Approximate P-values of Certain Tests Involving Hypotheses About Multiple Breaks

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Abstract

We provide formulae for calculating approximate p-values for the non-standard asymptotic null distributions of a variety of tests used for detecting multiple structural change in a wide range of models. Our approximations are based on simulated quantiles obtained from 100,000 replications, and the latter are more accurate than the quantiles reported in the literature by a factor of 10,000 replications. The p-value response surfaces are approximated using a parametric method proposed by Hansen (1997) and their use is illustrated with an example. Using our p-value response surfaces, it is shown that the use of Bai and Perron's (2003b) response surfaces for the critical values of these tests can lead to misleading inferences, and thus should be used with extreme caution.

JEL classification: C1

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1 Introduction

There has been considerable interest in developing inference techniques for econometric models that exhibit parameter variation at multiple unknown points. Leading statistics are the so-called Sup-, UDmax- and sequential tests that are used respectively to test stability versus a fixed number of breaks, stability against up to a fixed number of breaks, and ℓ breaks versus $\ell+1$ breaks. These statistics have been shown to have the same limiting distributions in linear models (with exogenous regressors) estimated by Ordinary Least Squares (Bai and Perron, 1998), linear models (with endogenous regressors) estimated via Two Stage Least Squares (Hall, Han, and Boldea, 2011), and nonlinear regression models estimated by nonlinear least squares (Boldea and Hall, 2010). The limiting distributions in question are non-standard but depend only on the number of unknown breaks, a trimming parameter, and the number of time varying regression parameters in each regime. To date, researchers wishing to use these inference procedures have only been able to perform tests using simulated critical values for conventional significance levels. While such critical values enable researchers to determine whether the hypothesis in question is rejected or not at the significance level in question, they do not reveal the extent to which a decision may be considered "marginal". The latter information is revealed by the p-value (or equivalently, the observed significance level) of the test; as a result, p-values are routinely reported in other settings in applied econometric analysis. The lack of p-values for the parameter variation tests described above can thus be argued as a lacuna in the literature.

In this paper, we provide response surfaces that allow for the easy yet accurate calculation of asymptotic p-values. Our results are based on simulated quantile functions that are more accurate by a factor of 10,000 replications than the majority of the critical values for the tests in question that have been published in the literature. A further advantage of our results is that they cover an extended set of values for q, thus making it possible to perform inference about parameter variation in a broader class of models than was previously possible.

The paper is organised as follows: The next section presents the tests, their asymptotic null distributions, and the models in which they can be applied, as well as discussing the limitations of simulated critical values available in the literature. Section 3 deploys the methodology used to obtain the response surfaces, while Section 4 illustrates their use, and analyses their performance. The tables of parameters used for the calculation of the p-values are given in the Appendix.

2 The model and hypothesis testing

Consider the multiple linear regression model with m breaks ($m+1$ regimes):

$$y_t = x_t'\beta + z_t'\delta_j + u_t, \qquad t = T_{j-1}+1, \ldots, T_j$$

for $j = 1, \ldots, m+1$ with $T_0 = 0$ and $T_{m+1} = T$. In this model, y_t is the dependent variable, x_t is a $p \times 1$, z_t is a $q \times 1$ vector of explanatory variables, and u_t is a mean zero error.[1] This specification allows for partial structural change where the variables in x_t are not subject to shifts but are estimated using the entire sample.[2] The remaining variables, contained in z_t, are subject to multiple breaks at unknown locations denoted $T_i = [T\lambda_i]$ for $0 < \lambda_1 < \ldots < \lambda_m < 1$, where [.] denotes the integer part of the quantity in brackets. The breaks are assumed to be asymptotically distinct and bounded from the boundaries of the sample with the admissible set of break locations defined as $\Lambda_\epsilon = \{(\lambda_1, ..., \lambda_k) : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1 - \epsilon\}$ for an arbitrary small positive number ϵ that serves the purpose of a sample trimming parameter.

The method of estimation for this model, proposed in Bai and Perron (1998), is one based on least squares. For each partition defined by a set of break points $(T_1, \ldots, T_m)$ denoted $\{T_j\}$, the least squares estimates of the model parameters, denoted as $\hat{\beta}(\{T_j\})$ and $\hat{\delta}(\{T_j\})$ respectively, are calculated by minimizing

$$S_T(T_1, \ldots, T_m) = \sum_{i=1}^{m+1} \sum_{t=T_{i-1}+1}^{T_i} (y_t - x_t'\beta - z_t'\delta_i)^2 .$$

The estimated break points $(\hat{T}_1, \ldots, \hat{T}_m)$ are then calculated by minimizing the sum of square residuals across all admissible partitions. This can be written as

$$(\hat{T}_1, \ldots, \hat{T}_m) = argmin_{T_1,\ldots,T_m} S_T(T_1, \ldots, T_m) \tag{1}$$

and the parameter estimates of the model will be the ones based on the resulting estimated set of break points. While this global minimization requires an exhaustive grid search, Bai and Perron (2003a) provide an algorithm that alleviates much of the computational cost by applying the principle of dynamic programming that results in least squares calculations in the order of at most $O(T^2)$ instead of the order $O(T^m)$ calculations that would be required for a full grid search in a model with m breaks.

[1] The error must also satisfy certain other conditions detailed in the sources mentioned below.

[2] If $p = 0$ then all parameters are regime specific and the model is said to exhibit pure structural change.

A variety of tests that can be used for different types of hypotheses of interest in this type of model were proposed in Bai and Perron (1998). To simplify the presentation, we describe the versions of these tests based on F-type statistics but it should be noted that all the tests described below can also be based on Wald-type statistics. The assumptions under which these tests are valid, some general and some depending on the specific hypothesis that is being considered, as well as the arguments for consistency, can be found in the aforementioned paper and will not be reproduced here for brevity.

For the simple hypothesis of 0 against k breaks, that is $H_0 : m = 0$ vs $H_1 : m = k$, the F-type tests are based on

$$F_T(\lambda_1, ..., \lambda_k; q) \;=\; \left\{ \frac{T - (k+1)q - p}{kq} \right\} \left\{ \frac{SSR_0 - SSR_k}{SSR_k} \right\}$$

for a particular set of break locations $\lambda_1, ..., \lambda_k$. SSR_0 and SSR_k are the sum of squared residuals under the null and alternative hypotheses, respectively. These statistics, calculated for all the admissible sets of $T_i = [T\lambda_i]$, $i = 1, \ldots, k$, can be used to construct the Sup-F statistic, $Sup - F_T(k; q)$, as

$$Sup - F_T(k; q, \epsilon) \;=\; sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F_T(\lambda_1, .., \lambda_k; q). \tag{2}$$

Since $Sup - F_T(k; q, \epsilon)$ is constructed to test the hypothesis of a specific number of breaks (k), it is useful to also consider a double maximum test that compares $Sup - F_T(k; q, \epsilon)$ statistics for different values of k starting from one to some ceiling K. This effectively changes the testing hypothesis to $H_0 : m = 0$ vs $H_1 : m \leq K$ and the appropriate statistic can be written as

$$UDmaxF_T(K; q, \epsilon) \;=\; \max_{1\leq k\leq K} \sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F_T(\lambda_1, ..., \lambda_k; q, \epsilon). \tag{3}$$

Alternatively, a weighted version of $UDmax$ may be used, motivated by the fact that as the number of breaks m increases, a fixed sample of data becomes less informative due to the estimation being based on smaller partitions. Bai and Perron (1998) note that since the critical values of the Sup-F tests decrease as m increases for a fixed q so will the marginal p-values, and propose to use the critical values as weights to ensure that the marginal p-values are equal across m. Denoting the asymptotic critical value of $Sup - F_T(k; q, \epsilon)$ as $c(q, \alpha, k, \epsilon)$ for a significance level α, the weights are defined as $w_1 = 1$ for $k = 1$ and $w_k = \frac{c(q,\alpha,1,\epsilon)}{c(q,\alpha,k,\epsilon)}$ for $k > 1$. Then, the

weighted double maximum test can be written as

$$WDmaxF_T(K;q,\epsilon) = \max_{1\leq k\leq K} w_k \times \sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F_T(\lambda_1,...,\lambda_k;q,\epsilon). \quad (4)$$

Another hypothesis of interest is that of $H_0 : m = \ell$ vs $H_1 : m = \ell + 1$. Statistics developed for this hypothesis may be used as part of a sequential strategy that starts by estimating one break and then tests for the presence of an additional break in the segments defined by the first break. Iterating this procedure by testing for three against two breaks and in general for $(\ell+1)$ against ℓ breaks can lead to an estimation of the total number of breaks in the model. To construct this statistic, start by denoting as $(\hat{T}_1,\ldots,\hat{T}_l)$ the ℓ fixed breaks that are calculated by globally minimizing the sum of squared residuals as in (1), and will form the null hypothesis. Then, for the model with $\ell+1$ breaks, the ℓ breaks are fixed at $(\hat{T}_1,\ldots,\hat{T}_l)$ and the location of the $(\ell+1)^{th}$ break is chosen by minimizing the residual sum of squares across all the admissible $l+1$ partitions. The statistic is then given by

$$F_T(\ell+1|\ell;q,\gamma) = \max_{1\leq i\leq l+1}\left\{\frac{SSR_\ell(\hat{T}_1,...,\hat{T}_\ell) - \inf_{\tau\in\Lambda_{i,\gamma}} SSR_{\ell+1}(\hat{T}_1,...,\hat{T}_{i-1},\tau,\hat{T}_i,...,\hat{T}_\ell)}{\hat{\sigma}^2}\right\} \quad (5)$$

where

$$\Lambda_{i,\gamma} = \{\tau : \hat{T}_{i-1} + (\hat{T}_i - \hat{T}_{i-1})\gamma \leq \tau \leq \hat{T}_i - (\hat{T}_i - \hat{T}_{i-1})\gamma\}$$

for an arbitrary small γ, and $\hat{\sigma}^2$ is a consistent estimate of the error variance under the null hypothesis.

The following Lemma presents the limiting distributions of the tests in (2)-(5) that are derived in Bai and Perron (1998).[3]

Lemma 1 *Let $W_q(\cdot)$ be a q-vector of independent Wiener processes on $[0,1]$. Under certain conditions and $m = 0$,*

(i) $Sup - F_T(k;q,\epsilon) \Longrightarrow sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F(\lambda_1,...,\lambda_k;q)$

(ii) $UDmaxF_T(K;q,\epsilon) \Longrightarrow \max_{1\leq k\leq K} \sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F(\lambda_1,...,\lambda_k;q)$

(iii) $WDmaxF_T(K;q) \Longrightarrow \max_{1\leq k\leq K} w_k \times \sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F(\lambda_1,...,\lambda_k;q)$

[3]It should be noted that these limiting distributions only apply for the statistics based on $F_T(\lambda_1,...,\lambda_k;q)$ if the regression error u_t is homoscedastic and serially uncorrelated. If the errors are heteroscedastic and/or serially correlated then the same limiting distributions hold for the analogous functions of Wald statistics that satisfactorily account for the error structure; see Bai and Perron (1998) for further details.

and under $m = \ell$,

(iv) $lim_{T\to\infty} P(F_T(\ell+1|\ell; q, \gamma) \leq x) = G_{q,\gamma}(x)^{\ell+1}$

where

$$F(\lambda_1, ..., \lambda_k; q) \stackrel{def}{=} \frac{1}{kq} \sum_{i=1}^{k} \frac{[\lambda_i W_q(\lambda_{i+1}) - \lambda_{i+1} W_q(\lambda_i)]' [\lambda_i W_q(\lambda_{i+1}) - \lambda_{i+1} W_q(\lambda_i)]}{\lambda_i \lambda_{i+1} (\lambda_{i+1} - \lambda_i)}$$

and $G_{q,\gamma}(x)$ *is the distribution function of* $\sup_{\gamma \leq \mu \leq 1-\gamma} \|W_q(\mu) - \mu W_q(1)\|^2 / (\mu(1-\mu))$.

Note that these distributions are non-standard and depend on three parameters: the number of breaks k or ℓ as appropriate, the number of parameters q, and the trimming parameter ϵ or γ as appropriate. Furthermore, since $G_{q,\gamma}(x)$ is effectively the limit distribution of $Sup - F_T(1; q)$, the distributions of all the statistics above can be constructed using an approximation of $sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F(\lambda_1, ..., \lambda_k; q\epsilon)$ for a set of values for k, q, and ϵ.

A number of studies have provided approximated percentiles for the above distributions using Monte Carlo simulations for a variety of parameter sets and simulation settings. We detail the setup of the most inclusive of these simulations here as it may be seen as a decisive factor to accessing the availability and accuracy of the critical values that can be found in the literature. Andrews (1993) simulates the distribution of interest for the case of one break, $q = 1, \ldots, 20$, and $\epsilon = 0.05, 0.10, \ldots, 0.50$[4] using a sample of $T = 3,600$ observations and $10,000$ replications, the latter being the number of simulated points from the distribution that is being approximated. Bai and Perron (1998) provide results for the much more computationally demanding cases of up to nine breaks, but for only up to ten model parameters (q) and one value of $\epsilon = 0.05$. Their results are calculated using a sample of $T = 1,000$ and $10,000$ replications. Andrews (2003) provides a corrigendum to his previously reported percentiles, this time calculated with $100,000$ replications instead of $10,000$ for increased accuracy. However this is only available for the computationally simple case of $k = 1$. Bai and Perron (2003b) expand their previous results to include the cases of $\epsilon = 0.10, 0.15, 0.20, 0.25$ and also provide response surfaces that can be used to approximate the asymptotic critical values. These are based on a method of nonlinear regressions where the dependent variable is the simulated critical value, with the model parameters q, k, $\ell + 1$, and ϵ, in various combinations used as regressors. As such, Bai and Perron's (2003b) results are useful in cases where the simulated critical values are not available for a particular case k, q, and ϵ.

[4]Such large values of ϵ are reported because Andrews (1993) allows for asymmetric trimming.

Hansen (1997) presents response surfaces for calulating approximate p-values of $Sup - F_T(1; q, \epsilon)$ statistic for up to $q = 40$ model parameters, and a wide range of trimming windows ϵ from 0.01 to 0.49, using samples of $T = 1,000$ with $50,000$ replications in the quantile function approximation. There are two key differences between this method and Bai and Perron's (2003b) response surface approximations: first, within Hansen's (1997) approach, the response surfaces are used to calculate p-values that provide more information in inference than critical values; second, in Hansen's (1997) method, the approximation is based on optimising a function for the specific simulated distribution of interest in terms of (q, $k = 1$, ϵ) instead of optimising over a large set of values for (q,k,ϵ). However, an important limitation of Hansen's (1997) response surfaces is that they only cover the case of testing no breaks against one break ($k = 1$).

Remark 1 *Comparison of the approximated percentiles from the above sources, as well as our own initial simulations of the distributions described above using the same specifications, shows noticeable differences between sources as they result in critical values that can differ in the first decimal point or even in multiple units depending on the case considered. Repeating a simulation case for different seed values in the random number generator can also illustrate this. The leading reason behind these differences, implied by Andrews (2003) and verified by our experiments, is the number of replications and sample size that is used in the simulations. At a high cost of computer resources accuracy can be improved for the cases of multiple breaks, as it is by Andrews (2003) for the one break case.*

Remark 2 *Recent developments in the literature have established that the distributions described in Lemma* 1 *can be applied to structural stability tests developed for a wider range of models than the OLS linear regression. This makes the availability of an updated set of asymptotic critical values and response surfaces for models with multiple breaks even more important to applied work. The following is a list of proposed tests and model specifications that use the non-standard distributions considered here to test for multiple breaks: Hall, Han, and Boldea (2011) in linear models with endogenous regressors estimated by two stage least squares; Boldea and Hall (2010) in nonlinear least squares models; Bai, Chen, Chong, and Wang (2008) in models under specification errors.*

Remark 3 *The current literature does not provide critical values for multiple breaks tests with* $q > 10$ *parameters, which can be restrictive in applied work.*

Motivated by the above remarks, in this paper we use Hansen's (1997) methodology to obtain response surfaces for the calculation of approximate p-values for the cases of multiple breaks for a large set of parameter values, using simulation settings that ensure improved accuracy. The next section discusses the setup for our simulations and methodology for the p-value approximations.

3 P-value function approximations

To simulate $sup_{(\lambda_1,...,\lambda_k)\in\Lambda_\epsilon} F(\lambda_1, ..., \lambda_k; q)$ from *Lemma* 1 we approximate the Weiner process, $W_q(\lambda)$, by the partial sums $T^{-\frac{1}{2}} \sum_{i=1}^{[T\lambda]} e_i$ with e_i $i.i.d.N(0, I_q)$ and $T = 3,600$. Matching Andrews's (2003) setup we iterate $100,000$ times to increase the accuracy over currently available results. For each iteration, we use Bai and Perron's (2003a) dynamic programming algorithm to calculate the supremum of $F(\lambda_1, ..., \lambda_k; q)$ with respect to $(\lambda_1, ..., \lambda_k)$ in the set Λ_ϵ. We include all combinations of $q = 1, 2, \ldots, 20$ model parameters, $\epsilon = 0.05, 0.10, \ldots, 0.25$ trimming values, and $k = 1, 2, \ldots, 9$ breaks up to the point where the trimming window will not lead to trivial break point estimations. So, for $\epsilon = 0.25$ we allow for a maximum of 2 breaks since a 3^{rd} break would result in breaks being fitted at 25%, 50%, and 75% of the sample by default. Similarly, we allow for up to 3 breaks for $\epsilon = 0.20$, 5 breaks for $\epsilon = 0.15$, 8 breaks for $\epsilon = 0.10$, and 9 breaks for $\epsilon = 0.05$.

This specification provides quantile approximations for the distributions of statistics up to $Sup - F_T(9; 20)$, and $F_T(9|8)$. For the $UDmax$ and $WDmax$ statistics we set the ceiling of allowed breaks to $K = 5$, where allowed by the ϵ value, and the maximum number of allowed breaks otherwise. So, for $\epsilon = 0.05, 0.10, 0.15$ we use $K = 5$, but for $\epsilon = 0.20$ and 0.25 we use $K = 3$ and $K = 2$, respectively.

Under these specifications, the wall clock time to complete the simulations is prohibitive on a desktop computer. To overcome this we used a Condor pool, a mechanism that utilises spare cycles in a collection of processor cores (96 cores in our case). To implement this solution we split the replications for each simulation experiment into a number of separate jobs, each job performing part of the overall calculation, and collated the data after the jobs were complete.

Based on these simulated quantile functions, we calculate approximate p-value response surfaces following Hansen's (1997) methodology. This provides computationally convenient and very accurate response surface approximations that are easy to use in applied work as they only

involve a ν degree polynomial and a standard chi-squared distribution.[5]

To describe this methodology, let $p(x) = P(X \geq x)$ denote the true p-value function of X. In our context, X is then the asymptotic distribution for either $Sup - F_T(k;q)$, $F_T(\ell+1|\ell)$, $UDmaxF_T(K;q)$, or $WDmaxF_T(K;q)$, and the approximated p-values of our Monte Carlo simulations can be denoted as $\hat{p}(x)$. The objective of Hansen's (1997) method is to use a parametric function to calculate response surfaces based on $\hat{p}(x)$. For this purpose he uses a function of the form

$$p(x|\theta,\eta) = 1 - \chi^2(\theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_\nu x^\nu|\eta) \tag{6}$$

where $\chi^2(\cdot|\eta)$ is the cumulative distribution function of the chi-squared distribution with η degrees of freedom and $\theta = (\theta_0, \theta_1, \ldots \theta_\nu)'$ is the collection of parameters in a ν degree polynomial. A chi-squared is chosen as the leading distribution for the similarity it presents to the limiting distributions of the statistics for different parameter values since, as Hansen observes, for $\epsilon = 0.5$ the limit of $Sup - F_T(1;q)$ simplifies to a χ^2_q distribution.

The parameters θ and η have to be chosen such as to make $p(x|\theta,\eta)$ as close as possible to $p(x)$ for all values of x in the range of the random variable X. Denoting the inverse of the true p-value function by $Q(\cdot)$, that is $Q(\cdot) = p(\cdot)^{-1}$, we have the following identity

$$\phi = p(Q(\phi)) \;\; \forall \phi \in [0,1].$$

Using this transformation, fitting (6) to the true p-values $p(x)$ is then equivalent to choosing θ and η so that ϕ is as close as possible to $p(Q(\phi)|\theta,\eta)$ for all values of $\phi \in [0,1]$. To increase the accuracy of the approximation for smaller p-values, that are of more interest in inference, Hansen (1997) further proposes a weighted criterion function, that based on the above will take the form of

$$\begin{pmatrix} \theta^* \\ \eta^* \end{pmatrix} = argmin_{(\theta,\eta)} \left[\int_0^1 |p(Q(\phi)|\theta,\eta) - \phi|^r \, \omega(\phi) d\phi \right]^{\frac{1}{r}} \tag{7}$$

where r is a positive integer and $\omega(\phi)$ is a weight function that assigns large weights to small p-values and small weights to large p-values. For a particular choice of $\omega(\phi)$ and r we follow

[5] This methodology has also been employed by Sen and Hall (1999) for the distributions of their overidentifying restrictions test (O_T) and by Sen (1999) for the Ghysels, Guay, and Hall's (1997) predictive test.

Hansen (1997) and use

$$\omega(\phi) = \begin{cases} 1, & 0 \leq \phi \leq 0.1 \\ \left[\frac{0.8-\phi}{0.7}\right]^2, & 0.1 < \phi \leq 0.8 \\ 0, & 0.8 < \phi \leq 1 \end{cases}$$

with $r = 8$. The choice of the weight function becomes obvious when considering that the p-values that are normally of interest are in the region of 0.01 to 0.1, and the value r is chosen to ensure that a few large approximation errors are penalised more than many small errors. By replacing the integral with a sum over a sufficiently fine grid $\{\phi\}_{i=0}^{M}$ and replacing the true and unknown quantile function $Q(\phi_i)$ with the approximate critical value $\hat{Q}(\phi_i)$ that corresponds to the $(1-\phi_i)\times 100\%$ significance level, the minimization problem of (7) becomes

$$\begin{pmatrix} \theta^* \\ \eta^* \end{pmatrix} = argmin_{(\theta,\eta)} \left[\sum_{i=0}^{M} \left| p(\hat{Q}(\phi_i)|\theta,\eta) - \phi_i \right|^r \omega(\phi_i)\right]^{\frac{1}{r}}. \tag{8}$$

We implement the above by setting $M = 999$ and $\phi_i = i/1000$ and applying to all the quantiles of the $Sup - F_T(k;q)$, $F_T(\ell+1|\ell)$, $UDmaxF_T(K;q)$ and $WDmaxF_T(K;q)$ test distributions that were obtained in our simulations. Additionally, the order of polynomial ν is chosen so that the maximum error in the fitted p-values over all ϕ_i is less than 0.01, that is $max_{1\leq i\leq M}\left|p(\hat{Q}(\phi_i)|\theta,\eta) - \phi_i\right| < 0.01$. Then, the approximate p-value function is obtained by replacing (θ,η) in (6) with (θ^*,η^*). Experimentation with different ν revealed that $\nu = 1$ is sufficient for all tests and all cases of numbers of parameters, numbers of breaks, and trimming windows. This can be verified by looking at the maximum and median errors at selected p-values for all model specifications for each of the four statistics. These results are presented in Table 1.

The optimal (θ^*,η^*) parameters are presented in Tables $2a-2i$ for $Sup-F_T(k;q)$, $3a-3h$ for $F_T(\ell+1|\ell)$, $4a$ for $UDmaxF_T(K;q)$, and $4b$ for $WDmaxF_T(K;q)$. Note that since $F_T(1|0) = Sup - F_T(1;q)$ by construction, the results for $F_T(1|0)$ are those given by Table $2a$.

4 Illustration and Performance

To illustrate the use of the results we calculate approximate p-values for the $Sup - F_T(k;q)$ and $F_T(\ell+1|\ell)$ tests in one of the equations estimated in the empirical application in Hall, Han, and Boldea (2011). In the context of exploring the stability of the New Keynesian Phillips Curve for US data using multiple break tests in the presence of endogenous regressors, Hall,

Han, and Boldea (2011) apply these tests on an output gap equation estimated by OLS. Output gap is estimated using a constant, the first lag of inflation, the short term interest rate, the unemployment rate, the growth rate of the money aggregate M2, and three lags of inflation first differences to account for serial correlation. That is eight explanatory variables in total and the estimation uses a trimming value of $\epsilon = 0.1$. Some statistics for tests applied to this equation are given as $Sup - F_T(1;8) = 50$, $F_T(2|1) = 30.53$, $F_T(3|2) = 23.1$, and $F_T(4|3) = 11.3$. The p-value function parameters for the case of $Sup - F_T(1;8)$ can then be found in Table 2a, in the column for $\epsilon = 0.1$ and the row for $q = 8$. Applying these parameter values to (6), gives a p-value of $1 - \chi^2(-5.94 + 1.13 \times 50|12.38) = 0.000001$ thus strongly rejecting the null of no breaks. For the test of two against one breaks, $F_T(2|1)$, the parameter values can be found in Table 3a, and applying them in the same way results in a p-value of $1 - \chi^2(-10.06 + 1.14 \times 30.53|11.38) = 0.012$, thus rejecting the null of one break at the 5% significance level but not at the 1% level. The parameters for $F_T(3|2)$, and $F_T(4|3)$ are the equivalent entries in Tables 3b and 3c and applying them to (6) gives $1 - \chi^2(-12.16 + 1.12 \times 23.1|10.44) = 0.21$ and $1 - \chi^2(-13.73 + 1.20 \times 11.38|11.58) = 1$, respectively; thus both tests fail to reject, and these rejections are clearly not marginal. The evidence therefore suggests the presence of a single break in the equation.

To analyse the performance of the results presented in this paper we compare the p-values obtained from our response surfaces with two different sets of critical values as input. The first set of critical values are those obtained from our simulations. The second set of critical values are those obtained using the response surface reported in Bai and Perron (2003b). To compare these results, we tabulate the difference between our approximate 5% p-values and the "nominal" 0.05 value across multiple breaks ($k = 1, ..., 9$) for a given number of parameters in the model ($q = 1,\ 5,\ 10$), and trimming parameter ($\epsilon = 0.05$). Together, we present the differences from 0.05 of the p-values calculated using Bai and Perron's (2003b) approximate critical values for the same cases, obtained as described in Section 4 of that paper. To calculate p-values based on these, we feed the critical values to (6) using the relevant parameter values from the Appendix. These calculations are presented in Table 5 for the cases of $q = 1,\ 5,\ 10,\ 15$ and 20. The comparison shows a clear advantage of our results: it can be seen that our p-values are remarkably close to 0.05; whereas the p-values implied by Bai and Perron's (2003b) response surface for critical values can show severe distortions, up to 0.5 (that is, an observed significance level of

approximately 54% when the critical value is supposed to yield a 5% test[6]) for $q = 1$, up to 0.14 for $q = 5$, and 0.036 for $q = 10$. While the distortions are decreasing as q increases toward 10, they increase again as q goes beyond 10 with size distortions up to 0.125 for $q = 15$ and up to 0.574 for $q = 20$. However, it should be noted that Bai and Perron's (2003b) approximations are based on simulations only for $q \leq 10$ and so the results for $q = 15, 20$ are based on extrapolations to settings different from those used to estimate their response surfaces.[7] It is also notable that the Bai and Perron (2003b) results show significant variation across different numbers of breaks (their errors generally increase with the number of breaks), a feature not present in the p-values derived in this paper.[8] The reviewers asked us to discuss the source of these distortions that arise from the use of Bai and Perron's (2003b) response surfaces. We can only offer the conjecture that they arise because the functional form is chosen to provide a global approximation of the relationship between the critical values and (q, k, ϵ), and such a global approximation involves a trade-off in accuracies for different values of (q, k, ϵ) with the $q = 1$, $\epsilon = 0.05$ and k large turning out to be combinations for which this global approximation does not fit well; see footnote 6. Furthermore, Bai and Perron (2003b) report goodness of fit measures that relate to accuracy in predicting the critical values and not to the distance between the actual significance level associated with the predicted critical value and the nominal significance level of the test. It is the latter (as reported in Table 5) that is more relevant to assessing the performance of the test based on the predicted critical values from their response surfaces. In contrast, our response surfaces yield p-values and are different for each (ϵ, k, q) so that the accuracy at one

[6]From the case of $q = 1$, it can be seen that this case occurs for $k = 9$. Further investigation reveals the main source of the distortion is Bai and Perron's (2003b) response surface: the appropriate critical value reported in Bai and Perron (1998) is 5.20 which translates, using our response surfaces, to a p-value of 0.0758; the critical value predicted by Bai and Perron's (2003b) response surface is 3.8023. In contrast, our simulations (based on an enhanced design) yield a critical value of 5.4210 for which our response surface gives a p-value of 0.05006.

[7]In the original version of this paper, we restricted attention to $q = 1, 5, 10$ but, given the nature of the results for those cases, a referee conjectured Bai and Perron's (2003b) response surfaces would work well for cases $q > 10$. While natural given results for $q = 1, 5, 10$, this conjecture is erroneous as our results demonstrate. We therefore included these cases to avoid the reader making the same mistake. We attribute the inaccuracy of Bai and Perron's (2003b) response surfaces to these cases to the fact that the surfaces were calculated using cases for which $q \leq 10$ only.

[8]For completeness, we note that we performed similar calculations for $\epsilon = 0.10, 0.15, 0.20, 0.25$ for $q = 1, 5, 10$. The distortions are smaller than those reported for $\epsilon = 0.05$ but are nevertheless non-trivial in a number of circumstances. These results are available from the authors upon request.

choice (ϵ, k, q) has no impact on the accuracy at another choice. This comparison indicates the value of using our response surfaces over extant methods for inference in these types of situation.

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Appendix

Table 1: Absolute errors in fitted distributions

$p-value$	$SupF$		$F(l+l\|l)$		$UDmax$		$WDmax$	
	$Median$	$Maximum$	$Median$	$Maximum$	$Median$	$Maximum$	$Median$	$Maximum$
.00	.0001	.0003	.0002	.0007	.0002	.0007	.0001	.0001
.01	.0003	.0012	.0008	.0033	.0011	.0032	.0014	.0033
.02	.0004	.0015	.0009	.0037	.0013	.0038	.0016	.0047
.03	.0004	.0014	.0009	.0037	.0014	.0034	.0016	.0043
.04	.0004	.0015	.0008	.0038	.0014	.0032	.0016	.0041
.05	.0004	.0015	.0008	.0034	.0014	.0031	.0015	.0045
.06	.0004	.0015	.0007	.0035	.0013	.0028	.0012	.0038
.07	.0004	.0014	.0007	.0036	.0011	.0027	.0011	.0047
.08	.0004	.0014	.0006	.0030	.0010	.0023	.0012	.0040
.09	.0004	.0014	.0007	.0033	.0007	.0020	.0010	.0041
.10	.0004	.0013	.0006	.0026	.0005	.0020	.0010	.0041
.15	.0003	.0014	.0007	.0033	.0007	.0025	.0013	.0038
.20	.0004	.0015	.0007	.0042	.0011	.0027	.0013	.0033
.25	.0004	.0017	.0006	.0035	.0013	.0032	.0014	.0043
.30	.0004	.0016	.0006	.0034	.0010	.0031	.0012	.0048
.40	.0004	.0018	.0007	.0035	.0005	.0014	.0013	.0045
.50	.0004	.0017	.0008	.0036	.0012	.0035	.0019	.0044
.60	.0005	.0016	.0008	.0036	.0013	.0042	.0016	.0055
.70	.0005	.0021	.0009	.0043	.0006	.0019	.0014	.0059

Table 2a: SupF Distributions, $k = 1$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$			$\epsilon = 0.20$			$\epsilon = 0.25$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-1.41	1.08	3.76	-1.17	1.05	3.34	-1.01	1.02	3.02	-0.86	1.00	2.78	-0.74	0.98	2.55
2	-2.34	1.11	5.53	-1.99	1.06	4.94	-1.70	1.03	4.56	-1.44	1.02	4.36	-1.26	0.99	4.01
3	-3.27	1.13	6.94	-2.66	1.10	6.56	-2.22	1.07	6.25	-1.84	1.06	6.05	-1.58	1.04	5.67
4	-4.08	1.16	8.46	-3.29	1.14	8.21	-2.77	1.11	7.81	-2.45	1.07	7.22	-2.08	1.05	6.87
5	-4.86	1.14	9.44	-4.10	1.11	8.88	-3.55	1.07	8.39	-3.00	1.05	8.11	-2.50	1.04	7.91
6	-5.81	1.14	10.25	-4.83	1.11	9.96	-3.89	1.12	10.34	-3.20	1.12	10.25	-2.65	1.10	1.10
7	-6.30	1.15	11.80	-5.26	1.11	11.20	-4.28	1.11	11.47	-3.66	1.08	10.98	-3.15	1.05	10.43
8	-7.23	1.14	12.42	-5.94	1.13	12.38	-4.97	1.11	12.28	-4.26	1.08	11.86	-3.74	1.05	11.28
9	-8.07	1.14	13.19	-6.57	1.13	13.43	-5.67	1.10	12.97	-4.63	1.11	13.42	-4.15	1.06	12.45
10	-8.90	1.11	13.42	-7.51	1.10	13.54	-6.32	1.09	13.78	-5.32	1.08	13.78	-4.24	1.08	14.05
11	-9.36	1.15	15.51	-7.82	1.13	15.34	-6.45	1.13	15.79	-5.57	1.09	15.07	-4.73	1.07	14.82
12	-10.56	1.09	14.42	-8.84	1.08	14.85	-7.73	1.05	14.43	-6.60	1.05	14.69	-5.65	1.03	14.53
13	-10.83	1.14	16.97	-9.07	1.11	16.79	-8.12	1.06	15.61	-6.7	1.06	16.26	-5.51	1.07	16.73
14	-8.36	1.13	17.72	-9.68	1.09	17.30	-8.36	1.06	16.90	-7.36	1.04	16.61	-5.82	1.08	18.08
15	-12.53	1.11	17.47	-10.74	1.07	17.00	-9.23	1.05	17.07	-7.82	1.06	17.84	-6.60	1.06	18.11
16	-13.26	1.08	17.40	-11.61	1.01	15.94	-10.00	1.03	17.20	-8.47	1.05	18.24	-7.42	1.02	17.63
17	-13.90	1.08	18.34	-11.72	1.07	18.92	-10.10	1.06	19.08	-8.46	1.07	20.14	-7.27	1.06	20.17
18	-14.83	1.07	18.44	-12.61	1.03	18.35	-11.06	1.02	18.60	-9.38	1.03	19.61	-7.70	1.05	20.93
19	-15.33	1.06	19.24	-13.13	1.02	18.70	-11.46	1.02	19.50	-9.62	1.04	20.96	-8.55	1.02	20.29
20	-16.16	1.06	19.56	-13.55	1.07	21.37	-11.53	1.08	22.45	-10.02	1.07	22.78	-8.50	1.07	23.09

Table 2b: SupF Distributions, $k = 2$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$			$\epsilon = 0.20$			$\epsilon = 0.25$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-5.19	2.32	8.09	-3.06	2.19	7.31	-2.08	2.09	6.34	-1.59	1.95	5.11	-1.19	1.90	4.24
2	-7.92	2.32	10.62	-5.17	2.22	10.06	-3.47	2.18	9.73	-2.46	2.12	8.81	-1.79	2.02	7.46
3	-10.35	2.20	11.49	-6.89	2.17	11.98	-4.60	2.19	12.43	-3.40	2.11	11.30	-2.38	2.07	10.28
4	-12.34	2.19	13.24	-8.50	2.19	14.25	-5.62	2.24	15.43	-3.77	2.20	14.91	-2.78	2.09	12.98
5	-14.08	2.18	14.97	-9.95	2.16	15.86	-7.10	2.14	16.31	-4.77	2.17	16.89	-2.57	2.21	17.17
6	-16.01	2.25	17.50	-11.22	2.25	19.09	-8.15	2.20	19.18	-6.18	2.12	17.95	-4.01	2.12	17.73
7	-17.49	2.26	19.74	-12.51	2.18	20.12	-9.00	2.18	21.15	-7.32	2.03	18.57	-5.28	1.99	17.73
8	-19.36	2.14	19.18	-14.35	2.03	19.00	-10.65	2.06	20.80	-8.13	2.04	20.70	-5.94	1.99	19.74
9	-20.97	2.19	21.63	-15.02	2.19	24.05	-11.02	2.19	25.31	-7.48	2.22	26.81	-4.76	2.19	26.22
10	-22.72	2.13	21.82	-16.63	2.16	24.91	-11.65	2.24	28.59	-8.27	2.20	28.64	-4.31	2.26	30.40
11	-24.32	2.06	21.59	-17.63	2.18	27.35	-13.92	2.04	25.41	-10.73	2.02	25.77	-7.27	2.04	26.71
12	-25.92	2.13	24.56	-19.66	2.02	24.80	-15.56	1.96	24.97	-12.49	1.95	25.16	-8.54	1.99	27.12
13	-27.50	2.16	26.68	-20.87	2.13	28.83	-15.70	2.13	31.02	-11.13	2.16	33.37	-8.48	2.07	31.16
14	-28.85	2.09	26.65	-22.03	2.02	27.74	-17.06	2.04	30.36	-11.72	2.17	35.69	-8.08	2.13	35.10
15	-30.75	2.00	25.10	-22.71	2.18	33.97	-16.26	2.25	38.83	-11.33	2.24	40.42	-7.95	2.16	38.59
16	-31.86	2.02	27.00	-24.20	2.07	32.19	-18.86	2.02	33.34	-14.37	2.03	35.12	-10.29	2.02	35.71
17	-33.45	2.03	28.50	-24.73	2.17	37.13	-19.48	2.08	36.74	-14.05	2.14	40.56	-10.05	2.10	40.26
18	-34.58	1.92	26.70	-26.76	2.03	33.95	-20.39	2.10	39.08	-15.11	2.10	41.06	-12.18	1.99	37.85
19	-35.69	1.89	27.16	-28.22	1.90	31.12	-21.65	2.03	38.46	-15.84	2.08	42.44	-12.17	2.02	40.98
20	-37.70	1.97	30.37	-29.17	2.03	36.76	-20.79	2.20	46.98	-16.72	2.09	44.26	-13.21	2.00	41.81

Table 2c: SupF Distributions, $k = 3$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$			$\epsilon = 0.20$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-6.01	3.37	11.95	-3.08	3.29	11.04	-2.25	3.04	8.42	-1.69	2.88	6.23
2	-10.04	3.29	14.94	-5.92	3.25	14.81	-3.62	3.20	13.75	-2.82	2.98	10.39
3	-13.07	3.31	18.34	-8.24	3.16	17.51	-5.44	3.10	16.57	-3.28	3.12	15.25
4	-15.85	3.23	20.46	-9.57	3.36	23.32	-5.20	3.43	24.37	-4.26	3.10	18.50
5	-17.63	3.47	26.97	-11.02	3.40	27.55	-8.59	3.05	22.23	-5.80	3.00	20.29
6	-21.00	3.37	27.66	-12.91	3.41	30.90	-9.57	3.16	26.85	-7.42	2.93	21.97
7	-23.07	3.35	30.45	-14.93	3.33	32.65	-10.33	3.20	31.02	-7.02	3.08	27.87
8	-25.42	3.43	34.56	-17.15	3.21	33.40	-11.16	3.22	34.69	-9.46	2.88	26.80
9	-26.61	3.64	42.26	-17.72	3.34	39.40	-12.35	3.19	37.39	-8.89	3.06	33.48
10	-30.66	3.23	35.25	-21.12	3.09	36.01	-13.58	3.21	40.68	-11.55	2.85	31.47
11	-33.11	3.13	35.05	-22.51	3.16	40.16	-14.86	3.16	42.64	-9.98	3.09	40.48
12	-35.16	3.15	37.93	-24.33	3.13	42.05	-17.59	3.06	42.00	-10.76	3.16	44.73
13	-37.66	3.24	41.97	-24.76	3.33	50.20	-18.03	3.20	48.38	-9.26	3.35	53.53
14	-39.57	3.25	44.82	-25.36	3.41	55.92	-18.69	3.21	51.96	-8.59	3.42	59.52
15	-41.97	3.17	44.49	-28.17	3.27	53.94	-18.83	3.28	57.51	-9.06	3.39	62.63
16	-43.78	2.86	38.05	-30.57	3.15	52.74	-19.18	3.31	62.00	-15.34	3.02	52.19
17	-45.72	3.17	49.32	-33.33	2.95	48.46	-25.99	2.82	46.76	-14.63	3.15	59.31
18	-48.01	2.92	43.42	-33.08	3.22	60.34	-26.04	2.95	53.67	-17.45	3.03	57.25
19	-49.73	2.84	42.89	-35.89	3.03	55.61	-23.20	3.20	67.17	-19.52	2.87	54.79
20	-52.17	3.04	50.67	-36.45	3.18	63.72	-23.87	3.24	71.46	-13.67	3.31	75.81

Table 2d: SupF Distributions, $k = 4$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-6.31	4.60	17.10	-2.95	4.38	14.24	-2.47	3.96	9.47
2	-11.33	4.46	21.22	-6.68	4.16	18.05	-4.11	4.09	15.44
3	-15.32	4.36	24.62	-9.13	4.09	22.23	-6.01	4.01	19.41
4	-18.80	4.28	27.77	-10.69	4.34	29.45	-6.62	4.21	26.15
5	-20.52	4.63	37.24	-11.44	4.51	37.17	-8.66	4.10	29.04
6	-23.68	4.71	42.45	-13.32	4.53	42.05	-10.88	3.99	31.20
7	-27.90	4.40	40.53	-16.21	4.42	44.12	-10.93	4.18	38.75
8	-29.74	4.65	49.46	-18.19	4.39	47.86	-10.75	4.30	45.93
9	-29.93	4.98	62.12	-18.78	4.44	54.12	-12.20	4.24	49.06
10	-36.15	4.44	52.23	-23.30	4.14	50.30	-16.36	4.00	46.50
11	-39.96	4.15	48.54	-24.71	4.26	56.73	-14.81	4.20	56.58
12	-41.45	4.49	60.50	-25.94	4.33	62.50	-16.70	4.23	60.30
13	-43.96	4.62	67.11	-27.97	4.38	67.17	-17.26	4.30	66.46
14	-47.61	4.33	62.45	-27.01	4.59	78.56	-19.52	4.16	66.58
15	-51.32	4.06	57.25	-33.35	4.12	66.77	-19.18	4.29	74.64
16	-53.71	4.01	58.80	-32.38	4.41	80.22	-17.28	4.48	86.24
17	-56.40	3.96	59.85	-36.95	4.15	74.35	-26.75	3.92	68.48
18	-58.88	4.01	64.08	-39.13	4.11	76.36	-27.05	3.99	74.65
19	-59.11	4.42	82.05	-37.29	4.31	88.96	-23.23	4.23	89.69
20	-63.31	4.25	78.24	-37.90	4.48	99.04	-27.58	4.08	86.01

Table 2e: SupF Distributions, $k = 5$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-6.95	5.49	19.54	-3.30	5.26	15.53	-2.20	4.78	8.86
2	-11.84	5.58	27.13	-7.24	5.06	20.65	-3.72	4.83	14.82
3	-15.49	5.78	35.52	-8.97	5.18	28.19	-5.26	4.79	19.76
4	-21.31	5.19	33.28	-11.35	5.31	35.00	-5.16	5.13	28.41
5	-21.94	5.87	49.15	-11.50	5.61	45.64	-6.29	5.16	34.00
6	-25.60	5.88	54.75	-14.33	5.52	49.78	-9.83	4.72	33.02
7	-31.25	5.44	51.16	-16.61	5.50	54.99	-11.17	4.79	38.12
8	-32.69	5.79	63.43	-20.66	5.23	54.53	-9.92	5.06	48.27
9	-27.66	6.70	93.08	-18.94	5.56	68.47	-10.30	5.16	55.24
10	-39.50	5.67	70.78	-23.40	5.31	67.01	-16.06	4.59	47.78
11	-45.00	5.29	65.11	-24.98	5.37	73.57	-14.84	4.90	59.03
12	-46.84	5.49	75.01	-30.09	5.05	69.06	-13.15	5.17	70.88
13	-45.99	6.06	96.55	-29.42	5.42	83.89	-12.58	5.32	80.00
14	-51.46	5.75	91.70	-25.36	5.83	103.75	-14.71	5.16	80.98
15	-57.68	5.21	78.73	-27.35	5.72	106.59	-16.66	5.10	83.84
16	-58.94	5.49	92.09	-34.38	5.40	98.93	-19.91	4.97	83.72
17	-64.70	5.09	81.92	-36.91	5.40	102.89	-22.89	4.86	84.01
18	-68.18	4.84	77.74	-42.06	5.15	97.81	-25.46	4.80	85.73
19	-66.71	5.56	107.76	-38.65	5.39	114.30	-21.03	5.02	100.86
20	-70.06	5.56	111.62	-37.12	5.68	131.05	-26.94	4.80	94.89

Table 2f: SupF Distributions, $k = 6$

	$\epsilon = 0.05$			$\epsilon = 0.10$		
q	θ_0	θ_1	η	θ_0	θ_1	η
1	-7.24	6.47	22.42	-3.99	6.00	15.46
2	-12.22	6.69	32.68	-7.24	6.08	23.63
3	-15.57	7.01	44.17	-9.09	6.14	31.90
4	-22.75	6.31	41.55	-11.70	6.28	39.65
5	-22.38	7.08	60.94	-10.24	6.81	55.14
6	-28.25	6.75	61.49	-17.16	6.13	50.21
7	-32.94	6.67	65.53	-17.11	6.49	62.97
8	-34.16	6.99	79.02	-21.06	6.22	64.02
9	-32.09	7.60	101.85	-18.41	6.69	81.74
10	-42.45	6.83	87.30	-24.25	6.33	78.78
11	-49.15	6.37	80.66	-26.28	6.33	84.79
12	-47.08	7.05	106.03	-32.02	5.98	80.48
13	-44.98	7.53	128.72	-28.86	6.54	102.30
14	-50.66	7.33	127.99	-25.11	6.85	120.57
15	-62.21	6.33	100.46	-30.00	6.59	118.34
16	-61.31	6.94	126.03	-36.85	6.29	112.43
17	-69.62	6.35	110.21	-38.42	6.36	120.20
18	-73.51	6.20	110.01	-43.53	6.15	117.34
19	-67.09	7.13	152.80	-35.19	6.73	149.05
20	-69.97	7.19	160.97	-31.19	7.04	170.03

Table 2g: SupF Distributions. $k = 7$

	$\epsilon = 0.05$			$\epsilon = 0.10$		
q	θ_0	θ_1	η	θ_0	θ_1	η
1	-8.08	7.17	23.23	-4.08	6.98	16.07
2	-12.31	7.78	37.87	-7.49	6.98	24.68
3	-17.16	7.80	47.07	-9.56	7.05	33.69
4	-24.42	7.16	46.28	-11.28	7.38	44.50
5	-22.19	8.33	73.08	-9.58	7.97	61.55
6	-28.40	8.01	75.22	-16.42	7.32	58.77
7	-30.84	8.22	87.47	-19.00	7.23	64.75
8	-34.86	8.16	94.27	-18.41	7.53	78.38
9	-32.95	8.74	117.45	-18.69	7.67	89.63
10	-45.23	7.83	100.34	-26.61	7.05	81.84
11	-52.67	7.35	93.96	-28.98	7.08	88.74
12	-43.70	8.66	140.63	-36.27	6.57	81.03
13	-45.38	8.74	152.33	-30.74	7.40	110.29
14	-53.94	8.33	145.31	-22.89	8.00	139.26
15	-69.56	6.85	103.40	-32.72	7.35	125.13
16	-65.72	7.95	145.04	-34.56	7.47	134.78
17	-71.20	7.73	143.63	-19.46	8.49	182.98
18	-78.58	7.21	131.01	-38.01	7.55	151.04
19	-51.71	9.31	230.80	-33.83	7.79	170.45
20	-71.62	8.46	195.76	-28.67	8.17	195.04

Table 2h: SupF Distributions. $k = 8$

$\epsilon = 0.05$			$\epsilon = 0.10$		
θ_0	θ_1	η	θ_0	θ_1	η
-8.74	7.95	24.30	-3.59	8.09	15.84
-12.36	8.85	42.54	-7.11	7.82	23.81
-17.03	8.94	54.17	-8.90	8.03	34.12
-24.55	8.36	55.55	-11.17	8.18	43.54
-22.79	9.42	81.86	-9.77	8.73	59.77
-29.47	9.07	84.82	-18.10	7.72	53.23
-33.21	9.03	93.61	-16.50	8.33	70.78
-37.74	9.00	101.32	-21.74	7.84	69.96
-38.06	9.39	120.01	-19.72	8.31	87.41
-50.94	8.31	100.42	-23.51	8.16	91.53
-53.52	8.62	115.01	-31.34	7.50	83.27
-43.60	9.83	161.54	-29.50	8.09	103.61
-36.25	10.55	197.66	-24.07	8.73	129.72
-56.51	9.36	163.33	-27.56	8.42	130.58
-71.50	8.16	130.66	-43.05	7.12	98.98
-69.34	8.94	163.69	-40.80	7.68	121.29
-66.95	9.38	189.36	-37.24	8.16	144.16
-73.45	9.17	188.75	-22.73	9.12	190.43
-59.52	10.14	243.14	-34.95	8.46	172.41
-66.70	10.01	244.45	12.34	10.57	283.52

Table 2i: SupF Distributions, $k = 9$

	$\epsilon = 0.05$		
q	θ_0	θ_1	η
1	-9.32	8.69	25.10
2	-12.20	9.93	47.07
3	-14.57	10.50	66.24
4	-24.91	9.42	62.67
5	-22.70	10.56	91.44
6	-29.64	10.16	95.11
7	-30.04	10.53	114.25
8	-38.89	10.00	112.09
9	-37.56	10.56	136.33
10	-53.14	9.23	111.33
11	-54.89	9.61	129.09
12	-34.69	11.59	202.38
13	-45.52	11.06	195.16
14	-55.04	10.61	189.53
15	-71.78	9.44	157.89
16	-74.81	9.69	173.25
17	-60.45	10.94	233.31
18	-67.03	10.79	236.23
19	-47.35	11.87	301.11
20	-43.71	12.18	328.79

Table 3a: $F_T(\ell + 1|\ell)$ **Distributions,** $\ell = 1$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$			$\epsilon = 0.20$			$\epsilon = 0.25$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-2.4	1.17	4.8	-2.00	1.14	4.44	-1.75	1.11	4.06	-1.48	1.11	3.93	-1.28	1.10	3.72
2	-3.82	1.25	7.07	-3.24	1.20	6.54	-2.78	1.19	6.35	-2.44	1.18	6.10	-2.16	1.15	5.67
3	-5.43	1.20	7.63	-4.61	1.20	7.61	-3.92	1.21	7.78	-3.37	1.22	7.83	-2.94	1.20	7.54
4	-6.72	1.25	9.21	-5.82	1.22	8.92	-5.13	1.19	8.60	-4.47	1.18	8.54	-3.91	1.18	8.46
5	-7.82	1.23	10.20	-6.85	1.18	9.54	-6.11	1.16	9.27	-5.29	1.19	9.77	-4.56	1.18	9.81
6	-9.15	1.17	10.02	-8.06	1.14	9.69	-7.16	1.17	10.33	-6.46	1.16	10.19	-5.76	1.14	9.85
7	-10.10	1.23	12.06	-8.76	1.21	11.98	-7.81	1.20	12.01	-6.92	1.19	12.04	-6.16	1.17	11.79
8	-11.33	1.15	11.18	-10.06	1.14	11.38	-8.93	1.14	11.74	-8.01	1.14	11.79	-7.13	1.15	12.10
9	-12.32	1.21	13.25	-10.91	1.16	12.55	-9.80	1.16	13.00	-8.96	1.14	12.63	-7.97	1.14	12.74
10	-13.29	1.16	13.07	-11.92	1.11	12.30	-10.81	1.12	12.82	-9.68	1.14	13.57	-8.51	1.17	14.53
11	-14.55	1.17	13.76	-12.83	1.16	14.18	-11.55	1.16	14.60	-10.18	1.20	15.93	-9.29	1.15	15.04
12	-15.34	1.10	13.07	-13.65	1.10	13.70	-12.43	1.09	13.77	-11.29	1.09	13.99	-10.40	1.04	13.10
13	-16.44	1.16	15.04	-14.56	1.15	15.55	-13.20	1.12	15.14	-11.92	1.14	16.16	-10.87	1.12	15.95
14	-17.29	1.15	15.70	-15.42	1.12	15.69	-13.85	1.15	17.12	-12.62	1.14	17.10	-11.67	1.11	16.41
15	-18.35	1.12	15.45	-16.36	1.11	15.77	-14.82	1.12	16.75	-13.62	1.10	16.50	-12.53	1.06	15.73
16	-19.02	1.09	15.46	-16.99	1.12	17.21	-15.64	1.02	14.69	-14.52	1.02	14.86	-13.30	1.01	14.92
17	-20.00	1.11	16.50	-17.92	1.07	16.31	-16.40	1.05	16.08	-15.14	1.04	16.07	-13.92	1.06	17.11
18	-20.95	1.09	16.36	-18.76	1.13	18.63	-17.04	1.13	19.41	-15.67	1.09	18.60	-14.44	1.06	17.96
19	-21.59	1.06	16.18	-19.32	1.12	19.30	-17.84	1.05	17.42	-16.48	1.02	16.92	-15.29	1.00	16.64
20	-22.57	1.05	16.40	-20.45	1.02	16.39	-18.89	1.02	16.92	-17.53	1.01	17.10	-16.22	1.03	17.91

Table 3b: $F_T(\ell+1|\ell)$ **Distributions,** $\ell = 2$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$			$\epsilon = 0.20$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-3.15	1.22	5.34	-2.66	1.20	5.08	-2.31	1.18	4.82	-2.00	1.18	4.69
2	-5.05	1.30	7.52	-4.30	1.27	7.25	-3.76	1.27	7.18	-3.36	1.25	6.86
3	-6.90	1.22	7.66	-6.04	1.22	7.69	-5.37	1.22	7.75	-4.79	1.23	7.88
4	-8.48	1.24	8.86	-7.54	1.22	8.55	-6.74	1.22	8.75	-6.08	1.22	8.72
5	-9.73	1.23	9.78	-8.64	1.22	9.70	-7.83	1.20	9.55	-7.06	1.23	10.11
6	-11.11	1.22	10.32	-9.95	1.16	9.61	-9.16	1.18	10.01	-8.45	1.14	9.42
7	-12.44	1.30	12.76	-11.10	1.25	12.01	-10.18	1.22	11.56	-9.30	1.19	11.25
8	-13.50	1.20	11.70	-12.16	1.12	10.44	-11.23	1.14	10.93	-10.34	1.12	10.77
9	-14.99	1.23	12.91	-13.43	1.17	12.05	-12.37	1.19	12.57	-11.38	1.14	11.83
10	-15.93	1.19	12.77	-14.40	1.15	12.33	-13.30	1.12	11.90	-12.34	1.15	12.89
11	-17.25	1.14	12.12	-15.66	1.17	13.34	-14.47	1.17	13.63	-13.32	1.20	14.78
12	-18.09	1.14	13.07	-16.39	1.11	12.93	-15.23	1.14	13.90	13.99	1.18	15.27
13	-19.30	1.15	13.83	-17.59	1.13	13.80	-16.22	1.12	14.19	-15.16	1.13	14.55
14	-20.27	1.12	13.76	-18.51	1.12	14.29	-17.24	1.15	15.56	-16.03	1.14	15.68
15	-21.45	1.12	14.21	-19.50	1.11	14.77	-18.15	1.13	15.69	-16.93	1.12	15.79
16	-22.29	1.12	15.08	-20.47	1.14	16.20	-18.70	1.08	15.05	-17.47	1.07	15.14
17	-23.40	1.12	15.42	-21.36	1.12	16.25	-19.69	1.08	15.81	-18.33	1.08	16.04
18	-24.34	1.10	15.29	-22.56	1.13	17.16	-20.96	1.13	17.70	-19.42	1.13	18.29
19	-24.91	1.07	15.11	-23.26	1.12	17.50	-21.40	1.06	16.22	-19.96	1.08	17.53
20	-25.70	1.04	14.72	-23.61	1.01	14.70	-22.08	1.00	14.81	-20.98	1.02	15.70

Table 3c: $F_T(\ell+1|\ell)$ **Distributions,** $\ell = 3$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-3.77	1.26	5.81	-3.22	1.25	5.60	-2.72	1.27	5.67
2	-6.05	1.31	7.60	-5.21	1.30	7.49	-4.64	1.30	7.50
3	-7.95	1.23	7.68	-7.12	1.21	7.39	-6.45	1.20	7.40
4	-9.63	1.23	8.47	-8.68	1.19	8.03	-7.94	1.23	8.58
5	-11.11	1.28	10.17	-9.98	1.22	9.42	-9.11	1.21	9.41
6	-12.59	1.27	10.81	-11.29	1.19	9.77	-10.56	1.20	9.96
7	-14.28	1.34	12.76	-12.79	1.26	11.74	-11.82	1.23	11.30
8	-15.30	1.31	13.34	-13.73	1.20	11.58	-12.75	1.15	10.82
9	-16.82	1.22	11.98	-15.18	1.19	11.81	-14.17	1.19	12.01
10	-17.73	1.19	12.20	-16.18	1.17	12.38	-15.07	1.16	12.40
11	-18.79	1.11	10.91	-17.58	1.16	12.46	-16.41	1.16	12.90
12	-19.83	1.13	12.12	-18.32	1.14	12.89	-17.24	1.16	13.56
13	-21.37	1.16	13.40	-19.63	1.13	13.19	-18.18	1.11	13.12
14	-22.09	1.10	12.50	-20.40	1.10	13.13	-19.37	1.12	13.89
15	-23.33	1.11	13.19	-21.63	1.12	13.97	-20.38	1.13	14.70
16	-24.51	1.13	14.50	-22.82	1.15	15.71	-21.04	1.12	15.36
17	-25.57	1.11	14.18	-23.78	1.13	15.48	-22.11	1.12	15.84
18	-26.50	1.09	14.36	-25.00	1.12	15.81	-23.55	1.14	16.78
19	-27.03	1.06	14.03	-25.70	1.11	16.12	-23.76	1.06	15.33
20	-28.00	1.05	14.26	-25.85	1.03	14.35	-24.20	1.00	13.94

Table 3d: $F_T(\ell+1|\ell)$ **Distributions,** $\ell = 4$

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-4.33	1.29	6.07	-3.73	1.29	5.94	-3.24	1.30	5.99
2	-6.85	1.33	7.64	-6.00	1.31	7.57	-5.41	1.31	7.46
3	-8.75	1.22	7.48	-7.89	1.20	7.30	-7.26	1.21	7.34
4	-10.56	1.29	9.20	-9.47	1.17	7.67	-8.85	1.21	8.24
5	-12.24	1.30	10.32	-11.02	1.24	9.47	-10.11	1.22	9.40
6	-13.82	1.33	11.70	-12.38	1.22	10.07	-11.64	1.22	9.99
7	-15.71	1.35	12.60	-14.08	1.27	11.52	-13.07	1.24	11.23
8	-16.80	1.32	13.04	-15.12	1.26	12.46	-13.99	1.22	11.84
9	-18.03	1.20	11.07	-16.49	1.19	11.41	-15.48	1.19	11.56
10	-19.06	1.18	11.53	-17.51	1.18	12.13	-16.54	1.21	12.87
11	-19.80	1.09	10.26	-18.88	1.14	11.63	-17.84	1.15	12.04
12	-21.20	1.13	11.90	-19.79	1.14	12.46	-18.75	1.16	13.11
13	-22.89	1.17	12.98	-21.13	1.14	12.80	-19.55	1.09	12.25
14	-23.41	1.09	11.88	-21.85	1.10	12.63	-20.70	1.09	12.49
15	-24.65	1.10	12.58	-23.25	1.13	13.66	-21.99	1.12	13.76
16	-26.19	1.14	13.98	-24.63	1.16	15.20	-22.93	1.15	15.54
17	-27.02	1.09	13.20	-25.48	1.12	14.49	-23.97	1.12	15.16
18	-28.10	1.09	13.72	-26.58	1.10	14.46	-25.38	1.13	15.59
19	-28.42	1.04	13.07	-27.36	1.10	15.07	-25.43	1.05	14.35
20	-29.77	1.06	13.99	-27.76	1.05	14.45	-25.79	1.00	13.32

Table 3e: $F_T(\ell+1|\ell)$ **Distributions,** $\ell = 5$

	$\epsilon = 0.05$			$\epsilon = 0.10$		
q	θ_0	θ_1	η	θ_0	θ_1	η
1	-4.82	1.31	6.22	-4.17	1.31	6.18
2	-7.50	1.33	7.61	-6.64	1.32	7.58
3	-9.36	1.21	7.31	-8.52	1.20	7.15
4	-11.39	1.34	9.82	-10.18	1.22	8.24
5	-13.19	1.32	10.39	-11.86	1.25	9.49
6	-14.92	1.40	12.62	-13.34	1.28	10.79
7	-16.85	1.35	12.17	-15.12	1.28	11.35
8	-18.00	1.32	12.74	-16.26	1.28	12.42
9	-18.91	1.18	10.45	-17.46	1.17	10.73
10	-20.05	1.17	11.05	-18.66	1.19	12.01
11	-20.40	1.06	9.60	-19.86	1.14	11.25
12	-22.38	1.14	11.78	-20.85	1.13	11.78
13	-23.94	1.15	12.30	-22.26	1.13	12.23
14	-24.46	1.09	11.54	-23.10	1.11	12.53
15	-25.89	1.11	12.43	-24.54	1.13	13.37
16	-27.52	1.14	13.59	-25.97	1.15	14.37
17	-28.09	1.08	12.54	-26.73	1.11	13.67
18	-29.18	1.08	12.85	-27.62	1.08	13.34
19	-29.53	1.04	12.37	-28.52	1.08	14.04
20	-31.17	1.07	13.71	-29.22	1.06	14.13

Table 3f: $F_T(\ell+1|\ell)$ **Distributions,** $\ell = 6$

	$\epsilon = 0.05$			$\epsilon = 0.10$		
q	θ_0	θ_1	η	θ_0	θ_1	η
1	-5.24	1.32	6.31	-4.57	1.33	6.36
2	-8.06	1.32	7.47	-7.20	1.32	7.56
3	-9.82	1.19	7.08	-9.02	1.19	7.05
4	-12.16	1.36	9.88	-10.84	1.26	8.70
5	-13.99	1.33	10.29	-12.56	1.26	9.50
6	-15.93	1.42	12.70	-14.20	1.33	11.50
7	-17.79	1.34	11.67	-16.02	1.29	11.39
8	-18.93	1.31	12.21	-17.24	1.29	12.32
9	-19.63	1.16	9.92	-18.17	1.15	10.12
10	-20.89	1.17	10.86	-19.60	1.19	11.77
11	-21.16	1.07	9.58	-20.60	1.13	10.82
12	-23.28	1.14	11.40	-21.76	1.13	11.52
13	-24.70	1.14	11.65	-23.10	1.12	11.68
14	-25.43	1.10	11.46	-24.26	1.13	12.71
15	-26.92	1.12	12.26	-25.52	1.12	12.84
16	-28.68	1.15	13.42	-27.14	1.16	14.07
17	-28.78	1.06	11.78	-27.68	1.10	13.05
18	-29.89	1.06	11.98	-28.48	1.07	12.64
19	-30.35	1.02	11.75	-29.23	1.05	12.89
20	-32.21	1.07	13.19	-30.42	1.07	13.89

Table 3g: $F_T(\ell+1|\ell)$ **Distributions,** $\ell = 7$

	$\epsilon = 0.05$			$\epsilon = 0.10$		
q	θ_0	θ_1	η	θ_0	θ_1	η
1	-5.63	1.31	6.26	-4.96	1.33	6.36
2	-8.52	1.32	7.41	-7.68	1.32	7.44
3	-10.25	1.20	7.14	-9.41	1.17	6.84
4	-12.83	1.37	9.93	-11.43	1.27	8.77
5	-14.66	1.32	10.00	-13.15	1.26	9.39
6	-16.81	1.43	12.51	-15.02	1.38	12.31
7	-18.55	1.33	11.17	-16.80	1.30	11.35
8	-19.70	1.30	11.66	-18.07	1.29	12.10
9	-20.27	1.16	9.62	-18.79	1.14	9.83
10	-21.58	1.17	10.63	-20.39	1.20	11.59
11	-22.14	1.10	10.11	-21.15	1.11	10.40
12	-24.11	1.15	11.28	-22.51	1.12	11.13
13	-25.26	1.12	11.11	-23.62	1.09	10.92
14	-26.20	1.10	11.24	-25.26	1.15	12.81
15	-27.86	1.13	12.23	-26.35	1.12	12.45
16	-29.60	1.15	13.09	-28.04	1.15	13.61
17	-29.18	1.04	10.98	-28.38	1.08	12.41
18	-30.66	1.06	11.66	-29.19	1.06	12.10
19	-31.09	1.02	11.40	-29.76	1.03	11.96
20	-32.98	1.06	12.70	-31.41	1.07	13.62

Table 3h: $F_T(\ell+1|\ell)$ Distributions, $\ell = 8$

	$\epsilon = 0.05$		
q	θ_0	θ_1	η
1	-5.99	1.29	6.08
2	-8.93	1.32	7.35
3	-10.66	1.20	7.13
4	-13.44	1.39	9.99
5	-15.24	1.32	9.79
6	-17.58	1.44	12.41
7	-19.17	1.31	10.74
8	-20.39	1.30	11.37
9	-20.81	1.15	9.40
10	-22.00	1.15	10.11
11	-23.09	1.14	10.67
12	-24.83	1.15	11.18
13	-25.67	1.10	10.55
14	-27.12	1.12	11.48
15	-28.73	1.14	12.25
16	-30.33	1.15	12.67
17	-29.48	1.02	10.37
18	-31.18	1.05	11.20
19	-31.68	1.02	11.10
20	-33.62	1.05	12.29

Table 4a: UDmax Distributions

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$			$\epsilon = 0.20$			$\epsilon = 0.25$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-4.73	1.45	4.36	-3.04	1.34	4.06	-2.12	1.26	3.76	-1.49	1.18	3.39	-1.06	1.11	2.97
2	-6.71	1.34	4.86	-4.68	1.24	4.73	-3.47	1.18	4.62	-2.56	1.14	4.52	-1.85	1.09	4.28
3	-8.19	1.23	4.99	-5.96	1.18	5.27	-4.55	1.15	5.51	-3.44	1.12	5.59	-2.52	1.08	5.47
4	-9.56	1.20	5.42	-7.20	1.15	5.85	-5.59	1.13	6.24	-4.32	1.10	6.36	-3.22	1.08	6.45
5	-10.68	1.15	5.72	-8.21	1.11	6.18	-6.48	1.07	6.45	-5.10	1.06	6.87	-3.85	1.07	7.30
6	-11.89	1.13	6.03	-9.28	1.09	6.66	-7.48	1.08	7.22	-5.94	1.07	7.76	-4.52	1.06	8.17
7	-12.88	1.10	6.32	-10.05	1.05	6.89	-8.18	1.05	7.64	-6.55	1.03	8.10	-5.04	1.02	8.60
8	-13.80	1.05	6.22	-11.01	1.03	7.10	-9.01	1.01	7.79	-7.34	1.01	8.49	-5.74	1.02	9.27
9	-14.98	1.06	6.81	-11.94	1.03	7.69	-9.87	1.02	8.58	-8.12	1.03	9.46	-6.36	1.02	10.16
10	-15.69	1.01	6.59	-12.80	1.00	7.72	-10.69	1.01	8.93	-8.77	1.01	9.87	-6.80	1.03	11.18
11	-16.64	1.00	6.91	-13.60	0.99	8.27	-11.41	1.01	9.73	-9.37	1.01	10.69	-7.36	1.02	11.76
12	-17.36	0.96	6.75	-14.32	0.96	8.08	-12.11	0.95	9.10	-10.23	0.96	10.03	-8.22	0.98	11.43
13	-18.44	0.97	7.26	-15.16	0.95	8.32	-12.83	0.94	9.27	-10.78	0.96	10.85	-8.69	0.98	12.25
14	-18.92	0.93	7.14	-15.63	0.91	8.29	-13.44	0.93	9.87	-11.39	0.94	10.90	-9.14	1.00	13.50
15	-19.89	0.93	7.36	-16.62	0.92	8.71	-14.33	0.95	10.55	-12.10	0.95	11.79	-9.94	0.96	13.18
16	-20.30	0.90	7.22	-17.20	0.90	8.85	-14.87	0.91	10.16	-12.71	0.94	12.03	-10.48	0.94	13.24
17	-21.44	0.91	7.67	-18.01	0.90	9.34	-15.43	0.89	10.27	-13.26	0.92	12.07	-11.00	0.95	14.11
18	-22.11	0.89	7.70	-18.64	0.88	9.28	-16.19	0.89	10.64	-13.93	0.91	12.45	-11.49	0.95	14.95
19	-22.16	0.85	7.25	-19.02	0.85	8.84	-16.83	0.89	11.09	-14.49	0.92	13.21	-12.13	0.92	14.59
20	-23.56	0.87	7.96	-20.00	0.86	9.50	-17.61	0.90	11.89	-15.26	0.93	14.00	-12.64	0.96	16.38

Table 4b: WDmax Distributions

	$\epsilon = 0.05$			$\epsilon = 0.10$			$\epsilon = 0.15$			$\epsilon = 0.20$			$\epsilon = 0.25$		
q	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-3.15	1.45	5.06	-2.47	1.38	4.59	-2.11	1.31	4.12	-1.53	1.24	3.71	-1.14	1.19	3.29
2	-5.05	1.34	6.03	-4.04	1.37	6.36	-3.33	1.33	6.15	-2.51	1.30	5.86	-1.84	1.17	5.02
3	-6.70	1.28	6.57	-5.49	1.23	6.39	-4.80	1.24	6.48	-3.55	1.19	6.42	-2.07	1.26	7.56
4	-8.35	1.43	9.06	-6.87	1.38	8.89	-5.74	1.44	9.97	-4.76	1.27	8.00	-3.01	1.26	8.70
5	-9.55	1.37	9.72	-8.13	1.21	7.92	-7.32	1.25	8.52	-5.18	1.30	10.05	-3.72	1.25	9.77
6	-10.83	1.38	10.91	-8.95	1.34	11.06	-7.86	1.28	10.38	-6.44	1.15	8.93	-4.63	1.12	9.26
7	-12.14	1.50	13.93	-10.14	1.40	12.99	-9.25	1.31	11.56	-6.25	1.37	14.13	-3.72	1.35	14.93
8	-13.19	1.36	12.65	-11.30	1.37	13.34	-9.58	1.46	15.75	-7.45	1.33	14.19	-3.30	1.43	18.48
9	-14.66	1.41	14.21	-12.51	1.41	14.98	-10.75	1.46	16.70	-6.99	1.43	18.27	-5.54	1.30	16.13
10	-15.12	1.13	9.96	-13.52	1.19	11.55	-11.98	1.30	14.56	-9.08	1.26	15.01	-4.75	1.37	19.82
11	-17.20	1.31	13.71	-14.25	1.46	18.58	-12.78	1.44	18.51	-9.10	1.37	18.97	-5.37	1.35	20.70
12	-16.90	1.09	10.47	-15.04	1.13	12.03	-13.48	1.20	14.31	-10.71	1.13	14.22	-8.72	1.06	13.41
13	-19.45	1.29	14.70	-17.23	1.25	14.64	-15.32	1.21	14.52	-12.29	1.08	12.87	-8.48	1.27	19.48
14	-19.70	1.26	15.46	-16.81	1.21	15.71	-15.13	1.20	16.14	-12.68	1.07	13.90	-9.59	1.10	16.29
15	-21.15	1.28	16.23	-17.79	1.23	16.90	-16.53	1.20	16.24	-12.37	1.26	20.32	-9.85	1.15	18.54
16	-22.33	1.30	17.45	-19.82	1.23	16.50	-17.98	1.35	20.84	-14.64	1.19	18.14	-11.07	1.16	19.25
17	-23.55	1.25	16.65	-20.90	1.27	18.31	-19.06	1.29	19.78	-15.09	1.24	20.41	-11.72	1.16	20.01
18	-25.01	1.26	17.13	-21.75	1.18	16.44	-20.21	1.26	19.49	-14.20	1.42	28.49	-12.53	1.17	20.98
19	-24.56	1.23	18.38	-21.76	1.10	15.48	-20.60	1.14	17.03	-16.77	1.05	16.31	-13.65	0.97	15.47
20	-26.56	1.20	17.01	-23.30	1.12	15.96	-21.75	1.18	18.40	-17.87	1.08	17.49	-14.14	1.11	20.23

Table 5: P-value accuracy comparison of Hall and Sakkas (2011) (HS) and Bai and Perron (2003b) (BP)

k	$q = 1$		$q = 5$		$q = 10$		$q = 15$		$q = 20$	
	BP	HS	BP	HS	BP	HS	BP	HS	BP	HS
1	.0383	.0485	.0540	.0494	.0526	.0497	.0768	.0496	.1620	.0492
2	.0524	.0490	.0653	.0499	.0622	.0494	.1096	.0487	.2993	.0490
3	.0487	.0494	.0608	.0495	.0562	.0491	.1130	.0494	.3588	.0497
4	.0577	.0497	.0624	.0499	.0532	.0492	.1156	.0496	.4059	.0495
5	.0767	.0494	.0684	.0496	.0527	.0495	.1198	.0503	.4460	.0497
6	.1161	.0496	.0815	.0497	.0550	.0493	.1271	.0508	.4876	.0495
7	.1912	.0498	.1034	.0497	.0610	.0501	.1388	.0507	.5305	.0502
8	.3269	.0504	.1383	.0499	.0714	.0503	.1543	.0501	.5767	.0498
9	.5419	.0501	.1913	.0501	.0861	.0505	.1751	.0497	.6242	.0494