Discrete special isothermic surfaces

(F Burstall, U Hertrich-Jeromin, W Rossman & S Santos)

Abstract. We discuss special isothermic nets of type N, a new class of discrete isothermic nets, generalizing isothermic nets with constant mean curvature in spaceforms. In the case $N = 2$ these are the discrete analogues of Bianchi's special isothermic surfaces that can be regarded as the origin of the rich transformation theory of isothermic surfaces. Accordingly, special isothermic nets come with Bäcklund transformations and a Lawson correspondence. The notion of complementary nets naturally occurs and sheds further light on the relation between geometry and integrability.

MSC 2010. *53A10*, *53C42*, 53A30, 37K25, 37K35

Keywords. isothermic surface; discrete isothermic net; Calapso transformation; Darboux transformation; Lawson correspondence; Bäcklund transformation; polynomial conserved quantity; constant mean curvature.

1. Introduction

We discuss a novel concept in discrete differential geometry: the concept of "polynomial conserved quantities". Associated with this concept is a new class of "special isothermic nets", which generalize discrete nets of constant mean curvature. These discrete nets come, as their smooth counterparts [7], with a "Lawson correspondence" as well as a Bäcklund transformation.

We work in the context of "integrable" or structure preserving discretizations. In particular, we consider discrete isothermic nets (see [3] or [11, Sect 5.7]), that is, discrete quadrilateral nets that discretize curvature line parametrizations of surfaces admitting conformal curvature line parameters. Here, we take a gauge theoretic viewpoint: as their smooth counterparts, discrete isothermic nets can be characterized by the existence of an "isothermic loop of flat connections", defined on a suitable (discrete) vector bundle over the surface. This gauge theoretic approach can be considered as the key to the rich transformation theory of (smooth as well as discrete) isothermic surfaces, cf [11, Chap 5], and provides a conceptual reason for the preservation of structure in this approach to discretization, cf [4].

"Polynomial conserved quantities" of a loop of (flat) connections are simply polynomial loops of parallel sections, that is, polynomials of the spectral parameter with coefficients in the underlying vector bundle that are parallel for each value of the spectral parameter, reminiscent of the finite gap integration scheme from integrable systems theory.

While (discrete) constant mean curvature surfaces in spaceforms admit a more direct approach via the (isothermic) transformation theory (as "special" isothermic nets), they also constitute the simplest non-trivial class of examples of (smooth or discrete) surfaces with polynomial conserved quantities, see [6] or [8]. More specifically: a discrete constant mean curvature surface can be characterized by the existence of a dual net in a concentric quadric and, in particular, the existence of a mean curvature sphere congruence that takes values in a linear sphere complex, see [5] and [8]. This mean curvature sphere congruence, being itself an isothermic surface, comes with its own associated isothermic loop of flat connections. Using a gauge transformation turns the constant parallel section for the loop of isothermic connections of the mean curvature sphere congruence, given by the fixed linear sphere complex, into a linear conserved quantity for the isothermic loop of flat connections of the original isothermic net.

Thus the (discrete) "special isothermic surfaces" discussed in this paper are generalizations of (discrete) constant mean curvature surfaces. They come with a similar transformation theory, as we shall see in Sects 2 and 3 of the present text. In Sect 4 of the paper we discuss special isothermic surfaces of type 2, that is, isothermic surfaces admitting a polynomial conserved quantity of degree 2: these are discrete versions of Bianchi's "special isothermic surfaces" [2] that are intimately related to deformations of quadrics — and that motivated much of the original research of the transformation theory of isothermic surfaces. In particular, we provide geometric characterizations of these type 2 special isothermic surfaces in terms of their transformations, similar to the classical characterizations in the smooth case [9, §§84ff] and [7].

This paper is a condensation of [6], all theoretical results of this text can be found there. However, here we focus on the results that are particular to the key ideas outlined above and hope to provide more concise and clear statements of the key results, as well as more succinct arguments of proof. Furthermore, we provide here a new and interesting example of a discrete special isothermic torus of type 2, that touches upon global questions relating to the transformations of isothermic surfaces or nets, cf [1].

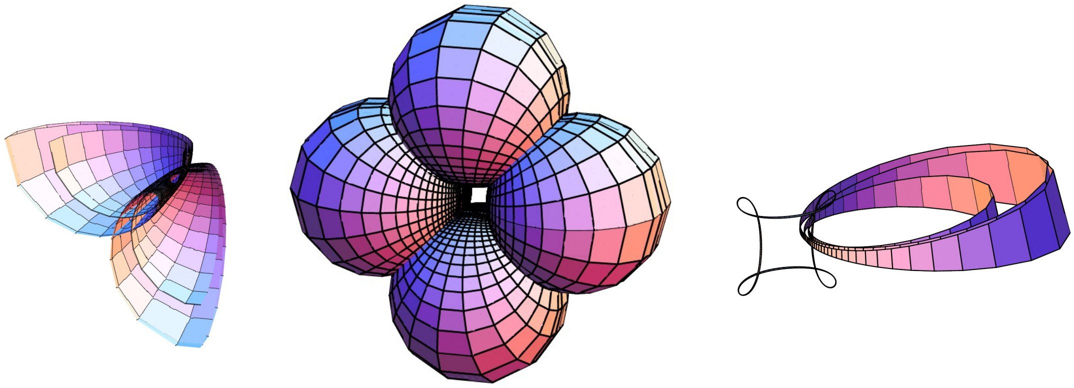

Fig 1: Discrete special isothermic surface of type 2, cf [1] or [11, §5.4.25]

Acknowledgements. The third author expresses his gratitude to Vienna University of Technology for financial support and their hospitality during the preparation of this paper.

The figures in this text were created using `Mathematica`.

2. Special isothermic nets

We consider *discrete isothermic nets* $f : \mathbb{Z}^2 \to S^3$ in the (conformal) 3-sphere, that is, nets with a real-valued function $(ij) \mapsto a_{ij}$ on (unoriented) edges, $a_{ij} = a_{ji}$, that is constant across elementary quadrilaterals, $a_{ij} = a_{kl}$ on any elementary quadrilateral $(ijkl)$ of $\mathbb{Z}^2$, and that factorizes the cross ratio of faces,

$$cr_{ijkl} := cr(f_i, f_j, f_k, f_l) = \tfrac{a_{ij}}{a_{il}},$$

cf [11, Sect 5.7] or [4, Sect 4.3]. For simplicity we restrict to $\mathbb{Z}^2$ as a domain, but most of the presented results will remain true when $\mathbb{Z}^2$ is replaced by a 2-dimensional quadrilateral cell complex.

A discrete isothermic net f comes with an associated "isothermic" loop of flat connections $(\Gamma^t)_{t\in\mathbb{R}}$ on the (trivial) vector bundle $\mathbb{Z}^2 \times \mathbb{R}^{4,1} \to \mathbb{Z}^2$ over $\mathbb{Z}^2$: we adopt the classical model of Möbius geometry, cf [11], where the conformal 3-sphere

$$S^3 \cong \mathcal{L}^4/\mathbb{R} = \{\mathrm{span}\{Y\} \subset \mathbb{R}^{4,1} \,|\, \langle Y, Y\rangle = 0\}$$

and $f : \mathbb{Z}^2 \to S^3$ is thought of as a null line bundle, $f = \mathrm{span}\{F\}$ for a lightcone-valued map F. When f is now isothermic, its associated *isothermic loop* of (flat) connections Γ^t is given by (see [6, Lemma 2.5] or [8, Cor 3.8])

$$\Gamma^t_{ij} : \mathbb{R}^{4,1}_j \to \mathbb{R}^{4,1}_i, \;\; \Gamma^t_{ij}(X) = \begin{cases} (1 - ta_{ij})X & \text{if } X \in f_i, \\ (1 - ta_{ij})^{-1}X & \text{if } X \in f_j, \\ X & \text{if } X \perp f_i \oplus f_j. \end{cases} \tag{2.1}$$

Each Γ^t is regarded as a *connection* since $\Gamma^t_{ij}\Gamma^t_{ji} = 1$; this connection is *flat* since it has trivial monodromy, $\Gamma^t_{ij}\Gamma^t_{jk}\Gamma^t_{kl}\Gamma^t_{li} = 1$, around each elementary quadrilateral $(ijkl)$.

As the $\Gamma^t \in SO(\mathbb{R}^{4,1})$ are flat they come with gauge transformations $T^t \in SO(\mathbb{R}^{4,1})$ that relate them to the trivial connection $\Gamma^0 = id$, $T^t_j = T^t_i\Gamma^t_{ij}$. These are the *Calapso transformations* of f. For any fixed t the *Calapso transform* $f^t = T^t f : \mathbb{Z}^2 \to S^3$ of f is a new isothermic net, see [11, Sect 5.7]. The Calapso transformations T^t govern the transformation theory of isothermic nets: any *Darboux transform* $\hat{f} : \mathbb{Z}^2 \to S^3$ of f is characterized by the existence of a parameter μ so that $T^\mu \hat{f} \equiv \text{const}$, that is, by the fact that $\hat{f} = \mathrm{span}\{\hat{F}\}$ is spanned by a Γ^μ-parallel section $\hat{F}$:

$$0 = d(T^\mu\hat{F})_{ij} = T^\mu_i(\Gamma^\mu_{ij}\hat{F}_j - \hat{F}_i) \;\;\Leftrightarrow\;\; \Gamma^\mu_{ij}\hat{F}_j = \hat{F}_i.$$

Here we let

$$dG_{ij} := G_j - G_i \text{ and } G_{ij} := \tfrac{1}{2}(G_j + G_i)$$

denote the discrete "derivative" and the edge function associated to a function G defined on the vertices of our base complex $\mathbb{Z}^2$, so that, for example, $d(GF) = G_{ij}dF_{ij} + dG_{ij}F_{ij}$.

These observations may motivate the following definition:

Def 1. *Let $f : \mathbb{Z}^2 \to S^3$ be an isothermic net and $\mathbb{R} \ni t \mapsto \Gamma^t$ be its associated isothermic loop of flat connections. A* polynomial conserved quantity *of f is a map*

$$\mathbb{R} \times \mathbb{Z}^2 \ni (t,i) \mapsto P_i(t) := Z_i t^N + Y_i t^{N-1} + \ldots + Q_i t^0 \in \mathbb{R}^{4,1}[t] \tag{2.2}$$

so that P is Γ^t-parallel for every fixed t, that is, $\Gamma^t_{ij}P_j(t) = P_i(t)$.

Clearly the space of polynomial conserved quantities of f is a vector space. Further, since Γ^t are metric connections, $d(|P(t)|^2)_{ij} = 0$, showing that the (real) polynomial $|P(t)|^2$ of degree $2N$ has constant coefficients, e.g., $|Z|^2 \equiv \text{const}$.

Expanding the condition on a polynomial conserved quantity to be a loop of parallel sections for the connections of an isothermic loop yields

$$dP_{ij} = \tfrac{ta_{ij}}{\langle F_i, F_j\rangle}\{\langle P_j(t), F_j\rangle F_i - \langle P_i(t), F_i\rangle F_j\}, \tag{2.3}$$

where $F : \mathbb{Z}^2 \to \mathcal{L}^4$ denotes some lightcone lift of the underlying isothermic net, $f = \text{span}\{F\}$. Comparing coefficients leads, in particular, to conclusions on the bottom and top coefficients Q and Z of a polynomial conserved quantity (2.2):

Q. We obtain $dQ_{ij} = 0$, implying that $Q = \text{const}$. Hence every polynomial conserved quantity produces a corresponding 3-dimensional quadric of constant sectional curvature $\kappa = -|Q|^2$, cf [11, Sect 1.4]:

$$\mathcal{Q}^3 := \{X \in \mathcal{L}^4 \mid \langle X, Q\rangle = -1\} \tag{2.4}$$

Z. Here we learn that $Z_i \perp f_i$ for all vertices $i \in \mathbb{Z}^2$, and that, for every edge (ij) of $\mathbb{Z}^2$,

$$Z_i + a_{ij}\tfrac{\langle Y_j, F_j\rangle}{\langle F_i, F_j\rangle}F_i = Z_j + a_{ij}\tfrac{\langle Y_i, F_i\rangle}{\langle F_i, F_j\rangle}F_j =: \kappa_{ij}.$$

As $Z_i \perp f_i$ implies $|Z_i|^2 \geq 0$, with equality iff $Z_i \in f_i$, while $|Z|^2 \equiv \text{const}$, we infer that Z is either a (special) lift of f or $|Z_i|^2 > 0$ at every vertex $i \in \mathbb{Z}^2$.

In the latter case we can, without loss of generality, assume that $|Z|^2 \equiv 1$ so that

$$Z : \mathbb{Z}^2 \to S^{3,1} := \{X \in \mathbb{R}^{4,1} \mid \langle X, X\rangle = 1\}$$

defines a (discrete) sphere congruence, cf [11, Chap 1] or [4, Sect 9.3], so that f envelops Z and κ_{ij} define the curvature spheres of the corresponding *principal contact element net* or *discrete Legendre map*, cf [4, Sect 3.5] or [8].

Def 2. *A polynomial conserved quantity* (2.2) *is* normalized *if* $|Z|^2 \equiv 1$. *A* special isothermic net of type N *is a discrete isothermic net f that admits a normalized polynomial conserved quantity of degree N, but not of any lower degree.*

Thus a special isothermic net $f : \mathbb{Z}^2 \to S^3$ comes with a naturally associated enveloped sphere congruence Z as well as a natural ambient spaceform $\mathcal{Q}^3$ given by the constant coefficient Q of its normalized polynomial conserved quantity, see (2.4).

Note that a special isothermic net f of type 0 must take values in a 2-sphere: if $Z = Q$ is constant it defines a fixed 2-sphere in S^3 that is enveloped by f, in particular, is incident with f.

A special isothermic net of type 1 has constant mean curvature in its associated space $\mathcal{Q}^3$ of constant curvature, where Z becomes its mean curvature sphere congruence, just as in the smooth case, cf [7, Prop 2.5]. This situation has been comprehensively discussed in [6, Sect 5].

Thus we will later be most interested in special isothermic nets of type 2, the discrete analogue of Bianchi's classical "special isothermic surfaces" see [2], cf [9, §84]. These isothermic surfaces admit beautiful geometric characterizations in terms of their (isothermic) transformations, which rely on configurations of circles associated to the surface or net: recall that a circle in the conformal S^3 can be identified with a Minkowski 3-space in $\mathbb{R}^{4,1}$, cf [11, Sect 1.2].

- Given a discrete net $f : \mathbb{Z}^2 \to S^3$, an incident sphere congruence $Z : \mathbb{Z}^2 \to S^{3,1}$, $Z \perp f$, defines a "contact element" or "normal direction" at every vertex f_i. If adjacent contact elements have a common sphere κ_{ij} then f is a "principal net" that is enveloped by the sphere congruence Z (and the corresponding contact element map is a discrete "principal contact element net" of [4, Sect 3.5] or "Legendre map" of [8]). This is the case if and only if there is an *edge circle* c_{ij} that intersects both spheres Z_i and Z_j orthogonally in f_i and f_j, respectively. Algebraically, these *orthogonal edge circles* of (f, Z) are given by

$$c_{ij} = \operatorname{span}\{f_i, f_j, Z_i, Z_j\} = \operatorname{span}\{f_i, f_j, Z_i\} = \operatorname{span}\{f_i, f_j, Z_j\}. \tag{2.5}$$

- Given, additionally, a second net $\hat{f} : \mathbb{Z}^2 \to S^3$ associated with f in a pointwise manner, a congruence of circles $\hat{c}$ can be constructed from the contact element net (f, Z) by requiring that $\hat{c}_i$ intersect Z_i orthogonally in f_i and also pass through $\hat{f}_i$. Algebraically, this *orthogonal vertex circle* congruence associated to the triple $(f, Z, \hat{f})$ is given by

$$\hat{c}_i = \operatorname{span}\{f_i, Z_i, \hat{f}_i\}. \tag{2.6}$$

- Finally, given two associated nets $f, \hat{f} : \mathbb{Z}^2 \to S^3$ so that endpoints of corresponding edges are concircular, we obtain a family of *edge circles* $\hat{c}_{ij}$ associated to the pair $(f, \hat{f})$ of nets,

$$\hat{c}_{ij} = \operatorname{span}\{f_i, f_j, \hat{f}_i\} = \operatorname{span}\{f_i, f_j, \hat{f}_j\}. \tag{2.7}$$

3. Transformations

Just as for constant mean curvature surfaces, the isothermic transformations descend to transformations of special isothermic surfaces of type N in general. In particular, the Calapso transformation of isothermic surfaces descends to a Lawson correspondence for special isothermic surfaces, and special Darboux transformations give rise to Bäcklund transformations.

We start by analyzing the interplay between the Calapso transformation and polynomial conserved quantities:

Thm & Def 3. *If f is special isothermic of type N then so are its Calapso transforms f^μ. We say that a special isothermic surface f and its Calapso transforms f^μ are related by* Lawson correspondence.

Proof. As the Calapso transform $f^\mu = T^\mu f$ of f is isothermic it comes with an isothermic loop of connections

$$\Gamma^{\mu,t}_{ij} = T^\mu_i \Gamma^{\mu+t}_{ij} (T^\mu_j)^{-1} \text{ with } T^{\mu,t} T^\mu = T^{\mu+t}$$

as corresponding Calapso transformations, see [11, §5.7.30]. Hence, if $P(t)$ is a polynomial conserved quantity of f, then

$$P^\mu(t) := T^\mu P(\mu + t)$$

defines a polynomial conserved quantity of f^μ of the same degree. ∎

Recall that a Darboux transformation $\hat{f} = \operatorname{span}\{\hat{F}\}$ of a discrete isothermic net $f : \mathbb{Z}^2 \to S^3$ is given by a Γ^μ-parallel lightcone section $\hat{F}$, see [11, Lemma 5.7.20]. Thus an isothermic net f and any of its Darboux transforms $\hat{f}$ have concircular edges, $\dim \operatorname{span}\{f_i, f_j, \hat{f}_i, \hat{f}_j\} = 3$, and therefore give rise to a family of edge circles (2.7) of the pair $(f, \hat{f})$ — which therefore qualifies as a Ribaucour pair in the Möbius geometric sense, see [11, Sect 8.3].

Lemma 4. *Let $\hat{f}$ be a Darboux transform of a special isothermic net f of type N, with normalized polynomial conserved quantity $P(t)$ of minimal degree N. Then $\hat{f}$ is special isothermic of type at most $N+1$, and of type at most N if $P(\mu) \perp \hat{f}$.*

Proof. For $p, \hat{p} \in S^3 = \mathcal{L}^4/\mathbb{R}$ and $q \in \mathbb{R}$ we let (cf (2.1))

$$\Gamma^q_{p,\hat{p}} : \mathbb{R}^{4,1} \to \mathbb{R}^{4,1}, \ \Gamma^q_{p,\hat{p}}(Y) := \begin{cases} qY & \text{if } Y \in p, \\ q^{-1}Y & \text{if } Y \in \hat{p}, \\ Y & \text{if } Y \perp p \oplus \hat{p}. \end{cases} \tag{3.1}$$

Then $\hat{T}^t = T^t \Gamma^{1-t/\mu}_{f,\hat{f}}$, see [6, Lemma 4.2] (cf [11, §5.7.35]), and

$$\hat{P}(t) := (t-\mu)\Gamma^{1-t/\mu}_{\hat{f},f} P(t) \tag{3.2}$$

yields a polynomial conserved quantity for $\hat{f}$: clearly $\hat{T}^t \hat{P}(t) \equiv \text{const}$ and $\hat{P}(t)$ is polynomial of degree at most $N+1$ by (2.3) since $Z \perp f$; further, $|\hat{P}(t)|^2 = (t-\mu)^2 |P(t)|^2$ so that $\hat{P}(t)$ is normalized as soon as $P(t)$ is.

If additionally $P(\mu) \perp \hat{f}$ at one hence all points, then μ is a zero of $\hat{P}$, $\hat{P}(\mu) = 0$, so that $\hat{P}(t)$ of (3.2) has a factor $(t-\mu)$ and $\Gamma^{1-t/\mu}_{\hat{f},f} P(t)$ defines a normalized polynomial conserved quantity of degree at most N for $\hat{f}$. ∎

As an example consider a Clifford torus $f : \mathbb{Z}^2 \to S^3$ in S^3: as a constant mean curvature net it is special isothermic of type 1. Hence, its Darboux transforms will generically be special isothermic of type 2, analogous to the smooth case [7, Thm 3.2]. Remarkably, there exist doubly periodic Darboux transforms $\hat{f} : \mathbb{Z}^2 \to S^3$ for certain (periodic) discrete Clifford tori $f : \mathbb{Z}^2 \to S^3$ and a suitable choice of the spectral parameter μ, just as in the smooth case, see [1] or [11, §5.4.25]. Fig 1 shows a discrete special isothermic torus of type 2 that is obtained as a Darboux transform of a 9-fold covering of a suitable (discrete) Clifford torus in S^3.

The second claim of Lemma 4 gives rise to the following definition:

Def 5. *A* Bäcklund transform *$\hat{f}$ of a special isothermic net f with normalized polynomial conserved quantity $P(t)$ of minimal degree is a Darboux transform so that $P(\mu) \perp \hat{f}$.*

It is also straightforward to verify symmetry of the Bäcklund transformation between special isothermic nets of type N, using that

$$\hat{P}(\mu) = \lim_{t\to\mu} \Gamma^{1-t/\mu}_{\hat{f},f} P(t) = P(\mu) - \tfrac{\langle P(\mu), F\rangle}{\langle \hat{F}, F\rangle} \hat{F} + \ldots F \perp F.$$

To understand the geometry of the Bäcklund transformation observe that the (reduced) polynomial conserved quantity

$$\hat{P}(t) = \Gamma^{1-t/\mu}_{\hat{f},f} P(t) = t^N \hat{Z} + t^{N-1}\hat{Y} + \ldots + \hat{Q}$$

of a Bäcklund transform $\hat{f}$ of a special isothermic net f of type N, cf (3.2), satisfies equations analogous to (2.3), but "in the transformation direction". Hence

(i) $\hat{Q} = Q$, so that $\hat{f}$ takes values in the same space of constant curvature as f does; and

(ii) $Z + \frac{1}{\mu}\frac{\langle \hat{Y}, \hat{F}\rangle}{\langle F, \hat{F}\rangle} F = \hat{Z} + \frac{1}{\mu}\frac{\langle Y, F\rangle}{\langle F, \hat{F}\rangle}\hat{F}$, showing that corresponding contact elements, (f, Z) and $(\hat{f}, \hat{Z})$, of f and $\hat{f}$ contain a common sphere, that is, the Bäcklund transformation qualifies as a Ribaucour transformation in the Lie geometric sense, see [4, Sect 3.5].

The Bäcklund transformation also satisfies the usual permutability theorem, $B_{\mu_0} B_{\mu_1} = B_{\mu_1} B_{\mu_0}$, where $B_\mu f$ denotes a Bäcklund transform with respect to spectral parameter μ:

Thm 6 (Bianchi permutability). *Let $\hat{f}^0, \hat{f}^1$ be Bäcklund transforms with parameters $\mu_0 \neq \mu_1$ of a special isothermic net f. Then there is a common Bäcklund transform f^{01} of $\hat{f}^i$ with parameter μ_{1-i}, $i = 0, 1$.*

Proof. We show that the corresponding permutability theorem for the Darboux transformation of isothermic nets, see [11, §5.7.28], descends to this theorem for the Bäcklund transformation of special isothermic nets.

Thus suppose that $\hat{f}^i$ are given by lightcone lifts $\hat{F}^i$ satisfying $T^{\mu_i}\hat{F}^i \equiv$ const and $\hat{F}^i \perp P(\mu_i)$, where $P(t)$ is a normalized degree N polynomial conserved quantity of f; let $\hat{P}^i(t) = \Gamma^{1-t/\mu_i}_{\hat{f}^i,f} P(t)$ denote the corresponding degree N polynomial conserved quantities of $\hat{f}^i$, $i = 0, 1$. We aim to show that

$$P^{01}(t) := \Gamma^{1-t/\mu_1}_{f^{01},\hat{f}^0}\hat{P}^0(t) = \Gamma^{1-t/\mu_0}_{f^{01},\hat{f}^1}\hat{P}^1(t)$$

yields a polynomial conserved quantity for the isothermic net $f^{01} = \Gamma^{\mu_1/\mu_0}_{\hat{f}^0,\hat{f}^1} f$, obtained from the Bianchi permutability theorem for the Darboux transformation of discrete isothermic nets. Now the key argument is that

$$\Gamma^{1-t/\mu_1}_{f^{01},\hat{f}^0}\Gamma^{1-t/\mu_0}_{\hat{f}^0,f} = \Gamma^{(1-t/\mu_0)/(1-t/\mu_1)}_{\hat{f}^0,\hat{f}^1} = \Gamma^{1-t/\mu_0}_{f^{01},\hat{f}^1}\Gamma^{1-t/\mu_1}_{\hat{f}^1,f}.$$

Firstly, this shows that $P^{01}(t)$ above is well defined. Secondly, we conclude that

$$P^{01}(\mu_0) = \lim_{t\to\mu_0}\Gamma^{(1-t/\mu_0)/(1-t/\mu_1)}_{\hat{f}^0,\hat{f}^1} P(t) = P(\mu_0) - \tfrac{\langle P(\mu_0),\hat{F}^1\rangle}{\langle \hat{F}^0,\hat{F}^1\rangle}\hat{F}^0 + \dots \hat{F}^1 \perp \hat{F}^1,$$

showing that $\hat{f}^0$ is a Bäcklund transform of f^{01}. By symmetry, $\hat{f}^1$ is a Bäcklund transform of f^{01} as well, which completes the argument. ∎

As in the case of the Darboux transformation, "higher dimensional" permutability theorems can now be proved by purely combinatorial arguments, for example, given three Bäcklund transforms of a special isothermic net, a "Bianchi cube" can be constructed in a unique way: repeated application of the above permutability theorem leads to a configuration of eight special isothermic nets associated with the vertices of a cube so that the edges of that cube signify Bäcklund transformations with parameters that are equal on opposite edges of the faces of the cube. In fact, a more structural analysis would have revealed that the above theorem could have been proven by using that "3D-consistency" of a "2D-system" implies higher dimensional consistency, cf [4], interpreting the above permutability theorem as "4D-consistency".

If $P(t)$ is a (normalized) polynomial conserved quantity of a special isothermic net f and $|P(\mu)|^2 = 0$ for some μ then $\hat{f} = \mathrm{span}\{P(\mu)\}$ yields a Bäcklund transform of f:

Def 7. *If $|P(\mu)|^2 = 0$ for the normalized polynomial conserved quantity $P(t)$ of a special isothermic net f then $\hat{f} := \mathrm{span}\{P(\mu)\}$ will be called a* complementary net *of f.*

Note that a special isothermic net f of type N has at most $2N$ complementary nets. In the case $N = 1$ of a net of constant mean curvature H in a quadric $\mathcal{Q}^3$ of constant sectional curvature κ, see (2.4), the condition $0 = |P(t)|^2 = t^2 - 2Ht - \kappa$ for a complementary net leads to the Lawson invariant $H^2 + \kappa$ as the quantity governing the existence of complementary nets. In fact, constant mean curvature surfaces with positive Lawson invariant can (in both the smooth and discrete cases) be characterized in terms of their complementary nets, see [6, Sect 4.3] and [5, Sect 4]: the simplest case is Bonnet's theorem on parallel constant mean curvature surfaces in the smooth case, and the original definition of constant mean curvature nets as particular isothermic nets via "parallel" Darboux transformations in the discrete case, cf [10, Sect 5].

4. Quadratic conserved quantities

We shall see that complementary nets play a similarly crucial role in the case of type 2 special isothermic nets, the discrete analogues of Bianchi's "special isothermic nets" [2], cf [9, §84]. We start by investigating the geometry of complementary nets of a type 2 special isothermic net f.

Suppose that

$$\hat{F}^n = P(\mu_n) = \mu_n^2 Z + \mu_n Y + Q \quad (n = 0, \dots, N)$$

yield (Γ^μ-parallel lightcone lifts of) $N + 1$ complementary nets of a special isothermic net f of type 2. If $N = 1$ we conclude that the planes (in the spaceform ambient geometry given by (2.4))

$$\hat{e}^n = \mathrm{span}\{F, Z, Q, \hat{F}^n\} = \mathrm{span}\{F, Z, Q, Y\}$$

of the orthogonal (vertex) circles (2.6) coincide. On the other hand, if f has $N+1=3$ complementary nets, then

$$\forall i \in \mathbb{Z}^2 : Q \in \operatorname{span}\{\hat{F}_i^n \,|\, n = 0, 1, 2\},$$

showing that the three complementary nets $\hat{f}^n$ of f must be (pointwise) collinear in the ambient spaceform geometry of f.

The following two theorems provide "reverse engineering", that is, construction of a polynomial conserved quantity from the respective geometric configurations.

Thm 8. *Let f be an isothermic net and $\hat{f}^n$, $n = 0, 1, 2$, three Darboux transforms of f. If $\hat{f}^n$ are pointwise collinear in a suitable ambient spaceform geometry* (2.4) *then f is, generically, special isothermic of type* 2.

In the second theorem an enveloped sphere congruence S replaces one of the Darboux transforms, in order to obtain the "orthogonal vertex circle" congruences $i \mapsto \hat{c}_i^n$ of (2.6).

Thm 9. *Let $f : Z^2 \to S^3$ be an isothermic net, $S : \mathbb{Z}^2 \to S^{3,1}$ an enveloped sphere congruence and $\hat{f}^n$, $n = 0, 1$, two Darboux transforms of f. If the planes $\hat{e}^n$ of the associated orthogonal vertex circles* (2.6) *coincide in a suitable spaceform geometry* (2.4) *then, generically, f has a quadratic conserved quantity.*

To prove both theorems we use the following more general lemma that provides a construction of a degree N polynomial conserved quantity $P(t)$ from (Γ^{μ_n}-parallel lifts $\hat{F}^n$ of) $N+1$ Darboux transforms $\hat{f}^n$, $n = 0, \dots, N$, for pairwise distinct μ_n of an isothermic net f, cf [6, Lemma 4.9]:

Lemma 10. *If, for suitable constants $\alpha_n \in \mathbb{R}$, $n = 0, \dots, N$,*

$$Z := \textstyle\sum_{n=0}^N \alpha_n \hat{F}^n \perp f, \ \text{ then } \ P(t) := \sum_{n=0}^N \alpha_n \hat{F}^n \prod_{m \neq n} (t - \mu_m)$$

is a degree N polynomial conserved quantity for f.
Further, Z is the top degree coefficient and $P(t)$ is normalized as soon as $|Z_i|^2 = 1$ at some $i \in \mathbb{Z}^2$; and, in this case, $\hat{f}^n$ are complementary nets of f.

Proof of Lemma 10. First note that $P(\mu_n) = \text{const}\, \hat{F}^n$, so that $\hat{f}^n$ are complementary nets as soon as $P(t)$ is a normalized polynomial conserved quantity. Clearly, Z is the top degree coefficient of the degree N polynomial $P(t)$.

We aim to show that $P(t)$ is Γ^t-parallel for fixed $t \in \mathbb{R}$, that is, for every edge (ij) of $\mathbb{Z}^2$

$$0 = \Gamma^t_{ij} P_j(t) - P_i(t),$$

which is a polynomial of degree N by (2.3) and the assumption that $Z \perp f$. As $P(\mu_n) = \text{const}\, \hat{F}^n$ this polynomial vanishes for $N+1$ parameter values $t = \mu_n$, $n = 0, \dots, N$, hence vanishes identically.

Finally, $|P(t)|^2 = t^{2N}|Z|^2 + \ldots + t^0|Q|^2$ is a polynomial with *constant* coefficients, as $P(t)$ are parallel with respect to the *metric* connections Γ^t. Thus, in particular, $|Z|^2 \equiv \text{const}$. ∎

We have not used that $\hat{F}^n$ be isotropic in order to derive that $P(t)$ is a polynomial conserved quantity. Thus the assumptions can accordingly be relaxed, at the cost of losing that $\hat{f}^n$ become complementary nets of f.

Proof of Thm 8. By the assumption of $\hat{f}^n$ being collinear in a suitable quadric (2.4) of constant curvature we learn that the vector $Q \in \mathbb{R}^{4,1} \setminus \{0\}$ defining $\mathcal{Q}^3$ can be written as

$$Q = \textstyle\sum_{n=0}^2 \beta_n \hat{F}^n$$

with suitable functions $\beta_n : \mathbb{Z}^2 \to \mathbb{R}$, $n = 0, 1, 2$, and Γ^{μ_n}-parallel lifts $\hat{F}^n$ of $\hat{f}^n$. First we show that the β_n are constant:

$$0 = dQ_{ij} = \textstyle\sum_{n=0}^2 d(\beta_n)_{ij} \hat{F}_i^n + (\beta_n)_j d\hat{F}_{ij}^n = \sum_{n=0}^2 d(\beta_n)_{ij} \hat{F}_i^n \bmod f_i \oplus f_j.$$

Thus assuming, for genericity, that the three families of edge circles (2.7) associated to the three pairs $(f, \hat{f}^n)$ do not become cospherical, $\hat{F}^n_i$ are linearly independent $\bmod f_i \oplus f_j$, and we conclude that $d(\beta_n)_{ij} = 0$. Now, cf (2.3),

$$0 = dQ_{ij} = \sum_{n=0}^{2} \beta_n d\hat{F}^n_{ij} = \frac{a_{ij}}{\langle F_i, F_j \rangle} \{ \langle Z, F \rangle_j F_i - \langle Z, F \rangle_i F_j \} \text{ for } Z := \sum_{n=0}^{2} \beta_n \mu_n \hat{F}^n$$

and any lift F of f, showing that $Z \perp f$. By genericity $Z \notin f$, hence $|Z|^2 > 0$ and without loss of generality $|Z_i|^2 = 1$ at some $i \in \mathbb{Z}^2$. The claim now follows with Lemma 10 above as soon as we assume, for genericity again, that the parameters μ_n are distinct. ■

In a completely analogous way one proves that two Darboux transforms that are antipodal in a suitable spaceform subgeometry lead to a normalized linear conserved quantity, thus to a characterization of constant mean curvature nets in spaceforms, cf [6, Thm 4.11] or [5, Sect 5].

Proof of Thm 9. Here we express the equality assumption on the planes of the orthogonal vertex circles $\hat{c}^n$, see (2.6), in a spaceform geometry (2.4) given by $Q \in \mathbb{R}^{4,1} \setminus \{0\}$ as

$$\hat{e}^0 = \operatorname{span}\{F, S, Q, \hat{F}^0\} = \operatorname{span}\{F, S, Q, \hat{F}^1\} = \hat{e}^1,$$

where, again F and $\hat{F}^n$ denote (Γ^{μ_n}-parallel) lifts of f and $\hat{f}^n$, respectively. Here we already make the (implicit) genericity assumption that the orthogonal vertex circles not be straight lines in $\mathcal{Q}^3$. Hence $\hat{e}^0 = \hat{e}^1$ yields the linear dependence

$$\alpha_0 \hat{F}^0 + \alpha_1 \hat{F}^1 + \alpha_\infty Q = \beta S + \gamma F =: Z$$

with suitable functions $\alpha_n, \beta, \gamma : \mathbb{Z}^2 \to \mathbb{R}$, where neither α_0 nor α_1 vanish as $\dim \hat{e}^n = 4$. We aim to show that α_n can be chosen constant in order to apply Lemma 10: note that Q is Γ^{μ_∞}-parallel with $\mu_\infty := 0$ since $dQ = 0$ and $\Gamma^0 = \mathrm{id}$.

Now

$$dZ_{ij} = d(\alpha_\infty)_{ij} Q + d(\alpha_0)_{ij} \hat{F}^0_{ij} + d(\alpha_1)_{ij} \hat{F}^1_{ij} + (\alpha_0)_{ij} d\hat{F}^0_{ij} + (\alpha_1)_{ij} d\hat{F}^1_{ij},$$

where $dZ_{ij}, d\hat{F}^n_{ij} \in c_{ij}$, the orthogonal edge circle of (2.5). With the further genericity assumption that these orthogonal edge circles not be straight in $\mathcal{Q}^3$ and their planes

$$e_{ij} = \operatorname{span}\{F_i, F_j, Q, S_i\} = \operatorname{span}\{F_i, F_j, Q, S_j\} \neq \hat{e}^n \ \ (n = 0, 1),$$

hence $\hat{F}^n_{ij} \notin e_{ij}$, we infer that $d\alpha_n = 0$ as soon as $d\alpha_{1-n} = 0$ as $0 = d(\alpha_0)_{ij} \hat{F}^0_{ij} + d(\alpha_1)_{ij} \hat{F}^1_{ij} \bmod e_{ij}$. As both, α_0 and α_1, have no zeroes, we may without loss of generality assume that either, hence both, are constant. Then $0 = d(\alpha_\infty)_{ij} Q + d(\alpha_0)_{ij} \hat{F}^0_{ij} + d(\alpha_1)_{ij} \hat{F}^1_{ij} \bmod c_{ij}$ implies that $d\alpha_\infty = 0$ as well.

With a final genericity assumption, that $\hat{f}^n$ be Darboux transforms for different parameters μ_n, Lemma 10 applies to yield existence of a quadratic conserved quantity for f. ■

A completely analogous line of arguments proves the existence of a (normalized) linear conserved quantity when two Darboux transforms $\hat{f}^n$, $n = 0, 1$, are given so that the associated congruences of orthogonal vertex circles (2.6) coincide, $\hat{c}^0 = \hat{c}^1$, cf [6, Thm 4.14].

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F Burstall
Department of Mathematical Sciences
University of Bath
Bath, BA2 7AY (United Kingdom)
Email: f.e.burstall@bath.ac.uk

U Hertrich-Jeromin
Vienna University of Technology
Wiedner Hauptstraße 8–10/104
A-1040 Vienna (Austria)
Email: uhj@geometrie.tuwien.ac.at

W Rossman
Department of Mathematics
Kobe University
Rokko, Kobe 657-8501 (Japan)
Email: wayne@math.kobe-u.ac.jp

S Santos
Departamento de Matemática, CMAF
Universidade de Lisboa
1749-016 Lisboa (Portugal)
Email: susantos@ptmat.fc.ul.pt