On a singular initial-value problem for the Navier-Stokes equations

L. E. Fraenkel and M. D. Preston

This paper presents a recent result for the problem introduced eleven years ago in [1], but described only briefly there. We shall prove the following, as far as space allows. The vorticity $\omega$ of a diffusing vortex circle in a viscous fluid has, for small values of a non-dimensional time, a second approximation $\omega_A + \omega_1$ that, although formulated for a fixed, finite Reynolds number $\lambda$ and exact for $\lambda = 0$ (then $\omega = \omega_A$), tends to a smooth limiting function as $\lambda \uparrow \infty$.

In §1 and §2 the necessary background and apparatus are described; §3 outlines the new result and its proof.

1 Introduction

In a certain weak sense, this paper is a continuation of [1]. However, no knowledge of [1] is required if the reader is willing to accept that a vorticity field in $\mathbb{R}^3$ (subject to mild restrictions, but not required to have any symmetry) has a centroid of vorticity moving with a velocity $U(t)$ that is given by an explicit formula when the vorticity $\omega(\cdot, t)$ throughout $\mathbb{R}^3$ is known. This result is essentially due to Saffman [2]; it was generalized a little (and perhaps clarified and sharpened) in [1].

We seek a solution of the Navier-Stokes equations with the initial condition illustrated in Figure 1: at time zero, vorticity $\omega$ is concentrated on, and is tangential to, a horizontal circle in $\mathbb{R}^3$. This initial vorticity induces an initial velocity field that has infinite kinetic energy. (The circle then diffuses and moves vertically, at first with infinite velocity; at all positive times $t > 0$ the kinetic energy is finite.)

More precisely, consider incompressible fluid occupying all of $\mathbb{R}^3$ and at rest at infinity there; let $x := (x_1, x_2, x_3)$ be such that the frame $(Ox_1, Ox_2, Ox_3)$ moves, relative to the motionless fluid at infinity, with the velocity $(0, 0, U(t))$ of the centroid of vorticity, the axes remaining parallel to their initial positions. The fluid velocity relative to this moving frame is written $\mathbf{v}(x, t)$ and the vorticity is

$$\omega := \text{curl} \mathbf{v} = \nabla \times \mathbf{v}.$$

Our time variable is $t = \nu T$, where $T$ denotes physical time and $\nu$ is the kinematic viscosity (a given positive constant). This choice of $t$ simplifies the heat operator in (1.3) below and simplifies most subsequent equations.

In writing $U(t) = (0, 0, U(t))$, we have restricted attention to the cylindrical symmetry implied by the initial condition

$$\omega(x, 0) = \kappa \delta(z) \delta(r - a) e^\phi,$$  \hspace{1cm} (1.1)
in which the circulation \( \kappa \) and the radius \( a \) are given positive constants, cylindrical co-ordinates \((z, r, \phi)\) are defined by \( x =: (r \cos \phi, r \sin \phi, z) \), the unit vector \( e^\phi := (-\sin \phi, \cos \phi, 0) \) and \( \delta \) denotes the Dirac generalized function.

In terms of the vorticity \( \omega \), the fluid velocity (relative to our moving frame) is

\[
\mathbf{v}(x, t) = -(0, 0, U(t)) + \nabla \times \int_{\mathbb{R}^3} \frac{1}{4\pi|x - x'|} \omega(x', t) \mathrm{d}x'.
\]  

(1.2)

With (1.1) and (1.2) understood, we seek \( \omega(x, t) \) such that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \omega = -\frac{1}{\nu} ((\mathbf{v} \cdot \nabla)\mathbf{\omega} - (\mathbf{\omega} \cdot \nabla)\mathbf{v}) \quad \text{in} \quad \mathbb{R}^3 \times (0, T)
\]

(1.3)

for some small \( T > 0 \).

Of course, it would be better to solve the problem (1.1) to (1.3) for all \( t > 0 \), but this is beyond us because we seek rather explicit answers. There are two excuses for considering only small \( t \), or, rather, small \( t/a^2 \), which is non-dimensional. First, once a solution for \( t > 0 \) has been established, the general theory of the Navier-Stokes equations implies a continuation of the solution to all time, thanks to finite energy for \( t > 0 \), cylindrical symmetry and absence of a swirl velocity (of a velocity component in the direction \( e^\phi \)). Secondly, if the viscosity \( \nu \) is small, which may be the case of primary interest, then the requirement that \( \nu T/a^2 \) be small does not demand that the physical time \( T \) be small.

In view of (1.1), we write

\[
\omega(x, t) =: \omega(z, r, t) e^\phi,
\]

and seek the solution of (1.1) to (1.3) in the scalar form \( \omega = \omega_A + \omega_1 + \rho \), where \( \omega_A \) is to be a first approximation for small \( t/a^2 \) and \( \omega_A + \omega_1 \) is to be a second (improved) approximation; the remainder \( \rho \) is to make \( \omega_A + \omega_1 + \rho \) an exact solution and is to be \( o(\omega_1) \) as \( t \downarrow 0 \). Here are some details.
(i) The non-linear terms on the right-hand side of (1.3) are expected to be small for small $t/a^2$, because $\omega$ should be approximately constant and large on small circles in a meridional plane ($\phi = \text{constant}$) centred at $(z, r) = (0, a)$, so that $\psi$ is approximately tangential to such circles and approximately of constant magnitude on each of them. (If the initial vortex circle were a straight line, then these non-linear terms would vanish.) If the right-hand member of (1.3) is neglected, there results the formal approximation

$$\omega_A(z, r, t) = \frac{\kappa}{4\pi t} \exp\left(-\frac{s^2}{4t}\right) B \left(\frac{ar}{2t}\right), \tag{1.4}$$

where $s := \{z^2 + (r - a)^2\}^{1/2}$ and $B$ is a known function such that $B(y) \to 1$ as $y \to \infty$; in fact,

$$B(y) := (2\pi y)^{1/2}e^{-y}I_1(y) \quad (0 \leq y < \infty), \tag{1.5}$$

where $I_1$ is the modified Bessel function of the first kind and of order 1 (as in [3], p.77).

(ii) The exponential in (1.4) prompts us to introduce inner variables

$$\sigma := \frac{s}{(4t)^{1/2}}, \quad \theta := \tan^{-1}\frac{r-a}{z}; \tag{1.6}$$

then the amplitude $\kappa/4\pi t$ in (1.4) prompts us to pose

$$\omega_1(z, r, t) = (4t)^{-1/2}\tilde{\omega}_1(\sigma, \theta). \tag{1.7}$$

It suffices to consider $\omega_1$ in an inner region: $t \downarrow 0$ with $\sigma$ fixed, so that $s \downarrow 0$, because in an outer region: $t \downarrow 0$ with $s \geq \text{constant} > 0$, not only $\omega_A$, but also $\omega$, are exponentially small.

The rest of this paper is devoted mainly to description of $\tilde{\omega}_1$; the Reynolds number

$$\lambda := \frac{\kappa}{2\pi \nu} \tag{1.8}$$

will be an important parameter.

(iii) The problem for the remainder $\rho$ was sketched in [1]; the function $\rho(z, r, t)$ must be shown to exist and to be suitably small on the whole set $\mathbb{R} \times [0,\infty) \times (0, \bar{t}]$. Considerable progress has been made with this problem since [1] was written, but this analysis (which can only estimate $\rho$) is too long and too elaborate to be described here.

2 The perturbation $\omega_1$ for fixed Reynolds number $\lambda$

With $\omega_1$ as in (1.7), we adopt the notation

(a) $(\sigma, \theta) \in E := (0, \infty) \times (-\pi, \pi]$,

(b) $\Delta_\sigma := \left(\frac{\partial}{\partial \sigma}\right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial \sigma} + \frac{1}{\sigma^2} \left(\frac{\partial}{\partial \theta}\right)^2$,

(c) $(A\tilde{\omega}_1)(\sigma_0, \theta_0) := \frac{1}{2\pi} \int \int_E \log \frac{1}{|\sigma e^{i\theta} - \sigma_0 e^{i\theta_0}|} \tilde{\omega}_1(\sigma, \theta) \sigma d\sigma d\theta$,

(d) $\omega_{A,0}(\sigma, t) := \frac{\kappa}{4\pi t} e^{-\sigma^2}$,
in which $A\tilde{\omega}_1$ is a stream function describing the plane flow induced by vorticity $\tilde{\omega}_1$; the approximation $\omega_{A,0}$ to $\omega_A$ is that appropriate to $t \downarrow 0$ with $\sigma$ fixed. We seek $\tilde{\omega}_1(\sigma,\theta)$ by linearizing (1.2) and (1.3) about $\omega_{A,0}$; the problem is then to solve the equation

$$-\left(\Delta_\sigma + 2\sigma \frac{\partial}{\partial \sigma} + 2\right)\tilde{\omega}_1 + \frac{\lambda}{\sigma^2} \frac{\partial}{\partial \theta} \tilde{\omega}_1 - 4\lambda e^{-\sigma^2} \frac{\partial}{\partial \theta}(A\tilde{\omega}_1) = \frac{\kappa \lambda}{\pi a} g(\sigma) \cos \theta \quad \text{on } E,$$

with side conditions

$$\tilde{\omega}_1(\sigma,\theta) \to 0 \quad \text{as } \sigma \downarrow 0 \text{ and as } \sigma \uparrow \infty.$$

The function $g$ is a known, smooth function such that

(a) $g(\sigma) = O(\sigma)$ as $\sigma \downarrow 0$;
(b) $g(\sigma) = O \left(\sigma \log \sigma e^{-\sigma^2}\right)$ as $\sigma \uparrow \infty$;

in fact,

(c) $g(\sigma) := \sigma e^{-\sigma^2} \left(\frac{3}{2} 1 - e^{-\sigma^2} + \left(\log \frac{1}{\sigma} - \int_\sigma^\infty \frac{e^{-\rho^2}}{\rho} d\rho\right) - \frac{1}{2} \left(\gamma_E + 1 - \log 2\right)\right)$,

where $\gamma_E = 0.5772...$ denotes Euler’s constant.

**Theorem 2.1.** For fixed $\lambda \in [0, \infty)$, the problem (2.2) and (2.3) for $\tilde{\omega}_1$ has a pointwise, unique solution; in particular, $\tilde{\omega}_1(\cdot, \theta) \in C^\infty(0, \infty)$, $\tilde{\omega}_1(0, \theta) = 0$ and $\tilde{\omega}_1(\sigma, \theta) = o(e^{-\sigma^2/2})$ as $\sigma \uparrow \infty$.

Here we have space only to sketch the main steps of the proof.

(i) Under the transformation

$$\tilde{\omega}_1(\sigma, \theta) = e^{-\sigma^2/2}q(\sigma, \theta),$$

equation (2.2) becomes

$$-\left(\Delta_\sigma - \sigma^2\right)q + \frac{\lambda}{\sigma^2} \frac{\partial q}{\partial \theta} - 4\lambda e^{-\sigma^2/2} T \left(e^{-\sigma^2/2}q\right) = \lambda e^{\sigma^2/2} f(\sigma, \theta) \quad \text{on } E,$$

where the operator $T := \left(\partial/\partial \theta\right) A$ and $f(\sigma, \theta) := (\kappa/\pi a) g(\sigma) \cos \theta$. Let

$$\xi(\sigma, \eta) := \sigma(\cos \theta, \sin \theta), \quad q_*\xi(\sigma, \eta) := q_*(\sigma \cos \theta, \sigma \sin \theta) := q(\sigma, \theta).$$

The condition in (2.3) for $\sigma \downarrow 0$ will be implicit in what follows; it was imposed only to make $q_*$ decent at the origin, because we shall find that $q$ is of the form $q_*(\sigma \cos \theta$ + $q_*(\sigma \sin \theta$. Henceforth the functions $q_*$ and $q$ will be identified wherever no confusion is possible. Similarly, the Cartesian-co-ordinate and polar-co-ordinate representations of other functions will be identified.

(ii) In the first instance we establish a weak solution of (2.6). Let the real Hilbert space $Z$ be the completion of the set $C^\infty_0(\mathbb{R}^2)$, of real-valued, infinitely
differentiable functions on $\mathbb{R}^2$ having compact support, in the norm implied by the inner product
\[
\langle u, v \rangle_Z := \iint_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \sigma^2 uv) \, d\xi d\eta.
\] (2.8)

We shall say that $q$ is a weak solution of (2.6) if (and only if)
\[
q \in Z \text{ and, for all test functions } u \in Z,
B(u, q) := \iint_{\mathbb{R}^2} \left( \nabla u \cdot \nabla q + \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial q}{\partial \theta} u - 4\lambda e^{-\sigma^2/2} T e^{-\sigma^2/2} \right) \, d\xi d\eta = \lambda \iint_{\mathbb{R}^2} e^{\sigma^2/2} f u \, d\xi d\eta.
\] (2.9)

(iii) Here is the key step of the proof.

**Lemma 2.2.** The bilinear form $B$ satisfies, for all $u$ and $v$ in $Z$,
\[
B(u, u) = \|u\|^2, \quad (2.10)
\]
\[
|B(u, v)| \leq (1 + k_B \lambda) \|u\| \|v\|, \quad (2.11)
\]
where $\| \cdot \| = \| \cdot \|_Z$ and $k_B$ is an absolute constant (independent of the variables, parameters and functions in question).

**Partial proof.** We shall prove only that (2.10) holds for all functions in $C^\infty_c(\mathbb{R}^2)$. The remainder of the proof is neither trivial nor immediate, but it is of a kind familiar in Sobolev-space theory and its application to partial differential equations.

In view of the definition of $B$ in (2.9), we wish to prove that, for all $\varphi \in C^\infty_c(\mathbb{R}^2)$,
\[
\iint_{\mathbb{R}^2} \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial \varphi}{\partial \theta} \, d\xi d\eta = 0
\]
and
\[
\iint_{\mathbb{R}^2} e^{-\sigma^2/2} \varphi T (e^{-\sigma^2/2} \varphi) \, d\xi d\eta = 0.
\]

The first of these is immediate because $\int_{-\pi}^{\pi} \varphi \frac{\partial \varphi}{\partial \theta} \, d\theta = 0$. For the second, let $A(e^{-\sigma^2/2} \varphi) =: \psi$; then $e^{-\sigma^2/2} \varphi = -\Delta \psi$ and we wish to prove that
\[
-\iint_{\mathbb{R}^2} (\Delta \psi) \frac{\partial \psi}{\partial \theta} \, d\xi d\eta = 0.
\]

Here it suffices to integrate over an open disc (or ball) $B(0, R)$ with centre the origin and radius $R$ so large that $B(0, R)$ contains the compact support of $\Delta \psi$. Thus the integral may be written
\[
-\iint_{\partial B(0, R)} \frac{\partial \psi}{\partial \sigma} \frac{\partial \psi}{\partial \theta} R d\theta + \iint_{B(0, R)} \nabla \psi \cdot \frac{\partial}{\partial \theta} \nabla \psi \, d\xi d\eta.
\]

That this last integral over $B(0, R)$ vanishes is immediate as before. The boundary integral is now independent of $R$ and vanishes because $\partial \psi/\partial \sigma$ and
\[ \partial \psi / \partial \theta \text{ are both } O(R^{-1}) \text{ as } R \uparrow \infty, \text{ by the definition (2.1)(c) of the operator } A. \]

(iv) Existence and uniqueness of a weak solution. The forcing function in (2.6) satisfies amply the condition

\[ \int_{\mathbb{R}^2} \sigma f(\sigma, \theta)^2 \frac{e^{\sigma^2} + \sigma^2}{1 + \sigma^2} \, d\xi d\eta < \infty, \quad (2.12) \]

because \( f(\sigma, \theta) = (\kappa / \pi a) g(\sigma) \cos \theta \) with \( g \) as in (2.4). This condition is sufficient to make the forcing integral in (2.9), namely,

\[ F(u) := \int_{\mathbb{R}^2} e^{\sigma^2/2} f u \, d\xi d\eta, \quad u \in Z, \]

a bounded linear functional evaluated at \( u \). In other words, \( F \) belongs to the dual space \( Z^* \) of \( Z \). The Lax-Milgram lemma now implies

**Lemma 2.3.** Equation (2.6) has a unique weak solution \( q \) and

\[ \frac{\lambda}{1 + k_B \lambda} \| F \|_{Z^*} \leq \| q \|_Z \leq \lambda \| F \|_{Z^*}. \quad (2.13) \]

(v) Regularity theory: pointwise estimates. We separate the variables \( \sigma \) and \( \theta \). Let \( Y \) denote the real Hilbert space of functions \( y : [0, \infty) \to \mathbb{R} \) such that the functions having values \( y(\sigma) \cos \theta \) or \( y(\sigma) \sin \theta \) belong to \( Z \). It can be proved that, equivalently, \( Y \) is the completion of the set

\[ D := \{ \zeta \in C_c^\infty[0, \infty) \mid \zeta(0) = 0 \}, \]

where the compact support of \( \zeta \) may extend to the origin, in the norm implied by the inner product

\[ \langle v, w \rangle_Y := \int_0^\infty \left( v' w' + \left( \frac{1}{\sigma^2} + \sigma^2 \right) vw \right) \sigma d\sigma, \]

where the \( (\cdot)' \) denotes differentiation.

It can then be proved that, if

(a) \[ Q(\sigma) := q_c(\sigma) + iq_s(\sigma), \quad \text{where } (q_c, q_s) \in Y^2; \quad (2.14) \]

(b) the operator \( T_1 \) is defined by

\[ (T_1 y)(\sigma) := \frac{1}{2} \int_0^\infty \left( \frac{\rho \wedge \sigma}{\sigma^2} \right) y(\rho) \rho d\rho \quad \text{for all } y \in Y, \quad (2.15) \]

where \( a \wedge b \) denotes the lower envelope, or lesser, of \( a \) and \( b \); and
(c) for all test functions \( v \in Y \),

\[ \int_0^\infty \left( v' Q' + \left( \frac{1}{\sigma^2} + \sigma^2 \right) vQ - i \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} vQ + i 4 \lambda e^{-\sigma^2/2} v T_1 (v^{-\sigma^2/2} Q) \right) \sigma d\sigma = \lambda \int_0^\infty e^{\sigma^2/2} f_c v \sigma d\sigma, \quad (2.16) \]
where $f_c(\sigma) := (\kappa/\pi a)g(\sigma)$;

\[ q(\sigma, \theta) := q_c(\sigma) \cos \theta + q_s(\sigma) \sin \theta; \quad (2.17) \]

then $q$ satisfies (2.9), so that the right-hand member of (2.17) is the unique weak solution of (2.6). Conversely, equations (2.9), (2.17) and (2.14) imply (2.16).

We now choose the test function in (2.16) to be a Green function of the operator

\[ -\left( \frac{d}{d\sigma} \right)^2 - \frac{1}{\sigma} \frac{d}{d\sigma} + \left( \frac{1}{\sigma^2} + \sigma^2 \right), \]

which results from insertion of (2.17) into (2.6). It is legitimate to choose

\[ v(\sigma) = K(\rho, \sigma) := \begin{cases} \frac{1}{\sigma} \sinh \frac{\sigma^2}{2} \frac{\rho}{\sigma} \exp \left( -\frac{\sigma^2}{2} \right) & \text{if } \sigma \leq \rho, \\ \frac{1}{\sigma} \exp \left( -\frac{\sigma^2}{2} \right) \frac{1}{\rho} \sinh \frac{\rho^2}{2} & \text{if } \sigma \geq \rho, \end{cases} \quad (2.18) \]

because $K(\rho, \cdot) \in Y$ for fixed $\rho \in (0, \infty)$. Then (2.16) yields, after an integration by parts, the integral equation

\[ Q(\rho) = \lambda \int_0^\infty K(\rho, \sigma) e^{\sigma^2/2} f_c(\sigma) \sigma d\sigma 
+ i \lambda \int_0^\infty K(\rho, \sigma) \left( \frac{1 - e^{-\sigma^2}}{\sigma^2} Q(\sigma) - 4e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} Q) \right) \sigma d\sigma. \quad (2.19) \]

Since Lemma 2.3 provides bounds for $\|q_c\|_Y$ and $\|q_s\|_Y$, the regularity of $Q$, and pointwise bounds, can be deduced from (2.19) and from Lemma 3.8 below without great difficulty.

3 The perturbation $\omega_1$ as $\lambda \uparrow \infty$

We return to equations (2.1) to (2.4) and define a stream function $\tilde{\psi}_1 := A\tilde{\omega}_1$. Then $\tilde{\omega}_1 = -\Delta \tilde{\psi}_1$ and (2.2) becomes

\[ \left( \Delta + \frac{\partial}{\partial \sigma} + 2 \right) \Delta \tilde{\psi}_1 - \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial}{\partial \theta} \Delta \tilde{\psi}_1 - 4\lambda e^{-\sigma^2} \frac{\partial}{\partial \theta} \tilde{\psi}_1 
= \frac{\kappa \lambda}{\pi a} g(\sigma) \cos \theta \quad \text{on } E. \quad (3.1) \]

In view of (2.5) and (2.17), the function $\tilde{\psi}_1$ has the form

\[ \tilde{\psi}_1(\sigma, \theta) = \tilde{\psi}_{1c}(\sigma) \cos \theta + \tilde{\psi}_{1s}(\sigma) \sin \theta. \quad (3.2) \]

We divide (3.1) by $\lambda(1 - e^{-\sigma^2})/\sigma^2$, write the $\cos \theta$ and $\sin \theta$ parts as separate equations and define, similarly to (2.14),

\[ \Psi(\sigma) = \tilde{\psi}_{1c}(\sigma) + i\tilde{\psi}_{1s}(\sigma). \quad (3.3) \]
With the notation

\[ (a) \quad \Delta_1 := \left( \frac{d}{d\sigma} \right)^2 + \frac{1}{\sigma} \frac{d}{d\sigma} - \frac{1}{\sigma^2}, \]

\[ (b) \quad \alpha(\sigma) := \frac{4\sigma^2}{e^{\sigma^2} - 1}, \]

\[ (c) \quad \beta(\sigma) := \frac{\sigma^2}{1 - e^{-\sigma^2}}, \]

\[ (d) \quad \mathcal{E} := \Delta_1 + 2\sigma \frac{d}{d\sigma} + 2, \]

the problem for \( \tilde{\omega}_1 \) is to solve the equation

\[ -i\frac{\beta(\sigma)}{\lambda} \mathcal{E}(\Delta_1 \Psi) + \{\Delta_1 + \alpha(\sigma)\} \Psi = -i \frac{K}{\pi a} \beta(\sigma) g(\sigma), \quad 0 < \sigma < \infty, \tag{3.5} \]

with the side conditions

as \( \sigma \downarrow 0, \quad (\Delta_1 \Psi)(\sigma) \to 0, \quad \Psi'(\sigma) = O(1) \quad \text{and} \quad \Psi(\sigma) = O(\sigma); \]

as \( \sigma \uparrow \infty, \quad (\Delta_1 \Psi)(\sigma) \to 0, \quad \Psi'(\sigma) = O(\sigma^{-2}) \quad \text{and} \quad \Psi(\sigma) = O(\sigma^{-1}). \tag{3.6} \]

Here the conditions on \( \Delta_1 \Psi \) come from (2.3); the conditions on \( \Psi' \) and \( \Psi \) are implied by \( \Psi = -T_1(\Delta_1 \Psi) \), with \( T_1 \) as in (2.15), and by conditions on \( \Delta_1 \Psi \) much weaker than those in Theorem 2.1.

For \( \lambda \uparrow \infty \), equation (3.5) with (3.6) seems to form a singular perturbation problem, since a small parameter multiplies the highest derivatives. Surprisingly, this turns out not to be the case; nevertheless there is work to be done.

Apparently, if \( \Psi_0(\sigma) := \lim_{\lambda \uparrow \infty} \Psi(\sigma; \lambda) \) exists, then it must satisfy

\[ \{\Delta_1 + \alpha(\sigma)\} \Psi_0 = -i \frac{K}{\pi a} \beta(\sigma) g(\sigma), \quad 0 < \sigma < \infty, \tag{3.7} \]

and the six side conditions (3.6). We proceed to explore this problem.

**Lemma 3.1.** The equation

\[ \{\Delta_1 + \alpha(\sigma)\} u = 0, \quad 0 < \sigma < \infty, \tag{3.8} \]

has solutions

\[ U(\sigma) := \frac{1}{\sigma} \left( 1 - e^{-\sigma^2} \right) \]

and

\[ V(\sigma) := \frac{1}{\sigma} - U(\sigma) \log \left( e^{\sigma^2} - 1 \right). \tag{3.10} \]

Here \( U \) is an eigensolution (with eigenvalue 0) in that it satisfies not only (3.8) but also all six side conditions (3.6).

**Proof.** This is a matter of direct calculation. \( \square \)

**Lemma 3.2.** The forcing function in (3.7) is orthogonal to the eigensolution \( U \) in the sense that

\[ \int_0^\infty U(\sigma) \beta(\sigma) g(\sigma) \sigma d\sigma = 0. \tag{3.11} \]
Hence the problem (3.7) and (3.6) has a (non-unique) solution

\[ \Psi_0(\sigma) = c_0 U(\sigma) + i \frac{K}{2\pi a} \int_0^\sigma \{ U(\rho)V(\sigma) - U(\sigma)V(\rho) \} \beta(\rho) g(\rho) \, d\rho \]  

(3.12)

for every \( c_0 \in \mathbb{C} \).

**Proof.** Again this is a matter of direct calculation, but the calculation is not short. With \( \beta \) defined by (3.4)(c) and \( g \) by (2.4)(c), the analytic proof of the orthogonality condition (3.11) seems to require a page. (However, with any machine capable of numerical integration, numerical verification of (3.11) is quick and easy.) We note that Liouville’s formula for Wronskians yields

\[ U(\sigma)V'(\sigma) - U'(\sigma)v(\sigma) = -\frac{2}{\sigma}, \quad 0 < \sigma < \infty. \]  

(3.13)

The following lemma is also relevant.

**Lemma 3.3.** Define, for suitable functions \( f \),

\[ (Gf)(\sigma) := \frac{1}{2} \int_0^\sigma \{ U(\rho)V(\sigma) - U(\sigma)V(\rho) \} f(\rho) \, d\rho, \quad 0 < \sigma < \infty, \]  

(3.14)

and

\[ \mathcal{J}(f) := \int_0^\infty V(\rho)f(\rho) \, d\rho. \]  

(3.15)

Assume that \( f \in C[0,\infty) \), that \( \int_0^\infty U(\rho)f(\rho) \, d\rho = 0 \), that \( f(\sigma) = O(\sigma) \) as \( \sigma \downarrow 0 \) and that \( f(\sigma) = O(\sigma^m e^{-\sigma^2}) \), with \( m \geq 1 \), as \( \sigma \uparrow \infty \). Then

\[ \{ \Delta_1 + \alpha(\sigma) \} (Gf)(\sigma) = -f(\sigma) \quad \text{in } (0,\infty); \]  

(3.16)

as \( \sigma \downarrow 0 \),

\[ (Gf)(\sigma) = O(\sigma^3) \quad \text{and} \quad (\Delta_1 Gf)(\sigma) = O(\sigma); \]  

(3.17)

as \( \sigma \uparrow \infty \),

\[ (Gf)(\sigma) = -\frac{1}{2} \mathcal{J}(f) \sigma^{-1} + O(\sigma^m e^{-\sigma^2}), \]  

(3.18)

\[ (Gf)'(\sigma) = \frac{1}{2} \mathcal{J}(f) \sigma^{-2} + O(\sigma^{m-1} e^{-\sigma^2}), \]  

(3.19)

\[ (Gf)''(\sigma) = -\mathcal{J}(f) \sigma^{-3} + O(\sigma^m e^{-\sigma^2}), \]  

(3.20)

and

\[ (\Delta_1 Gf)(\sigma) \quad \text{and} \quad (\mathcal{E} Gf)(\sigma) \quad \text{are} \quad O(\sigma^m e^{-\sigma^2}). \]  

(3.21)

**Proof.** Equation (3.16) follows from the definition of \( Gf \) and two differentiations. That \( (Gf)(\sigma) = O(\sigma^3) \) as \( \sigma \downarrow 0 \) also follows from the definition; then the differential equation (3.16) shows that \( (\Delta_1 Gf)(\sigma) = O(\sigma) \) as \( \sigma \downarrow 0 \). In order to prove (3.18) and (3.19), we note that \( U(\sigma) \sim 1/\sigma \) and \( V(\sigma) \sim -\sigma \) as \( \sigma \uparrow \infty \), whence

\[ \int_0^\sigma U(\rho)f(\rho) \, d\rho = \left( \int_0^\infty - \int_\sigma^\infty \right) U(\rho)f(\rho) \, d\rho = 0 + O(\sigma^{m-1} e^{-\sigma^2}), \]  

\[ \int_0^\sigma V(\rho)f(\rho) \, d\rho = \left( \int_0^\infty - \int_\sigma^\infty \right) V(\rho)f(\rho) \, d\rho = \mathcal{J}(f) + O(\sigma^m e^{-\sigma^2}), \]  

9
from which (3.18) and (3.19) follow.

The differential equation (3.16) and the estimate (3.18) of $\mathcal{G}f$ imply that $(\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma^m e^{-\sigma^2})$ as $\sigma \uparrow \infty$. What has been proved now implies the estimates of $(\mathcal{G}f)'(\sigma)$ and $(\mathcal{E}\mathcal{G}f)(\sigma)$ for $\sigma \uparrow \infty$.

The result (3.12) prompts two questions. How (if at all) is $c_0$ to be evaluated? How smooth is $\Psi_0$? Analogues of both these questions will have to be answered more generally for each function $\Psi_n$ in an identity

$$
\Psi(\sigma; \lambda) = \sum_{n=0}^{N} \lambda^{-n} \Psi_n(\sigma) + R_N(\sigma; \lambda).
$$

Here we anticipate later results and note that, with $J$ as in (3.15),

$$
c_0 = i \frac{K}{2\pi a} J(\beta g) = i \frac{K}{a}(0.11527...).
$$

This follows from the equation governing $\Psi_1$, which requires an orthogonality condition involving $\Psi_0$.

Because of the function $Q =: Q(\cdot; \lambda)$ defined by (2.14) and (2.16), we now define $Q_0 = q_{0c} + iq_{0s}$ by

$$
Q_0(\sigma) := -e^{\sigma^2/2}(\Delta_1 \Psi_0)(\sigma) = e^{\sigma^2/2} \left( 4c_0 e^{-\sigma^2} - \alpha(\sigma) (\mathcal{G}h_0)(\sigma) - h_0(\sigma) \right),
$$

where $h_0 := -i(\kappa/\pi a)\beta g$. Evidently $q_{0c} = 0$.

It will emerge from Theorem 3.5 that $Q_0$ is the limit of $Q(\cdot; \lambda)$ as $\lambda \uparrow \infty$. In Figures 2 and 3, $Q_0$ is compared with $Q(\cdot; \lambda)$ for large $\lambda$; these values of $Q(\cdot; \lambda)$ were obtained by numerical solution of the equation

$$
-(\Delta_1 - \sigma^2)Q - i\lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} Q + i4\lambda e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} Q) = \frac{k\lambda}{\pi a} e^{\sigma^2/2} g(\sigma).
$$

This equation is equivalent to (2.6), because of (2.17); it is also the pointwise form of (2.16); with the condition that $Q(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, it has a pointwise, unique solution. Figures 2 and 3 are consistent with the result of Theorem 3.5 that, as $\lambda \uparrow \infty$,

$$
q_{c}(\cdot; \lambda) = O(\lambda^{-1}) \quad \text{and} \quad q_{s}(\cdot; \lambda) - q_{0s} = O(\lambda^{-2}).
$$

**Definition.** We shall say that a function $\varphi : [0, \infty) \to \mathbb{C}$ is satisfactory on $[0, k)$ if (and only if) there exist coefficients $b_n$ and a number $k > 0$ such that

$$
\varphi(\sigma) := \sum_{n=0}^{\infty} b_n \sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < k.
$$

**Lemma 3.4.** (i) The function $\beta g$ is satisfactory on $[0, (2\pi)^{1/2})$.

(ii) If $f$ is satisfactory on $[0, (2\pi)^{1/2})$, then so is $\mathcal{G}f$.

*Proof.* (i) We note that

$$
\beta(\sigma) g(\sigma) = \sigma e^{-\sigma^2} \left( \frac{3}{2} + \frac{\sigma^2}{1 - e^{-\sigma^2}} \left( \int_{0}^{1} \frac{1 - e^{-\rho^2}}{\rho} d\rho + C_0 + C_1 \right) \right),
$$
where

\[ C_0 := -\int_1^\infty \frac{e^{-\rho^2}}{\rho} \, d\rho, \quad C_1 := -\frac{1}{2} \left( \gamma_E + 1 - \log 2 \right), \]

and that the function \( w \) defined by

\[ w(z) = \frac{z}{1 - e^{-z}} \quad \text{if} \quad z \in \mathbb{C} \setminus \{0\} \setminus \{\text{poles}\} \quad \text{and} \quad w(0) = 1, \]

is holomorphic for \(|z| < 2\pi\).

(ii) Let

\[ W(\sigma) := \frac{1}{\sigma} - U(\sigma) \log \frac{e^{\sigma^2} - 1}{\sigma^2}, \]

where the limiting value of \((e^{\sigma^2} - 1)/\sigma^2\) is taken at \(\sigma = 0\). Then

\[ V(\sigma) = W(\sigma) - 2U(\sigma) \log \sigma \]
and

\[
(\mathcal{G}f)(\sigma) = \frac{1}{2} W(\sigma) \int_0^{\sigma} U(\rho) f(\rho) \rho d\rho - \frac{1}{2} U(\sigma) \int_0^{\sigma} W(\rho) f(\rho) \rho d\rho + \sigma^2 U(\sigma) \int_0^1 (\log t) U(\sigma t) f(\sigma t) t \, dt.
\]

The functions with values \(\sigma^2 W(\sigma), U(\sigma)\) and \(f(\sigma)\) are all satisfactory on \([0, (2\pi)^{1/2})\), so that \(\mathcal{G}f\) inherits this property.

**Theorem 3.5.** The perturbation \(\tilde{\omega}_1\) has a representation

\[
\tilde{\omega}_1(\sigma; \theta; \lambda) = \cos \theta \left\{ \lambda^{-1} \zeta_1(\sigma) + \lambda^{-3} \zeta_4(\sigma) + \zeta_5(\sigma; \lambda) \right\} + \sin \theta \left\{ \zeta_0(\sigma) + \lambda^{-2} \zeta_2(\sigma) + \zeta_4(\sigma; \lambda) \right\}
\]

(3.25) in which, for \(m = 0, 1, 2, 3\) and \(n = 4, 5\),
(a) the functions \(\zeta_m\) and \(\zeta_n(\cdot; \lambda)\) belong to \(C^\infty[0, \infty)\) and are satisfactory on \([0, (2\pi)^{1/2})\);
(b) as \(\sigma \uparrow \infty\), \(\zeta_m(\sigma) = O(\sigma^{2m+1} e^{-\sigma^2})\) and \(\zeta_n(\sigma; \lambda) = o(e^{-\sigma^2/2})\) for fixed \(\lambda\);
(c) as \(\lambda \uparrow \infty\), \(\zeta_n(\sigma; \lambda) = O(\lambda^{-n})\), uniformly over \(\sigma \in [0, \infty)\).

The proof will be by means of further lemmas. Let \(\Psi := \tilde{\psi}_{1c} + i\tilde{\psi}_{1s}\), as before, and let \(\Omega := -\Delta_1 \Psi\), so that \(\Omega = \tilde{\omega}_{1c} + i\tilde{\omega}_{1s}\). Our plan is to construct identities

\[
\Psi(\sigma; \lambda) = \sum_{n=0}^{N} \lambda^{-n} \Psi_n(\sigma) + R_N(\sigma; \lambda),
\]

(3.26)

\[
\Omega(\sigma; \lambda) = \sum_{n=0}^{N} \lambda^{-n} \Omega_n(\sigma) + r_N(\sigma; \lambda),
\]

(3.27)

in which estimates of the remainders \(R_N\) and \(r_N\) can be crude. In fact, we shall prove only that \(R_N\) and \(r_N\) are \(O(\lambda^{-N})\), but this is sufficient for (3.25) if \(N \geq 6\).

The terms of the expansion of \(\Psi\) are to satisfy

\[
\{\Delta_1 + \alpha(\sigma)\} \Psi_n = h_n, \quad n = 0, 1, \ldots, N,
\]

(3.28)

where

\[
h_0(\sigma) = -i \frac{K}{\pi a} \beta(\sigma) g(\sigma),
\]

(3.29)

\[
h_n := i \beta(\Delta_1 \Psi_{n-1}) \quad \text{for} \quad n = 1, \ldots, N,
\]

(3.30)

and

\[
-i \lambda^{-1} \beta(\sigma) \delta(\Delta_1 R_N) + \{\Delta_1 + \alpha(\sigma)\} R_N = i \lambda^{-N-1} \beta(\sigma) \delta(\Delta_1 \Psi_N);
\]

(3.31)

then the right-hand member of (3.26) will satisfy the equation (3.5) governing \(\Psi\).

Since \(\Omega = -\Delta_1 \Psi\) and \(\Psi = T_1 \Omega\), this scheme corresponds to

\[
-\Omega_n + \alpha(\sigma) T_1 \Omega_n = h_n, \quad n = 0, 1, \ldots, N,
\]

(3.32)

\[
i \lambda^{-1} \beta(\sigma) \epsilon_r N - r_N + \alpha(\sigma) T_1 r_N = -i \lambda^{-N-1} \beta(\sigma) \epsilon \Omega_N,
\]

(3.33)

where \(h_n = -i \beta(\sigma) \epsilon \Omega_{n-1}\) for \(n = 1, \ldots, N\).
Lemma 3.6. In order that equation (3.28), with the side conditions (3.6), have a solution, it is necessary that

\[ \int_{0}^{\infty} U(\sigma) h_n(\sigma) \sigma d\sigma = 0, \quad n = 0, 1, \ldots, N; \]  

equivalently, that

\[ \int_{0}^{\infty} \sigma^2 g(\sigma) d\sigma = 0 \quad \text{if} \quad n = 0, \]  

\[ \int_{0}^{\infty} \sigma^2 (\mathcal{E} \Omega_{n-1})(\sigma) d\sigma = 0 \quad \text{if} \quad n = 1, \ldots, N. \]  

Proof. Let \( M := \Delta_1 + \alpha(\sigma) \). Assume that \( u \) and \( v \) are in \( C^2(0, \infty) \), that \( \sigma u(\sigma)v'(\sigma) \rightarrow 0 \) as \( \sigma \downarrow 0 \) and as \( \sigma \uparrow \infty \) and that \( \sigma u'(\sigma)v(\sigma) \rightarrow 0 \) as \( \sigma \downarrow 0 \) and as \( \sigma \uparrow \infty \). Then integration by parts yields

\[ \int_{0}^{\infty} u(Mv) \sigma d\sigma = \int_{0}^{\infty} (Mu)v \sigma d\sigma. \]  

Now let \( u = U \) and \( v = \Psi_n \). If \( \Psi_n \) satisfies (3.6), then the foregoing hypotheses are satisfied. If also \( M\Psi_n = h_n \), then

\[ \int_{0}^{\infty} U h_n \sigma d\sigma = \int_{0}^{\infty} (M U) \sigma d\sigma = 0. \]  

Equations (3.35) and (3.36) follow from the identity \( U(\sigma) \beta(\sigma) = \sigma \) and from the definitions of \( h_n \). \qed

If \( h_n \) satisfies not only the orthogonality condition (3.34), but also the other hypotheses on \( f \) in Lemma 3.3 (and this will be the case), then the differential equation (3.28), with side conditions (3.6), has solutions

\[ \Psi_n = c_n U - G h_n, \quad n = 0, 1, \ldots, N, \]  

whence

\[ \Omega_n(\sigma) = -(\Delta_1 \Psi_n)(\sigma) = 4c_n \sigma e^{-\sigma^2} - \alpha(\sigma) \beta(\sigma) h_n(\sigma) - h_n(\sigma), \]  

for every \( c_n \in \mathbb{C} \).

In order to evaluate \( c_0, \ldots, c_N \) and in order to discuss \( r_N \), we extend the definition (3.30) to \( h_{N+1} \). Recall from Lemma 3.2 that for \( n = 0 \) the orthogonality condition (3.34) has already been established.

Lemma 3.7. For \( n = 0, 1, \ldots, N \), the necessary condition \( \int_{0}^{\infty} U h_{n+1} \sigma d\sigma = 0 \) implies that \( c_n = -\frac{1}{2} J(h_n) \), where \( J(\cdot) \) is defined by (3.15).

Proof. Extended to \( \Omega_N \), the orthogonality condition (3.36) states that, for \( n = 0, 1, \ldots, N \),

\[ 0 = \int_{0}^{\infty} \sigma^2 (\mathcal{E} \Omega_n) d\sigma = -4 \int_{0}^{\infty} \sigma^2 \Omega_n d\sigma, \]

by an integration by parts for which it suffices that \( \Omega_n \in C^2(0, \infty) \), that \( \Omega_n'(\sigma) = o(\sigma^{-2}) \) both as \( \sigma \downarrow 0 \) and as \( \sigma \uparrow \infty \), that \( \Omega_n = o(\sigma^{-1}) \) as \( \sigma \downarrow 0 \) and that \( \Omega_n = o(\sigma^{-1}) \) as \( \sigma \uparrow \infty \).
Next, we observe that, if $\Psi_n \in C^2(0, \infty)$ and if both $\sigma^2 \Psi'_n(\sigma)$ and $\sigma \Psi_n(\sigma)$ have limits both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, then

$$0 = \int_0^\infty \sigma^2(\Delta_1 \Psi_n) d\sigma = \left[\sigma^2 \Psi'_n - \sigma \Psi_n\right]_0^\infty,$$

in which limiting values are implied on the right-hand side. In view of (3.39), the orthogonality condition is now

$$c_n \left[\sigma^2 U' - \sigma U\right]_0^\infty = \left[\sigma^2 (G h_n)' - \sigma (G h_n)\right]_0^\infty.$$

Referring to the definition of $U$ in (3.9) and to the description of $G f$ in Lemma 3.3, one is led to $c_n = -\frac{1}{2} J(h_n)$. \hfill \square

It is time to relate the $\zeta_m$ and $\zeta_n(\cdot; \lambda)$ in Theorem 3.5 to the $\Omega_n$ and $r_N$ in (3.27). We noted after (3.23) that $Q_0$ is imaginary. Since $\Omega(\sigma) = \exp(-\sigma^2/2)Q(\sigma)$, the function $\Omega_0$ is imaginary. Then, since $h_1 = -i\beta E \Omega_0$, the function $h_1$ and the coefficient $c_1$ are real. Equation (3.40) shows that $\Omega_1$ is real. An easy induction now shows that $\Omega_n$ is imaginary if $n$ is even and $\Omega_n$ is real if $n$ is odd.

Accordingly, if $N$ is odd, then

$$\zeta_1 = \Omega_1, \quad \zeta_3 = \Omega_3 \quad \text{and} \quad \zeta_5(\cdot; \lambda) = \lambda^{-5} \Omega_5 + \ldots + \lambda^{-N+1} \Omega_N + \text{Re} \ r_N(\cdot; \lambda),$$
$$\zeta_0 = -i \Omega_0, \quad \zeta_2 = -i \Omega_2 \quad \text{and} \quad \zeta_4(\cdot; \lambda) = -i \left(\lambda^{-4} \Omega_4 + \ldots + \lambda^{-N+1} \Omega_{N-1}\right) + \text{Im} \ r_N(\cdot; \lambda).$$

If $N$ is even, then there is a similar array.

Because of the explicit formula (3.40) for $\Omega_n$ (with $h_n = -i\beta E \Omega_{n-1}$, with $c_n = -\frac{1}{2} J(h_n)$) and with the operator $G$ described by Lemmas 3.3 and 3.4), enough may have been said about $\Omega_n$ to justify the claims made for $\zeta_0$ to $\zeta_3$ in Theorem 3.5. For example, the result

$$\zeta_m(\sigma) = O(\sigma^{2m+4} e^{-\sigma^2}) \quad \text{as} \quad \sigma \uparrow \infty$$

follows for $m = 0$ from $\beta(\sigma) \sim \sigma^2$ and from the overestimate $g(\sigma) = O(\sigma^2 e^{-\sigma^2})$, which imply that $h_0$ and $\Omega_0$ are $O(\sigma^4 e^{-\sigma^2})$. Then repeated use of $h_{n+1} = -i\beta E \Omega_n$ leads to (3.42).

On the other hand, the remainder $r_N$ requires further discussion. Under the transformations

$$r_N(\sigma) = e^{-\sigma^2/2} P_N(\sigma) = e^{-\sigma^2/2} \left\{p_{Nc}(\sigma) + ip_{Ns}(\sigma)\right\}, \quad (3.43)$$
$$p_N(\sigma, \theta) := p_{Nc}(\sigma) \cos \theta + p_{Ns}(\sigma) \sin \theta, \quad (3.44)$$

equation (3.33) becomes

$$-(\Delta_\sigma - \sigma^2)p_N + \frac{\lambda}{\beta(\sigma)} \frac{\partial}{\partial \theta} p_N - 4\lambda e^{-\sigma^2/2} T(e^{-\sigma^2/2} p_N) = \lambda^{-N} e^{\sigma^2/2} f_N(\sigma, \theta),$$

where

$$f_N(\sigma, \theta) = \text{Re} \left\{(E \Omega_N)(\sigma)e^{-i\theta}\right\}. \quad (3.45)$$

$$f_N(\sigma, \theta) = \text{Re} \left\{\left(E \Omega_N(\sigma)\right)e^{-i\theta}\right\}. \quad (3.46)$$
The operator on the left-hand side of (3.45) is that in (2.6); as in §2, it follows that equation (3.45) has a unique weak solution bounded by
\[ \|p_N\|_Z \leq A_N\lambda^{-N}, \] (3.47)
where \( A_N \) depends only on \( N \).

Choosing the test function in the definition of weak solution as in (2.18), we obtain the equation
\[ P_N(\rho) = \lambda^{-N} \int_0^\infty K(\rho, \sigma) e^{\sigma^2/2}(E\Omega N)(\sigma) \sigma d\sigma \]
(3.48)
This leads without difficulty to a pointwise solution \( P_N \in C^2[0, \infty) \) such that \( P_N(0) = 0 \) and such that \( P_N(\sigma) \to 0 \) as \( \sigma \to \infty \). (Correspondingly, \( r_N(\sigma) \) is \( o(e^{-\sigma^2/2}) \) as \( \sigma \to \infty \).) Equation (3.48) and the bound (3.47) now imply that \( P_N(\sigma; \lambda) \) is \( O(\lambda^{-N+1}) \) uniformly over \( \sigma \in [0, \infty) \).

Moreover, the equation
\[-(\Delta - \sigma^2)P_N - \frac{i\lambda}{\beta(\sigma)}P_N + 4i\lambda e^{-\sigma^2/2}T_1(e^{-\sigma^2/2}P_N) = \lambda^{-N}e^{\sigma^2/2}E\Omega N,\]
may be written as
\[ P''_N = -\frac{1}{\sigma}P'_N + \left( \frac{1}{\sigma^2} + \sigma^2 \right)P_N - ... - \lambda^{-N}e^{\sigma^2/2}E\Omega N. \]

The right-hand member of this is in \( C^1[0, \infty) \) for any \( k > 0 \), say in \( C^1[1, \infty) \). Repetition of this step shows that \( P_N \in C^\infty[0, \infty) \).

It remains to prove that \( P_N \) is better than \( C^2 \) at and near the origin. We return to equation (3.33) for \( r_N \) and to the equation
\[ E\Omega + \frac{i\lambda}{\beta(\sigma)}\Omega - 4i\lambda e^{-\sigma^2/2}T_1\Omega = -\frac{\kappa\lambda g}{\pi a} \]
for \( \Omega = \tilde{\omega}_1c + i\tilde{\omega}_1s \); our final lemma applies to both \( r_N \) and \( \Omega \).

**Lemma 3.8.** Assume that the equation
\[ Eu + \frac{i\lambda}{\beta(\sigma)}u - 4i\lambda e^{-\sigma^2}T_1u = \lambda f \] (3.49)
has a solution \( u \in C^2[0, \infty) \) such that \( u(0) = 0 \) and such that \( u(\sigma) \) is \( o(e^{-\sigma^2/2}) \) as \( \sigma \to \infty \). Assume also that \( u \) is unique because it is the transformed version of a solution in the Hilbert space \( Z \).

Then \( u \) is satisfactory on \([0, (2\pi)^{1/2})\) whenever \( f \) has this property.

**Proof.** We shall prove that there are coefficients \( a_n \) such that
\[ u(\sigma) = \sum_{n=0}^{\infty} a_n\sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < b \]
if \( b \in (0, (2\pi)^{1/2}) \). Here we are not constructing a series solution \textit{ab initio} in the usual way; rather, we are establishing a regularity property of a known,
unique solution. Therefore we may regard \( a_0 = u'(0) \) and \( \int_0^\infty u d\sigma \) as known; we proceed to calculate the other coefficients in terms of these. The equation is satisfied, subject to convergence of the series, if for \( n = 0, 1, 2, ... \)

\[
4(n+1)(n+2)a_{n+1} = -4(n+1)a_n - \sum_{j=0}^{n} (B_{n-j}a_j + A_{n-j}\tau_j) + \lambda f_n, \quad (3.50)
\]

where

\[
B_m = \frac{i\lambda (-1)^m}{(m+1)!}, \quad A_m = \frac{4i\lambda (-1)^{m+1}}{m!},
\]

\[
\tau_0 = \frac{1}{2} \int_0^\infty u d\sigma, \quad \tau_m = -\frac{1}{4} a_{m-1} \frac{m}{m(m+1)} \text{ for } m \geq 1,
\]

and

\[
f(\sigma) = \sum_{n=0}^{\infty} f_n \sigma^{2n+1} \quad \text{for } 0 \leq \sigma < (2\pi)^{1/2}.
\]

Hence there is a constant \( C = C(b) \) such that \( |f_n| \leq Cb^{-2n} \).

Now, for every \( p \in \{1, 2, 3, ...\} \) there is a number \( \Gamma_p = \Gamma_p(b, \lambda) \) such that \( |a_n| \leq \Gamma_p b^{-2n} \) for \( n = 0, 1, ..., p \). We may suppose that \( \Gamma_p \geq \kappa/\alpha \). Then (3.50) implies that

\[
|a_{n+1}| \leq \Gamma_p b^{-2n-2} \varphi(n, \lambda) \quad \text{for } n \leq p,
\]

where

\[
\varphi(n, \lambda) := \frac{2\pi}{4(n+1)(n+2)} \left( 4(n+1) + \lambda (1+\pi) e^{2\pi} + \frac{4\lambda |\tau_0| (2\pi)^n}{n!} + \lambda C \right).
\]

For fixed \( \lambda \), we choose \( p \) so large that \( \varphi(p, \lambda) \leq 1 \) and so large that \( \varphi(n, \lambda) \) decreases for \( n \geq p \). Then \( |a_m| \leq \Gamma_p b^{-2m} \) not merely for \( m \leq p \), but also for \( m \geq p + 1 \).

References

