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A geometric realisation of 0-Schur and 0-Hecke algebras

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ABSTRACT

We define a new product on orbits of pairs of flags in a vector space over a field k, using open orbits in certain varieties of pairs of flags. This new product defines an associative $\mathbb{Z}$-algebra, denoted by $G(n, r)$. We show that $G(n, r)$ is a geometric realisation of the 0-Schur algebra $S_0(n, r)$ over $\mathbb{Z}$, which is the q-Schur algebra $S_q(n, r)$ at $q = 0$. A pair of flags naturally determines a pair of projective resolutions for a quiver of type $\mathbb{A}$ with linear orientation, and we study q-Schur algebras from this point of view. This allows us to understand the relation between q-Schur algebras and Hall algebras and to construct bases of q-Schur algebras. Using the geometric realisation, we construct idempotents and multiplicative bases for 0-Schur algebras. We also give a geometric realisation of 0-Hecke algebras and a presentation of the q-Schur algebra over a ground ring, where q is not invertible.

0. Introduction

Let k be a finite or an algebraically closed field and $\mathcal{F}$ the variety of partial n-step flags in an r-dimensional vector space V over k. Denote by $|k|$ the cardinality of k. In [1], using the double flag variety $\mathcal{F}\times\mathcal{F}$, Beilinson, Lusztig and MacPherson gave a geometric construction of some finite dimensional quotients of the quantised enveloping algebra $U_q(gl_n)$. In [11], Du remarked that the quotients are isomorphic to the q-Schur algebras defined by Dipper and James in [7]. So in this paper the q-Schur algebras $S_q(n, r)$ are defined as the quotients constructed in [1], which we recall below.

Note that the natural $\mathrm{GL}(V)$-action on V induces a $\mathrm{GL}(V)$-action on the flag variety $\mathcal{F}$ and a diagonal $\mathrm{GL}(V)$-action on the double flag variety $\mathcal{F}\times\mathcal{F}$. Denote by $[f, g]$ the $\mathrm{GL}(V)$-orbit of $(f, g) \in \mathcal{F}\times\mathcal{F}$ and by $\mathcal{F}\times\mathcal{F}/\mathrm{GL}(V)$ the set of $\mathrm{GL}(V)$-orbits on $\mathcal{F}\times\mathcal{F}$. Let Δ and π be the maps

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$$\begin{array}{ccc} \mathcal{F}\times\mathcal{F}\times\mathcal{F} & \xrightarrow{\Delta} & (\mathcal{F}\times\mathcal{F})\times(\mathcal{F}\times\mathcal{F}) \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{F}\times\mathcal{F} & & \end{array}$$

given by $\Delta(f, f', f'') = ((f, f'), (f', f''))$ and $\pi(f, f', f'') = (f, f'')$.

Let $\mathbb{Z}[q]$ be the ring of polynomials in q over $\mathbb{Z}$, where q is an indeterminate. The q-Schur algebra $S_q(n, r)$ is a free $\mathbb{Z}[q]$-module with basis $\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$ and with multiplication

$$[f, g][h, l] = \sum_{[f',l']} F_{f,g,h,l,f',l'} [f', l'],$$

where $F_{f,g,h,l,f',l'}$ is the polynomial in $\mathbb{Z}[q]$ such that $F_{f,g,h,l,f',l'}(|k|)$ is the cardinality of the set

$$\pi^{-1}(f', l') \cap \Delta^{-1}([f, g] \times [h, l])$$

for any finite field k. The multiplication implies in particular that if g and h are not in the same $\mathrm{GL}(V)$-orbit, then $[f, g][h, l] = 0$.

The main goal of this paper is to give a geometric realisation of the 0-Schur algebra $S_0(n, r)$, which is the q-Schur algebra $S_q(n, r)$ at $q = 0$. We define a new $\mathbb{Z}$-algebra $G(n, r)$ with basis $\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$ by defining the product of $[f, f']$ and $[f', f'']$ to be the unique open orbit (see Section 6) in

$$\pi\Delta^{-1}([f, f'] \times [f', f'']).$$

The definition of the new product is similar to the one defined by Reineke [20] for Hall algebras and the following main result generalises Theorem 2.3 in [23].

Theorem 1. *As $\mathbb{Z}$-algebras, $G(n, r)$ is isomorphic to $S_0(n, r)$.*

Remark 2. Throughout, $S_q(n, r)$ is a $\mathbb{Z}[q]$-algebra, $G(n, r)$ and $S_0(n, r)$ are $\mathbb{Z}$-algebras and $S_v(n, r) = S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v)$ is a $\mathbb{Q}(v)$-algebra, where $v^2 = q$. In these cases, we don't always emphasise the ground rings $\mathbb{Z}[q]$, $\mathbb{Z}$ and $\mathbb{Q}(v)$. We will occasionally consider q-Schur algebras over other ground rings, which will then be specified. For instance, if R is a commutative $\mathbb{Z}[q]$-algebra, then by applying the functor $- \otimes_{\mathbb{Z}[q]} R$ we obtain a q-Schur algebra

$$S_q(n, r) \otimes_{\mathbb{Z}[q]} R$$

with the ground ring R. We also say that this q-Schur algebra is an R-algebra to emphasise the ground ring R.

We intend to understand the algebras from the viewpoint of representation theory of quivers. Where it is possible, we give explanations using representations. In particular, we view a flag as a projective representation of the linear quiver of type $\mathbb{A}_n$. In this way, a pair of flags (f, g) naturally determines a pair of projective resolutions $f \cap g \subseteq f$ and $f \cap g \subseteq g$. We will show that a pair of flags and its corresponding pair of projective resolutions uniquely determine each other and give a criterion, using representations, for when two pairs of flags are in the same orbit. Applying the criterion, we construct two new bases for $S_q(n, r)$,

$$\{[f, f + g][f + g, f] \mid [f, g] \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)\}$$

and

$$\{[f, f \cap g][f \cap g, f] \mid [f, g] \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)\}.$$

Using the new bases and the main result, we give new presentations of q-Schur algebras over a commutative $\mathbb{Z}[q]$-algebra $\mathcal{Q}$, where q is not invertible. Using open orbits, we construct families of idempotents and an ideal $M(n,r) \subseteq G(n,r)$, which splits off as a direct factor of the algebra $G(n,r)$. In the case $n = r$, we obtain a geometric realisation of the 0-Hecke algebra $\mathcal{H}_0(n)$.

The remainder of the paper is organised as follows. In Section 1, we explain the construction of $S_q(n,r)$ by Beilinson, Lusztig and MacPherson in more detail. In particular, we recall the description of $\mathrm{GL}(V)$-orbits in $\mathcal{F} \times \mathcal{F}$ using matrices and the fundamental multiplication rules. In Section 2, we give a new description of the $\mathrm{GL}(V)$-orbits in $\mathcal{F} \times \mathcal{F}$ using representations of linear quivers of type $\mathbb{A}_n$. In Section 3, we recall the definition of the positive and negative parts of the q-Schur algebras and their relationship to the Hall algebras. In Section 4, we construct new bases of $S_q(n,r)$. In Section 5, we describe q-Schur algebras over $\mathcal{Q}$ using quivers and relations and obtain presentations of the algebras, modified from the presentations given in [9] by Doty and Giaquinto. We define the generic algebra in Section 6 and show that it is isomorphic to the 0-Schur algebra in Section 7. In Section 8, we consider the degeneration order of orbits in $\mathcal{F} \times \mathcal{F}$, and use open orbits to construct idempotents for the 0-Schur algebra in Section 9. Finally, we discuss 0-Hecke algebras in Section 10.

1. Flag varieties and q-Schur algebras

In this section, we fix notation and recall some definitions and results of Beilinson, Lusztig and MacPherson on q-Schur algebras in [1].

Let $n, r \geq 1$ be integers and V an r-dimensional vector space over a field k. Denote by $\mathcal{F}$ the set of all n-steps flags in V. Let f and f' be flags in $\mathcal{F}$ with

$$f : \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

and

$$f' : \{0\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = V.$$

We say that f' is a subflag of f, denoted by $f' \subseteq f$, if for all i,

$$U_i \subseteq V_i.$$

Denote the intersection of f and f' by $f \cap f'$, which is the flag

$$\{0\} \subseteq V_1 \cap U_1 \subseteq \cdots \subseteq V_n \cap U_n = V,$$

and the sum of f and f' by $f + f'$, which is the flag

$$\{0\} \subseteq V_1 + U_1 \subseteq \cdots \subseteq V_n + U_n = V.$$

Let $\alpha_i = \dim V_i - \dim V_{i-1}$ for $i = 1, \cdots, n$. Then

$$\alpha = (\alpha_1, \cdots, \alpha_n)$$

is a decomposition of r into n parts. Two flags are in the same $\mathrm{GL}(V)$-orbit if and only if they have the same decomposition, in this case, we write

$$f \simeq g.$$

Let $\Lambda(n,r)$ denote the set of all decompositions of r in n parts, and let $\mathcal{F}_\alpha \subseteq \mathcal{F}$ denote the orbit corresponding to $\alpha \in \Lambda(n,r)$.

The natural $\mathrm{GL}(V)$-action on $\mathcal{F}$ induces a diagonal $\mathrm{GL}(V)$-action on $\mathcal{F} \times \mathcal{F}$. Associate a matrix $A = A(f,f') = (A_{ij})$ to the pair of flags (f,f') with

$$\begin{aligned} A_{ij} &= \dim\bigl(V_{i-1} + V_i \cap V'_j\bigr) - \dim\bigl(V_{i-1} + V_i \cap V'_{j-1}\bigr) \\ &= \dim V_i \cap V'_j - \dim\bigl(V_i \cap V'_{j-1} + V_{i-1} \cap V'_j\bigr). \end{aligned}$$

This defines a bijection between the $\mathrm{GL}(V)$-orbits in $\mathcal{F} \times \mathcal{F}$ and $n \times n$ matrices of non-negative integers with the sum of entries equal to r. We denote the $\mathrm{GL}(V)$-orbit of (f,f') by $[f,f']$ and by e_A if we want to emphasise the matrix $A = A(f,f')$. If two pairs of flags (f,f') and (g,g') belong to the same $\mathrm{GL}(V)$-orbit, we write

$$\bigl(f,f'\bigr) \simeq \bigl(g,g'\bigr).$$

Let $e_A, e_{A'}, e_{A''} \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$ and $(f_1,f_2) \in e_{A''}$. Let

$$S\bigl(A,A',A''\bigr) = \bigl\{f \in \mathcal{F} \mid (f_1,f) \in e_A,\ (f,f_2) \in e_{A'}\bigr\}.$$

Following Proposition 1.1 in [1], there exists a polynomial $g_{A,A',A''} \in \mathbb{Z}[q]$, such that

$$g_{A,A',A''}\bigl(|k|\bigr) = \bigl|S\bigl(A,A',A''\bigr)\bigr|$$

for any finite field k. The projection

$$\mathcal{F} \times \mathcal{F} \times \mathcal{F} \to \mathcal{F}$$

onto the middle factor maps $\Delta^{-1}(e_A \times e_{A'}) \cap \pi^{-1}(f_1,f_2)$ bijectively onto $S(A,A',A'')$, and so these two sets have the same cardinality.

Define

$$e_A e_{A'} = \sum_{e_{A''} \in \mathcal{F}\times\mathcal{F}/\mathrm{GL}(V)} g_{A,A',A''} e_{A''}.$$

This gives an associative algebra over $\mathbb{Z}[q]$ with basis $\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$. Denote this algebra by $S_q(n,r)$. Du proved in [11] that $S_q(n,r)$ is isomorphic to the q-Schur algebras defined by Dipper and James in [7].

Although, in general it is difficult to compute the polynomial $g_{A,A',A''}$, the following lemma from [1], dealing with special A and A', gives clear multiplication rules. Also, Deng and Yang give a recursive formula of $g_{A,A',A''}$ using Hall polynomials for any A and A' [12].

Let

$$[m] = \frac{q^m - 1}{q - 1} = q^{m-1} + \cdots + q + 1$$

for $m \in \mathbb{N}$ and let $E_{i,j}$ be the (i,j) elementary matrix.

Lemma 1.1. *(See [1].) Assume that $1 \leq h < n$. Let $e_A \subseteq \mathcal{F}_\beta \times \mathcal{F}_\gamma$. Assume that $e_B \subseteq \mathcal{F}_\alpha \times \mathcal{F}_\beta$ and $e_C \subseteq \mathcal{F}_\delta \times \mathcal{F}_\beta$ such that $B - E_{h,h+1}$, $C - E_{h+1,h}$ are diagonal matrices. Then the following multiplication formulas hold in $S_q(n,r)$,*

$$e_B e_A = \sum_{\{p \mid A_{h+1,p} > 0\}} q^{\sum_{j>p} A_{h,j}} [A_{h,p} + 1] e_X,$$

$$e_C e_A = \sum_{\{p \mid A_{h,p} > 0\}} q^{\sum_{j<p} A_{h+1,j}} [A_{h+1,p} + 1] e_Y,$$

where $X = A + E_{h,p} - E_{h+1,p}$ and $Y = A - E_{h,p} + E_{h+1,p}$.

Note that the classical Schur algebra [13] can be obtained by evaluating $q = 1$, i.e.,

$$S(n,r) = S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q]/(q-1),$$

the 0-Schur algebra $S_0(n,r)$, obtained by evaluating $q = 0$, is

$$S_0(n,r) = S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q]/(q)$$

and

$$S_v(n,r) = S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v),$$

where $v^2 = q$.

2. Representations of linear quivers

In this section, we describe orbits of pairs of flags using representations of the linear quiver Λ of type $\mathbb{A}_n$,

$$\Lambda: \ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n.$$

We first recall a few definitions regarding representations of quivers. Given a quiver Q with the set of vertices $Q_0 = \{1, \ldots, n\}$ and the set of arrows Q_1, we denote a representation of Q by

$$X = \left(\{X_i\}_{i \in Q_0}, \{X_{i,j}\}_{i \to j \in Q_1}\right),$$

where each X_i is a vector space and each $X_{i,j}$ is a linear map from X_i to X_j. A homomorphism $h : X \to Y$ between two representations X and Y is a collection of linear maps $\{h_i : X_i \to Y_i\}_{i=1}^n$, satisfying

$$h_j X_{i,j} = Y_{i,j} h_i$$

for all arrows $i \to j \in Q_1$. A homomorphism h is an isomorphism if all the h_i are bijective. We write

$$X \simeq Y,$$

if X and Y are isomorphic. The direct sum $X \oplus Y$ of representations X and Y is the representation with

$$(X \oplus Y)_i = X_i \oplus Y_i \quad \text{and} \quad (X \oplus Y)_{i,j} = X_{i,j} \oplus Y_{i,j}.$$

A non-zero representation is indecomposable if it is not isomorphic to a direct sum of non-zero representations. We denote the dimension vector of X by

$$\underline{\dim}\, X = (\dim X_1, \ldots, \dim X_n).$$

Given a vector $d \in \mathbb{Z}^n_{\geq 0}$, define the representation variety of Q, parameterising representations of Q of dimension vector d, to be

$$\operatorname{Rep}(Q, d) = \bigoplus_{i \to j \in Q_1} \operatorname{Hom}\bigl(k^{d_i}, k^{d_j}\bigr).$$

The group $G = \prod_i \operatorname{GL}(d_i)$ acts on $\operatorname{Rep}(Q, d)$ by conjugation, i.e. for $g = (g_1, \ldots, g_n) \in G$ and $X = (X_{1,2}, \ldots, X_{n-1,n}) \in \operatorname{Rep}(Q, d)$,

$$(g \cdot X)_{i \to j} = g_j X_{i,j} g_i^{-1}.$$

Then the G-orbits in $\operatorname{Rep}(Q, d)$ are in one-to-one correspondence to the isomorphism classes of representations of Q with dimension vector d. In this paper, we only consider quivers of type $\mathbb{A}$, so there are a unique closed orbit and a unique open orbit in $\operatorname{Rep}(Q, d)$.

Now let M_{ij}, for $j \geq i$, be the indecomposable representation of Λ supported on the interval of vertices $[i, j] = \{i, \cdots, j\}$, with vector spaces in the support equal to k and all non-zero maps equal to the identity. Any representation M is isomorphic to the direct sum of some M_{ij}, i.e.

$$M \simeq \bigoplus_{i,j} (M_{ij})^{\alpha_{ij}},$$

where each α_{ij} is a non-negative integer. For each vertex i, let $S_i = M_{ii}$ and $P_i = M_{in}$ be the simple and indecomposable projective representation associated to i, respectively. A representation P is projective if and only if each map $P_{i,i+1}$ is injective.

For any vector $\alpha \in \mathbb{Z}^n_{\geq 0}$, let $P(\alpha)$ be the projective representation defined by

$$P(\alpha) = \bigoplus_i (P_i)^{\alpha_i}.$$

Then

$$P(\alpha)_1 \subseteq P(\alpha)_2 \subseteq \cdots \subseteq P(\alpha)_n$$

is a flag in $P(\alpha)_n$. Any projective representation is isomorphic to $P(\alpha)$ for some α and so we can view a projective representation as a flag. Conversely, an n-step flag

$$\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

can be naturally viewed as a projective representation of Λ

$$V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n,$$

with the natural inclusion $V_i \hookrightarrow V_{i+1}$ the linear map on the arrow $i \to i + 1$.

Two flags are in the same $\operatorname{GL}(V)$-orbit if and only if they are isomorphic as representations. So if two flags are in the same $\operatorname{GL}(V)$-orbit, we also say that they are isomorphic. If f is a flag in U and f' is a flag in U', then $f \oplus f'$ denotes the flag in $U \oplus U'$ with vector space at ith step equal to $U_i \oplus U'_i$.

A pair of flags (g, f) with $g \subseteq f$ can be viewed as a projective resolution

$$0 \to g \to f \to f/g \to 0.$$

If $(f_1, f_2), (f_1', f_2') \in \mathcal{F} \times \mathcal{F}$, with $f_1 \subseteq f_2$ and $f_1' \subseteq f_2'$, then $(f_1, f_2) \simeq (f_1', f_2')$, i.e. they are in the same $\mathrm{GL}(V)$-orbit, if and only if $f_2/f_1 \simeq f_2'/f_1'$ and $f_2 \simeq f_2'$, as representations of Λ. This fact generalises to arbitrary pairs.

Lemma 2.1. *Let $(f_1, f_2), (f_1', f_2') \in \mathcal{F} \times \mathcal{F}$. The following are equivalent.*

i) $(f_1, f_2) \simeq (f_1', f_2')$.
ii) $(f_1 + f_2)/f_i \simeq (f_1' + f_2')/f_i'$ *for* $i = 1, 2$, *and* $f_1 + f_2 \simeq f_1' + f_2'$.
iii) $(f_i, f_1 + f_2) \simeq (f_i', f_1' + f_2')$ *for* $i = 1, 2$.

Proof. The implication from i) to ii) is trivial.

We prove that ii) implies i). By ii), $f_i/(f_1 \cap f_2) \simeq f_i'/(f_1' \cap f_2')$, and $f_1 \cap f_2 \simeq f_1' \cap f_2'$.

Let $g_i : f_i/f_1 \cap f_2 \to f_i'/f_1' \cap f_2'$ be isomorphisms. Consider the following diagram,

$$\begin{tikzcd}
f_1 + f_2 \arrow[r, "\pi"] \arrow[d, "\exists h"] & (f_1/f_1 \cap f_2) \oplus (f_2/f_1 \cap f_2) \arrow[d, "\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right)"] \\
f_1' + f_2' \arrow[r, "\pi'"] & (f_1'/f_1' \cap f_2') \oplus (f_2'/f_1' \cap f_2'),
\end{tikzcd}$$

where π and π' are natural projections. Since $f_1 + f_2$ and $f_1' + f_2'$ are isomorphic projective representations, there is an isomorphism h such that the above diagram commutes. Thus $h(f_1) \subseteq (\pi')^{-1}(f_1'/f_1' \cap f_2') = f_1'$. Therefore $h(f_1) = f_1'$. Similarly $h(f_2) = f_2'$. Hence (f_1, f_2) and (f_1', f_2') are in the same orbit. This proves i).

iii) is a reformulation of ii). So the proof is done. $\square$

Note that the isomorphism of two pairs of flags can also be characterised using the inclusions $f_1 \cap f_2 \subseteq f_i$ and the isomorphisms $(f_1 + f_2)/f_i \simeq f_j/(f_1 \cap f_2)$ for $i \neq j \in \{1, 2\}$.

The lemma shows that a pair of flags in $\mathcal{F} \times \mathcal{F}$ and a triple $(\alpha, [M], [N])$, where $\alpha \in \Lambda(n, r)$ and $[M]$, $[N]$ are isomorphism classes of representations M and N of Λ with a surjection $P(\alpha) \to M \oplus N$, mutually determine each other. In fact, given such a triple $(\alpha, [M], [N])$, we can construct a corresponding pair (g_1, g_2) as follows. Recall that a surjective homomorphism $\psi : P \to M$ from a projective representation P is a projective cover if $\ker \psi \subseteq \operatorname{rad} P$ where $\operatorname{rad} P$ denotes the Jacobson radical of P. As inner direct sum, we have

$$P(\alpha) = f_1 \oplus f_2 \oplus c$$

such that $\psi_{|f_1}$ is a projective cover of M and $\psi_{|f_2}$ is a projective cover of N, and then

$$\ker \psi = f_1' \oplus f_2' \oplus c$$

with $f_i' = \ker \psi_{|f_i}$. Now let

$$(g_1, g_2) = \big(f_1 \oplus f_2' \oplus c, f_1' \oplus f_2 \oplus c\big).$$

Then

i) $g_1 + g_2 = f_1 \oplus f_2 \oplus c$ and $g_1 \cap g_2 = f_1' \oplus f_2' \oplus c$.
ii) $f_1' \subseteq \operatorname{rad} f_1$ and $f_2' \subseteq \operatorname{rad} f_2$.
iii) $g_1 + g_2/g_2 \simeq g_1/g_1 \cap g_2 \simeq M$ and $g_1 + g_2/g_1 \simeq g_2/g_1 \cap g_2 \simeq N$.

Given a pair of flags $(f, f') \in \mathcal{F} \times \mathcal{F}$, we have the following description of the matrix corresponding to the orbit $[f, f']$.

Lemma 2.2. *Let $e_A = [f, f'] = [f_1 \oplus f_2' \oplus c, f_1' \oplus f_2 \oplus c]$ as above. Then*

i) *if $i < j$, then A_{ij} is the multiplicity of $M_{i,j-1}$ as a direct summand in f_1/f_1',*
ii) *if $i > j$, then A_{ij} is the multiplicity of $M_{j,i-1}$ as a direct summand in f_2/f_2'.*
iii) *A_{ii} is the multiplicity of M_{in} as a direct summand in c.*

Proof. By definition, A_{ij} is equal to the dimension of the space

$$\frac{(f_{1,i} \oplus f_{2,i}' \oplus c_i) \cap (f_{1,j}' \oplus f_{2,j} \oplus c_j)}{(f_{1,i} \oplus f_{2,i}' \oplus c_i) \cap (f_{1,j-1}' \oplus f_{2,j-1} \oplus c_{j-1}) + (f_{1,i-1} \oplus f_{2,i-1}' \oplus c_{i-1}) \cap (f_{1,j}' \oplus f_{2,j} \oplus c_j)}.$$

For $i < j$,

$$\begin{aligned} A_{ij} &= \dim \frac{f_{1,i} \cap f_{1,j}' \oplus f_{2,i}' \oplus c_i}{f_{1,i} \cap f_{1,j-1}' + f_{2,i}' + c_i + f_{1,i-1} \cap f_{1,j}'} \\ &= \dim \frac{f_{1,i} \cap f_{1,j}'}{f_{1,i} \cap f_{1,j-1}' + f_{1,i-1} \cap f_{1,j}'}, \end{aligned}$$

which is the multiplicity of $M_{i,j-1}$ as a direct summand in f_1/f_1'. This proves i). Similarly, ii) holds.

For $i = j$,

$$A_{ii} = \dim \frac{f_{1,i}' \oplus f_{2,i}' \oplus c_i}{f_{1,i-1} \cap f_{1,i}' + f_{2,i}' \cap f_{2,i-1} + f_{2,i}' + f_{1,i}' + c_{i-i}}.$$

As $f_1' \subseteq f_1$ and $f_2' \subseteq f_2$ are minimal projective resolutions, we have inclusion of vector spaces $f_{1,i}' \subseteq f_{1,i-1}$ and $f_{2,i}' \subseteq f_{2,i-1}$. Therefore

$$A_{ii} = \dim \frac{c_i}{c_{i-1}},$$

which is the multiplicity of M_{in} as a direct summand in c. $\square$

3. The non-negative q-Schur algebras

In this section, we describe the non-negative part of a q-Schur algebra as a Hall algebra of projective resolutions of representations of the linear quiver Λ, defined in Section 2. We also include some easy lemmas on the computation of Hall numbers for the linear quiver which are needed in subsequent sections.

An orbit $[f, f'] \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$ with $f' \subseteq f$ decomposes as

$$[f, f'] = [c \oplus g, c \oplus g']$$

where $f/f' \simeq g/g'$ and $g' \subseteq \operatorname{rad} g$. That is, such an orbit is determined by the minimal projective resolution of f/f'

$$0 \to g' \to g \to f/f' \to 0$$

and a projective representation c such that

$$f \simeq g \oplus c.$$

The non-negative $\mathbb{Z}[q]$-subalgebra $S_q^+(n,r)$ is the subalgebra of $S_q(n,r)$ with basis consisting of all orbits $[f,f']$ with $f' \subseteq f$. Similarly, the non-positive q-Schur algebra $S_q^-(n,r)$ has the corresponding basis of all orbits $[f',f]$ with $f' \subseteq f$.

Let M, N and L be representations of Λ. Recall that the Hall polynomial $h^L_{MN} \in \mathbb{Z}[q]$ defined by Ringel [21] is the polynomial such that $h^L_{MN}(|k|)$ is equal to the number of subrepresentations $X \subseteq L$ such that

$$X \simeq N \quad \text{and} \quad L/X \simeq M$$

for any finite field k.

Lemma 3.1. *Let $f_1 \supseteq f_2 \supseteq f_3$ be flags and let $e_A = [f_1, f_2]$, $e_{A'} = [f_2, f_3]$, $e_{A''} = [f_1', f_3']$ with $f_1' \supseteq f_3'$, $f_1' \simeq f_1$ and $f_3' \simeq f_3$, $M = f_1/f_2$, $N = f_2/f_3$ and $L = f_1'/f_3'$. Then*

$$g_{A,A',A''} = h^L_{MN}.$$

Proof. Denote the set $\{X \subseteq L \mid X \simeq N, L/X \simeq M\}$ by U. Note that

$$g_{A,A',A''}\big(|k|\big) = \big|S\big(A,A',A''\big)\big| \quad \text{and} \quad h^L_{MN}\big(|k|\big) = |U|$$

for any finite field k. We need only to show that

$$\big|S\big(A,A',A''\big)\big| = |U|.$$

We will define two mutually inverse maps between U and $S(A,A',A'')$. Given $f_2' \in S(A,A',A'')$, we have the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & f_3' \arrow[r] \arrow[d] & f_3' \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_2' \arrow[r] \arrow[d] & f_1' \arrow[r] \arrow[d, "\pi"] & f_1'/f_2' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_2'/f_3' \arrow[r] & L \arrow[r] & f_1'/f_2' \arrow[r] & 0,
\end{tikzcd}$$

with $f_1/f_2 \simeq f_1'/f_2' \simeq M$ and $f_2/f_3 \simeq f_2'/f_3' \simeq N$.

Define maps

$$S\big(A,A',A''\big) \to U, \qquad f_2' \mapsto \pi\big(f_2'\big)$$

and

$$U \to S(A, A', A''), \qquad X \mapsto \pi^{-1}(X).$$

It is easy to check that these two maps are mutually inverse, and so the equality follows. □

Denote the (non-twisted) Ringel–Hall algebra [21] by $H_q(\Lambda)$. That is, $H_q(\Lambda)$ is the free $\mathbb{Z}[q]$-module with basis isomorphism classes $[M]$ of representations of Λ and multiplication

$$[M][N] = \sum_L h^L_{MN}[L].$$

Mapping representations to choices of projective resolutions induces an algebra homomorphism

$$\Theta^+ : H_q(\Lambda) \to S_q^+(n, r), \qquad [M] \mapsto \sum_{\{[f,f'] \mid f' \subseteq f, f/f' \simeq M\}} [f, f']$$

with kernel spanned by those $[M]$ with the number of indecomposable direct summands bigger than r [14]. There is a similar map $\Theta^- : H_q(\Lambda) \to S_q^-(n, r)$.

As a consequence, we have the following special case of Corollary 4.5 in [1] (see also Proposition 14.1 in [10]). The assumptions are as in Lemma 3.1.

Corollary 3.2. *We have*

$$g_{A+D, A'+D, A''+D} = h^L_{M,N},$$

for any diagonal matrix $D = \mathrm{diag}(\alpha_1, \cdots, \alpha_n)$ *with* $\alpha_i \geq 0$.

This yields a different proof of Theorem 14.27 in [10], which we restate as follows.

Theorem 3.3. *As* $\mathbb{Z}[q]$*-algebras, the Hall algebra* $H_q(\Lambda)$ *is isomorphic to the algebra with basis consisting of all formal sums*

$$\left\{ \sum_{[f/g]=[M]} [f, g] \;\middle|\; M \text{ is a representation of } \Lambda \right\},$$

and with multiplication induced by the multiplication in q-Schur algebras.

As before, let M_{ij} be the indecomposable representation of Λ supported on the interval $[i, j]$. Let $M_{ij} \leq M_{i'j'}$ if $j < j'$ or $i \leq i'$ when $j = j'$. This is a total order on the indecomposable representations of Λ. Observe that if $M_{ij} \leq M_{i'j'}$ then

$$\mathrm{Ext}^1(M_{i'j'}, M_{ij}) = 0,$$

i.e., any extension of $M_{i'j'}$ by M_{ij} splits. The next lemma follows easily from Lemma 3.1 and the corresponding computations in the Ringel–Hall algebra $H_q(\Lambda)$.

Lemma 3.4. *Suppose that* g *is a subflag of* f *and* $f/g = \bigoplus_{ij} M_{ij}^{m_{ij}}$. *Then there exists a filtration*

$$f = f_n \supset f_{n-1} \supset \cdots \supset f_1 = g \supset 0$$

with indecomposable factors $f_i/f_{i-1} \leq f_{i+1}/f_i$ *for all* i*, such that*

$$[f_n, f_{n-1}] \cdots [f_2, f_1] = [f, g] \prod [m_{ij}]!$$

and m_{ij} *is the multiplicity of* M_{ij} *as a subfactor in the filtration.*

4. Bases of $S_q(n, r)$

In this section, we describe a basis for $S_q(n, r)$ using the non-negative and non-positive subalgebras defined in the previous section. Let

$$\mathcal{B} = \{[f_1, f_1 + f_2][f_1 + f_2, f_2] \mid [f_1, f_2] \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)\}.$$

Lemma 2.1 shows that the map $\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V) \to \mathcal{B}$ given by

$$[f_1, f_2] \mapsto [f_1, f_1 + f_2][f_1 + f_2, f_2]$$

is well-defined. We will prove that $\mathcal{B}$ is a $\mathbb{Z}[q]$-basis of $S_q(n, r)$. Note that there is a similar basis $\mathcal{B}'$ of $S_q(n, r)$ consisting of elements of the form $[f_1, f_1 \cap f_2][f_1 \cap f_2, f_2]$.

Lemma 4.1. *Let* $(f_1, f_2) \in \mathcal{F} \times \mathcal{F}$ *and let* $e_A = [f_1, f_1 + f_2]$ *and* $e_{A'} = [f_1 + f_2, f_2]$*. Then*

$$[f_1, f_1 + f_2][f_1 + f_2, f_2] = [f_1, f_2] + \sum_{\{e_{A''} = [f'_1, f'_2] \mid f'_1 + f'_2 \subsetneq f_1 + f_2\}} g_{A,A',A''} [f'_1, f'_2].$$

Proof. Suppose that $[f'_1, f'_2]$ is one of the terms with a non-zero coefficient in the sum

$$[f_1, f_1 + f_2][f_1 + f_2, f_2] = \sum_{A''} g_{A,A',A''} e_{A''}.$$

Then there exists an $f \in \mathcal{F}$ such that $(f'_1, f) \simeq (f_1, f_1 + f_2)$ and $(f, f'_2) \simeq (f_1 + f_2, f_2)$. Thus $f'_1, f'_2 \subseteq f$ and so $f'_1 + f'_2 \subseteq f$. Note that if $f'_1 + f'_2 = f$, then $(f'_1, f) = (f'_1, f'_1 + f'_2) \simeq (f_1, f_1 + f_2)$ and $(f, f'_2) = (f'_1 + f'_2, f'_2) \simeq (f_1 + f_2, f_2)$. Therefore $(f'_1 + f'_2)/f'_i \simeq (f_1 + f_2)/f_i$ for $i = 1, 2$. By Lemma 2.1, $(f'_1, f'_2) \simeq (f_1, f_2)$. Moreover $g_{A,A',A''} = 1$ for $e_A = [f_1, f_2]$. So the lemma follows. □

There is a similar formula for the product $[f_1, f_1 \cap f_2][f_1 \cap f_2, f_2]$.

Theorem 4.2. *The set* $\mathcal{B}$ *is a* $\mathbb{Z}[q]$*-basis of* $S_q(n, r)$*.*

Proof. By Lemma 4.1,

$$[f_1, f_2] = [f_1, f_1 + f_2][f_1 + f_2, f_2] - \sum_{\{e_{A''} = [f'_1, f'_2] \mid f'_1 + f'_2 \subsetneq f_1 + f_2\}} g_{A,A',A''} [f'_1, f'_2].$$

We prove by induction on the dimension of $f_1 + f_2$, considered as a representation of Λ, that $[f_1, f_2]$ is a $\mathbb{Z}[q]$-linear combination of elements in $\mathcal{B}$. If $f_1 = f_2$, then $f_1 + f_2 = f_1$ and

$$[f_1, f_2] = [f_1, f_1][f_2, f_2] = [f_1, f_1 + f_2][f_1 + f_2, f_2],$$

and so it is done. In general, for each term $[f'_1, f'_2]$ in the sum, $f'_1 + f'_2$ is a proper subrepresentation of $f_1 + f_2$. By induction, $[f'_1, f'_2]$ can be written as a $\mathbb{Z}[q]$-linear combination of elements in $\mathcal{B}$ and therefore so can $[f_1, f_2]$. This proves that $\mathcal{B}$ spans $S_q(n,r)$ as a $\mathbb{Z}[q]$-module. On the other hand, note that the map from $\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$ to $\mathcal{B}$, sending $[f_1, f_2]$ to $[f_1, f_1 + f_2][f_1 + f_2, f_2]$, is surjective. Therefore $\mathcal{B}$ is a $\mathbb{Z}[q]$-basis of $S_q(n,r)$. $\square$

By Lemma 2.2, each basis element in $\mathcal{B}$ has a decomposition

$$\left[c \oplus f_1' \oplus f_2, c \oplus f_1 \oplus f_2\right]\left[c \oplus f_1 \oplus f_2, c \oplus f_1 \oplus f_2'\right]$$

where $f_1' \subseteq \operatorname{rad} f_1$ and $f_2' \subseteq \operatorname{rad} f_2$.

Lemma 4.3. *Let f_1, g and f_2 be flags in $\mathcal{F}$ with $f_1 \subseteq g$ and $f_2 \subseteq g$. Then $(f_1, g) \simeq (h_1, h_1 + h_2)$ and $(g, f_2) \simeq (h_1 + h_2, h_2)$ for a pair of flags $(h_1, h_2) \in \mathcal{F} \times \mathcal{F}$ if and only if there is a surjective map $g \to g/f_1 \oplus g/f_2$.*

Proof. Assume that $\psi : g \to g/f_1 \oplus g/f_2$ is surjective. Let $h_1 = \psi^{-1}(g/f_2)$ and $h_2 = \psi^{-1}(g/f_1)$. Then $(h_1, g) \simeq (f_1, g)$ and $(g, h_2) \simeq (g, f_2)$ and $g = \psi^{-1}(g/f_1 \oplus g/f_2) = h_1 + h_2$.

The converse holds, since the map $\pi : h_1 + h_2 \to (h_1 + h_2)/h_1 \oplus (h_1 + h_2)/h_2$ is surjective. $\square$

The lemma shows that a surjective map $g \to g/f_1 \oplus g/f_2$ implies $[f_1, g][g, f_2] \in \mathcal{B}$, but the converse is not true. An example is given below.

Example 4.4. Let $n = 3$, $r = 2$, $V = \operatorname{span}\{x_1, x_2\}$, $f_i : 0 \subseteq kx_i \subseteq V$ for $i = 1, 2$, and $g : V \subseteq V \subseteq V$. Then

$$[f_1, g][g, f_2] = [f_1, f_2] + [f_1, f_1] = [f_1, f_1 + f_2][f_1 + f_2, f_2] \in \mathcal{B},$$

with no surjective map $g \to g/f_1 \oplus g/f_2$. In this case $g \not\simeq f_1 + f_2$.

5. Quiver and relations for q-Schur algebras

In this section, we present an algebra using quivers and binomial relations, which will be shown to be the 0-Schur algebra in Section 7. This will lead to presentations of the q-Schur algebras over a ground ring, where q is not invertible. Also, following from the relations, the 0-Schur algebra has a multiplicative basis of paths, which will be constructed geometrically in Sections 6 and 7.

5.1. The quiver $\Sigma(n,r)$

Let $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith unit vector in $\mathbb{Z}^n$. Let $\Sigma(n,r)$ be the quiver with vertices K_α and arrows $E_{i,\alpha}$ and $F_{i,\alpha}$,

$$\begin{tikzcd} K_{\alpha+\epsilon_i-\epsilon_{i+1}} \arrow[r, bend left, "F_{i,\alpha+\epsilon_i-\epsilon_{i+1}}"] & K_\alpha \arrow[l, bend left, "E_{i,\alpha}"] \end{tikzcd}$$

where $\alpha, \alpha + \epsilon_i - \epsilon_{i+1} \in \Lambda(n,r)$. The vertices can be drawn on a simplex, where the vertices K_α with $\alpha_i = 0$ for some $i = 0$ are on the boundary, and vertices K_α with $\alpha_i \neq 0$ for all i are in the interior of the simplex.

For a commutative ring R, denote by $R\Sigma(n,r)$ the path R-algebra of $\Sigma(n,r)$, which is the free R-module with basis all paths in $\Sigma(n,r)$, and multiplication given by composition of paths. The vertices K_d form an orthogonal set of idempotents in $R\Sigma(n,r)$ and the composition of two paths p and q is pq, if q ends where p starts, and zero otherwise. To simplify our formulas we define

$$
\begin{gathered}
E_{i,\alpha}=0 \quad \text{if } \alpha\notin \Lambda(n,r),\ i=n,\ \text{or } \alpha_{i+1}=0,\\
F_{i,\alpha}=0 \quad \text{if } \alpha\notin \Lambda(n,r),\ i=n,\ \text{or } \alpha_{i}=0,\\
K_{\alpha}=0 \quad \text{if } \alpha\notin \Lambda(n,r),
\end{gathered}
$$

and finally,

$$E_i=\sum_\alpha E_{i,\alpha} \quad \text{and} \quad F_i=\sum_\alpha F_{i,\alpha}.$$

Recall that a relation in $R\Sigma(n,r)$ is an R-linear combination of paths with common starting and ending vertex

$$\rho=\sum_i r_ip_i,$$

where $r_i\in R$ and p_i is a path. Let $I(n,r)\subseteq \mathbb{Z}[q]\Sigma(n,r)$ be the ideal generated by the relations

$$
\begin{gathered}
P_{ij,\alpha}=K_{\alpha+p_{ij}}P_{ij}K_\alpha,\\
N_{ij,\alpha}=K_{\alpha-p_{ij}}N_{ij}K_\alpha, \quad \text{and}\\
C_{ij,\alpha}=K_{\alpha+\epsilon_i+\epsilon_{j+1}-\epsilon_{i+1}-\epsilon_j}C_{ij}K_\alpha,
\end{gathered}
$$

where

$$
\begin{aligned}
p_{ij}&=\begin{cases}2\epsilon_i+\epsilon_j-2\epsilon_{i+1}-\epsilon_{j+1}, & \text{if } |i-j|=1,\\ \epsilon_i+\epsilon_j-\epsilon_{i+1}-\epsilon_{j+1}, & \text{if } |i-j|>1;\end{cases}\\
P_{ij}&=\begin{cases}E_i^2E_j-(q+1)E_iE_jE_i+qE_jE_i^2, & \text{if } i=j-1,\\ qE_i^2E_j-(q+1)E_iE_jE_i+E_jE_i^2, & \text{if } i=j+1,\\ E_iE_j-E_jE_i, & \text{otherwise;}\end{cases}\\
N_{ij}&=\begin{cases}qF_i^2F_j-(q+1)F_iF_jF_i+F_jF_i^2, & \text{if } i=j-1,\\ F_i^2F_j-(q+1)F_iF_jF_i+qF_jF_i^2, & \text{if } i=j+1,\\ F_iF_j-F_jF_i, & \text{otherwise;}\end{cases}
\end{aligned}
$$

and

$$C_{ij}=E_iF_j-F_jE_i-\delta_{ij}\sum_\alpha \frac{q^{\alpha_i}-q^{\alpha_{i+1}}}{q-1}K_\alpha.$$

Let

$$e_{i,\alpha}=\left[f,f'\right], \qquad f_{i,\alpha+\epsilon_i-\epsilon_{i+1}}=\left[f',f\right] \quad \text{and} \quad k_\alpha=[h,h],$$

where $(f,f')\in\mathcal{F}\times\mathcal{F}$ with $f'\subseteq f$, $f/f'\simeq S_i$, and $f',h\in\mathcal{F}_\alpha$.

Lemma 5.1.1. *There is a homomorphism of $\mathbb{Z}[q]$-algebras*

$$\phi:\mathbb{Z}[q]\Sigma(n,r)/I(n,r)\to S_q(n,r)$$

defined by

$$\phi(E_{i,\alpha})=e_{i,\alpha}, \phi(F_{i,\alpha})=f_{i,\alpha} \quad \textit{and} \quad \phi(K_\alpha)=k_\alpha.$$

Proof. By Lemma 1.1, the relations P_{ij}, N_{ij} and C_{ij} hold in $S_q(n,r)$, and so ϕ is an algebra homomorphism. $\square$

We remark that the relations P_{ij} and N_{ij} hold in $S_q(n,r)$ also follows from Lemma 3.1 and the proposition in Section 2 of [22], and that the lemma can also be deduced from Lemma 5.6 in [1].

The homomorphism ϕ is not surjective in general, since for instance $[m]$ is not invertible in $\mathbb{Z}[q]$. So ϕ does not give a presentation of the q-Schur algebra over $\mathbb{Z}[q]$.

5.2. Change of rings

We need the following change of rings lemma for presentations of algebras using quivers with relations. The proof is similar to an argument at the end of Chapter 5 in [15]. Let $\psi : R \to S$ be a homomorphism of commutative rings, which gives S the structure of an R-algebra. Let Σ be a quiver, and let $I \subseteq R\Sigma$ be an ideal. There are induced maps of R-algebras $\psi : R\Sigma \to S\Sigma$ and $R\Sigma/I \to S\Sigma/S\psi(I)$, where $S\psi(I) \subseteq S\Sigma$ is the ideal generated by $\psi(I)$.

Lemma 5.2.1. *The induced map $(R\Sigma/I) \otimes_R S \to S\Sigma/S\psi(I)$ is an isomorphism of S-algebras.*

Proof. The natural isomorphism $R \otimes_R S \to S$ of S-algebras induces an S-algebra isomorphism

$$m : R\Sigma \otimes_R S \to S\Sigma.$$

Applying the functor $- \otimes_R S$ to the short exact sequence

$$0 \to I \xrightarrow{i} R\Sigma \to R\Sigma/I \to 0$$

gives us the exact sequence

$$I \otimes_R S \xrightarrow{j} S\Sigma \to (R\Sigma/I) \otimes_R S \to 0$$

where $j = m \circ (i \otimes \mathrm{Id}_S)$, which shows that

$$(R\Sigma/I) \otimes_R S \simeq S\Sigma/\operatorname{im}\big(m \circ (i \otimes \mathrm{Id}_S)\big).$$

As $\operatorname{im}(m \circ (i \otimes \mathrm{Id}_S)) = S\psi(I)$, the proof is complete. $\square$

5.3. q-Schur algebras over $\mathbb{Q}(v)$

Let v be an indeterminate with $v^2 = q$ and

$$S_v(n,r) = S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v).$$

Lemma 5.3.1. *There is an isomorphism of $\mathbb{Q}(v)$-algebras $\mathbb{Q}(v)\Sigma(n,r)/\mathbb{Q}(v)I(n,r) \to S_v(n,r)$ with $E_{i,\alpha} \mapsto e_{i,\alpha}$, $F_{i,\alpha} \mapsto f_{i,\alpha}$ and $K_\alpha \mapsto k_\alpha$.*

Proof. Let $\tilde{E}_i = \sum_\alpha v^{-\alpha_i+1} e_{i,\alpha}$, $\tilde{F}_i = \sum_\alpha v^{-\alpha_{i+1}+1} f_{i,\alpha}$ and $\tilde{K}_\alpha = k_\alpha$, and by abuse of notation, in this proof we let $E_i = \sum_\alpha e_{i,\alpha}$, $F_i = \sum_\alpha f_{i,\alpha}$ and $K_\alpha = k_\alpha$. Then both $\{\tilde{E}_i, \tilde{F}_j, \tilde{K}_\alpha\}$ and $\{E_i, F_j, K_\alpha\}$ generate $S_v(n,r)$. Moreover, by a straightforward computation, $\tilde{E}_i$, $\tilde{F}_j$, $\tilde{K}_\alpha$ satisfy the defining relations in Theorem 4' in [9] by Doty and Giaquinto if and only if E_i, F_j, K_α satisfy the relations P_{ij}, N_{ij} and C_{ij}. Therefore we have the isomorphism as required. $\square$

Proposition 5.3.2. *The induced map $\phi \otimes \mathrm{Id}_{\mathbb{Q}(v)} : \mathbb{Z}[q]\Sigma(n,r)/I(n,r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v) \to S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v)$ is a $\mathbb{Q}(v)$-algebra isomorphism, where ϕ is as in Lemma 5.1.1.*

Proof. By Lemma 5.2.1, the natural inclusion $\mathbb{Z}[q] \to \mathbb{Q}(v)$ induces an isomorphism

$$\mathbb{Z}[q]\Sigma(n,r)/I(n,r) \otimes_{Z[q]} \mathbb{Q}(v) \simeq \mathbb{Q}(v)\Sigma(n,r)/\mathbb{Q}(v)I(n,r),$$

which composed with the isomorphism in Lemma 5.3.1 is $\phi \otimes \mathrm{Id}_{\mathbb{Q}(v)}$. Thus the proposition follows. □

Since q is invertible in $\mathbb{Q}(v)$ and thus in $S_v(n,r)$, we cannot evaluate $q = 0$ in $S_v(n,r)$. We will modify the ground ring in the next subsection, so that q can be evaluated at 0.

5.4. A presentation of q-Schur algebra over $\mathcal{Q}$

Let $\mathcal{Q}$ be the ring obtained from $\mathbb{Z}[q]$ by inverting all polynomials of the form $1 + qf(q)$. In particular, all $[m]$ for $m \in \mathbb{N}$ are invertible. We have

$$\mathbb{Z}[q] \subseteq \mathcal{Q} \subseteq \mathbb{Q}(q)$$

and q is not invertible in $\mathcal{Q}$. So we can evaluate $q = 0$.

Proposition 5.4.1. *The induced map $\phi \otimes \mathrm{Id}_{\mathcal{Q}} : \mathbb{Z}[q]\Sigma(n,r)/I(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \to S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ is a surjective $\mathcal{Q}$-algebra homomorphism.*

Proof. The image of $\phi \otimes \mathrm{Id}_{\mathcal{Q}}$ is the subalgebra of $S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ generated by the set of all $e_{i,\alpha}$, $f_{i,\alpha}$ and k_α. Lemma 3.4 shows that the $\mathbb{Z}[q]$-subalgebra of $S_q^+(n,r)$ generated by all $e_{i,\alpha}$ and k_α contains all

$$[f,g] \prod [m_{ij}]!$$

where $g \subseteq f$ and m_{ij} is the multiplicity of M_{ij} as a direct summand in f/g. Since $[m]$ is invertible in $\mathcal{Q}$ for any m, the image contains $S_q^+(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$. Similarly, the image contains $S_q^-(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$. By Theorem 4.2,

$$\mathcal{B} = \big\{[f_1, f_1 + f_2][f_1 + f_2, f_2] \mid [f_1, f_2] \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)\big\}$$

is a $\mathbb{Z}[q]$-basis of $S_q(n,r)$, and thus a $\mathcal{Q}$-basis of $S_q^+(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$. Thus the map is surjective. □

By Lemma 5.2.1, $\mathcal{Q}\Sigma(n,r)/\mathcal{Q}I(n,r) \simeq \mathbb{Z}[q]\Sigma(n,r)/I(n,r) \otimes_{Z[q]} \mathcal{Q}$, and so the following theorem gives a presentation of q-Schur algebras over $\mathcal{Q}$ and will be proven in Section 7.

Theorem 5.4.2. *The induced map $\phi \otimes \mathrm{Id}_{\mathcal{Q}} : \mathbb{Z}[q]\Sigma(n,r)/I(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \to S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ is a $\mathcal{Q}$-algebra isomorphism.*

6. The generic algebras

In this section let k be algebraically closed. We define a generic multiplication of orbits in $\mathcal{F} \times \mathcal{F}$ and obtain an associative $\mathbb{Z}$-algebra $G(n,r)$, which we call a generic algebra. This multiplication generalises the one for positive 0-Schur algebras in [23] and is similar to the product defined by Reineke [20] for Hall algebras. We also give generators for $G(n,r)$ and find a standard decomposition of each basis element $[f,g]$ into a product of the generators.

Let $\Delta : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to (\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F})$ be the morphism given by

$$\Delta(p_1, p_2, p_3) = \bigl((p_1, p_2), (p_2, p_3)\bigr).$$

Let

$$\pi : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to \mathcal{F} \times \mathcal{F}$$

be the projection onto the left and right components. The map π is open, and Δ is a closed embedding.

Given two orbits e_A and $e_{A'}$, define

$$S\bigl(A, A'\bigr) = \pi\Delta^{-1}(e_A \times e_{A'}).$$

That is, $S(A, A')$ is the union of the orbits with non-zero coefficients in the product $e_A \cdot e_{A'}$ in $S_q(n, r)$.

Lemma 6.1. *The closure of $S(A, A')$ in $\mathcal{F} \times \mathcal{F}$ is irreducible.*

Proof. Let $[f_1, f_2] = e_A$, $[f_3, f_4] = e_{A'}$, and $S = \Delta^{-1}(e_A \times e_{A'})$. We first show that S is irreducible. If $f_2 \not\simeq f_3$ then S is empty, and we are done. So we may assume that $f_2 = f_3$. Let $(p_1, p_2, p_3) \in S$ then there exists $g \in \mathrm{GL}(V)$ such that $(p_2, p_3) = g(f_3, f_4)$ and $g(g^{-1}p_1, f_3, f_4) = (p_1, p_2, p_3)$, where $\mathrm{GL}(V)$ acts diagonally. Since $(g^{-1}p_1, f_3) \simeq (f_1, f_3)$, there is an $a \in \mathrm{Aut}(f_3)$ such that $g^{-1}p_1 = af_1$. Hence S is the image of the morphism

$$\mathrm{Aut}(f_3) \times \mathrm{GL}(V) \to \mathcal{F} \times \mathcal{F} \times \mathcal{F}$$

given by

$$(a, g) \mapsto (gaf_1, gf_3, gf_4)$$

and is therefore irreducible. Now $S(A, A') = \pi(S)$, and so its closure is irreducible. $\square$

Since there are only finitely many orbits in $S(A, A')$, as a consequence of Lemma 6.1, we have the following corollary.

Corollary 6.2. *There is a unique open $\mathrm{GL}(V)$-orbit in $S(A, A')$.*

We define a new multiplication

$$e_A \star e_{A'} = e_{A''}$$

if $S(A, A')$ is non-empty and $e_{A''}$ is the open orbit in $S(A, A')$, and

$$e_A \star e_{A'} = 0$$

if $S(A, A')$ is empty. Denote by $G(n, r)$ the free $\mathbb{Z}$-module with basis $\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$.

Proposition 6.3. *The free $\mathbb{Z}$-module $G(n, r)$ with the product $\star$ is an associative $\mathbb{Z}$-algebra.*

Proof. We need only to show that $\star$ is associative. That is, for any $\mathrm{GL}(V)$-orbits $[f_1, f_2]$, $[f_3, f_4]$, $[f_5, f_6] \in \mathcal{F}$, we have

$$\big([f_1, f_2] \star [f_3, f_4]\big) \star [f_5, f_6] = [f_1, f_2] \star \big([f_3, f_4] \star [f_5, f_6]\big).$$

Following the definition, we see that if one side of the equality is zero, then so is the other side. We now suppose that both sides are not zero, that is, $f_2 \simeq f_3$ and $f_4 \simeq f_5$. By the $\mathrm{GL}(V)$-action on $\mathcal{F}$, we may assume that $f_2 = f_3$ and $f_4 = f_5$. Denote the sets

$$\big\{(p_1, p_2, p_3, p_4) \mid (p_1, p_2) \simeq (f_1, f_2), (p_2, p_3) \simeq (f_3, f_4), (p_3, p_4) \simeq (f_5, f_6)\big\},$$
$$\big\{(p_1, p_3, p_4) \mid \exists p \text{ such that } (p_1, p) \simeq (f_1, f_2), (p, p_3) \simeq (f_3, f_4), (p_3, p_4) \simeq (f_5, f_6)\big\},$$
$$\big\{(p_1, p_2, p_4) \mid \exists p \text{ such that } (p_1, p_2) \simeq (f_1, f_2), (p_2, p) \simeq (f_3, f_4), (p, p_4) \simeq (f_5, f_6)\big\},$$

and

$$\big\{(p_1, p_4) \mid \exists p, p' \text{ such that } (p_1, p) \simeq (f_1, f_2), \big(p, p'\big) \simeq (f_3, f_4), \big(p', p_4\big) \simeq (f_5, f_6)\big\}$$

by T_1, T_2, T_3 and T_4, respectively. We have natural surjections

$$\pi_{ij} : T_i \to T_j$$

for $(i, j) = (1, 2), (1, 3), (2, 4), (3, 4)$. Similar to the proof of Lemma 6.1, we see that T_1 is irreducible, and so the closures of all the T_i are irreducible. In particular, there is a unique open orbit $\mathcal{O}$ in T_4. Then $\pi_{24}^{-1}(\mathcal{O})$ intersects with the open subset of T_2, consisting of triples (p_1, p_3, p_4) with $[p_1, p_3]$ open in $S(A, A')$. That is, $([f_1, f_2] \star [f_3, f_4]) \star [f_5, f_6]$ is the open orbit $\mathcal{O}$ in T_4. Similarly, $[f_1, f_2] \star ([f_3, f_4] \star [f_5, f_6])$ is also the open orbit $\mathcal{O}$. Therefore the equality holds and so $\star$ is associative. $\square$

The following is a direct consequence of the definition of the product in $G(n, r)$.

Corollary 6.4. *The set $\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$ is a multiplicative basis of $G(n, r)$.*

In addition to the basis of $G(n, r)$ consisting of orbits $[f_1, f_2]$ we can also consider bases analogous to the bases $\mathcal{B}$ and $\mathcal{B}'$ defined in Section 4 for the q-Schur algebras. We show that these three bases of $G(n, r)$ coincide.

Lemma 6.5. *Let $(f_1, f_2) \in \mathcal{F} \times \mathcal{F}$. Then*

$$[f_1, f_1 + f_2] \star [f_1 + f_2, f_2] = [f_1, f_2] = [f_1, f_1 \cap f_2] \star [f_1 \cap f_2, f_2].$$

Proof. We prove the first equality. Let $e_A = [f_1, f_1 + f_2]$ and $e_{A'} = [f_1 + f_2, f_2]$. We prove that the orbit $[f_1, f_2]$ is open in $S(A, A')$. For any $(f_1', f_2') \in S(A, A')$, $f_1' + f_2'$ is isomorphic to a subflag of $f_1 + f_2$. By Lemma 4.1, for $(f_1', f_2') \in S(A, A')$, we have $(f_1', f_2') \simeq (f_1, f_2)$ if and only if $f_1' + f' \simeq f_1 + f_2$. That the dimension of $f_1' + f_2'$ is maximal is an open condition. Therefore $e_A \star e_{A'} = [f_1, f_2]$.

Similarly, $[f_1, f_1 \cap f_2] \star [f_1 \cap f_2, f_2] = [f_1, f_2]$. $\square$

We now prove that the $\mathbb{Z}$-algebra $G(n, r)$ is generated by the orbits $e_{i,\alpha}$, $f_{i,\alpha}$ and k_α. Recall that a representation X is said to be a generic extension of N by M, if the stabiliser of X is minimal among all representations that are extensions of N by M.

Lemma 6.6. *(See [23].) Let $f \supseteq g \supseteq h$ be flags. Then $[f,h] = [f,g] \star [g,h]$ if and only if f/h is a generic extension of f/g by g/h.*

For an interval $[i,j]$ in $\{1,\cdots,n\}$ and $\alpha \in \Lambda(n,r)$ with $\alpha - \epsilon_{j+1}$ non-negative, let

$$e(i,j,\alpha) = e_{i,\alpha+\epsilon_{i+1}-\epsilon_{j+1}} \star \cdots \star e_{j,\alpha}.$$

Similarly, let $f(i,j,\alpha) = f_{j,\alpha-\epsilon_i+\epsilon_j} \star \cdots \star f_{i,\alpha}$ for $\alpha - \epsilon_i$ non-negative.

Lemma 6.7. *Let $f \supseteq h$ be flags with $h \in \mathcal{F}_\alpha$ and $f/h \simeq M_{ij}$. Then $[f,h] = e(i,j,\alpha)$ and $[h,f] = f(i,j,\alpha + \epsilon_i - \epsilon_{j+1})$.*

Proof. If $i = j$, then $[f,h] = e_{i,\alpha}$. Now assume $j > i$. Then there is $f \supseteq g \supseteq h$ with $f/g \simeq M_{i,j-1}$ and $g/h \simeq M_{jj}$. Since f/h is a generic extension of f/g by the simple representation g/h, the lemma follows from Lemma 6.6 by induction. □

Using the order $\leq$ on representations defined in Section 3, we can write each orbit $[f,g]$ with $f \supseteq g$ as a product over indecomposable summands of f/g.

Lemma 6.8. *Let $f \supseteq g$ be flags with $f/g \simeq \bigoplus_{i=1}^t M_i$ and $M_i \leq M_{i+1}$. Then there is a filtration $f = f_t \supset f_{t-1} \supset \cdots \supset f_0 = g \supset 0$ with indecomposable factors $M_i = f_i/f_{i-1}$ and $[f,g] = [f_t, f_{t-1}] \star \cdots \star [f_1, f_0]$.*

Proof. The lemma follows from the vanishing of extension groups along the filtration and Lemma 6.6. □

Lemma 6.9. *The $\mathbb{Z}$-algebra $G(n,r)$ is generated by the orbits $e_{i,\alpha}$, $f_{i,\alpha}$ and k_α.*

Proof. Lemma 6.7 and Lemma 6.8 imply that any orbit $[f,g]$ with $f \supseteq g$ is in the subalgebra of $G(n,r)$ generated by $e_{i,\alpha}$ and k_α. Similarly, any orbit $[f,g]$ with $f \subseteq g$ is generated by $f_{i,\alpha}$ and k_α. The lemma now follows from Lemma 6.5. □

Following Lemmas 6.5, 6.7 and 6.8, we obtain the following basis of $G(n,r)$ in terms the generators $e_{i,\alpha}$ and $f_{i,\alpha}$.

Lemma 6.10. *The $\mathbb{Z}$-algebra $G(n,r)$ has a basis consisting of all k_α and all non-zero monomials*

$$e(i_s,j_s,\alpha_s) \star \cdots \star e(i_1,j_1,\alpha_1) \star f\big(i'_1,j'_1,\alpha'_1\big) \star \cdots \star f\big(i'_t,j'_t,\alpha'_t\big),$$

where $M_{i_l j_l} \leq M_{i_{l+1} j_{l+1}}$, $M_{i'_l j'_l} \leq M_{i'_{l+1} j'_{l+1}}$ and $\alpha_1 \geq \sum_l \epsilon_{j_l+1} + \sum_l \epsilon_{j'_l+1}$.

Proof. First observe that any basis element $[f,g] = [f, f \cap g][f \cap g, g]$ can be written as a monomial described in the statement. So we need only show that for any such monomial

$$e(i_s,j_s,\alpha_s) \star \cdots \star e(i_1,j_1,\alpha_1) \star f\big(i'_1,j'_1,\alpha'_1\big) \star \cdots \star f\big(i'_t,j'_t,\alpha'_t\big)$$

there is a unique orbit $[f,g]$ such that

$$[f, f \cap g] = e(i_s,j_s,\alpha_s) \star \cdots \star e(i_1,j_1,\alpha_1) \quad \text{and} \quad [f \cap g, g] = f\big(i'_1,j'_1,\alpha'_1\big) \star \cdots \star f\big(i'_t,j'_t,\alpha'_t\big).$$

Write $\alpha_1 = \beta + \alpha' + \alpha''$, where $\alpha' = \sum_l \epsilon_{j_l+1}$ and $\alpha'' = \sum_l \epsilon_{j'_l+1}$. Consider $P(\alpha_1)$ as a flag in V, and decompose as $P(\alpha_1) = P(\beta) \oplus P(\alpha') \oplus P(\alpha'')$. Let $Q(\alpha')$ and $Q(\alpha'')$ be minimal flags containing $P(\alpha')$ and $P(\alpha'')$, respectively, such that

$$Q(\alpha')/P(\alpha') \simeq \bigoplus M_{i_l j_l} \quad \text{and} \quad Q(\alpha'')/P(\alpha'') \simeq \bigoplus M_{i'_l j'_l}.$$

Let $f = P(\beta) \oplus Q(\alpha') \oplus P(\alpha'')$ and $g = P(\beta) \oplus P(\alpha') \oplus Q(\alpha'')$, then $[f, g]$ is an orbit as required. The uniqueness is determined by the two quotients $f/f \cap g$ and $g/f \cap g$ and α_1. $\square$

We compute the multiplication in $G(n, r)$ of an arbitrary element with a generator.

Lemma 6.11. *Let $e_A \subseteq \mathcal{F}_\alpha \times \mathcal{F}_\beta$.*

i) *If $\alpha_{i+1} > 0$, then $e_{i,\alpha} \star e_A = e_X$ where $X = A + E_{i,p} - E_{i+1,p}$ and $p = \max\{j \mid A_{i+1,j} > 0\}$.*
ii) *If $\alpha_i > 0$, then $f_{i,\alpha} \star e_A = e_Y$ where $Y = A - E_{i,p} + E_{i+1,p}$ and $p = \min\{j \mid A_{i,j} > 0\}$.*

Proof. We prove i). By Lemma 1.1, the orbit e_X has a non-zero coefficient in the product $e_{i,\alpha} \cdot e_A$ in $S_q(n, r)$. Now, by Lemma 2.2 in [1], among all terms $A + E_{i,j} - E_{i+1,j}$ with $A_{i+1,j} > 0$, the elements in the orbit e_X has the smallest stabiliser, and so $e_{i,\alpha} \star e_A = e_X$.

The proof of ii) is similar. $\square$

7. A geometric realisation of the 0-Schur algebra

In this section we first give a presentation of $G(n, r)$ using quivers and relations. Then we show that $S_0(n, r)$ and $G(n, r)$ are isomorphic as $\mathbb{Z}$-algebras by an isomorphism which is the identity on the closed orbits $e_{i,\alpha}$, $f_{i,\alpha}$ and k_α. Finally, we prove Theorem 5.4.2.

7.1. A presentation of $G(n, r)$

Let $\Sigma(n, r)$, E_i and F_i be as in Section 5. Let

$$P_{ij}(0) = \begin{cases} E_i^2 E_j - E_i E_j E_i, & \text{if } i = j - 1, \\ -E_i E_j E_i + E_j E_i^2, & \text{if } i = j + 1, \\ E_i E_j - E_j E_i, & \text{otherwise;} \end{cases}$$

$$N_{ij}(0) = \begin{cases} -F_i F_j F_i + F_j F_i^2, & \text{if } i = j - 1, \\ F_i^2 F_j - F_i F_j F_i, & \text{if } i = j + 1, \\ F_i F_j - F_j F_i, & \text{otherwise;} \end{cases}$$

and

$$C_{ij}(0) = E_i F_j - F_i E_j - \delta_{ij} \sum_\alpha \lambda_{ij}(\alpha) \cdot K_\alpha,$$

where

$$\lambda_{ij}(\alpha) = \begin{cases} 1, & \text{if } \alpha_i > \alpha_{i+1} = 0, \\ -1, & \text{if } \alpha_{i+1} > \alpha_i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

That is, $P_{ij}(0)$, $N_{ij}(0)$ and $C_{ij}(0)$ are obtained by evaluating P_{ij}, N_{ij} and C_{ij} at $q = 0$. Let $I_0(n, r) \subseteq \mathbb{Z}\Sigma(n, r)$ be the ideal generated by $P_{ij,\alpha}(0)$, $N_{ij,\alpha}(0)$, and $C_{ij,\alpha}(0)$, which are obtained by evaluating $P_{ij,\alpha}$, $N_{ij,\alpha}$ and $C_{ij,\alpha}$ at $q = 0$.

Lemma 7.1.1. *$\mathbb{Z}\Sigma(n, r)/I_0(n, r)$ has a multiplicative basis of paths in $\Sigma(n, r)$.*

Proof. The lemma holds since each relation $P_{ij,\alpha}(0)$, $N_{ij,\alpha}(0)$, and $C_{ij,\alpha}(0)$ is a binomial in $E_{i,\alpha}$, $F_{i,\alpha}$ and K_α. This is obvious for $P_{ij,\alpha}(0)$, $N_{ij,\alpha}(0)$. For $C_{ij,\alpha}(0)$, if the coefficient of K_α is non-zero then either $C_{ij,\alpha}(0) = K_\alpha E_i F_j K_\alpha - K_\alpha$ or $C_{ij,\alpha}(0) = K_\alpha F_i E_j K_\alpha - K_\alpha$. □

For an interval $[i,j]$ in $\{1, \cdots, n\}$ and $\alpha \in \Lambda(n,r)$ with $\alpha - \epsilon_{j+1}$ non-negative, let

$$E(i,j,\alpha) = E_{i,\alpha+\epsilon_{i+1}-\epsilon_{j+1}} \cdots E_{j,\alpha}$$

and $F(i,j,\alpha) = F_{j,\alpha-\epsilon_i+\epsilon_j} \cdots F_{i,\alpha}$ for $\alpha - \epsilon_i$ non-negative. The $E(i,j,\alpha)$ and $F(i,j,\alpha)$ are analogous to $e(i,j,\alpha)$ and $f(i,j,\alpha)$, respectively, defined in Section 6.

Theorem 7.1.2. *The map $\eta : \mathbb{Z}\Sigma(n,r)/I_0(n,r) \to G(n,r)$ given by $\eta(E_{i,\alpha}) = e_{i,\alpha}$, $\eta(F_{i,\alpha}) = f_{i,\alpha}$ and $\eta(K_\alpha) = k_\alpha$ is an isomorphism of $\mathbb{Z}$-algebras.*

Proof. By Lemma 6.11, it is straightforward to check that $e_{i,\alpha}$, $f_{i,\alpha}$, and k_α satisfy the relations $P_{ij,d}(0)$, $N_{ij,d}(0)$, and $C_{ij,d}(0)$. Thus η is well-defined. Also, Lemma 6.9 implies that the map is surjective. It remains to prove that η is injective.

We claim that, modulo the relations in $I_0(n,r)$, any path p in $\Sigma(n,r)$ is either equal to k_α or a path of the form

$$E(i_s,j_s,\alpha_s) \cdots E(i_1,j_1,\alpha_1) F\big(i'_1,j'_1,\alpha'_1\big) \cdots F\big(i'_t,j'_t,\alpha'_t\big),$$

satisfying the conditions in Lemma 6.10. Note that such a path is mapped onto one of monomial basis elements in Lemma 6.10, and so the injectivity of η follows.

We prove the claim by induction on the length of p. If p has length less than or equal to one, it is equal to k_α or one of the arrows $F_{i,\alpha}$ and $E_{i,\alpha}$, and so the claim follows. Assume that p has length greater than one. Then we have

$$p = p' F_{i,\beta} \quad \text{or} \quad p = p' E_{i,\beta}$$

where p' is a non-trivial path of smaller length, and so by induction has the required form

$$p' = EF = E(i_s,j_s,\alpha_s) \cdots E(i_1,j_1,\alpha_1) F\big(i'_1,j'_1,\alpha'_1\big) \cdots F\big(i'_t,j'_t,\alpha'_t\big),$$

where E and F are products of the $E(i_a,j_a,\alpha_a)$ and $F(i_b,j_b,\alpha'_b)$, respectively.

We first consider $p = p'E_{i,\beta}$. If p' contains no $F_{j,\alpha}$, then the claim follows using the relations $P_{ab,\alpha}(0)$. Otherwise, by the relations $C_{ab,\alpha}(0)$, either $p = EF'$ with the length of F' smaller than that of F or $p = EE_{i,\alpha_1-\epsilon_i+\epsilon_{i+1}}F'$ with each factor $F(i'_l,j'_l,\alpha'_l)$ in F replaced with a factor $F(i'_l,j'_l,\beta'_l)$. In the first case, the claim follows by induction. Otherwise, by the relations $P_{ab,\alpha}(0)$, there are two possibilities. First, there exists a minimal m with $j_m = i-1$. Then $EE_{i,\alpha_1-\epsilon_i+\epsilon_{i+1}}F'$ is equal to

$$E(i_s,j_s,\alpha_s) \cdots E(i_{m+1},j_{m+1},\alpha_{m+1}) E(i_m,j_m+1,\beta_m) E(i_{m-1},j_{m-1},\beta_{m-1}) \cdots E(i_1,j_1,\beta_1) F'.$$

We have

$$\beta_1 = \alpha_1 - \epsilon_i + \epsilon_{i+1} \geq \sum_l \epsilon_{j_l+1} - \epsilon_i + \epsilon_{i+1} + \sum_l \epsilon_{j'_l+1} = \sum_{l \neq m} \epsilon_{j_l+1} + \epsilon_{j_m+2} + \sum_l \epsilon_{j'_l+1}.$$

Moreover, again using the relations $P_{ab,\alpha}(0)$, the factors can be reordered (up to change of α_l, β_m) to obtain a path of the required form.

Second, there is no such m with $j_m = i-1$. Then $EE_{i,\alpha_1-\epsilon_i+\epsilon_{i+1}}F'$ is equal to

$$E(i_s, j_s, \alpha_s)\cdots E(i_m, j_m, \alpha_m)E(i, i, \beta_m)E(i_{m-1}, j_{m-1}, \beta_{m-1})\cdots E(i_1, j_1, \beta_1)F',$$

with $j_{m-1} \leq i$ and $j_m > i$. In order to show that this path is of the required form, we need only to prove the inequality

$$\beta_1 = \alpha_1 - \epsilon_i + \epsilon_{i+1} \geq \sum_l \epsilon_{j_l+1} + \epsilon_{i+1} + \sum_l \epsilon_{j'_l+1}.$$

Clearly, the inequality holds for each component different from i. Since there are no m with $j_m = i-1$, the sum $\sum_l \epsilon_{j_l+1}$ contain no ϵ_i. Since $FE_{i,\beta} = E_{i,\beta_1}F'$ with the length of F' equal to that of F, we must have $(\alpha_1 - \epsilon_i)_i \geq (\sum_l \epsilon_{j'_l+1})_i$ and so the inequality follows.

Finally, we consider $p = p'F_{i,\beta}$, where p' is a path of the required form $p' = EF$ as above. If there are no factor $E_{j,\alpha}$ in p', then the claim follows from the relations $N_{ab,\alpha}(0)$. Otherwise $p = E'E_{j,\alpha_1}FF_{i,\beta}$, which following $C_{ab,\alpha}(0)$ is either $p = E'F'$ with F' not longer than F, or $p = E'F'E_{j,\beta}$. In the first case, the length of the path $E'F'$ is smaller than p in $\Sigma(n,r)$ and so the claim follows by induction; in the second case, the claim is proved above. In either case, the claim holds. □

7.2. A geometric realisation of $S_0(n,r)$

We now prove the main result.

Theorem 7.2.1. *The map*

$$\psi : G(n,r) \to S_0(n,r)$$

defined by $\psi(e_{i,\alpha}) = e_{i,\alpha}$, $\psi(f_{i,\alpha}) = f_{i,\alpha}$ and $\psi(k_\alpha) = k_\alpha$ is an isomorphism of $\mathbb{Z}$-algebras.

Proof. From Proposition 5.4.1, we have the surjective $\mathcal{Q}$-algebra homomorphism

$$\mathcal{Q}\Sigma(n,r)/\mathcal{Q}I(n,r) \to S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q},$$

which, since $\mathcal{Q}/q\mathcal{Q} \simeq \mathbb{Z}$, induces a surjective $\mathbb{Z}$-algebra homomorphism

$$\big(\mathcal{Q}\Sigma(n,r)/\mathcal{Q}I(n,r)\big) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \to S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q}.$$

Following the definition of $S_0(n,r)$ and the isomorphisms

$$\mathcal{Q}/q\mathcal{Q} \simeq \mathbb{Z}[q]/q\mathbb{Z}[q] \simeq \mathbb{Z},$$

we have

$$S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} = S_0(n,r)$$

and by Lemma 5.2.1

$$\big(\mathcal{Q}\Sigma(n,r)/\mathcal{Q}I(n,r)\big) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \simeq \big(\mathbb{Z}\Sigma(n,r)/I_0(n,r)\big).$$

So there is a surjective $\mathbb{Z}$-algebra homomorphism

$$\mathbb{Z}\Sigma(n,r)/I_0(n,r) \to S_0(n,r)$$

given by

$$E_{i,\alpha} \mapsto e_{i,\alpha}, \qquad F_{i,\alpha} \mapsto f_{i,\alpha}, \qquad K_{i,\alpha} \mapsto k_{i,\alpha}.$$

The theorem now follows from Theorem 7.1.2, since $G(n,r) = S_0(n,r)$ as $\mathbb{Z}$-modules. $\square$

Via the isomorphism in Theorem 7.2.1, the presentation of $G(n,r)$ in Section 7.1 becomes a presentation of $S_0(n,r)$. We remark that Deng and Yang [6] have independently given a similar presentation for $S_0(n,r)$, using a different approach.

Corollary 7.2.2. *Let ψ be the map in Theorem 7.2.1. The set $\psi(\mathcal{F} \times \mathcal{F}/\mathrm{GL}(V))$ is a multiplicative basis for $S_0(n,r)$.*

7.3. Proof of Theorem 5.4.2

By Proposition 5.4.1, the map $\phi \otimes \mathrm{Id}_{\mathcal{Q}}$ induces a short exact sequence

$$0 \to K \to \mathcal{Q}\Sigma(n,r)/\mathcal{Q}I(n,r) \to S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \to 0.$$

Since $S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ is a free $\mathcal{Q}$-module, applying $- \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q}$ gives the exact sequence

$$0 \to K \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \to \mathcal{Q}\Sigma(n,r)/\mathcal{Q}I(n,r) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \to S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \to 0.$$

As in the proof of Theorem 7.2.1, we have isomorphisms

$$S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} = S_0(n,r)$$

and

$$\big(\mathcal{Q}\Sigma(n,r)/\mathcal{Q}I(n,r)\big) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \simeq \big(\mathbb{Z}\Sigma(n,r)/I_0(n,r)\big).$$

Furthermore, via these two isomorphisms the map $\phi \otimes \mathrm{Id}_{\mathcal{Q}} \otimes \mathrm{Id}_{\mathcal{Q}/q\mathcal{Q}}$ is the composition of the isomorphism

$$\mathbb{Z}\Sigma(n,r)/I_0(n,r) \simeq G(n,r)$$

in Theorem 7.1.2 and the isomorphism

$$G(n,r) \simeq S_0(n,r)$$

in Theorem 7.2.1. Therefore

$$K \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} = K/qK = 0.$$

Now by Nakayama's lemma (see Theorem 2.2 in [18]), there is an element $r = 1 + qf(q) \in \mathcal{Q}$ such that $rK = 0$. Since r is invertible in $\mathcal{Q}$, we have $K = 0$. Thus $\phi \otimes \mathrm{Id}_{\mathcal{Q}}$ is an isomorphism.

8. The degeneration order on pairs of flags

In this section, let k be algebraically closed. We describe the degeneration order on $\mathrm{GL}(V)$-orbits in $\mathcal{F} \times \mathcal{F}$ using quivers and the symmetric group S_r.

Let $\Gamma = \Gamma(n)$ be the quiver of type $\mathbb{A}_{2n-1}$,

$$\Gamma:\ 1_L \longrightarrow 2_L \longrightarrow \cdots \longrightarrow n \longleftarrow \cdots \longleftarrow 2_R \longleftarrow 1_R$$

constructed by joining two linear quivers $\Lambda_L = \Lambda$ and $\Lambda_R = \Lambda$ at the vertex n. Often it will be clear from the context which side of Γ we are considering, and then we drop the subscripts on the vertices.

A pair $(f, f') \in \mathcal{F} \times \mathcal{F}$ is a representation of Γ, where f is supported on Λ_L, f' is supported on Λ_R. Conversely, any representation M of Γ that is projective when restricted to both Λ_L and Λ_R and has $\dim M_n = r$ determines uniquely an orbit of pair of flags $[f, f'] \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$. Moreover, two pairs of flags are isomorphic if and only if the corresponding representations are isomorphic.

For integers $i, j \in \{1, \cdots, n\}$, let N_{ij} be the indecomposable representation of Γ which is equal to the indecomposable projective representations M_{in} and M_{jn} when restricted to Λ_L and Λ_R, respectively. A representation N of Γ which is projective when restricted to Λ_L and Λ_R, and $\dim N_n = r$, decomposes up to isomorphism as

$$N \simeq \bigoplus_{l=1}^{r} N_{i_l j_l}.$$

We assume that $j_1 \leq j_2 \leq \cdots \leq j_r$.

Let $\leq_{deg}$ denote the degeneration order on isomorphism classes of representations of Γ. That is, $M \leq_{deg} N$ for two representations M and N, if N is contained in the closure of the orbit of M in the space of all representations. The degeneration order on pairs of flags is also denoted by $\leq_{deg}$, since there is a degeneration between two pair of flags if and only if there is a degeneration between the corresponding representations of Γ.

Since Γ is a Dynkin quiver, by a result of Bongartz [2], the degeneration $\leq_{deg}$ is the same as the degeneration $\leq_{ext}$ given by a sequence of extensions. That is, if there is an extension

$$0 \longrightarrow N' \longrightarrow M \longrightarrow N'' \longrightarrow 0,$$

then $M \leq_{ext} N' \oplus N''$, and more generally $\leq_{ext}$ is the transitive closure.

The symmetric group S_r of permutations of the set $\{1, \cdots, r\}$ acts on representations with a decomposition $N = \bigoplus_{l=1}^{r} N_{i_l j_l}$ by

$$\sigma N = \bigoplus_{l=1}^{r} N_{i_{\sigma l} j_l}$$

for $\sigma \in S_r$.

The following facts are the key lemmas on degenerations in $\mathcal{F} \times \mathcal{F}$. For the sake of completeness we include a brief sketch of the proofs.

Lemma 8.1. *Let $N = \bigoplus_{l=1}^{r} N_{i_l j_l}$ be a decomposition as above, and let (t, s) with $t < s$ be a transposition. Then $N <_{deg} (t, s)N$ if and only if $i_t > i_s$.*

Proof. Assume that $i_t > i_s$. There is a short exact sequence

$$0 \longrightarrow N_{i_t j_s} \longrightarrow N_{i_t j_t} \oplus N_{i_s j_s} \longrightarrow N_{i_s j_t} \longrightarrow 0.$$

So

$$N_{i_t j_t} \oplus N_{i_s j_s} \leq_{deg} N_{i_t j_s} \oplus N_{i_s j_t}$$

and thus

$$N <_{deg} (t,s)N.$$

Conversely, assume that $i_t \leq i_s$. By comparing the dimensions of the stabilisers of N and $(t,s)N$ we see that $N \not<_{deg} (t,s)N$. □

We say that a degeneration $M \leq_{deg} N$ is minimal if $M \not\simeq N$ and $M \leq_{deg} X \leq_{deg} N$ implies $X \simeq M$ or $X \simeq N$.

Lemma 8.2. *Let $N = \bigoplus_{l=1}^{r} N_{i_l j_l}$ and $M \leq_{deg} N$ be minimal. Then there exists a transposition (t,s) such that $M \simeq (t,s)N$.*

Proof. Since $M \leq_{deg} N$ is minimal, there is a non-split extension

$$0 \longrightarrow N' \longrightarrow M \longrightarrow N'' \longrightarrow 0,$$

where $N \simeq N' \oplus N''$. We may choose summands $N_{i_s j_s}$ and $N_{i_t j_t}$ of N' and N'', respectively, such that taking pushout along the projection $N' \to N_{i_s j_s}$ and then pullback along the inclusion $N_{i_t j_t} \to N''$ gives us a non-split extension

$$0 \longrightarrow N_{i_s j_s} \longrightarrow M' \longrightarrow N_{i_t j_t} \longrightarrow 0.$$

This extension is of the form of the extension in the proof of Lemma 8.1. Hence

$$M' <_{deg} N_{i_s j_s} \oplus N_{i_t j_t},$$

and so

$$M' \oplus \left(N'/N_{i_s j_s}\right) \oplus \left(N''/N_{i_t j_t}\right) <_{deg} \left(N'/N_{i_s j_s}\right) \oplus \left(N''/N_{i_t j_t}\right) \oplus N_{i_s j_s} \oplus N_{i_t j_t} \simeq N.$$

By the construction of M',

$$M \leq_{deg} M' \oplus \left(N'/N_{i_s j_s}\right) \oplus \left(N''/N_{i_t j_t}\right),$$

so by the minimality of the degeneration,

$$M \simeq M' \oplus \left(N'/N_{i_s j_s}\right) \oplus \left(N''/N_{i_t j_t}\right),$$

and so the lemma follows. □

There is a unique closed orbit in $\mathcal{F}_\alpha \times \mathcal{F}_\beta$. We describe a corresponding representation.

Lemma 8.3. *The orbit of a pair of flags corresponding to a representation N is closed, if and only if $N \simeq \bigoplus_{l=1}^{r} N_{i_l j_l}$ with $i_l \leq i_{l+1}$ for all $l = 1, \cdots, r-1$.*

Proof. A representation $N = \bigoplus_{l=1}^{r} N_{i_l j_l}$ with $i_l \leq i_{l+1}$ does not have any proper degenerations, according to Lemma 8.1. Hence N and thus the corresponding pair of flags have closed orbits.

Conversely, if $i_l > i_{l+1}$ for some l, then N has a degeneration again by Lemma 8.1, and so the orbit of N is not closed. □

Alternatively, we may prove the lemma by observing that among all representations of the form $N = \bigoplus_{l=1}^{r} N_{i_l j_l}$ the representation with $i_l \leq i_{l+1}$ has a stabiliser of maximal dimension, and so this representation has a closed orbit. The stabiliser in this case is a parabolic in $\mathrm{GL}(V)$.

There is a unique open orbit in $\mathcal{F}_\alpha \times \mathcal{F}_\beta$ with a corresponding representation given as follows. The proof is similar to the proof of the previous lemma.

Lemma 8.4. *The orbit of a pair of flags corresponding to a representation N is open, if and only if $N \simeq \bigoplus_{l=1}^{r} N_{i_l j_l}$ with $i_l \geq i_{l+1}$ for all $l = 1, \cdots, r-1$.*

Similar to the closed orbit, a representation of the form $N \simeq \bigoplus_{l=1}^{r} N_{i_l j_l}$ with $i_l \geq i_{l+1}$ has a stabiliser of minimal dimension, and so the orbit is open. The stabiliser in this case is the intersection of two opposite parabolics in $\mathrm{GL}(V)$. Such stabilisers are called seaweeds [5] (see also [16]). The stabiliser of an arbitrary pair of flags is equal to the intersection of two parabolics in $\mathrm{GL}(V)$.

Let $o_{\alpha,\beta}$ denote the unique open orbit and $k_{\alpha,\beta}$ the unique closed orbit in $\mathcal{F}_\alpha \times \mathcal{F}_\beta$. Then $k_\alpha = k_{\alpha,\alpha}$ and we let $o_\alpha = o_{\alpha,\alpha}$. For $\tau \in S_r$, denote by $\tau o_{\alpha,\beta}$ the orbit of pairs of flags corresponding to the representation τN, where $N = \bigoplus_{l=1}^{r} N_{i_l j_l}$ with $i_{l+1} \leq i_l$ is the representation corresponding to $o_{\alpha,\beta}$. Similarly, denote by $\tau k_{\alpha,\beta}$ the orbit corresponding to τN, where $N = \bigoplus_{l=1}^{r} N_{i_l j_l}$ with $i_{l+1} \geq i_l$ is the representation corresponding to $k_{\alpha,\beta}$.

9. Idempotents from open orbits

Let $M(n,r)$ be the $\mathbb{Z}$-submodule of $G(n,r)$ with basis the open orbits in $\mathcal{F} \times \mathcal{F}$. In this section we prove that $M(n,r)$ is a subalgebra $G(n,r)$ that is also a direct factor. We also show that $M(n,r)$ is isomorphic to the $\mathbb{Z}$-algebra of $|\Lambda(n,r)| \times |\Lambda(n,r)|$-matrices with integer entries, where $|\Lambda(n,r)|$ is the cardinality of $\Lambda(n,r)$.

We start with two lemmas relating degeneration and multiplication in $G(n,r)$. Let $\leq_{deg}$ be the degeneration order on orbits in $(\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F})$ with the action of $\mathrm{GL}(V) \times \mathrm{GL}(V)$.

Lemma 9.1. *If $e_B \times e_{B'} \leq_{deg} e_A \times e_{A'}$, then $e_B \star e_{B'} \leq_{deg} e_A \star e_{A'}$*

Proof. Since $e_B \times e_{B'} \subseteq \overline{e_A \times e_{A'}}$ we have $S(B, B') \subseteq \overline{S(A, A')}$. By Corollary 6.2, we have that $\overline{S(A, A')}$ is the orbit closure of $e_A \star e_{A'}$ and $\overline{S(B, B')}$ is the orbit closure of $e_B \star e_{B'}$, the lemma follows. □

We have the following key lemma on degeneration and multiplication in $G(n,r)$.

Lemma 9.2. *Let $\sigma \in S_r$, $e_{B'} \subseteq \mathcal{F}_\alpha \times \mathcal{F}_\beta$ and $e_B \subseteq \mathcal{F}_\gamma \times \mathcal{F}_\delta$. Then $e_{B'} \star (\sigma o_{\beta,\gamma}) \star e_B \leq_{deg} \sigma o_{\alpha,\delta}$.*

Proof. By Lemma 9.1, it suffices to consider the case where e_B and $e_{B'}$ are closed orbits. By Lemma 8.3, we may choose the representation

$$\bigoplus_{l=1}^{r} N_{j_l k_l},$$

where $k_{l+1} \geq k_l$ and $j_{l+1} \geq j_l$ for the orbit e_B. Similarly, $o_{\beta,\gamma}$ is the orbit corresponding to the representation

$$\bigoplus_{l=1}^{r} N_{i_l j_l},$$

where $i_l \geq i_{l+1}$ by Lemma 8.4. Then the coefficient of $\sigma o_{\beta,\delta}$ in the product $(\sigma o_{\beta,\gamma}) \cdot e_B$ in $S_q(n,r)$ is non-zero, and so

$$(\sigma o_{\beta,\gamma}) \star e_B \leq_{deg} \sigma o_{\beta,\delta}.$$

Similarly,

$$e_{B'} \star \sigma o_{\beta,\delta} \leq_{deg} \sigma o_{\alpha,\delta}.$$

By Lemma 9.1,

$$e_{B'} \star \sigma o_{\beta,\gamma} \star e_B \leq_{deg} e_{B'} \star \sigma o_{\beta,\delta} \leq_{deg} \sigma o_{\alpha,\delta},$$

as required. □

Corollary 9.3. *Let $\sigma \in S_r$, $e_{B'} \subseteq \mathcal{F}_\alpha \times \mathcal{F}_\beta$ and $e_B \subseteq \mathcal{F}_\gamma \times \mathcal{F}_\delta$. Then $e_{B'} \star (\sigma k_{\beta,\gamma}) \star e_B \leq_{deg} \sigma k_{\alpha,\delta}$.*

Proof. The corollary follows from the previous lemma, since

$$\sigma k_{\alpha,\delta} = \sigma \iota o_{\alpha,\beta},$$

where $\iota(i) = n - i + 1$. □

Corollary 9.4. *Let $e_{B'} \subseteq \mathcal{F}_\alpha \times \mathcal{F}_\beta$ and $e_B \subseteq \mathcal{F}_\gamma \times \mathcal{F}_\delta$. Then $e_{B'} \star o_{\beta,\gamma} \star e_B = o_{\alpha,\delta}$. In particular, $o_{\alpha,\beta} \star o_{\beta,\gamma} = o_{\alpha,\gamma}$.*

Proof. By the lemma we know that $e_{B'} \star o_{\beta,\gamma} \star e_B \leq_{deg} o_{\alpha,\delta}$. Since $o_{\alpha,\delta}$ is the unique dense open orbit in $\mathcal{F}_\alpha \times \mathcal{F}_\delta$, the equality follows. □

Corollary 9.5. *$M(n,r)$ is an ideal in $G(n,r)$.*

Proof. The previous corollary shows that the $\mathbb{Z}$-submodule $M(n,r) \subseteq G(n,r)$ is closed under multiplication from both sides with elements from $G(n,r)$, and so it is an ideal. □

Lemma 9.6. *$\{o_\alpha\}_\alpha \cup \{k_\alpha - o_\alpha\}_\alpha$ is a set of pairwise orthogonal idempotents in $G(n,r)$.*

Proof. By Corollary 9.4, $(o_\alpha)^2 = o_\alpha$, $(k_\alpha - o_\alpha)o_\alpha = o_\alpha - o_\alpha = 0$, $o_\alpha(k_\alpha - o_\alpha) = o_\alpha - o_\alpha = 0$, and $(k_\alpha - o_\alpha)^2 = (k_\alpha - o_\alpha - o_\alpha + o_\alpha) = k_\alpha - o_\alpha$. All other orthogonality relations follow from the definition of multiplication in $S_q(n,r)$. □

Let $M(\Lambda(n,r))$ be the $\mathbb{Z}$-algebra of $|\Lambda(n,r)| \times |\Lambda(n,r)|$-matrices with integer entries. Let

$$\omega_0 : M(n,r) \to M\big(\Lambda(n,r)\big)$$

be the $\mathbb{Z}$-linear map where $\omega_0(o_{\alpha,\beta}) = E_{\alpha,\beta}$ is the (α,β)-elementary matrix in $M(\Lambda(n,r))$.

Lemma 9.7. *The map $\omega_0 : M(n,r) \to M(\Lambda(n,r))$ is a $\mathbb{Z}$-algebra isomorphism.*

Proof. The result is an immediate consequence of Corollary 9.4. □

Lemma 9.8. *The map $\omega : G(n,r) \to M(n,r)$ defined by*

$$\omega(e_A) = o_{\alpha,\beta}$$

for all $e_A \subseteq \mathcal{F}_\alpha \times \mathcal{F}_\beta$ is a surjective $\mathbb{Z}$-algebra homomorphism.

Proof. The map is clearly a surjective $\mathbb{Z}$-module homomorphism. Let $e_A \subseteq \mathcal{F}_\alpha \times \mathcal{F}_\beta$ and $e_B \subseteq \mathcal{F}_\beta \times \mathcal{F}_\gamma$. Then $\omega(e_A \star e_B)$ is the unique open orbit in $\mathcal{F}_\alpha \times \mathcal{F}_\gamma$, which is equal to $\omega(e_A) \star \omega(e_B)$, by Corollary 9.4. Moreover, $\omega(1_{G(n,r)}) = \omega(\sum_\alpha k_\alpha) = \sum_\alpha o_\alpha = 1_{M(n,r)}$. This completes the proof of the lemma. □

We can now prove the main result of this section, which implies that $M(n,r)$ is a direct factor of the $\mathbb{Z}$-algebra $G(n,r)$.

Theorem 9.9. *We have an isomorphism of $\mathbb{Z}$-algebras $G(n,r) \to M(n,r) \times (G(n,r)/M(n,r))$ given by $e_A \mapsto (\omega(e_A), \overline{e_A})$.*

Proof. By Corollary 9.4, we have

$$M(n,r) = \Big(\sum_\alpha o_\alpha\Big) G(n,r) \Big(\sum_\alpha o_\alpha\Big).$$

Now, $1_{G(n,r)} = \sum_\alpha k_\alpha$, and again by Corollary 9.4, $\sum_\alpha o_\alpha$ is a central idempotent in $G(n,r)$. This proves that $M(n,r)$ is a direct factor of $G(n,r)$, and so the theorem follows. □

Let $\widetilde{\mathbb{A}}_n$ denote the preprojective algebra of type $\mathbb{A}_n$. See [4] for the definition and properties of preprojective algebras.

Corollary 9.10. $S_0(2,r) \simeq M(2,r) \times \widetilde{\mathbb{A}}_{r-1}$

Proof. We need to show that

$$\Big(\sum_\alpha k_\alpha - o_\alpha\Big) G(n,r) \Big(\sum_\alpha k_\alpha - o_\alpha\Big) \simeq \widetilde{\mathbb{A}}_{r-1}.$$

First observe that $(\sum_\alpha k_\alpha - o_\alpha) G(n,r) (\sum_\alpha k_\alpha - o_\alpha)$ is generated by $e_{1,\alpha} - o_{\alpha-\epsilon_2+\epsilon_1,\alpha}$, $f_{1,\alpha} - o_{\alpha+\epsilon_2-\epsilon_1,\alpha}$ and $k_\alpha - o_\alpha$. A direct computation shows that the generators satisfy the preprojective relations. By comparing dimensions we get the required isomorphism. □

Let $\beta = (n_1, \cdots, n_l)$ and $\gamma = (r_1, \cdots, r_l)$ be decompositions of n and r, respectively, into l parts, where $n_i > 0$. Let $m_i = \sum_{j=1}^{i-1} n_j$, where $m_1 = 0$.

There is a map ϕ_j of flags of length n_j to flags of length n given by $\phi_j(f)_l = 0$ for $l \le m_j$, $\phi_j(f)_l = f_{l-m_j}$ for $m_j < l \le m_{j+1}$ and $\phi_j(f)_l = f_{n_j}$ for $l > m_{j+1}$, where f_i denotes the vector space at the ith-step of the flag f. The corresponding map on orbits of pairs of flags

$$[f, f'] \mapsto [\phi_j(f), \phi_j(f')]$$

is also denoted by ϕ_j.

Let

$$\phi_{\beta,\gamma} : G(n_1, r_1) \times \cdots \times G(n_l, r_l) \to G(n, r)$$

be the $\mathbb{Z}$-linear map defined by

$$(N_1, \cdots, N_l) \mapsto \phi_1(N_1) \oplus \cdots \oplus \phi_l(N_l).$$

Lemma 9.11. *The map*

$$\phi_{\beta,\gamma} : G(n_1, r_1) \times \cdots \times G(n_l, r_l) \to G(n, r)$$

is an injective $\mathbb{Z}$-algebra homomorphism. Moreover, $\phi_{\beta,\gamma}(N_1, \cdots, N_l) \leq_{deg} \phi_{\beta,\gamma}(N'_1, \cdots, N'_l)$ if and only if $N_i \leq_{deg} N'_i$ for all i.

Proof. Since $\phi_{\beta,\gamma}$ is injective on basis elements, it is an injective $\mathbb{Z}$-linear map. By Lemma 2.2, in terms of matrices, the map is given by

$$\phi_{\beta,\gamma}(e_{A_1}, \cdots, e_{A_l}) = e_{A_1 \oplus \cdots \oplus A_l}.$$

Following Lemma 6.11, the map $\phi_{\beta,\gamma}$ preserves multiplication and thus is an injective $\mathbb{Z}$-algebra homomorphism.

Let

$$N = \phi_{\beta,\gamma}(N_1, \cdots, N_l) \quad \text{and} \quad N' = \phi_{\beta,\gamma}\big(N'_1, \cdots, N'_l\big),$$

and $N \leq_{deg} N'$. We may assume that the degeneration is minimal. By Lemma 8.2, $N' = (t, s)N$ for a transposition (t, s). Then the transposition (t, s) must act within one N_i, since the off-diagonal blocks of the matrices of both N and N' are zero, and so

$$(t, s)N = \phi_{\beta,\gamma}\big(N_1, \cdots, N_{i-1}, \big(t', s'\big)N_i, N_{i+1}, \cdots, N_l\big)$$

for a transposition (t', s'). This shows that $N_i \leq_{deg} N'_i$ for all i.

The converse also follows from Lemma 8.2. □

Let β, γ and m_i be as above. Let $\alpha^i = (\alpha_{m_i+1}, \cdots, \alpha_{m_{i+1}})$ be a decomposition of r_i into n_i parts. Then $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a decomposition in $\Lambda(n, r)$. Let

$$o_{(\alpha,\beta)} = \phi(o_{\alpha^1} \cdots, o_{\alpha^l}).$$

By Lemma 6.11 and Corollary 9.4, we have the following.

Lemma 9.12. *The orbit $o_{(\alpha,\beta)}$ is an idempotent.*

We call $o_{\alpha,\beta}$ an idempotent orbit. Note that $k_\alpha = o_{(\alpha,(1,\cdots,1))}$ and that $o_\alpha = o_{(\alpha,n)}$, where n denotes the trivial decomposition of n into 1 part. For a given α, if k_α is in the interior of the quiver $\Sigma(n, r)$ viewed as an $(n - 1)$-simplex, there is exactly one idempotent orbit for each decomposition β and two different decompositions give two different idempotents, so there are 2^{n-1} idempotent orbits in $k_\alpha G(n, r)k_\alpha$, If k_α is on the boundary, but in the interior of a t-simplex, then there are 2^t idempotent orbits. In particular, in the interior of a line, i.e. the 1-faces, there are the two idempotent orbits k_α and o_α, and for the vertices of the simplex, i.e. the 0-faces, there is a unique idempotent orbit $k_\alpha = o_\alpha$.

Lemma 9.13. *If $o_{(\alpha,\beta)} \leq_{deg} N$, then $o_{(\alpha,\beta)} \star N = N \star o_{(\alpha,\beta)} = o_{(\alpha,\beta)}$.*

Proof. By Lemma 9.1,

$$o_{(\alpha,\beta)} = o_{(\alpha,\beta)} \star o_{(\alpha,\beta)} \leq_{deg} N \star o_{(\alpha,\beta)} \leq_{deg} k_\alpha \star o_{(\alpha,\beta)} = o_{(\alpha,\beta)}.$$

So

$$N \star o_{(\alpha,\beta)} = o_{(\alpha,\beta)}.$$

Similarly,

$$o_{(\alpha,\beta)} \star N = o_{(\alpha,\beta)}. \qquad \square$$

10. Geometric realisation of 0-Hecke algebras

In this section, let $n = r$ and $\alpha = (1, \cdots, 1)$. In this case $\mathcal{F}_\alpha$ is the complete flag variety and the idempotent k_α is the unique interior vertex in the quiver $\Sigma(n,n)$. Let

$$H_0(n) = k_\alpha S_0(n,n) k_\alpha,$$

which is a $\mathbb{Z}$-algebra. It is known that $H_0(n) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the 0-Hecke algebra $\mathcal{H}_0(n)$ (see [3,8,17,19]). We prove this fact below, using $G(n,n)$ and thus give a geometric construction of 0-Hecke algebras.

From the previous section, we have 2^{n-1} distinct idempotents $o_{(\alpha,\beta)}$, one for each decomposition $\beta = (n_1, \cdots, n_l)$ of n with $n_i > 0$.

Let

$$t_i = (i, i+1) k_\alpha.$$

We have

$$t_i = o_{(\alpha,\beta)},$$

where $\beta = (n_1, \cdots, n_{r-1})$ with $n_i = 2$ and $n_j = 1$ for $j \neq i$, and so t_i is an idempotent. Also

$$t_i = f_{i,\alpha+\epsilon_i-\epsilon_{i+1}} \star e_{i,\alpha} = e_{i,\alpha-\epsilon_i+\epsilon_{i+1}} \star f_{i,\alpha}.$$

Although it can be deduced from a bubble sort algorithm that the t_i generate $H_0(n)$ as a $\mathbb{Z}$-algebra, we will give an explicit construction of each of the basis element e_A in $H_0(n)$, using the multiplication in $H_0(n)$.

Lemma 10.1. *Suppose $i < j$. Then $t_i \star t_{i+1} \star \cdots \star t_{j-1} = (i, i+1, \cdots, j) k_\alpha$, where $(i, i+1, \cdots, j)$ is a cycle in S_n.*

Proof. It follows from the fact that $t_i = f_{i,\alpha+\epsilon_i-\epsilon_{i+1}} \star e_{i,\alpha}$ (or $= e_{i,\alpha-\epsilon_i+\epsilon_{i+1}} \star f_{i,\alpha}$) and the fundamental multiplication rules in Lemma 6.11. $\square$

Let σ be a permutation. Let

$$t^\sigma = t^{\sigma,n} \star \cdots \star t^{\sigma,1}$$

be defined by

$$t^{\sigma,1} = t_1 \star t_2 \star \cdots \star t_{\sigma^{-1}(1)-1}$$

and then

$$t^{\sigma,i} = t_i \star t_{i+1} \star \cdots \star t_{\tau_{i-1}\sigma^{-1}(i)-1},$$

where τ_{i-1} is given by

$$\tau_{i-1} k_\alpha = t^{\sigma,i-1} \star \cdots \star t^{\sigma,1}.$$

Theorem 10.2. *With the notation above, $t^\sigma = \sigma k_\alpha$. Consequently, the set of all t^σ for $\sigma \in S_n$ is a multiplicative basis of $H_0(n)$.*

Proof. By the previous lemma, $t^{\sigma,1} = (1, \cdots, \sigma^{-1}(1))k_\alpha$. As a representation $\tau_1 k_\alpha = t^{\sigma,1}$ has the summand $N_{1,\sigma^{-1}(1)}$, which is fixed by any t_i for $i > 1$, and therefore by $t^{\sigma,i}$ for $i > 1$. By induction $\tau_i k_\alpha$ has the summands $N_{j,\sigma^{-1}(j)}$ for $j = 1, \cdots, i$, which are fixed by $t^{\sigma,j}$ for $j > i$. Therefore $t^\sigma = \sigma k_\alpha$. □

We construct the idempotents $o_{(\alpha,\beta)}$ using the generators t_i. Let $[i, j]$ be an interval in $[1, \cdots, n]$. Define $t^{[i,j]}$ by induction as follows. Let $t^{[i,i]} = k_d$ and

$$t^{[i,j]} = t^{[i+1,j]} \star t_i \star \cdots \star t_{j-1}.$$

To each decomposition $\beta = (n_1, \cdots, n_l)$, let

$$t^\beta = t^{[m_1+1,m_2]} \star \cdots \star t^{[m_l+1,m_{l+1}]},$$

where $m_i = \sum_{j=1}^{i-1} n_j$ and $m_1 = 0$.

Corollary 10.3. *We have $t^\beta = o_{(\alpha,\beta)}$.*

Recall that the 0-Hecke algebra $\mathcal{H}_0(n)$ is a $\mathbb{C}$-algebra generated by $\mathcal{T}_i$ for $i = 1, \cdots, n-1$ with generating relations

i) $\mathcal{T}_i^2 = -\mathcal{T}_i$,
ii) $\mathcal{T}_i \mathcal{T}_{i+1} \mathcal{T}_i = \mathcal{T}_{i+1} \mathcal{T}_i \mathcal{T}_{i+1}$ and
iii) $\mathcal{T}_i \mathcal{T}_j = \mathcal{T}_j \mathcal{T}_i$ for $|i - j| > 1$.

The algebra $\mathcal{H}_0(n)$ is a specialisation of a Hecke algebra at $q = 0$ and has dimension $n!$.

Theorem 10.4. *As $\mathbb{C}$-algebras, $H_0(n) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathcal{H}_0(n)$*

Proof. Let

$$h : \mathcal{H}_0(n) \to H_0(n) \otimes_{\mathbb{Z}} \mathbb{C}$$

be given by $h(\mathcal{T}_i) = -t_i$. A direct computation in $H_0(n)$ shows that $-t_i$ satisfy the 0-Hecke relations i), ii) and iii) above, so the map is well defined. The two algebras have the same dimension over $\mathbb{C}$, and so it suffices to have that the map is surjective, which is indeed true by Theorem 10.2. So the two algebras are isomorphic. □

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