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Optimal LQ feedforward tracking with preview: Practical design for rigid body motion control

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Abstract

For reference-tracking motion control, preview-based linear quadratic (LQ) design methods provide an effective means to balance tracking performance with available actuation capacity. This paper considers a control structure for which the optimal feedforward controller is independent of the feedback controller. In this way, explicit implementation formulas for feedforward controllers are derived that can be applied to a range of rigid-body motion systems. Key aspects of the optimal LQ solutions are identified, particularly how the choice of design weightings affect steady-state error for polynomial tracking. A redesign procedure for finite preview-time is proposed that preserves exact polynomial tracking properties and control bandwidth of the optimal solutions. Comparative experimental results are presented for a motor-driven linear motion stage.

Keywords: optimal control, linear quadratic, tracking control, reference preview, motion control

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1. Introduction

The two-degrees-of-freedom (feedback + feedforward) optimal LQ tracking control problem has been studied in various contexts over the past few decades. Two alternative approaches for calculating solutions have been established, which are

1. Based on variational methods and involve solving matrix Riccati equations - see [1] for continuous-time design or [2] for discrete-time design for finite preview.
2. Based on polynomial formulations and involve solving spectral factorization problems - discrete-time reference preview cases are covered in [3] and [4].

For an arbitrary reference signal, the optimal ‘deterministic’ control involves a feedback component that is the solution to the LQ regulation-only problem combined with a feedforward component that involves a LTI operation on future values of the reference [4]. Implementation thus requires prior knowledge or *preview* of the reference signal.

For servo motion systems, LQ designs provide an effective means to balance tracking performance with available actuation capacity. Nonetheless, application of the feedforward element of the LQ solutions in real-world motion control systems is seemingly still scarce (although application to vehicle active suspension has been considered in a number of research studies, e.g. [5, 6, 7]). A lack of adoption may, in part, be due to the intricacies of the design calculations, which in general require specialized numerical routines. In addition though, there is little published information concerning practical design of optimal feedforward controllers or suitable implementations, which must usually involve some approximation of the ideal solution. A practical approach using a basis-function optimization for non-preview tracking was presented in [8].

This paper considers a control architecture for which the optimal feedforward solution is independent of the feedback controller. Therefore, the solutions derived can be usefully applied with any feedback controller. In contrast to previous studies on optimal inversion [9, 10, 11], the design approach uses a polynomial/spectral factorization formulation, and this reveals key characteristics of the optimal feedforward solution in relation to actual motion performance; namely, control bandwidth and polynomial

tracking properties. Explicit formulas for real-time implementation can also be derived and are tabulated here for a range of SISO system models. The paper also deals with implementation issues and considers suitable approximations for practical use. A redesign procedure for finite preview is proposed that aims to preserve the aforementioned key characteristics. This approach is distinct from previous designs for finite preview where the focus has been on retaining optimality with respect to the defined quadratic cost (see [2] and references therein).

2. Optimal LQ tracking control problem

A standard two-degrees-of-freedom (2-d.o.f) tracking control structure is shown in Fig. 1a, where a feedforward controller H operating on the reference r is combined with a feedback controller K operating on the output y . The design objective is to choose H and K so that the tracking error $e = r - y$ is minimized (in some sense) but also excessive control effort is avoided. Physically, there is a need to avoid problems such as saturation of actuators, mechanical overloading of the system, excessive energy consumption, overheating, etc. The SISO LQ control problem [1, 12, 13] encompasses these objectives, albeit in an indirect manner, by adopting a cost index to be minimized in the form

$$J = \int_0^\infty e(t)^2 + \gamma^2 u(t)^2 dt. \quad (1)$$

The value of the weighting factor γ must be chosen to achieve a suitable balance between error minimization and level of control effort. It will be shown that the value of γ ultimately determines the bandwidth of the optimal feedforward controller.

2.1. Considered control structure

Outside the LQ framework, it is typical for a sequential design to be undertaken where the feedback design is based on an initial feedforward controller, or vice-versa. The former approach is dealt with in [14], while an overview of 2-d.o.f. design architectures for hard-disk drive servos is given in [15]. Here, we focus on the implementation shown in Fig. 1b, since this allows an optimal design of the feedforward controller F

based on the plant model alone. The implementation involves simultaneous computation of a predicted state/output trajectory matching the optimal input and thereby preserves optimality for operation under disturbance-free conditions. This approach was considered previously for combining state feedback with exact inverse models in [9, 16]. For deviation of the measured output from the optimal situation, e.g. due to nonzero disturbance w , nonzero initial conditions or model error, the feedback controller K acts to attenuate error. Here, $\tilde{G}$ is the system model that, ideally, should replicate G . In practice, differences may exist due to imperfect knowledge and nonlinear effects.

2.2. Mathematical notation

LTI signal mappings will be expressed in terms of the derivative operator q , i.e. $\dot{y} = qy$. Accordingly, $G(q)$ is a causal LTI system with impulse response $g(t)$ (with causality implying that $g(t) = 0, t < 0$). Input-output relations may then be expressed

$$y(t) = G(q)u(t) = (g * u)(t) = \int_{-\infty}^{\infty} g(\tau)u(t - \tau)d\tau. \quad (2)$$

Time-reversed anticausal counterparts may also be defined for a signal $x(t)$ according to $\bar{x}(t) = x(-t)$. Then, $\bar{y}$ and $\bar{u}$ are related according to

$$\bar{y} = \bar{G}\bar{u} = \bar{g} * \bar{u}, \quad \bar{G}(q) = G(-q). \quad (3)$$

Thus, the operator $G(-q)$ represents an anticausal LTI system with impulse response $\bar{g}$. These mappings may be expressed in the Laplace domain as

$$Y(s) = G(s)U(s), \quad \bar{Y}(s) = \bar{G}(s)\bar{U}(s). \quad (4)$$

In this notation, $G(s) = \mathcal{L}g(t)$ with $\mathcal{L}$ being the two-sided Laplace operator. In general, $\bar{G}(s) = \mathcal{L}g(-t) = G(-s)$. However, it should be recognized that $x(t) = \mathcal{L}^{-1}X(s)$ can be non-unique unless it is also known whether $x(t)$ is causal or anticausal (thereby specifying the region of convergence for s). To avoid possible confusion, the use of transfer function notation will be reserved for causal systems.

With the defined notation, the autocorrelation function of a signal may be expressed

$$S_{xx}(t) = \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau = (\bar{x} * x)(t). \quad (5)$$

3. Optimal controller solutions

3.1. Continuous-time feedforward filter solutions

Using the notation defined in Section 2.2, the cost index (1) can be written

$$J = S_J(0), \quad S_J(t) = S_{ee}(t) + \gamma^2 S_{uu}(t) = \bar{e} * e + \gamma^2 \bar{u} * u. \quad (6)$$

Assuming the system is initially at rest, $e = r - g * u$ can be substituted to obtain

$$S_J(t) = \bar{u} * (\bar{g} * g + \gamma^2 \delta(t)) * u - \bar{u} * \bar{g} * r - \bar{r} * g * u + \bar{r} * r, \quad (7)$$

where $\delta(t)$ is the (Dirac) unit impulse function. By solving the associated spectral factorization problem (an example of which is given in Appendix A), a causal function $c(t)$ can be obtained for which $C(q)$ has all zeros in the open left-half complex plane and which satisfies

$$\bar{e} * c = \bar{g} * g + \gamma^2 \delta(t) \Leftrightarrow C(-q)C(q) = G(-q)G(q) + \gamma^2. \quad (8)$$

Note that the anticausal counterpart $C(-q)$ has all zeros in the open right-half plane. Following from (7) and (8),

$$S_J(t) = \bar{C}\bar{u} * (Cu) - \bar{u} * (\bar{G}r) - (G\bar{r}) * u + \bar{r} * r. \quad (9)$$

where, for brevity, q is now omitted from the notation. ‘Completing-the-square’ gives

$$S_J(t) = (\bar{C}\bar{u} - C^{-1}G\bar{r}) * (Cu - \bar{C}^{-1}\bar{G}r) - (C^{-1}G\bar{r}) * (\bar{C}^{-1}\bar{G}r) + \bar{r} * r. \quad (10)$$

The cost $J = S_J(0)$ is thus seen to be minimum when $Cu - \bar{C}^{-1}\bar{G}r = 0$, or

$$u^* = C^{-1}\bar{C}^{-1}\bar{G}r. \quad (11)$$

The solution (11) defines only the feedforward component of the optimal control: accounting for nonzero initial conditions in (6) provides also the optimal feedback (regulator) part of the solution, which is not used here. For further context, the polynomial formulation of the discrete-time LQ regulation problem, and its relation to the state-space Ricatti equation approach, is covered in [12]. The LQ control problem was extended for preview-based tracking, and a 2-d.o.f. controller solution derived, in [4, 13]. In the 2-d.o.f. case, the ‘stable’ spectral factor corresponds to the optimally-assigned closed loop dynamics, while the ‘anti-stable’ factor appears (as an anticausal operation) in the feedforward control. The solution (11) is the feedforward element appearing in the full LQ solution. It also mirrors the frequency-domain feedforward solution, referred to as an optimal inverse, in [9]. In what follows, the polynomial solution (11) is examined further with regard to realization aspects and achievable motion performance for a selection of rigid-body system types.

The optimal controller (11) involves a noncausal filter

$$F := C^{-1}\overline{C}^{-1}\overline{G} \quad (12)$$

This filter solution is independent of the characteristics of r but requires knowledge of future values. It is worth remarking here that the same filter properties are not obtained when the tracking control problem is formulated as a LQ disturbance rejection/regulation problem [1] but, without a model for the reference behaviour, are essential for optimality. The main issue now is one of implementation as although C^{-1} can be realized as a stable causal IIR filter, the operation $\overline{C}^{-1}\overline{G}$ cannot. Defining

$$D := GC^{-1}$$

and noting that $d(t)$ is causal and stable ($d(t) \rightarrow 0$ as $t \rightarrow \infty$), (11) may be realized as

$$u = C^{-1}v, \quad v = \overline{d} * r, \quad (13)$$

where $\overline{d} * r$, is a ‘look-ahead’ prefiltering that can be computed directly as

$$v(t) = (\overline{d} * r)(t) = \int_0^\infty d(\tau)r(t + \tau)d\tau. \quad (14)$$

Table 1: Continuous-time LQ feedforward filter solutions for rigid-body motion control, to be implemented according to equations (13) and (14)

CASE	1	2*	3	4*
System model $G(s)$	$\frac{1}{Ms}$	$\frac{1}{M(s+b)}$	$\frac{1}{Ms^2}$	$\frac{1}{Ms(s+b)}$
Example situation	velocity control (no friction)	velocity control (viscous friction)	position control (no friction)	position control (viscous friction)
Filter parameter a (corner frequency)	$1/\sqrt{\gamma M}$	$1/\sqrt{\gamma M}$	$1/\sqrt{\gamma M}$	$1/\sqrt{\gamma M}$
Control weighting function $W(s)$	1	$\frac{s}{s+b}$	1	$\frac{s}{s+b}$
IIR filter component $C^{-1}(s)$	$\frac{aMs}{s+a}$	$\frac{aM(s+b)}{s+a}$	$\frac{a^2Ms^2}{s^2+\sqrt{2}as+a^2}$	$\frac{a^2Ms(s+b)}{s^2+\sqrt{2}as+a^2}$
Look-ahead filter function $d(t)$	ae^{-at}		$ae^{-\frac{a}{\sqrt{2}}t} \sin \frac{a}{\sqrt{2}}t$	
Overall filter frequency response $F(j\omega)$	$\frac{Ma^2(j\omega+b)}{\omega^2+a^2}$		$\frac{-Ma^4\omega(\omega-jb)}{\omega^4+a^4}$	
Overall filter impulse response $f(t)$	$\frac{Ma}{2}(b - \text{sign}(t)a)e^{-a t }$		$\frac{Ma^2}{2\sqrt{2}}e^{-\frac{a}{\sqrt{2}} t } \left(a \sin \frac{a}{\sqrt{2}} t - \sqrt{2}b \sin \frac{a}{\sqrt{2}}t - a \cos \frac{a}{\sqrt{2}}t \right)$	
Zero steady-state error for $r(t) = t^p$	$p \leq 1$		$p \leq 3$	

*cost function includes control weighting function to ensure zero steady-state tracking error

For real-time calculation with an arbitrary reference signal r , this integral must be approximated numerically over a finite interval (preview window). The effect of discretization (finite sampling time) can be accounted for via the discrete-time formulations presented in Section 4. The issue of how to minimize the effect of a truncated preview window is covered in Section 6. In examining the basic qualities of the control solutions, we will first focus on the continuous-time solutions for the four basic models considered in Table 1. These are typical linearized models of mechanical servo systems encountered in machine automation and robotics when general position- or speed-controlled motion is required. A derivation of the formulas for Case 4 is provided as an example in Appendix A. Although the polynomial formulation can be applied to higher order (and MIMO) systems, solving the spectral factorization problem will then, in general, require numerical computation.

3.2. Filter properties

The filter solutions given in Table 1 all involve a zero-phase, and therefore noncausal, low-pass filter $D\bar{D}$ (having corner frequency a) in series with the plant inverse dynamics:

$$F = C^{-1}\bar{D} = \tilde{G}^{-1}D\bar{D}. \quad (15)$$

These solutions can be implemented as shown in Fig. 2, where the feedforward filter F generates the optimal control input u^* and the zero-phase filter $D\bar{D}$ generates the corresponding optimal output y^* . A key property for achieving optimal performance is that the noncausal filtering allows an attenuation of high frequency components without introducing error due to phase lag. This is unattainable by causal LTI filtering. Note that the implementation involves $C^{-1} = D\tilde{G}^{-1}$, which is stable by construction. Therefore, for nonminimum-phase systems, the unstable poles of $\tilde{G}^{-1}$ are canceled by the zeros of D . The feedback controller acts on the error $y^* - y$, which remains zero if the plant is modelled perfectly ($\tilde{G} = G$) and no other disturbances are present.

Although the filter corner frequency a is determined by the value of the weighting factor γ , knowing the form of the solutions, it is perhaps more reasonable for a designer to select the value of a directly to match the desired control bandwidth.

3.3. Polynomial reference tracking behaviour

As tracking behaviour for polynomial reference signals is a key aspect of control performance, this will be considered further. Polynomial tracking properties can be influenced by modifying the cost function (6) to include a filtered version of the control input. Accordingly, u is replaced in equation (6) by $\tilde{u}$, for which $\tilde{U}(s) = W(s)U(s)$. If $W(s)$ is chosen as a high-pass filter, this will prevent the penalization of low frequency control input components that are important for eliminating tracking error. In this case, the controller solutions involve $C(q)$ satisfying

$$C(-q)C(q) = G(-q)G(q) + \gamma^2 W(-q)W(q). \quad (16)$$

For the mapping $e = \mathcal{S}r$, the sensitivity function $\mathcal{S}$ can be expressed by causal-anticausal operations as $\mathcal{S} = 1 - GF = 1 - D\bar{D}$. Then, from (16),

$$\mathcal{S} = \gamma^2 DG^{-1}W\overline{WG^{-1}\bar{D}}. \quad (17)$$

Therefore, $\mathcal{S}$ will have m zero-valued zeros with multiplicity given by

$$m = 2n + 2l \quad (18)$$

where n is the number of zero-valued plant poles (appearing as zeros of $G^{-1}(s)$) and l is the number of zero-valued zeros of $W(s)$. As $\mathcal{S}$ involves m derivative operations on r , zero steady-state error is achieved for a polynomial reference $r(t) = t^p$ if $p < m$.

For Cases 2 and 4 in Table 1, a high-pass filter $W(s) = s/(s + b)$ has been included in the design so that $l = 1$. This weighting only has a significant effect on the filter frequency response $F(j\omega)$ at frequencies lower than the value of b , and thus has a negligible effect on the filter bandwidth if $b \ll a$ (see Fig. 3a). The filter impulse response function $f(t)$ for each case can be seen in Fig. 3b. For Case 1, the function is antisymmetric, while for Case 3 it is symmetric. The effect of a finite plant pole (Cases 2 and 4) is to introduce some asymmetry in the function. The sensitivity function for the cases in Table 1 is

$$\mathcal{S}(q) = D(q)\bar{D}(q) \left(\frac{q}{a}\right)^{\frac{m}{2}} \left(-\frac{q}{a}\right)^{\frac{m}{2}} = \frac{q^m}{q^m + (-1)^{\frac{m}{2}} a^m}, \quad (19)$$

where $m = 2$ and 4 for the first and second order models respectively.

Example tracking behaviour, obtained by numerical simulation for $G(s) = \frac{10}{s^2 + 10s}$, is shown in Fig. 4. Here, $r(t)$ is chosen as a parabolic interval with stationary intervals either side. Discontinuities in $\dot{r}(t)$ between these intervals introduce localized error in $e(t)$. This error diminishes in magnitude and duration with increasing value of a , and with a corresponding increase in peak control effort. During the parabolic interval, the tracking error converges to zero irrespective of the value of a .

Table 2: Discrete-time LQ feedforward filter solutions (for sample period T)

CASE	1	2*	3	4*
System model $G(z)$	$\frac{T}{M(z-1)}$	$\frac{(1-A)}{Mb(z-A)}$ $A = e^{-bT}$	$\frac{T^2(z+1)}{2M(z-1)^2}$	$\frac{k(z+B)}{Mb^2(z-1)(z-A)}$ $A = e^{-bT}, \quad k = bT - 1 + A$ $B = (1 - A(1 + bT))/k$
Filter parameters (for corner frequency a)	$r = e^{-aT}$		$\theta = a/\sqrt{2}$	
			$P = 2 - \cos \theta T$	$P = -\frac{(1+B^2)}{2B} \cos \theta T$ $+ \frac{1+B}{2B} \sqrt{(1-B)^2 \cos^2 \theta T + 4B}$
			$R = P - \sqrt{P^2 - 1}$	
Weighting factor γ	$\frac{T\sqrt{r}}{M(1-r)}$	$\frac{2(1-A)\sqrt{r}}{Mb(1+A)(1-r)}$	$\frac{T^2}{4M(1-\cos \theta T)}$	$\frac{2RT(1-A)}{Mb(1+A)(1-2\cos \theta T R + R^2)}$
IIR filter component $C^{-1}(z)$	$\frac{M(1-r)(z-1)}{T(z-r)}$	$\frac{Mb(1-r)(z-A)}{(1-A)(z-r)}$	$\frac{4MR(1-\cos \theta T)(z-1)^2}{T^2(z^2 - 2\cos \theta T Rz + R^2)}$	$\frac{Mb(1-2\cos \theta T R + R^2)(z-A)(z-1)}{T(1-A)(z^2 - 2\cos \theta T Rz + R^2)}$
$D(z)$	$\frac{1-r}{z-r}$		$\frac{2R(1-\cos \theta T)(z+1)}{(z^2 - 2\cos \theta T Rz + R^2)}$	$\frac{(1-2\cos \theta T R + R^2)(z+B)}{(1+B)(z^2 - 2\cos \theta T Rz + R^2)}$
Look-ahead filter coefficients d_n for $\bar{D}(z) = \sum_{n=1}^{\infty} d_n z^n$	$\frac{1-r}{r} r^n$		$\frac{2(1-\cos \theta T)}{\sin \theta T} R^{n-1} \times$ $(R \sin n\theta T + \sin((n-1)\theta T))$	$\frac{(1-2\cos \theta T R + R^2)}{(1+B)R \sin \theta T} R^{n-1} \times$ $(R \sin n\theta T + B \sin((n-1)\theta T))$

4. Discrete-time realizations

Derivation of an optimal discrete-time solution can follow a similar procedure to that in Section 3. In this case, signals are expressed in terms of the forward shift operator z so that $G(z)$ may be treated as a causal power series in z^{-1} while $\bar{G}(z) = G(z^{-1})$ is the anticausal counterpart. The main design formulas are given in Table 2. The considered plant models are all zero-order-hold equivalents of those given in Table 1. For these cases, the solutions involve $C(z)$ having all zeros inside the unit circle and satisfying the equation $C(z^{-1})C(z) = G(z^{-1})G(z) + \gamma^2 W(z^{-1})W(z)$. For Cases 3 and 4, the given formulas are based on $C(z)$ having complex conjugate zeros and are therefore valid only for $T \leq \sqrt{2}\pi/a$, where T is the sampling period. To simplify the formulas, some intermediate filter parameters are defined. For all cases, the filter corner frequency a should be considered as the specified design parameter. Formulas for calculation of the remaining filter parameters (and corresponding weighting factor γ) are given. The

feedforward filter is to be implemented according to

$$v_k = \sum_{n=1}^{\infty} d_n r((n+k)T), \quad (20)$$

$$u = C^{-1}(z)v. \quad (21)$$

Use of these discrete-time solutions is more desirable for longer sampling periods, but is also important when the effects of truncating the look-ahead filter calculation (finite preview window) are to be accounted for explicitly in the design.

Computational load associated with real-time calculation of the control signal will impose an upper limit on the length of preview window. For some applications, however, the length of window may be dictated by a finite preview time for the reference signal, or the desire to minimize time-lag between reference construction and system operation. The simplest ways to modify the controller solution for finite preview is to truncate the series expansion for $\overline{D}(z)$ to a finite number of terms. Although this will lead to suboptimality with regard to the original cost index, there are other aspects to the resulting performance changes that may in practice be more important. One aspect, evident in Fig. 5a, is the increased high frequency gain of the feedforward filter that occurs with shorter preview times. Errors are also evident at low frequencies. These have a detrimental effect on the low frequency roll-off of the sensitivity function, as seen in Fig. 5b. The exact effects of truncation are clearly quite complex and although a rule-of-thumb may be given that the duration of the preview window be chosen $\gtrsim 5/a$, generally speaking, a numerical investigation would be appropriate within a design procedure.

The degradation of low frequency roll-off seen in Fig. 5b occurs because a simple truncation of $\overline{D}(z)$ does not preserve the unity-valued zeros of $\mathcal{S}(z)$. As explained in Section 3.3 for the continuous-time case, this will have important consequences for polynomial tracking performance. To improve behaviour, the filter coefficients can be rescaled in order to recover unity steady-state gain for $\overline{D}(z)$. However, even with this corrective measure, the tracking performance will be degraded. Figure 6 shows time-step simulation results for different preview times. Although the control input signals

are barely distinguishable, reduced preview leads to increased tracking error during the parabolic reference interval. A redesign method that overcomes this problem is described in Section 6.

5. Experimental case study

This section describes experimental tests on a linear motion stage comprised of load-mass mounted on a linear guide and driven by a brushed d.c. motor through a ball screw (Fig. 7). The motor is driven by a PWM electrical drive with internal current-control circuitry. An incremental encoder fitted to the motor is used for position measurement. The system typifies an industrial motion control set-up, as utilized in manufacturing automation and general machine positioning stages. Further specifications are given in Table 3. A linearized model of the system, obtained by time-response identification procedures, is

$$G(s) = \frac{2.04}{s^2 + 1.6s}. \quad (22)$$

For this model, the control input is the motor current in amps while the output is the linear displacement of the mass in meters. All designs are based on a target tracking control bandwidth of 40 rad/s. The following PD feedback controller is also used, specified to give similar closed loop bandwidth:

$$K(s) = \frac{30s + 1200}{0.0025s + 1}. \quad (23)$$

All implementations are based on a sample period of $T = 0.001$ seconds, with discrete-time transfer functions calculated as zero-order-hold equivalents.

5.1. Achieving near-optimal performance

In theory, achieving optimal performance requires complete preview of the reference signal. In practice, however, increasing preview beyond a certain time (typically between $10/a$ and $20/a$) will not give perceivable benefits. The main limiting factor then becomes non-ideal system behaviour (nonlinear effects, time-varying dynamics, limited signal/sensor resolution, etc.). For the experimental system, the main issue

Table 3: Key parameters for experimental set-up

parameter	value	units
ball screw pitch	0.004	m
ball screw length	0.45	m
motor torque constant	60.3×10^{-3}	Nm/A
total inertia (at motor)	18.6×10^{-6}	kgm ²
encoder resolution	3.9×10^{-6}	m

was friction from the linear guide and ball screw. Compensation for nonlinear friction was implemented as an additional feedforward component $u_{fc} = k_{fc} \text{sign}(\dot{y}^*)$, where y^* is the filtered reference signal used for feedback. The implementation follows the general form shown in Fig. 8 with $P_1 = P_2 = D\bar{D}$. The total control input is given by $u = u^* + u_{fb} + u_{fc}$ where u_{fb} is output from the PD controller (23) operating on $e_{fb} = y^* - y$. Experimental results for an optimal controller are shown in Fig. 9. The motion profile here involves a parabolic segment (similar to the simulation cases). A preview time of 1 second ($40/a$) was used. Small deviation from optimal conditions is indicated by the plots of $y - y^*$ and $u - u^*$. Steady-state tracking error is within ± 50 microns for the parabolic interval. The additional non-optimal components are clearly evident in the control signal, but remain small compared with peak values of u^* .

5.2. Comparison with non-optimal feedforward controllers

Performance of the optimal LQ controller can be compared with standard causal feedforward techniques based on plant inversion combined with low-pass filtering, as shown in Table 4. To allow valid comparisons, causal feedforward filters are considered having bandwidths that match the optimal controller. Controllers C3 and C4 are causal approximations of the optimal controller using techniques previously considered for approximating unstable inverse dynamics of nonminimum-phase systems (as covered in [18]). It should be recognised, however, that noncausality is an intrinsic property of the optimal controller and does not arise from unstable inverse dynamics of the system.

Results for the parabolic motion profile are shown in Fig. 10a. Controller C2 uses an unfiltered reference for error feedback ($P_1 = 1$) and hence gives the best tracking performance of the nonoptimal designs. However, an unwanted consequence of this is that large control input values occur during the initial acceleration and final deceleration phases of motion (saturating at an imposed limit of 3 Amps). The optimal controller C1 achieves lower levels of both tracking error and motor current. Steady-state tracking error is slightly better for controller C1 but does not exceed ± 50 microns in both cases. Controllers C3 and C4, which use a filtered reference for feedback, produce a control input signal similar in form to the optimal case but the phase lag associated with causal filtering results in large tracking error during the parabolic motion interval.

For smooth reference input signals, it might be expected that the optimal LQ controller would have less benefit compared with controller C2 because the reference is then less likely to cause input saturation problems. Figure 10b shows results for a cubic polynomial reference producing a rapid rest-to-rest translation. In this case, controller C2 does not produce saturation of the motor current. However, the optimal LQ controller still gives noticeable improvements in terms of reducing peak values of both tracking error and control input. It also gives lower steady-state tracking error.

Although an integral term is not included in the feedback control for these experiments, further improvements in polynomial-tracking performance should not be expected unless the feedback control acts directly on $r - y$. This is the case for controller C2 only. Even then, error convergence will depend on the closed loop dynamics and stability issues may limit achievable performance. In short, the LQ approach described here allows reference signals to be pre-filtered while ensuring good polynomial-tracking capabilities with arbitrary feedback controllers.

6. FIR redesign for polynomial tracking

In order to preserve polynomial tracking properties with finite preview, the unity-valued zeros of the discrete-time sensitivity function $\mathcal{S}$ may be divided out prior to

Table 4: Feedforward controller designs used for performance comparisons (see Fig. 8)

	Controller	P_1	P_2
C1	Optimal LQ control	$D(z)D(z^{-1})$	$D(z)D(z^{-1})$
C2	Filtered inverse	1	$D(z)$
C3	Zero-amplitude-error causal approximation	$D(z)^2z$	$D(z)^2z$
C4	Unstable-pole-ignore causal approximation	$D(z)(z^{-1} + B)/(1 + B)$	$D(z)(z^{-1} + B)/(1 + B)$

truncation and then reintroduced afterwards. To do this, the factorization (17) is first considered:

$$\mathcal{S}(z) = 1 - D(z)\overline{D}(z) = \gamma^2 C^{-1}(z)W(z)\overline{W}(z)\overline{C}^{-1}(z). \quad (24)$$

Defining the causal (low-pass) transfer function and corresponding series expansion

$$Q(z) = \gamma C^{-1}(z)W(z)(z - 1)^{-\frac{m}{2}} = \sum_{n=0}^{\infty} q_n z^{-n}, \quad (25)$$

we may write

$$D(z)\overline{D}(z) = 1 - Q(z)\overline{Q}(z)(z - 1)^{\frac{m}{2}}(z^{-1} - 1)^{\frac{m}{2}}. \quad (26)$$

A truncated series for $Q(z)$ can be computed from (25) as

$$Q_R(z) = \sum_{n=0}^R q_n z^{-n}. \quad (27)$$

A P^{th} order FIR approximation $D_P(z)$ then follows by solving the spectral factorization problem for

$$D_P(z^{-1})D_P(z) = 1 - Q_{P-\frac{m}{2}}(z)Q_{P-\frac{m}{2}}(z^{-1})(z - 1)^{\frac{m}{2}}(z^{-1} - 1)^{\frac{m}{2}}. \quad (28)$$

Rather than calculating $D_P(z)$ explicitly, however, the feedforward controller can be implemented directly as the zero-phase FIR filter $\overline{D}_P(z)D_P(z)$ in series with the (stable) plant inverse:

$$F_P(z) = \tilde{G}^{-1}(z)D_P(z)\overline{D}_P(z). \quad (29)$$

This filter solution will preserve the m unity-valued zeros, as well as the zero-phase properties, of $\mathcal{S}$. An issue for realization, however, is that the poles of $\tilde{G}^{-1}(z)$ may

no longer be canceled by the zeros of $D_P(z)$. Thus, for nonminimum-phase systems it would be necessary to ensure that any unstable plant zeros are also preserved as zeros of $D_P(z)$ so that an exact cancelation can be assumed in the realization of $\tilde{G}^{-1}(z)D_P(z)$. The situation does not apply to the present cases and so is not considered further here.

A standard technique in FIR filter design is to use a windowing function to smooth discontinuities that occur when an IIR series is truncated [17]. The main effect from smoothing is to improve high frequency attenuation properties compared with an abrupt truncation (square windowing function). This effect becomes more pronounced as the truncated series is made shorter, i.e. for lower values of P . For the redesign method suggested here, a one-sided windowing function can be applied to the series $Q(z)$ when formulating the zero-phase FIR filter :

$$Q_R(z) = \sum_{n=0}^R h(n)q_n z^{-n}. \quad (30)$$

Use of a smooth windowing function $h(n)$ is important here owing to the derivative action associated with reintroducing unity-valued zeros.

6.1. Experimental results

The redesign procedure was undertaken for the experimental system, as prescribed by equations (25)- (30). A preview time of 0.15 seconds was selected, with corresponding filter order $P = 150$. A one-sided 33% Tukey window was applied to achieve a smooth truncation of $Q(z)$ (see [17]). Experimental results shown in Fig. 11 confirm the effectiveness of the redesign in preserving the polynomial tracking performance of the optimal solution. The redesigned filter gives improved steady-state error for the parabolic interval compared with the direct truncation of $D\bar{D}$ (even with rescaling to preserve unity steady-state gain). Control input signals for the two cases are very similar, although the redesign does result in increased high frequency gain, which is evident as small additional spikes in the control signal.

7. Conclusions

This paper has presented explicit formulas for implementation of optimal LQ feed-forward controllers which, it is hoped, will provide a useful reference for practitioners and researchers involved in motion control applications. Furthermore, a number of important features of the controllers have been identified and discussed. These are summarized as follows:

1. The optimal output signal y^* can be generated in real-time by a zero-phase low-pass filter operating on the target reference.
2. Reference preview makes the zero-phase filtering realizable using FIR approximations.
3. Additional control action based on feedback of the error $y^* - y$ improves robustness to model error and unknown disturbances.
4. Polynomial tracking properties, which are dependent on the choice of design weightings, can be preserved in finite preview implementations through a suitable redesign procedure.
5. If the control/filter bandwidth is fixed a priori then model parameters appear simply in the controller formulation in the form of an inverse dynamic model. This approach could be applied (suboptimally) to nonlinear/time-varying systems when an inverse dynamic model is available, e.g. for robotic systems having configuration-dependent mass parameters, or similar.

These features can also be replicated in non-optimal design and point, more generally, to useful practice in the design of feedforward controllers for output reference tracking.

Appendix A. Example solution procedure for Case 4

Considering the cost function (6) with weighted control

$$\tilde{U}(s) = W(s)U(s), \quad W(s) = \frac{s}{s + b},$$

then, from (16) with $G(s) = \frac{1}{Ms(s+b)}$,

$$C(-s)C(s) = \frac{1 + \gamma^2 M^2 s^4}{M^2 s^2 (s^2 - b^2)} = \frac{s^4 + a^4}{M^2 a^4 s^2 (s^2 - b^2)},$$

where $a = 1/\sqrt{\gamma M}$. The roots of $s^4 + a^4$ can be calculated in order to separate LHP and RHP zeros:

$$C(s) = \frac{s^2 + \sqrt{2}as + a^2}{Ma^2 s(s+b)}, \quad \overline{C}(s) = \frac{s^2 - \sqrt{2}as + a^2}{Ma^2 s(s-b)}$$

From (11), the optimal feedforward filter is $F(s) = C^{-1}(s)\overline{D}(s)$, where

$$C^{-1}(s) = \frac{Ma^2 s(s+b)}{s^2 + \sqrt{2}as + a^2}, \quad D(s) = C(s)G^{-1}(s) = \frac{a^2}{s^2 + \sqrt{2}as + a^2}.$$

The look-ahead filter function is given by

$$d(t) = \mathcal{L}^{-1}D(s) = \sqrt{2}a e^{-\frac{a}{\sqrt{2}}t} \sin \frac{a}{\sqrt{2}}t.$$

The overall filter impulse response can be calculated as $f = x * \overline{d}$ where $x(t) = \mathcal{L}^{-1}C^{-1}(s)$.

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Figure 1: Tracking control implementations: a) Standard 2-d.o.f. LQ control structure, b) Closed-loop structure where feedforward filter F is designed for open-loop system (based on system model $\tilde{G}$)

Figure 2: Implementation of optimal feedforward controller where the zero-phase filter $D\bar{D}$ is combined with the inverse system model $\tilde{G}^{-1}$. Structure is equivalent to Fig. 1b with $F = \tilde{G}^{-1}D\bar{D}$.

(a)

(b)

Figure 3: Feedforward filter characteristics for Cases 1 to 4 in Table 1: (a) Frequency response magnitude $|F(j\omega)|$ (b) Impulse response $f(t)$ (with $b = a/10$ for cases 2 and 4)

Figure 4: Tracking performance for Case 4 with $G(s) = \frac{10}{s^2 + 10s}$. The reference has parabolic interval $r(t) = -t^2 + 5t - 5.25$. Simulation results are for controller designs with different values of γ giving filter corner frequencies: (A) $a = 10$ rad/s, (B) $a = 15$ rad/s, (C) $a = 30$ rad/s

Figure 5: Effect of truncated preview window on frequency response characteristics when using simple truncation of filter coefficients for $\bar{D}(z)$ (Case 4 with $b = a/10$): (a) Feedforward filter F (b) Sensitivity function S

Figure 6: Effect of truncated preview window on tracking error when using simple truncation of filter coefficients for $\overline{D}(z)$. Design with $a = 40$ rad/s. Reference preview times are: $4/a = 0.1$ sec, $6/a = 0.15$ sec and $8/a = 0.2$ sec.

Figure 7: Photograph of experimental set-up.

Figure 8: General tracking control implementation where prefiltering of the reference is combined with a feedforward controller incorporating a model of the inverse system $\tilde{G}^{-1}$. Feedforward compensation for nonlinear friction effects is also included.

Figure 9: Experimental tracking performance using optimal feedforward filter design with $a = 40$ rad/s. Preview time is 1 second: look-ahead filter $\bar{D}(z)$ truncated to 1000 terms.

Figure 10: Experimental comparison of tracking performance for feedforward controllers in Table 4. Controller C1 is the optimal LQ controller: (a) reference has parabolic interval, (b) reference has cubic polynomial interval

Figure 11: Experimental results for finite preview time $6/a = 0.15$ sec when using (A) simple truncation of filter coefficients for $D(z)\bar{D}(z)$, (B) FIR redesign of $D(z)\bar{D}(z)$ for zero steady-state tracking error.