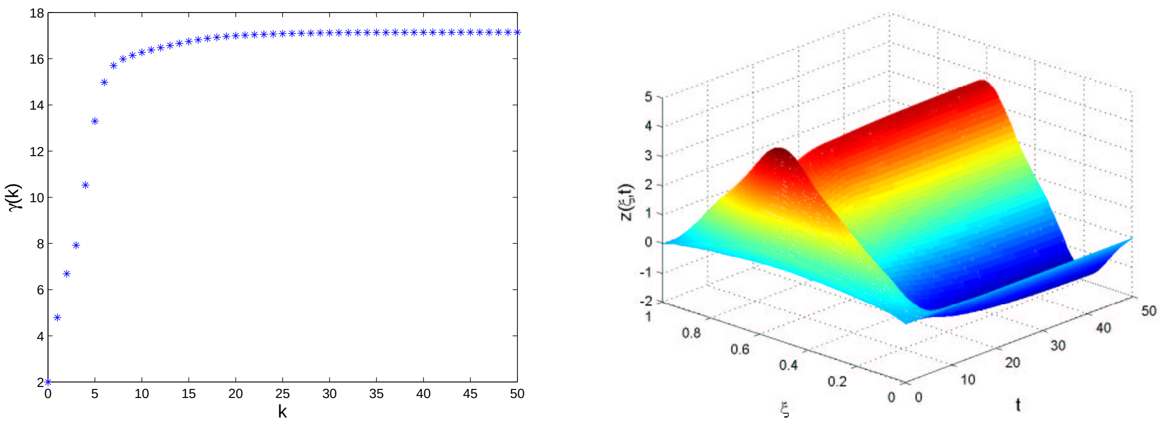

Figure 3.3: Sequence γ and temperature $z(\xi, t)$.

as is illustrated by Figure 3.1. Since $w^0 = 0$ and $Az^0 \in X$, it follows from statement (5) in Theorem 3.2 that

$$\lim_{t \to \infty} y(t) = r = \begin{pmatrix} 1 \\ 2 \end{pmatrix} ,$$

as is illustrated by Figure 3.2. The sequence γ and the evolution of the temperature profile are shown in Figure 3.3.

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