On a singular initial-value problem for the Navier-Stokes equations

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This paper presents a recent result for the problem introduced eleven years ago in [1], but described only briefly there. We shall prove the following, as far as space allows. The vorticity ω of a diffusing vortex circle in a viscous fluid has, for small values of a non-dimensional time, a second approximation $\omega_A + \omega_1$ that, although formulated for a fixed, finite Reynolds number λ and exact for $\lambda = 0$ (then $\omega = \omega_A$), tends to a smooth limiting function as $\lambda \uparrow \infty$.

In §1 and §2 the necessary background and apparatus are described; §3 outlines the new result and its proof.

1 Introduction

In a certain weak sense, this paper is a continuation of [1]. However, no knowledge of [1] is required if the reader is willing to accept that a vorticity field in $\mathbb{R}^3$ (subject to mild restrictions, but not required to have any symmetry) has a *centroid of vorticity* moving with a velocity $\mathbf{U}(t)$ that is given by an explicit formula when the vorticity $\boldsymbol{\omega}(\cdot, t)$ throughout $\mathbb{R}^3$ is known. This result is essentially due to Saffman [2]; it was generalized a little (and perhaps clarified and sharpened) in [1].

We seek a solution of the Navier-Stokes equations with the initial condition illustrated in Figure 1: at time zero, vorticity $\boldsymbol{\omega}$ is concentrated on, and is tangential to, a horizontal circle in $\mathbb{R}^3$. This initial vorticity induces an initial velocity field that has infinite kinetic energy. (The circle then diffuses and moves vertically, at first with infinite velocity; at all positive times $t > 0$ the kinetic energy is finite.)

More precisely, consider incompressible fluid occupying all of $\mathbb{R}^3$ and at rest at infinity there; let $x := (x_1, x_2, x_3)$ be such that the frame (Ox_1, Ox_2, Ox_3) moves, relative to the motionless fluid at infinity, with the velocity $(0, 0, U(t))$ of the centroid of vorticity, the axes remaining parallel to their initial positions. The *fluid velocity* relative to this moving frame is written $\mathbf{v}(x, t)$ and the *vorticity* is

$$\boldsymbol{\omega} := \text{curl } \mathbf{v} = \nabla \times \mathbf{v}.$$

Our *time variable* is $t = \nu T$, where T denotes physical time and ν is the kinematic viscosity (a given positive constant). This choice of t simplifies the heat operator in (1.3) below and simplifies most subsequent equations.

In writing $\mathbf{U}(t) = (0, 0, U(t))$, we have restricted attention to the cylindrical symmetry implied by the initial condition

$$\boldsymbol{\omega}(x, 0) = \kappa \delta(z) \delta(r - a) \mathbf{e}^{\phi}, \tag{1.1}$$

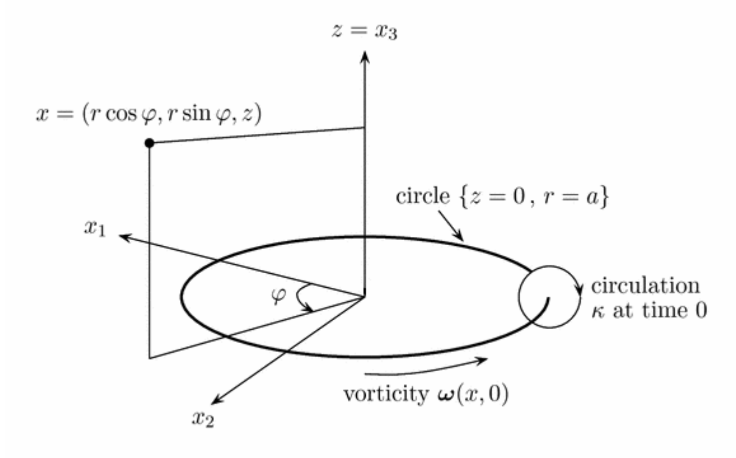

Figure 1: The initial condition.

in which the circulation κ and the radius a are given positive constants, cylindrical co-ordinates (z, r, ϕ) are defined by $x =: (r\cos\phi, r\sin\phi, z)$, the unit vector $\mathbf{e}^\phi := (-\sin\phi, \cos\phi, 0)$ and δ denotes the Dirac generalized function.

In terms of the vorticity $\boldsymbol{\omega}$, the fluid velocity (relative to our moving frame) is

$$\mathbf{v}(x,t) = -(0,0,U(t)) + \nabla \times \int_{\mathbb{R}^3} \frac{1}{4\pi|x-x'|}\boldsymbol{\omega}(x',t)\mathrm{d}x'. \tag{1.2}$$

With (1.1) and (1.2) understood, we seek $\boldsymbol{\omega}(x,t)$ such that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\boldsymbol{\omega} = -\frac{1}{\nu}\left((\mathbf{v}.\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}.\nabla)\mathbf{v}\right) \quad \text{in} \quad \mathbb{R}^3 \times (0, \bar{t}\,) \tag{1.3}$$

for some small $\bar{t} > 0$.

Of course, it would be better to solve the problem (1.1) to (1.3) for all $t > 0$, but this is beyond us because we seek rather explicit answers. There are two excuses for considering only small t, or, rather, small t/a^2, which is non-dimensional. First, once a solution for $t > 0$ has been established, the general theory of the Navier-Stokes equations implies a continuation of the solution to all time, thanks to finite energy for $t > 0$, cylindrical symmetry and absence of a swirl velocity (of a velocity component in the direction $\mathbf{e}^\phi$). Secondly, if the viscosity ν is small, which may be the case of primary interest, then the requirement that $\nu T/a^2$ be small does not demand that the physical time T be small.

In view of (1.1), we write

$$\boldsymbol{\omega}(x,t) =: \omega(z,r,t)\mathbf{e}^\phi,$$

and seek the solution of (1.1) to (1.3) in the scalar form $\omega = \omega_A + \omega_1 + \rho$, where ω_A is to be a first approximation for small t/a^2 and $\omega_A + \omega_1$ is to be a second (improved) approximation; the remainder ρ is to make $\omega_A + \omega_1 + \rho$ an exact solution and is to be $o(\omega_1)$ as $t \downarrow 0$. Here are some details.

(i) The non-linear terms on the right-hand side of (1.3) are expected to be small for small t/a^2, because $\boldsymbol{\omega}$ should be approximately constant and large on small circles in a meridional plane (ϕ = constant) centred at $(z,r)=(0,a)$, so that $\mathbf{v}$ is approximately tangential to such circles and approximately of constant magnitude on each of them. (If the initial vortex circle were a straight line, then these non-linear terms would vanish.) If the right-hand member of (1.3) is neglected, there results the formal approximation

$$\omega_A(z,r,t)=\frac{\kappa}{4\pi t}\exp\left(-\frac{s^2}{4t}\right)\left(\frac{a}{r}\right)^{1/2}B\left(\frac{ar}{2t}\right), \tag{1.4}$$

where $s:=\{z^2+(r-a)^2\}^{1/2}$ and B is a known function such that $B(y)\to 1$ as $y\to\infty$; in fact,

$$B(y):=(2\pi y)^{1/2}\mathrm{e}^{-y}I_1(y)\quad(0\le y<\infty), \tag{1.5}$$

where I_1 is the modified Bessel function of the first kind and of order 1 (as in [3], p.77).

(ii) The exponential in (1.4) prompts us to introduce *inner variables*

$$\sigma:=\frac{s}{(4t)^{1/2}},\quad \theta:=\tan^{-1}\frac{r-a}{z}; \tag{1.6}$$

then the amplitude $\kappa/4\pi t$ in (1.4) prompts us to pose

$$\omega_1(z,r,t)=(4t)^{-1/2}\tilde{\omega}_1(\sigma,\theta). \tag{1.7}$$

It suffices to consider ω_1 in an *inner region*: $t\downarrow 0$ with σ fixed, so that $s\downarrow 0$, because in an *outer region*: $t\downarrow 0$ with $s\ge$ *constant* >0, not only ω_A, but also ω, are exponentially small.

The rest of this paper is devoted mainly to description of $\tilde{\omega}_1$; the *Reynolds number*

$$\lambda:=\frac{\kappa}{2\pi\nu} \tag{1.8}$$

will be an important parameter.

(iii) The problem for the remainder ρ was sketched in [1]; the function $\rho(z,r,t)$ must be shown to exist and to be suitably small on the whole set $\mathbb{R}\times[0,\infty)\times(0,\bar{t}]$. Considerable progress has been made with this problem since [1] was written, but this analysis (which can only estimate ρ) is too long and too elaborate to be described here.

2 The perturbation ω_1 for fixed Reynolds number λ

With ω_1 as in (1.7), we adopt the notation

$$\begin{aligned}
&\text{(a)}\quad (\sigma,\theta)\in E:=(0,\infty)\times(-\pi,\pi],\\
&\text{(b)}\quad \Delta_\sigma:=\left(\frac{\partial}{\partial\sigma}\right)^2+\frac{1}{\sigma}\frac{\partial}{\partial\sigma}+\frac{1}{\sigma^2}\left(\frac{\partial}{\partial\theta}\right)^2,\\
&\text{(c)}\quad (A\tilde{\omega}_1)(\sigma_0,\theta_0):=\frac{1}{2\pi}\iint_E\log\frac{1}{|\sigma\mathrm{e}^{i\theta}-\sigma_0\mathrm{e}^{i\theta_0}|}\,\tilde{\omega}_1(\sigma,\theta)\,\sigma\mathrm{d}\sigma\mathrm{d}\theta,\\
&\text{(d)}\quad \omega_{A,0}(\sigma,t):=\frac{\kappa}{4\pi t}\mathrm{e}^{-\sigma^2},
\end{aligned} \tag{2.1}$$

in which $A\tilde{\omega}_1$ is a stream function describing the plane flow induced by vorticity $\tilde{\omega}_1$; the approximation $\omega_{A,0}$ to ω_A is that appropriate to $t \downarrow 0$ with σ fixed. We seek $\tilde{\omega}_1(\sigma,\theta)$ by linearizing (1.2) and (1.3) about $\omega_{A,0}$; the problem is then to solve the equation

$$-\left(\Delta_\sigma + 2\sigma\frac{\partial}{\partial\sigma} + 2\right)\tilde{\omega}_1 + \lambda\frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2}\frac{\partial}{\partial\theta}\tilde{\omega}_1 - 4\lambda\mathrm{e}^{-\sigma^2}\frac{\partial}{\partial\theta}(A\tilde{\omega}_1) = \frac{\kappa\lambda}{\pi a}g(\sigma)\cos\theta \quad \text{on } E, \tag{2.2}$$

with side conditions

$$\tilde{\omega}_1(\sigma,\theta) \to 0 \quad \text{as } \sigma \downarrow 0 \text{ and as } \sigma \uparrow \infty. \tag{2.3}$$

The function g is a known, smooth function such that

(a) $g(\sigma) = O(\sigma)$ as $\sigma \downarrow 0$;

(b) $g(\sigma) = O\left(\sigma\log\sigma\,\mathrm{e}^{-\sigma^2}\right)$ as $\sigma \uparrow \infty$;

in fact,

(c) $$g(\sigma) := \sigma\mathrm{e}^{-\sigma^2}\left(\frac{3}{2}\frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2} + \left(\log\frac{1}{\sigma} - \int_\sigma^\infty \frac{\mathrm{e}^{-\rho^2}}{\rho}\mathrm{d}\rho\right) - \frac{1}{2}(\gamma_E + 1 - \log 2)\right), \tag{2.4}$$

where $\gamma_E = 0.5772...$ denotes Euler's constant.

Theorem 2.1. *For fixed $\lambda \in [0,\infty)$, the problem (2.2) and (2.3) for $\tilde{\omega}_1$ has a pointwise, unique solution; in particular, $\tilde{\omega}_1(\cdot,\theta) \in C^\infty[0,\infty)$, $\tilde{\omega}_1(0,\theta) = 0$ and $\tilde{\omega}_1(\sigma,\theta) = o(\mathrm{e}^{-\sigma^2/2})$ as $\sigma \uparrow \infty$.*

Here we have space only to sketch the main steps of the proof.

(i) Under the transformation

$$\tilde{\omega}_1(\sigma,\theta) = \mathrm{e}^{-\sigma^2/2}q(\sigma,\theta), \tag{2.5}$$

equation (2.2) becomes

$$-\left(\Delta_\sigma - \sigma^2\right)q + \lambda\frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2}\frac{\partial q}{\partial\theta} - 4\lambda\mathrm{e}^{-\sigma^2/2}T\left(\mathrm{e}^{-\sigma^2/2}q\right) = \lambda\mathrm{e}^{\sigma^2/2}f(\sigma,\theta) \quad \text{on } E, \tag{2.6}$$

where the operator $T := (\partial/\partial\theta)A$ and $f(\sigma,\theta) := (\kappa/\pi a)g(\sigma)\cos\theta$. Let

$$(\xi,\eta) := \sigma(\cos\theta,\sin\theta), \qquad q_*(\xi,\eta) = q_*(\sigma\cos\theta,\sigma\sin\theta) := q(\sigma,\theta). \tag{2.7}$$

The condition in (2.3) for $\sigma \downarrow 0$ will be implicit in what follows; it was imposed only to make q_* decent at the origin, because we shall find that q is of form $q_c(\sigma)\cos\theta + q_s(\sigma)\sin\theta$. Henceforth the functions q_* *and* q *will be identified* wherever no confusion is possible. Similarly, the Cartesian-co-ordinate and polar-co-ordinate representations of other functions will be identified.

(ii) In the first instance we establish a weak solution of (2.6). Let the *real Hilbert space* Z be the completion of the set $C_c^\infty(\mathbb{R}^2)$, of real-valued, infinitely differentiable functions on $\mathbb{R}^2$ having compact support, in the norm implied by the inner product

$$\langle u, v\rangle_Z := \iint_{\mathbb{R}^2} \big(\nabla u \cdot \nabla v + \sigma^2 uv\big)\, \mathrm{d}\xi \mathrm{d}\eta. \tag{2.8}$$

We shall say that q is a *weak solution of* (2.6) if (and only if) $q \in Z$ and, for all test functions $u \in Z$,

$$\begin{aligned} B(u,q) := \iint_{\mathbb{R}^2} \bigg(\nabla u \cdot \nabla q + \sigma^2 uq + \lambda \frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2} u \frac{\partial q}{\partial \theta} \\ -4\lambda \mathrm{e}^{-\sigma^2/2} u\, T\left(\mathrm{e}^{-\sigma^2/2} q\right)\bigg)\, \mathrm{d}\xi \mathrm{d}\eta = \lambda \iint_{\mathbb{R}^2} \mathrm{e}^{\sigma^2/2} f u \, \mathrm{d}\xi \mathrm{d}\eta. \end{aligned} \tag{2.9}$$

(iii) Here is the key step of the proof.

Lemma 2.2. *The bilinear form B satisfies, for all u and v in Z,*

$$B(u,u) = \|u\|^2, \tag{2.10}$$

$$|B(u,v)| \le (1 + k_B \lambda)\|u\|\|v\|, \tag{2.11}$$

where $\|\cdot\| = \|\cdot\|_Z$ and k_B is an absolute constant (independent of the variables, parameters and functions in question).

Partial proof. We shall prove only that (2.10) holds for all functions in $C_c^\infty(\mathbb{R}^2)$. The remainder of the proof is neither trivial nor immediate, but it is of a kind familiar in Sobolev-space theory and its application to partial differential equations.

In view of the definition of B in (2.9), we wish to prove that, for all $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2} \frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2} \varphi \frac{\partial \varphi}{\partial \theta} \mathrm{d}\xi \mathrm{d}\eta = 0$$

and

$$\iint_{\mathbb{R}^2} \mathrm{e}^{-\sigma^2/2} \varphi\, T(\mathrm{e}^{-\sigma^2/2}\varphi)\, \mathrm{d}\xi \mathrm{d}\eta = 0.$$

The first of these is immediate because $\int_{-\pi}^{\pi} \varphi \frac{\partial \varphi}{\partial \theta} \mathrm{d}\theta = 0$. For the second, let $A(\mathrm{e}^{-\sigma^2/2}\varphi) =: \psi$; then $\mathrm{e}^{-\sigma^2/2}\varphi = -\Delta\psi$ and we wish to prove that

$$-\iint_{\mathbb{R}^2} (\Delta\psi) \frac{\partial \psi}{\partial \theta} \mathrm{d}\xi \mathrm{d}\eta = 0.$$

Here it suffices to integrate over an open disc (or ball) $\mathcal{B}(0,R)$ with centre the origin and radius R so large that $\mathcal{B}(0,R)$ contains the compact support of $\Delta\psi$. Thus the integral may be written

$$-\int_{\partial\mathcal{B}(0,R)} \frac{\partial \psi}{\partial \sigma} \frac{\partial \psi}{\partial \theta} R \mathrm{d}\theta + \iint_{\mathcal{B}(0,R)} \nabla\psi \cdot \frac{\partial}{\partial \theta} \nabla \psi \, \mathrm{d}\xi \mathrm{d}\eta.$$

That this last integral over $\mathcal{B}(0,R)$ vanishes is immediate as before. The boundary integral is now independent of R and vanishes because $\partial\psi/\partial\sigma$ and

$\partial\psi/\partial\theta$ are both $O(R^{-1})$ as $R\uparrow\infty$, by the definition (2.1)(c) of the operator A. □

(iv) Existence and uniqueness of a weak solution. The forcing function in (2.6) satisfies amply the condition

$$\iint_{\mathbb{R}^2} \frac{e^{\sigma^2} f(\sigma,\theta)^2}{1+\sigma^2} d\xi d\eta < \infty, \tag{2.12}$$

because $f(\sigma,\theta)=(\kappa/\pi a)g(\sigma)\cos\theta$ with g as in (2.4). This condition is sufficient to make the forcing integral in (2.9), namely,

$$F(u) := \iint_{\mathbb{R}^2} e^{\sigma^2/2} f u \, d\xi d\eta, \quad u \in Z,$$

a bounded linear functional evaluated at u. In other words, F belongs to the dual space Z^* of Z. The Lax-Milgram lemma now implies

Lemma 2.3. *Equation* (2.6) *has a unique weak solution* q *and*

$$\frac{\lambda}{1+k_B\lambda}\|F\|_{Z^*} \le \|q\|_Z \le \lambda\|F\|_{Z^*}. \tag{2.13}$$

(v) Regularity theory: pointwise estimates. We separate the variables σ and θ. Let Y denote the real Hilbert space of functions $y:[0,\infty)\to\mathbb{R}$ such that the functions having values $y(\sigma)\cos\theta$ or $y(\sigma)\sin\theta$ belong to Z. It can be proved that, equivalently, Y is the completion of the set

$$D := \{\zeta \in C_c^\infty[0,\infty) \,|\, \zeta(0)=0\},$$

where the compact support of ζ may extend to the origin, in the norm implied by the inner product

$$\langle v,w\rangle_Y := \int_0^\infty \left(v'w' + \left(\frac{1}{\sigma^2}+\sigma^2\right) vw \right) \sigma d\sigma,$$

where the $(\cdot)'$ denotes differentiation.

It can then be proved that, if

(a)

$$Q(\sigma) := q_c(\sigma) + iq_s(\sigma), \quad \text{where } (q_c,q_s)\in Y^2; \tag{2.14}$$

(b) the operator T_1 is defined by

$$(T_1 y)(\sigma) := \frac{1}{2}\int_0^\infty \left(\frac{\rho}{\sigma}\wedge\frac{\sigma}{\rho}\right) y(\rho)\,\rho d\rho \quad \text{for all } y\in Y, \tag{2.15}$$

where $a\wedge b$ denotes the lower envelope, or lesser, of a and b;

(c) for all test functions $v\in Y$,

$$\int_0^\infty \left(v'Q' + \left(\frac{1}{\sigma^2}+\sigma^2\right) vQ - i\lambda\frac{1-e^{-\sigma^2}}{\sigma^2} vQ + i4\lambda e^{-\sigma^2/2} v T_1(e^{-\sigma^2/2}Q) \right)\sigma d\sigma$$
$$= \lambda \int_0^\infty e^{\sigma^2/2} f_c v \,\sigma d\sigma, \tag{2.16}$$

where $f_c(\sigma) := (\kappa/\pi a)g(\sigma)$;

(d)

$$q(\sigma,\theta) := q_c(\sigma)\cos\theta + q_s(\sigma)\sin\theta; \tag{2.17}$$

then q satisfies (2.9), so that the right-hand member of (2.17) is the unique weak solution of (2.6). Conversely, equations (2.9), (2.17) and (2.14) imply (2.16).

We now choose the test function in (2.16) to be a Green function of the operator

$$-\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^2 - \frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma} + \left(\frac{1}{\sigma^2} + \sigma^2\right),$$

which results from insertion of (2.17) into (2.6). It is legitimate to choose

$$v(\sigma) = K(\rho,\sigma) := \begin{cases} \frac{1}{\sigma}\sinh\frac{\sigma^2}{2} . \frac{1}{\rho}\exp\left(-\frac{\rho^2}{2}\right) & \text{if } \sigma \le \rho, \\ \frac{1}{\sigma}\exp\left(-\frac{\sigma^2}{2}\right) . \frac{1}{\rho}\sinh\frac{\rho^2}{2} & \text{if } \sigma \ge \rho, \end{cases} \tag{2.18}$$

because $K(\rho,\cdot) \in Y$ for fixed $\rho \in (0,\infty)$. Then (2.16) yields, after an integration by parts, the integral equation

$$\begin{aligned} Q(\rho) =& \lambda \int_0^\infty K(\rho,\sigma)\mathrm{e}^{\sigma^2/2} f_c(\sigma)\,\sigma\mathrm{d}\sigma \\ &+ i\lambda \int_0^\infty K(\rho,\sigma)\left(\frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2}Q(\sigma) - 4\mathrm{e}^{-\sigma^2/2}T_1(\mathrm{e}^{-\sigma^2/2}Q)\right)\sigma\mathrm{d}\sigma. \end{aligned} \tag{2.19}$$

Since Lemma 2.3 provides bounds for $\|q_c\|_Y$ and $\|q_s\|_Y$, the regularity of Q, and pointwise bounds, can be deduced from (2.19) and from Lemma 3.8 below without great difficulty.

3 The perturbation ω_1 as $\lambda \uparrow \infty$

We return to equations (2.1) to (2.4) and define a stream function $\tilde{\psi}_1 := A\tilde{\omega}_1$. Then $\tilde{\omega}_1 = -\Delta_\sigma \tilde{\psi}_1$ and (2.2) becomes

$$\begin{aligned} \left(\Delta_\sigma + 2\sigma\frac{\partial}{\partial\sigma} + 2\right)\Delta_\sigma\tilde{\psi}_1 - \lambda\frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2}\frac{\partial}{\partial\theta}\Delta_\sigma\tilde{\psi}_1 - 4\lambda\mathrm{e}^{-\sigma^2}\frac{\partial\tilde{\psi}_1}{\partial\theta} \\ = \frac{\kappa\lambda}{\pi a}g(\sigma)\cos\theta \quad \text{on } E. \end{aligned} \tag{3.1}$$

In view of (2.5) and (2.17), the function $\tilde{\psi}_1$ has the form

$$\tilde{\psi}_1(\sigma,\theta) = \tilde{\psi}_{1c}(\sigma)\cos\theta + \tilde{\psi}_{1s}(\sigma)\sin\theta. \tag{3.2}$$

We divide (3.1) by $\lambda(1-\mathrm{e}^{-\sigma^2})/\sigma^2$, write the $\cos\theta$ and $\sin\theta$ parts as separate equations and define, similarly to (2.14),

$$\Psi(\sigma) = \tilde{\psi}_{1c}(\sigma) + i\tilde{\psi}_{1s}(\sigma). \tag{3.3}$$

With the notation

$$
\begin{aligned}
&\text{(a)}\quad \Delta_1 := \left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^2 + \frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma} - \frac{1}{\sigma^2},\\
&\text{(b)}\quad \alpha(\sigma) := \frac{4\sigma^2}{\mathrm{e}^{\sigma^2}-1},\\
&\text{(c)}\quad \beta(\sigma) := \frac{\sigma^2}{1-\mathrm{e}^{-\sigma^2}},\\
&\text{(d)}\quad \mathcal{E} := \Delta_1 + 2\sigma\frac{\mathrm{d}}{\mathrm{d}\sigma} + 2,
\end{aligned}
\tag{3.4}
$$

the problem for $\tilde{\omega}_1$ is to solve the equation

$$
-i\frac{\beta(\sigma)}{\lambda}\mathcal{E}(\Delta_1\Psi) + \{\Delta_1 + \alpha(\sigma)\}\,\Psi = -i\frac{\kappa}{\pi a}\beta(\sigma)g(\sigma), \quad 0<\sigma<\infty, \tag{3.5}
$$

with the side conditions

$$
\begin{aligned}
&\text{as } \sigma\downarrow 0, \quad (\Delta_1\Psi)(\sigma)\to 0, \quad \Psi'(\sigma) = O(1) \quad \text{and} \quad \Psi(\sigma) = O(\sigma);\\
&\text{as } \sigma\uparrow \infty, \quad (\Delta_1\Psi)(\sigma)\to 0, \quad \Psi'(\sigma) = O(\sigma^{-2}) \quad \text{and} \quad \Psi(\sigma) = O(\sigma^{-1}).
\end{aligned}
\tag{3.6}
$$

Here the conditions on $\Delta_1\Psi$ come from (2.3); the conditions on Ψ' and Ψ are implied by $\Psi = -T_1(\Delta_1\Psi)$, with T_1 as in (2.15), and by conditions on $\Delta_1\Psi$ much weaker than those in Theorem 2.1.

For $\lambda\uparrow\infty$, equation (3.5) with (3.6) seems to form a singular perturbation problem, since a small parameter multiplies the highest derivatives. Surprisingly, this turns out not to be the case; nevertheless there is work to be done.

Apparently, if $\Psi_0(\sigma) := \lim_{\lambda\uparrow\infty}\Psi(\sigma;\lambda)$ exists, then it must satisfy

$$
\{\Delta_1 + \alpha(\sigma)\}\,\Psi_0 = -i\frac{\kappa}{\pi a}\beta(\sigma)g(\sigma), \quad 0<\sigma<\infty, \tag{3.7}
$$

and the six side conditions (3.6). We proceed to explore this problem.

Lemma 3.1. *The equation*

$$
\{\Delta_1 + \alpha(\sigma)\}\,u = 0, \quad 0<\sigma<\infty, \tag{3.8}
$$

has solutions

$$
U(\sigma) := \frac{1}{\sigma}\left(1-\mathrm{e}^{-\sigma^2}\right) \tag{3.9}
$$

and

$$
V(\sigma) := \frac{1}{\sigma} - U(\sigma)\log\left(\mathrm{e}^{\sigma^2}-1\right). \tag{3.10}
$$

Here U is an eigensolution (with eigenvalue 0) in that it satisfies not only (3.8) *but also all six side conditions* (3.6).

Proof. This is a matter of direct calculation. □

Lemma 3.2. *The forcing function in* (3.7) *is orthogonal to the eigensolution U in the sense that*

$$
\int_0^\infty U(\sigma)\beta(\sigma)g(\sigma)\,\sigma\mathrm{d}\sigma = 0. \tag{3.11}
$$

Hence the problem (3.7) *and* (3.6) *has a (non-unique) solution*

$$\Psi_0(\sigma) = c_0 U(\sigma) + i\frac{\kappa}{2\pi a}\int_0^\sigma \{U(\rho)V(\sigma) - U(\sigma)V(\rho)\}\,\beta(\rho)g(\rho)\,\rho\mathrm{d}\rho \tag{3.12}$$

for every $c_0 \in \mathbb{C}$.

Proof. Again this is a matter of direct calculation, but the calculation is not short. With β defined by (3.4)(c) and g by (2.4)(c), the analytic proof of the orthogonality condition (3.11) seens to require a page. (However, with any machine capable of numerical integration, numerical verification of (3.11) is quick and easy.) We note that Liouville's formula for Wronskians yields

$$U(\sigma)V'(\sigma) - U'(\sigma)v(\sigma) = -\frac{2}{\sigma}, \quad 0 < \sigma < \infty. \tag{3.13}$$

The following lemma is also relevant. □

Lemma 3.3. *Define, for suitable functions* f,

$$(\mathcal{G}f)(\sigma) := \frac{1}{2}\int_0^\sigma \{U(\rho)V(\sigma) - U(\sigma)V(\rho)\}\,f(\rho)\,\rho\mathrm{d}\rho, \quad 0 < \sigma < \infty, \tag{3.14}$$

and

$$J(f) := \int_0^\infty V(\rho)f(\rho)\,\rho\mathrm{d}\rho. \tag{3.15}$$

Assume that $f \in C[0,\infty)$, *that* $\int_0^\infty U(\rho)f(\rho)\,\rho\mathrm{d}\rho = 0$, *that* $f(\sigma) = O(\sigma)$ *as* $\sigma \downarrow 0$ *and that* $f(\sigma) = O(\sigma^m \mathrm{e}^{-\sigma^2})$, *with* $m \geq 1$, *as* $\sigma \uparrow \infty$. *Then*

$$\{\Delta_1 + \alpha(\sigma)\}\,(\mathcal{G}f)(\sigma) = -f(\sigma) \quad \textit{in } (0,\infty); \tag{3.16}$$

as $\sigma \downarrow 0$,

$$(\mathcal{G}f)(\sigma) = O(\sigma^3) \quad \textit{and} \quad (\Delta_1\mathcal{G}f)(\sigma) = O(\sigma); \tag{3.17}$$

as $\sigma \uparrow \infty$,

$$(\mathcal{G}f)(\sigma) = -\frac{1}{2}J(f)\sigma^{-1} + O(\sigma^m \mathrm{e}^{-\sigma^2}), \tag{3.18}$$

$$(\mathcal{G}f)'(\sigma) = \frac{1}{2}J(f)\sigma^{-2} + O(\sigma^{m-1}\mathrm{e}^{-\sigma^2}), \tag{3.19}$$

$$(\mathcal{G}f)''(\sigma) = -J(f)\sigma^{-3} + O(\sigma^m \mathrm{e}^{-\sigma^2}), \tag{3.20}$$

and

$$(\Delta_1\mathcal{G}f)(\sigma) \quad \textit{and} \quad (\mathcal{E}\mathcal{G}f)(\sigma) \quad \textit{are} \quad O(\sigma^m \mathrm{e}^{-\sigma^2}). \tag{3.21}$$

Proof. Equation (3.16) follows from the definition of $\mathcal{G}f$ and two differentiations. That $(\mathcal{G}f)(\sigma) = O(\sigma^3)$ as $\sigma \downarrow 0$ also follows from the definition; then the differential equation (3.16) shows that $(\Delta_1\mathcal{G}f)(\sigma) = O(\sigma)$ as $\sigma \downarrow 0$. In order to prove (3.18) and (3.19), we note that $U(\sigma) \sim 1/\sigma$ and $V(\sigma) \sim -\sigma$ as $\sigma \uparrow \infty$, whence

$$\int_0^\sigma U(\rho)f(\rho)\,\rho\mathrm{d}\rho = \left(\int_0^\infty - \int_\sigma^\infty\right) U(\rho)f(\rho)\,\rho\mathrm{d}\rho = 0 + O(\sigma^{m-1}\mathrm{e}^{-\sigma^2}),$$
$$\int_0^\sigma V(\rho)f(\rho)\,\rho\mathrm{d}\rho = \left(\int_0^\infty - \int_\sigma^\infty\right) V(\rho)f(\rho)\,\rho\mathrm{d}\rho = J(f) + O(\sigma^{m+1}\mathrm{e}^{-\sigma^2}),$$

from which (3.18) and (3.19) follow.

The differential equation (3.16) and the estimate (3.18) of $\mathcal{G}f$ imply that $(\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma^m e^{-\sigma^2})$ as $\sigma \uparrow \infty$. What has been proved now implies the estimates of $(\mathcal{G}f)''(\sigma)$ and $(\mathcal{E}\mathcal{G}f)(\sigma)$ for $\sigma \uparrow \infty$. □

The result (3.12) prompts two questions. How (if at all) is c_0 to be evaluated? How smooth is Ψ_0? Analogues of both these questions will have to be answered more generally for each function Ψ_n in an identity

$$\Psi(\sigma; \lambda) = \sum_{n=0}^{N} \lambda^{-n} \Psi_n(\sigma) + R_N(\sigma; \lambda).$$

Here we anticipate later results and note that, with J as in (3.15),

$$c_0 = i\frac{\kappa}{2\pi a} J(\beta g) = i\frac{\kappa}{a}(0.11527...). \tag{3.22}$$

This follows from the equation governing Ψ_1, which requires an orthogonality condition involving Ψ_0.

Because of the function $Q =: Q(\cdot; \lambda)$ defined by (2.14) and (2.16), we now define $Q_0 = q_{0c} + iq_{0s}$ by

$$\begin{aligned} Q_0(\sigma) &:= -e^{\sigma^2/2}(\Delta_1 \Psi_0)(\sigma) \\ &= e^{\sigma^2/2}\left(4c_0\sigma e^{-\sigma^2} - \alpha(\sigma)(\mathcal{G}h_0)(\sigma) - h_0(\sigma)\right), \end{aligned} \tag{3.23}$$

where $h_0 := -i(\kappa/\pi a)\beta g$. Evidently $q_{0c} = 0$.

It will emerge from Theorem 3.5 that Q_0 is the limit of $Q(\cdot; \lambda)$ as $\lambda \uparrow \infty$. In Figures 2 and 3, Q_0 is compared with $Q(\cdot; \lambda)$ for large λ; these values of $Q(\cdot; \lambda)$ were obtained by numerical solution of the equation

$$-(\Delta_1 - \sigma^2)Q - i\lambda\frac{1 - e^{-\sigma^2}}{\sigma^2}Q + i4\lambda e^{-\sigma^2/2}T_1(e^{-\sigma^2/2}Q) = \frac{\kappa\lambda}{\pi a}e^{\sigma^2/2}g(\sigma). \tag{3.24}$$

This equation is equivalent to (2.6), because of (2.17); it is also the pointwise form of (2.16); with the condition that $Q(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, it has a pointwise, unique solution. Figures 2 and 3 are consistent with the result of Theorem 3.5 that, as $\lambda \uparrow \infty$,

$$q_c(\cdot; \lambda) = O(\lambda^{-1}) \quad \text{and} \quad q_s(\cdot; \lambda) - q_{0s} = O(\lambda^{-2}).$$

Definition. We shall say that a function $\varphi : [0, \infty) \to \mathbb{C}$ is *satisfactory on* $[0, k)$ if (and only if) there exist coefficients b_n and a number $k > 0$ such that

$$\varphi(\sigma) := \sum_{n=0}^{\infty} b_n \sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < k.$$

Lemma 3.4. *(i) The function βg is satisfactory on* $[0, (2\pi)^{1/2})$.

(ii) If f is satisfactory on $[0, (2\pi)^{1/2})$, *then so is* $\mathcal{G}f$.

Proof. (i) We note that

$$\beta(\sigma)g(\sigma) = \sigma e^{-\sigma^2}\left(\frac{3}{2} + \frac{\sigma^2}{1 - e^{-\sigma^2}}\left(\int_\sigma^1 \frac{1 - e^{-\rho^2}}{\rho}d\rho + C_0 + C_1\right)\right),$$

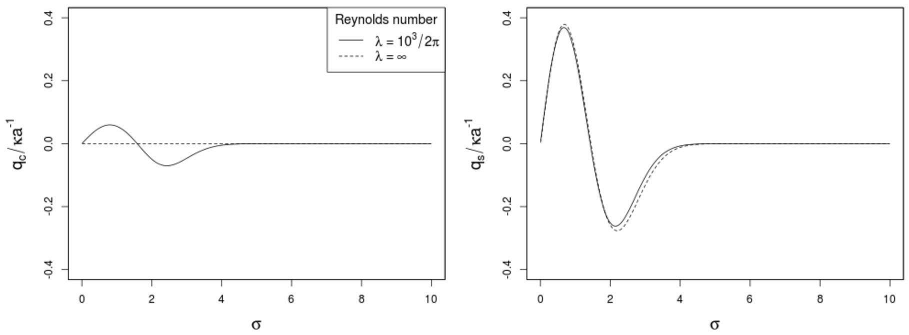

Figure 2: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^3/2\pi$ and $\lambda = \infty$.

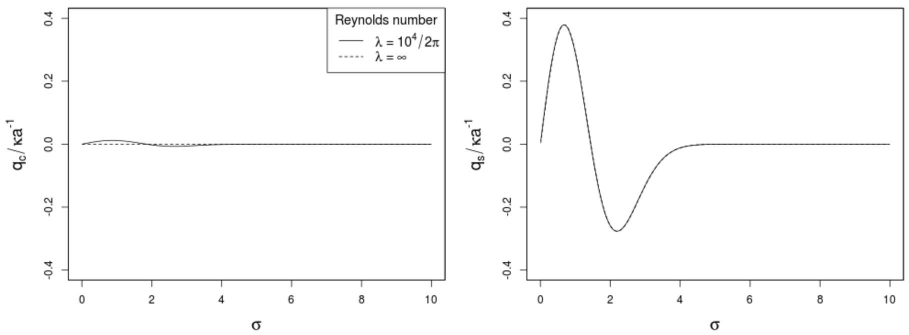

Figure 3: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^4/2\pi$ and $\lambda = \infty$.

where

$$C_0 := -\int_1^\infty \frac{\mathrm{e}^{-\rho^2}}{\rho}\mathrm{d}\rho, \quad C_1 := -\frac{1}{2}\left(\gamma_E + 1 - \log 2\right),$$

and that the function w defined by

$$w(z) = \frac{z}{1-\mathrm{e}^{-z}} \quad \text{if} \quad z \in \mathbb{C} \setminus \{0\} \setminus \{\text{poles}\} \quad \text{and} \quad w(0) = 1,$$

is holomorphic for $|z| < 2\pi$.

(ii) Let

$$W(\sigma) := \frac{1}{\sigma} - U(\sigma) \log \frac{\mathrm{e}^{\sigma^2} - 1}{\sigma^2},$$

where the limiting value of $(\mathrm{e}^{\sigma^2} - 1)/\sigma^2$ is taken at $\sigma = 0$. Then

$$V(\sigma) = W(\sigma) - 2U(\sigma)\log\sigma$$

and

$$\begin{aligned}(\mathcal{G}f)(\sigma) = &\frac{1}{2}W(\sigma)\int_0^\sigma U(\rho)f(\rho)\,\rho\mathrm{d}\rho - \frac{1}{2}U(\sigma)\int_0^\sigma W(\rho)f(\rho)\,\rho\mathrm{d}\rho \\ &+ \sigma^2 U(\sigma)\int_0^1 (\log t)U(\sigma t)f(\sigma t)\,t\mathrm{d}t.\end{aligned}$$

The functions with values $\sigma^2 W(\sigma)$, $U(\sigma)$ and $f(\sigma)$ are all satisfactory on $[0, (2\pi)^{1/2})$, so that $\mathcal{G}f$ inherits this property. □

Theorem 3.5. *The perturbation $\tilde{\omega}_1$ has a representation*

$$\begin{aligned}\tilde{\omega}_1(\sigma,\theta;\lambda) = &\cos\theta\left\{\lambda^{-1}\zeta_1(\sigma) + \lambda^{-3}\zeta_3(\sigma) + \zeta_5(\sigma;\lambda)\right\} \\ &+ \sin\theta\left\{\zeta_0(\sigma) + \lambda^{-2}\zeta_2(\sigma) + \zeta_4(\sigma;\lambda)\right\}\end{aligned} \tag{3.25}$$

in which, for $m = 0, 1, 2, 3$ and $n = 4, 5$,
(a) the functions ζ_m and $\zeta_n(\cdot;\lambda)$ belong to $C^\infty[0,\infty)$ and are satisfactory on $[0, (2\pi)^{1/2})$;
(b) as $\sigma \uparrow \infty$, $\zeta_m(\sigma) = O(\sigma^{2m+4}\mathrm{e}^{-\sigma^2})$ and $\zeta_n(\sigma;\lambda) = o(\mathrm{e}^{-\sigma^2/2})$ for fixed λ;
(c) as $\lambda \uparrow \infty$, $\zeta_n(\sigma;\lambda) = O(\lambda^{-n})$, uniformly over $\sigma \in [0,\infty)$.

The proof will be by means of further lemmas. Let $\Psi := \tilde{\psi}_{1c} + i\tilde{\psi}_{1s}$, as before, and let $\Omega := -\Delta_1\Psi$, so that $\Omega = \tilde{\omega}_{1c} + i\tilde{\omega}_{1s}$. Our plan is to construct identities

$$\Psi(\sigma;\lambda) = \sum_{n=0}^{N} \lambda^{-n}\Psi_n(\sigma) + R_N(\sigma;\lambda), \tag{3.26}$$

$$\Omega(\sigma;\lambda) = \sum_{n=0}^{N} \lambda^{-n}\Omega_n(\sigma) + r_N(\sigma;\lambda), \tag{3.27}$$

in which estimates of the remainders R_N and r_N can be crude. In fact, we shall prove only that R_N and r_N are $O(\lambda^{1-N})$, but this is sufficient for (3.25) if $N \geq 6$.

The terms of the expansion of Ψ are to satisfy

$$\{\Delta_1 + \alpha(\sigma)\}\,\Psi_n = h_n, \quad n = 0, 1, ..., N, \tag{3.28}$$

where

$$h_0(\sigma) = -i\frac{\kappa}{\pi a}\beta(\sigma)g(\sigma), \tag{3.29}$$

$$h_n := i\beta\mathcal{E}(\Delta_1\Psi_{n-1}) \quad \text{for} \quad n = 1, ..., N, \tag{3.30}$$

and

$$-i\lambda^{-1}\beta(\sigma)\mathcal{E}(\Delta_1 R_N) + \{\Delta_1 + \alpha(\sigma)\}\,R_N = i\lambda^{-N-1}\beta(\sigma)\mathcal{E}(\Delta_1\Psi_N); \tag{3.31}$$

then the right-hand member of (3.26) will satisfy the equation (3.5) governing Ψ.

Since $\Omega = -\Delta_1\Psi$ and $\Psi = T_1\Omega$, this scheme corresponds to

$$-\Omega_n + \alpha(\sigma)T_1\Omega_n = h_n, \quad n = 0, 1, ..., N, \tag{3.32}$$

$$i\lambda^{-1}\beta(\sigma)\mathcal{E}r_N - r_N + \alpha(\sigma)T_1 r_N = -i\lambda^{-N-1}\beta(\sigma)\mathcal{E}\Omega_N, \tag{3.33}$$

where $h_n = -i\beta(\sigma)\mathcal{E}\Omega_{n-1}$ for $n = 1, ..., N$.

Lemma 3.6. *In order that equation* (3.28)*, with the side conditions* (3.6)*, have a solution, it is necessary that*

$$\int_0^\infty U(\sigma)h_n(\sigma)\sigma\mathrm{d}\sigma = 0, \quad n = 0, 1, ..., N; \tag{3.34}$$

equivalently, that

$$\int_0^\infty \sigma^2 g(\sigma)\mathrm{d}\sigma = 0 \quad \text{if} \quad n = 0, \tag{3.35}$$

$$\int_0^\infty \sigma^2(\mathcal{E}\Omega_{n-1})(\sigma)\mathrm{d}\sigma = 0 \quad \text{if} \quad n = 1, ..., N. \tag{3.36}$$

Proof. Let $\mathcal{M} := \Delta_1 + \alpha(\sigma)$. Assume that u and v are in $C^2(0,\infty)$, that $\sigma u(\sigma)v'(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$ and that $\sigma u'(\sigma)v(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$. Then integration by parts yields

$$\int_0^\infty u(\mathcal{M}v)\,\sigma\mathrm{d}\sigma = \int_0^\infty (\mathcal{M}u)v\,\sigma\mathrm{d}\sigma. \tag{3.37}$$

Now let $u = U$ and $v = \Psi_n$. If Ψ_n satisfies (3.6), then the foregoing hypotheses are satisfied. If also $\mathcal{M}\Psi_n = h_n$, then

$$\int_0^\infty Uh_n\,\sigma\mathrm{d}\sigma = \int_0^\infty U(\mathcal{M}\Psi_n)\,\sigma\mathrm{d}\sigma = \int_0^\infty (\mathcal{M}U)\Psi_n\,\sigma\mathrm{d}\sigma = 0. \tag{3.38}$$

Equations (3.35) and (3.36) follow from the identity $U(\sigma)\beta(\sigma) = \sigma$ and from the definitions of h_n. □

If h_n satisfies not only the orthogonality condition (3.34), but also the other hypotheses on f in Lemma 3.3 (and this will be the case), then the differential equation (3.28), with side conditions (3.6), has solutions

$$\Psi_n = c_n U - \mathcal{G}h_n, \quad n = 0, 1, ..., N, \tag{3.39}$$

whence

$$\Omega_n(\sigma) = -(\Delta_1\Psi_n)(\sigma) = 4c_n\sigma\mathrm{e}^{-\sigma^2} - \alpha(\sigma)(\mathcal{G}h_n)(\sigma) - h_n(\sigma), \tag{3.40}$$

for every $c_n \in \mathbb{C}$.

In order to evaluate $c_0, ..., c_N$ and in order to discuss r_N, we extend the definition (3.30) to h_{N+1}. Recall from Lemma 3.2 that for $n = 0$ the orthogonality condition (3.34) has already been established.

Lemma 3.7. *For* $n = 0, 1, ..., N$*, the necessary condition* $\int_0^\infty Uh_{n+1}\sigma\mathrm{d}\sigma = 0$ *implies that* $c_n = -\frac{1}{2}J(h_n)$*, where* $J(\cdot)$ *is defined by* (3.15)*.*

Proof. Extended to Ω_N, the orthogonality condition (3.36) states that, for $n = 0, 1, ..., N$,

$$0 = \int_0^\infty \sigma^2(\mathcal{E}\Omega_n)\mathrm{d}\sigma = -4\int_0^\infty \sigma^2\Omega_n\mathrm{d}\sigma,$$

by an integration by parts for which it suffices that $\Omega_n \in C^2(0,\infty)$, that $\Omega_n'(\sigma) = o(\sigma^{-2})$ both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, that $\Omega_n = o(\sigma^{-1})$ as $\sigma \downarrow 0$ and that $\Omega_n = o(\sigma^{-3})$ as $\sigma \uparrow \infty$.

Next, we observe that, if $\Psi_n \in C^2(0,\infty)$ and if both $\sigma^2\Psi_n'(\sigma)$ and $\sigma\Psi_n(\sigma)$ have limits both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, then

$$0 = \int_0^\infty \sigma^2(\Delta_1\Psi_n)\mathrm{d}\sigma = \left[\sigma^2\Psi_n' - \sigma\Psi_n\right]_0^\infty ,$$

in which limiting values are implied on the right-hand side. In view of (3.39), the orthogonality condition is now

$$c_n\left[\sigma^2U' - \sigma U\right]_0^\infty = \left[\sigma^2(\mathcal{G}h_n)' - \sigma(\mathcal{G}h_n)\right]_0^\infty .$$

Referring to the definition of U in (3.9) and to the description of $\mathcal{G}f$ in Lemma 3.3, one is led to $c_n = -\frac{1}{2}J(h_n)$. □

It is time to relate the ζ_m and $\zeta_n(\cdot;\lambda)$ in Theorem 3.5 to the Ω_n and r_N in (3.27). We noted after (3.23) that Q_0 is imaginary. Since $\Omega(\sigma) = \exp(-\sigma^2/2)Q(\sigma)$, the function Ω_0 is imaginary. Then, since $h_1 = -i\beta\mathcal{E}\Omega_0$, the function h_1 and the coefficient c_1 are real. Equation (3.40) shows that Ω_1 is real. An easy induction now shows that Ω_n *is imaginary if* n *is even and* Ω_n *is real if* n *is odd.*

Accordingly, if N is odd, then

$$\begin{aligned}&\zeta_1 = \Omega_1,\ \zeta_3 = \Omega_3 \text{ and } \zeta_5(\cdot;\lambda) = \lambda^{-5}\Omega_5 + ... + \lambda^{-N}\Omega_N + \mathrm{Re}\ r_N(\cdot;\lambda),\\ &\zeta_0 = -i\Omega_0,\ \zeta_2 = -i\Omega_2 \text{ and } \zeta_4(\cdot;\lambda) = -i\left(\lambda^{-4}\Omega_4 + ... + \lambda^{-N+1}\Omega_{N-1}\right)\\ &\qquad\qquad + \mathrm{Im}\ r_N(\cdot;\lambda).\end{aligned} \tag{3.41}$$

If N is even, then there is a similar array.

Because of the explicit formula (3.40) for Ω_n (with $h_n = -i\beta\mathcal{E}\Omega_{n-1}$, with $c_n = -\frac{1}{2}J(h_n)$ and with the operator $\mathcal{G}$ described by Lemmas 3.3 and 3.4), enough may have been said about Ω_n to justify the claims made for ζ_0 to ζ_3 in Theorem 3.5. For example, the result

$$\zeta_m(\sigma) = O(\sigma^{2m+4}\mathrm{e}^{-\sigma^2}) \quad \text{as } \sigma \uparrow \infty \tag{3.42}$$

follows for $m = 0$ from $\beta(\sigma) \sim \sigma^2$ and from the overestimate $g(\sigma) = O(\sigma^2\mathrm{e}^{-\sigma^2})$, which imply that h_0 and Ω_0 are $O(\sigma^4\mathrm{e}^{-\sigma^2})$. Then repeated use of $h_{n+1} = -i\beta\mathcal{E}\Omega_n$ leads to (3.42).

On the other hand, the remainder r_N requires further discussion. Under the transformations

$$r_N(\sigma) = \mathrm{e}^{-\sigma^2/2}P_N(\sigma) = \mathrm{e}^{-\sigma^2/2}\left\{p_{Nc}(\sigma) + ip_{Ns}(\sigma)\right\}, \tag{3.43}$$

$$p_N(\sigma,\theta) := p_{Nc}(\sigma)\cos\theta + p_{Ns}(\sigma)\sin\theta, \tag{3.44}$$

equation (3.33) becomes

$$-(\Delta_\sigma - \sigma^2)p_N + \frac{\lambda}{\beta(\sigma)}\frac{\partial}{\partial\theta}p_N - 4\lambda\mathrm{e}^{-\sigma^2/2}T(\mathrm{e}^{-\sigma^2/2}p_N) = \lambda^{-N}\mathrm{e}^{\sigma^2/2}f_N(\sigma,\theta), \tag{3.45}$$

where

$$f_N(\sigma,\theta) = \mathrm{Re}\left\{(\mathcal{E}\Omega_N)(\sigma)\mathrm{e}^{-i\theta}\right\}. \tag{3.46}$$

The operator on the left-hand side of (3.45) is that in (2.6); as in §2, it follows that equation (3.45) has a unique weak solution bounded by

$$\|p_N\|_Z \leq \frac{\kappa}{a} A_N \lambda^{-N}, \tag{3.47}$$

where A_N depends only on N.

Choosing the test function in the definition of weak solution as in (2.18), we obtain the equation

$$\begin{aligned} P_N(\rho) = \lambda^{-N} \int_0^\infty K(\rho,\sigma)\, e^{\sigma^2/2} (\mathcal{E}\Omega_N)(\sigma)\, \sigma d\sigma \\ + i\lambda \int_0^\infty K(\rho,\sigma) \left(\frac{P_N(\sigma)}{\beta(\sigma)} - 4e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} P_N) \right) \sigma d\sigma. \end{aligned} \tag{3.48}$$

This leads without difficulty to a pointwise solution $P_N \in C^2[0,\infty)$ such that $P_N(0) = 0$ and such that $P_N(\sigma) \to 0$ as $\sigma \uparrow \infty$. (Correspondingly, $r_N(\sigma)$ is $o(\exp(-\sigma^2/2))$ as $\sigma \uparrow \infty$.) Equation (3.48) and the bound (3.47) now imply that $P_N(\sigma;\lambda)$ is $O(\lambda^{-N+1})$ uniformly over $\sigma \in [0,\infty)$. Moreover, the equation

$$-(\Delta_1 - \sigma^2) P_N - \frac{i\lambda}{\beta(\sigma)} P_N + 4i\lambda e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} P_N) = \lambda^{-N} e^{\sigma^2/2} \mathcal{E}\Omega_N,$$

may be written as

$$P_N'' = -\frac{1}{\sigma} P_N' + \left(\frac{1}{\sigma^2} + \sigma^2 \right) P_N - ... - \lambda^{-N} e^{\sigma^2/2} \mathcal{E}\Omega_N.$$

The right-hand member of this is in $C^1[k,\infty)$ for any $k > 0$, say in $C^1[1,\infty)$. Therefore $P_N'' \in C^1[1,\infty)$. Repetition of this step shows that $P_N \in C^\infty[1,\infty)$.

It remains to prove that P_N is better than C^2 at and near the origin. We return to equation (3.33) for r_N and to the equation

$$\mathcal{E}\Omega + \frac{i\lambda}{\beta(\sigma)} \Omega - 4i\lambda e^{-\sigma^2} T_1 \Omega = -\frac{\kappa\lambda}{\pi a} g$$

for $\Omega = \tilde{\omega}_{1c} + i\tilde{\omega}_{1s}$; our final lemma applies to both r_N and Ω.

Lemma 3.8. *Assume that the equation*

$$\mathcal{E}u + \frac{i\lambda}{\beta(\sigma)} u - 4i\lambda e^{-\sigma^2} T_1 u = \lambda f \tag{3.49}$$

has a solution $u \in C^2[0,\infty)$ *such that* $u(0) = 0$ *and such that* $u(\sigma)$ *is* $o(\exp(-\sigma^2/2))$ *as* $\sigma \uparrow \infty$. *Assume also that* u *is unique because it is the transformed version of a solution in the Hilbert space* Z.

Then u *is satisfactory on* $[0,(2\pi)^{1/2})$ *whenever* f *has this property.*

Proof. We shall prove that there are coefficients a_n such that

$$u(\sigma) = \sum_{n=0}^{\infty} a_n \sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < b$$

if $b \in (0,(2\pi)^{1/2})$. Here we are not constructing a series solution *ab initio* in the usual way; rather, we are establishing a regularity property of a known, unique solution. Therefore we may regard $a_0 = u'(0)$ and $\int_0^\infty u \mathrm{d}\sigma$ as known; we proceed to calculate the other coefficients in terms of these. The equation is satisfied, subject to convergence of the series, if for $n = 0, 1, 2, ...$

$$4(n+1)(n+2)a_{n+1} = -4(n+1)a_n - \sum_{j=0}^{n} (B_{n-j}a_j + A_{n-j}\tau_j) + \lambda f_n, \quad (3.50)$$

where

$$B_m = \frac{i\lambda(-1)^m}{(m+1)!}, \quad A_m = \frac{4i\lambda(-1)^{m+1}}{m!},$$
$$\tau_0 = \frac{1}{2}\int_0^\infty u \mathrm{d}\sigma, \quad \tau_m = -\frac{1}{4}\frac{a_{m-1}}{m(m+1)} \text{ for } m \geq 1,$$

and

$$f(\sigma) = \sum_{n=0}^{\infty} f_n \sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < (2\pi)^{1/2}.$$

Hence there is a constant $C = C(b)$ such that $|f_n| \leq Cb^{-2n}$.

Now, for every $p \in \{1, 2, 3, ...\}$ there is a number $\Gamma_p = \Gamma_p(b, \lambda)$ such that $|a_n| \leq \Gamma_p b^{-2n}$ for $n = 0, 1, ..., p$. We may suppose that $\Gamma_p \geq \kappa/\alpha$. Then (3.50) implies that

$$|a_{n+1}| \leq \Gamma_p b^{-2n-2} \varphi(n, \lambda) \quad \text{for} \quad n \leq p,$$

where

$$\varphi(n, \lambda) := \frac{2\pi}{4(n+1)(n+2)} \left(4(n+1) + \lambda(1+\pi)\mathrm{e}^{2\pi} + \frac{4\lambda|\tau_0|(2\pi)^n}{n!} + \lambda C \right).$$

For fixed λ, we choose p so large that $\varphi(p, \lambda) \leq 1$ and so large that $\varphi(n, \lambda)$ decreases for $n \geq p$. Then $|a_m| \leq \Gamma_p b^{-2m}$ not merely for $m \leq p$, but also for $m \geq p + 1$. □

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