G_2–MANIFOLDS AND ASSOCIATIVE SUBMANIFOLDS VIA SEMI-FANO 3-FOLDS

ALESSIO CORTI, MARK HASKINS, JOHANNES NORDSTRÖM, AND TOMMASO PACINI

Abstract. We construct many new topological types of compact G_2–manifolds, *ie* Riemannian 7-manifolds with holonomy group G_2. To achieve this we extend the twisted connected sum construction first developed by Kovalev and apply it to the large class of asymptotically cylindrical Calabi-Yau 3-folds built from semi-Fano 3-folds (a subclass of weak Fano 3-folds) studied in [21]. In many cases we determine the diffeomorphism type of the underlying smooth 7-manifolds completely; we find that many 2-connected 7-manifolds can be realised as twisted connected sums in a variety of ways, raising questions about the global structure of the moduli space of G_2–metrics. Many of the G_2–manifolds we construct contain compact rigid associative 3-folds, which play an important role in the the higher-dimensional enumerative geometry (gauge theory/calibrated submanifolds) approach to defining deformation invariants of G_2–metrics. By varying the semi-Fanos used to build different G_2–metrics on the same 7-manifold we can change the number of rigid associative 3-folds we produce.

1. Introduction

Compact G_2–manifolds, that is Riemannian 7-manifolds whose holonomy group is the compact exceptional Lie group G_2, play a distinguished role in both geometry and theoretical physics: in geometry they provide one of very few sources of (nonflat“) compact Ricci-flat metrics; in theoretical physics they occur in M-theory in 11 dimensions in the same way that Calabi–Yau 3-folds appear in String Theory in 10 dimensions, namely as the simplest compactifying spaces that preserve supersymmetry. At present only two constructions of compact G_2–manifolds are known: Joyce's original pioneering construction via "orbifold resolutions" [45, 46], and the so-called twisted connected sum construction. Kovalev, based on a suggestion of Donaldson, developed the twisted connected sum construction [48] as a way to obtain compact G_2–manifolds by combining pairs of (exponentially) asymptotically cylindrical (ACyl) Calabi–Yau 3-folds. Loosely speaking, this method seeks to construct G_2–manifolds that contain a sufficiently long almost cylindrical neck-like region; in this sense it resembles familiar "stretching the neck" constructions in a number of other geometric PDE problems.

In this paper we provide a significant extension of the twisted connected sum construction of G_2–manifolds. Our extension allows us to prove many new results about compact G_2–manifolds and leads to some new perspectives for future research in the area. Some of the main contributions of the paper are:

(i) We correct, clarify and extend several aspects of the K3 "matching problem" that occurs as a key step in the twisted connected sum construction.
(ii) We show that the large class of ACyl Calabi–Yau 3-folds built from semi-Fano 3-folds (a subclass of weak Fano 3-folds) can be used as components in the twisted connected sum construction; Kovalev used ACyl Calabi–Yau 3-folds constructed from (the much smaller class of) Fano 3-folds.

Key words and phrases. Differential geometry, Einstein and Ricci-flat manifolds, special and exceptional holonomy, noncompact Calabi-Yau manifolds, compact G_2–manifolds, Fano and weak Fano varieties, lattice polarised K3 surfaces, calibrated submanifolds, associative submanifolds, differential topology.

(iii) We construct many new topological types of compact G_2–manifolds by applying the twisted connected sum construction to ACyl Calabi–Yau 3-folds of semi-Fano type.
(iv) We obtain much more precise topological information about twisted connected sum G_2–manifolds; one application is the determination for the first time of the diffeomorphism type of many compact G_2–manifolds.
(v) We describe "geometric transitions" between G_2–metrics on different 7-manifolds mimicking "flopping" behaviour among semi-Fano 3-folds and "conifold transitions" between Fano and semi-Fano 3-folds.
(vi) We construct many G_2–manifolds that contain rigid compact associative 3-folds.
(vii) We prove that many smooth 2-connected 7-manifolds can be realised as twisted connected sums in numerous ways; by varying the semi-Fano 3-folds used to build different G_2–metrics on the same 7-manifold we can change the number of rigid associative 3-folds produced by our method.

The last point leads to speculation that the moduli space of G_2–metrics on a given 7-manifold may consist of many different connected components, and opens up many further questions for future study. For instance, the higher-dimensional enumerative invariants proposed in [30, 31] may provide ways to detect G_2–metrics on a given 7-manifold that are not deformation equivalent.

We now describe some of the key components of the paper in more detail.

Twisted connected sums and hyper-Kähler rotations. In order to construct a metric with Riemannian holonomy the full group G_2 the underlying compact 7-manifold M must have finite fundamental group. Given a pair of ACyl Calabi–Yau 3-folds V_+ and V_- we need a way to glue the two noncompact 7-manifolds $M_+ = \mathbb{S}^1 \times V_+$ and $M_- = \mathbb{S}^1 \times V_-$ to get such a compact 7-manifold. By construction the ends of our ACyl Calabi–Yau 3-folds will have the form $\mathbb{R}^+ \times \mathbb{S}^1 \times S_\pm$ where $S_\pm$ are smooth K3 surfaces. The obvious connected sum construction would yield a manifold with infinite fundamental group. Instead we choose to identify the cross-section of our ends $\mathbb{T}^2 \times S_\pm$ using a diffeomorphism which exchanges the two circle factors of $\mathbb{T}^2$. However, in order to get a well-defined G_2–structure on M we also need to identify the two K3 surfaces $S_\pm$ using a special diffeomorphism $\mathfrak{r} : S_+ \to S_-$. Both asymptotic K3 surfaces $S_\pm$ inherit hyper-Kähler structures from the geometry at infinity of $V_\pm$, which can be defined in terms of a Ricci-flat metric and a triple of parallel complex structures $I_\pm, J_\pm, K_\pm$. We need to construct a diffeomorphism $\mathfrak{r}$ which is an isometry and satisfies

$$\mathfrak{r}^* I_- = J_+, \quad \mathfrak{r}^* J_- = I_+, \quad \text{and hence} \quad \mathfrak{r}^* K_- = -K_+.$$

We call such a map a *hyper-Kähler rotation.*

Even given a plentiful supply of ACyl Calabi–Yau 3-folds it is non-trivial to find *pairs* of $V_\pm$ for which such a hyper-Kähler rotation $\mathfrak{r}$ exists; we often refer to this as solving the *matching problem.* Once we have constructed a hyper-Kähler rotation $\mathfrak{r}$ for a pair of ACyl Calabi–Yau 3-folds $V_\pm$ we can use $\mathfrak{r}$ to form a twisted connected sum 7-manifold $M_{\mathfrak{r}}$, and build on it a closed G_2–structure which has small torsion. The perturbation theory for closed G_2–structures developed by Joyce then shows that we can always choose an appropriate small perturbation to produce a metric with holonomy G_2 on $M_{\mathfrak{r}}$.

Thus two main steps are needed to implement the twisted connected sum construction:

(i) Construct exponentially ACyl Calabi–Yau structures on suitable quasiprojective 3-folds.
(ii) Solve the matching problem, *ie* understand how to find pairs of exponentially ACyl Calabi–Yau 3-folds for which there exists a hyper-Kähler rotation.

We explain below in more detail how together with [21] and [40] this paper addresses problems with both steps (i) and (ii) in Kovalev's original paper [48] and therefore puts the twisted connected sum construction on a firm foundation; it also extends substantially the settings in which solutions to (i) and (ii) can be constructed.

Exponentially ACyl Calabi–Yau 3-folds. There are two ingredients, one analytic and one complex algebraic, for producing exponentially ACyl Calabi–Yau 3-folds. The analytic ingredient is to solve a complex Monge-Ampère equation on suitable smooth quasiprojective varieties and to obtain sufficiently strong estimates for these solutions. The proof of the exponential asymptotics of solutions to the complex Monge-Ampère equation in [48] is not valid, but a complete, short self-contained proof of the existence of exponentially ACyl Calabi–Yau metrics was given recently in [40]. With a suitable analytic existence theory in place the remaining complex algebraic task is to find a (large) supply of suitable quasiprojective varieties.

ACyl Calabi–Yau 3-folds from Fano and weak Fano 3-folds. Recall a smooth *Fano* 3-fold F is a smooth projective 3-fold for which $-K_F$ is ample or positive. There are exactly 105 deformation families of smooth Fano 3-folds: complex projective space $\mathbb{P}^3$, smooth quadrics, cubics and quartics in $\mathbb{P}^4$ being the simplest examples. For all but two of these, the base locus of a generic anticanonical pencil is a smooth curve, and by blowing up this curve and removing a smooth anticanonical divisor one obtains suitable quasiprojective varieties. We call the 3-folds obtained this way ACyl Calabi–Yau 3-folds of *Fano type*; these form the building blocks used in Kovalev's original twisted connected sum construction [48].

A smooth *weak Fano* 3-fold Y is a smooth projective 3-fold for which $-K_Y$ is big and nef (but not ample). Differential geometers are encouraged to think of a big and nef line bundle as the algebro-geometric formulation of the line bundle admitting a hermitian metric whose curvature is sufficiently semi-positive. There are at least *hundreds of thousands* of deformation families of smooth weak Fano 3-folds and their topology is more varied than for Fano 3-folds; while many examples are now known, unlike the Fano case there is at present no classification theory for weak Fano 3-folds, except under very special geometric assumptions. In our paper [21] we proved that one can construct suitable quasiprojective 3-folds from any weak Fano 3-fold satisfying the (very mild) assumption that the base locus of a generic anticanonical pencil is a smooth curve (as already needed in the Fano case); combining this weak Fano construction with the analytic existence results from [40] we thereby increased the number of known ACyl Calabi–Yau 3-folds from a few hundred to several hundred thousand. We call these ACyl Calabi–Yau 3-folds of *weak Fano type.*

Solving the matching problem. With a plentiful supply of exponentially ACyl Calabi–Yau 3-folds now at hand, to complete the twisted connected sum construction it remains to solve (ii): find hyper-Kähler rotations.

The basic strategy for constructing hyper-Kähler rotations is not to find them between the asymptotic K3s of a given pair of ACyl Calabi–Yau 3-folds, but rather to show that within a pair of *deformation families* of ACyl Calabi–Yaus there exist *some* pairs that can be matched. It is important to understand that as one deforms the ACyl Calabi–Yau structure on V, the complex structures that can appear on the asymptotic K3 are special. A key deformation invariant of an ACyl Calabi–Yau V is its *polarising lattice* N, *ie* the image of $H^2(V)$ in the K3 lattice $L := H^2(S)$. The Picard group of the asymptotic K3 S always contains N; S is thus an N*-polarised* K3 surface.

Given sufficient understanding of which elements of the moduli space of N-polarised K3s appear as asymptotic K3s in a deformation family of ACyl Calabi–Yau 3-folds, an application of the Global Torelli theorem lets us reduce the problem of constructing hyper-Kähler rotations to (arithmetic) questions about the existence of embeddings of lattices in the K3 lattice, outlined below. In [48] Kovalev developed an approach along these lines to proving the existence of hyper-Kähler rotations between pairs of ACyl Calabi–Yau 3-folds of Fano type. Unfortunately in almost all cases his argument relies on Lemma 6.47 in [48] which is false. In this paper we therefore provide a self-contained treatment of the construction of hyper-Kähler rotations.

If we successfully find a hyper-Kähler rotation between a pair of ACyl Calabi–Yau 3-folds $V_\pm$ then one can view their polarising lattices $N_\pm$ as a pair of distinguished sublattices inside the K3 lattice L. The configuration of this pair is however not determined a priori by the deformation types of $V_\pm$, and so one can consider the problem of seeking a hyper-Kähler rotation compatible with a specified configuration. For a given pair of families of ACyl Calabi–Yau 3-folds some choices of configuration may lead to a solvable hyper-Kähler rotation problem, while others may not. Should hyper-Kähler rotations compatible with different configurations exist they can give rise to topologically distinct G_2–manifolds built from the same pair of families of ACyl Calabi–Yau 3-folds. At present we do not understand in a systematic way all possible ways to match a given pair of families of ACyl Calabi–Yau 3-folds $V_\pm$; however, we will exhibit examples where several different matchings exist and lead to topologically distinct G_2–manifolds. In simple cases we do understand all ways to match a given pair.

Our main strategy for matching a given pair of deformation families of ACyl Calabi–Yau 3-folds is "orthogonal gluing", *ie* we consider configurations where the polarising lattices intersect orthogonally (the reflections in the subspaces they span commute). Finding hyper-Kähler rotations compatible with such configurations turns out to make only very reasonable demands of the deformation theory: that a *generic* N-polarised K3 surface appear as the asymptotic K3 of some element of the relevant family of ACyl Calabi–Yau 3-folds. Given that information, the problem of finding hyper-Kähler rotations therefore reduces to finding orthogonal embeddings of the polarising lattices into L, with an additional condition on ample cones.

The last condition automatically holds if we embed the perpendicular direct sum of the polarising lattices primitively in L: we refer to this special case as "primitive perpendicular gluing". Given precise knowledge about the pairs of polarising lattices $N_\pm$ we can appeal to the general theory of lattice embeddings to determine precisely when such a primitive embedding exists; in particular this is always possible if the sum of the ranks of the polarising lattices is at most 11. The resulting G_2–manifolds are often topologically simple enough, namely 2-connected, that it is feasible to determine the diffeomorphism type as we describe below.

Orthogonal gluing where the polarising lattices have non-trivial intersection produces manifolds M with second Betti number $b^2(M) > 0$. For such configurations to exist requires some compatibility between the polarising lattices, which is not always satisfied: see Example 6.8. Also the condition on ample cones is not automatic, as illustrated in Example 8.3.

We will also explain an approach to finding hyper-Kähler rotations compatible with non-orthogonal configurations, but this requires more precise information about K3 moduli spaces that is usually very expensive to obtain. We refer to this approach as "handcrafted gluing". Constructing G_2–manifolds with very particular topological or geometric properties sometimes requires the use of (the more labour-intensive) handcrafting, as we discuss briefly below.

Deformation theory for weak and semi-Fano 3-folds. The matching strategies explained above rely on information about the deformation theory of ACyl Calabi–Yau 3-folds and their asymptotic K3s. For ACyl Calabi–Yaus of weak Fano type, this can be understood in terms of the deformation theory for pairs (Y, S) where $Y \in \mathcal{Y}$ is the relevant deformation type of smooth weak Fano 3-folds and $S \in |-K_Y|$ is a smooth anticanonical divisor. In [21, §6] we showed that the deformation theory of such pairs is well-behaved for the subclass of *semi-Fano* 3-folds. There are still hundreds of thousands of deformation families of semi-Fano 3-folds.

A semi-Fano[1] 3-fold is a weak Fano 3-fold on which we impose an extra assumption on the geometry of its anticanonical morphism, namely that it contracts no divisor to a point. This assumption guarantees that certain cohomology vanishing theorems that are true for Fano 3-folds (but false for general weak Fano 3-folds) still hold; one consequence is that the polarising lattice of an ACyl Calabi–Yau of semi-Fano type is the Picard group of the semi-Fano.

The deformation theory results from [21, §6] are exactly what is needed to apply the orthogonal gluing strategy to construct hyper-Kähler rotations between ACyl Calabi–Yau 3-folds of semi-Fano type. In particular, we can use primitive perpendicular gluing to "mass-produce" twisted connected sum G_2–manifolds: considering pairs of ACyl Calabi–Yau 3–folds built out of Fano or semi-Fano 3-folds of rank at most two or from toric semi-Fano 3-folds yields *at least 50 million pairs* that can be matched.

The topology of twisted connected sums. We compute detailed topological information about twisted connected sum G_2–manifolds in terms of data for the constituent ACyl Calabi–Yaus (which we computed for many examples in [21]) and the configuration of polarising lattices. In particular we determine the integral cohomology, including the torsion in H^3 and H^4, and the characteristic class p_1. Computing characteristic classes of a manifold constructed by gluing can be quite difficult, but the twisting in the twisted connected sum construction is sufficiently mild to make it manageable.

By distinguishing between examples with equal Betti numbers but different torsion or different p_1 we can prove the existence of many new compact G_2–manifolds, and in some cases the invariants we compute even determine their homeomorphism or diffeomorphism type. Even in the simplest case where we use a pair of ACyl Calabi–Yau 3-folds constructed from rank one Fano 3-folds—as considered in Kovalev's original twisted connected sum construction—our refined topological results give new information. In Section 7 we show for instance

- The simplest matching, *ie* using primitive perpendicular gluing, between such ACyl Calabi–Yau 3-folds leads to 2-connected G_2–manifolds with torsion-free cohomology, in which case b^3 is the only Betti number to consider; 46 different values of b^3 are realised this way.
- By distinguishing between examples with the same Betti numbers but different p_1 we show that at least 82 different smooth 7-manifolds are realised this way.
- The invariants we compute determine the homeomorphism types, so precisely 82 homeomorphism classes of 7-manifolds are realised. 79 of these admit a unique smooth structure.
- One particular smooth 7-manifold is realised as a twisted connected sum of 7 different pairs of such ACyl Calabi–Yau 3-folds.
- Other ways to match such ACyl Calabi–Yau 3-folds exist and lead to simply-connected 7-manifolds with $H^2 = 0$ but with non-trivial torsion in H^3 (and hence non-trivial second homotopy group); at least 41 other topological types of G_2–manifold arise this way.

The last point makes concrete a fact already mentioned: it is often possible to arrange different matchings between the same pair of ACyl Calabi-Yaus and thereby obtain topologically distinct 7-manifolds from that pair. The point concerning homeomorphism and diffeomorphism types uses the classification theory for 2-connected 7-manifolds developed by Wall and Wilkens [77], and recently completed by Crowley [22] and Crowley and the third author [25].

More generally, we find that all the "mass-produced" primitive perpendicular gluings mentioned above are 2-connected with torsion-free H^4, so they too have their homeomorphism

[1]There seems to be no established terminology for this particular subclass of weak Fano 3-folds, so the term *semi-Fano* is our invention: it is intended to suggest that a semi-Fano 3-fold has *semi*-small anticanonical morphism. Warning: semi-Fano has also been used to mean something even weaker than weak Fano, *ie* a complex manifold for which $-K_Y$ is nef (but not necessarily big), but this terminology is not well-established.

types determined by the invariants we have computed. By contrast, Joyce's "orbifold resolution" constructions typically yield 7-manifolds with relatively large second Betti number (only a single example in Joyce's book [46] has $b^2 = 0$, see Remark 4.28), leaving them out of reach of current classification results.

For the majority of the mass-produced 2-connected examples, the underlying topological 7-manifold admits a unique smooth structure, so we actually determine their diffeomorphism type; these are the first compact G_2–manifolds for which the diffeomorphism type is known. These smooth 7-manifolds have simple topological realisations as connected sums of an appropriate number of copies of $\mathbb{S}^3 \times \mathbb{S}^4$ with a nontrivial $\mathbb{S}^3$-bundle over $\mathbb{S}^4$. This is one of only a few instances of geometrically interesting 2-connected 7-manifolds for which the computations needed to determine the diffeomorphism classification have been performed.

In a minority of cases we find that the underlying topological 7-manifold admits different (in fact precisely two) smooth structures. To pin down the diffeomorphism type in this case requires the calculation of a generalisation of the classical Eells–Kuiper invariant, recently introduced in [25]. We believe that perpendicular gluing can only ever realise one of the two smooth structures, and that constructing 2-connected twisted connected sums that are homeomorphic but not diffeomorphic requires handcrafting. This will be discussed elsewhere.

Different G_2*–metrics on the same manifold?* The moduli space of torsion-free G_2–structures $\mathcal{M}$ on a compact G_2–manifold M, *ie* the space of torsion-free G_2–structures modulo the action of diffeomorphisms, is an orbifold of dimension $b^3(M)$. By contrast with this simple local structure of $\mathcal{M}$ almost nothing is currently known about its global structure, *eg* the connectedness of $\mathcal{M}$. Whenever $b^3(M) > 1$ then any given G_2–metric has nontrivial (*ie* nonhomothetic) local moduli. If the original G_2–metric is obtained by a gluing construction one might expect that every sufficiently close G_2–metric is also obtained by gluing: for G_2–manifolds obtained by gluing a pair of ACyl manifolds (as is the case in the twisted connected sum construction) this was proven in [63].

We already pointed out that matchings of different pairs of ACyl Calabi-Yaus constructed from rank one Fano 3-folds can give the same smooth 7-manifold. The points they define in the moduli space $\mathcal{M}$ of that manifold cannot be close, because the characteristic long neck allows us to recognise the topology of the two constituent "halves".

This phenomenon is actually very common. Recall from above that applying primitive perpendicular gluing to pairs of ACyl Calabi–Yau 3–folds built out of Fano or semi-Fano 3-folds of rank at most two or from toric semi-Fano 3-folds yielded over 50 million matching pairs, and that all the resulting G_2–manifolds are 2-connected with torsion-free cohomology. Combining the classification theory of 2-connected 7-manifolds with knowledge of the geography of these examples (in particular restrictions on the possible values of b^3 and on p_1) shows that the number of diffeomorphism types realised is *much* smaller; it follows that some smooth 2-connected 7-manifolds must arise as twisted connected sums in *many* different ways.

We are led to ask:

Question. When do these G_2–metrics on the same 7-manifold belong to different connected components of the moduli space of G_2–metrics?

Motivated by the constructions in this paper Crowley and the third author considered an elementary approach to distinguishing between components of the moduli space, using homotopy theory of G_2–structures. They prove [24, Theorem 1.7 and Corollary 1.12] that for all the diffeomorphic G_2–manifolds constructed explicitly in this paper one can always choose the diffeomorphism so that their G_2–structures are homotopic, *ie* connected by a continuous path of G_2–structures without any constraint on the torsion. A refinement of this approach using eta invariants can in other situations distinguish between different connected components of $\mathcal{M}$ even when the G_2–structures are homotopic, but this too appears unable to distinguish between twisted connected sum metrics.

A more sophisticated (though more speculative) approach to this question would be to develop the higher-dimensional enumerative invariants envisioned in the papers of Donaldson–Thomas [30] and Donaldson–Segal [31]. The basic idea is that one should try to define an invariant of G_2–metrics by "counting" some combination of G_2–instantons and associative 3-folds, discussed further below.

Rigid associative 3-folds and rigid holomorphic curves in semi-Fano 3-folds. G_2–manifolds have two natural classes of calibrated submanifolds: 3-dimensional *associative* submanifolds and 4-dimensional *coassociative* submanifolds. Relatively few examples of compact associative 3-folds in compact G_2–manifolds are known; part of the difficulty is that—unlike that of its calibrated cousins: special Lagrangians or coassociatives—the deformation theory of compact associative 3-folds can be obstructed. In many of the G_2–manifolds we construct we can exhibit a finite number of rigid—and therefore unobstructed—associative 3-folds diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$; the use of ACyl Calabi–Yau 3-folds constructed from semi-Fano (rather than Fano) 3-folds is crucial here as we now explain.

The key point is the close relation between holomorphic curves in a Calabi–Yau 3-fold V and associative 3-folds in the product $\mathbb{S}^1 \times V$: if C is a real surface in V then $\mathbb{S}^1 \times C$ is associative in $\mathbb{S}^1 \times V$ if and only if C is a holomorphic curve in V; moreover, $\mathbb{S}^1 \times C$ is rigid as an associative 3-fold if and only if C is rigid as a holomorphic curve. Algebraic geometry provides many tools to understand the deformation theory of C and hence of $\mathbb{S}^1 \times C$ as an associative 3-fold; for a general associative 3-fold we have no such tools at present. We show that each closed rigid holomorphic curve C in one of our ACyl Calabi–Yau 3-folds can be perturbed to yield a compact rigid associative 3-fold diffeomorphic to $\mathbb{S}^1 \times C$ in our twisted connected sum G_2–manifold for all sufficiently long "neck lengths".

If Y is a Fano 3-fold then any compact holomorphic curve C meets any anticanonical divisor (because $-K_Y \cdot C > 0$). Because of the way we obtain our ACyl Calabi–Yau 3-folds V from Y, *ie* by blowing up the base locus of a generic anticanonical pencil in Y and then removing a smooth anticanonical divisor, the compact curve $C \subset Y$ therefore gives rise to a noncompact holomorphic curve in V. However, because of the weakening of $-K_Y$ to being big and nef semi-Fano 3-folds can contain special curves C for which $K_Y \cdot C = 0$; such compact curves $C \subset Y$ therefore give rise to compact holomorphic curves in the ACyl Calabi–Yaus V constructed from Y. Moreover, in many cases C is a smooth rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$: in this case C is infinitesimally rigid, *ie* has no infinitesimal (holomorphic) deformations. We can use these special rigid K_Y-trivial curves to construct *compact rigid* holomorphic curves in V and hence rigid associative 3-folds in the resulting twisted connected sum G_2–manifolds.

While the enumerative invariants of G_2–metrics remain speculative, a better understanding of compact rigid associative 3-folds appears to be one important component in this programme. Counting only G_2–instantons does not have nice invariance properties under deformations, but when G_2–instantons "bubble off" then, according to the fundamental analysis of Tian [72], they do so along associative 3-cycles. Walpuski [75] has recently shown how one can reverse this, constructing G_2–instantons that "bubble" at suitable rigid associative submanifolds.

The compact associative 3-folds we construct are the first compact associative 3-folds in compact G_2–manifolds that are proven to be rigid, and provide a natural testing ground for further development of the enumerative invariants. In our examples of diffeomorphic 2-connected twisted connected sums, we can vary the number of rigid associatives that we can construct by changing the pairs of semi-Fanos used. While we do not claim that these are the only rigid associatives of these G_2–metrics, it still suggests the possibility that they can be distinguished by enumerative invariants.

G_2*–transitions.* In the geometry of Calabi–Yau 3-folds, especially in some of their applications to String Theory, an important role is played by so-called *geometric transitions.* The simplest and most important such transitions are *flops* and *conifold transitions.* These two types of transitions also appear in the context of semi-Fano 3-folds; many smooth semi-Fano 3-folds can be flopped to yield other smooth semi-Fano 3-folds (which typically are not deformation equivalent to the original semi-Fano 3-fold). However, unlike the Calabi–Yau setting where the condition $c_1 = 0$ is preserved, a conifold transition that begins with a Fano 3-fold F will yield only a semi-Fano 3-fold Y. We can construct ACyl Calabi–Yau metrics on 3-folds constructed from both the Fano F and the semi-Fano Y, and then try to match both types of ACyl Calabi–Yau to some other given (deformation family of) ACyl Calabi–Yau structure. This gives rise to the idea of related G_2–manifolds or G_2*–transitions.* For the moment we present G_2–transitions as a convenient organisational principle that explains certain features of the geography of twisted connected sum G_2–manifolds. However, there is the future prospect of realising these G_2–transitions at the level of metric geometry; we explain some of the technical difficulties that would need to be overcome to achieve this.

Connections to M-theory. G_2–manifolds play a similar role in M-theory as Calabi–Yau 3-folds do in String Theory. Two questions of significance for M-theory concern the existence of coassociative K3-fibrations and singular G_2–spaces.

Any twisted connected sum G_2–manifold is K3-fibred—essentially because the building blocks from which we construct our ACyl Calabi–Yau 3-folds are K3-fibred. Generically the only singular fibres of a building block, and therefore of our G_2–manifolds, are A_1 singularities. Because of subtleties due to the singular fibres it is still unknown if these topological "almost" coassociative K3-fibrations can be made into coassociative K3-fibrations as expected in [1, 37].

To obtain realistic particle physics (*ie* non-abelian gauge groups and chiral fermions) from M-theory on G_2–manifolds it appears necessary to consider singular G_2–spaces with very particular kinds of singularity, as explained in [1, 2, 7, 9]. For some recent physical predictions from M-theory on G_2–spaces see [3–5]; see also [6, 11, 27, 38, 65, 68] for some other aspects of M-theory on G_2–spaces. In the present paper we consider only smooth compact G_2–manifolds (apart from the discussion in the G_2–transitions section where we discuss potential ways to realise singular G_2–spaces as degenerate limits of our constructions). There are potential extensions of the present constructions that might allow the construction of blocks fibred by generically singular K3 fibres. However it is not clear that these could give rise to G_2–spaces with the sort of singularity structure apparently required.

It would be interesting to know the following: does the presence of torsion in H^3 or H^4 of a compact G_2–manifold have any significance in M-theory? What if any significance do the G_2–transitions discussed in Section 8 have in M-theory? Does the existence of many potentially different G_2–metrics on the same smooth 7-manifold have any M-theory interpretation?

Structure of paper. We now describe the structure of the rest of the paper.

Section 2 reviews basic facts about G_2–holonomy manifolds, Calabi–Yau 3-folds and hyper-Kähler K3 surfaces. We include this standard material to make the paper more accessible to readers with backgrounds in algebraic geometry or topology and also to establish the notation and the conventions we adopt. The reader familiar with the basics of G_2–holonomy metrics can safely skip most of this section.

Section 3 describes Kovalev's twisted connected sum construction and how it reduces the problem of finding G_2–metrics to the problem of constructing hyper-Kähler rotations between a given pair of (deformation families) of ACyl Calabi–Yau 3-folds. It also explains how to construct a large number of ACyl Calabi–Yau 3-folds using smooth weak Fano 3-folds.

Section 4 develops tools to compute topological invariants of compact twisted connected sum G_2–manifolds. We apply these topological results to study the diffeomorphism type of concrete G_2–manifolds constructed in Section 7, but our methods apply to twisted connected sum manifolds more generally. It is particularly important that there is a simple sufficient condition for a twisted connected sum to be 2-connected; this allows us to construct a large number of 2-connected G_2–manifolds in Section 8.

Section 5 deals with the construction of associative submanifolds in our twisted connected sum G_2–manifolds by exploiting the close connection to holomorphic curves in ACyl Calabi–Yau 3-folds explained above. The main result is Proposition 5.15: each closed rigid holomorphic curve C in one of our ACyl Calabi–Yau 3-folds can be perturbed to a compact rigid associative 3-fold diffeomorphic to $\mathbb{S}^1 \times C$ in our twisted connected sum G_2–manifold for all sufficiently long "neck lengths". With a little more work, we also show how to produce closed associative 3-folds, including some non-rigid ones, in twisted connected sum G_2–manifolds from certain closed special Lagrangian 3-folds in our ACyl Calabi–Yau 3-folds.

Section 6 deals with the so-called "matching problem", *ie* the construction of pairs of ACyl Calabi–Yau 3-folds with a hyper-Kähler rotation. We concentrate mainly on two of the matching strategies mentioned above: "orthogonal gluing" and its special case "primitive perpendicular gluing". These approaches to the matching problem require some well-known facts about moduli spaces of lattice polarised K3 surfaces, the global Torelli theorem in this context and some results from deformation theory proved in [21]; we review these very briefly.

In Section 7 we make some illustrative examples of twisted connected sum G_2–manifolds constructed mainly using primitive perpendicular and orthogonal gluing of ACyl Calabi–Yau 3-folds of semi-Fano type. We employ the tools developed in Section 4 to compute the topology of these examples and in many cases determine their diffeomorphism type. By the results of Section 5 many of these G_2–manifolds contain compact rigid associative 3-folds. We also give a single example to illustrate "handcrafted nonorthogonal gluing" and its potential complexities.

In Section 8 we describe the more general possibilities and limitations of the construction and make some comments about the "geography" of examples achievable by matching currently known pairs; we concentrate on the 2-connected case where existing tools allow us to determine the diffeomorphism type of the majority of such twisted connected sums. We also explain what questions remain in the 2-connected case and discuss the prospects for diffeomorphism classification beyond the 2-connected world. Finally we describe a way to organise various different twisted connected sum G_2–manifolds constructed by matching pairs of Fano or semi-Fano 3-folds related via flops or conifold transitions; by analogy we term these G_2*–transitions*.

Acknowledgements. The authors would like to thank Bobby Acharya, Kevin Buzzard, Paolo Cascini, Tom Coates, Diarmuid Crowley, Igor Dolgachev, Simon Donaldson, Bert van Geemen, Anne-Sophie Kaloghiros, Al Kasprzyk and Viacheslav Nikulin. Computations related to toric semi-Fanos were performed in collaboration with Tom Coates and Al Kasprzyk and were carried out on the Imperial College mathematics cluster and the Imperial College High Performance Computing Service; we thank Simon Burbidge, Matt Harvey, and Andy Thomas for technical assistance. Part of these computations were performed on hardware supported by AC's EPSRC grant EP/I008128/1. MH would like to thank the EPSRC for their continuing support of his research under Leadership Fellowship EP/G007241/1, which also provided postdoctoral support for JN. TP gratefully acknowledges the financial support provided by a Marie Curie European Reintegration Grant.

2. Preliminaries: G_2 and $SU(n)$ geometry

In this section we collect some basic facts and definitions concerning the linear algebra and geometry associated to the Lie groups G_2 and $SU(n)$. The material in this section is standard and the reader may find proofs of various quoted facts in the articles by Bryant [14] and Harvey-Lawson [39] and the books by Joyce [46] and Salamon [69]. We include this material to establish our conventions and notation and to make the paper more self-contained and accessible to topologists and algebraic geometers.

The octonions, a cross product on $\mathbb{R}^7$ and the group G_2. One way to define G_2 is as the automorphism group of $\mathbb{O}$, the normed algebra of octonions. The automorphisms preserve the splitting $\mathbb{O} = \mathbb{R} \oplus \operatorname{Im} \mathbb{O}$ and act trivially on $\mathbb{R}$, so can therefore be identified with a subgroup of $GL(7, \mathbb{R})$. Since the inner product on $\operatorname{Im} \mathbb{O}$ is defined in terms of the normed algebra structure it is preserved by the automorphisms. We will see below that the automorphisms also preserve orientation, so G_2 can be embedded in $SO(7)$.

If we choose an isometry $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$ then we can define a vector product on $\mathbb{R}^7$ by

$$u \times v = \operatorname{Im} uv.$$

The algebra structure on $\mathbb{R} \oplus \operatorname{Im} \mathbb{O}$ can be recovered from the vector product $\times$ and the standard inner product g_0 by

$$(x, u)(y, v) = (xy - g_0(u, v),\ xv + yu + u \times v).$$

An equivalent definition of G_2 is therefore that it is the subgroup of $GL(7, \mathbb{R})$ that preserves both g_0 and $\times$. From g_0 and $\times$ we can define the trilinear form

$$\varphi_0(u, v, w) = g_0(u \times v, w). \tag{2.1}$$

In fact this is alternating, so $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$. With a standard choice of isometry $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$ that we fix once and for all (our convention is the same as that used by *eg* Joyce [46, §10]) we can write

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}. \tag{2.2}$$

For any $\varphi \in \Lambda^3(\mathbb{R}^7)^*$ one can algebraically define a form $\mathrm{vol}_\varphi \in \Lambda^7(\mathbb{R}^7)^*$ (see Hitchin [43, §7]), and we call φ *non-degenerate* if $\mathrm{vol}_\varphi \neq 0$. Then the bilinear form g_φ determined by

$$g_\varphi(v, w)\, \mathrm{vol}_\varphi = \frac{1}{6}(v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi$$

is non-degenerate, and its induced volume form is vol_φ. For φ_0 we can compute that $g_{\varphi_0} = g_0$, so the metric can be recovered from φ_0, and hence so can the vector product $\times$. Thus the stabiliser of φ_0 in $GL(7, \mathbb{R})$ preserves g_0 and $\times$, and must equal G_2. This gives yet another possible definition of G_2. Since it is in terms of an alternating 3-form it is a useful one for the purposes of differential geometry.

The set of 3-forms that are equivalent to φ_0, and whose associated orientation, symmetric bilinear form and cross product are thus isomorphic to the standard one, is in fact open in $\Lambda^3(\mathbb{R}^7)^*$.

Proposition 2.3.

(i) G_2 *is a compact 2-connected Lie group of dimension* 14.
(ii) *The stabiliser in* G_2 *of a non-zero vector in* $\mathbb{R}^7$ *is isomorphic to* $SU(3)$.
(iii) G_2 *acts transitively on the unit sphere* $\mathbb{S}^6 \subset \mathbb{R}^7$.
(iv) *The* $GL(7, \mathbb{R})$*-orbit of* φ_0 *is open in* $\Lambda^3(\mathbb{R}^7)^*$.

Proof. Since $\dim \Lambda^3(\mathbb{R}^7)^* = 35$ and $\dim \mathrm{GL}(7,\mathbb{R}) = 49$, we must have $\dim \mathrm{G}_2 \geq 14$ with equality if and only if the orbit of φ_0 is open.

We will prove below that the stabiliser in G_2 of e_1 can be identified with SU(3) in a natural way. Because $\dim \mathrm{SU}(3) = 8$, the G_2–orbit of e_1 must have dimension ≥ 6. Since the orbit is contained in $\mathbb{S}^6$ equality must hold. Consequently the G_2–orbit of e_1 is exactly $\mathbb{S}^6$, all unit vectors have isomorphic stabilisers, $\dim \mathrm{G}_2$ is exactly 14, and the $\mathrm{GL}(7,\mathbb{R})$-orbit of φ_0 is open. The fibration $\mathrm{SU}(3) \to \mathrm{G}_2 \to \mathbb{S}^6$ shows that G_2 is 2-connected. $\square$

Remark. The set of non-degenerate 3-forms on $\mathbb{R}^7$ is in fact the union of four connected components: two $\mathrm{GL}(7,\mathbb{R})$-orbits, each of which splits into two components inducing opposite orientation. The orbit not containing φ_0 consists of those non-degenerate 3-forms whose induced bilinear form has signature $(3,4)$.

The Hodge dual $*\varphi_0$ of φ_0 is a 4-form ψ_0

$$\psi_0 = -dx^{1247} - dx^{1256} - dx^{1346} + dx^{1357} + dx^{2345} + dx^{2367} + dx^{4567}. \tag{2.4}$$

We can use ψ_0 and the metric to obtain an alternating vector-valued 3-form $\chi_0 : \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by

$$g_0(u, \tfrac{1}{2}\chi_0(v,w,x)) = \psi_0(u,v,w,x) \quad \text{for all} \quad u,v,w,x \in \mathbb{R}^7. \tag{2.5}$$

Remark 2.6. The stabiliser of ψ_0 in $\mathrm{GL}(7,\mathbb{R})$ is the subgroup $\mathbb{Z}_2 \times \mathrm{G}_2$, where $\mathbb{Z}_2$ is generated by $-\mathrm{Id}$. We can therefore recover φ_0 from ψ_0, modulo orientation.

Lemma 2.7. *For all $u, v, w \in \mathbb{R}^7$*

$$\|u \times v\|^2 = \|u\|^2\|v\|^2 - g_0(u,v)^2, \tag{2.8a}$$

$$u \times (v \times w) + (u \times v) \times w = 2g_0(u,w)v - g_0(u,v)w - g_0(w,v)u, \tag{2.8b}$$

$$\varphi_0(u,v,w)^2 + \tfrac{1}{4}|\chi_0(u,v,w)|^2 = |u \wedge v \wedge w|^2. \tag{2.8c}$$

Proof. See [14, p. 540], [13, 2.2] and [39, Thm. IV.1.6] for proofs of (2.8a), (2.8b) and (2.8c) respectively. $\square$

***G*–structures on vector spaces.** Let V be an n-dimensional real vector space. Let P denote the set of ordered bases of V; equivalently, the set of isomorphisms $\beta : \mathbb{R}^n \to V$. We call P the set of frames of V. P has a free transitive right $\mathrm{GL}(n,\mathbb{R})$-action determined by composition of maps:

$$g \cdot \beta := \beta \circ g.$$

We can thus think of P as a principal $\mathrm{GL}(n,\mathbb{R})$-fibre bundle over a point.

Definition 2.9. Let G be a subgroup of $\mathrm{GL}(n,\mathbb{R})$. A *$G$–structure* on V is a G–subbundle of P, *i.e.* an orbit of the induced action of G on P. The space of all G–structures can be identified with the quotient space P/G.

The above definition makes it clear that if H is a subgroup of G, an H-structure automatically defines a G–structure.

G_2–structures on a vector space. The subgroups G of interest in this paper arise as isotropy groups of algebraic structures on $\mathbb{R}^n$. In such cases one can give an alternative definition of G–structure, which we exemplify in the case $G = \mathrm{G}_2$.

Definition 2.10. Let V be a real vector space of dimension 7. We call $\varphi \in \Lambda^3 V^*$ a G_2*–structure* (or G_2*–form*) if there is a linear isomorphism $V \cong \mathbb{R}^7$ identifying φ with φ_0.

Since $G_2 \subset SO(7)$, a G_2–structure on V induces an inner product and an orientation. We often find it convenient to restrict attention to G_2–structures that agree with a given orientation.

Definition 2.11. Let V be a real oriented 7-dimensional vector space. We call $\varphi \in \Lambda^3 V^*$ a *positive 3-form* if there is an oriented linear isomorphism $V \cong \mathbb{R}^7$ identifying φ with φ_0. Let $\Lambda^3_+ V^* \subset \Lambda^3 V^*$ denote the set of positive forms.

Note that $\Lambda^3_+ V^*$ is open in $\Lambda^3 V^*$ by 2.3(iv). By Remark 2.6, we could study G_2–structures on an oriented vector space equivalently in terms of the Hodge duals of the positive 3-forms.

Remark. Our definition of 'positive' agrees with that of Joyce [46], while Hitchin [43] uses 'positive' where we use 'G_2–form'.

$SU(n)$*–structures.* Let $z^1, \dots, z^n$ be standard coordinates on $\mathbb{C}^n$, and

$$\begin{aligned} \Omega_0 &= dz^1 \wedge \cdots \wedge dz^n, \\ \omega_0 &= \tfrac{i}{2}(dz^1 \wedge d\bar{z}^1 + \cdots + dz^n \wedge d\bar{z}^n). \end{aligned}$$

These are, respectively, the standard complex volume form and Kähler form, and are invariant under the action of $SU(n)$. In fact, their stabiliser in $GL(2n, \mathbb{R})$ is precisely $SU(n)$. For Ω_0 on its own determines $\Lambda^{1,0}_{\mathbb{C}}(\mathbb{C}^n)^*$ (as the kernel of $\alpha \mapsto \Omega_0 \wedge \alpha$) and hence the complex structure on $\mathbb{C}^n$, so the stabiliser of Ω_0 in $GL(2n, \mathbb{R})$ is precisely $SL(n, \mathbb{C})$.

By analogy with Definition 2.10, we can think of any complex n-form Ω that is $GL(2n, \mathbb{R})$-equivalent to Ω_0 (*ie* any decomposable form such that $\Omega \wedge \overline{\Omega} \neq 0$) as defining an $SL(n, \mathbb{C})$-structure, and any pair (Ω, ω) of a decomposable complex n-form and a non-degenerate real 2-form such that

$$\Omega \wedge \omega = 0, \tag{2.13a}$$

$$(-1)^{\frac{n(n-1)}{2}} \left(\tfrac{i}{2}\right)^n \Omega \wedge \overline{\Omega} = \frac{\omega^n}{n!}, \tag{2.13b}$$

as an $SU(n)$-structure. (2.13a) encodes that ω is $(1,1)$ with respect to the complex structure defined by Ω, while (2.13b) is a normalisation condition that the natural volume forms defined by ω and Ω are equal, or equivalently that $|\Omega|^2 = 2^n$ (see Hitchin [42, §2]).

$SU(3)$*–structures.* We have a particular interest in the case of complex dimension three since $SU(3)$ is the stabiliser in G_2 of a vector in $\mathbb{R}^7$. Let us now give the previously promised proof of this fact.

Proof of Proposition 2.3(ii). Let S be the stabiliser of the basis vector $e_1 \in \mathbb{S}^6 \subset \mathbb{R}^7$. Since $G_2 \subset SO(7)$, S maps the orthogonal complement $e_1^\perp$ to itself. $e_1^\perp$ can be identified with $\mathbb{C}^3$ by introducing complex coordinates $z^1 = x^2 + ix^3$, $z^2 = x^4 + ix^5$, $z^3 = x^6 + ix^7$. The action of S on $\mathbb{C}^3$ evidently preserves the forms

$$\begin{aligned} e_1 \lrcorner \varphi_0 = dx^{23} + dx^{45} + dx^{67} &= \omega_0, \\ \varphi_0|_{e_1^\perp} = dx^{246} - dx^{257} - dx^{347} - dx^{356} &= \operatorname{Re}\Omega_0, \\ -e_1 \lrcorner \psi_0 = -dx^{247} - dx^{256} - dx^{346} + dx^{357} &= \operatorname{Im}\Omega_0, \end{aligned}$$

so S is contained in $SU(3)$. Conversely

$$\varphi_0 = dx^1 \wedge \omega_0 + \operatorname{Re}\Omega_0 \tag{2.15}$$

implies that $SU(3)$ preserves φ_0, so S is precisely $SU(3)$. □

It follows that any SU(3)–structure on a real vector space V of dimension 6 (together with a covector dt on $\mathbb{R}$ defining orientation and length) determines a G_2–structure on $\mathbb{R}\oplus V$. Moreover we see from the proof how to express the relationship between the structures in terms of the forms. If the SU(3)–structure on V is defined by (Ω, ω) then the induced G_2–structure on $\mathbb{R} \oplus V$ has G_2–form

$$\varphi = dt \wedge \omega + \operatorname{Re}\Omega. \tag{2.16}$$

Similarly, the Hodge dual 4-form ψ of φ takes the form

$$\psi = \tfrac{1}{2}\omega^2 - dt \wedge \operatorname{Im}\Omega. \tag{2.17}$$

Another way to think of the relationship is that the orthogonal complement to a unit vector u in a vector space with G_2–structure inherits (in addition to the metric) two structures from the cross product: using Lemma 2.7, $I_u : v \mapsto u \times v$ defines an orthogonal complex structure on $u^\perp$, while the restriction/projection of the cross product to $u^\perp$ defines a bilinear map that is I_u-antilinear in each factor and which is equivalent to a complex volume form (because the complex dimension is 3). See also p. 41.

Remark 2.18. Complex volume forms in dimension three have some special properties. Hitchin [43, §2] explains that the stabiliser of $\operatorname{Re}\Omega_0$ alone in $\mathrm{GL}_+(6,\mathbb{R})$ is $\mathrm{SL}(3,\mathbb{C})$. The $\mathrm{GL}(6,\mathbb{R})$-orbit of $\operatorname{Re}\Omega_0$ in $\Lambda^3(\mathbb{R}^6)^*$ is therefore open by dimension counting: $\dim \mathrm{GL}(6,\mathbb{R}) - \dim \mathrm{SL}(3,\mathbb{C}) = 36 - 16 = 20 = \dim \Lambda^3(\mathbb{R}^6)^*$. For any 3-form α in this open set there is a unique real 3-form β such that $\alpha + i\beta$ is decomposable and the induced $\mathrm{SL}(3,\mathbb{C})$-structure has the standard orientation. For a real vector space of dimension 6, an $\mathrm{SL}(3,\mathbb{C})$-structure is therefore equivalent to a choice of orientation together with a 3-form equivalent to $\operatorname{Re}\Omega_0$ (reversing the orientation while keeping the 3-form fixed corresponds to replacing the complex structure by its conjugate).

SU(2)*–structures.* The case of complex dimension two also plays an important role in the paper. Let $\omega_0^I := \omega_0$ be the standard Kähler form on $\mathbb{C}^2$, and write the holomorphic volume form Ω_0 as $\omega_0^J + i\,\omega_0^K$. As suggested by the notation, ω_0^J and ω_0^K define g_0-orthogonal complex structures J and K on $\mathbb{R}^4$ by the relations $\omega_0^J(x,y) = g_0(Jx,y)$ and $\omega_0^K(x,y) = g_0(Kx,y)$. In real coordinates (x^i) where $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$

$$\omega_0^I = dx^{12} + dx^{34}, \quad \omega_0^J = dx^{13} - dx^{24}, \quad \omega_0^K = dx^{14} + dx^{23}.$$

When we identify $\mathbb{C}^2$ with the quaternions $\mathbb{H}$ by $(x^1 + ix^2, x^3 + ix^4) \mapsto x^1 + ix^2 + jx^3 + kx^4$ the complex structures I, J, K correspond to left multiplication by the standard orthonormal triple i, j, k of imaginary quaternions. This identifies SU(2) with the automorphism group Sp(1) of $\mathbb{H}$. Furthermore, any unit imaginary quaternion defines an orthogonal complex structure, so SU(2) preserves a whole $\mathbb{S}^2$ of complex structures.

We can therefore think of an SU(2)–structure on a 4-dimensional vector space in two different ways: either as a pair (ω, Ω) as before, or as a choice of an ordered triple of 2-forms $(\omega^I, \omega^J, \omega^K)$ equivalent to $(\omega_0^I, \omega_0^J, \omega_0^K)$, *ie* satisfying

$$(\omega^I)^2 = (\omega^J)^2 = (\omega^K)^2,$$
$$\omega^I \wedge \omega^J = \omega^J \wedge \omega^K = \omega^K \wedge \omega^I = 0.$$

These two definitions of SU(2)–structures are equivalent, setting $\omega = \omega^I$ and $\Omega = \omega^J + i\,\omega^K$. However, the first highlights a preferred complex structure I, while the second emphasises the two-sphere of complex structures. We will switch back and forth between these two points of view.

If we want to choose an SU(2)–structure compatible with a particular inner product and orientation we first choose ω^I in the $\mathbb{S}^2$ of 2-forms such that $(\omega^I)^2 = 2\,\mathrm{vol}$, and then ω^J among the $\mathbb{S}^1$ of such forms that are perpendicular to ω^I (and ω^K is then determined by $K = IJ$). All in all, there is therefore an SO(3)–family of SU(2)–structures inducing the same inner product and orientation.

Remark. Any complex 2-form Ω on a real vector space of dimension 4 such that $\Omega \wedge \overline{\Omega} \neq 0$ and $\Omega^2 = 0$ is decomposable, and thus determines an $\mathrm{SL}(2,\mathbb{C})$-structure.

Calibrations in $\mathbb{R}^7$. Let (V, g) be an inner product space. A k-form $\alpha \in \Lambda^k V^*$ is said to be a *calibration* if, for every oriented k-plane π in V, we have $\alpha_{|\pi} \leq vol_\pi$. The oriented k-planes π for which $\alpha_{|\pi} = vol_\pi$ are said to be *calibrated.*

A G_2–form φ and its Hodge dual ψ define calibrations with respect to the metric g_φ.

Lemma 2.19.

(i) *The 3-form φ_0 and the 4-form $\psi_0 = *\varphi_0$ defined in (2.2) and (2.4) respectively are calibrations on $(\mathbb{R}^7, g_0)$.*

(ii) *If u, v, w is an orthonormal triple of vectors in $\mathbb{R}^7$, then $\varphi_0(u, v, w) = 1$ if and only if $w = u \times v$.*

(iii) *If u, v, w, x is an orthonormal quadruple of vectors in $\mathbb{R}^7$ then $\psi_0(u, v, w, x) = 1$ if and only if $u = \frac{1}{2}\chi_0(v, w, x)$.*

Proof. For any orthonormal quadruple $u, v, w, x \in \mathbb{R}^7$ using Cauchy-Schwarz, (2.8a) and (2.8c) we have

$$\varphi_0(u, v, w) = g_0(u \times v, w) \leq |u \times v|\,|w| = 1, \tag{2.20}$$

and

$$|\psi_0(u, v, w, x)| = |g_0(u, \tfrac{1}{2}\chi_0(v, w, x))| \leq |u|\,\left|\tfrac{1}{2}\chi_0(v, w, x)\right| \leq 1. \tag{2.21}$$

If $w = u \times v$ then $\varphi_0(u, v, w) = g_0(u \times v, u \times v) = 1$. Conversely, if $\varphi_0(u, v, w) = 1$, then equality must hold throughout (2.20) and in particular in the Cauchy-Schwarz inequality. Hence $w = \lambda u \times v$ for some $\lambda \in \mathbb{R}$. But $1 = \varphi_0(u, v, \lambda u \times v) = \lambda g_0(u \times v, u \times v) = \lambda$, hence we must have $w = u \times v$.

Similarly we have equality in (2.21) if and only if $u = \lambda \frac{1}{2}\chi_0(v, w, x)$ for some $\lambda \in \mathbb{R}$ and $\left|\frac{1}{2}\chi_0(v, w, x)\right| = 1$. Hence equality holds in (2.21) if and only if $u = \pm\frac{1}{2}\chi_0(v, w, x)$, and clearly we have $\psi_0(\pm\frac{1}{2}\chi_0(v, w, x), v, w, x) = \pm 1$. □

Definition 2.22. An oriented 3-plane π in $\mathbb{R}^7$ calibrated by φ_0 is called an *associative* plane. An oriented 4-plane π in $\mathbb{R}^7$ calibrated by ψ_0 is called a *coassociative* plane.

Lemma 2.23.

(i) *A 3-plane π is associative (for one choice of orientation) if and only if $\chi_{0|\pi} = 0$.*

(ii) *Any 2-plane is contained in a unique associative 3-plane.*

Proof. (i) follows directly from (2.8c) and the fact that φ_0 is a calibration.
(ii) Let $\{u, v\}$ be an orthonormal basis for the 2-plane. Then $\{u, v, u \times v\}$ is an oriented orthonormal basis for an associative 3-plane. Suppose π is any associative 3-plane containing the 2-plane $\langle u, v\rangle_{\mathbb{R}}$. Then we can choose an oriented orthonormal basis $\{u, v, w\}$ for π extending $\{u, v\}$. Hence by Lemma 2.19 we must have $w = u \times v$. □

Relation to calibrations on $\mathbb{C}^3$. There are also standard calibrations on $\mathbb{C}^n$, given by powers of the standard Kähler form and real parts of normalised $(n, 0)$-forms. The fact that $\frac{1}{k!}\omega_0^k$ is a calibration for each k, and that the calibrated subspaces are precisely the complex k-planes, is known as *Wirtinger's inequality.* The other type of calibration is described by the following lemma.

Lemma 2.24.

(i) *The n-forms* $\mathrm{Re}(e^{i\theta}\Omega_0)$ *are calibrations on* $(\mathbb{C}^n, g_0)$ *for each* $\theta \in \mathbb{R}$.

(ii) *A real n-plane $L \subset \mathbb{C}^n$ is calibrated by* $\mathrm{Re}\,\Omega_0$ *(for one choice of orientation) if and only if* $\omega_{0|L} = \mathrm{Im}\,\Omega_{0|L} = 0$.

Proof. If $a_1, \ldots, a_n \in \mathbb{C}^n$ is an orthonormal basis for a real n-plane $L \subset \mathbb{C}^n$ then (switching between regarding a_i as complex and real column vectors)

$$\left|\Omega_{0|L}\right|^2 = \left|\det_{\mathbb{C}}(a_1, \ldots, a_n)\right|^2 = \det_{\mathbb{R}}(a_1, Ja_1, \ldots, a_n, Ja_n) \leq 1.$$

Equality holds if and only if the unit vectors a_i and Ja_i are all orthogonal, *ie* when JL is the orthogonal complement to L, or equivalently when $\omega_{0|L} = 0$. Thus $\left|\mathrm{Re}(e^{i\theta}\Omega_0)_{|L}\right| \leq 1$ with equality if and only if L is Lagrangian and $\mathrm{Im}(e^{i\theta}\Omega_0)_{|L} = 0$. □

Note that for each Lagrangian plane $L \subset \mathbb{C}^n$ there is a θ (unique modulo 2π) such that L is calibrated by $\mathrm{Re}(e^{i\theta}\Omega_0)$.

Definition 2.25. We call the planes calibrated by $\mathrm{Re}(e^{i\theta}\Omega_0)$ *special Lagrangian with phase* θ, or simply *special Lagrangian* if $\theta = 0$.

Now consider $\mathbb{C}^3$ with its standard SU(3)–structure (Ω_0, ω_0) as a hyperplane in $\mathbb{R}^7 \cong \langle e_1 \rangle \oplus \mathbb{C}^3$ with the standard product G_2–structure $\varphi_0 = dx^1 \wedge \omega_0 + \mathrm{Re}\,\Omega_0$ given in (2.15).

Lemma 2.26.

(i) *Let $\ell \subset \mathbb{C}^3$ be a real 2-plane. Then $\langle e_1 \rangle \oplus \ell$ is associative in $\mathbb{R}^7$ if and only if ℓ is a complex line.*

(ii) *Let $L \subset \mathbb{C}^3$ be a real 3-plane. Then L is associative in $\mathbb{R}^7$ if and only if L is special Lagrangian.*

Proof. (i) $\varphi_{0|\langle e_1\rangle \oplus \ell} = \omega_{0|\ell}$, so $\langle e_1 \rangle \oplus \ell$ is calibrated by φ_0 if and only if ℓ is calibrated by ω_0. (ii) $\varphi_{0|L} = \mathrm{Re}\,\Omega_{0|L}$, so L is calibrated by φ_0 if and only if L is calibrated by $\mathrm{Re}\,\Omega_0$. □

We can also think of (i) the following way. Let V be a 7-dimensional vector space with a G_2–structure, $u \in V$ a unit vector, and consider the orthogonal complement $u^\perp$ with its induced SU(3)-structure (2.16). The complex structure on $u^\perp$ is $I_u : v \mapsto u \times v$. So for $v \in u^\perp$, the unique associative 3-plane in V containing both u and v is $\langle u, v, I_u v\rangle_{\mathbb{R}}$, which is the direct sum of $\langle u \rangle$ and the unique complex line in $u^\perp$ containing v.

***G*–structures and manifolds with special holonomy.** Let M be a smooth n-dimensional manifold. Let $GL(M)$ denote the principal $\mathrm{GL}(n, \mathbb{R})$-bundle of linear frames on M.

Definition 2.27. Let G be a subgroup of $\mathrm{GL}(n, \mathbb{R})$. A G–structure on M is a G–subbundle of $GL(M)$. Equivalently, it is a smooth section of the quotient bundle $GL(M)/G$.

The G–structures of interest to us can equivalently be defined in terms of a choice of special algebraic structure on M.

G_2*–structures and manifolds with holonomy* G_2.

Definition 2.28. For an oriented manifold M of dimension 7, let $\Lambda^3_+ T^*M \subset \Lambda^3 T^*M$ be the smooth subbundle of positive 3-forms, in the sense of Definition 2.11. A G_2*–structure* on M (compatible with its orientation) is a smooth section of $\Lambda^3_+ T^*M$, *ie* a smooth 3-form φ such that for each $x \in M$ there is an oriented isomorphism $(T_xM, \varphi) \cong (\mathbb{R}^7, \varphi_0)$.

It follows from Proposition 2.3(ii) that $\Lambda^3_+ T^*M$ is an open subset of $\Lambda^3 T^*M$; in particular, any small perturbation of a G_2–structure φ is again a G_2–structure.

Remark 2.29. The existence of G_2–structures on a manifold is a topological question. G_2 is simply connected by Proposition 2.3, so $G_2 \hookrightarrow SO(7)$ lifts to $G_2 \hookrightarrow \mathrm{Spin}(7)$, and any G_2–structure induces a spin structure. In fact, the converse also holds: a 7-manifold M admits G_2–structures if and only if it is orientable and spin (*cf* Gray [35, Theorem 3.2]).

A G_2–structure φ induces a Riemannian metric g_φ on M, and hence also a Levi-Civita connection ∇_φ and a Hodge star $*_\varphi$. We may drop the subscripts if the G_2–structure is clear from the context. The canonical 4-form $\psi = *\varphi$ is also important.

Definition 2.30. A G_2–structure defined by a positive 3-form φ is *torsion-free* if $\nabla_\varphi \varphi = 0$.

Remark. There is a notion of the *intrinsic torsion* of a G–structure on M for a general Lie subgroup $G \subseteq GL(n, \mathbb{R})$ (see *eg* Joyce [46, §2.6]). A G_2–structure has zero intrinsic torsion in this sense if and only if it is torsion-free according to Definition 2.30.

It follows immediately from the definition of holonomy that if (M^7, g) is a Riemannian manifold, then $\mathrm{Hol}(g)$ is a subgroup of G_2 if and only if there is a torsion-free G_2–structure φ on M such that $g = g_\varphi$.

Definition 2.31. A G_2*–manifold* is a manifold M^7 equipped with a torsion-free G_2–structure φ and the associated Riemannian metric g_φ. We say that (M, φ) is a manifold *with holonomy* G_2 or *has holonomy* G_2 if $\mathrm{Hol}(g_\varphi) = G_2$.

Holonomy G_2 is a much stronger condition on M than the existence of a G_2–structure, involving the metric. For example, any such metric is Ricci-flat (Salamon [69, Proposition 11.8]). On the basis of Berger's classification of holonomy groups one can prove the following, see Joyce [46, p. 245].

Proposition 2.32. *A compact* G_2*–manifold has holonomy* G_2 *if and only if* $\pi_1(M)$ *is finite.*

Using Hodge theory and the decomposition of the exterior algebra of any G_2–manifold into irreducible G_2–representations one can prove the following additional restrictions on the topology of any compact G_2–manifold (M, φ, g) manifold with $\mathrm{Hol}(g) = G_2$.

Proposition 2.33 ([46, p. 246])**.** *Let* (M, φ, g) *be a compact* G_2*–manifold with* $\mathrm{Hol}(g) = G_2$*, and* $p_1(M) \in H^4(M; \mathbb{Z})$ *the first Pontrjagin class. Then*

(i) $(\alpha \cup \alpha \cup [\varphi])[M] < 0$ *for every nonzero* $\alpha \in H^2(M; \mathbb{R})$*.*
(ii) $(p_1(M) \cup [\varphi])[M] < 0$*. In particular* $p_1(M) \neq 0$*.*

By considering how $d\varphi$ and $d\psi$ are obtained algebraically from $\nabla_\varphi \varphi$ one can deduce the following characterisation of torsion-free G_2–structures.

Theorem 2.34 ([69, Lemma 11.5])**.** *A smooth positive 3-form* φ *is torsion-free if and only if* $d\varphi = 0$ *and* $d^*_\varphi \varphi = 0$ *(or equivalently* $d\psi = 0$*).*

Remark. Given a Riemannian manifold whose holonomy is contained in the group G_2, there may be several compatible torsion-free G_2–structures. For general $H \subseteq G \subseteq K$, parallel G–subbundles of a connection with holonomy H on a principal K-bundle correspond to $\{k \in K : kHk^{-1} \subseteq G\}$. Since G_2 is the stabiliser of a unique element of $\mathbb{P}(\Lambda^3(\mathbb{R}^7)^*)$ it equals its own stabiliser in $SO(7)$ (or indeed in $GL_+(7, \mathbb{R})$), so a metric with holonomy *exactly* G_2 has a unique compatible torsion-free G_2–structure (up to orientation).

A Riemannian manifold has holonomy contained in G_2 if and only if it admits a parallel spinor for some spin structure. Wang [76] gives an explicit way to construct a parallel positive 3-form from a parallel spinor.

Remark 2.35. We call a G_2–structure defined by a closed positive 3-form φ a *closed* G_2*–structure*. Joyce [46, Thm. 11.6.1] gave sufficient conditions under which a closed G_2–structure with small torsion can be perturbed to a torsion-free G_2–structure within its cohomology class.

$\mathrm{SU}(n)$*–structures and Calabi–Yau manifolds.* Let M be a real $2n$-dimensional manifold with an $\mathrm{SU}(n)$–structure. Then M is equipped with an almost complex structure I, a real non-degenerate 2-form ω equivalent to a hermitian metric g, and an $(n,0)$-form Ω of constant norm $2^{n/2}$.

If $d\Omega = 0$ then the complex structure is integrable, and Ω is holomorphic. In particular, the canonical bundle of M is trivial, so $c_1(M) = 0 \in H^2(M;\mathbb{Z})$. If also $d\omega = 0$, then M is a Kähler manifold. In particular $\nabla\omega = 0$, so $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$. The fact that Ω is holomorphic of constant norm forces that also $\nabla\Omega = 0$, so actually the holonomy must reduce further to $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$.

Definition 2.36. We call an $\mathrm{SU}(n)$–structure *torsion-free* or a *Calabi–Yau structure* if $\nabla\Omega = \nabla\omega = 0$ with respect to the induced metric. We call M^{2n} equipped with a torsion-free $\mathrm{SU}(n)$–structure (Ω,ω) and its associated metric a *Calabi–Yau manifold*. We say that (M^{2n},Ω,ω) is a manifold *with holonomy* $\mathrm{SU}(n)$ or *has holonomy* $\mathrm{SU}(n)$ if its holonomy is exactly $\mathrm{SU}(n)$.

Remark 2.37. Yau's proof [79] of the Calabi conjecture shows that any compact Kähler manifold M with $c_1(M) = 0 \in H^2(M;\mathbb{R})$ admits Ricci-flat Kähler metrics. Ricci-flat Kähler manifolds are also often referred to as Calabi–Yau manifolds, which is not quite equivalent to our definition: the vanishing of the Ricci curvature implies that the canonical bundle is flat so that the restricted holonomy (*ie* the group generated by parallel transport around contractible closed curves in M, or equivalently the identity component of $\mathrm{Hol}(g)$) is contained in $\mathrm{SU}(n)$, but if M is not simply connected then there need not be any global holomorphic section.

Now let M^6 be a manifold with an $\mathrm{SU}(3)$–structure (g,I,ω,Ω). Then the product manifold $\mathbb{S}^1 \times M$ has a natural product G_2–structure. The pointwise model (2.16) shows that in terms of the forms the G_2–structure is given by

$$\varphi = d\theta \wedge \omega + \mathrm{Re}\,\Omega, \tag{2.38}$$

where θ is the natural coordinate on $\mathbb{S}^1$. The induced metric is the product metric, and for any $v \in TM$, $\frac{\partial}{\partial\theta} \times v = Iv$.

Lemma 2.39. *If* (M^6,g,I,ω,Ω) *is a Calabi–Yau 3-fold then the product manifold* $\mathbb{S}^1 \times M$ *with the above* G_2*–structure is a* G_2*–manifold.*

Observe that $\mathbb{S}^1 \times M$ is not a manifold with holonomy G_2: its holonomy equals $\mathrm{Hol}(M) \subseteq \mathrm{SU}(3) \subset G_2$.

Hyper-Kähler $K3$ *surfaces.* Recall that a *K3 surface* is a smooth compact complex surface (S,I) which is simply connected and whose canonical bundle is holomorphically trivial, *i.e.* $\pi_1(S) = 0$ and $K_S \simeq \mathcal{O}_S$. By definition, S has a non-vanishing holomorphic 2-form Ω. Siu [70] proved that any K3 surface admits Kähler metrics, and by Yau's solution to the Calabi conjecture there exists a unique Ricci-flat Kähler metric ω in every Kähler class and thus Calabi–Yau structures (ω,Ω). The pointwise considerations on p. 13 show that a manifold with holonomy $\mathrm{SU}(2) = \mathrm{Sp}(1)$ has an $\mathbb{S}^2$ of integrable complex structures. A Calabi–Yau structure (ω,Ω) compatible with the metric corresponds to a choice of oriented orthonormal triple I,J,K in this $\mathbb{S}^2$, *ie* complex structures satisfying the usual quaternionic relations. The structure, including the metric, can be recovered from the associated Kähler forms $\omega^I,\omega^J,\omega^K$ by

$$\omega = \omega^I, \qquad \Omega = \omega^J + i\,\omega^K.$$

We call a K3 surface S with the structure $(\omega^I, \omega^J, \omega^K)$ a *hyper-Kähler K3 surface.*

Any two K3 surfaces are related by complex deformation. In particular, there is up to diffeomorphism a unique K3 surface S. It has $b_2(S) = 22$, and we will often refer to $H^2(S;\mathbb{Z})$ with its intersection form as the *K3 lattice* L. It is the unique even unimodular lattice of signature $(3, 19)$, *ie*

$$L = 2E_8(-1) \perp 3\,U, \tag{2.40}$$

where E_8 denotes the unique even unimodular positive definite lattice of rank 8 and U the standard hyperbolic lattice. We denote by $\mathrm{O}(L)$ the group of isometries of the K3 lattice L. A *marking* of a complex K3 surface (S, I) is an isometry $L \cong H^2(S;\mathbb{Z})$.

3. The twisted connected sum construction of G_2–manifolds

In this section we describe the main steps of our construction of compact G_2–manifolds. Starting from suitable algebraic varieties we first construct asymptotically cylindrical Calabi–Yau 3-folds. Given a suitably compatible pair of such manifolds we then form a "twisted connected sum" 7-manifold by gluing. The procedure is essentially the same as used by Kovalev [48], but as we will describe we change the algebraic starting point to use *semi-Fano 3-folds* rather than Fano 3-folds. The issue of how to satisfy the compatibility condition between the ACyl Calabi–Yau manifolds is discussed in detail in §6. Throughout this section all homology and cohomology groups are over $\mathbb{Z}$ unless explicitly stated otherwise.

Asymptotically cylindrical Calabi–Yau 3-folds. We begin with a review of the definition of asymptotically cylindrical Calabi–Yau 3-folds and an analytic existence result; the latter reduces the analytic problem of finding asymptotically cylindrical Calabi–Yau 3-folds to a problem purely in complex projective geometry.

Definition 3.1. Let $(S^4, I_S, g_S, \omega_S, \Omega_S)$ be a hyper-Kähler K3 surface. We call the complex 3-fold $V_\infty := \mathbb{R}^+ \times \mathbb{S}^1 \times S$ endowed with the $\mathbb{R}^+$-translation invariant Calabi–Yau structure

$$\begin{aligned} I_\infty &:= I_{\mathbb{C}} + I_S, \\ g_\infty &:= dt^2 + d\vartheta^2 + g_S, \\ \omega_\infty &:= dt \wedge d\vartheta + \omega_S, \\ \Omega_\infty &:= (d\vartheta - i dt) \wedge \Omega_S, \end{aligned} \tag{3.2}$$

(where t and ϑ denote the standard variables on $\mathbb{R}^+$ and $\mathbb{S}^1$) a *Calabi–Yau cylinder.* The phase in the expression for Ω_∞ is unimportant but has been chosen to put (3.12) in a convenient form.

Definition 3.3. Let $(V, g, I, \omega, \Omega)$ be a complete Calabi–Yau 3-fold. We say that V is an *asymptotically cylindrical* (or *ACyl* for short) Calabi–Yau 3-fold if there exist (i) a compact set $K \subset V$, (ii) a Calabi–Yau cylinder V_∞ and (iii) a diffeomorphism $\eta : V_\infty \to V \backslash K$ such that for some $\lambda > 0$ and all $k \geq 0$,

$$\begin{aligned} \eta^*\omega - \omega_\infty &= d\varrho, \quad \text{for some } \varrho \text{ such that } |\nabla^k \varrho| = O(e^{-\lambda t}) \\ \eta^*\Omega - \Omega_\infty &= d\varsigma, \ \text{ for some } \varsigma \text{ such that } |\nabla^k \varsigma| = O(e^{-\lambda t}) \end{aligned}$$

for sufficiently large t. Here ∇ and $|\cdot|$ are defined using the metric g_∞ on V_∞. We will refer to $V_\infty = \mathbb{R}^+ \times \mathbb{S}^1 \times S$ as the *asymptotic end* of V and to the hyper-Kähler K3 surface $(S, I_S, g_S, \omega_S, \Omega_S)$ as the *asymptotic K3 surface* of V.

Remark. Our definition asks that $\eta^*\omega$ be cohomologous to ω_∞ on the asymptotic end of V. However, as long as $|\eta^*\omega - \omega_\infty| \to 0$, this is automatic. The main point of the definition is thus to impose the existence of specific ϱ and ς with the stated rate of decay.

Since the complex structures on both $\mathbb{R}^+\times\mathbb{S}^1\times S$ and V are determined by the corresponding complex volume forms, similar estimates automatically hold for $|\nabla^k(\eta^*I - I_\infty)|$. The same is true for the metrics.

Remark. We could consider a more general definition of an ACyl Calabi–Yau 3-fold in which the cross-section of the asymptotic cylinder is not a priori assumed to split as a product $\mathbb{S}^1 \times S$. Such ACyl Calabi–Yau 3-folds do exist, but we are not yet able to use them to construct compact G_2–manifolds. See [40] for further discussion of this and other related issues.

Our examples of ACyl Calabi–Yau 3-folds arise by application of the following ACyl version of the Calabi–Yau theorem sharpening an earlier result of Tian-Yau [73, Thm 5.2]. The statement is taken from [21, Theorem 2.6]. For details of the proof see [40].

Theorem 3.4. *Let Z be a closed Kähler 3-fold with a morphism $f : Z \to \mathbb{P}^1$, with a reduced smooth K3 fibre S that is an anticanonical divisor, and let $V = Z\setminus S$. If Ω_S is a non-vanishing holomorphic 2-form on S, ω_S a Ricci-flat Kähler metric satisfying the normalisation condition* (2.13b)*, and $[\omega_S] \in H^{1,1}(S)$ is the restriction of a Kähler class on Z, then there is an ACyl Calabi–Yau structure (ω, Ω) on V whose asymptotic limit on $\mathbb{R}^+\times\mathbb{S}^1\times S$ is the product structure* (3.2)*.*

Remark. Arguments similar to Lemma 3.6 below show that the hypotheses of Theorem 3.4 imply $H_1(Z)$ finite and $H^{2,0}(Z) = 0$, so Z must be projective.

In the statement above we use the fact that the fibration structure of Z implies that $V := Z \setminus S$ has an obvious topological end $\mathbb{R}^+ \times \mathbb{S}^1 \times S$. We call (Z, S) a *building block* if it satisfies some additional topological conditions. These assumptions will simplify the calculation of the topological invariants of V in §4.

Definition 3.5. A *building block* is a nonsingular algebraic 3-fold Z together with a projective morphism $f\colon Z \to \mathbb{P}^1$ satisfying the following assumptions:

(i) the anticanonical class $-K_Z \in H^2(Z)$ is indivisible,
(ii) $S = f^\star(\infty)$ is a nonsingular K3 surface and $S \in |-K_Z|$.

Identify $H^2(S)$ with the K3 lattice L (2.40) (*ie* choose a marking for S), and let N denote the image of $H^2(Z) \to H^2(S)$.

(iii) The inclusion $N \hookrightarrow L$ is primitive, that is, L/N is torsion-free.
(iv) The group $H^3(Z)$—and thus also $H^4(Z)$—is torsion-free.

Lemma 3.6. *If Z is a building block then*

(i) $\pi_1(Z) = 0$*. In particular, $H^*(Z)$ and $H_*(Z)$ are torsion-free.*
(ii) $H^{2,0}(Z) = 0$*, so $N \subseteq \operatorname{Pic} S$.*

Proof. (i) is [21, Lemma 5.2]. For (ii), Serre duality implies $H^{2,0}(Z) \cong H^1(K_Z)^*$, which vanishes by the long exact sequence of $0 \to K_Z \to \mathcal{O}_Z \to \mathcal{O}_S \to 0$ together with the fact that $H^1(\mathcal{O}_Z) \cong H^{1,0}(Z) = 0$. □

Remark. $N \subset L$ inherits the structure of a lattice from the K3 lattice L. Because of 3.6(ii) we call N the *polarising lattice* of the building block Z. The lattice N plays a key role in this paper as we explain shortly.

Further topological properties of building blocks are recalled in Section 4. For now, let us remark that $V = Z \setminus S$ is always simply-connected, so that any ACyl Calabi–Yau metric on V has holonomy exactly SU(3).

Most of the building blocks we use in this paper arise from *semi-Fano* 3-folds, as we discuss below in Proposition 3.17. We say that the resulting ACyl Calabi–Yau 3-folds are of *semi-Fano type*; see Definition 3.18 for a precise definition.

Examples of ACyl Calabi–Yau 3-folds have been constructed previously by similar methods, using building blocks obtained from genuine Fanos by Kovalev [48] or from K3s with non-symplectic involution by Kovalev-Lee [49] (see Remark 3.20). We will call these ACyl Calabi–Yau 3-folds of *Fano type* and *non-symplectic type* respectively. While there are 105 deformation families of smooth Fano 3-folds and 75 deformation classes of K3 surfaces with non-symplectic involution, deformation families of semi-Fano 3-folds are much more plentiful and therefore so are ACyl Calabi–Yau 3-folds of semi-Fano type.

Since S has trivial normal bundle in Z, there is an inclusion $S \hookrightarrow V$ well-defined up to homotopy, and

$$K := \ker(\rho), \quad \text{for the restriction map } \rho : H^2(V) \to H^2(S) \tag{3.7}$$

is an important parameter of the block Z. K turns out to contribute to the rank of H^2 of twisted connected sums M built from Z, so to get 2-connected M we need $K = 0$. While this holds for semi-Fano blocks by Proposition 3.17, we do not make it part of the definition of a building block in order to accommodate *eg* non-symplectic blocks.

The gluing procedure. We can now outline Kovalev's construction of compact G_2–manifolds by combining a pair of *compatible* asymptotically cylindrical Calabi–Yau 3-folds. We call this the *twisted connected sum* construction of compact G_2–manifolds and refer to the resulting G_2–manifolds as twisted connected sums. We emphasise at the outset that finding compatible pairs of asymptotically cylindrical Calabi–Yau 3-folds is perhaps the most involved part of the whole construction.

Let $V_\pm$ be two asymptotically cylindrical Calabi–Yau 3-folds with structures $(g_\pm, I_\pm, \omega_\pm, \Omega_\pm)$. Then as in (3.2) the asymptotic end of $V_\pm$ is of the form $V_{\infty,\pm} = \mathbb{R}^+ \times \mathbb{S}^1 \times S_\pm$ where $S_\pm$ is the asymptotic hyper-Kähler K3 surface of $V_\pm$. Using maps $\eta_\pm$ as in Definition 3.3 to identify the ends $V_{\infty,\pm}$ with $\mathbb{R}^+ \times \mathbb{S}^1 \times S_\pm$, on each end we can write

$$\omega_\pm = \omega_{\infty,\pm} + d\varrho_\pm,$$
$$\Omega_\pm = \Omega_{\infty,\pm} + d\varsigma_\pm.$$

Let $\rho = \rho(s) : \mathbb{R} \to [0, 1]$ denote a smooth function satisfying $\rho(s) \equiv 0$ for $s \leq 0$ and $\rho(s) \equiv 1$ for $s \geq 1$. For fixed $T \gg 0$, consider the same manifolds $V_\pm$ endowed with forms $\omega_{T,\pm}$, $\Omega_{T,\pm}$ obtained by the following perturbation on the ends:

$$\omega_{T,\pm} := \omega_\pm - d(\rho(t - T + 1)\varrho_\pm), \tag{3.8a}$$

$$\Omega_{T,\pm} := \Omega_\pm - d(\rho(t - T + 1)\varsigma_\pm). \tag{3.8b}$$

Both forms are closed and in the interval $t \in [T - 1, T]$ they interpolate between the ACyl SU(3)-structure $(\omega_\pm, \Omega_\pm)$ on $V_\pm$ and the product SU(3)-structure $(\omega_{\infty,\pm}, \Omega_{\infty,\pm})$ on the ends $V_{\infty,\pm}$. The C^k norms of $\omega_{T,\pm} - \omega_\pm$ and $\Omega_{T,\pm} - \Omega_\pm$ are $O(e^{-\lambda T})$.

Now consider the product (asymptotically cylindrical) 7-manifolds $M_\pm = \mathbb{S}^1 \times V_\pm$. We let θ denote the standard variable on the new $\mathbb{S}^1$ factor, reserving the notation ϑ for the copy of $\mathbb{S}^1$ contained in the ends of $V_\pm$. We endow $\mathbb{S}^1 \times V_\pm$ with the 3-forms (*cf* (2.38))

$$\varphi_{T,\pm} := d\theta \wedge \omega_{T,\pm} + \operatorname{Re} \Omega_{T,\pm}.$$

For T large the forms $\varphi_{T,\pm}$ are small perturbations of the G$_2$–structures on $\mathbb{S}^1 \times V_\pm$ defined by the original Calabi–Yau structures on $V_\pm$ as in (2.38), so they are again G$_2$–structures.

To form the twisted connected sum of M_+ and M_- we require a certain compatibility condition of the pair of asymptotic K3 surfaces $S_\pm$ of $V_\pm$. The asymptotic limit of $V_\pm$ defines a Calabi–Yau structure $(\omega_\pm, \Omega_\pm)$ on $S_\pm$ and a preferred complex structure $I_\pm$ on $S_\pm$. However, recall from p.17 that $S_\pm$ admits an $\mathbb{S}^2$ of complex structures, and that setting

$$\omega_\pm = \omega^I_\pm, \quad \Omega_\pm = \omega^J_\pm + i\,\omega^K_\pm, \tag{3.9}$$

defines a hyper-Kähler structure $(\omega^I_\pm, \omega^J_\pm, \omega^K_\pm)$. These are Kähler forms with respect to complex structures $I_\pm$, $J_\pm$ and $K_\pm$ respectively; the special status of $I_\pm$ is reflected by the ordering. The compatibility condition we need for our pair of ACyl Calabi–Yau 3-folds $V_\pm$ is the existence of the following special type of map between their asymptotic hyper-Kähler K3 surfaces.

Definition 3.10. Consider two hyper-Kähler K3 surfaces $S_\pm$. A map $\mathfrak{r} : S_+ \to S_-$ is a *hyper-Kähler rotation* if $\mathfrak{r}^* g_- = g_+$, $\mathfrak{r}^* I_- = J_+$ and $\mathfrak{r}^* J_- = I_+$; the hyper-Kähler relationship $IJ = K$ then implies that $\mathfrak{r}^* K_- = -K_+$. Equivalently, $\mathfrak{r}^*\omega^I_- = \omega^J_+$, $\mathfrak{r}^*\omega^J_- = \omega^I_+$ and $\mathfrak{r}^*\omega^K_- = -\omega^K_+$.

As soon as we are given a pair of ACyl Calabi–Yau 3-folds $V_\pm$ for which we can establish the existence of a hyper-Kähler rotation $\mathfrak{r}$ between the asymptotic hyper-Kähler K3 surfaces $S_\pm$ then we can glue the two 7-manifolds $M_\pm = \mathbb{S}^1 \times V_\pm$ together by their ends, as follows. On the region defined by $t \in (T, T+1)$ consider the diffeomorphism

$$\begin{aligned} F:\ \mathbb{S}^1 \times V_{\infty,+} \cong \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1 \times S_+ \ &\longrightarrow\ \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1 \times S_- \cong \mathbb{S}^1 \times V_{\infty,-}, \\ (\theta,\, t,\, \vartheta,\, x) \ &\longmapsto (\vartheta,\, 2T+1-t,\, \theta,\, \mathfrak{r}(x)). \end{aligned} \tag{3.11}$$

Notice that by (3.8) we are working on regions where $(\Omega_{T,\pm}, \omega_{T,\pm})$ are the standard product structures (3.2). Thus, using (3.9), the G$_2$–structures on these regions can be written

$$\begin{aligned} \varphi_{T,\pm} &= d\theta \wedge \omega_{\infty,\pm} + \operatorname{Re}\Omega_{\infty,\pm} \\ &= d\theta \wedge dt \wedge d\vartheta + d\theta \wedge \omega^I_\pm + d\vartheta \wedge \omega^J_\pm + dt \wedge \omega^K_\pm. \end{aligned} \tag{3.12}$$

The compatibility condition for $\mathfrak{r}$ given in 3.10 implies immediately that $F^*\varphi_{T,-} = \varphi_{T,+}$. Now truncate each $\mathbb{S}^1 \times V_\pm$ at $t = T+1$ to form a pair of compact manifolds $M_\pm(T)$ with boundaries $\mathbb{S}^1 \times \mathbb{S}^1 \times S_\pm$. Using F we can glue these manifolds together at the boundary to form a 'twisted connected sum' $M_\mathfrak{r} = M_+(T) \cup_F M_-(T)$. This is a smooth compact 7-manifold (independent of T up to diffeomorphism but depending on the choice of the hyper-Kähler rotation $\mathfrak{r}$), which admits a closed G$_2$–structure $\varphi_{T,\mathfrak{r}}$ defined by setting its restriction to $M_\pm(T)$ to equal $\varphi_{T,\pm}$. With respect to the metric of $\varphi_{T,\mathfrak{r}}$, $M_\mathfrak{r}$ contains an approximately cylindrical neck of length $\sim 2T$. The torsion of $\varphi_{T,\mathfrak{r}}$ (which is measured by $d^*\varphi_{T,\mathfrak{r}}$ according to Theorem 2.34) is $O(e^{-\lambda T})$. Kovalev [48, Theorem 5.34] uses this to prove that for T sufficiently large there are nearby torsion-free G$_2$–structures (one could also apply more general results of Joyce, see Remark 2.35).

Theorem 3.13. *Let $(V_\pm, \omega_\pm, \Omega_\pm)$ be two asymptotically cylindrical Calabi–Yau 3-folds whose asymptotic ends are of the form $\mathbb{R}^+ \times \mathbb{S}^1 \times S_\pm$ for a pair of hyper-Kähler K3 surfaces $S_\pm$, and suppose there exists a hyper-Kähler rotation $\mathfrak{r} : S_+ \to S_-$. Define closed* G$_2$*–structures $\varphi_{T,\mathfrak{r}}$ on the twisted connected sum $M_\mathfrak{r}$ as above. For sufficiently large T there is a torsion-free perturbation of $\varphi_{T,\mathfrak{r}}$ within its cohomology class.*

Whenever the $V_\pm$ in the theorem have holonomy SU(3), [40, Proposition 2.15] implies that their fundamental groups are finite and generated by the $\mathbb{S}^1$ factors in the cylindrical ends. Because $\pi_1((\mathbb{S}^1 \times V_+) \cap (\mathbb{S}^1 \times V_-)) \cong \pi_1(\mathbb{T}^2 \times S)$ surjects onto both $\pi_1(\mathbb{S}^1 \times V_+)$ and $\pi_1(\mathbb{S}^1 \times V_-)$, van Kampen implies that $\pi_1(M_\mathfrak{r})$ is isomorphic to the quotient of $\pi_1(\mathbb{T}^2 \times S) \cong \mathbb{Z}^2$ by the product of the two kernels, and hence to $\pi_1(V_+) \times \pi_1(V_-)$. In particular $\pi_1(M_r)$ is finite, so the holonomy of the metric defined by the torsion-free G_2–structure on M_r is exactly G_2 by Proposition 2.32. Any ACyl Calabi–Yau 3-fold V of semi-Fano or Fano type is simply connected and therefore twisted connected sums using them are also simply connected. The 74 deformation families of ACyl Calabi–Yau 3-folds of non-symplectic type are also simply connected [49, Lem 4.2].

ACyl Calabi–Yau 3-folds from semi-Fano 3-folds. It remains to explain how we can construct ACyl Calabi–Yau 3-folds suited to the twisted connected sum construction from *semi-Fano* 3-folds. To this end we now recall from [21, §4] the definition and a few of the basic properties of semi-Fano 3-folds; we refer the reader to [21] for proofs of the facts recalled here and for a much more comprehensive treatment of semi-Fano 3-folds, including relevant algebro-geometric background.

A semi-Fano 3-fold is a particular type of *weak Fano* 3-fold, a generalisation of a Fano 3-fold in which the positivity of $-K_Y$ is replaced with a sufficiently strong notion of semi-positivity.

Definition 3.14. A *weak Fano 3-fold* is a nonsingular projective complex 3-fold Y such that the anticanonical sheaf $-K_Y$ is a nef and big line bundle, *ie* $-K_Y \cdot C \geq 0$ for any compact algebraic curve $C \subset Y$ and $(-K_Y)^3 > 0$. For any weak Fano 3-fold Y the integer $(-K_Y)^3$ is an even integer which we write $2g-2$; $(-K_Y)^3 = 2g-2$ is called the *anticanonical degree* of Y and g the *genus* of Y.

The *index* of a weak Fano 3-fold Y is the integer $r = \operatorname{div} c_1(Y)$, *ie* the greatest divisor of $c_1(Y) \in H^2(Y)$.

From the classification of Fano 3-folds we know that there are exactly 105 deformation families of smooth Fano 3-folds. For weak Fano 3-folds we still know that there are only finitely many deformation families. However, there are many more deformation families of weak Fano 3-folds as explained in [21] and a classification of all weak Fano 3-folds looks a long way off.

If Y is a weak Fano 3-fold then for n sufficiently large the linear system $|-nK_Y|$ is basepoint-free. It follows that

$$R(Y, -K_Y) := \oplus_{n\geq 0} H^0(Y; -nK_Y)$$

is a finitely generated ring called the *anticanonical ring* of Y. We call the birational morphism $\varphi\colon Y \to X := \operatorname{Proj} R(Y, -K_Y)$ attached to $|-K_Y|$ the *anticanonical morphism* of Y and X the *anticanonical model* of Y. X is a singular Fano 3-fold with mild (at worst Gorenstein canonical) singularities and $\varphi\colon Y \to X$ is a crepant resolution of X, *ie* $\varphi^* K_X = K_Y$.

Conversely, if Y is a projective crepant resolution $\varphi\colon Y \to X$ of a Fano 3-fold X with Gorenstein canonical singularities then Y is a weak Fano 3-fold whose anticanonical model is X. In other words, one way to exhibit weak Fano 3-folds is to find projective crepant resolutions of Gorenstein canonical Fano 3-folds. For instance a sufficiently general quartic $X \subset \mathbb{P}^4$ that contains a projective plane Π is a suitable singular Fano 3-fold; X has exactly 9 singular points, all ordinary nodes contained in Π and admits a projective crepant (in fact small) resolution $\varphi\colon Y \to X$, obtained by blowing up the plane Π: Y is a weak Fano 3-fold which we use later in the paper—see Example 7.3.

A key fact about any smooth weak Fano 3-fold Y is that a general anticanonical divisor $S \in |-K_Y|$ is a nonsingular K3 surface. From now on we make the following extra assumption about all the weak Fano 3-folds we will use in this paper.

Assumption: the linear system $|-K_Y|$ contains two nonsingular members S_0, S_∞ intersecting transversally.

The few weak Fano 3-folds for which this assumption is not satisfied are classified: see [21, §4] and references therein for further details. It fails for precisely 2 of the 105 families of genuine Fano 3-folds.

Proposition 3.15 ([44, Theorem 2.4.5])**.** *The assumption holds for all Fano 3-folds except*

(i) *the product of a degree 1 del Pezzo surface (blow-up of $\mathbb{P}^2$ in 8 points) with $\mathbb{P}^1$, which has Picard rank 10, and*

(ii) *the blow-up of a degree 1 del Pezzo 3-fold in the intersection of two hyperplane divisors (number 1 in the Mori-Mukai list of Fano 3-folds of Picard rank 2 [55, Table 2]).*

Under the assumption above a generic pencil in $|-K_Y|$ has a base locus which is a smooth curve (of genus $g = g(Y)$). Hence from Y we can construct a smooth projective 3-fold Z fibred over $\mathbb{P}^1$ by (generically) smooth anticanonical K3 fibres by blowing up the base locus of a generic pencil $|S_0, S_\infty| \subset |-K_Y|$: see [21, Proposition 4.24]. Therefore by Theorem 3.4 we can construct ACyl Calabi–Yau structures on $V := Z \setminus S$.

However, for the purposes of this paper it is convenient to restrict to ACyl Calabi–Yau structures obtained from a subclass of weak Fano 3-folds which we call *semi-Fano* 3-folds. There is still a large number of deformation families of semi-Fano 3-folds.

Definition 3.16. Let Y be a weak Fano 3-fold and $\varphi\colon Y \to X$ its anticanonical morphism. If φ is *semi-small*, we call Y a *semi-Fano* 3-fold, *ie* the anticanonical morphism $\varphi\colon Y \to X$ can contract divisors to curves, or curves to points, but not divisors to points.

From any semi-Fano 3-fold Y satisfying our assumption above we can obtain a building block.

Proposition 3.17 ([21, Props. 4.24 & 5.7])**.** *Let Y be a semi-Fano 3-fold with $H^3(Y)$ torsion-free, $|S_0, S_\infty| \subset |-K_Y|$ a generic pencil with (smooth) base locus C, $S \in |S_0, S_\infty|$ generic, and Z the blow-up of Y at C. Then S is a smooth K3 surface, its proper transform in Z is isomorphic to S, and (Z, S) is a building block in the sense of Definition 3.5. Furthermore*

(i) *the image N of $H^2(Z) \to H^2(S)$ equals that of $H^2(Y) \to H^2(S)$;*

(ii) $\mathrm{Amp}_Y \subseteq \mathrm{Amp}_Z$, *where* Amp_Y *and* Amp_Z *denote the* images *in $N_{\mathbb{R}} \subseteq H^{1,1}(S)$ of the Kähler cones of Y and Z;*

(iii) $H^2(Y) \to H^2(S)$ *is injective, and $K = 0$ in* (3.7)*.*

Definition 3.18. We will refer to a building block (Z, S) arising from Proposition 3.17 as a *building block of semi-Fano type*. By Theorem 3.4 we can obtain ACyl Calabi–Yau structures (ω, Ω) on $V := Z \setminus S$ and we call (V, ω, Ω) an ACyl Calabi–Yau 3-fold of *semi-Fano type*.

Remark 3.19. The significance of 3.17(ii) is that Theorem 3.4 ensures that exactly the classes in Amp_Z can be represented by the asymptotic limit of an ACyl Calabi–Yau Kähler form ω on V. Note that Amp_Y and Amp_Z are typically proper subcones of the Kähler cone of S, even when Y is Fano (*cf* Remark 6.16). We need to pay attention to this in the matching argument in §6.

Sometimes one can get different building blocks from the same semi-Fano by blowing up base loci of non-generic anticanonical pencils (*cf* Examples 7.8, 7.9, 7.11). In this case extra work is required both to check that the topological conditions of a building block are satisfied, and to apply the matching arguments from §6. To avoid ambiguity, the term semi-Fano type will always refer to blow-ups of generic pencils as in Proposition 3.17, and we will warn explicitly in the few cases where we use non-generic pencils.

Remark. We do not know any example of a semi-Fano 3-fold Y with torsion in $H^3(Y)$, but cannot in general prove $H^3(Y)$ is torsion-free: see [21, §5] for further remarks in this direction.

This assumption is used to prove that $H^3(Z)$ is torsion-free as required in Definition 3.5(iv). Note that this condition is only used in order to simplify the calculation of the full integral cohomology. Dropping it would not affect the more crucial matching arguments, but in the absence of known examples with torsion in $H^3(Z)$ we do not concern ourselves with this generality.

If Z is obtained—in the manner of Proposition 3.17—from a weak Fano Y which is not semi-Fano then the natural map $H^2(Y) \to H^2(S)$ cannot be injective, since the class of any contracted divisor lies in the kernel. It is also not clear that the map has to have primitive image; in particular Z might not be a building block in the sense of Definition 3.5 because property (iii) could fail.

By varying the choice of semi-Fano 3-fold Y within its deformation type $\mathcal{Y}$, the choice of generic pencil $|S_0, S_\infty| \subset |-K_Y|$ and the choice of a generic $S \in |S_0, S_\infty|$ we can obtain families of ACyl Calabi–Yau structures on the same smooth 6-manifold V. Varying the ACyl Calabi–Yau structure on V this way allows us to obtain different asymptotic hyper-Kähler K3 surfaces S.

This observation is crucial when we come to construct pairs of compatible ACyl Calabi–Yau 3-folds. Given a *fixed* pair of ACyl Calabi–Yau 3-folds $V_\pm$ in general it will not be possible to construct any hyper-Kähler rotation $\mathfrak{r}$ between the asymptotic K3 surfaces $S_\pm$. However, for ACyl Calabi–Yau 3-folds of semi-Fano type we will prove that in many cases it is possible to deform the pair of ACyl Calabi–Yau structures on the 3-folds $V_\pm$ as above, so that within these deformation families a compatible pair does exist. To achieve this it is important to understand which hyper-Kähler K3 surfaces can arise as the asymptotic K3 surface of ACyl Calabi–Yau 3-folds of semi-Fano type.

If Y is a semi-Fano 3-fold then Proposition 3.17(i) shows that the polarising lattice N of a semi-Fano type block obtained from Y is isomorphic to $H^2(Y)$. Therefore the asymptotic K3 surface S of an ACyl Calabi–Yau 3-fold of semi-Fano type obtained from any deformation of Y has a primitive sublattice isomorphic to $\operatorname{Pic} Y = H^2(Y)$ in $\operatorname{Pic} S$.

So $\operatorname{rk}\operatorname{Pic} S \geq b^2(Y)$, whereas a generic (projective) K3 surface S has $\operatorname{rk}\operatorname{Pic} S = 1$. In other words, the larger $b^2(Y)$ is the more special the K3 surfaces that can arise as asymptotic K3 surfaces obtained from a fixed deformation type $\mathcal{Y}$ of semi-Fanos via Proposition 3.17. The moduli theory of K3 surfaces whose Picard group contains a given sublattice N—so-called *lattice polarised K3 surfaces*—is well-understood and was reviewed in our previous paper [21]. We will need to know that the generic lattice polarised K3 surface of a given type occurs as the asymptotic hyper-Kähler K3 surface of some ACyl Calabi–Yau structure obtained from a given deformation type $\mathcal{Y}$ of semi-Fano 3-folds via Proposition 3.17. The proof of this fact relies on semi-Fano 3-folds enjoying a better deformation theory than general weak Fano 3-folds: see [21, §6]. The improved deformation theory uses the stronger cohomology vanishing theorems available for semi-Fano 3-folds.

We will explain the above more precisely when we explain how to construct compatible pairs of ACyl Calabi–Yau 3-folds $V_\pm$ by orthogonal matching in Section 6.

Remark 3.20. Another kind of building blocks was defined by Kovalev and Lee [49] from K3 surfaces S with non-symplectic involution, *ie* with an involution τ acting as -1 on $H^{2,0}(S)$. In Nikulin's classification, 1 of the 75 families of non-symplectic involutions acts freely. In the other 74 cases, resolving the singular set of $(S \times \mathbb{P}^1)/(\tau, -1)$ by blow-up defines a simply-connected building block Z, which we say is of *non-symplectic type*.

The polarising lattice of Z is the τ-invariant part N of $H^2(S)$. N characterises τ in the sense that a generic N-polarised K3 admits an equivalent involution. The matching arguments we will use for families of semi-Fano blocks can therefore also be used for families of non-symplectic blocks. The image $\mathrm{Amp}_Z \subset H^{1,1}(S)$ of the Kähler cone of Z is the full Kähler cone of S [49, Prop. 4.1]. $\mathrm{rk}\, K$ (as defined in (3.7)) is twice the number of fixed components of τ, so at least 2 [49, (4.3)].

Semi-Fano 3-folds from nodal Fano 3-folds. While our general theory will allow us to find compatible pairs of ACyl Calabi–Yau 3-folds of semi-Fano type, most of the specific semi-Fano 3-folds we use to build concrete G_2–manifolds in this paper satisfy additional properties which we now describe.

An important special class of semi-Fano 3-folds are those for which the anticanonical morphism $\varphi\colon Y \to X$ is not just semi-small but *small*, *ie* contracts only finitely many curves. A special case—and for this paper by far the most important case—is when X is a *nodal Fano* 3-fold, *ie* X has only finitely many singular points each (locally analytically) equivalent to the 3-fold ordinary double point. In this case any small resolution Y of X replaces each node with a smooth rational curve $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Most of the semi-Fano 3-folds Y we consider in detail in this paper will arise from projective small resolutions of nodal Fano 3-folds.

Remark. When the semi-Fano Y arises as a projective small resolution $\varphi\colon Y \to X$ of a nodal Fano X then each exceptional curve C of φ gives rise to a *compact* rigid curve C in the ACyl Calabi–Yau 3-fold $V = Z \setminus S$ constructed from Y using Proposition 3.17. These compact rigid curves in V will allow us to construct compact rigid associative 3-folds in twisted connected sum G_2–manifolds built using V.

Whenever the anticanonical morphism φ of a semi-Fano 3-fold Y is small we have the following additional features:

(i) The anticanonical model X is a Fano 3-fold with Gorenstein *terminal* (and therefore isolated) singularities.
(ii) The small projective morphism $\varphi\colon Y \to X$ can be *flopped*. Flopping yields other smooth semi-Fano 3-folds Y' with the same anticanonical model X and whose anticanonical morphism $\varphi'\colon Y' \to X$ is also small.
(iii) X is smoothable by a flat deformation and hence is a degeneration of a nonsingular Fano 3-fold X_t. In particular, the Picard ranks and the Fano indices of X and X_t are equal.

The topologies of the smooth 3-folds Y and X_t and the singular 3-fold X are closely related. The following is explained in much greater detail in our previous paper [21]. In the current paper we will need some of the facts below in our discussion of G_2*–transitions* in Section 8 but not elsewhere in the paper.

Since X is singular in general it need not satisfy Poincaré duality. One way to define the *defect* of X is as the following measure of failure of Poincaré duality on X,

$$\sigma(X) := \mathrm{rk}\, H_4(X) - \mathrm{rk}\, H^2(X). \tag{3.21}$$

The existence of a *projective* small resolution $\varphi\colon Y \to X$ can be shown to force the defect $\sigma(X)$ to be positive. Also for any small resolution $\varphi\colon Y \to X$ we have

$$b^2(Y) = b^2(X) + \sigma(X) = b^2(X_t) + \sigma(X). \tag{3.22}$$

In particular, if we start from a smooth Fano X_t, degenerate to the singular Fano X and then resolve to obtain the smooth semi-Fano 3-fold Y then necessarily $b^2(Y) > b^2(X_t)$. For instance, if Y is a small resolution of a generic quartic containing a plane Π then one can show that $\sigma(X) = 1$, $b^2(X_t) = b^2(X) = 1$ and hence $b^2(Y) = 2$. Therefore the asymptotic K3 surfaces in ACyl Calabi–Yau 3-folds of semi-Fano type Y are more special than those in ACyl

Calabi–Yau 3-folds of Fano type X_t. One can interpret this as saying that finding an ACyl Calabi–Yau 3-fold compatible with an ACyl Calabi–Yau 3-fold of semi-Fano type Y should be harder than finding one compatible with an ACyl Calabi–Yau 3-fold of Fano type X_t.

If X is a nodal 3-fold with e nodes and defect σ one can show that the third Betti numbers b^3 of Y, X and X_t are related as follows

$$b^3(X) = b^3(X_t) + \sigma - e, \qquad b^3(Y) = b^3(X_t) - 2e + 2\sigma. \tag{3.23}$$

Since one always has $\sigma \leq e$ the second equation shows that $b^3(Y) \leq b^3(X_t)$.

To summarise, in passing from the smooth Fano X_t to the smooth semi-Fano Y b^2 must increase whereas b^3 typically decreases. We will discuss the significance of these facts for G_2–manifolds arising as twisted connected sums of ACyl Calabi–Yau 3-folds of semi-Fano type in Section 8.

4. Topology of the G_2–manifolds

In this section, we collect some tools to compute topological invariants of G_2–manifolds that are obtained by gluing asymptotically cylindrical Calabi-Yaus. All homology and cohomology groups in this section are over $\mathbb{Z}$ unless explicitly stated otherwise. Theorem 4.8 computes the integral cohomology groups of our twisted connected sum G_2–manifolds and proves under our assumptions that they are all simply-connected. In general there can be torsion in $H^3(M)$ and $H^4(M)$. We review the almost-diffeomorphism classification of closed 2-connected 7-manifolds (including cases in which almost-diffeomorphism can be replaced with diffeomorphism), as it applies in particular to 7-manifolds M that are 2-connected with torsion-free $H^4(M)$. Lemma 4.27 gives a simple sufficient condition for a twisted connected sum G_2–manifold M to have that property. A key role is played by the divisibility of the first Pontrjagin class $p_1(M)$, and Proposition 4.20 relates this to the divisibility of the second Chern class c_2 of our building blocks, as studied in [21].

Cohomology of the building blocks. Here we recall notation and some computations of cohomology groups from [21, §5]. First recall the definition of a building block from 3.5. We denoted there by N the image of $H^2(Z) \to H^2(S) = L$. We regard N as a lattice with the quadratic form inherited from L. In examples, N is almost never unimodular, so the natural inclusion $N \hookrightarrow N^*$ is not an isomorphism. We write

$$T = N^\perp = \{l \in L \mid <l, n> = 0 \ \ \forall\, n \in N\}.$$

(T stands for "transcendental"; in examples, N and T are the Picard and transcendental lattices of a lattice polarized K3 surface.) By hypothesis 3.5(iii) N is primitive, and because L is unimodular we find $L/T \simeq N^*$.

Let $V = Z \setminus S$, and recall from (3.7) that K denotes the kernel of the natural restriction map

$$\rho\colon H^2(V) \to L.$$

It follows from (ii) of the following lemma that the image of ρ equals N.

Lemma 4.1 ([21, Lemma 5.3])**.** *Let $f\colon Z \to \mathbb{P}^1$ be a building block. Then:*

(i) $\pi_1(V) = 0$ *and* $H^1(V) = 0$*;*

(ii) *the class* $[S] \in H^2(Z)$ *fits in a split exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{[S]} H^2(Z) \to H^2(V) \to 0,$$

hence $H^2(Z) \simeq \mathbb{Z}[S] \oplus H^2(V)$*, and the restriction homomorphism* $H^2(Z) \to L$ *factors through* $\rho\colon H^2(V) \to L$*;*

(iii) *there is a split exact sequence*

$$0 \to H^3(Z) \to H^3(V) \to T \to 0,$$

hence $H^3(V) \simeq H^3(Z) \oplus T$;

(iv) *there is a split exact sequence*

$$0 \to N^* \to H^4(Z) \to H^4(V) \to 0,$$

hence $H^4(Z) \simeq H^4(V) \oplus N^*$;

(v) $H^5(V) = 0$.

In particular, $H^*(V)$ *is torsion-free.*

Corollary 4.2 ([21, Corollary 5.4])**.** *Let* $f\colon Z \to \mathbb{P}^1$ *be a building block. Since the normal bundle of* S *in* Z *is trivial, we get a natural inclusion of* $S \times \mathbb{S}^1 \subset V$. *Denote by* $\mathbf{a}^0 \in H^0(\mathbb{S}^1)$, $\mathbf{a}^1 \in H^1(\mathbb{S}^1)$ *the standard generators. The natural restriction homomorphisms:*

$$\beta^m\colon H^m(V) \to H^m(S \times \mathbb{S}^1) = \mathbf{a}^0 H^m(S) \oplus \mathbf{a}^1 H^{m-1}(S)$$

are computed as follows:

(i) $\beta^1 = 0$;

(ii) $\beta^2\colon H^2(V) \to H^2(S \times \mathbb{S}^1) = \mathbf{a}^0 H^2(S)$ *is the homomorphism* $\rho\colon H^2(V) \to L$;

(iii) $\beta^3\colon H^3(V) \to H^3(S \times \mathbb{S}^1) = \mathbf{a}^1 H^2(S)$ *is the composition* $H^3(V) \twoheadrightarrow T \subset L$;

(iv) *the natural surjective restriction homomorphism* $H^4(Z) \to H^4(S) = \mathbb{Z}$ *factors through* $\beta^4\colon H^4(V) \to H^4(S \times \mathbb{S}^1) = \mathbf{a}^0 H^4(S) = \mathbb{Z}$, *and there is a split exact sequence:*

$$0 \to K^* \to H^4(V) \xrightarrow{\beta^4} H^4(S) \to 0.$$

Lemma 4.1 and Corollary 4.2 are closely related to the long exact sequences for cohomology of Z relative to S and V relative to its boundary $\mathbb{S}^1 \times S$, respectively.

$$H^k_{cpt}(V) \to H^k(Z) \to H^k(S) \to H^{k+1}_{cpt}(V) \tag{4.3}$$

$$H^k_{cpt}(V) \overset{j^k}{\to} H^k(V) \overset{\beta^k}{\to} H^k(\mathbb{S}^1 \times S) \overset{\partial}{\to} H^{k+1}_{cpt}(V) \tag{4.4}$$

In particular note that $H^4_{cpt}(V) \hookrightarrow H^4(Z)$. Also $H^4_{cpt}(V) \cong N^* \oplus K^*$, where the term $N^* \cong L/T$ is the image of $H^3(\mathbb{S}^1 \times S)$ under ∂. Its image in $H^4(Z)$ is precisely the N^* appearing in 4.1(iv).

Cohomology of the 7-manifolds. We are interested in smooth 7-manifolds M constructed as follows. Start with two building blocks (Z_+, S_+), (Z_-, S_-) and a hyper-Kähler rotation $\mathfrak{r}\colon S_+ \to S_-$. Let $\mathbb{S}(S_\pm) = S_\pm \times \mathbb{S}^1_\pm \subset V_\pm$ denote the unit normal bundles of $S_\pm$ in $Z_\pm$. We glue $M_+ = V_+ \times \mathbb{S}^1_-$ with $M_- = V_- \times \mathbb{S}^1_+$ identifying the ends via the diffeomorphism of $\mathbb{S}(S_+) \times \mathbb{S}^1_- = S_+ \times \mathbb{T}^2$ with $\mathbb{S}(S_-) \times \mathbb{S}^1_+ = S_- \times \mathbb{T}^2$ that identifies S_+ with S_- by the hyper-Kähler rotation $\mathfrak{r}$ and exchanges the two factors of $\mathbb{T}^2$ (see (3.11)). For the purposes of this section $\mathfrak{r}$ is fixed and, using this identification, we let S denote $S_+ = S_-$.

We now compute the cohomology groups of M in terms of the cohomology groups of $Z_\pm$, the restrictions $\rho_\pm\colon H^2(V_\pm) \to L$, their kernels $K_\pm$ and their images $N_\pm \subset L$, which are primitive sublattices by assumption. Our main tool is the Mayer-Vietoris exact sequence for the decomposition $M = M_+ \cup M_-$ along the common intersection $W = S \times \mathbb{S}^1_+ \times \mathbb{S}^1_-$:

$$H^{m-1}(M_+) \oplus H^{m-1}(M_-) \to H^{m-1}(W) \overset{\delta}{\to} H^m(M) \overset{\rho^m}{\to} H^m(M_+) \oplus H^m(M_-) \overset{\gamma^m}{\to} H^m(W) \tag{4.5}$$

We write $\gamma^m = \gamma^m_+ \oplus \gamma^m_-\colon H^m(M_+) \oplus H^m(M_-) \to H^m(W)$.

Lemma 4.1 implies that $H^m(M_\pm)$, thus $\mathrm{Im}(\rho^m)$, is torsion-free. Sequence (4.5) thus yields isomorphisms

$$H^m(M) \simeq \mathrm{Im}(\rho^m) \oplus \ker(\rho^m) \simeq \ker(\gamma^m) \oplus \mathrm{coker}(\gamma^{m-1}). \tag{4.6}$$

The key task is to describe the homomorphisms γ^m in terms of $\beta^m_\pm\colon H^m(V_\pm) \to H^m(\mathbb{S}^1 \times S_\pm)$ and $\rho_\pm\colon H^2(V_\pm) \to L$.

Lemma 4.7. *Let $Z_\pm \to \mathbb{P}^1$ be building blocks; let $M_\pm$ and M be as above. We use the self-explanatory notation:*

$$H^m(M_+) = \mathbf{a}^0_- H^m(V_+) \oplus \mathbf{a}^1_- H^{m-1}(V_+)$$
$$H^m(M_-) = \mathbf{a}^0_+ H^m(V_-) \oplus \mathbf{a}^1_+ H^{m-1}(V_-)$$

and

$$H^m(W) = \mathbf{a}^0_+\mathbf{a}^0_- H^m(S) \oplus \mathbf{a}^1_+\mathbf{a}^0_- H^{m-1}(S) \oplus \mathbf{a}^0_+\mathbf{a}^1_- H^{m-1}(S) \oplus \mathbf{a}^1_+\mathbf{a}^1_- H^{m-2}(S).$$

The homomorphisms $\gamma^m\colon H^m(M_+) \oplus H^m(M_-) \to H^m(W)$ that occur in the Mayer-Vietoris sequence are computed as follows:

(i) $H^1(M_+) \oplus H^1(M_-) = \mathbf{a}^1_- H^0(V_+) \oplus \mathbf{a}^1_+ H^0(V_-)$,
$H^1(W) = \mathbf{a}^0_+\mathbf{a}^1_- H^0(S) \oplus \mathbf{a}^1_+\mathbf{a}^0_- H^0(S)$, *and*
$$\gamma^1 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} : H^0(V_+) \oplus H^0(V_-) \to H^0(S) \oplus H^0(S)$$
is the natural isomorphism.

(ii) $H^2(M_+) \oplus H^2(M_-) = \mathbf{a}^0_- H^2(V_+) \oplus \mathbf{a}^0_+ H^2(V_-)$,
$H^2(W) = \mathbf{a}^0_+\mathbf{a}^0_- H^2(S) \oplus \mathbf{a}^1_+\mathbf{a}^1_- H^0(S) = L \oplus \mathbb{Z}[S]$, *and*
$$\gamma^2 = \begin{pmatrix} \rho_+ & \rho_- \\ 0 & 0 \end{pmatrix} : H^2(V_+) \oplus H^2(V_-) \to L \oplus \mathbb{Z}[S].$$

(iii) $H^3(M_+) \oplus H^3(M_-) = \mathbf{a}^0_- H^3(V_+) \oplus \mathbf{a}^1_- H^2(V_+) \oplus \mathbf{a}^0_+ H^3(V_-) \oplus \mathbf{a}^1_+ H^2(V_-)$,
$H^3(W) = \mathbf{a}^1_+\mathbf{a}^0_- H^2(S) \oplus \mathbf{a}^0_+\mathbf{a}^1_- H^2(S)$, *and*
$$\gamma^3 = \begin{pmatrix} \beta^3_+ & 0 & 0 & \rho_- \\ 0 & \rho_+ & \beta^3_- & 0 \end{pmatrix} : H^3(V_+) \oplus H^2(V_+) \oplus H^3(V_-) \oplus H^2(V_-) \to L \oplus L;$$

(iv) $H^4(M_+) \oplus H^4(M_-) = \mathbf{a}^0_- H^4(V_+) \oplus \mathbf{a}^1_- H^3(V_+) \oplus \mathbf{a}^0_+ H^4(V_-) \oplus \mathbf{a}^1_+ H^3(V_-)$,
$H^4(W) = \mathbf{a}^0_+\mathbf{a}^0_- H^4(S) \oplus \mathbf{a}^1_+\mathbf{a}^1_- H^2(S) = H^4(S) \oplus L$, *and*
$$\gamma^4 = \begin{pmatrix} \beta^4_+ & 0 & \beta^4_- & 0 \\ 0 & \beta^3_+ & 0 & \beta^3_- \end{pmatrix} : H^4(V_+) \oplus H^3(V_+) \oplus H^4(V_-) \oplus H^3(V_-) \to H^4(S) \oplus L.$$

Proof. This is an elementary application of the Künneth formula once all the notation has been unravelled. □

Theorem 4.8.

(i) $\pi_1(M) = 0$ *and* $H^1(M) = 0$;
(ii) $H^2(M) = \ker\Big[H^2(V_+) \oplus H^2(V_-) \to N_+ + N_-\Big] \cong H^2(M) \simeq (N_+ \cap N_-) \oplus K_+ \oplus K_-$;
(iii) $H^3(M) \simeq \mathbb{Z}[S] \oplus (L/_{N_+ + N_-}) \oplus (N_- \cap T_+) \oplus (N_+ \cap T_-) \oplus H^3(Z_+) \oplus H^3(Z_-) \oplus K_+ \oplus K_-$;
(iv) $H^4(M) \simeq H^4(S) \oplus (T_+ \cap T_-) \oplus (L/_{N_- + T_+}) \oplus (L/_{N_+ + T_-}) \oplus H^3(Z_+) \oplus H^3(Z_-) \oplus K^*_+ \oplus K^*_-$.

Proof. Since $\pi_1(V_\pm) = 0$, the van Kampen theorem for the decomposition $M = M_+ \cup M_-$ along the common intersection $W = S \times \mathbb{T}^2$ immediately implies that $\pi_1(M) = 0$.

We know that γ^0 is surjective and γ^1 injective, hence (i). Since γ^1 is surjective, $H^2(M) = \ker(\gamma^2) = \ker\big(H^2(V_+) \oplus H^2(V_-) \to N_+ + N_-\big)$. Thus, we have an exact sequence:

$$0 \to K_+ \oplus K_- \to H^2(M) \to N_+ \cap N_- \to 0,$$

which is split since $N_+ \cap N_-$ is torsion-free (ii). To show (iii) note first that, from the description of γ^2, it is clear that

$$\operatorname{coker}(\gamma^2) = \mathbb{Z}[S] \oplus (L/_{N_+ + N_-}).$$

Now $\ker(\gamma^3)$ is a direct sum of two pieces

$$\ker_\pm = \ker\Big[\begin{pmatrix}\beta^3_\pm & \rho_\mp\end{pmatrix} : H^3(V_\pm) \oplus H_\mp \to L\Big].$$

Each of these kernels is computed by a split exact sequence:

$$0 \to H^3(Z_\pm) \oplus K_\mp \to \ker_\pm \to N_\mp \cap T_\pm \to 0$$

and (iii) follows from (4.6). To show (iv) note first that, from the description of γ^3, it is clear that

$$\operatorname{coker}(\gamma^3) = (L/_{N_+ + T_-}) \oplus (L/_{N_- + T_+}). \tag{4.9}$$

Now $\ker(\gamma^4)$ is the direct sum of two pieces

$$\begin{aligned} &\ker\Big[\begin{pmatrix}\beta^4_+ & \beta^4_-\end{pmatrix} : H^4(V_+) \oplus H^4(V_-) \to H^4(S)\Big] \\ \oplus &\ker\Big[\begin{pmatrix}\beta^3_+ & \beta^3_-\end{pmatrix} : H^3(V_+) \oplus H^3(V_-) \to L\Big]. \end{aligned} \tag{4.10}$$

The first of these kernels is isomorphic to $H^4(S) \oplus K^*_+ \oplus K^*_-$; the second is isomorphic to $(T_+ \cap T_-) \oplus H^3(Z_+) \oplus H^3(Z_-)$, and (iv) again follows from (4.6). $\square$

From Theorem 4.8 we can immediately identify the torsion part of the cohomology.

Corollary 4.11.
(i) $\operatorname{Tor} H^3(M) \simeq \operatorname{Tor}(L/_{N_+ + N_-})$;
(ii) $\operatorname{Tor} H^4(M) \simeq \operatorname{Tor}(L/_{N_- + T_+}) \oplus \operatorname{Tor}(L/_{N_+ + T_-})$.

Remark. If $H^3(Z)$ is not torsion-free, then Corollary 4.2 remains true, except that $0 \to \bar{K} \to H^4(V) \xrightarrow{\beta^4} H^4(S) \to 0$, with natural isomorphisms $K \cong \operatorname{Hom}(\bar{K}, \mathbb{Z})$ and $\operatorname{Tor} \bar{K} \cong \operatorname{Tor} H^4(Z)$. Theorem 4.8 remains true too except that appearances of $K^*_\pm$ should be replaced by $\bar{K}_\pm$ (but proving that the short exact sequences used in the proof split becomes a bit more complicated).

Remark 4.12. A further cohomological invariant one can assign to a closed odd-dimensional manifold M^{2n-1} is its *torsion-linking form*, which is a non-degenerate finite bilinear form $\operatorname{Tor} H^n(M) \times \operatorname{Tor} H^n(M) \to \mathbb{Q}/\mathbb{Z}$ (symmetric when n is even). One can show that the two summands in 4.11(ii) are isotropic with respect to the torsion-linking form. This implies that for manifolds M^7 of this twisted connected sum type, the *isomorphism class* of the torsion-linking form is determined by the isomorphism class of $H^4(M)$. The vast majority of the twisted connected G$_2$–manifolds that we construct in this paper will have torsion-free cohomology and therefore the torsion-linking form plays essentially no role in this paper. For this reason we omit the proof of the facts stated above.

Gluing classes in $H^4(Z_\pm)$*.* The Mayer–Vietoris theorem says that if we try to glue a pair of classes in $H^4(M_+)$ and $H^4(M_-)$ having the same image in $H^4(W)$ to a class in $H^4(M)$ then there is an ambiguity given by the image of $\delta : H^3(W) \to H^4(M)$. However, in this particular construction there *is* an unambiguous way to glue a pair of classes in $H^4(Z_+)$ and $H^4(Z_-)$, which will be important for describing the characteristic classes of M. Define

$$H^4(Z_+) \oplus_0 H^4(Z_-) \;=\; \left\{([\alpha_+],[\alpha_-]) \in H^4(Z_+) \oplus H^4(Z_-) : [\alpha_+]_{|S} = [\alpha_-]_{|S} \in H^4(S)\right\}.$$

Definition 4.13. We define a homomorphism

$$Y : H^4(Z_+) \oplus_0 H^4(Z_-) \to H^4(M)$$

as follows. Recall that $S = f_\pm^{-1}(\infty)$ for a fibration $f_\pm \colon Z_\pm \to \mathbb{P}^1$. Let $\Delta \subset \mathbb{P}^1$ be a trivialising neighbourhood of ∞ for $f_\pm$, and let $U_\pm = f_\pm^{-1}(\Delta) \cong \Delta \times S \subset Z_\pm$. ($\Delta \backslash \{0\} \times S$ correspond to the cylindrical ends $\mathbb{R}^+ \times \mathbb{S}^1 \times S$ of $V_\pm$, mapping $\mathbb{R}^+ \times \mathbb{S}^1 \to \Delta \backslash \{0\}$ by $(t,\vartheta) \mapsto z = e^{-t-i\vartheta}$). Let $p_\pm : U_\pm \to S$ be the projection for the local trivialisation. For $([\alpha_+],[\alpha_-]) \in H^4(Z_+) \oplus_0 H^4(Z_-)$, let $[\beta]$ be their common image in $H^4(S)$. Then we may choose the cocycles $\alpha_\pm \in C^4(Z_\pm;\mathbb{Z})$ so that the restriction of $\alpha_\pm$ to $U_\pm$ equals $p_\pm^*\beta$. The pull-backs of $\alpha_\pm$ to $\mathbb{S}^1 \times V_\pm$ have the same restriction to the cylindrical end, and patch to a cocycle on M. We set $Y([\alpha_+],[\alpha_-])$ to be the class represented by that cocycle.

Let $N'_\mp$ be the image of $N_\mp$ in $N_\pm^* = L/T_\pm$. Recall from Lemma 4.1(iv) that $N_\pm^* \hookrightarrow H^4(Z_\pm)$. The image lies in the kernel of restriction to $V_\pm$ and hence also of restriction to S, so $N_\pm^* \hookrightarrow H^4(Z_+) \oplus_0 H^4(Z_-)$.

Lemma 4.14. *$Y : H^4(Z_+) \oplus_0 H^4(Z_-) \to H^4(M)$ maps onto the terms*

$$H^4(S) \oplus (L/_{N_-+T_+}) \oplus (L/_{N_++T_-}) \oplus K_+^* \oplus K_-^* \subseteq H^4(M).$$

in the expression 4.8(iv) *for $H^4(M)$, with kernel $N'_+ \oplus N'_-$.*

Proof. It follows from (4.3) that $0 \to H^4_{cpt}(V_\pm) \to H^4(Z_\pm) \to H^4(S) \to 0$ is split exact. Hence so is

$$0 \to H^4_{cpt}(V_+) \oplus H^4_{cpt}(V_-) \to H^4(Z_+) \oplus_0 H^4(Z_-) \to H^4(S) \to 0.$$

If $[\alpha] \in H^4_{cpt}(V_\pm)$ then $\mathbf{a}^0_\mp[\alpha] \in H^4_{cpt}(M_\pm)$ can be pushed forward to a class in $H^4(M)$. Denoting this map by $i_\pm : H^4_{cpt}(V_\pm) \to H^4(M)$, we obtain a commutative diagram

$$\begin{tikzcd}
H^3(\mathbb{S}^1 \times S) \oplus H^3(\mathbb{S}^1 \times S) \arrow[r, "\partial_+ \oplus \partial_-"] \arrow[dd, "\cong"'] & H^4_{cpt}(V_+) \oplus H^4_{cpt}(V_-) \arrow[rr, "j^4_+ \oplus j^4_-"] \arrow[dd, "i_+ + i_-"'] \arrow[dr, hook] & & H^4(V_+) \oplus H^4(V_-) \arrow[dd, hook] \\
& & H^4(Z_+) \oplus_0 H^4(Z_-) \arrow[ur] \arrow[dl, "Y"'] & \\
H^3(\mathbb{T}^2 \times S) \arrow[r, "\delta"] & H^4(M) \arrow[rr, "\rho^4"] & & H^4(M_+) \oplus H^4(M_-)
\end{tikzcd}$$

where the top row is the direct sum of the sequences (4.4) of relative cohomology for V_+ and V_-, and the bottom row is the Mayer–Vietoris sequence (4.5). Recall from (4.4) that

$$H^4_{cpt}(V_\pm) \cong \ker j^4_\pm \oplus \operatorname{Im} j^4_\pm \cong N_\pm^* \oplus K_\pm^*.$$

We now claim that $i_\pm : H^4_{cpt}(V_\pm) \to H^4(M)$ maps onto the terms $L/_{N_\mp+T_\pm} \oplus K_\pm^*$ in 4.8(iv), with kernel $N'_\pm$.

In the proof of Theorem 4.8(iv) we decomposed $H^4(M) \cong \ker\rho^4 \oplus \operatorname{Im}\rho^4 \cong \operatorname{coker}\gamma^3 \oplus \ker\gamma^4$. Correspondingly $\operatorname{Im} i_\pm$ splits as a direct sum of $\operatorname{Im}(\rho^4 \circ i_\pm) \subseteq \operatorname{Im}(\rho^4)$ and $\ker(\rho^4_{|\operatorname{Im} i_\pm}) \subseteq \ker\rho^4$.

Since $\operatorname{Im} j^4_\pm = \ker \beta^4_\pm \subseteq H^4(V_\pm)$, we find that $\rho^4 \circ i_\pm : H^4_{cpt}(V_\pm) \to H^4(M_+) \oplus H^4(M_-)$ has image precisely $\ker \beta^4_\pm \subset \ker \gamma^4$, and kernel equal to $\ker j^4_\pm$.

Further $\ker j^4_\pm = \operatorname{Im} \partial_\pm$, so $\ker(\rho^4_{|\operatorname{Im} i_\pm})$ is the image of $i_\pm \circ \partial_\pm : H^3(\mathbb{S}^1 \times S) \to H^4(M)$. That map equals the restriction of $\delta : H^3(\mathbb{T}^2 \times S) \to H^4(M)$ to $\mathbf{a}^1_\pm \mathbf{a}^0_\mp H^2(S)$. In the expression (4.9) for $\operatorname{Im} \delta \cong \operatorname{coker}(\gamma^3)$, we can identify the term $L/_{N_\mp + T_\pm}$ as the image of $i_\pm \circ \partial_\pm$. Finally $\ker i_\pm \subseteq \operatorname{Im} \partial_\pm \cong N^*_\pm \cong L/T_\pm$ corresponds to $(N_\mp + T_\pm)/T_\pm \cong N'_\pm$.

From the claim it is now easy to see that $\rho^4 \circ Y$ maps onto the first term in (4.10), accounting for the terms $H^4(S) \oplus K^*_+ \oplus K^*_-$, while $\ker(\rho^4_{|\operatorname{Im} Y}) = \ker(\rho^4_{|\operatorname{Im} i_+}) \oplus \ker(\rho^4_{|\operatorname{Im} i_-})$. Meanwhile $\ker Y$ is the sum of the kernels of $i_\pm$. $\square$

Characteristic classes of twisted connected sums. We now consider how to determine the characteristic classes of a twisted connected sum in terms of related data on the building blocks $Z_\pm$. We begin with a summary of the characteristic classes of relevance for a closed oriented spin 7-manifold M.

Oriented characteristic classes. The characteristic classes of (the tangent bundle of) an oriented 7-manifold M are the Stiefel–Whitney classes $w_2(M), \dots, w_7(M)$ and the first Pontrjagin class $p_1(M)$. First we want to show that all the Stiefel–Whitney classes vanish for any oriented spin 7-manifold and hence that the only oriented characteristic class of interest for a G_2–manifold is $p_1(M)$. We will use some standard facts about characteristic classes, which can be found in Milnor–Stasheff [54]. First of all, for any vector bundle $E \to M$ the Stiefel–Whitney class $w_k(E) \in H^k(M; \mathbb{Z}/2\mathbb{Z})$ can be determined from $\{w_{2^i}(E) : 2^i \leq k\}$ using the Steenrod square operations $\operatorname{Sq}^k : H^i(M; \mathbb{Z}/2\mathbb{Z}) \to H^{i+k}(M; \mathbb{Z}/2\mathbb{Z})$ [54, page 94], *eg*

$$w_3 = \operatorname{Sq}^1 w_2 + w_1 w_2. \tag{4.15}$$

Hence all Stiefel–Whitney classes of an oriented rank 7 bundle are determined algebraically by w_2 and w_4. Further, Wu's formula [54, Theorem 11.14] expresses the Stiefel–Whitney classes of a closed n-dimensional manifold M as

$$w_k = \sum_{i=0}^{k} \operatorname{Sq}^{k-i} v_i, \tag{4.16}$$

where the Wu class $v_k(M)$ can be defined as the Poincaré dual to $\operatorname{Sq}^k \colon H^{n-k}(M; \mathbb{Z}/2\mathbb{Z}) \to H^n(M; \mathbb{Z}/2\mathbb{Z})$. Applying (4.16) recursively, we find for any closed oriented manifold that $v_1 = w_1 = 0$, $v_2 = w_2$, combining with (4.15) gives $v_3 = 0$, and

$$v_4 = w_4 + w_2^2 \tag{4.17}$$

since $\operatorname{Sq}^2 a = a^2$ for any $a \in H^2(M; \mathbb{Z}/2\mathbb{Z})$. Because Sq^k vanishes on $H^i(M; \mathbb{Z}/2\mathbb{Z})$ for $i < k$, Wu classes above the middle dimension always vanish (*cf* [54, page 132]), so $w_4 = w_2^2$ for any closed orientable 7-manifold M. If M is spin, then $w_2 = 0$, and hence all other Stiefel–Whitney classes vanish too.

Spin characteristic classes. The Stiefel–Whitney and Pontrjagin classes are all stable, *ie* they are invariant under addition of trivial bundles. The stable characteristic classes of an oriented vector bundle are pull-backs of elements of $H^*(\mathrm{BSO})$ under a classifying map $M \to \mathrm{BSO}$, where BSO is the classifying space for the stable special orthogonal group $\mathrm{SO} = \lim_{n\to\infty} \mathrm{SO}(n)$. If the vector bundle is spin then the classifying map can be lifted to BSpin, and we can possibly define further characteristic classes by considering $H^*(\mathrm{BSpin})$. BSO and BSpin have isomorphic cohomology groups over $\mathbb{Q}$ or mod p with p an odd prime, but over $\mathbb{Z}$ and mod 2 there is extra subtlety that was first studied by Thomas [71]. The lowest non-trivial cohomology group is $H^4(\mathrm{BSpin})$, and for our purposes it suffices to consider a generator

$$p_{1/2} \in H^4(\mathrm{BSpin}).$$

The following well-known lemma implies that if there is no 2-torsion in $H^4(M)$ then $p_{1/2}(M)$ is determined from the Pontrjagin class $p_1(M)$. Since we are mostly concerned with the case when $H^4(M)$ is torsion-free, for simplicity we choose to phrase our subsequent main discussion in terms of $p_1(M)$, addressing the refinements concerning $p_{1/2}(M)$ in supplementary remarks.

Lemma 4.18 (*cf* [71, (1.5),(1.6)], [15, Lemma 2.4])**.** *For any spin bundle, $p_1 = 2p_{1/2}$ and $w_4 = p_{1/2} \bmod 2$.*

Since as we explained above $w_4 = 0$ for any closed spin 7-manifold we deduce the following:

Corollary 4.19. *If M is a closed spin 7-manifold then $p_{1/2}(M)$ is even, and hence $p_1(M)$ is divisible by 4.*

Remark. $p_{1/2}(E)$ of a spin vector bundle is the primary obstruction to stable trivialisability of E: E is stably trivial over the 7-skeleton of the base if and only if $p_{1/2}(E) = 0$, *cf* [25, §2.1].

Computing p_1 of twisted connected sums. The restrictions $p_1(\mathbb{S}^1 \times V_\pm)$ of $p_1(M)$ to $\mathbb{S}^1 \times V_\pm$, *ie* $\rho^4(p_1(M))$ in the notation of (4.5), do not determine $p_1(M)$ since the Mayer-Vietoris boundary map $H^3(W) \to H^4(M)$ is non-trivial. Another point of view is that the isomorphism class of a vector bundle on M is not determined by the isomorphism classes of its restrictions to V_+ and V_-: it also depends on (the homotopy class of) the isomorphism one uses to glue the bundles together on the overlap. However, it turns out that we can determine $p_1(M)$ from $p_1(Z_\pm)$, using the map Y from Definition 4.13.

Recall that $p_1(Z) = -2c_2(Z) + c_1(Z)^2$ for any complex manifold Z. If Z is a building block then $c_1(Z)^2 = 0$, so $p_1(Z) = -2c_2(Z)$. The image of $c_2(Z_\pm)$ in $H^4(S)$ is $c_2(S)$, so in particular $(c_2(Z_+), c_2(Z_-)) \in H^4(Z_+) \oplus_0 H^4(Z_-)$, and $Y(c_2(Z_+), c_2(Z_-))$ is defined.

Proposition 4.20. *Let M be a twisted connected sum of the building blocks Z_+ and Z_-. Then*

$$p_1(M) = -2Y(c_2(Z_+), c_2(Z_-)).$$

Proof. We need to find a suitable cocycle representing $p_1(M)$. Let $E_k(\mathbb{R})$ be the tautological bundle over $\mathrm{BSO}(k) = \widetilde{Gr}_k(\mathbb{R}^\infty)$, the Grassmannian of oriented k-planes. A classifying map for TM is a map $g : M \to \widetilde{Gr}_7(\mathbb{R}^\infty)$ such that there is a vector bundle isomorphism $G : TM \to g^*E_7(\mathbb{R})$. By definition, there is a cocycle $\wp_1 \in C^4(\widetilde{Gr}_7(\mathbb{R}^\infty); \mathbb{Z})$ such that $p_1(M) = [g^*\wp_1]$ for any classifying map g.

Consider $Z_\pm$ as the union of $V_\pm = Z_\pm \setminus S$ and $U_\pm \cong \Delta \times S$, and define a complex vector bundle $R_\pm$ over $Z_\pm$ by gluing $TV_\pm$ and $TU_\pm$ as follows: on the overlap $\mathbb{R}^+ \times \mathbb{S}^1 \times S \cong \Delta^* \times S$, $(t, \vartheta) \mapsto z = x+iy = e^{-t-i\vartheta}$, map TS to TS by the identity, and $T(\mathbb{R}\times\mathbb{S}^1)$ to $T\Delta^*$ by $\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial x}$, $\frac{\partial}{\partial \vartheta} \mapsto \frac{\partial}{\partial y}$. Identifying a complex vector bundle with the $(1,0)$-part of its complexification, this is the complex linear isomorphism that maps $\frac{\partial}{\partial w} \mapsto \frac{\partial}{\partial z}$, where we let $\frac{\partial}{\partial w}$ denote the basis vector field $\frac{1}{2}(\frac{\partial}{\partial t} - i\frac{\partial}{\partial \vartheta})$ of $T^{1,0}(\mathbb{R} \times \mathbb{S}^1)$. In contrast, $TZ_\pm$ is formed by gluing $TV_\pm$ and $TU_\pm$ by the derivative of $(t, \vartheta) \mapsto z$, which maps $\frac{\partial}{\partial w} \mapsto -z\frac{\partial}{\partial z}$. For comparison, if we glue the trivial complex line bundle $\underline{\mathbb{C}}$ over $V_\pm$ to $\underline{\mathbb{C}}$ over $U_\pm$ by $u \mapsto -z^{-1}u$ over $\Delta^* \times S$, then the result is $[-S]$, the line bundle over $Z_\pm$ with divisor $-S$. Now

$$R_\pm \oplus \underline{\mathbb{C}} \;\cong\; TZ_\pm \oplus [-S],$$

because both bundles are the result of gluing $TV_\pm \oplus \underline{\mathbb{C}}$ to $TU_\pm \oplus \underline{\mathbb{C}}$ by homotopic maps; at $(z,p) \in \Delta^* \times S$, the difference of the gluing maps sends $(v,w,u) \in TS \oplus T\Delta \oplus \underline{\mathbb{C}}$ to $(v, zw, z^{-1}u)$, and any $\Delta^* \to SU(2)$ is homotopic to a constant since $SU(2)$ is simply-connected. Because p_1 is additive and $p_1([-S]) = [-S]^2 = 0$, we find

$$p_1(R_\pm) = p_1(Z_\pm) = -2c_2(Z_\pm).$$

Let $f : S \to \widetilde{Gr}_4(\mathbb{R}^\infty)$ be a classifying map for the (real) vector bundle TS, with an isomorphism $F : TS \to f^*E_4(\mathbb{R})$. Identifying $\mathbb{R}^2 \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$ embeds $\widetilde{Gr}_4(\mathbb{R}^\infty) \hookrightarrow \widetilde{Gr}_6(\mathbb{R}^\infty)$, so that the restriction of $E_6(\mathbb{R})$ to $\widetilde{Gr}_4(\mathbb{R}^\infty)$ splits as $\underline{\mathbb{R}}^2 \oplus E_4(\mathbb{R})$. Then $f^*E_6(\mathbb{R}) \cong \underline{\mathbb{R}}^2 \oplus f^*E_4(\mathbb{R})$.

As in Definition 4.13, let $p_\pm : U_\pm \to S$ denote the projection onto the second factor of $U_\pm = \Delta \times S$. Then $f \circ p_\pm : U_\pm \to \widetilde{Gr}_6(\mathbb{R}^\infty)$ is a classifying map for $TU_\pm$, and the isomorphism $TU_\pm \cong f_\pm^* E_6(\mathbb{R})$ can be taken to be $\mathrm{Id}_{\underline{\mathbb{R}}^2} \oplus F$, *ie* mapping the TS factor to $f^*E_4(\mathbb{R})$ by F, and the $T\Delta$ factor to $\underline{\mathbb{R}}^2$ by the identity map. Because $\widetilde{Gr}_6(\mathbb{R}^\infty)$ is the universal classifying space $BSO(6)$, there is no obstruction to extending $f \circ p_\pm$ to a classifying map $f_\pm : Z_\pm \to \widetilde{Gr}_6(\mathbb{R}^\infty)$ for $R_\pm$ with an isomorphism $F_\pm : R_\pm \cong f_\pm^* E_6(\mathbb{R})$ whose restriction over $U_\pm$ equals $\mathrm{Id}_{\underline{\mathbb{R}}^2} \oplus F$.

$$\begin{tikzcd}
U_+ \arrow[rr, hook] \arrow[d, "p_+"'] & & Z_+ \arrow[d, "f_+"] & V_+ \arrow[l, hook'] & \mathbb{S}^1 \times V_+ \arrow[l] \arrow[dl, "g_+"'] \arrow[d, hook] \\
S \arrow[r, "f"] & \widetilde{Gr}_4(\mathbb{R}^\infty) \arrow[r, hook] & \widetilde{Gr}_6(\mathbb{R}^\infty) \arrow[r, hook] & \widetilde{Gr}_7(\mathbb{R}^\infty) & M \arrow[l, "g"'] \\
U_- \arrow[rr, hook] \arrow[u, "p_-"] & & Z_- \arrow[u, "f_-"'] & V_- \arrow[l, hook'] & \mathbb{S}^1 \times V_- \arrow[l] \arrow[ul, "g_-"] \arrow[u, hook']
\end{tikzcd}$$

Define $g_\pm : \mathbb{S}^1 \times V_\pm \to \widetilde{Gr}_7(\mathbb{R}^\infty)$ as the composition

$$\mathbb{S}^1 \times V_\pm \to V_\pm \to Z_\pm \xrightarrow{f_\pm} \widetilde{Gr}_6(\mathbb{R}^\infty) \to \widetilde{Gr}_7(\mathbb{R}^\infty),$$

and $g : M \to \widetilde{Gr}_7(\mathbb{R}^\infty)$ by patching $g_\pm$; this is possible because on the neck region $\mathbb{R} \times \mathbb{T}^2 \times S$, both g_+ and g_- equal the composition of f with projection to S.

Define $G_\pm : T(\mathbb{S}^1 \times V_\pm) \to g_\pm^* E_7(\mathbb{R}) \cong \underline{\mathbb{R}} \oplus g_\pm^* E_6(\mathbb{R})$ by $(u_3 \frac{\partial}{\partial\theta}, v) \mapsto (u_3, F_\pm(v))$ for $u_3 \in \mathbb{R}$, $v \in TV_\pm$. The gluing map in the definition of $R_\pm$ has been chosen precisely to ensure that the restriction of $G_\pm$ over the cylindrical end $\mathbb{R}^+ \times \mathbb{T}^2 \times S$ is translation-invariant:

$$\begin{gathered}
T(\mathbb{R}^+ \times \mathbb{T}^2) \times TS \to \underline{\mathbb{R}}^3 \oplus f^*E_4(\mathbb{R}), \\
(u_1 \tfrac{\partial}{\partial t} + u_2 \tfrac{\partial}{\partial \vartheta} + u_3 \tfrac{\partial}{\partial\theta}, v) \mapsto (u_1, u_2, u_3, F(v)).
\end{gathered}$$

Recall that in this section we have identified $S_+ = S_-$ in order to treat the hyper-Kähler rotation $\mathfrak{r} : S_+ \to S_-$ as Id_S. Therefore the gluing map (3.11) used to define M takes the form

$$\begin{gathered}
\mathbb{R} \times \mathbb{T}^2 \times S \to \mathbb{R} \times \mathbb{T}^2 \times S, \\
(t, \vartheta, \theta, x) \mapsto (2T{+}1{-}t, \theta, \vartheta, x),
\end{gathered}$$

and on the neck region $G_- \circ G_+^{-1} : (u_1, u_2, u_3, v) \mapsto (-u_1, u_3, u_2, v)$, *ie* it is just a *constant* rotation of the $\underline{\mathbb{R}}^3$ factor in $\underline{\mathbb{R}}^3 \oplus f^*E_4(\mathbb{R})$. By picking a path from this rotation to the identity we can interpolate between G_+ and G_- to define an isomorphism $G : TM \to g^*E_7(\mathbb{R})$. Hence g is a classifying map for TM, and

$$p_1(M) = [g^*\wp_1] = Y([f_+^*\wp_1], [f_-^*\wp_1]) = Y(p_1(Z_+), p_1(Z_-)). \qquad \square$$

Remark 4.21. Note that $c_1(R_\pm) = 0$; indeed the gluing map in the construction matches the non-vanishing complex 3-forms $\Omega_\pm$ and $dz \wedge (\omega^J_\pm + i\omega^K_\pm)$ over $V_\pm$ and $U_\pm$. In particular $R_\pm$ is a spin bundle, and its spin characteristic class $p_{1/2}(R_\pm)$ equals $-c_2(R_\pm) = -c_2(Z_\pm)$. Carrying out the proof of Proposition 4.20 using classifying maps to BSpin(k) instead of $\widetilde{Gr}_k(\mathbb{R}^\infty)$ therefore proves the more refined statement that $p_{1/2}(M) = -Y(c_2(Z_+), c_2(Z_-))$.

Remark. If we work with real coefficients then the relation $p_1(M) = Y(p_1(Z_+), p_1(Z_-))$ is more conveniently proved using Chern–Weil theory. It is clear how to define Y as a map on de Rham cohomology $H^4_{dR}(Z_+) \oplus_0 H^4_{dR}(Z_-) \to H^4_{dR}(M)$. For a Riemannian metric g on M, a certain quadratic polynomial function of the curvature of g defines a differential form $p_1(g) \in \Omega^4(M)$ representing $p_1(M) \in H^4_{dR}(M)$.

Let g_S be a metric on S, and $g_\pm$ a metric on $V_\pm$ that equals $dt^2 + d\vartheta^2 + g_S$ on the cylindrical end. Let $g'_\pm$ be a metric on $Z_\pm$ that equals $g_\pm$ outside a neighbourhood of S and is a product metric on $\Delta \times S$, equal to $|dz|^2 + g_S$ near S. Then $p_1(g'_\pm) = p_1(|dz|^2) + p_1(g_S) = p_1(g_S)$ on $\Delta \times S$, so the differential forms $p_1(g_\pm)$ and $p_1(g'_\pm)_{|V_\pm}$ are equal. Finally let g on M be a patching of $d\theta^2 + g_\pm$ on $\mathbb{S}^1 \times V_\pm$. Then $p_1(M) = [p_1(g)] = Y([p_1(g'_+)], [p_1(g'_-)]) = Y(p_1(Z_+), p_1(Z_-))$.

Smooth type of connected-sum $\mathbf{G_2}$–manifolds. Many of the G_2–manifolds we construct in this paper are 2-connected; in this case we can compute classifying topological invariants and in many cases determine the diffeomorphism type of the underlying smooth 7-manifold. These are the first compact manifolds with holonomy G_2 for which the diffeomorphism type of the underlying 7-manifold has been determined. We will see in §7 that in many cases we can get 7-manifolds with the same invariants by taking the twisted connected sum of completely unrelated pairs of building blocks, and can thus construct different metrics with holonomy G_2 on the same underlying smooth 7-manifold. Judicious choices of pairs of building blocks allow us to vary the number of compact associative 3-folds we can exhibit in different G_2-holonomy metrics on the same smooth 7-manifold.

Let us first review the classification theory of smooth 2-connected 7-manifolds; we concentrate on the simplest case, namely where the cohomology is torsion-free. Lemma 4.27 gives sufficient conditions on a twisted connected sum manifold M to ensure that M is 2-connected with torsion-free cohomology, and therefore the classification theory discussed below applies to M.

Almost-diffeomorphism classification of smooth closed 2-connected 7-manifolds. Two smooth manifolds M, N are *almost-diffeomorphic* if there is a homeomorphism $M \to N$ that is smooth away from a finite set of points; this is equivalent to M being diffeomorphic to $N \# \Sigma$ for some homotopy sphere Σ. Recall that by the h-cobordism theorem, any homotopy sphere of dimension $n > 4$ is a smooth manifold homeomorphic but not necessarily diffeomorphic to $\mathbb{S}^n$; under connected sums the homotopy spheres form a finite abelian group denoted Θ_n. The group Θ_7 of homotopy 7-spheres is $\mathbb{Z}/28\mathbb{Z}$. It turns out that the classification of smooth 2-connected 7-manifolds is the same up to homeomorphism as up to almost-diffeomorphism; in particular there are at most 28 smooth structures on any 2-connected topological 7-manifold.

Let M be a smooth connected closed 7-manifold that is 2-connected, *ie* $\pi_1(M)$ and $\pi_2(M)$ are trivial. Then $H_1(M) \cong H_2(M) = 0$ by the Hurewicz theorem, so $H^1(M) = H^2(M) = 0$ by universal coefficients and $H^5(M) = H^6(M) = 0$ by Poincaré duality. So apart from $H^0(M) \cong H^7(M) \cong \mathbb{Z}$ the only non-vanishing cohomology groups are $H^3(M)$, which is torsion-free, and $H^4(M)$, whose free part is isomorphic to $H^3(M)$. If $H^4(M)$ is torsion-free then all the information about the cohomology of M reduces to the integer $b^3(M) = b^4(M)$.

Another invariant of M is the first Pontrjagin class $p_1(M) \in H^4(M)$. If $H^4(M)$ is torsion-free then the position of $p_1(M)$ in $H^4(M)$ up to isomorphism is determined by the greatest divisor $\operatorname{div} p_1(M) \in \mathbb{N}$; recall from Corollary 4.19 that this is always divisible by 4. For our purposes, the following special case of the classification results of Wilkens [77, Theorem 3] will suffice.

Theorem 4.22. *Smooth closed 2-connected 7-manifolds M with $H^4(M)$ torsion-free are classified up to almost-diffeomorphism by the isomorphism class of the pair $(H^4(M), p_1(M))$, or equivalently by the non-negative integers $b^4(M)$ and $\operatorname{div} p_1(M)$. Moreover, any pair of non-negative integers of the form $(k, 4m)$ is realised as $k = b^4(M)$ and $4m = \operatorname{div} p_1(M)$ for some smooth closed 2-connected 7-manifold M.*

By Novikov [64] rational Pontrjagin classes are natural under homeomorphisms. In the absence of torsion in H^4, so are the integral classes, *ie* $p_1(M) = f^* p_1(N)$ for any homeomorphism $f\colon M \to N$. Since the classifying almost-diffeomorphism invariants are also invariant under homeomorphism, it follows that the classification up to homeomorphism is the same.

Remark 4.23. When $H^4(M)$ has torsion, the invariants in Theorem 4.22 need to be amended. Instead of $p_1(M)$, one should use the spin characteristic class $p_{1/2}(M) \in H^4(M)$. The torsion-linking form $b : TH^4(M) \times TH^4(M) \to \mathbb{Q}/\mathbb{Z}$ mentioned in Remark 4.12 is another obvious invariant; Wilkens showed that the isomorphism class of the triple $(H^4(M), b, p_{1/2}(M))$ classifies M up to almost-diffeomorphism when $H^4(M)$ has no 2-torsion. Crowley [22, Theorem B] showed that when $H^4(M)$ has 2-torsion one obtains classifying invariants by replacing b with a "family of quadratic refinements". (All triples of invariants are realised subject only to the constraint that $p_{1/2}(M)$ is divisible by 2.)

Concrete realisations of 2-connected smooth 7-manifolds. We can give concrete descriptions of many 2-connected smooth 7-manifolds using $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ and connected sums thereof. The trivial bundle $\mathbb{S}^3 \times \mathbb{S}^4$ gives a 2-connected 7-manifold with torsion-free cohomology; clearly, it has $H^3(M) = H^4(M) = \mathbb{Z}$ and vanishing first Pontrjagin class $p_1(M)$ (since $\mathbb{S}^3 \times \mathbb{S}^4$ is clearly stably parallelisable; indeed it is even parallelisable because $\mathbb{S}^3$ is parallelisable, and only a rank one trivial factor needs to be added to trivialise $T\mathbb{S}^4$). The k-fold connected sum $k(\mathbb{S}^3 \times \mathbb{S}^4)$ gives a 2-connected 7-manifold with $H^3(M) = H^4(M) = \mathbb{Z}^k$ with $p_1(M) = 0$ (since connected sums of stably parallelisable manifolds are stably parallelisable, and Pontrjagin classes are stable).

Via the usual 'clutching' construction for bundles over a sphere, equivalence classes of linear $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ are in one-to-one correspondence with $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Convenient generators for $\pi_3(\mathrm{SO}(4))$ are given by

$$\rho(u)v = uvu^{-1}, \quad \sigma(u)v = uv;$$

here we have identified $\mathbb{S}^3$ with the unit quaternions and composition denotes quaternionic multiplication. Identifying the pair of integers (m, n) with the element $m\rho + n\sigma \in \pi_3(\mathrm{SO}(4))$ hence determines a real rank 4 vector bundle $\xi_{m,n}$ over $\mathbb{S}^4$ and its corresponding 3-sphere bundle $M_{m,n} := S(\xi_{m,n}) \to \mathbb{S}^4$, with projection map π.

By the homotopy long exact sequence of a fibration, any $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ is 2-connected. Together with the fact that $H^4(M_{m,n}) = \mathbb{Z}/n\mathbb{Z}$ (using the Gysin sequence and that the Euler number of the bundle is $e(\xi_{m,n}) = n$) this determines all the homology groups of the bundle. For the $\mathbb{S}^3$-bundles $M_{m,0}$ with Euler number 0 we have (*cf* Crowley and Escher [23, Fact 3.1])

$$H^3(M_{m,0}) \cong H^4(M_{m,0}) \cong \pi^* H^4(\mathbb{S}^4) \cong \mathbb{Z}; \qquad p_1(M_{m,0}) = 4m\kappa_4 \in \mathbb{Z};$$

where $\kappa_4 := \pi^* \iota_4 \in \pi^* H^4(\mathbb{S}^4)$ is the generator of $H^4(M_{m,0}) \cong \mathbb{Z}$ and ι_4 denotes a generator of $H^4(\mathbb{S}^4) \cong \mathbb{Z}$.

Remark 4.24. The connected sum $M_m^k := M_{m,0} \# (k-1)(\mathbb{S}^3 \times \mathbb{S}^4)$ is a 2-connected smooth 7-manifold with torsion-free cohomology, $b^3(M_m^k) = b^4(M_m^k) = k$ and $\operatorname{div} p_1(M_m^k) = 4m$; taking a further connected sum with any exotic 7-sphere $\Sigma \in \Theta_7 \cong \mathbb{Z}/28\mathbb{Z}$ yields another 2-connected 7-manifold with the same invariants which may or may not be (oriented) diffeomorphic to M_m^k.

Almost-diffeomorphism to diffeomorphism classification. In general, finding the number of (oriented) diffeomorphism classes in the almost diffeomorphism class of a 2-connected 7-manifold M is equivalent to identifying the *inertia subgroup*

$$I(M) \subseteq \Theta_7 := \{\Sigma \in \Theta_7 |\, M\#\Sigma \text{ is oriented-diffeomorphic to } M\}.$$

Theorem 4.25 ([78, Theorem 1])**.** *Let M be a closed 2-connected 7-manifold. If $H^4(M)$ has no 2- or 7-torsion and d is the greatest divisor of $p_1(M)$, then the inertia subgroup $I(M) \subseteq \Theta_7$ consists of the elements of Θ_7 divisible by $d/8$. (If $p_1(M)$ is a torsion element then we interpret d to be 0, and $I(M)$ is trivial.)*

So, for example, if $\gcd(p_1(M), 8{\cdot}28)$ divides 8 then $I(M) = \Theta_7$ and any manifold almost-diffeomorphic to M is actually diffeomorphic to M. If there is torsion in $H^4(M)$ then one can still say that $I(M) \subseteq (d_\pi/4)\Theta_7$ where d_π is the greatest divisor of $p_1(M)$ modulo torsion [78, Corollary to Proposition 5], but the precise value of $I(M)$ may depend on the torsion linking form [25, Example 5.2].

If M has holonomy G_2 then, by Proposition 2.33(ii), $p_1(M)$ is never a torsion class even if $H^4(M)$ has torsion.

Remark 4.26. Eells and Kuiper [32] defined a $\mathbb{Z}/28\mathbb{Z}$ valued invariant for (in particular) closed simply-connected spin 7-manifolds M with $b^4(M) = 0$ (*ie* $H^4(M)$ finite). This invariant classifies the elements of Θ_7, and can be used to detect the connected sum action of Θ_7 and thus distinguish between the diffeomorphism types in an almost-diffeomorphism class. In [25], this invariant is generalised to the case when $b^4(M) > 0$, in such a way that it distinguishes between all smooth structures on M when M is 2-connected.

Application to twisted connected sums. We now consider compact G_2–manifolds M constructed as a twisted connected sum from a pair of building blocks Z_+, Z_- from the point-of-view of their diffeomorphism and almost-diffeomorphism type. To begin with we deduce from our results on the cohomology of twisted connected sum manifolds a simple sufficient condition for M to be 2-connected and for $H^4(M)$ to be torsion-free. Combined with our calculation of $p_1(M)$ we can then apply the classification Theorem 4.22.

Lemma 4.27 (2-connected twisted connected sums with torsion-free H^4)**.**

(i) *If $K_\pm = 0$ (*ie $H^2(V_\pm) \to H^2(S)$ *is injective; recall* (3.7)*), $N_+ \cap N_- = 0$ and the inclusion $N_+ + N_- \subset L$ is primitive then M is 2-connected.*
(ii) *If $N_+ \perp N_-$, then $H^4(M)$ is torsion-free.*

Proof.
(i) We know from Theorem 4.8 that $\pi_1(M) = 0$. Theorem 4.8(ii) implies that $H^2(M) = 0$ and Corollary 4.11(i) that $H^3(M)$ is torsion-free. So $\pi_2(M) \cong H_2(M) = 0$.
(ii) Follows from 4.11(ii). ☐

The twisted connected sum construction relies on being able to find pairs of suitably compatible ACyl Calabi-Yau 3-folds $V_\pm = Z_\pm \setminus S_\pm$. We will often refer to finding such compatible pairs as *solving the matching problem.* We will see (*cf* Proposition 6.18) that the easiest way to find solutions to the matching problem involves

- using building blocks of semi-Fano type which automatically (Proposition 3.17) have $K = 0$;

- applying results of Nikulin [61] to embed the orthogonal direct sum $N_+ \perp N_-$ primitively in the K3 lattice L ("primitive perpendicular gluing").

This will allow us to obtain a large class of examples of compact G$_2$–manifolds that are 2-connected and have $H^4(M)$ torsion-free. When $K_\pm = 0$ and $N_+ \perp N_-$, Theorem 4.8 implies that

$$b^3(M) = b^4(M) = b^3(Z_+) + b^3(Z_-) + 23.$$

So by Theorem 4.22, to understand the almost-diffeomorphism type of such M it remains only to determine the divisibility of $p_1(M)$.

Remark. If M is 2-connected but $H^4(M)$ has torsion then we could still apply the almost-diffeomorphism classification theory of Wilkens and Crowley as in Remark 4.23. Recall from Remark 4.12 that the isomorphism class of the torsion-linking form of a twisted connected sum G$_2$-manifold is determined by the isomorphism class of $H^4(M)$. Hence for 2-connected twisted connected sums the isomorphism class of the pair $(H^4(M), p_1(M))$ is sufficient to determine the almost-diffeomorphism class, except possibly when $H^4(M)$ has 2-torsion.

Remark 4.28. Of all the G$_2$–manifolds constructed by Joyce's orbifold desingularisation methods [45, 46] only one example has $b^2 = 0$; in particular, none of the other Joyce G$_2$–manifolds are 2-connected. Since the diffeomorphism classification of general simply-connected smooth 7-manifolds is still unsolved, the determination of the diffeomorphism type of Joyce's G$_2$–manifolds remains a challenge. The example with $b^2 = 0$ is found in [46, Thm 12.5.7], and has $b^3 = 215$. It is in fact topologically a twisted connected sum of blocks of the type described in the following remark.

Remark 4.29. The non-symplectic type blocks described in Remark 3.20 always have $\operatorname{rk} K \geq 2$ in (3.7). Hence by Theorem 4.8(ii) any twisted connected sum G$_2$–manifold M constructed using at least one such building block has $b^2(M) \geq 2$; in particular, the diffeomorphism classification of such twisted connected sum G$_2$–manifolds also remains open. The non-trivial K of these blocks arises from resolving singularities by blow-ups; in some cases it is possible to desingularise by smoothing instead to obtain blocks with $K = 0$. While the details are beyond the present scope, Joyce's example with $b^2 = 0$ can be seen to be recovered topologically by using such blocks from K3s with non-symplectic involution with fixed lattice $U(2)$, *ie* double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a smooth curve of bidegree $(4, 4)$.

Let $N'_\mp$ be the image of $N_\mp$ in $N^*_\pm = L/T_\pm \subset H^4(Z)$ as before. From Proposition 4.20 and Lemma 4.14 we immediately deduce

Corollary 4.30. *Let M be a twisted connected sum of the building blocks Z_+ and Z_-. Then*

$$\operatorname{div} p_1(M) = 2 \gcd(c_2(Z_+) \bmod N'_-,\ c_2(Z_-) \bmod N'_+).$$

In particular any common divisor of $2c_2(Z_+)$ and $2c_2(Z_-)$ also divides $p_1(M)$, and if $N_+ \perp N_-$ then

$$\operatorname{div} p_1(M) = 2 \gcd(c_2(Z_+), c_2(Z_-)).$$

Here the 'greatest common divisor' of $c_2(Z_+)$ and $c_2(Z_-)$ should simply be interpreted as the greatest integer by which both are divisible in the respective $\mathbb{Z}$-modules $H^4(Z_\pm)$ (and $H^4(Z_\pm) \bmod N'_\mp$).

For the building blocks used in this paper, we already computed the greatest divisors of $c_2(Z)$ in [21], see Table 2. In examples of twisted connected sums where N_+ and N_- are not perpendicular (so that $N'_\pm$ are non-trivial), we need more detailed information about $c_2(Z)$. When $N'_\pm$ is primitive, corresponding to $H^4(M)$ being torsion-free, [21, Lemma 5.18] can be applied to give the information we need for building blocks constructed from semi-Fanos using

Proposition 3.17. (In general, it is a little easier to compute $\operatorname{div} p_1(M)$ modulo the torsion in $H^4(M)$.)

Lemma 4.31. $\operatorname{div} p_1(M) \in \{4, 8, 12, 16, 24, 48\}$ *for any twisted connected-sum* G_2*–manifold* M.

Proof. Since M is spin, $p_1(M)$ is divisible by 4 according to Corollary 4.19 (we can also deduce this from Corollary 4.30 and $c_2(Z)$ being even for any building block Z [21, Lemma 5.10]). On the other hand, M contains a K3 surface S with trivial normal bundle, so the image of $p_1(M)$ in $H^4(S) \cong \mathbb{Z}$ is $p_1(S) = -2c_2(S) \cong -2\chi(S) = -48$. □

Remark. The examples in Table 3 show that the restrictions in Lemma 4.31 are the only constraints on the possible greatest divisors of p_1 of twisted connected sum G_2–manifolds.

Corollary 4.32. *For a* 2*-connected twisted connected sum* G_2*–manifold* M *with* $H^4(M)$ *torsion-free either*

(i) *The inertia group* $I(M) = \Theta_7$ *and hence the almost diffeomorphism class of* M *consists of a single diffeomorphism class; this holds when* $\operatorname{div} p_1(M) \in \{4, 8, 12, 24\}$*; or*

(ii) *The inertia group* $I(M)$ *consists of all even elements in* $\Theta_7 \simeq \mathbb{Z}/28\mathbb{Z}$ *and hence the almost diffeomorphism class of* M *contains exactly two diffeomorphism classes; this holds when* $\operatorname{div} p_1(M) \in \{16, 48\}$.

In particular, knowing only $b^4(M)$ *determines the diffeomorphism type of* M *up to* 8 *possibilities.*

Proof. Follows immediately from Theorem 4.25 and Lemma 4.31. □

5. Construction of associative submanifolds

Let (M, g) be a Riemannian manifold. A k-form α on M is a *calibration* if $d\alpha = 0$ and, for all $x \in M$ and every oriented k-plane π in T_xM, we have $\alpha_{|\pi} \leq \mathrm{vol}_\pi$. An oriented submanifold $i : A \hookrightarrow M$ is *calibrated* if, for all $x \in A$, $\pi_x := i_*(T_xA)$ attains the equality: $\alpha_{|\pi_x} = vol_{\pi_x}$. The fundamental property of any calibrated submanifold is that it minimises volume in its homology class [39, Thm. II.4.2].

It follows from Lemma 2.19 that, on any G_2–manifold (M, φ), the (parallel) 3-form φ is a calibration. The corresponding calibrated 3-dimensional submanifolds are known as *associative*.

In this section we explain that if the ACyl Calabi–Yau 3-folds $V_\pm$ used in the twisted connected-sum construction of G_2–manifolds M described in §3 contain appropriate *compact* calibrated submanifolds, then these will give rise to associative submanifolds of M. More precisely, if $C \subset V_\pm$ is a holomorphic curve, then $\mathbb{S}^1 \times C$ is an associative in $\mathbb{S}^1 \times V_\pm$, and if $L \subset V_\pm$ is special Lagrangian, then $\{\theta\} \times L$ is associative. We will prove that under certain conditions it is possible to perturb these to manifolds that are associative with respect to the torsion-free G_2–structure on M, when the neck-length parameter in the construction is sufficiently large.

Geometry of associative submanifolds. This subsection recalls basic features of the geometry of associative submanifolds. Let A be an associative submanifold in a G_2–manifold (M, φ). Let NA denote the normal bundle of A, and let ∇ denote the Levi-Civita connection defined by the metric g on M. Recall that for $x \in A$ the projections $T_xM \to T_xA$ and $T_xM \to N_xA$ corresponding to the orthogonal splitting $T_xM = T_xA \oplus N_xA$ define connections on the bundles TA, NA. When necessary we will distinguish these via the notation $\nabla^\top$, $\nabla^\perp$.

The cross product $TA \times NA \to NA$ gives the normal bundle a Clifford bundle structure. Together with the connection $\nabla^\perp$ this defines a natural *Dirac operator* $\not{D} : \Gamma(NA) \to \Gamma(NA)$.

For $v \in \Gamma(NA)$ we can express $\not{D}v$ as follows. For any $x \in A$ let e_1, e_2, e_3 denote a positive orthonormal basis of T_xM, and let

$$\not{D}v(x) := \sum_{i=1}^{3} e_i \times (\nabla^{\perp}_{e_i} v). \tag{5.1}$$

$\not{D}$ is a first order differential operator. One can check that it is elliptic and formally self-adjoint, *i.e.*

$$\int_A <\not{D}v, w> dvol = \int_A <v, \not{D}w> dvol.$$

Remark. $\not{D}$ is in fact a twisted Dirac operator, in the sense that $NA \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic as a Clifford bundle to a twisted spinor bundle $S \otimes E$. For the relation $TA \cong \Lambda^2_+ NA$ implies that for any spin structure P on A (which exists because A is 3-dimensional and orientable) there is a lift of the SO(4)-structure of NA to a Spin(4)-structure, so that P is associated via the projection of Spin(4) $\cong$ Spin(3) $\times$ Spin(3) to one factor. Then $NA \otimes_{\mathbb{R}} \mathbb{C}$ is the tensor product of the two vector bundles associated to the spin representations of Spin(4), and one of these is the spinor bundle S associated to P. See McLean [53, §5] and Lawson-Michelsohn [51].

The Dirac operator plays an important role in the deformation theory of associative submanifolds. Given the G$_2$–structure φ, we can define a global vector-valued 3-form χ on M modelled on (2.5):

$$g(u, \tfrac{1}{2}\chi(v,w,z)) = \psi(u,v,w,z) \quad \text{for all } u,v,w,z \in T_xM, \tag{5.2}$$

where $\psi = *\varphi$. Then $A \subset M$ is associative if and only if the normal vector field $F(A,\varphi) = \chi(TA) \in \Gamma(NA)$ vanishes, where TA is interpreted as a simple unit norm section of $\Lambda^3 TM$ over A. Recall that we can parametrise the deformations of A as follows. Let *exp* denote the exponential map on M. Then all (small) deformations of A, up to reparametrisation, can be obtained as $A_v = i_v(A)$ for some $v \in \Gamma(NA)$ close to the zero section, where $i_v : A \to M$ is defined by

$$i_v(x) := exp_x(v(x)).$$

Given v, $F(A_v)$ defines a section of NA_v. In other words, if we let $\mathcal{N}$ be the vector bundle over $\Gamma(NA)$ whose fibre over v is $\Gamma(NA_v)$, then F is a section of $\mathcal{N}$.

The associative deformations of A are parametrized by the zero set of F in a small neighbourhood U of the zero section in $\Gamma(NA)$. We say that A is *isolated* if $F^{-1}(0) = \{0\}$, *i.e.* if there do not exist other associative submanifolds attainable as small deformations of A.

Because $F(0) = 0$, the differential $DF_0 : \Gamma(NA) \to \Gamma(NA)$ is defined naturally (without any connection on $\mathcal{N}$), and it is precisely equal to $\not{D}$ (see [53, §5] or [34, Theorem 2.1]). We call the kernel of $\not{D}$ the *infinitesimal deformation space* of A, and say that A is *rigid* if this space vanishes.

We could attempt to study the set $F^{-1}(0)$ via the Implicit Function Theorem. It is first necessary to pass to the Banach space completions of the relevant spaces and maps, *eg* using Sobolev spaces. If A is closed then the standard theory of elliptic operators shows that $\not{D}$ extends to a Fredholm operator. It follows that if A is rigid then it is also isolated. As $\not{D}$ is formally self-adjoint it has index 0, and the *obstruction space* coker $\not{D}$ vanishes if and only if A is rigid. Therefore we can use the Implicit Function Theorem to prove smoothness of the deformation space of a closed associative only when the space is in fact discrete.

Persistence of associatives. We prove that any rigid associative submanifold A will persist under small deformations of the ambient G_2–structure.

Theorem 5.3. *Let A be a closed associative in a G_2–manifold (M, φ). If $\ker \not{D} = 0$ then for any small deformation of the G_2–structure, there is a unique small deformation of A which is associative with respect to the new G_2–structure.*

When we apply Theorem 5.3 (proven below) we will often first replace M by an open neighbourhood of A in order to avoid regions where the G_2–structure has torsion. Even if the obstruction space $\ker \not{D}$ is non-zero, A may be "unobstructed in a family". Infinitesimal deformations of the G_2–structure on M correspond simply to 3-forms, and so the derivative of $\varphi' \mapsto F(A, \varphi')$ at φ is a map $\Omega^3(M) \to \Gamma(NA)$. Let $R_{A,\varphi} : \Omega^3(M) \to \ker \not{D}$ denote the composition with the projection $\Gamma(NA) \to \operatorname{coker} \not{D} \cong \ker \not{D}$.

Theorem 5.4. *Let A be a closed associative in a G_2–manifold (M, φ), and $\{\varphi_s : s \in \mathcal{G}\}$ an m-dimensional family of deformations of φ such that $R_{A,\varphi} : T\mathcal{G} \to \ker \not{D}$ is an isomorphism. Then there is a ball $B \subset \mathbb{R}^m$, a family of perturbations A_b of A parametrised by $b \in B$ (a smooth function $A \times B \to M$) and $f : B \to \mathcal{G}$, such that each A_b is associative with respect to $f(b)$. The same conclusion holds with $\mathcal{G}$ replaced by any sufficiently small deformation to a family of G_2–structures $\mathcal{G}'$ (not necessarily containing φ).*

The perturbation A_b is rigid as an $f(b)$-associative unless b is a critical point of f. Theorem 5.3 is of course a special case of Theorem 5.4. It can be proved with less cumbersome notation.

Proof of Theorem 5.3. As above, mapping $v \in L^p_{k+1}(NA)$ to the image $A_v = i_v(A)$ identifies a neighbourhood U of A in the space of L^p_{k+1}-submanifolds of M with a neighbourhood of the origin in $L^p_{k+1}(NA)$. Choose a trivialisation of the bundle $\mathcal{N}$ over U, *ie* isomorphisms $\Gamma(NA_v) \cong \Gamma(NA)$ for each v. Let $\{\varphi_t : t \in (-\epsilon, \epsilon)\}$ be a 1-parameter family of G_2–structures (containing $\varphi = \varphi_0$). Consider the map

$$U \times (-\epsilon, \epsilon) \to L^p_k(NA), \quad (A', t) \mapsto F(A', \varphi_t).$$

By hypothesis, the derivative at $(A, 0)$ is bijective on the first factor. By the Implicit Function Theorem, a neighbourhood of $(A, 0)$ in the pre-image of 0 is the graph of a function $t \mapsto A'(t)$, *ie* for each perturbation φ_t of the G_2–structure there is a unique L^p_{k+1}-perturbation A_v of A that is associative with respect to φ_t. Because the deformation operator $\not{D}$ is elliptic, v is a solution of a non-linear elliptic equation, and is smooth by elliptic regularity. □

Proof of Theorem 5.4. Let $\{\varphi_{s,t} : s \in \mathcal{G}, t \in (-\epsilon, \epsilon)\}$ be a one-parameter family of deformations of $\mathcal{G}$ (with $\varphi_{s_0,0}$ corresponding to the initial G_2–structure φ on M, with respect to which A is associative). With U as before, consider the map

$$U \times \mathcal{G} \times (-\epsilon, \epsilon) \to L^p_k(NA), \quad (A', s, t) \mapsto F(A', \varphi_{s,t})$$

The derivative $T_AU \times T_{s_0}\mathcal{G} \times \mathbb{R} \to L^p_k(NA)$ at $(A, s_0, 0)$ equals $\not{D}$ on $T_AU = L^p_{k+1}(NA)$, while the composition of the derivative with the projection to $\operatorname{coker} \not{D}$ equals $R_{A,\varphi}$ on the $T_{s_0}\mathcal{G}$ factor. Hence the derivative is an isomorphism transverse to $\ker \not{D} \oplus \{0\} \oplus \mathbb{R}$. By the Implicit Function Theorem, a neighbourhood of $(A, s_0, 0)$ in the pre-image of 0 is a graph over $B \times (-\epsilon', \epsilon')$, for some small ball $B \subset \ker \not{D}$. For each fixed $t \in (-\epsilon', \epsilon')$, this defines a family of deformations $\{A_b : b \in B\}$ and a map $f : B \to \mathcal{G}' = \{\varphi_{s,t} : s \in \mathcal{G}\}$. □

In the situation where we want to use the unobstructedness in a family, there is an obvious family $\mathcal{A}$ of initial associatives, and we can perturb the whole family.

Corollary 5.5. *Suppose that $\mathcal{A}$ is a smooth compact (possibly with boundary) m-dimensional family of closed associatives in a G_2–manifold (M, φ), and that $\{\varphi_s : s \in \mathcal{G}\}$ is an m-dimensional family of deformations of φ such that $R_{A,\varphi} : T\mathcal{G} \to \ker \not{D}$ is an isomorphism for each $A \in \mathcal{A}$. Then for any sufficiently small deformation of $\mathcal{G}$ to a family of G_2–structures $\mathcal{G}'$, there is a small deformation $\mathcal{A}'$ of $\mathcal{A}$ and a smooth map $f : \mathcal{A}' \to \mathcal{G}'$ such that each $A' \in \mathcal{A}'$ is associative with respect to $f(A')$.*

Proof. For each $A \in \mathcal{A}$, Theorem 5.4 describes how to deform a neighbourhood of A, provided that $\mathcal{G}'$ is a sufficiently small deformation of $\mathcal{G}$. Because $\mathcal{A}$ is compact it can be covered by finitely many such neighbourhoods. □

Notice that we can think of $\mathcal{G}'$ as an open subset of $\mathbb{R}^m$ where m is equal to the the dimension of $\mathcal{A}'$. Hence when $\mathcal{A}'$ is compact without boundary, then $f : \mathcal{A}' \to \mathcal{G}'$ will definitely have some critical points, so some elements of $\mathcal{A}'$ are not rigid.

Associative submanifolds and complex curves. Let (V, Ω, ω) be a Calabi–Yau 3-fold, and consider $\mathbb{S}^1 \times V$ with the torsion-free G_2–structure $\varphi = d\theta \wedge \omega + \operatorname{Re}\Omega$ as described in (2.38). Let C be a complex curve in V. Then Lemma 2.26(i) implies that $\mathbb{S}^1 \times C$ is an associative submanifold. The aim of this section is to relate the properties of the associative submanifold to those of the complex curve.

Recall that a Calabi–Yau 3-fold V carries a global holomorphic $(3,0)$-form Ω. We will denote its real part by α and its imaginary part by β, *i.e.* $\Omega = \alpha + i\beta$. We can define a bilinear map $TV \times TV \to TV$, $(a, b) \mapsto a \times b$ via the formula

$$g(a \times b, c) = \alpha(a, b, c), \tag{5.6}$$

where g is the Calabi–Yau metric of V. Of course, this coincides with the projection onto TV of the cross product on $\mathbb{S}^1 \times V$. The fact that Ω is I-linear and g is Hermitian implies that $\times$ is I-antilinear in each factor. $\times$ has the usual property of a cross product that $a \times b$ is perpendicular to both a and b, but it is only nonzero when a and b are linearly independent as complex vectors; we call $\times$ a complex cross product. In particular, for a complex curve $C \subset V$ the complex cross product gives a complex linear map

$$TC \times NC \to \overline{NC}. \tag{5.7}$$

Moreover, because Ω is parallel

$$\nabla(a \times b) = \nabla a \times b + a \times \nabla b. \tag{5.8}$$

Let us now review some well-known facts concerning holomorphic vector fields. Given any complex manifold (V, I) with real tangent bundle TV, recall the isomorphism of complex vector bundles $(TV, I) \cong T^{1,0}V$ given by

$$X \mapsto X - iIX. \tag{5.9}$$

Recall also that any holomorphic bundle $E \to V$ has a natural *Cauchy-Riemann* operator $\overline{\partial} : \Gamma(E) \to \Omega^{0,1}(E)$ whose kernel consists of the holomorphic sections of E. A Hermitian metric h on E defines a *Chern connection* $\widetilde{\nabla} : \Gamma(E) \to \Omega^1(E)$: it is uniquely characterized by the properties $\widetilde{\nabla} h = 0$ and $\widetilde{\nabla}^{0,1} = \overline{\partial}$, where $\widetilde{\nabla}^{0,1} := \frac{1}{2}(\widetilde{\nabla} + i\widetilde{\nabla}_I)$ is the $(0,1)$-component under the splitting $\Omega^1(E) = \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$. Because g is a Kähler metric on V, the Chern connection on TV coincides with the Levi-Civita connection ∇. Hence the Chern connection on NC coincides with the projection $\nabla^\perp$. In particular the Cauchy-Riemann operator on NC is just the $(0,1)$-part of $\nabla^\perp$. We can use this fact and the complex Clifford structure (5.7) to define an operator

$$\not{D}^c : \Gamma(NC) \to \Gamma(\overline{NC})$$

whose kernel is exactly the space of holomorphic normal vector fields: for $v \in \Gamma(NA)$ and $x \in C$ pick any unit vector $a \in T_xC$ and set

$$\not{D}^c v(x) := a \times (\nabla_a + I\nabla_{Ia})^{\perp} v.$$

One can check that it is in fact independent of the choice of a. It defines a complex first-order linear elliptic operator, which we will refer to as the *complex Dirac operator* on NC. Using (5.8) and I-antilinearity of the cross product we find

$$<\not{D}^c v, w> = \operatorname{div}_C(v \times w) + <v, \not{D}^c w>$$

where div_C denotes the divergence operator on vector fields tangent to C, defined via an orthonormal basis of TC by $\operatorname{div}_C X = <\nabla_{e_i} X, e_i>$. Under integration, the divergence term vanishes, so $\not{D}^c$ is formally self-adjoint:

$$\int_C <\not{D}^c v, w>\, dvol = \int_C <v, \not{D}^c w>\, dvol.$$

Let us now return to the product G_2–manifold $\mathbb{S}^1 \times V$ and the associative submanifold $\mathbb{S}^1 \times C$. We can identify the normal bundle of $\mathbb{S}^1 \times C \subset \mathbb{S}^1 \times V$ with the normal bundle of $C \subset V$; notice however that any section v will depend on both the θ coordinate and the coordinate on C. Choose a point $(\theta, x) \in \mathbb{S}^1 \times V$. Set $e_1 := \frac{\partial}{\partial\theta}$ and let $e_2 = a$ be any unit vector on T_xC so that $e_3 = Ia$. Then (5.1) becomes

$$\not{D} v = \tfrac{\partial}{\partial\theta} \times \dot{v} + a \times (\nabla_a v)^{\perp} + Ia \times (\nabla_{Ia} v)^{\perp}.$$

where $\dot{v}$ denotes the derivative with respect to θ. As seen at (2.38), $\frac{\partial}{\partial\theta} \times \dot{v} = I\dot{v}$. Using that $\nabla^{\perp}$ is I-linear and the cross product is I-antilinear we can then rewrite $\not{D} v$ as follows:

$$\not{D} v = I\dot{v} + a \times (\nabla_a v + \nabla_{Ia} I v)^{\perp} = I\dot{v} + \not{D}^c v. \tag{5.10}$$

Normal holomorphic vector fields represent the infinitesimal deformations of C as a complex curve in V. The curve C is said to be *rigid* if it has no infinitesimal holomorphic deformations. In the previous section we saw that the solutions to $\not{D} v = 0$ (for the Dirac operator defined in (5.1)) correspond to the infinitesimal (associative) deformations of an associative submanifold of a G_2–manifold.

Lemma 5.11. *For the associative submanifold $\mathbb{S}^1 \times C \subset \mathbb{S}^1 \times V$, the kernel of $\not{D}$ is the pull-back of the kernel of $\not{D}^c$. Thus $\mathbb{S}^1 \times C$ is rigid if and only if the complex curve C is rigid.*

Proof. The facts that $\not{D}^c$ is formally self-adjoint, $v \mapsto \dot{v}$ is formally skew-adjoint and commutes with $\not{D}^c$, and I is skew-adjoint and anti-commutes with $\not{D}^c$ imply

$$<\not{D}^c v, I\dot{v}>_{L^2} = 0.$$

Therefore (5.10) implies that $\|\not{D} v\|^2_{L^2} = \|\not{D}^c v\|^2_{L^2} + \|\dot{v}\|^2_{L^2}$. □

Associative submanifolds and special Lagrangians. Let (V, Ω, ω) be a Calabi–Yau 3-fold, and consider as before $\mathbb{S}^1 \times V$ with the torsion-free G_2–structure $\varphi = d\theta \wedge \omega + \operatorname{Re}\Omega$ described in (2.38). If $L \subset V$ is a special Lagrangian 3-fold then Lemma 2.26(ii) implies that $L_\theta = \{\theta\} \times L$ is associative in $\mathbb{S}^1 \times V$ for any $\theta \in \mathbb{S}^1$. We assume that L is closed.

We want to describe the relation between the deformation theory of the associative L_θ and the special Lagrangian L. Note that since we can deform L_θ simply by changing $\theta \in \mathbb{S}^1$, it is *never* rigid, and the obstruction space $\operatorname{coker} \not{D}$ is always non-trivial. We will therefore study the map $\Omega^3(\mathbb{S}^1 \times V) \to \operatorname{coker} \not{D}$ in order to apply Theorem 5.4 later.

Let us first recall the deformation theory of a closed special Lagrangian $L \subset V$ [53, §3]. According to Lemma 2.24(ii), for L to be special Lagrangian is equivalent to $\omega_{|L} = \operatorname{Im}\Omega_{|L} = 0$.

The Lagrangian condition implies that we can identify the normal bundle NL with T^*L by $\sigma \mapsto \sigma \lrcorner \omega$. We can therefore parametrise small deformations of L by small $\alpha \in \Omega^1(L)$.

Since ω and $\operatorname{Im}\Omega$ are closed, the cohomology classes represented by their restrictions to L are homotopy invariant, so the restrictions are exact for all deformations of L. The special Lagrangian deformations of L are therefore parametrised by the zero set of a map

$$\Omega^1(L) \to d\Omega^1(L) \times d\Omega^2(L).$$

The linearisation of this map at 0 (corresponding to L) is $D_L : \alpha \mapsto (d\alpha, d{*}\alpha)$. This is surjective, with kernel $\mathcal{H}^1(L)$, the space of harmonic 1-forms on L. Thus the deformations of L are always unobstructed, and form a smooth manifold near L of dimension $b^1(L)$.

Now consider the associative $L_\theta = \{\theta\} \times L$. Its normal bundle NL_θ in $\mathbb{S}^1 \times V$ is a direct sum of the trivial bundle spanned by $\frac{\partial}{\partial\theta}$ and the normal bundle NL of L in V. We can identify it with $\Lambda^0 T^*L \oplus \Lambda^1 T^*L$. Then the Dirac operator $\not{D} : \Gamma(NL_\theta) \to \Gamma(NL_\theta)$ is interpreted as

$$\begin{aligned} \Omega^0(L) \times \Omega^1(L) &\to \Omega^0(L) \times \Omega^1(L), \\ (f, \alpha) &\mapsto (d^*\alpha,\, df + {*}d\alpha) \end{aligned} \tag{5.12}$$

(see Gayet [34, Proposition 4.7]). The kernel consists of the harmonic forms. In particular, the infinitesimal deformation space of L_θ consists of the infinitesimal special Lagrangian deformations of L in V together with translations of θ. (Note that on the second factor, (5.12) equals ${*}D_L$, which is of course consistent with the fact that L_θ is associative if and only if L is special Lagrangian.)

Lemma 5.13. *Let $L \subset V$ be a closed special Lagrangian submanifold. For the associative submanifold $L_\theta \subset \mathbb{S}^1 \times V$, the kernel of $\not{D}$ is the direct sum of the kernel of D_L and the span of $\frac{\partial}{\partial\theta}$.*

Now we study the map from infinitesimal deformations of the G_2–structure, parametrised by $\Omega^3(\mathbb{S}^1 \times V)$, to the obstruction space $\operatorname{coker}\not{D}$. In the identification of $\not{D}$ with (5.12), $\operatorname{coker}\not{D}$ corresponds to $\mathcal{H}^0(L) \oplus \mathcal{H}^1(L)$. The map from $\Omega^3(\mathbb{S}^1 \times V)$ to $\operatorname{coker}\not{D}$ is the composition of a point-wise map $\Lambda^3 T^*_x(\mathbb{S}^1 \times V) \to \Lambda^0 T^*_x L \oplus \Lambda^1 T^*_x L$ and the projection to the harmonic forms. We are interested primarily in torsion-free deformations of $\mathbb{S}^1 \times V$, and (at least for V compact/ACyl with $b^1(V) = 0$) up to diffeomorphism and rescaling of the $\mathbb{S}^1$ factor they are all products.

Lemma 5.14. *Let (σ, τ) be an infinitesimal deformation of the $\mathrm{SU}(3)$-structure (Ω, ω), and let φ_t be a 1-parameter family of G_2–structures with $\frac{d\varphi_t}{dt} = d\theta \wedge \tau + \operatorname{Re}\sigma$. Then $\frac{d}{dt}F(L_\theta, \varphi_t)_{|t=0} \in \Gamma(NL_\theta) \cong \Omega^0(L) \times \Omega^1(L)$ corresponds to $({*}(\operatorname{Im}\sigma_{|L}), {*}(\tau_{|L}))$, and the image in $\operatorname{coker}\not{D} \cong \mathcal{H}^0(L) \oplus \mathcal{H}^1(L)$ to the de Rham projection.*

Proof. Keeping the 3-fold L_θ fixed, $\frac{d}{dt}F(L_\theta, \varphi_t)_{|t=0}$ is a linear function of $\frac{d\varphi_t}{dt}_{|t=0}$. Therefore we may assume that $\varphi_t = d\theta \wedge \omega_t + \operatorname{Re}\Omega_t$ where (Ω_t, ω_t) is an $\mathrm{SU}(3)$–structure deformation of (Ω, ω) tangent to (σ, τ). Then by (2.17) the dual 4-form is $\psi_t = \frac{1}{2}\omega_t^2 - d\theta \wedge \operatorname{Im}\Omega_t$. Recalling from after (5.2) how F is defined in terms ψ, one can check that the image of $F(L_\theta, \varphi_t)$ under $\Gamma(NL_\theta) \cong \Omega^0(L) \oplus \Omega^1(L)$ is $({*}(\operatorname{Im}\Omega_{t|L}), {*}(\omega_{t|L}))$, which implies the result. □

In particular, consider the case when L is a rational homology 3-sphere, *ie* $b^1(L) = 0$, so that L is rigid as a special Lagrangian. If (Ω_t, ω_t) is a 1-parameter family of deformations of the Calabi–Yau structure on V and $\int_L \frac{d\operatorname{Im}\Omega_t}{dt} \neq 0$, then the $\mathbb{S}^1$-family of associatives $\{L_\theta : \theta \in \mathbb{S}^1\}$ is unobstructed with respect to the 1-parameter family $\varphi_t = d\theta \wedge \omega_t + \operatorname{Re}\Omega_t$, in the sense of Corollary 5.5.

Associatives in twisted connected sums. We now put together the results of the section to identify the data we can use to construct associatives in twisted connected-sum G_2–manifolds. As in Theorem 3.13, let $(V_\pm, \omega_\pm, \Omega_\pm)$ be two asymptotically cylindrical Calabi–Yau 3-folds with asymptotic ends of the form $\mathbb{R}^+ \times \mathbb{S}^1 \times S_\pm$ for a pair of hyper-Kähler K3 surfaces $S_\pm$, and $\mathrm{r} : S_+ \to S_-$ a hyper-Kähler rotation. Let M_r be the twisted connected sum of $\mathbb{S}^1 \times V_\pm$, and $\tilde{\varphi}_{T,\mathrm{r}}$ the torsion-free G_2–structure with 'neck length' $2T$ defined in Theorem 3.13.

Proposition 5.15. *Let $C \subset V_+$ be a closed rigid holomorphic curve. Then for sufficiently large T, there is a small deformation of the image of $\mathbb{S}^1 \times C \subset \mathbb{S}^1 \times V_+$ in M_r that is associative with respect to $\tilde{\varphi}_{T,\mathrm{r}}$, and this associative is rigid.*

Proof. By Lemma 5.11, $\mathbb{S}^1 \times C \subset \mathbb{S}^1 \times V_+$ is a rigid associative.

Recall that the G_2–structure $\varphi_{T,\mathrm{r}}$ with small torsion defined before Theorem 3.13 is exactly the product G_2–structure on the complement of $\{t > T\}$ in $\mathbb{S}^1 \times V_+$, and hence near $\mathbb{S}^1 \times C$ when T is large. The C^k norms of the difference between $\varphi_{T,\mathrm{r}}$ and $\tilde{\varphi}_{T,\mathrm{r}}$ are of order $O(e^{-\lambda t})$, so Theorem 5.3 implies that $\mathbb{S}^1 \times C$ can be perturbed to an associative with respect to $\tilde{\varphi}_{T,\mathrm{r}}$ for any sufficiently large T. □

Constructing associatives from closed special Lagrangians $L \subset V_\pm$ requires a little bit more work since, as pointed out above, the associatives $L_\theta = \{\theta\} \times L \subset \mathbb{S}^1 \times V_\pm$ are never rigid. We restrict our attention to the case when $b^1(L) = 0$, so that L is rigid as a special Lagrangian and the obstruction space of L_θ is 1-dimensional. We can find a 1-parameter family of torsion-free deformations of $\mathbb{S}^1 \times V_\pm$ such that the family $\{L_\theta : \theta \in S^1\}$ is unobstructed (in the sense required to apply Corollary 5.5) if the class of L in the homology of $V_\pm$ relative to its boundary is non-zero. (In this section, all homology and cohomology refers to $\mathbb{R}$ coefficients.)

Lemma 5.16. *Let L^m be a closed submanifold of an ACyl manifold V^n with cross-section X. If $[L] \in H_m(V, X)$ is non-zero then there is an exponentially decaying harmonic m-form β on V such that $\int_L \beta \neq 0$.*

Proof. The image of the Poincaré dual of L under $H^{n-m}_{cpt}(V) \to H^{n-m}(V)$ is non-zero, and represented by an exponentially decaying harmonic form α (Lockhart [52, Theorems 7.6 & 7.9]). Take $\beta = *\alpha$. □

Corollary 5.17. *Let $L \subset V_+$ be a compact special Lagrangian with $b^1(L) = 0$, such that $[L] \neq 0 \in H_3(V_+, \mathbb{S}^1 \times S_+)$. Then there is a 1-parameter family of deformations $\varphi_{t,+}$ of the product G_2–structure on $\mathbb{S}^1 \times V_+$, all with the same asymptotic limit as φ_+, with respect to which $\{L_\theta : \theta \in \mathbb{S}^1\}$ is unobstructed in the sense of Corollary 5.5.*

Proof. Take $\beta \in \Omega^3(V_+)$ as in the previous lemma. There is a unique complex 3-form σ on V_+ such that $\operatorname{Im} \sigma = \beta$ and σ is an infinitesimal deformation of Ω as an $SL(3, \mathbb{C})$-structure (*cf* Remark 2.18). Because the map $\beta \mapsto \sigma$ is $SU(3)$-equivariant it maps harmonic forms to harmonic forms. Because $b^1(V_+) = 0$, $(\sigma, 0)$ is an infinitesimal deformation of (Ω, ω) as an $SU(3)$-structure.

$\operatorname{Re} \sigma$ is an infinitesimal deformation of the product G_2–structure $\operatorname{Re} \Omega + d\theta \wedge \omega$, and because it is harmonic it can be integrated to a 1-parameter family of torsion-free deformations $\varphi_{t,+}$ [62, Proposition 6.18]. It follows from Lemma 5.14 that $\{L_\theta : \theta \in \mathbb{S}^1\}$ is unobstructed in this family. □

Since each $\varphi_{t,+}$ ($t \in [-\epsilon, \epsilon]$) has the same asymptotic limit as φ_+, the hyper-Kähler rotation r matches $\varphi_{t,+}$ and φ_-. Thus for T sufficiently large we can define a 1-parameter family of torsion-free G_2–structures $\{\tilde{\varphi}_{t,T,\mathrm{r}} : t \in [-\epsilon, \epsilon]\}$. Corollary 5.5 implies

Proposition 5.18. *Let $L \subset V_+$ be a compact special Lagrangian with $b^1(L) = 0$, such that $[L] \neq 0 \in H_3(V_+, \mathbb{S}^1 \times S_+)$. Then for T large enough there is a smooth map $f : \mathbb{S}^1 \to [-\epsilon, \epsilon]$ and a deformation $\{L'_\theta : \theta \in \mathbb{S}^1\}$ in M_{r} such that each L'_θ is associative with respect to $\tilde{\varphi}_{f(\theta),T,\mathrm{r}}$.*

f has at least 2 critical points, which correspond to associatives that are not rigid.

Remark 5.19. If V_+ is compactifiable in the sense that $V_+ \cong Z_+ \backslash S_+$ for a K3 divisor S_+ with trivial normal bundle in a compact complex manifold Z_+, then $H_3(Z_+) \hookrightarrow H_3(Z_+, \Delta \times S_+) \cong H_3(V_+, \mathbb{S}^1 \times S_+)$ since $H_3(S_+) = 0$. If Z_+ is in turn a blow-up of a weak Fano Y, then the preimage of any closed homologically non-trivial $L \subset Y$ not meeting the blow-up locus or S_+ will represent a non-trivial class in $H_3(V_+, \mathbb{S}^1 \times S_+)$.

6. The matching problem

Recall that Theorem 3.13 allows us to form a twisted connected sum G_2–manifold from any pair of ACyl Calabi–Yau 3-folds $V_\pm$ satisfying a compatibility condition on their asymptotic hyper-Kähler K3 surfaces $S_\pm$. In [21], we constructed large numbers of suitable ACyl Calabi–Yau 3-folds, applying Theorem 3.4—the ACyl version of the Calabi–Yau theorem—to building blocks $Z_\pm$ obtained from semi-Fano 3-folds as in Proposition 3.17. In fact, as we already remarked, varying various choices made in the construction produces families of ACyl Calabi–Yau structures on the same underlying smooth 6-manifold $Z \setminus S$. To complete the construction of G_2–manifolds, it remains to explain how to find *compatible pairs* of such ACyl Calabi–Yau 3-folds; this will require us to exploit the freedom we have to vary the ACyl Calabi–Yau structures on both building blocks.

We reformulate the compatibility condition in terms of existence of “matching data” between a pair of building blocks, which are certain triples of cohomology classes in $L_{\mathbb{R}} \cong H^2(S;\mathbb{R})$. The definition of the matching data is linked to the moduli theory of algebraic K3 surfaces. This formulation will help us prove the existence of many pairs of compatible ACyl Calabi–Yau 3-folds given some additional algebraic geometry input. We remark at the outset that the same pair of deformation families of building blocks $\mathcal{Z}_\pm$ may be matched in different ways and hence give rise to several different twisted connected sum G_2–manifolds.

In this section we describe one convenient strategy for finding matching data which we term “orthogonal gluing”. Given some additional input about the deformation theory of the building blocks used to construct the ACyl Calabi–Yau 3-folds, orthogonal gluing allows us to reduce the problem of finding compatible pairs of ACyl Calabi–Yau 3-folds $V_\pm = Z_\pm \setminus S_\pm$ almost entirely to arithmetic questions about the pair of polarising lattices $N_\pm$ of the building blocks $Z_\pm$. For ACyl Calabi–Yau 3-folds of semi-Fano type the deformation theory we need was developed in [21, §6]. In §7 we use orthogonal gluing to find many compatible pairs of ACyl Calabi–Yau 3-folds.

At the end of §7 we also discuss so-called “handcrafted nonorthogonal gluing”. This allows matching in situations impossible to achieve using orthogonal gluing; the price ones pays is that the method is much more labour-intensive as it requires more precise information about K3 moduli spaces.

Reformulating the existence of hyper-Kähler rotations. Let us first recall the set-up for the gluing Theorem 3.13. $V_\pm$ is a pair of ACyl Calabi–Yau 3-folds with asymptotic limits $\mathbb{R}^+ \times \mathbb{S}^1 \times S_\pm$. The $S_\pm$ are K3 surfaces, with preferred complex structure $I_\pm$, Kähler form $\omega^I_\pm$ and holomorphic volume form $\Omega_\pm$. Because this is a hyper-Kähler structure, there are further complex structures $J_\pm$ and $K_\pm$, with Kähler forms $\omega^J_\pm$ and $\omega^K_\pm$ ($\Omega_\pm = \omega^J_\pm + i\omega^K_\pm$). The compatibility condition for V_+ and V_- is that $S_\pm$ are related by a *hyper-Kähler rotation* as in Definition 3.10: we need an orientation-preserving isometry $\mathrm{r}\colon S_+ \to S_-$ such that $\mathrm{r}^*(I_-) = J_+$

and $\mathsf{r}^*(J_-) = I_+$ (with the isometry condition this implies $\mathsf{r}^*(K_-) = -K_+$). Equivalently, $\mathsf{r}^*\omega_-^I = \omega_+^J$, $\mathsf{r}^*\omega_-^J = \omega_+^I$ and $\mathsf{r}^*\omega_-^K = -\omega_+^K$.

We use the Torelli theorem to reduce this relation to the action on cohomology.

Lemma 6.1. *Let $h : H^2(S_-;\mathbb{Z}) \to H^2(S_+;\mathbb{Z})$ be an isometry, extend it to $H^2(S_-;\mathbb{R}) \to H^2(S_+;\mathbb{R})$, and suppose that*

$$h[\omega_-^I] = [\omega_+^J], \quad h[\omega_-^J] = [\omega_+^I] \quad \textit{and} \quad h[\omega_-^K] = -[\omega_+^K].$$

Then there is a hyper-Kähler rotation $\mathsf{r} : S_+ \to S_-$ such that $\mathsf{r}^* = h$.

Proof. Consider the complex structure J_- on S_-. $\omega_-^I - i\omega_-^K$ is a holomorphic 2-form with respect to J_-. Therefore h maps $H^{2,0}(S_-, J_-)$ to $H^{2,0}(S_+, I_+)$, *ie* it is a Hodge isometry between the complex K3 surfaces (S_-, J_-) and (S_+, I_+). Moreover, the Kähler class $[\omega_-^J]$ is mapped to the Kähler class $[\omega_+^I]$. Therefore the strong Torelli theorem [10, Chapter VIII, Section 11] implies that there is a holomorphic map $\mathsf{r}\colon (S_+, I_+) \to (S_-, J_-)$ such that $\mathsf{r}^* = h$. Since the holomorphic 2-forms are uniquely determined by their de Rham cohomology classes, $\mathsf{r}^*\omega_-^I = \omega_+^J$ and $\mathsf{r}^*\omega_-^K = -\omega_+^K$. Further $\mathsf{r}^*\omega_-^J = \omega_+^I$, by uniqueness of a Ricci-flat Kähler metric in its cohomology class. Thus r is a hyper-Kähler rotation. □

It is useful to rephrase the previous lemma in the language of the moduli theory of K3 surfaces. Recall that a *marking* of a complex K3 surface (S, I) is an isometry $L \cong H^2(S;\mathbb{Z})$. $H^{2,0}(S) \subset H^2(S;\mathbb{C})$ can be identified with an oriented real 2-plane in $H^2(S;\mathbb{R})$, and its image in $L_\mathbb{R}$ is the *period* of the marked K3 surface.

Proposition 6.2. *Let (k_0, k_+, k_-) be an orthonormal triple of positive classes in $L_\mathbb{R}$. Let $(S_\pm, I_\pm)$ be complex K3 surfaces with markings $h_\pm : L \to H^2(S_\pm;\mathbb{Z})$ such that $\langle k_\mp, \pm k_0\rangle$ is the period point, and $h_\pm(k_\pm)$ is a Kähler class on $S_\pm$. Let $h = h_+ \circ h_-^{-1} : H^2(S_-;\mathbb{Z}) \to H^2(S_+;\mathbb{Z})$. Then there exist unique hyper-Kähler structures $(\omega_\pm^I, \omega_\pm^J, \omega_\pm^K)$ on $S_\pm$ with $[\omega_\pm^I] = h_\pm(k_\pm)$ and $\omega_\pm^J + i\omega_\pm^K$ holomorphic with respect to $I_\pm$, such that there is a hyper-Kähler rotation $\mathsf{r} : S_+ \to S_-$ with $\mathsf{r}^* = h$.*

Proof. The Kähler class $h_\pm(k_\pm)$ contains a unique Ricci-flat Kähler metric $\omega_\pm^I$. Up to complex scalar multiplication, there is a unique 2-form $\omega_\pm^J + i\omega_\pm^K$ on $S_\pm$ that is holomorphic with respect to $I_\pm$. Since $\langle k_\mp, \pm k_0\rangle$ is the period with respect to the marking $h_\pm$, we can normalise it so that $[\omega_\pm^J] = h_\pm(k_\mp)$ and $[\omega_\pm^K] = h_\pm(\pm k_0)$. Then $(\omega_\pm^I, \omega_\pm^J, \omega_\pm^k)$ are hyper-Kähler structures. This choice of normalisation is the only one for which $h[\omega_-^I] = [\omega_+^J]$, $h[\omega_-^J] = [\omega_+^I]$ and $h[\omega_-^K] = -[\omega_+^K]$, which is equivalent to the existence of a hyper-Kähler rotation with $\mathsf{r}^* = h$. □

Matching data for pairs of building blocks. The asymptotic hyper-Kähler K3 surfaces $S_\pm$ of our ACyl Calabi–Yau 3-folds $V_\pm$ come with a preferred complex structure $I_\pm$ and Kähler form $\omega_\pm^I$ defined by the asymptotic limit of the Calabi–Yau structure. We need to take this fact into account when we attempt to construct hyper-Kähler rotations between S_+ and S_-.

Definition 6.3. A set of *matching data for a pair of building blocks* $(Z_\pm, S_\pm)$ is a triple (k_+, k_-, k_0) of classes in $L_\mathbb{R}$ for which there are markings $h_\pm : L \to H^2(S_\pm;\mathbb{Z})$ such that $\langle k_\mp, \pm k_0\rangle$ is the period point of the marked K3 $(S_\pm, I_\pm, h_\pm)$, and $h_\pm(k_\pm)$ is the restriction to $S_\pm$ of a Kähler class on $Z_\pm$.

With this terminology, the following is an immediate consequence of Theorem 3.4 and Proposition 6.2.

Corollary 6.4. *If there is matching data for the pair of building blocks $(Z_\pm, S_\pm)$ then $V_\pm = Z_\pm \setminus S_\pm$ admit compatible ACyl Calabi–Yau structures,* ie *there exists a hyper-Kähler rotation*

$\mathsf{r}\colon S_+ \to S_-$. *Hence the twisted connected sum 7-manifold* M_{r} *formed by gluing* $\mathbb{S}^1 \times V_\pm$ *using the diffeomorphism specified in* (3.11) *admits* G_2*–metrics as described in Theorem 3.13.*

The markings $h_\pm$ in Definition 6.3 are not necessarily unique. Hence nor is $h = h_+ \circ h_-$ or the hyper-Kähler rotation r, and the choices can affect the topology of the G_2–manifold M_{r}. We can relate this to the computations of §4 as follows.

Recall that it is part of Definition 3.5 that if (Z, S) is a building block and $H^2(S;\mathbb{Z}) \cong L$ is a marking, then the image $N \subset L$ of $H^2(Z;\mathbb{Z})$ is primitive. According to Lemma 3.6, $N \subseteq \operatorname{Pic} S$. This means that S is a marked N*-polarised K3 surface.* Since $H^{2,0}(S)$ is perpendicular to $H^{1,1}(S)$, the period of such a marked N-polarised K3 surface is perpendicular to N; it must lie in the *Griffiths domain* D_N of oriented positive 2-planes $\Pi \subset N^\perp \subset L_\mathbb{R}$. D_N can be considered as an open subset (determined by the positivity) of $\mathbb{P}(\Lambda^2(N^\perp \otimes \mathbb{C}))$, and hence as a complex manifold. For a deformation family $\mathcal{Z}$ of building blocks, all members have the same polarising lattice N and the primitive embedding $N \hookrightarrow L$ is well-defined up to the action of $O(L)$.

Let (k_+, k_-, k_0) be a set of matching data for a pair of blocks $(Z_\pm, S_\pm)$. A choice of markings $h_\pm : L \to H^2(S_\pm;\mathbb{Z})$ in Definition 6.3 determines embeddings $N_\pm \hookrightarrow L$ of the polarising lattices. While each embedding is unique up to the action of $O(L)$, the *pair* is not, *eg* $N_+ \cap N_-$ could vary. Since $N_+ \cap N_-$ is a summand in $H^2(M;\mathbb{Z})$, the choice of markings can affect the topology of the G_2–manifold produced in Corollary 6.4. We say that the matching data is *adapted* to a given pair of embeddings $N_\pm \hookrightarrow L$ if the markings $h_\pm$ can be taken to be $N_\pm$-polarised. A necessary condition is that $\langle k_\mp, \pm k_0\rangle \perp N_\pm$.

Given a semi-Fano 3-fold Y, we can blow up to get a building block (Z, S) according to Proposition 3.17. In this case the polarising lattice $N \subset L$ of the block is simply given by the primitive embedding $H^2(Y;\mathbb{Z}) \to H^2(S;\mathbb{Z}) \cong L$, isometric with respect to the form $(-K_Y) \cdot D_1 \cdot D_2$ on $H^2(Y;\mathbb{Z})$. By deforming the semi-Fano 3-fold Y and varying the choice of smooth section $S \in |-K_Y|$ we obtain a family of building blocks $\mathcal{Z}$, all with the same topology and polarising lattice $N \cong H^2(Y;\mathbb{Z})$. In this way we can obtain ACyl Calabi–Yau manifolds $V = Z \setminus S$ with potentially different asymptotic hyper-Kähler K3 surfaces S.

Given a pair $\mathcal{Z}_\pm$ of such families of building blocks, we can now approach the problem of using them to construct a compact G_2–manifold as follows.

1. Choose embeddings $N_\pm \hookrightarrow L$ in the $O(L)$-orbits of primitive isometric embeddings determined by $\mathcal{Z}_\pm$. Let $T_\pm = N_\pm^\perp \subset L$ denote the orthogonal complements.
2. Consider triples (k_+, k_-, k_0) such that $k_\pm \in N_\pm(\mathbb{R}) \cap T_\mp(\mathbb{R})$ and $k_0 \in T_+(\mathbb{R}) \cap T_-(\mathbb{R})$. Then $\langle k_\mp, \pm k_0\rangle$ lives in the Griffiths period domain $D_{N_\pm}$ for $N_\pm$-polarised K3 surfaces. Find a triple that forms matching data, adapted to the chosen embeddings $N_\pm \hookrightarrow L$, for some $(Z_\pm, S_\pm) \in \mathcal{Z}_\pm$.
3. Apply Corollary 6.4 to construct matching ACyl Calabi–Yau structures on $V_\pm = Z_\pm \setminus S_\pm$.
4. Apply Theorem 3.13 to glue $\mathbb{S}^1 \times V_\pm$ to a compact G_2–manifold M.

Remark. In a sense this scheme reverse engineers the process described in §3: in effect, we first identify what hyper-Kähler K3 to aim for, and then construct ACyl Calabi–Yau 3-folds with that asymptotic K3 (up to hyper-Kähler rotation).

We can then use the results in §4 to compute topological invariants of M. Note that the cohomology of M depends only on the cohomology of the building blocks and on the choice of the pair of embeddings $N_\pm \hookrightarrow L$. In many cases we can determine the diffeomorphism type of M. Also, if either building block $(Z_\pm, S_\pm)$ contains rigid complex curves (*eg* if Z is a building block of semi-Fano type where the semi-Fano Y is obtained as a small resolution of a nodal Fano) then Proposition 5.15 shows that M contains corresponding rigid associative submanifolds.

So the key problem that remains to be addressed is to find the matching data in Step 2. This is a difficult problem in general and in most cases we do not currently understand all possible ways to match a given pair of deformation families of building blocks $\mathcal{Z}_\pm$. In this section we describe a general method which we call *orthogonal gluing* that yields large numbers of matching ACyl Calabi–Yau structures.

Orthogonal gluing. Let $\mathcal{Z}_\pm$ be a pair of families of building blocks, obtained from semi-Fano 3-folds $Y_\pm$ of given deformation types $\mathcal{Y}_\pm$. We describe a method that provides matching data for a large class of such pairs.

- Choose the pair of primitive embeddings $N_\pm \hookrightarrow L$ so that N_+ and N_- *intersect orthogonally*, *ie* $N_\pm(\mathbb{R}) = (N_\pm(\mathbb{R}) \cap N_\mp(\mathbb{R})) \oplus (N_\pm(\mathbb{R}) \cap T_\mp(\mathbb{R}))$ (in other words, the reflections in $N_\pm(\mathbb{R})$ commute).
- In addition, arrange that some elements of $N_\pm(\mathbb{R}) \cap T_\mp(\mathbb{R})$ correspond to Kähler classes of some $Y_\pm$ under some marking of $S_\pm \subset Y_\pm$ (as pointed out in Remark 3.19, this is more restrictive than asking for Kähler classes on $S_\pm$.)
- Show that for a generic positive $k_0 \in T_+(\mathbb{R}) \cap T_-(\mathbb{R})$, choosing generic positive $k_\pm \in N_\pm(\mathbb{R}) \cap T_\mp(\mathbb{R})$ gives $\langle k_\mp, \pm k_0 \rangle \in D_{N_\pm}$ that are periods of some $S_\pm \subset Y_\pm \in \mathcal{Y}_\pm$, and use this to prove that $k_\pm$ can be taken to correspond to Kähler classes on $Y_\pm$.
- Blow up $Y_\pm$ according to Proposition 3.17 to form building blocks $(Z_\pm, S_\pm)$. The facts that the K3 fibres $S_\pm$ are isomorphic to the chosen K3 divisors in the semi-Fanos $Y_\pm$ and that the image of the Kähler cone of $Z_\pm$ contains that of $Y_\pm$ imply that (k_+, k_-, k_0) is a set of matching data for this pair of building blocks.

Lattice push-outs and embeddings. To complete the first step, we first try to find an "orthogonal push-out" W of N_+ and N_-, and then try to embed W in L so that the inclusions $N_\pm \hookrightarrow L$ are primitive. The last condition is obviously satisfied if the embedding of W itself is primitive, and the existence of such embeddings can often be deduced from results of Nikulin [61].

Definition 6.5. Let R, N_+, N_- be nondegenerate lattices, and assume given primitive inclusions $R \hookrightarrow N_+$, $R \hookrightarrow N_-$. An *orthogonal pushout* $W = N_+ \perp_R N_-$ is a nondegenerate lattice, with a diagram of primitive inclusions, where

$$\begin{tikzcd} & N_+ \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & W \\ & N_- \arrow[ur] & \end{tikzcd}$$

- $R = N_+ \cap N_-$, $W = N_+ + N_-$;
- $N_+^\perp \subset N_-$, $N_-^\perp \subset N_+$.

Note that W is unique, though it does not always exist, *eg* see Example 6.8.

Remark 6.6. In all cases in this paper, N_+ and N_- have signature $(1, r_+ - 1)$ and $(1, r_- - 1)$ and the "intersection" R is negative definite of rank ρ. This ensures that the orthogonal pushout W has signature $(2, r_+ + r_- - \rho - 2)$.

The perpendicular direct sum $N_+ \perp N_-$ is always an orthogonal push-out with $R = N_+ \cap N_- = 0$. We will refer to gluing that uses this push-out as *perpendicular gluing*, while the term *orthogonal gluing* allows push-outs with non-trivial intersection R. Some statements simplify for perpendicular gluing (*eg* the computation of $\operatorname{div} p_1(M)$ in Corollary 4.30), but most nice properties are enjoyed by orthogonal gluing too. Most important is the matching method in Proposition 6.18, but there is also a convenient Betti number formula.

Lemma 6.7. *Any* G_2*–manifold* M *constructed by orthogonal gluing of blocks* $Z_\pm$ *satisfies*

$$b^2(M) + b^3(M) = b^3(Z_+) + b^3(Z_-) + 2\operatorname{rk} K_+ + 2\operatorname{rk} K_- + 23.$$

Proof. The orthogonality assumption implies that $N_+ + N_- = (N_+ \cap N_-) \oplus (T_+ \cap N_-) \oplus (N_+ \cap T_-)$ (over $\mathbb{R}$), so the ranks of the terms $L/_{N_+ + N_-}$, $N_+ \cap N_-$, $T_+ \cap N_-$ and $N_+ \cap T_-$ in Theorem 4.8 add up to 22. □

Remark. Note the previous formula is not always valid if M is not constructed by orthogonal gluing: see Example *No 11* in Section 7.

In general, it is not difficult to state a simple criterion for the existence of orthogonal pushouts in terms of *discriminant groups* $N_\pm^*/N_\pm$; we do not need to do so here. Instead we demonstrate by the next example that they do not always exist.

Example 6.8. We show a simple situation where the orthogonal pushout does not exist. Indeed, consider the two (isomorphic) lattices N_+, N_- with quadratic form

$$\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$$

(this is the Picard lattice of a general quartic K3 surface containing a line). Let us try to form an orthogonal pushout along the common sublattice R perpendicular to the basis vector $\mathbf{e}_1$ of square-norm 4. Now R is generated by the vector $\mathbf{e}_2' = (-1, 4)$ of square-norm -36. Using the rational basis $\mathbf{e}_1, \mathbf{e}_2'$, we can say

$$N_+ = N_- = \mathbb{Z}^2 + \frac{1}{4}(1,1)\mathbb{Z} \supset \mathbb{Z}^2 \quad \text{with quadratic form} \quad \begin{pmatrix} 4 & 0 \\ 0 & -36 \end{pmatrix}.$$

Thus, if the orthogonal pushout $W = N_+ \perp_R N_-$ exists, then

$$W = \mathbb{Z}^3 + \frac{1}{4}(1,1,0)\mathbb{Z} + \frac{1}{4}(0,1,1)\mathbb{Z} \supset \mathbb{Z}^3 \quad \text{with quadratic form} \quad \begin{pmatrix} 4 & 0 & 0 \\ 0 & -36 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Note, however, that, in this lattice:

$$<\tfrac{1}{4}(1,1,0), \tfrac{1}{4}(0,1,1)> = \begin{pmatrix} \frac14 & \frac14 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -36 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ \frac14 \\ \frac14 \end{pmatrix} = \begin{pmatrix} \frac14 & \frac14 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \\ 1 \end{pmatrix} = -\frac{9}{4}$$

is not an integer, that is, W is not an integral lattice.

Once we have an orthogonal push-out W, we look for an embedding W in L such that the inclusions $N_\pm \hookrightarrow L$ are primitive. In many cases, the following result guarantees the existence of a primitive lattice embedding $W \subset L$. Here $\ell(W)$ denotes the minimal number of generators for the discriminant group W^*/W of a non-degenerate lattice W; in particular $\ell(W) \leq \operatorname{rk} W$.

Theorem 6.9. *Let W be an even non-degenerate lattice of signature (t_+, t_-), and L an even unimodular lattice of indefinite signature (ℓ_+, ℓ_-). There exists a primitive embedding $W \hookrightarrow L$ if $t_+ \leq \ell_+$, $t_- \leq \ell_-$ and*
 (i) $2 \operatorname{rk} W \leq \operatorname{rk} L$, *or*
 (ii) $\operatorname{rk} W + \ell(W) < \operatorname{rk} L$

Proof. (i) is Nikulin [61, Theorem 1.12.4], while (ii) is [61, Corollary 1.12.3] (see also Dolgachev [29, Theorem 1.4.6]). □

If there is a primitive embedding of W into the (unimodular) lattice L, with orthogonal complement $T = W^\perp$, then $W^* \cong L/T$ and $T^* \cong L/W$ imply that $W^*/W \cong L/(W \perp T) \cong T^*/T$, *ie* the discriminant groups are isomorphic. In particular $\ell(W) \leq \operatorname{rk} T$, so

$$\operatorname{rk} W + \ell(W) \leq \operatorname{rk} L \tag{6.10}$$

is a *necessary* condition for W to be primitively embeddable in L.

In our application L will be the K3 lattice and W will be the orthogonal pushout of a pair of lattices $N_\pm$—the polarising lattices of a pair of building blocks $Z_\pm$. Therefore $(\ell_+, \ell_-) = (3, 19)$, while $t_+ = 2$ and $\operatorname{rk} W \leq \operatorname{rk} N_+ + \operatorname{rk} N_-$ with equality if and only if $W = N_+ \perp N_-$. Hence a sufficient condition for the existence of a primitive embedding $W \hookrightarrow L$ is that

$$\operatorname{rk} N_+ + \operatorname{rk} N_- \leq 11. \tag{6.11}$$

Sometimes we will look more closely at the discriminant groups, and apply 6.9(ii). Note that the discriminant group of $N_+ \perp N_-$ is simply the product of the discriminant groups of the two terms. In particular $\ell(N_+ \perp N_-) \leq \ell(N_+) + \ell(N_-)$ (but equality need not hold, *eg* if the discriminants are coprime).

Remark 6.12. For any lattice N, N^* has a natural quadratic form, given in terms of a basis by the inverse of the matrix of the form on N. The restriction to N is the original form on N, so if N is even then the discriminant group N^*/N has a well-defined $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form. For any overlattice W' of $N_+ \perp N_-$ such that $N_\pm \hookrightarrow W'$ are primitive, the images of W' in $N_\pm^*/N_\pm$ are anti-isometric with respect to the discriminant forms. This sets up a correspondence between such overlattices and pairs of anti-isometric subgroups of the discriminant groups. We will sometimes use this to find overlattices, and since the overlattice has smaller discriminant group they can be easier to embed in the K3 lattice L.

The method of the proof of Theorem 6.9 is to show that given W, there exists a lattice T with anti-isometric discriminant group. Then the maximal overlattice of $W \perp T$ is unimodular, and isometric to L by the classification of unimodular indefinite lattices.

Dolgachev [29, Theorem 1.4.8], following Nikulin [61, 1.14.1-2], also gives a sufficient condition for the primitive embedding to be unique.

Theorem 6.13. *If in addition* $\operatorname{rk} W + \ell(W) + 2 \leq \operatorname{rk} L$ *then the primitive embedding from Theorem 6.9 is unique up to automorphisms of* L*.*

Deformation theory and matching. In order to find matching data for building blocks of semi-Fano type we use the deformation theory input provided by Proposition 6.15. See Beauville [12] for a more detailed review of the relevant deformation theory in the context of Fano 3-folds and [21] for the extension to the semi-Fano case needed here.

Definition 6.14. Fix an abstract lattice N, and an element $A \in N$ with $A^2 = 2g - 2 > 0$.

- An *N-polarised semi-Fano 3-fold* is a semi-Fano 3-fold Y together with an isometry $N \cong \operatorname{Pic}(Y)$ sending A to $-K_Y$.
- A *family of N-polarised semi-Fano 3-folds* is a smooth projective morphism $f\colon \mathcal{Y} \to B$ of noetherian schemes, all of whose fibres Y_b are semi-Fano 3-folds; and a sheaf isometry $g\colon N \to \underline{\operatorname{Pic}}(\mathcal{Y}/B)$ such that for each $b \in B$, Y_b together with $g_b : N \to \operatorname{Pic}(Y_b)$ is an N-polarised semi-Fano.
- Two N-polarised semi-Fano 3-folds Y_1, Y_2 are *deformation equivalent* if there is a connected scheme B, a family $f\colon \mathcal{Y} \to B$ and $b_1, b_2 \in B$ such that $Y_1 = f^{-1}(b_1)$, $Y_2 = f^{-1}(b_2)$.
- A *deformation type* is a deformation equivalence class $\mathcal{Y}$ of semi-Fano 3-folds.

For a smooth $S \in |-K_Y|$ in an N-polarised semi-Fano 3-fold Y, the composition $N \cong H^2(Y;\mathbb{Z}) \to H^2(S;\mathbb{Z}) \cong L$ defines a primitive embedding $N \hookrightarrow L$. For a deformation type $\mathcal{Y}$ of N-polarised semi-Fanos, this gives an embedding $N \hookrightarrow L$ that is well-defined up to the action of $\mathrm{O}(L)$. The precise definition of the deformation type $\mathcal{Y}$ is actually not that crucial in this paper. For the application, it suffices to know that given a semi-Fano 3-fold Y, there is a collection $\mathcal{Y}$ of semi-Fano 3-folds with the same topology, so that the following result holds.

Proposition 6.15 ([21, Proposition 6.9]). *Fix a primitive lattice $N \subset L$, and let D_N be the Griffiths domain $\{\Pi \in \mathbb{P}(\Lambda^2(N^\perp \otimes \mathbb{C})) : \Pi \wedge \bar{\Pi} > 0\}$. Let $\mathcal{Y}$ be a deformation type of N-polarised semi-Fano 3-folds Y such that for $S \subset Y$ a smooth anticanonical K3 divisor the restriction map $\operatorname{Pic} Y \to H^2(S;\mathbb{Z})$ is equivalent (for the chosen polarisation $N \cong \operatorname{Pic} Y$ and some isomorphism $H^2(S;\mathbb{Z}) \cong L$) to the inclusion $N \hookrightarrow L$. Then there exist*

- *a $U_{\mathcal{Y}} \subseteq D_N$ with complement a locally finite union of complex analytic submanifolds of positive codimension*
- *an open subcone $\mathrm{Amp}_{\mathcal{Y}}$ of the positive cone of $N_{\mathbb{R}}$*

with the following property: for any $\Pi \in U_{\mathcal{Y}}$ and $k \in \mathrm{Amp}_{\mathcal{Y}}$ there is $Y \in \mathcal{Y}$, a smooth $S \in |-K_Y|$ and a marking $h : L \to H^2(S;\mathbb{Z})$ such that $h(\Pi) = H^{2,0}(S)$, and $h(k)$ is the restriction to S of a Kähler class on Y.

Remark 6.16. It is important to distinguish $\mathrm{Amp}_{\mathcal{Y}}$ from the cone $\mathrm{Amp}_S \subset N_{\mathbb{R}}$ of Kähler classes on S. For example, if Y is semi-Fano (but not Fano) with small anticanonical morphism then $-K_Y$ is not a Kähler class on Y but it is when restricted to a generic S. $\mathrm{Amp}_{\mathcal{Y}}$ can be a proper subcone of Amp_S also for genuine Fanos when the Picard rank is ≥ 2, *eg* $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

If we apply Proposition 3.17 to $\mathcal{Y}$ to construct a family of semi-Fano type blocks $\mathcal{Z}$, then it is immediate that $\mathcal{Z}$ has the following property (with $\mathrm{Amp}_{\mathcal{Z}} = \mathrm{Amp}_{\mathcal{Y}}$).

Definition 6.17. Let $N \subset L$ be a primitive sublattice, and $\mathrm{Amp}_{\mathcal{Z}}$ an open subcone of the positive cone in $N_{\mathbb{R}}$. We say that a family of building blocks $\mathcal{Z}$ is $(N, \mathrm{Amp}_{\mathcal{Z}})$-*generic* if there is $U_{\mathcal{Z}} \subseteq D_N$ with complement a locally finite union of complex analytic submanifolds of positive codimension with the property that: for any $\Pi \in U_{\mathcal{Z}}$ and $k \in \mathrm{Amp}_{\mathcal{Z}}$ there is a building block $(Z, S) \in \mathcal{Z}$ and a marking $h : L \to H^2(S;\mathbb{Z})$ such that $h(\Pi) = H^{2,0}(S)$, and $h(k)$ is the image of the restriction to S of a Kähler class on Z.

Given an embedding in L of the orthogonal push-out we can now solve the matching problem for semi-Fano 3-folds.

Proposition 6.18. *Let $N_\pm \subset L$ be primitive sublattices of signature $(1, r_\pm - 1)$, and let $\mathcal{Z}_\pm$ be $(N_\pm, \mathrm{Amp}_{\mathcal{Z}_\pm})$-generic families of building blocks. Suppose that*

(i) *$R = N_+ \cap N_-$ is negative definite of rank ρ,*
(ii) *$W = N_+ + N_-$ is an orthogonal pushout.*

Denote by $T_\pm = N_\pm^\perp$ the transcendental lattices, and let $W_\pm = T_\mp \cap N_\pm \subset N_\pm$ be the perpendicular of $N_\mp$ in $N_\pm$. Assume also that

(iii) *$W_\pm \cap \mathrm{Amp}_{\mathcal{Z}_\pm} \neq \emptyset$.*

Then there exist $(Z_\pm, S_\pm) \in \mathcal{Z}_\pm$ and $N_\pm$-polarised markings $h_\pm : L \to H^2(S_\pm;\mathbb{Z})$ with period points $\langle k_\mp, \pm k_0\rangle$, for an orthonormal triple of positive classes (k_+, k_-, k_0) in $L_{\mathbb{R}}$ such that $k_\pm \in \mathrm{Amp}_{\mathcal{Z}_\pm}$, ie *$(k_+, k_-, k_0)$ is a set of matching data adapted to the chosen pair of embeddings $N_\pm \hookrightarrow L$.*

Proof. Let $T = W^\perp$. $W_\pm(\mathbb{R})$ and $T(\mathbb{R})$ are real vector spaces of signature $(1, r_\pm - \rho - 1)$ and $(1, 21 - r)$ respectively, where $r = \operatorname{rk} W = r_+ + r_- - \rho$. A priori, $k_\pm$ and k_0 must belong to the positive cones $W_\pm(\mathbb{R})^+$ and $T(\mathbb{R})^+$ respectively. Consider the real manifold

$$D = \mathbb{P}\big(W_+(\mathbb{R})^+\big) \times \mathbb{P}\big(W_-(\mathbb{R})^+\big) \times \mathbb{P}\big(T(\mathbb{R})^+\big)\,.$$

Below, we need the open subset $\mathcal{A} = \{(\ell_+, \ell_-, \ell) \in D : \ell_\pm \subset \mathrm{Amp}_{\mathcal{Z}_\pm}\}$. By hypothesis (iii), $\mathcal{A}$ is nonempty. We have two Griffiths period domains

$$D_{N_\pm} = \{\Pi^2 \subset T_\pm(\mathbb{R}) \mid \langle \bullet, \bullet\rangle_{|\Pi^2} > 0\},$$

and projections

$$\mathrm{pr}_\pm\colon D \to D_{N_\pm},\ (\ell_+, \ell_-, \ell) \mapsto \langle \ell_\mp, \pm\ell\rangle.$$

As we stated previously, $D_{N_\pm}$ can be regarded as an open subset of $\mathbb{P}(N_\pm^\perp \otimes \mathbb{C})$; if α, β is an oriented orthonormal basis of $\Pi \in D_{N_\pm}$ then $\Pi \mapsto \langle \alpha + i\beta\rangle \in \mathbb{P}(N_\pm^\perp \otimes \mathbb{C})$. Given a choice α and β, we can identify $T_\Pi D_{N_\pm}$ with pairs (u, v) of vectors in $\Pi^\perp \subseteq T_\pm(\mathbb{R})$. Then the complex structure on $T_\Pi D_{N_\pm}$ is given by $J : (u, v) \mapsto (-v, u)$.

Observe that the real analytic embedded submanifold $\mathbb{P}\big(W_\mp(\mathbb{R})^+\big) \times \mathbb{P}\big(T(\mathbb{R})^+\big) \hookrightarrow D_{N_\pm}$ is totally real: the tangent space $\mathcal{T}$ at $\Pi = \langle w, t\rangle$, $w \in W_\mp$, $t \in T(\mathbb{R})$ corresponds to (u, v) such that $u \in w^\perp \subseteq W(\mathbb{R})$ and $v \in t^\perp \subseteq T_\mp(\mathbb{R})$, so $J(\mathcal{T})$ is transverse to $\mathcal{T}$. Now the key point is that the condition that N_+ and N_- intersect orthogonally ensures that this totally real submanifold has *maximal dimension*: $\dim_\mathbb{C} D_{N_\pm} = 20 - r_\pm$, and

$$\dim_\mathbb{R} \mathbb{P}\big(W_\mp(\mathbb{R})^+\big) \times \mathbb{P}\big(T(\mathbb{R})^+\big) = \big(r_\mp - \rho - 1\big) + \big(22 - r - 1\big) = 20 - r_\pm.$$

In particular, the submanifold is Zariski dense (in a complex analytic sense), so it must intersect the subset $U_{\mathcal{Z}_\pm} \subset D_{N_\pm}$ from Definition 6.17. Actually, we need to use a stronger consequence: the complement of the preimage of $U_{\mathcal{Z}_\pm}$ in $\mathbb{P}\big(W_\mp(\mathbb{R})^+\big) \times \mathbb{P}\big(T(\mathbb{R})^+\big)$ is a locally finite union of real analytic subsets of positive codimension. Because $\mathrm{pr}_\pm$ is a projection of a product manifold onto a factor the same is true for $\mathrm{pr}_\pm^{-1}(U_{\mathcal{Z}_\pm}) \subset D$. To conclude the proof, take $(\ell_+, \ell_-, \ell) \in \mathcal{A} \cap \mathrm{pr}_+^{-1}(U_{\mathcal{Z}_+}) \cap \mathrm{pr}_-^{-1}(U_{\mathcal{Z}_-})$, and let $k_\pm \in \ell_\pm, k_0 \in \ell$ be unit vectors. □

Proposition 6.18 fulfils the plan for finding compatible semi-Fano type ACyl Calabi–Yau 3-folds outlined at the start of the orthogonal gluing subsection. The non-symplectic type blocks of Kovalev and Lee [49] also satisfy the condition in Definition 6.17, as do some families of blocks obtained by resolving non-generic anticanonical pencils on semi-Fanos, *eg* Example 7.9. So we can solve the matching problem for these kinds of blocks by the same method.

Remark 6.19. Note that in perpendicular gluing, hypothesis (iii) is automatically satisfied. This condition may look innocuous, but it adds an extra layer of difficulty to the problem of finding suitable orthogonal but non-perpendicular pushouts.

For families of non-symplectic type blocks, we may take $\mathrm{Amp}_\mathcal{Z}$ in Definition 6.17 to be the full Kähler cone Amp_S of a generic N-polarised K3 surface (*cf* Remark 3.20). Modulo choice of markings, this consists of all positive classes in $N_\mathbb{R}$ that are orthogonal to all -2 classes in N. For these blocks, hypothesis (iii) is therefore equivalent to R not containing any -2 classes. This is always a necessary condition, but for semi-Fano type blocks it is not sufficient, *cf* Example 8.3.

7. Examples: G_2–manifolds

Our aim in this section is to present in detail concrete examples of G_2–manifolds that illustrate the main points of what is achievable by our constructions. In Section 8 we will give a more systematic overview of the range of examples one can construct using these methods and some remarks on the basic "geography" of the examples. For each example in this section we compute the integral cohomology groups, the number of associative submanifolds arising from the construction, and the first Pontrjagin class. Many of the examples are 2-connected, and for most of these we can determine the diffeomorphism type completely using the classification theorems 4.22 and 4.25.

All examples except *No 11* are constructed using perpendicular or orthogonal gluing. We mostly stick to the building blocks of semi-Fano type that we described in detail in our earlier paper [21, §7]; these building blocks are described briefly in Examples 7.1–7.12. *No 11* uses "handcrafted nonorthogonal gluing". This method allows us to construct examples not possible using orthogonal gluing; the main drawback is that the method requires much more explicit information about K3 moduli spaces than orthogonal gluing. This can make constructing such examples a very labour-intensive process. Here we give only the simplest possible example to illustrate how the method works and its potential subtleties.

We close the section with a pair of examples (Examples 7.16 and 7.17) in which we can construct families of associative 3-folds (recall Proposition 5.18) because of the existence of suitable special Lagrangian submanifolds of the building blocks.

Building blocks. A small number of representative examples of building blocks (Z, S), together with computations of their topological and geometric invariants, is given in [21, §7]. Here we give a very brief description of these examples: see also Tables 1, 2 and 4. In each case the polarising lattice N (the image of $H^2(Z) \to H^2(S)$) has a unique primitive embedding in L; except in Example 7.7 this is a direct consequence of Theorem 6.13.

The building blocks Z in Examples 7.1–7.7 are of Fano or semi-Fano type, *ie* Z is the blow-up of a smooth Fano or semi-Fano Y in the base locus of a generic anticanonical pencil on Y (recall Proposition 3.17). Below we will list the Fano or semi-Fano Y we use to construct the building block Z.

Example 7.1. Take Y to be a Fano "of the first species", *ie* a member of one of the 17 deformation families of smooth Fano 3-folds with Picard rank 1. The building blocks Z which arise this way—which we call *building blocks of rank one Fano type*—are listed in Table 1. In the descriptions of our examples of twisted connected sums, "7.1^r_d" refers to the building block obtained from the rank 1 Fano Y with index r and degree $-(\frac{1}{r}K_Y)^3 = d$. The polarising lattice is $N = \langle rd \rangle$.

Y	r	$-K_Y^3$	$b^3(Y)$	$b^3(Z)$	$\operatorname{div} c_2(Z)$
$\mathbb{P}^3$	4	4^3	0	66	2
$Q_2 \subset \mathbb{P}^4$	3	$3^3 \cdot 2$	0	56	2
$V_1 \to W_4$	2	2^3	42	52	8
$V_2 \to \mathbb{P}^3$	2	$2^3 \cdot 2$	20	38	4
$Q_3 \subset \mathbb{P}^4$	2	$2^3 \cdot 3$	10	36	24
$V_{2\cdot 2} \subset \mathbb{P}^5$	2	$2^3 \cdot 4$	4	38	4
$V_5 \subset \mathbb{P}^6$	2	$2^3 \cdot 5$	0	42	8
$V_2 \to \mathbb{P}^3$	1	2	104	108	2
$Q_4 \subset \mathbb{P}^4$	1	4	60	66	4
$V_{2\cdot 3} \subset \mathbb{P}^5$	1	6	40	48	6
$V_{2\cdot 2\cdot 2} \subset \mathbb{P}^6$	1	8	28	38	8
$V_{10} \subset \mathbb{P}^7$	1	10	20	32	2
$V_{12} \subset \mathbb{P}^8$	1	12	14	28	12
$V_{14} \subset \mathbb{P}^9$	1	14	10	26	2
$V_{16} \subset \mathbb{P}^{10}$	1	16	6	24	8
$V_{18} \subset \mathbb{P}^{11}$	1	18	4	24	6
$V_{22} \subset \mathbb{P}^{13}$	1	22	0	24	2

Table 1. Building blocks Z from Fanos Y with Picard rank 1

Example 7.2. Similarly, we can take Y to be any of the Fano 3-folds of Picard rank ≥ 2 classified by Mori-Mukai [55–59]. We list some building blocks of this type separately in Table 4. In our final Table 5 of examples of G_2–manifolds, the notation $Z = Ex\ 7.2_n$ signifies the building block Z of Fano type obtained from the *rank 2 Fano* 3-fold Y listed as no. n in the table in [59] (and also in our Table 4). We call these *building blocks of rank* 2 *Fano type*.

Examples 7.3–7.7 are building blocks of semi-Fano type where the semi-Fano Y is obtained as a projective small resolution of a Fano 3-fold X with nodal singularities. For a given X there may be several non-isomorphic small resolutions Y, but they all have the same cohomology. However, the value of $\operatorname{div} c_2(Z)$ may depend on the choice of small resolution $Y \to X$.

Example 7.3. Fix a 2-plane $\Pi \subset \mathbb{P}^4$ and let $\Pi \subset X \subset \mathbb{P}^4$ be a general quartic 3-fold containing Π. Let Y be one of the two projective small resolutions of X.

Example 7.4. Fix a quadric surface $Q = Q_2^2 \subset \mathbb{P}^4$ and let $Q \subset X \subset \mathbb{P}^4$ be a general quartic 3-fold containing Q. Let Y be one of the two projective small resolutions of X.

Example 7.5. Fix a cubic scroll surface $\mathbb{F} \subset \mathbb{P}^4$ and let $\mathbb{F} \subset X \subset \mathbb{P}^4$ be a general quartic 3-fold containing $\mathbb{F}$. Let Y be one of the two projective small resolutions of X.

Example 7.6. Fix the complete intersection of two quadrics $F = F_{2,2} \subset \mathbb{P}^4$ and let $F \subset X \subset \mathbb{P}^4$ be a general quartic 3-fold containing F. Let Y be one of the two projective small resolutions of X.

Example 7.7. The Burkhardt quartic 3-fold is the hypersurface

$$X = \left(x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3) + 3x_1x_2x_3x_4 = 0\right) \subset \mathbb{P}^4.$$

X contains 40 planes, has 45 ordinary nodes as singularities, defect $\sigma = 15$ (recall (3.21)), and admits projective small resolutions. (See Finkelnberg's thesis [33] for these and other facts on the Burkhardt quartic.) We take Y to be one particular projective small resolution of X previously studied by Finkelnberg in [33]. The polarising lattice N has rank 16 and discriminant group $(\mathbb{Z}/3\mathbb{Z})^5$. Its orthogonal complement $T \subset L$ is the rank 6 lattice $A_2(-1) \perp 2U(3)$, where $A_2(-1)$ and $U(3)$ denote the rank 2 lattices with intersection forms

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

respectively. In [21, Example 7.7] we deduce the uniqueness of the embedding $N \subset L$ from that of $T \subset L$.

The next two examples arise by blowing up the base locus of a *non-generic* anticanonical pencil on $\mathbb{P}^3$, *ie* they do not come from an application of Proposition 3.17. In these cases extra work is required both to verify that the topological conditions of a building block (recall Definition 3.5) are satisfied and that the matching arguments of §6 can be applied.

Example 7.8. Consider the non-generic AC (anti-canonical) pencil $|S_0, S_\infty| \subset |\mathcal{O}(4)|$, where

$$S_0 = (x_0x_1x_2x_3 = 0)$$

is the sum of the four coordinate planes, and S_∞ is a nonsingular quartic surface meeting all coordinate planes transversely. The base curve of the pencil is the union $C = \sum_{i=0}^{3} \Gamma_i$ of the four nonsingular curves $\Gamma_i = (x_i = 0) \cap S_\infty$. Let Z be obtained from $Y = \mathbb{P}^3$ by blowing up the four base curves one at a time. Any smooth quartic K3 appears as a fibre of a building block of this kind, so even though we are using non-generic pencils we can apply the same orthogonal gluing argument as for building blocks obtained by resolving generic pencils.

Example 7.9. Fix two general conics $C_1, C_2 \subset \mathbb{P}^3$, and take a generic pencil of quartic K3 surfaces containing both C_1 and C_2. The base locus C consists of C_1, C_2 and a degree 12 curve C_3 (of genus 15) meeting each of C_1 and C_2 in 10 points. Let Z be the result of first blowing up C_1, then the proper transform of C_3, and then the proper transform of C_2, and let S be the proper transform of a smooth element of the chosen pencil on $\mathbb{P}^3$. (Z, S) is a building block, with 20 $(-1,-1)$ curves corresponding to the double points of C. (Blowing up the components of C in a different order changes Z by flopping some of the 20 exceptional curves, but does not change the data listed in Table 2.)

S contains the pair of conics C_1, C_2, so these represent classes in $N = \operatorname{Pic} S$. Together with the hyperplane class A they are the basis of a subgroup $N \subset \operatorname{Pic} S$, and in this basis the quadratic form on N is

$$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

We check by hand that this family of blocks satisfies the conditions of Definition 6.17, so that the orthogonal matching Proposition 6.18 can be applied to it. The main point is that a generic N-polarised K3 S appears as the fibre in some block (Z, S) in the family.

Let D_N be the Griffiths domain for N. It is explained in [21, Example 7.9] that a generic marked K3 S with period in D_N embeds as a quartic in $\mathbb{P}^3$, and contains a pair of conics. We can then form a block (Z, S) by blowing up the intersection of S with a generic quartic containing those two conics. Thus there is a $U_{\mathcal{Z}} \subset D_N$ with complement a locally finite union of complex analytic subsets of positive codimension, such that for any $\Pi \in U_{\mathcal{Z}}$ there is a building block (Z, S) in our family, with Π the period of a marking for S.

Next, let E_i be the exceptional divisor in Z over C_i (E_i is isomorphic to the projectivisation of the normal bundle of C_i, blown up at points corresponding to intersections with those components of C blown up after C_i). The pull-back H to Z of the hyperplane class on $\mathbb{P}^3$ is nef, but it fails to be positive on the fibres of E_i. On the other hand, S is positive on almost all of the fibres. For small $\lambda_0 > 0$, $H + \lambda_0 S$—which has image A in $H^2(S)$—is nef and positive on all curves except the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ curves over the 20 intersection points of $C_1 \sqcup C_2$ with C_3. By adding $-\lambda_1 E_1 + \lambda_2 E_2$ for small $\lambda_i > 0$ we get a Kähler class, with image $A - \lambda_1 C_1 + \lambda_2 C_2$ in $H^2(S)$. Therefore there is an open subcone $\mathrm{Amp}_{\mathcal{Z}}$ of the positive cone in $N_{\mathbb{R}}$ that can be taken as restrictions of Kähler classes on Z (A spans an edge of $\mathrm{Amp}_{\mathcal{Z}}$).

Thus the family $\mathcal{Z}$ is $(N, \mathrm{Amp}_{\mathcal{Z}})$-generic, and can be used in orthogonal gluing.

Examples 7.10 and 7.11 are obtained from the same toric semi-Fano 3-fold Y by blowing up a generic AC pencil and a nongeneric AC pencil on Y respectively.

Example 7.10. Let X be the terminal Gorenstein toric Fano 3-fold with Fano polytope the reflexive polytope in $\operatorname{Hom}(\mathbb{C}^\times, \mathbb{T})$ with vertices

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 \end{pmatrix}.$$

This is polytope 1942 in the Sage implementation of Kreuzer and Skarke's database of 4319 reflexive polytopes in 3 dimensions. Let Y be a projective small resolution of X, and Z the blow-up of Y in the base locus of a generic AC pencil.

Example 7.11. We construct the building block Z by blowing up a different (non-generic) pencil on the toric semi-Fano 3-fold Y used in the previous example. The pencil we use is described in [21, Example 7.11]. One can show that any generic anticanonical divisor in Y

What	$-K_Y^3$	$H^2(Z)$	N	K	$H^3(Z)$	$\operatorname{div} c_2(Z)$	e
Ex 7.3	4	$\mathbb{Z}^3$	$\begin{pmatrix}-2 & 1\\ 1 & 4\end{pmatrix}$	0	$\mathbb{Z}^{50}$	2, 4	9
Ex 7.4	4	$\mathbb{Z}^3$	$\begin{pmatrix}-2 & 2\\ 2 & 4\end{pmatrix}$	0	$\mathbb{Z}^{44}$	2	12
Ex 7.5	4	$\mathbb{Z}^3$	$\begin{pmatrix}-2 & 3\\ 3 & 4\end{pmatrix}$	0	$\mathbb{Z}^{34}$	2, 4	17
Ex 7.6	4	$\mathbb{Z}^3$	$\begin{pmatrix}0 & 4\\ 4 & 4\end{pmatrix}$	0	$\mathbb{Z}^{36}$	4	16
Ex 7.7	4	$\mathbb{Z}^{17}$	$E_6^*(-3) \perp E_8(-1) \perp U$	0	$\mathbb{Z}^6$	2	45
Ex 7.8	64	$\mathbb{Z}^5$	$\langle 4\rangle$	$\mathbb{Z}^3$	$\mathbb{Z}^{24}$	2	24
Ex 7.9	64	$\mathbb{Z}^4$	$\begin{pmatrix}-2 & 0 & 2\\ 0 & -2 & 2\\ 2 & 2 & 4\end{pmatrix}$	0	$\mathbb{Z}^{30}$	2	20
Ex 7.10	22	$\mathbb{Z}^{11}$	$E_8(-1) \perp \langle 8\rangle \perp \langle -16\rangle$	0	$\mathbb{Z}^{24}$	2	9
Ex 7.11	22	$\mathbb{Z}^{23}$	$E_8(-1) \perp \langle 8\rangle \perp \langle -16\rangle$	$\mathbb{Z}^{12}$	0	2	33
Ex 7.12	4	$\mathbb{Z}^3$	$\langle 4\rangle \perp \langle -2\rangle$	0	$\mathbb{Z}^{46}$	2	12

Table 2. A small number of examples of building blocks (reproduced from [21, Table 2]). In the rightmost column of this table e refers to the number of rigid $\mathbb{P}^1$'s in Z.

appears as a fibre in a building block of this kind, so we can apply the orthogonal gluing argument when attempting to find matchings involving this block.

The final example comes from a semi-Fano 3-fold whose anticanonical morphism is not small. Even though it is not constructed as a small resolution of a nodal variety, it still contains some curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Example 7.12. Let $X \subset \mathbb{P}^4$ be a generic quartic containing a double line, Y the crepant resolution of X, and Z the blow-up of Y in the base locus of a generic AC pencil. The exceptional set of $Y \to X$ is a conic bundle with 6 degenerate fibres. Each degenerate fibre consists of two $\mathbb{P}^1$s intersecting in a single point. Each of these 12 $\mathbb{P}^1$s has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Examples of compact G_2–manifolds from orthogonal gluing. We start with pairs of building blocks $Z_\pm$ taken from the examples listed above and construct compact G_2–manifolds from such pairs by using orthogonal gluing to solve the matching problem. We summarise the invariants of the resulting G_2–manifolds in Table 5.

More specifically given a pair of building blocks $Z_\pm$ with corresponding polarising lattices $N_\pm$ first we make a choice of an *orthogonal push-out* $W = N_+ \perp_R N_-$ of the pair $N_\pm$ as in Definition 6.5; for a given pair of lattices $N_\pm$ there is often some choice in this. Recall that *perpendicular gluing* refers to the special case when we choose $W = N_+ \perp N_-$, *ie* $R = N_+ \cap N_- = 0$. In order to satisfy conditions (i) and (ii) of Proposition 6.18, we then find an embedding $W \hookrightarrow L$ such that the inclusions $N_\pm \hookrightarrow L$ are primitive. Usually we achieve this by applying Theorem 6.9 to find a primitive embedding $W \hookrightarrow L$; we refer to this as *primitive orthogonal gluing* or *primitive perpendicular gluing*. In the perpendicular case Proposition 6.18 then produces matching data, and therefore compact G_2–manifolds by appeal to Theorem 3.13 and Corollary 6.4. In the non-perpendicular case, we also need to calculate the Kähler cones of $Z_\pm$ to verify condition 6.18(iii).

The topology of the resulting G_2–manifold depends only on the blocks and the choice of push-out. The integral cohomology groups can readily be computed using Theorem 4.8 and the data in Tables 1, 2 and 4.

The following observation is helpful for identifying the torsion in H^3 and H^4.

Lemma 7.13. *Let L be a unimodular lattice, $N_+, N_- \subset L$ two primitive submodules and T_+, T_- their perpendicular complements in L. Then*

$$L/(N_+ + N_-) = \operatorname{coker}(N_+ \to T_-^*) = \operatorname{coker}(N_- \to T_+^*)\,,$$
$$L/(N_+ + T_-) = \operatorname{coker}(N_+ \to N_-^*) = \operatorname{coker}(T_- \to T_+^*)\,.$$

In the case of perpendicular gluing p_1 is also straightforward to compute; Corollary 4.30 tells us that it suffices to know the greatest divisors of c_2 of the building blocks, which we also included in the tables. For non-perpendicular gluing, we have to work a little bit harder to compute p_1, using some of the details of the c_2 calculation from [21, §5].

The simplest building blocks to match are the 17 families of building blocks of rank one Fano type described in Example 7.1 and summarised in Table 1. G_2–manifolds obtained by matching pairs of rank one Fanos already appear in [48, §8] but we can now give much more precise information about the topology of G_2–manifolds constructed this way including, in most cases, their diffeomorphism type. The most straightforward way to achieve matching in this case is to use primitive perpendicular gluing, *ie* to choose a primitive lattice embedding of the rank two lattice $W = N_+ \perp N_-$ into the K3 lattice L. Existence and uniqueness (up to lattice automorphisms of L) of this embedding follow from Theorems 6.9 and 6.13. However even in this case there are other ways to achieve matching which lead to G_2–manifolds with the same Betti numbers but different integral cohomology groups; see example *No 1* below. For now though we restrict attention to matching by primitive perpendicular gluing and consider the topology of the resulting compact G_2–manifolds.

Perpendicular gluing of pairs of rank 1 smooth Fano 3-folds. By Lemma 4.27 any twisted connected sum G_2-manifold M arising by primitive perpendicular gluing of blocks of semi-Fano or Fano type is 2-connected (recall from Proposition 3.17 that $K = 0$ for any block of semi-Fano type) and has $H^4(M)$ torsion-free. Hence the almost-diffeomorphism classification of Theorem 4.22 applies to M. Recall also that from Lemma 4.31 we have $\operatorname{div} p_1(M) \in \{4, 8, 12, 16, 24, 48\}$ for any twisted connected sum G_2-manifold and that Corollary 4.32 restricts the number of diffeomorphism types in a given almost diffeomorphism class according to $\operatorname{div} p_1(M)$. In particular, there are at most 8 diffeomorphism classes realising the same value of $b^3(M)$.

The data of all possible $153 = \frac{1}{2} \cdot 18 \cdot 17$ such matching pairs is collected in Table 3. We summarise some of the main features apparent from this table.

(i) 46 different values of $b^3(M)$ arise with $71 \le b^3(M) \le 239$.
(ii) All six permitted integers $\{4, 8, 12, 16, 24, 48\}$ occur as $\operatorname{div} p_1(M)$ for some M in Table 3.
(iii) 82 different almost-diffeomorphism types occur.
(iv) By Corollary 4.32 the diffeomorphism type is uniquely determined except in the 14 cases in which $\operatorname{div} p_1(M) = 16$ and the 1 case in which $\operatorname{div} p_1(M) = 48$.

b	#	$\operatorname{div}(p_1)$					
		4	8	12	16	24	48
48	6	4	0	1	1	0	0
50	3	3	0	0	0	0	0
52	4	2	1	1	0	0	0
54	1	1	0	0	0	0	0
56	4	3	0	0	0	1	0
58	1	1	0	0	0	0	0
60	4	2	0	1	1	0	0
62	10	7	2	0	1	0	0
64	5	4	0	0	0	1	0
66	6	2	3	0	1	0	0
68	2	2	0	0	0	0	0
70	4	3	1	0	0	0	0
72	4	2	0	1	0	0	1
74	5	2	2	0	1	0	0
76	10	2	5	1	2	0	0
78	2	1	0	0	1	0	0
80	8	4	3	0	1	0	0
82	1	1	0	0	0	0	0
84	4	2	0	1	1	0	0
86	3	3	0	0	0	0	0
88	2	1	0	0	1	0	0
90	10	6	3	0	1	0	0
92	3	3	0	0	0	0	0
94	6	4	1	0	1	0	0
96	1	0	0	1	0	0	0
98	3	3	0	0	0	0	0
100	1	1	0	0	0	0	0
102	2	1	1	0	0	0	0
104	8	4	3	0	1	0	0
108	3	2	1	0	0	0	0
112	1	1	0	0	0	0	0
114	2	2	0	0	0	0	0
118	2	1	1	0	0	0	0
122	2	2	0	0	0	0	0
132	6	5	1	0	0	0	0
134	1	1	0	0	0	0	0
136	1	1	0	0	0	0	0
140	1	1	0	0	0	0	0
144	1	1	0	0	0	0	0
146	3	3	0	0	0	0	0
150	1	1	0	0	0	0	0
156	1	1	0	0	0	0	0
160	1	1	0	0	0	0	0
164	1	1	0	0	0	0	0
174	2	2	0	0	0	0	0
216	1	1	0	0	0	0	0
Total	153	101	28	7	14	2	1

TABLE 3. Betti numbers and almost-diffeomorphism types of 2-connected twisted connected sum G_2–manifolds M constructed by perpendicular gluing from pairs of rank 1 Fano 3-folds. $b^3(M) = b^4(M) = b + 23$; # gives number of instances of a given value of b, further broken down according to divisibility of $p_1(M)$ on right of table.

(v) There are exactly two ways to construct a 2-connected G_2-manifold with $b^3(M) = 76 + 23 = 99$ and $\operatorname{div} p_1(M) = 16$: either take both blocks from the family Example 7.1^2_4, or match 7.1^2_1 to 7.1^1_{16}. By Corollary 4.32 there are precisely two diffeomorphism classes in the almost diffeomorphism type of such a 2-connected 7-manifold M. A natural question is therefore: are these two almost diffeomorphic twisted connected sum G_2-manifolds diffeomorphic or not? To answer it requires the calculation of the generalised Eells-Kuiper invariant [25] discussed in Remark 4.26. We believe that perpendicular gluing will only ever realise one of the two smooth structures.

(vi) There are many ways to use primitive perpendicular gluing of different pairs of building blocks of rank one Fano type to produce *diffeomorphic* 2-connected G_2-manifolds; in Table 3 we simply look at any of the four columns where $\operatorname{div} p_1 \mid 24$ and find an entry in any row in that column which is greater than 1. There are many such entries in the table. Of the 46 values of b^3 that occur in Table 3, 15 of those can occur for a unique choice of pair of rank 1 Fanos. For the remaining 31 values of b^3 we see that except for four cases ($b = 78, 88, 102, 118$ in the table; recall $b^3 = b + 23$ there) we can find at least two different pairs of rank one Fano building blocks that yield diffeomorphic 2-connected 7-manifolds with $b^3(M) = b^3$.

One concrete way to get distinct pairs of building blocks of rank one Fano type which yield diffeomorphic G$_2$-manifolds is to take the pair (a) $(7.1^1_{22}, 7.1^1_{22})$ or the pair (b) $(7.1^1_{22}, 7.1^1_{18})$ These both yield a 2-connected G$_2$-manifold M with $b^3 = 48+23 = 71$ and $\operatorname{div} p_1 = 4$. (The pairs $(7.1^1_{22}, 7.1^1_{16})$ and $(7.1^1_{18}, 7.1^1_{16})$ are the two other pairs yielding the same 7-manifold M.) By Remark 4.24 M is diffeomorphic to the connected sum of $M_{1,0}$ with 70 copies of $\mathbb{S}^3 \times \mathbb{S}^4$ where $M_{1,0}$ denotes the unique $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ with Euler number 0 and $p_1(M) = 4 \cdot 1 \in H^4(M_{1,0}) \cong \mathbb{Z}$.

Detailed examples. We now describe in detail a small number of examples to illustrate some of the main points. Consulting the overview given at the beginning of Section 8 may also benefit the reader.

The first example shows one way in which it is possible to produce different G$_2$–manifolds from the same pair of building blocks $Z_\pm$.

No 1. We take both Z_+ and Z_- to be building blocks of Fano type obtained from a smooth quartic in $\mathbb{P}^4$ (Example 7.1^1_4). Table 3 already includes the twisted connected sum of these two blocks given by embedding $W = N_+ \perp N_- \cong \langle 4\rangle \perp \langle 4\rangle$ primitively in L; the entry has $b = 132$, $\operatorname{div} p_1 = 8$. However, we can also consider a *non-primitive* embedding of W in L for which the resulting inclusions $N_\pm \hookrightarrow L$ are still primitive: W is isometric to the index 2 sublattice $\{(x,y) : x = y \bmod 2\}$ of $\langle 2\rangle \perp \langle 2\rangle$, so a primitive embedding of the latter in L (which exists by Theorem 6.9, or indeed by inspection) gives an embedding $W \hookrightarrow L$ with cotorsion $L/W \cong \mathbb{Z}/2\mathbb{Z}$. Using this "non-primitive" perpendicular matching we get a twisted connected sum with the same Betti numbers and $\operatorname{div} p_1$ as before, but now $\operatorname{Tor} H^3(M) \cong \mathbb{Z}/2\mathbb{Z}$ (recall Corollary 4.11). In particular, although Theorem 4.8 shows that M is simply-connected and has $H^2(M) = 0$ it is no longer 2-connected.

Remark 7.14. In a similar way, one can get alternative perpendicular matchings with torsion in H^3 for many other pairs of building blocks, whether of rank 1 Fano type or otherwise. Whether there exist suitable overlattices of $N_+ \perp N_-$ reduces to a problem about the discriminant groups of N_+ and N_-, as discussed in Remark 6.12. Carrying out such an analysis for the rank 1 pairs allows us to realise $\mathbb{Z}/k\mathbb{Z}$ as $\operatorname{Tor} H^3(M)$ of twisted connected sums for $2 \le k \le 5$, and a total of 41 triples of invariants $(b^4(M), \operatorname{div} p_1(M), k)$ (in addition to the 82 with $H^3(M)$ torsion-free).

Our remaining examples use building blocks from Tables 2 and 4.

No 2. We take Z_+ to be the building block from Example 7.3, and Z_- from Examples 7.3–7.6 and use primitive perpendicular gluing to achieve matching. In all these cases the polarising lattices $N_\pm$ have signature $(1,1)$ and hence $W := N_+ \perp N_-$ has signature $(2,2)$. Therefore by Theorems 6.9 and 6.13 W admits a primitive embedding $W \hookrightarrow L$ which is unique up to automorphisms of L. Now we apply Proposition 6.18 to solve the matching problem noting that hypothesis (iii) is automatically satisfied because we are using perpendicular gluing. Observe that when we choose Z_- from 7.3, 7.5 or 7.6, $p_1(M)$, and hence the diffeomorphism type of M, depends on the choice of resolution used for the semi-Fanos.

No 3. We match blocks from Example 7.8 and Example 7.1^1_4 by primitive perpendicular gluing. Because Example 7.8 has $\operatorname{rk} K = 3$, the twisted connected sum G$_2$–manifold has $b^2(M) = 3$.

No 4. We use perpendicular gluing to match the semi-Fano type blocks $Z_\pm$ from Examples 7.10 and 7.12 respectively. In this case we cannot appeal to Theorem 6.9(i) to guarantee we can embed $W = N_+ \perp N_-$ in L because $\operatorname{rk} W = 2 + 10 = 12 > 22/2$. However, the discriminant group W^*/W is $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is generated by 4 elements. So $\ell(W) = 4$ and we can apply 6.9(ii) to get a primitive embedding (and it is unique by Theorem 6.13).

Consulting Table 2 we see that there are 9 rigid $\mathbb{P}^1$'s in Z_+ and 12 in Z_-. Using Proposition 5.15 we thus find 21 associative $\mathbb{S}^1 \times \mathbb{S}^2$ in M. The 12 from Z_- come in pairs that are close together, as they arise from pairs of $\mathbb{P}^1 \subset Z_-$ that intersect. However, there is no a priori reason that the associatives in M should intersect after the perturbation in Proposition 5.15.

No 5–6. In these examples, we use perpendicular gluing to match a block Z_+ arising from the Burkhardt quartic (Example 7.7) with a block Z_- of Fano type arising from a Fano 3-fold of Picard rank 1 (Example 7.1). Let r and d be the rank and degree of the Fano 3-fold used.

The polarising lattice N_+ of the Burkhardt quartic block Z_+ has rank 16, while N_- is generated by a single vector of square-norm $m = rd$. Note that because $\operatorname{rk} N_- = 1$, we must a priori choose the embeddings $N_\pm \hookrightarrow L$ to be perpendicular to have any chance of finding matching data, since this involves finding a (Kähler) class in N_- that is orthogonal to N_+. So we seek embeddings of $W := N_+ \perp \langle m \rangle$ in the K3 lattice L, so that each of the two sublattices $N_\pm$ is primitive in L; recall however, that we do not insist that the embedding of the whole lattice W is primitive in L. Because of the high rank of W some work is required to demonstrate existence of such an embedding and for this we will need to use precise information about the lattice N_+. Recall from Example 7.7 that N_+ has a unique primitive embedding in L; its orthogonal complement in L is $T = A_2(-1) \perp 2U(3)$. The problem is therefore equivalent to finding a primitive vector $x \in T$ of square-norm m, so that we can take the image of N_- to be $\langle x \rangle$. (Theorem 6.9 is of no use for finding the primitive embedding $N_- \hookrightarrow T$ since T is not unimodular.)

The discriminant group of W is simply the product of the discriminant groups of the two orthogonal summands
$$W^*/W \cong (\mathbb{Z}/3\mathbb{Z})^5 \times \mathbb{Z}/m\mathbb{Z}.$$
Consider first the case when $3 \mid m$. Then $\ell(W) = 6$. Since $\operatorname{rk} W = 17$, (6.10) is not satisfied, and there can be no *primitive* embedding $W \hookrightarrow L$. On the other hand, we can certainly find a primitive vector x of square-norm m in $U(3) \subset T$, and thus we get embeddings $W \hookrightarrow L$ that allow us to match Example 7.7 to 7.1^1_6, 7.1^1_{12}, 7.1^1_{18}, 7.1^2_3 or 7.1^3_2. We label these examples *No 6a-e*. In all five cases $\operatorname{Tor} L/W \cong \mathbb{Z}/3\mathbb{Z}$ by Lemma 7.13, so the resulting G_2–manifolds have $\operatorname{Tor} H^3(M) \cong \mathbb{Z}/3\mathbb{Z}$.

If m is not divisible by 3, then $(\mathbb{Z}/3\mathbb{Z})^5 \times \mathbb{Z}/m\mathbb{Z} \cong (\mathbb{Z}/3\mathbb{Z})^4 \times \mathbb{Z}/3m\mathbb{Z}$ and $\ell(W) = 5$. Therefore we are just on the borderline where the existence of a primitive embedding $W \hookrightarrow L$ is not excluded by (6.10), but also not guaranteed by Theorem 6.9. In fact, all elements of $A_2(-1) \perp 2U(3)$ have square-norm 0 or 1 mod 3, so if $m = 2$ mod 3 there is no suitable embedding $W \hookrightarrow L$, and therefore we cannot match Example 7.7 with 7.1^1_2, 7.1^1_8, 7.1^1_{14}, 7.1^2_1 or 7.1^2_4 at all. On the other hand, $A_2(-1)$ does contain a primitive vector of square-norm -2 and $U(3)$ contains vectors of square-norm $3k$ for any k; thus, if $m = 3k-2$ we can find the desired primitive $x \in T$, and the resulting embedding $W \hookrightarrow L$ is primitive by Lemma 7.13. Hence we can match 7.7 to 7.1^1_4, 7.1^1_{10}, 7.1^1_{16}, 7.1^1_{22}, 7.1^2_2, 7.1^2_5 and 7.1^4_1 using primitive perpendicular gluing to get 2-connected G_2–manifolds, which we label *No 5a-g*.

Since $\operatorname{div} c_2(Z_+) = 2$, all the G_2–manifolds we get this way have $\operatorname{div} p_1(M) = 4$. Note that *No 5a* and *5g* are both 2-connected with $b^3(M) = 95$, so are diffeomorphic by Theorem 4.22. *No 5c, 5d* and *6c* all have $b^3(M) = 53$, but *No 6c* has $\operatorname{Tor} H^3(M) = \mathbb{Z}/3\mathbb{Z}$ so is not diffeomorphic to the first two.

No 7. We match two copies $Z_\pm$ of blocks from Example 7.11 using perpendicular gluing. Let $N_0 = \langle 8 \rangle \perp \langle -16 \rangle$, and let $N_\pm$ be two copies of the polarising lattice $E_8(-1) \perp N_0$ of the block. We need to construct an embedding of $N_+ \perp N_-$ in the K3 lattice L. First we embed $2N_0$ in $3U$ by the matrix

$$\begin{pmatrix} 4 & 0 & 0 & -4 \\ 1 & 0 & 0 & 1 \\ 0 & -8 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Note that each of the two copies of N_0 is embedded primitively, but $3U/2N_0 \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/8\mathbb{Z})$. (For a finite index overlattice of $2N_0$ to be primitively embeddable in $3U$ its discriminant group can have at most 2 generators according to (6.10); since the discriminant group of $2N_0$ is $(\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z})^2$ such an overlattice must have index at least 8, so there is no way to embed $2N_0$ into $3U$ with smaller cotorsion.)

Next we embed $N_+ \perp N_-$ in $L = 3U \perp 2E_8(-1)$ by embedding $N_0 \perp N_0$ in $3U$ as above, the first copy of $E_8(-1)$ in the first copy of $E_8(-1)$, and the second in the second. By Corollary 4.11 $\operatorname{Tor} H^3$ of the glued G_2–manifold is $\mathbb{Z}/8\mathbb{Z}$. Since N_+ is embedded perpendicular to N_- there will be no torsion in H^4 of the G_2–manifolds.

No 8: orthogonal gluing with large cotorsion. We use a pair of building blocks $Z_\pm$ of semi-Fano type obtained from the construction of Example 7.6, *ie* starting with a quartic 3-fold containing a quartic del Pezzo surface $F = F_{2,2}$ (the complete intersection of two quadrics). We aim to use “non-primitive” perpendicular gluing to achieve “cotorsion” as large as possible. The polarising lattice $N_+ \cong N_-$ is the integral lattice with matrix

$$\begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix}$$

and discriminant $(\mathbb{Z}/4\mathbb{Z})^2$. We construct an explicit embedding of $W = N_+ \perp N_-$ in L with cotorsion $L/W = (\mathbb{Z}/4\mathbb{Z})^2$—the largest compatible with the requirement that both $N_\pm \subset L$ be primitive embeddings. Consider the lattice

$$W \cong \mathbb{Z}^4, \quad \text{with intersection matrix} \quad B = \begin{pmatrix} 4 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}.$$

Then embed W in $2U$ via the matrix

$$\begin{pmatrix} 2 & 0 & -2 & 0 \\ 1 & 1 & 0 & -1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

We can check that the embedding is isometric, that the restrictions to $N_\pm$ are primitive and that $(2U)/W \cong (\mathbb{Z}/4\mathbb{Z})^2$. Next, compose with the obvious primitive embedding $2U \hookrightarrow L$.

More abstractly, we could use Nikulin's theory of lattices [61, §1]. N is anti-isometric to itself, and hence so is the form on its discriminant group $(\mathbb{Z}/4\mathbb{Z})^2$. Therefore Remark 6.12 immediately provides overlattices W' of W with W'/W any of the six subgroups of $(\mathbb{Z}/4\mathbb{Z})^2$.

Similar principles are at work in *No 7* (there N_0 is anti-isometric to itself).

No 9: orthogonal gluing with nontrivial intersection. For this family of examples we glue orthogonally (but not perpendicularly) building blocks $Z_\pm$ of rank two Fano type, *cf* Example 7.2. Note that we could of course choose a primitive embedding of the signature $(2,2)$ lattice $N_+ \perp N_-$ into L and therefore match $Z_\pm$ by perpendicular gluing. As we have seen this would yield 2-connected 7-manifolds with torsion-free H^4. Instead here we choose to use orthogonal gluing where the intersection $R = N_+ \cap N_-$ has rank one; this will give rise to a series of examples with $H^2(M) \cong \mathbb{Z}$ and illustrates again how the same pair of building blocks—matched in different ways—yields different smooth 7-manifolds.

We will use the rank two Fanos which are No 2, 6, 10, 12, 21 and 24 from the Mori-Mukai list. Table 4 summarises the information we need about these rank two Fanos; the Picard lattices N of Y are computed in a basis L, M of supporting divisors, *ie* the (closure of the) ample cone of Y is spanned by L and M.

No	$-K_Y^3$	$H^2(Z)$	N	$H^3(Y)$	$H^3(Z)$	$\operatorname{div} c_2(Z)$ mod $A^\perp$
2	6	$\mathbb{Z}^3$	$\begin{pmatrix}0 & 2\\ 2 & 2\end{pmatrix}$	$\mathbb{Z}^{40}$	$\mathbb{Z}^{48}$	6
6	12	$\mathbb{Z}^3$	$\begin{pmatrix}2 & 4\\ 4 & 2\end{pmatrix}$	$\mathbb{Z}^{18}$	$\mathbb{Z}^{32}$	12
10	16	$\mathbb{Z}^3$	$\begin{pmatrix}8 & 4\\ 4 & 0\end{pmatrix}$	$\mathbb{Z}^{6}$	$\mathbb{Z}^{24}$	8
12	20	$\mathbb{Z}^3$	$\begin{pmatrix}4 & 6\\ 6 & 4\end{pmatrix}$	$\mathbb{Z}^{6}$	$\mathbb{Z}^{28}$	4
21	28	$\mathbb{Z}^3$	$\begin{pmatrix}6 & 8\\ 8 & 6\end{pmatrix}$	0	$\mathbb{Z}^{30}$	4
24	30	$\mathbb{Z}^3$	$\begin{pmatrix}2 & 5\\ 5 & 2\end{pmatrix}$	0	$\mathbb{Z}^{32}$	12

TABLE 4. Some building blocks from rank 2 Fanos

In all cases we choose $A = L + M$ as our ample class in the lattice (this coincides with $-K_Y$ except for No 24, where $-K_Y = 2L + M$) and push out along a common $R = A^\perp$. To verify that the pushout exists, we present N as an overlattice of $\langle A\rangle \perp R$:

$$\begin{pmatrix}0 & 2\\ 2 & 2\end{pmatrix} = \frac{1}{3}(1,1)\mathbb{Z} + \mathbb{Z}^2 \text{ in } \begin{pmatrix}6 & 0\\ 0 & -6\end{pmatrix} \qquad \begin{pmatrix}2 & 4\\ 4 & 2\end{pmatrix} = \frac{1}{2}(1,1)\mathbb{Z} + \mathbb{Z}^2 \text{ in } \begin{pmatrix}12 & 0\\ 0 & -4\end{pmatrix}$$
$$\begin{pmatrix}8 & 4\\ 4 & 0\end{pmatrix} = \frac{1}{4}(3,1)\mathbb{Z} + \mathbb{Z}^2 \text{ in } \begin{pmatrix}16 & 0\\ 0 & -16\end{pmatrix} \qquad \begin{pmatrix}4 & 6\\ 6 & 4\end{pmatrix} = \frac{1}{2}(1,1)\mathbb{Z} + \mathbb{Z}^2 \text{ in } \begin{pmatrix}20 & 0\\ 0 & -4\end{pmatrix}$$
$$\begin{pmatrix}6 & 8\\ 8 & 6\end{pmatrix} = \frac{1}{2}(1,1)\mathbb{Z} + \mathbb{Z}^2 \text{ in } \begin{pmatrix}28 & 0\\ 0 & -4\end{pmatrix} \qquad \begin{pmatrix}2 & 5\\ 5 & 2\end{pmatrix} = \frac{1}{2}(1,1)\mathbb{Z} + \mathbb{Z}^2 \text{ in } \begin{pmatrix}14 & 0\\ 0 & -6\end{pmatrix}$$

We see that we can form G_2–manifolds M with $H^2(M) = R = N_+ \cap N_- \cong \mathbb{Z}$ by matching any pair taken from Nos 6, 12 and 21, matching 10 to itself, and 2 to 24. In each case the image of $N_\pm$ in $N_\mp^*$ is primitive, so there is no contribution to the torsion of $H^4(M)$.

To compute $p_1(M)$, Corollary 4.30 explains that we need to find the greatest divisor of $c_2(Z_\pm)$ modulo the image of $R = A^\perp$ in $N_\pm^* \subset H^4(Z_\pm)$. By [21, Lemma 5.18], this is the greatest common divisor of 24 and $c_2(Y_\pm) + c_1(Y_\pm)^2$ modulo the image of R in $N_\pm^* \cong H^4(Y)$. The latter is determined by the restriction of $c_2(Y_\pm) + c_1(Y_\pm)^2$ to divisors in the orthogonal complement to R, *ie* just to A itself. For the cases where we use $A = -K_Y$, the relation $c_2(Y)(-K_Y) = \chi(K3) = 24$ implies that $\operatorname{div}(c_2(Z) \bmod A^\perp) = \gcd(24, -K_Y^3)$.

For No 24 we must do a little more work. This Fano 3-fold is a generic bidegree (1,2) divisor in $\mathbb{P}^2 \times \mathbb{P}^2$. The class A has bidegree (1,1). The projection of a generic divisor in the class to the second $\mathbb{P}^2$ factor contracts 7 (-1) curves, so $c_2(A) = c_2(\mathbb{P}^2) + 7 = 10$ while $c_1(A)^2 = c_1(\mathbb{P}^2)^2 - 7 = 2$. Using [21, Lemma 5.15], we deduce that the evaluation of $c_2(Y) + c_1(Y)^2$ on A is $10 - 2 + 28 = 36$, so $\operatorname{div}(c_2(Z) \bmod A^\perp) = 12$.

No 10: torsion in H^4. In this example, we use orthogonal gluing with non-trivial intersection arranged so that there is some torsion in H^4 of the twisted connected sum. We take both Z_+ and Z_- to be building blocks from Example 7.9, that is $\mathbb{P}^3$ blown up at the components C_1, C_3, C_2 (in that order) of the base locus C of a pencil of quartics containing a fixed pair of conics C_1, C_2.

In the notation from Example 7.9, the triple $A - C_1 - C_2$, $C_1 - C_2$, A spans an index 2 sublattice $N' \subset N$ with intersection form

$$N' = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

$N = N' + \frac{1}{2}(1,1,1)$, and we can form an orthogonal push-out $W = N_+ \perp_R N_-$, identifying the sublattices $R \cong \langle -8 \rangle$ spanned by $A - C_1 - C_2$ in each copy $N_\pm$. Note that the image of $N_\pm$ in $N_\mp^*$ has cotorsion $\mathbb{Z}/2\mathbb{Z}$. Therefore the twisted connected sum has $\operatorname{Tor} H^4(M) \cong (\mathbb{Z}/2\mathbb{Z})^2$ by Lemma 7.13.

To apply Proposition 6.18 to find matching data, we need to check that $\operatorname{Amp}_{\mathcal{Z}} \cap W \neq \emptyset$, where $W = R^\perp = \langle A, C_1 - C_2 \rangle \subset N_\mathbb{R}$. From the analysis of $\operatorname{Amp}_{\mathcal{Z}}$ in Example 7.9, we see that $A + \lambda(-C_1 + C_2) \in \operatorname{Amp}_{\mathcal{Z}} \cap W$ for small $\lambda > 0$.

Because $H^4(M)$ has only 2-torsion, and $p_1(M)$ is divisible by 4 a priori, $\operatorname{div} p_1(M)$ is the same as the greatest divisor of the image of $p_1(M)$ in the free part of $H^4(M)$. To compute the latter, it suffices by Proposition 4.20 and Lemma 4.14 to find $\operatorname{div} c_2(Z_\pm)$ modulo the primitive overlattice of the image of N in $H^4(Z_\pm)$. This amounts to evaluating $c_2(Z_\pm)$ on divisors representing classes in $H^2(Z_\pm)$ whose image in $H^2(S)$ is orthogonal to R. The group of such divisors is spanned by S itself, $E_1 + E_2 + E_3$ and $E_1 - E_3$. $c_2(Z_\pm)$ evaluated on S is 24 as usual. On the other two basis elements we see from [21, Proposition 5.11] that it evaluates to 64 and 20, respectively. Thus the greatest common divisor is 4, and $\operatorname{div} p_1(M) = 8$.

Handcrafted nonorthogonal K3 gluing. Gluing by means of the orthogonal pushout construction is convenient because it reduces the problem of finding compatible pairs of ACyl Calabi–Yau 3-folds $V_\pm$ arising from a given pair of deformation types of building blocks $\mathcal{Z}_\pm$, essentially to arithmetic considerations involving the pair of polarising lattices $N_\pm$ of the two families. This allows us to produce large numbers of compact G$_2$–manifolds with relatively little labour; see Section 8 for a discussion of many further such examples.

In most cases arising in practice, we expect to be able to glue much more generally. The complication is that without the assumption of orthogonal intersection of the Picard lattices, there are fewer degrees of freedom in the problem of finding adapted matching data considered after Corollary 6.4, and one needs more precise information about the moduli of K3s in the building blocks than is provided by Proposition 6.15. In the following general scheme, the extra information is essentially obtained "by hand" in step 3.

1) Choose semi-Fano deformation types $\mathcal{Y}_\pm$ with Picard lattices $N_\pm$. Also choose $H_\pm \in N_\pm$ that correspond to ample (Kähler) classes on the semi-Fanos ($H_\pm \in \mathrm{Amp}_{\mathcal{Y}_\pm}$ in the terminology of Proposition 6.15). In the end, we plan to glue blocks $Z_\pm$ obtained from semi-Fanos from these families by blowing up AC curves.
2) Choose a lattice $W = N_+ \oplus N_-$ where N_+, N_- *are not necessarily orthogonal*, but where W has signature $(2, r-2)$ and $H_+ \in N_-^\perp$ and $H_- \in N_+^\perp$. Embed W primitively in L.
3) Let $\Lambda_+ = H_-^\perp \subset W$ and $\Lambda_- = H_+^\perp \subset W$. Construct projective models for $\Lambda_\pm$-polarised K3s to show that the generic K3s can still be found as hyperplane sections of semi-Fanos from the starting classes $\mathcal{Y}_\pm$.
4) Among the semi-Fano type building blocks constructed from $\mathcal{Y}_\pm$ we can therefore find a *subfamily* that is $(\Lambda_\pm, \mathrm{Amp}_{\mathcal{Y}_\pm})$-generic in the sense of Definition 6.17 (except that the cone $\mathrm{Amp}_{\mathcal{Y}_\pm}$ is not open in $\Lambda_\pm(\mathbb{R})$, but that is unimportant here). Since we made sure that $\mathrm{Amp}_{\mathcal{Y}_\pm} \cap \Lambda_\mp^\perp$ is non-empty, Proposition 6.18 shows that we can glue.

Remark. Note that even though the K3 fibres have $\mathrm{Pic}\, S_\pm = \Lambda_\pm$, the images of $H^2(Z_\pm)$ in $H^2(S_\pm)$ are still $N_\pm$; the topology of the twisted connected sum involves the embeddings of $N_\pm$ in L and not $\Lambda_\pm$.

In the construction of the projective models we use the following well-known:

Lemma 7.15 ([66, Chapter 3])**.** *Let S be a K3 surface, and A a nef line bundle on S with $A^2 > 0$ (that is, A is nef and big).*
(I) *Either:*
- *$|A|$ has no fixed part, or:*
- *$|A|$ is monogonal, that is: $A = aE + \Gamma$ where $E^2 = 0$, $E \cdot \Gamma = 1$, $\Gamma^2 = -2$, and $a \geq 1$.*

(II) *Assume that $|A|$ is not monogonal. Then $|A|$ is base point free and either:*
- *the morphism given by $|A|$ is birational onto its image and an isomorphism away from a finite union of -2 curves, or*
- *$|A|$ is hyperelliptic, that is, one of the following cases:* (a) *$A^2 = 2$ and S is a double cover of $\mathbb{P}^2$;* (b) *$A = 2B$ with $B^2 = 2$ and S is a double cover of the Veronese surface; or* (c) *S has an elliptic pencil E with $A \cdot E = 2$.*

No 11. We plan to glue two building blocks obtained by blowing up AC curves on two semi-Fano 3-folds $Y_\pm$ that are small resolutions of quartic 3-folds $X_\pm$ containing a $2 \cdot 2$ complete intersection in $\mathbb{P}^4$ (Example 7.6). In a basis $A = -K_Y$, E the Picard lattice N of $Y_\pm$ has quadratic form

$$\begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix}.$$

Note that A is not ample on Y. We change basis to $H = A + E$, E so the intersection form is

$$\begin{pmatrix} 12 & 4 \\ 4 & 0 \end{pmatrix}.$$

We glue together two copies $N_\pm$ of N into a lattice W with basis H_+, E_+, H_-, E_- and intersection form

$$\begin{pmatrix} 12 & 4 & 0 & 0 \\ 4 & 0 & 0 & 1 \\ 0 & 0 & 12 & 4 \\ 0 & 1 & 4 & 0 \end{pmatrix}.$$

Note that W has signature $(2, 2)$.

Let $\Lambda_+ = H_-^\perp$ and $\Lambda_- = H_+^\perp$: we want to glue Λ_+-polarised K3 surfaces with Λ_--polarised K3 surfaces. Write $\Lambda = \Lambda_+ \cong \Lambda_-$: we will show that a generic Λ-polarized K3 surface S—that is, one for which $\operatorname{Pic} S = \Lambda$—is always the hyperplane section of a quartic X as above.

Λ has basis $H = H_+,\ E = E_+,\ \Sigma = -H_- + 3E_-$ and the intersection form in this basis is

$$\begin{pmatrix} 12 & 4 & 0 \\ 4 & 0 & 3 \\ 0 & 3 & -12 \end{pmatrix}.$$

To study this, it is best to change basis back to $A = H - E,\ E,\ \Gamma = H - E - \Sigma$:

$$\begin{pmatrix} 4 & 4 & 7 \\ 4 & 0 & 1 \\ 7 & 1 & -2 \end{pmatrix}.$$

First we show that the class A is ample and not hyperelliptic, and thus embeds S as a quartic surface in $\mathbb{P}^3$ containing a $2 \cdot 2$ curve. Indeed we claim:

(i) There is no vector $v \in \Lambda$ with $A \cdot v = 1$ and $v^2 = 0$;
(ii) There is no vector $v \in \Lambda$ with $A \cdot v = 2$ and $v^2 = 0$;
(iii) There is no vector $v \in \Lambda$ with $A \cdot v = 0$ and $v^2 = -2$,

Indeed, write $v = (x, y, z)$.

For the first we know $4x + 4y + 7z = 1$ and $4x^2 + 8xy + 14xz + 2yz - 2z^2 = 0$. Use the linear equation to get x in terms of y and z, then substitute and clean up. We end up with

$$16y^2 + 48yz + 57z^2 = 1.$$

This is easy to rule out, because the conic is positive definite. In fact we can just complete the square and write the left hand side as

$$(4y + 6z)^2 + 21z^2 = 1.$$

This immediately gives $z = 0$ (otherwise the left hand side is ≥ 21), and hence $16y^2 = 1$ which is impossible. Similarly, a counter-example to the second claim gives

$$16y^2 + 48yz + 57z^2 = 4,$$

which yields to the same technique. The third gives

$$16y^2 + 48yz + 57z^2 = 8,$$

which again cannot work for the same reasons.

Consider now the moduli stack $\mathfrak{K}^{\Lambda,A}$ of (Λ, A)-polarised K3 surfaces introduced in [21, §6]. This involves a choice of a certain partition $\Delta^+ \sqcup \Delta^-$ of the set $\Delta = \{\delta \in N \mid \delta^2 = -2\}$; by (iii), we can take $\Delta^+ = \{\delta \in \Delta \mid A \cdot \delta > 0\}$. It follows from Lemma 7.15 above that if S is a generic surface of the moduli stack—that is, one for which $\operatorname{Pic} S = \Lambda$ exactly—then A is ample on S and that it embeds S as a quartic in $\mathbb{P}^3$.

All this goes to show that S embeds in $\mathbb{P}^3$ as a nonsingular K3 with an equation of the form

$$a_2 b_2 + c_2 d_2 = 0,$$

where a_2, b_2, c_2, d_2 are degree 2 homogeneous forms in $x_0, \dots, x_3$. Now view $\mathbb{P}^3$ as $\{x_4 = 0\} \subset \mathbb{P}^4$: it is elementary to see that, if $\tilde{a}_2, \tilde{b}_2, \tilde{c}_2, \tilde{d}_2$ are general forms in $x_0, \dots, x_4$ that give a, b, c, d when restricted to $\mathbb{P}^3$, then

$$X = \{\tilde{a}_2 \tilde{b}_2 + \tilde{c}_2 \tilde{d}_2 = 0\} \subset \mathbb{P}^4$$

is a 3-fold of the required type containing S as a hyperplane section.

To work out p_1, we need to understand $c_2(Z_+)$ modulo the image of N_- in $N_+^* \hookrightarrow H^4(Z_+)$ and vice versa. By [21, Lemma 5.18], we can compute it by taking the greatest common divisor

No	Z_+	Z_-	b^2	b^3	TH^3	TH^4	a_0	what	p_1
1	$Ex\ 7.1^1_4$	$Ex\ 7.1^1_4$	0	155	2		0	$\langle 4\rangle \perp \langle 4\rangle$	8
$2a$	$Ex\ 7.3$	$Ex\ 7.3$	0	123			18	$N_+ \perp N_-$	$4, 8$
$2b$	$Ex\ 7.3$	$Ex\ 7.4$	0	117			21	$N_+ \perp N_-$	4
$2c$	$Ex\ 7.3$	$Ex\ 7.5$	0	107			26	$N_+ \perp N_-$	$4, 8$
$2d$	$Ex\ 7.3$	$Ex\ 7.6$	0	109			25	$N_+ \perp N_-$	$4, 8$
3	$Ex\ 7.1^1_4$	$Ex\ 7.8$	3	116			24	$\langle 4\rangle \perp N_-$	4
4	$Ex\ 7.12$	$Ex\ 7.10$	0	93			21	$N_+ \perp N_-$	4
$5a$	$Ex\ 7.7$	$Ex\ 7.1^1_4$	0	95			45	$N_+ \perp \langle 4\rangle$	4
$5b$	$Ex\ 7.7$	$Ex\ 7.1^1_{10}$	0	61			45	$N_+ \perp \langle 10\rangle$	4
$5c$	$Ex\ 7.7$	$Ex\ 7.1^1_{16}$	0	53			45	$N_+ \perp \langle 16\rangle$	4
$5d$	$Ex\ 7.7$	$Ex\ 7.1^1_{22}$	0	53			45	$N_+ \perp \langle 22\rangle$	4
$5e$	$Ex\ 7.7$	$Ex\ 7.1^2_2$	0	67			45	$N_+ \perp \langle 4\rangle$	4
$5f$	$Ex\ 7.7$	$Ex\ 7.1^2_5$	0	71			45	$N_+ \perp \langle 10\rangle$	4
$5g$	$Ex\ 7.7$	$Ex\ 7.1^4_1$	0	95			45	$N_+ \perp \langle 4\rangle$	4
$6a$	$Ex\ 7.7$	$Ex\ 7.1^1_6$	0	77	3		45	$N_+ \perp \langle 6\rangle$	4
$6b$	$Ex\ 7.7$	$Ex\ 7.1^1_{12}$	0	57	3		45	$N_+ \perp \langle 12\rangle$	4
$6c$	$Ex\ 7.7$	$Ex\ 7.1^1_{18}$	0	53	3		45	$N_+ \perp \langle 18\rangle$	4
$6d$	$Ex\ 7.7$	$Ex\ 7.1^2_3$	0	65	3		45	$N_+ \perp \langle 6\rangle$	4
$6e$	$Ex\ 7.7$	$Ex\ 7.1^3_2$	0	85	3		45	$N_+ \perp \langle 6\rangle$	4
7	$Ex\ 7.11$	$Ex\ 7.11$	24	47	8		66	$N_+ \perp N_-$	4
8	$Ex\ 7.6$	$Ex\ 7.6$	0	95	$4{\cdot}4$		32	$N_+ \perp N_-$	8
$9a$	$Ex\ 7.2_2$	$Ex\ 7.2_{24}$	1	82			0	$N_+ \perp_{\langle -6\rangle} N_-$	12
$9b$	$Ex\ 7.2_6$	$Ex\ 7.2_6$	1	86			0	$N_+ \perp_{\langle -4\rangle} N_-$	24
$9c$	$Ex\ 7.2_{10}$	$Ex\ 7.2_{10}$	1	70			0	$N_+ \perp_{\langle -16\rangle} N_-$	16
$9d$	$Ex\ 7.2_{12}$	$Ex\ 7.2_{12}$	1	78			0	$N_+ \perp_{\langle -4\rangle} N_-$	8
$9e$	$Ex\ 7.2_{21}$	$Ex\ 7.2_{21}$	1	82			0	$N_+ \perp_{\langle -4\rangle} N_-$	8
$9f$	$Ex\ 7.2_6$	$Ex\ 7.2_{12}$	1	82			0	$N_+ \perp_{\langle -4\rangle} N_-$	8
$9g$	$Ex\ 7.2_6$	$Ex\ 7.2_{21}$	1	84			0	$N_+ \perp_{\langle -4\rangle} N_-$	8
$9h$	$Ex\ 7.2_{12}$	$Ex\ 7.2_{21}$	1	80			0	$N_+ \perp_{\langle -4\rangle} N_-$	8
10	$Ex\ 7.9$	$Ex\ 7.9$	1	82		2^2	40	$N_+ \perp_{\langle -8\rangle} N_-$	8
11	$Ex\ 7.6$	$Ex\ 7.6$	0	93			32	non-orth	48

TABLE 5. A small number of examples of G_2–manifolds. a_0 is the number of rigid associative 3-folds diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ we can exhibit.

of 24 and $c_2(Y_1)+c_1(Y_1)^2$ evaluated on divisors in $N_+ \cap N_-^\perp$. In this case, $N_+ \cap N_-^\perp$ is generated by $H_+ = A_+ + E_+$. The restriction of $c_2(Y_1)+c_1(Y_1)^2$ to the first term is $\chi(K3) + (-K_Y)^3 = 24+4 = 28$ (since A is just $-K_Y$), while it is computed in [21, Example 7.6] that the restriction to E_+ is 44. Hence $\operatorname{div}(c_2(Z_+) \bmod \operatorname{Im}(N_-)) = \gcd(24, 28+44) = 24$.

It is straightforward to assemble the remaining topological information for the entry in Table 5. Note that the usual relation for $b^2(M)+b^3(M)$ (Lemma 6.7) is not satisfied since the gluing is not orthogonal; in particular the value of $b^3(M)$ is different from *No 8*, even though that example uses the same building blocks.

Obstructed associatives. Let us illustrate how one can apply Proposition 5.18 to construct families of associatives – including some obstructed ones – in compact G$_2$–manifolds from special Lagrangian rational homology spheres in building blocks. The easiest way to exhibit concrete examples of the latter is to use real algebraic geometry; complex conjugation on an algebraic variety is an antiholomorphic involution, and the fixed set of an antiholomorphic involution on a Calabi–Yau manifold is special Lagrangian (with some phase). If we construct a building block from a real (semi) Fano Y by blowing up a real anticanonical curve, then the building block also has a real structure. A component L of the real locus of Y not meeting the chosen anticanonical divisor gives rise to a special Lagrangian in the ACyl Calabi–Yau V. To apply Proposition 5.18 we require that $b^1(L) = 0$ and that $[L] \in H_3(Y;\mathbb{R})$ is non-zero (see Remark 5.19).

Given a building block with a suitable special Lagrangian, we still need to match it to another building block to construct a G$_2$–manifold. The 'orthogonal gluing' argument is unfortunately not very compatible with the use of real algebraic geometry, so it is not so easy to write down a list of building blocks containing special Lagrangian spheres and claim that each can be matched with a list of building blocks. Instead we limit ourselves to showing that for some examples we can find at least some matching.

When we construct an $\mathbb{S}^1$-family of associatives like this, with a map f from $\mathbb{S}^1$ to a 1-parameter family of G$_2$–structures that specifies by which G$_2$–structure the members of the family are calibrated, critical points of f correspond to associatives with obstructions. If the entire $\mathbb{S}^1$-family is associative with respect to the same G$_2$–structure (*ie* f is constant) then they are all obstructed, but one would expect that the critical points of f are isolated. As one moves in the 1-parameter family of G$_2$–structures and approaches a local extreme value, two associatives move together, coincide as a single obstructed associative, and then disappear (or vice versa).

Example 7.16. We consider a particular block Z_+ from Example $7.1\frac{1}{4}$. Let Y be the quartic 3-fold in $\mathbb{C}P^4$ defined by $Q(X) = -X_0^4 + X_1^4 + X_2^4 + X_3^4 + X_4^4 = 0$. Its real locus L is homeomorphic to $\mathbb{S}^3$ and does not meet the anticanonical divisor $X_0 = 0$. If we blow up the intersection of $X_0 = 0$ and another real hyperplane section of Y to form Z_+, then Z_+ has a real structure and an anticanonical divisor S_+ that does not meet the real locus $\mathbb{S}^3$. We can give $V_+ = Z_+ \backslash S_+$ an ACyl Calabi–Yau metric that is invariant under the real structure σ, and it then contains a special Lagrangian $L \cong \mathbb{S}^3$.

More precisely, up to sign V_+ has a unique holomorphic volume form $\Omega = \alpha + i\beta$ such that $\sigma^*\Omega = \overline{\Omega}$, and L (correctly oriented) is calibrated by α. On the cylindrical end $\mathbb{R}^+ \times \mathbb{S}^1 \times S$ we can write the Kähler form as $dt \wedge d\vartheta + \omega^I$ and $\alpha = d\vartheta \wedge \omega^J + dt \wedge \omega^K$. The real structure σ on S_+ preserves ω^K and reverses ω^I and ω^J. On the other hand, the involution $(X_1 : X_2 : X_3 : X_4) \mapsto (X_2 : X_1 : X_3 : X_4)$ has fixed set of dimension 1, so defines a non-symplectic isometry τ on S_+, *ie* $\tau^*\omega^I = \omega^I$ while $\tau^*(\omega^J + i\omega^K) = -(\omega^J + i\omega^K)$. Under a hyper-Kähler rotation $S_+ \to S_-$ (Definition 3.10), $\tau\sigma$ therefore corresponds to a non-symplectic involution. The fixed set of $\tau\sigma$ is homeomorphic to 2 copies of $\mathbb{S}^2$ (any point in the fixed set can be written uniquely as $(e^{i\theta} : e^{-i\theta} : x_3 : x_4)$, with $\theta \in [0,\pi)$, $x_3, x_4 \in \mathbb{R}$ and $x_3^2 + x_4^2 = 2\,\mathrm{Re}(e^{i4\theta})$), and the quotient $(S_- \times \mathbb{C})/\tau\sigma$ can be resolved by blow-up to form an ACyl Calabi–Yau V_- of non-symplectic type (*cf* Remark 3.20) that is compatible with V_+.

To apply Proposition 5.18, we need to check that L is not homologous to 0 in Y. If it is, then so is its preimage $\hat{L}$ in H_4 of the unit normal bundle of Y in $\mathbb{C}P^4$. For any homogeneous polynomial P of degree 3, $\frac{P}{Q^2}(X_0 dX_1 dX_2 dX_3 dX_4 - X_1 dX_0 dX_2 dX_3 dX_4 + \ldots)$ defines a meromorphic 4-form on $\mathbb{C}P^4$ with poles only on Y. Using the affine chart $X_0 = 1$, its integral over $\hat{L}$ reduces to $\int_L P(X) X \lrcorner dX_1 dX_2 dX_3 dX_4$, which is non-zero if we choose *eg* $P = X_0(X_1^2 + X_2^2 + X_3^2 + X_4^2)$.

Remark. For L to be homologically non-zero in Y it was sufficient to find one meromorphic form with poles on Y and non-zero integral over $\hat{L}$. In fact this condition is also necessary (Griffiths [36]).

Remark. In the example above, the compact G_2–manifold (M, φ_0) has a G_2–involution $\tilde{\sigma}$, which acts on the first half $\mathbb{S}^1 \times V$ by $(-1, \sigma)$. $\tilde{\sigma}$ acts as a reflection on the $\mathbb{S}^1$-family of associatives in $\mathbb{S}^1 \times M$. We can also choose the 1-parameter family of G_2–structures $\{\varphi_t : t \in (-\epsilon, \epsilon)\}$ in which the associative family becomes unobstructed so that $\tilde{\sigma}^*\varphi_t = \varphi_{-t}$. The map $f : \mathbb{S}^1 \to (-\epsilon, \epsilon)$ must be $\tilde{\sigma}$-equivariant. That f maps the fixed points of the reflection in $\mathbb{S}^1$ to φ_0 corresponds to the fact that the fixed locus of $\tilde{\sigma}$ in the G_2–manifold is associative. But considering only the fixed locus we cannot tell whether it is rigid or not. By considering the whole $\mathbb{S}^1$-family we see that it contains some obstructed associatives.

Example 7.17. Let $Q_0 \subset \mathbb{C}P^4$ be a real quartic 3-fold with a single nodal singularity. The singular point must then be real. Suppose that the local model of the singularity is $x^2 + y^2 + z^2 + w^2 = 0$ (in real coordinates). Then the real locus of a (real) deformation Q of Q_0 that smooths the singularity to $x^2 + y^2 + z^2 + w^2 = \epsilon > 0$ contains an $\mathbb{S}^3$. Because this is the vanishing cycle of a quartic with a single node it is homologically non-zero.

8. Review and Outlook

We now move from concrete examples to a more general discussion of the possibilities of the construction and some of the prospects for future developments.

Overview. At this point it will probably be helpful to give an overview of what has been achieved so far and also to reflect on some of the lessons learned from the examples given in the previous section. We begin by recalling the main degrees of freedom in the construction.

First we have the *choice of building blocks*: according to Proposition 3.17, we can for almost (recall the Assumption before 3.16) any Fano or semi-Fano 3-fold blow up a generic anti-canonical pencil to get a building block of Fano or semi-Fano type (in the sense of Definition 3.18), which has $K = 0$ (recall (3.7)). For some Fanos and semi-Fanos we can instead choose to blow up a nongeneric anticanonical pencil to obtain building blocks, *eg* Examples 7.8, 7.9 and 7.11 give building blocks obtained from nongeneric anticanonical pencils on $\mathbb{P}^3$, $\mathbb{P}^3$ and a particular toric semi-Fano 3-fold respectively. As illustrated by these examples, depending on the pencil being blown up these blocks may or may not have $K = 0$. In such cases care must be taken to ensure that the blocks satisfy the conditions in Definition 6.17, which are used in our matching arguments (and also the topological properties assumed in our calculations of the cohomology of M). The 74 blocks of non-symplectic type constructed by Kovalev-Lee all have $K \neq 0$ (however, recall Remark 4.29).

Choosing a pair of deformation types of building blocks $\mathcal{Z}_\pm$ fixes the pair of polarising lattices $N_\pm$. Let $n_\pm = \operatorname{rk} N_\pm$. Choosing the building blocks also fixes the number $e_\pm$ of compact rigid curves in $V_\pm = Z_\pm \setminus S_\pm$. By Theorem 4.8(ii) $b^2(M) \geq \operatorname{rk} K_+ + \operatorname{rk} K_-$. In particular, to obtain G_2–manifolds with $b^2(M) = 0$ (*eg* if we want to construct 2-connected manifolds) we must choose blocks with $K_\pm = 0$.

Next we choose the *method of matching*: perpendicular gluing, orthogonal gluing or handcrafted gluing. For simplicity and because it is difficult at this stage to say anything very systematic about handcrafted gluing here we stick to commentary on perpendicular or orthogonal gluing.

Primitive perpendicular gluing. Whenever $N_+ \perp N_-$ can be primitively embedded in the K3 lattice L then we can match the building blocks $\mathcal{Z}_\pm$ by primitive perpendicular matching; this always yields a 2-connected 7-manifold with torsion-free cohomology to which we may apply the general classification theory described in §4. $N_+ \perp N_-$ always embeds primitively in L if

$$n_+ + n_- \leq 11 \tag{8.1}$$

(see (6.11)). If we are able to compute the lattices $N_\pm$ in detail (and not just their ranks $n_\pm$) then we can determine their discriminant groups and hence determine $\ell = \ell(N_+ \perp N_-)$; by 6.9(ii) $N_+ \perp N_-$ admits a primitive embedding in L if

$$n_+ + n_- + \ell < 22. \tag{8.2}$$

See *No 4* for an example satisfying the second inequality but not the first.

To summarise: when it applies primitive perpendicular gluing requires little effort and yields 2-connected 7-manifolds with torsion-free cohomology; it therefore allows us to produce many G_2–manifolds for which we understand the diffeomorphism type.

Non-primitive perpendicular gluing. If $N_+ \perp N_-$ admits a primitive embedding in L then $n_+ + n_- + \ell \leq 22$. For some pairs of blocks (*eg* the Burkhardt block matched to any semi-Fano or Fano block with Picard rank greater than 1) this inequality is violated and hence $N_+ \perp N_-$ admits no primitive embeddings in L. Nevertheless, $N_+ \perp N_-$ may admit non-primitive embeddings in L which are primitive when restricted to both N_+ and N_- (as exhibited in *No 6*). In this case the resulting G_2–manifoldis simply-connected with $H^2 = 0$ and $\operatorname{Tor} H^3 \neq 0$ (and therefore $H_2 = \pi_2 \neq 0$). Even if $N_+ \perp N_-$ does admit a primitive embedding in L, it may still admit non-primitive embeddings in L which are primitive on each factor $N_\pm$. In this case different matchings of the same blocks can produce both 2-connected and non-2-connected G_2–manifolds with the same Betti numbers, distinguished by the torsion in H^3. As explained in Remark 6.12, the problem of finding suitable non-primitive embeddings $N_+ \perp N_- \hookrightarrow L$ reduces to a fairly manageable analysis of the discriminant groups of $N_\pm$; this is used in *No 1* and *8* and Remark 7.14. Later in this section we describe the prospects of extending the known smooth classification results to 1-connected 7-manifolds with $\pi_2(M)$ a finite cyclic group.

Orthogonal gluing. If $\min(n_-, n_+) > 1$ then we may also attempt to use orthogonal but not perpendicular gluing; this will always yield manifolds with $b^2 > 0$. In this case we encounter two additional problems. The first is the arithmetic problem of finding a non-trivial lattice R that can be primitively embedded in both N_+ and N_-, such that the push-out $W = N_+ \perp_R N_-$ is an integral lattice. For instance, if we want $R = \langle -m \rangle$ then we look for primitive vectors $x_\pm \in N_\pm$ of square-norm $-m$, and Example 6.8 demonstrates that we need that if the image of the orthogonal projection of $N_\pm$ to $\langle x_\pm \rangle$ is $\frac{1}{d_\pm}\langle x_\pm \rangle$, then $d_+ d_- \mid m$. The second problem is to satisfy condition (iii) in Proposition 6.18, that the orthogonal complement of R in $N_\pm$ contains some classes that are ample on the building blocks. For non-symplectic type blocks it would be enough to check that R contains no -2 classes (*cf* Remark 6.19), but the semi-Fano case is more subtle.

Example 8.3. The polarising lattice N of the block Z in Example 7.4 can be presented as

$$\begin{pmatrix} -2 & 2 \\ 2 & 4 \end{pmatrix} = \frac{1}{2}(1,1)\mathbb{Z} + \mathbb{Z}^2 \quad \text{in} \quad \begin{pmatrix} -12 & 0 \\ 0 & 4 \end{pmatrix},$$

and there is an orthogonal push-out $W = N \perp_R N$ with $R = \langle -12 \rangle$. However, the pre-image in $\operatorname{Pic} Y$ of the square-norm 4 vector $x \in N$ that is orthogonal to R is exactly $-K_Y$ of the quartic semi-Fano Y that the block is obtained from. Since Y is not a genuine Fano, $-K_Y$ is not an ample class on Y; indeed, the anticanonical morphism contracts 12 curves. Any pre-image of x in $\operatorname{Pic} Z$ evaluates to 0 on the 12 exceptional curves, so cannot be ample on Z. Therefore there can be no matchings compatible with this W.

For a given pair of blocks, there need not be any suitable R. For blocks with $\operatorname{rk} N = 2$ there are not very many degrees of freedom in choosing R, but we found some solutions in *No 9.* There we used rank 2 Fano type blocks, and it was not too hard to compute the ample cones. We expect that solutions become more plentiful as the rank increases, but on the other hand the ample cones are more complicated to describe, and it is less practical to search for solutions by hand. The supply of toric semi-Fano blocks described below should be suitable for an automated search.

Twisted connected sums that use perpendicular gluing (whether primitive or not) always have $H^4(M)$ torsion-free. To get non-trivial torsion in $H^4(M)$ from orthogonal gluing, we need to find a non-trivial orthogonal push-out $W = N_+ \perp_R N_-$, but with conditions on R that are more restrictive than merely ensuring that W is an integral lattice. For example, if $R = \langle -m \rangle$ then in the notation used above the condition for W to be an integral lattice is that $d_+d_- \mid m$, but Lemma 7.13 shows that $\operatorname{Tor} H^4(M) \cong (\mathbb{Z}/k\mathbb{Z})^2$ where $k = \frac{m}{d_+d_-}$ (*eg* a matching with the data $W = N \perp_R N$ from Example 8.3 would give $\operatorname{Tor} H^4(M) \cong (\mathbb{Z}/3\mathbb{Z})^2$). Again we expect that solutions are easier to find when $\operatorname{rk} N_\pm$ are larger; in *No 10*, our only explicit example with non-trivial torsion in $H^4(M)$, we used blocks with rank 3.

Mass-production and Geography. In this section we describe some general features of the G_2–manifolds that can be mass-produced using our methods.

G_2–manifolds from pairs of smooth Fano 3-folds. We previously described in detail the 2-connected manifolds that can be constructed as twisted connected sums of rank 1 Fano type building blocks; we now outline what can be achieved if we drop the rank 1 assumption. Orthogonal, but non-perpendicular, matching of building blocks of rank two Fano type was considered in *No 9* in the previous section, but the easiest way to mass-produce examples is to consider primitive perpendicular gluing again.

Consulting the Mukai–Mori classification shows that out of $5356 = \frac{1}{2} \times (104 \times 103)$ possible pairs of Fano 3-folds satisfying our standing assumption (recall 3.15), 5280 satisfy the (crude) rank condition (8.1) and therefore admit a primitive embedding of $N_+ \perp N_-$ in L. We call the 76 pairs that fail to satisfy the previous inequality the exceptional Fano pairs. With more work one could compute the Picard lattices of all the smooth Fano 3-folds and determine their discriminant groups in order to check whether the refined rank/discriminant condition (8.2) is satisfied; this would yield further matching pairs. (Recall from the Mukai–Mori classification that every Fano 3-fold F has Picard rank $\rho(F) \leq 10$ and if $\rho = \rho(F) \geq 6$ then F is biholomorphic to a product $\mathbb{P}^1 \times S_{11-\rho}$; here S_d denotes a del Pezzo surface of degree d with $1 \leq d \leq 5$ and is obtained by blowing up $\mathbb{P}^2$ in $9 - d$ sufficiently general points. Therefore all 76 of the exceptional Fano pairs include at least one product Fano 3-fold $\mathbb{P}^1 \times S_d$. For simplicity, we shall not consider these exceptional pairs any further in this paper.)

All 5280 G_2–manifolds produced are 2-connected and have torsion-free cohomology; therefore they are classified up to almost-diffeomorphism by $b^3(M)$ and $\operatorname{div} p_1(M)$. 67 values of $b^3(M)$ are realised by primitive perpendicular gluing of nonexceptional pairs of Fanos (versus 46 values from pairs of rank 1 Fanos). We have $b^3(M) = 23 + b$ for some even integer b where either $b = 216$ or $38 \leq b \leq 174$; in the latter case all even values of b within the range are realised except for $b = 126, 168$ or 170. In particular, the smallest value of $b^3(M)$ produced this way is $38 + 23 = 61$.

We have not studied systematically the values of $\operatorname{div} p_1(M)$ arising from pairs of higher rank Fanos but this would be possible (though time-consuming) by adapting the methods used elsewhere in the paper and in [21, §5]. However, Corollary 4.32 implies that there are at most 8 possible diffeomorphism types with the same value of b^3. So while there are over

5 000 Fano pairs we can match using perpendicular gluing, there are at the very most 536 diffeomorphism types of G_2–manifold that can be realised this way. In other words, there are many ways of finding different pairs of perpendicularly glued ACyl Calabi–Yau 3-folds of Fano type which yield G_2–metrics on the *same* smooth 7-manifold. For example, there are 290 different matching Fano-type pairs that give rise to a smooth 2-connected 7-manifold M with $b^3(M) = 87$. Since the metrics "see" the long cylindrical neck, they cannot be isometric unless the building blocks are diffeomorphic. If they belong to the same component of the G_2 moduli space then the path connecting them therefore cannot merely be some small perturbation.

Beyond the 2-connected world we could also seek Fano type matching pairs using non-primitive perpendicular gluing. This yields G_2–manifolds with the same Betti numbers and p_1 as the 2-connected examples constructed via primitive perpendicular gluing, but which have nontrivial $\operatorname{Tor} H^3$. As we have already discussed such non-primitive embeddings of $N_+ \perp N_-$ are related to its overlattices and therefore to properties of the discriminant groups of $N_\pm$. Given the variety of ways to find matching Fano pairs yielding the same value of b^3 it seems likely that non-primitive perpendicular gluing will yield a considerably greater number of topological types, distinguished by $\operatorname{Tor} H^3$. We will not pursue this any further in the current paper.

G_2–manifolds from toric semi-Fanos. One very abundant source of semi-Fano 3-folds are the *toric* semi-Fano 3-folds. We refer the reader to [21, §8] for a more detailed review of their construction and properties. The anticanonical model of a smooth toric weak Fano 3-fold is a toric Fano 3-fold with at worst Gorenstein canonical singularities. Such toric Fano 3-folds correspond (uniquely up to isomorphism) to combinatorial objects called *reflexive polytopes.* Kreuzer-Skarke gave an algorithm to classify reflexive polytopes and showed that there are 4319 3-dimensional reflexive polytopes of which 18 correspond to smooth toric Fanos and 82 to terminal toric Fanos (the latter have only ordinary double point, ODP, singularities). Every Gorenstein canonical toric Fano 3-fold admits at least one and often many projective crepant resolutions; moreover, all these crepant resolutions are toric and so can be enumerated purely combinatorially.

Using this fact we have together with Tom Coates and Al Kasprzyk enumerated all (smooth) toric semi-Fano 3-folds up to isomorphism. For instance, we found that there are 1009 non-isomorphic toric semi-Fano 3-folds whose anticanonical morphism is small; these correspond to projective small resolutions of the 82 terminal reflexive polytopes. Because any toric semi-Fano 3-fold with small anticanonical morphism is rigid, these all give rise to non deformation equivalent toric semi-Fano 3-folds.

While not every toric semi-Fano 3-fold is rigid many of them are and rigidity/nonrigidity depends only on the Fano polytope and not the choice of projective crepant resolution; in total there are 526 230 non-isomorphic toric semi-Fano 3-folds of which 435 459 are rigid (including the 18 smooth toric Fanos and the 1 009 arising from the 82 terminal reflexive polytopes already mentioned). Thus we have at least 435 459 deformation types of toric semi-Fanos. (For the remaining non-rigid toric semi-Fanos more work would be needed to understand how many new deformation types these realise.)

Now consider all pairs of blocks of Fano-type or rigid toric semi-Fano type. Already 39 584 matching pairs are obtained by primitive perpendicular gluing of pairs that satisfy the crude inequality (8.1) (or 15 027 pairs if we only include rigid toric semi-Fanos with small anticanonical morphism). On the other hand no new values of $b^3(M)$ are achieved this way. In other words, in the 2-connected world the main effects of allowing toric semi-Fanos (with small anticanonical morphism) in addition to Fano 3-folds are: (i) to increase significantly the number of different ways of using the twisted connected sum construction to produce G_2–metrics on

Polytope	$\rho(N)$	A_N	$\ell(N)$	no. of resolutions	g
3282	14	$2 \cdot 3^2 \cdot 4$	2	46720	8
3267	14	$2 \cdot 5 \cdot 8$	2	44120	8
3415	15	$2^2 \cdot 16$	3	35775	7
3452	15	$2 \cdot 3^3$	3	34118	7
3297	14	$3 \cdot 27$	2	24216	8
2989	13	$4 \cdot 19$	1	23400	9
3033	13	$3 \cdot 32$	1	16092	9
3013	13	$2 \cdot 5 \cdot 9$	1	13770	9
3026	13	$8 \cdot 11$	1	12771	9
2986	13	$3^2 \cdot 8$	2	12528	9
3018	13	$3 \cdot 4 \cdot 7$	1	8770	9
2683	12	$3 \cdot 29$	1	8280	10

TABLE 6. The top dozen rigid semi-small Gorenstein Fano 3-polytopes ordered by the number of nonisomorphic projective crepant resolutions they admit. $\rho(N)$, A_N and $\ell(N)$ denote the rank of the polarising lattice N, the orders of cyclic factors in the discriminant group of N, and its discriminant rank, respectively, and g denotes the anticanonical genus of the polytope; the number in column one refers to the index of the polytope in the Sage implementation of the Kreuzer-Skarke list of reflexive 3-polytopes.

the same smooth 7-manifold and (ii) to produce many smooth 7-manifolds on which we have G_2–metrics with different numbers of obvious rigid associatives.

The reason why the number of Fano/rigid toric semi-Fano type pairs satisfying (8.1) is small compared to the numbers of deformation types of rigid toric semi-Fanos is that over 400 000 of these deformation types have Picard rank 11 or greater and therefore can never lead to a pair satisfying (8.1). Many more matching pairs would be obtained if we computed the discriminant groups of the toric Picard lattices and applied (8.2)—the toric semi-Fanos with large Picard rank tend to have many nonisomorphic flops and their discriminant groups are typically relatively small; a more systematic study of G_2–manifolds obtained via toric semi-Fano type blocks will be described elsewhere [20].

For now we content ourselves with a demonstration of the plethora of matching pairs that can be obtained using information about the discriminant groups of the polarising lattices of rigid toric semi-Fanos. For other reasons Rohsiepe computed the discriminant groups for the polarising lattices corresponding to any reflexive 3-polytope [67]. Table 6 lists the dozen "most prolific" toric semismall reflexive polytopes, *ie* the semismall reflexive polytopes with the most non-isomorphic crepant projective resolutions, along with the rank of the polarising lattice N, its discriminant group A_N, its discriminant rank $\ell(N)$, the number of non-isomorphic projective crepant resolutions and the anticanonical genus of the polytope.

For all but two of the dozen polytopes (numbers 3 415 and 3 452) it follows from the data in Table 6 that $W = N \perp N_-$ satisfies (8.2) for *any* rank two polarising lattice N_-. In fact, using the complete criterion for embeddings given by Nikulin [61, Theorem 1.12.2] one can show that if N is the polarising lattice corresponding to polytope 3 415 then, because its discriminant form splits off an orthogonal $\mathbb{Z}/2\mathbb{Z}$ summand (the form is $\mathrm{diag}(1/2, 1/2, 1/16)$), one can still primitively embed $N \perp N_-$ in L for any rank two polarising lattice N_-. Thus each deformation type of rank 2 block in combination with the rigid semi-Fanos generated by these 11 polytopes yields almost 250 000 matching pairs. (In fact, it seems likely that a systematic study of the discriminant groups associated with rigid toric semi-Fanos will show that almost all of them can be primitively perpendicularly matched to any block of rank at most two).

We know that there are precisely 17 deformation types of smooth Fano 3-folds of rank 1. We also know from [21, §8] that there at least 186 deformation types of rank 2 semi-Fano or Fano 3-folds, but the precise number of deformation types has yet to be determined. It follows that just these eleven prolific rigid semismall polytopes along with known blocks of rank at most two generate over 50 000 000 ($246\,442 \times (17+186) = 50\,027\,726$) matching pairs via primitive perpendicular gluing.

The geography of 2-connected twisted connected sums. Let M be a 2-connected twisted connected sum G_2–manifold. All such examples constructed in this paper so far have $55 \leq b^3 \leq 239$. If M is obtained from perpendicular gluing of blocks of Fano or semi-Fano type then there is an absolute lower bound for b^3 of $31 = 22+1+4+4$ (because any Fano or semi-Fano 3-fold Y has anticanonical genus g at least 2 and $b^3(Z) = b^3(Y)+2g(Y)$). To achieve this lower bound we would need to find a semi-Fano 3-fold Y with $b^3(Y) = 0$ and $g(Y) = 2$, *ie* Y should be a resolution of a singular sextic double solid. Recently Arap, Cutrone and Marshburn [8] claimed the existence of such a smooth semi-Fano 3-fold with small anticanonical morphism and Picard rank $\rho = 2$; we have not verified this example in detail ourselves.

Assuming the existence of such a smooth semi-Fano then (because $\rho = 2$) we can immediately match such a block to itself by primitive perpendicular gluing and thus exhibit a 2-connected twisted connected sum G_2–manifoldwith $b^3 = 31$, the smallest possible value of b^3. Moreover, because Y has such small Picard rank we can also primitively perpendicularly match it to many other blocks of Fano or semi-Fano type. Hence the existence of this extremal Y gives rise to a sequence of 7 new values of b^3 less than 55 and also gives $b^3 = 149$ which was previously a "gap" in the sequence of values of b^3.

The ongoing classification programme for rank 2 weak Fanos (see [21, §8] for an overview) looks likely to produce other rank two weak Fanos Y whose corresponding block Z has small b^3, *eg* there is potentially a rank 2 small resolution Y of a terminal quartic with $b^3(Y) = 0$ and hence $b^3(Z) = 6$. In this way it seems quite likely that essentially all odd numbers between 31 and 189 should be realised as b^3 of some 2-connected twisted connected sum. (The existence of the quartic semi-Fano described above would only leave gaps at 37 and 39).

It is somewhat curious that the building blocks we know with largest $b^3 = 108$ come from smooth sextic double solids, so that both the smallest and largest values of b^3 for 2-connected twisted connected sums arise from sextic double solids. Among Fano type blocks, there is a big gap down to the next highest value 66 for b^3. There are a few other blocks that can be used to construct 2-connected twisted connected sums with $197 < b^3 < 239$, *eg* some of the non-symplectic smoothing blocks described in Remark 4.29 or blocks obtained from a small resolution of a nodal sextic double solid with relatively few nodes (*eg* a block from a sextic double solid with 15 nodes [19, Example 1.5] has $b^3 = 80$).

Examples with positive b^2. We have already seen how (non-perpendicular) orthogonal gluing can be used to construct G_2–manifolds with $b^2 > 0$. However, for the two reasons we observed at the beginning of this section it can be somewhat labour-intensive to implement. Perhaps a more economical way to mass-produce G_2–manifolds with $b^2 > 0$ is to use perpendicular gluing and to choose at least one block with $K \neq 0$, *eg* Example 7.11 has $\mathrm{rk}\, K = 12$ and arises by blowing up a nongeneric anticanonical pencil on a toric semi-Fano 3-fold.

One uniform source of building blocks with $K \neq 0$ are the 74 non-symplectic type blocks described in Remark 3.20. For example, there is a K3 surface S with non-symplectic involution whose action on $H^2(S)$ fixes $N_+ = 2E_8(-1) \perp U$, and from which one may construct a building block Z_+ with $\operatorname{rk} K_+ = 20$ and $b^3(Z_+) = 8$. If Z_- is any Fano or semi-Fano type block whose polarising lattice N_- has rank ≤ 2, then N_- can be embedded primitively in $2U$ by Theorem 6.9; thus $N_+ \perp N_-$ can be embedded in L, and we can use perpendicular gluing to construct a G_2–manifold M with $b^2(M) = 20$, $b^3(M) = b^3(Z_-)+51$. As there are many Fano and semi-Fano 3-folds with Picard rank ≤ 2, we can ask how many different values of $b^3(M)$ are realised this way. Considering only the Fano-type blocks we obtain 18 values of $b^3(Z_-) \in \{24, \dots, 66\} \cup \{108\}$ and where $\{46, 54, 60, 62\}$ are omitted. Since the classification of rank two weak Fano 3-folds is still in progress we cannot currently say precisely what values can be obtained for semi-Fano type blocks. However, matching allowing rank two semi-Fano type blocks should enable us to obtain further examples with small values of $b^3(Z_-)$, potentially as low as 4. Of course using a semi-Fano block also allows us to construct numerous G_2–manifolds with $b^2 > 0$ that contain rigid associative 3-folds. There are also non-symplectic type blocks Z_+ with $\operatorname{rk} K_+$ taking any even value between 0 and 20, such that N_+ embeds in $2E_8(-1) \oplus U$ and which therefore can be matched using perpendicular gluing to any Fano or semi-Fano type block of rank up to 2 as above.

One could also use primitive perpendicular gluing to match Example 7.8 (which has $\operatorname{rk} K = 3$ and $N = \langle 4 \rangle$) with any Fano block or any semi-Fano block of rank up to 10. There will therefore be many G_2–manifolds with $b^2 = 3$. Finally with further work one could calculate $\operatorname{div} p_1$ for all the examples above and thereby distinguish further topologically distinct G_2–manifolds.

Prospects of further smooth classification of simply-connected spin 7-manifolds. For the large number of examples of 2-connected twisted connected sums with $H^4(M)$ torsion-free, we can use the classification result 4.22 to determine the almost-diffeomorphism type. Except for the relatively few examples where $\operatorname{div} p_1(M) = 16$ or 48 we pin down the diffeomorphism class completely; as explained in Remark 4.26, in those two cases one needs to compute a generalised Eells-Kuiper invariant to eliminate the remaining ambiguity in the smooth structure.

We have however also constructed (and explained how to construct many more) examples with relatively simple cohomology, but with $\pi_2(M)$ non-trivial. The classification results for simply-connected spin but not 2-connected 7-manifolds available in the literature mostly require (at least) that $H^4(M)$ is finite. With cues from Diarmuid Crowley, we speculate about what analogues one can hope to prove when $H^4(M)$ is infinite but torsion-free.

$\pi_2(M)$ finite cyclic. Using non-primitive but perpendicular gluing, we can find many examples with $H^2(M) = 0$ and $\operatorname{Tor} H^3(M)$ a cyclic group $\mathbb{Z}/k\mathbb{Z}$. Then $\pi_2(M) \cong \mathbb{Z}/k\mathbb{Z}$. As before, the isomorphism class of the pair $(H^4(M), p_1(M))$ is an obvious homeomorphism invariant, and when $H^4(M)$ is torsion-free it is equivalent to $(b^4(M), \operatorname{div} p_1(M))$. Now we have an additional invariant given by the square $z^2 \in H^4(M; \mathbb{Z}/k\mathbb{Z})$ modulo $((\mathbb{Z}/k\mathbb{Z})^*)^2$ of a generator $z \in H^2(M; \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$. We expect that (for a fixed k) the class of the triple $(H^4(M), p_1(M), z^2)$ determines the almost-diffeomorphism type of M. If k is prime, and $x \in H^4(M)$ is a primitive element of which $p_1(M)$ is a multiple, this means specifying $b^4(M)$, $\operatorname{div} p_1(M)$, whether $z^2 = 0$, if not whether it is a multiple of the mod k reduction of x, and if so whether the coefficient is a quadratic residue mod k.

$\pi_2(M)$ infinite cyclic. In *No 9* we gave examples with $H^*(M)$ torsion-free, and $H^2(M) \cong \mathbb{Z}$. Then $\pi_2(M) \cong \mathbb{Z}$. If $z \in H^2(M)$ is a generator, then the isomorphism class of the triple $(H^4(M), p_1(M), z^2)$ is an obvious invariant. In the setting where $H^4(M)$ is finite instead of

torsion-free, and generated by z^2 and $p_{1/2}(M)$, Kreck and Stolz [50] proved a classification result in terms of a triple of invariants $s_1, s_2, s_3 \in \mathbb{Q}/\mathbb{Z}$ (when $H^4(M) = 0$, s_1 corresponds to the Eells-Kuiper invariant). By analogy we expect that $(H^4(M), p_1(M), z^2)$ may not suffice to determine even the homotopy type of M on its own, but that it may be possible that together with some generalisations of s_2 and s_3 it determines the almost-diffeomorphism type (and a generalised Eells-Kuiper invariant would pin down the precise diffeomorphism type).

Formality and torsion-free $\pi_2(M)$. Hepworth [41] generalised the work of Kreck and Stolz to the case when $\pi_2(M) \cong \mathbb{Z}^k$ (but still under assumptions requiring $H^4(M)$ to be finite). Some of the generalised Kreck-Stolz invariants can in this case be interpreted in terms of the Massey product structure on the cohomology of M. We expect that the classification problem when $\pi_2(M) \cong \mathbb{Z}^k$ with $H^4(M)$ torsion-free should also be greatly simplified if we restrict to the case when all Massey products vanish; this happens in particular if M is *formal*. Deligne *et al* [28] showed that any Kähler manifold is formal, and it is an interesting problem whether the same is true for G_2–manifolds. Cavalcanti [16] shows that any simply-connected 7-manifold M with $b^2(M) \leq 1$ is formal (so we did not need to consider Massey products when $\pi_2(M)$ is cyclic), and that $b^2(M) \leq 2$ suffices for formality if M is a G_2–manifold. Formality of G_2–manifolds is also studied by Verbitsky [74].

G_2–transitions. In this section we make some more speculative remarks on how compact G_2–manifolds constructed by gluing different ACyl Calabi–Yau 3-folds may be seen as related. The basic idea is that well-known transitions for Calabi–Yau 3-folds, *ie* flops and conifold transitions, can yield via the twisted connected sum construction analogous G_2–transitions. We begin by recalling the basic features of these well-known 3-fold transitions.

3-fold transitions: flops and conifold transitions. Recall that the ordinary double point (ODP)

$$\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$$

admits two (isomorphic) projective small resolutions (each of which replaces the singularity at the origin with a $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$) and a smoothing $\{z_1^2+z_2^2+z_3^2+z_4^2 = t\}$ which replaces the singularity with a Lagrangian 3-sphere. Let X be a nodal Fano 3-fold, *ie* X has only finitely many singular points each (locally analytically) modelled on the ODP. By Namikawa's deformation results [60] X is globally smoothable, *ie* X is smoothable to a family of smooth Fano 3-folds F_t. Suppose that X also admits a projective small resolution Y. Then Y is a smooth semi-Fano 3-fold and the transition from the smooth semi-Fano Y to a smooth Fano F is called a *conifold transition.* (Conifold transitions have traditionally been studied in the context of 3-folds with $c_1 = 0$, *ie* the Calabi–Yau case; unlike the Fano condition the condition $c_1 = 0$ is preserved under conifold transitions.) We can often also *flop* a given projective small resolution Y of a nodal 3-fold X; this has the effect of changing the choice of which of the two projective small resolutions of the ODP is used at some of the nodes. In the semi-Fano world flopping Y yields other smooth semi-Fano 3-folds which also have X as their anticanonical models.

Flopping will lead to 6-manifolds with the same integral cohomology groups but typically with different cohomology rings. The topological effect of a conifold transition is to replace a finite number of two-spheres (with normal bundle of a given type) with the same number of three-spheres (also with normal bundle of a given type). Care must be exercised in understanding how this sort of topological surgery changes the topology—in particular the cycles being created or destroyed need not be homologically independent.

Remark 8.4. There is also a conjectural picture of conifold transitions as metric transitions and not just topological or complex-geometric transitions. Recall that the 3-fold ordinary double point admits a Ricci-flat Kähler (KRF) cone metric; the term *conifold* often refers to the ODP endowed with this KRF metric. The smoothings and the small resolutions of the ODP admit Ricci-flat Kähler metrics asymptotic to the conifold metric.

Suppose X_0 is a nodal projective 3-fold with trivial canonical bundle. It has long been conjectured that the nodal variety X_0 should admit a Ricci-flat Kähler metric, smooth away from the nodes and asymptotic to the conifold metric at each node. This is however still unproven. If X_0 smoothes to a family of smooth 3-folds X_t with $c_1 = 0$ then by Yau's result X_t admits Ricci-flat Kähler metrics in each Kähler class. *Given* the existence of a conically singular KRF metric on X_0 one can instead use gluing methods to construct smooth KRF metrics on X_t, *ie* one glues in an appropriately scaled copy of the smoothing of the ODP endowed with its asymptotically conical KRF metric [17, 18]. This yields a 1-parameter family of KRF metrics that converges as $t \to 0$ to the conically singular metric on X_0. Similarly, if X_0 admits a projective small resolution Y then one could use gluing methods to construct 1-parameter families of smooth KRF metrics on Y that degenerate back to the conically singular KRF metric on X_0. See Karigiannis [47] for a G_2 analogue of these gluing constructions.

We emphasise that this metric picture remains conjectural since the existence of the conically singular KRF metric on X_0 remains open.

Related G_2*–manifolds.* Suppose that we have a conifold transition $Y \to X \to F$ between a smooth semi-Fano Y and a smooth Fano 3-fold F via the nodal Fano 3-fold X. Using Proposition 3.17 we can generate building blocks Z_Y and Z_F and hence via Theorem 3.4 also (families of) ACyl Calabi–Yau manifolds V_Y and V_F. Suppose that $\mathcal{Z}_-$ is another family of building blocks chosen so that there are ACyl Calabi–Yau structures in its deformation family compatible with some V_Y and V_F. Then we can construct the resulting twisted connected sum G_2–manifolds M_Y and M_F and regard them as related G_2–manifolds. We could of course replace the conifold transition above with a flop $Y \to Y'$ and proceed as in the previous case to obtain related G_2–manifolds M_Y and $M_{Y'}$. We use the term G_2*–transition* to describe either of these operations.

Remark 8.5. By analogy with Remark 8.4 one might hope for a stronger metric counterpart of this relation between M_Y and M_F. The ultimate aim would be to find families of G_2–metrics on M_Y and M_F that converge to the same singular G_2–space—with transverse conifold singularities along $\mathbb{S}^1$s—but that is currently out of reach. We will discuss the difficulties later. For the time being we use the relation between M_Y and M_F (or $M_{Y'}$) of being descended from related pairs of ACyl Calabi–Yau 3-folds as an organising principle.

At the level of the 3-folds passing from the original Fano F to the semi-Fano 3-fold Y has three principal effects:

(i) b^2 increases when passing from F to Y (recall 3.22 and the fact that the existence of a projective small resolution of X forces its defect $\sigma(X)$ to be positive). Hence the K3 surfaces $S_Y \in |-K_Y|$ appearing in any semi-Fano $Y \in \mathcal{Y}$ are more special than the K3 surfaces $S_F \in |-K_F|$ for any Fano $F \in \mathcal{F}$.

(ii) b^3 typically decreases when passing from F to Y, often by more than the increase in b^2, but it may also stay constant (recall 3.23);

(iii) Y unlike F contains compact rigid rational curves $C_1, \dots, C_e$ which do not intersect smooth anticanonical divisors $S_Y \in |-K_Y|$. Each such curve C_i gives rise to a compact rigid rational curve in the associated ACyl Calabi–Yau structures on $Z_Y \setminus S_Y$.

One can think of the defect σ of the nodal degeneration X as giving a way to stratify the possible nodal degenerations of the original family of smooth Fanos $\mathcal{F}$, *ie* we can order our hierarchy according to the defect of the degeneration X: by (3.22) this is the same as ordering by the rank of $\operatorname{Pic} Y$.

Each of the three effects above has significance for obtaining G_2–manifolds by matching with the family of blocks $\mathcal{Z}_-$.

(i') The possible asymptotic K3 surfaces S_Y of ACyl Calabi–Yau structures obtained from a semi-Fano Y are more special than those S_F obtained via the original Fano F. We interpret this as follows: it should be harder to match in the deformation family of ACyl Calabi–Yau structures obtained from the semi-Fano 3-fold Y than for those obtained from the original Fano F.

(ii') *Assuming* that we can use perpendicular gluing to achieve matching of ACyl Calabi–Yau structures obtained from both the semi-Fano Y or from the original Fano F, then we will usually obtain topologically distinct 2-connected G_2–manifolds M_Y and M_F. At the level of complex 3-folds passing from F to Y decreases b^3 at the expense of increasing b^2; however, at the level of G_2–manifolds this transition decreases $b^3(M)$ while maintaining $b^2(M) = 0$. In this sense one can think of the transition from F to Y as yielding 2-connected G_2–manifolds which are topologically smaller.

(iii') The e rigid rational curves C_i give rise to e new compact associative 3-folds in M_Y compared to M_F.

Moreover, by passing to (deformation types of) semi-Fanos associated with different nodal degenerations of F one can obtain G_2–manifolds with successively smaller and smaller topology: see below for concrete examples obtained by degenerating quartics. In this sense G_2 conifold transitions create a “hierarchy” of related G_2–manifolds. Similarly different nodal degenerations X of F allow us to vary the number e. In some cases by choosing different nodal degenerations we can vary e without changing b^3 of the resulting 2-connected G_2–manifold. This gives one way to exhibit G_2–metrics on the same underlying smooth 7-manifold with different numbers of obvious compact rigid associative 3-folds.

Remark. More generally, if after making a transition from a Fano F to a semi-Fano Y we can no longer match by perpendicular gluing but can instead match using the more general orthogonal gluing then $b^2(M)$ can increase under the transition from F to Y. However, by Lemma 6.7 the sum $b^2(M) + b^3(M)$ cannot increase when passing from F to Y and usually must decrease.

Matching quartic type blocks. We now give a concrete illustration of the general discussion above using ACyl Calabi–Yau 3-folds associated with various quartic 3-folds. Eight families of ACyl Calabi–Yau 3-folds associated with various quartics appear already in this paper: one family of Fano type obtained from any smooth quartic (Example $7.1\frac{1}{4}$), six families of semi-Fano type obtained from projective small resolutions of defect 1 nodal quartics (Examples 7.3–7.6—recall that in two of these four examples different choices of small resolution lead to non deformation equivalent semi-Fanos) and the family of semi-Fano type obtained from a projective small resolution of the Burkhardt quartic (Example 7.7). The polarising lattices N in these cases are respectively: $\langle 4 \rangle$, the rank 2 lattices and the rank 16 lattice listed in Table 2. In all cases we have $b^3(Z) = b^3(Y) + 6$; for the smooth quartic we have $b^3(Y) = 60$; for the four examples with $\operatorname{rk} N = 2$ applying (3.23) we see that $b^3(Y)$ decreases linearly with the number of nodes e in the anticanonical model X and for the Burkhardt quartic we have $b^3(Y) = 0$. (So one might hope to use the Burkhardt block to produce G_2–manifolds with small b^3. However, its large Picard rank makes it difficult to match via perpendicular gluing as we explained earlier.)

If for the moment we remove the Burkhardt example from consideration there are 15 pairs $N_\pm$ of lattices we can choose. In these 15 cases $N_+ \perp N_-$ is a lattice of signature $(2,0)$, $(2,1)$ or $(2,2)$, which therefore may be primitively embedded in the K3 lattice L. Hence we can match all 15 pairs by primitive perpendicular gluing to obtain a series of 2-connected G_2–manifolds with torsion-free cohomology and $\operatorname{div} p_1 = 4$ or 8. The 14 values of b^3 realised this way are

$$b^3 \in \{91, 93, 95, 101, 103, 107, 109, 111, 117, 123, 125, 133, 139, 155\}. \tag{8.6}$$

The number of rigid associative 3-folds diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ we can realise in these G_2–manifolds is

$$a_0 \in \{0, 9, 12, 16, 17, 18, 21, 24, 25, 26, 28, 29, 32, 33, 34\}.$$

These examples illustrate (ii'): passing from an initial family of Fano-type blocks (which corresponds to the G_2–manifold with $b^3 = 155$) to related semi-Fano type blocks via nodal degeneration and projective small resolution (conifold transitions) leads—via the twisted connected sum construction—to a family of related but "smaller" G_2–manifolds.

These examples also illustrate how we can exhibit G_2–metrics on the same 7-manifold with different numbers of obvious rigid associative 3-folds: matching Example 7.3 with itself or $7.1\frac{1}{4}$ with 7.5 both yield manifolds with $b^3 = 123 = 50 + 50 + 23 = 34 + 66 + 23$; in both cases depending on the choice of small resolutions made we can achieve either $\operatorname{div} p_1(M) = 4$ or 8, and in each case the almost-diffeomorphism type contains a unique diffeomorphism type. For the first matching pair we have $a_0 = 18 = 9{+}9$ whereas for the second we have $a_0 = 17 = 17{+}0$.

Remark 8.7. It is natural to wonder whether the fact that there are different numbers of obvious rigid associative 3-folds for G_2–metrics on the same 7-manifold M can be used to show these metrics are not in the same connected component of the moduli space of G_2–metrics on M.

These examples also illustrate another somewhat subtle point related to flops and their effect on the topology of the associated G_2–manifolds. In Examples 7.3 and 7.5 flopping leads to two non-diffeomorphic blocks of semi-Fano type (arising from different projective small resolutions of the same nodal quartic X). These blocks are distinguished by $\operatorname{div} c_2(Z)$ which is 2 or 4 depending on the choice of small resolution made. If we match these blocks to Example 7.4 then, because that block has $\operatorname{div} c_2(Z) = 2$, Corollary 4.30 implies that we obtain diffeomorphic G_2–manifolds irrespective of our choice of small resolution. On the other hand Example $7.1\frac{1}{4}$ has $\operatorname{div} c_2(Z) = 4$, so if we match with that then the diffeomorphism type of the resulting G_2–manifolds does depend on our choice of small resolution.

Now suppose we choose N_+ to be the rank 16 polarising lattice of the Burkhardt example. Then because of the high rank of N_+, primitive perpendicular matching is now much more difficult to achieve. Nevertheless, *No 5a* showed that it is possible to match blocks of Burkhardt type with blocks obtained from smooth quartics using primitive perpendicular gluing. This yields a 2-connected 7-manifold M^4_{95} with torsion-free cohomology, $\operatorname{div} p_1 = 4$, $b^3 = 95$ (by the classification theory there is a unique such smooth 7-manifold) and containing (at least) 45 rigid associative $\mathbb{S}^1 \times \mathbb{S}^2$s.

Remark 8.8. It is not possible to achieve primitive perpendicular gluing using the Burkhardt block and any of the other quartic-related blocks. In fact, if N_- is any polarising lattice of rank greater than 1 then it is not possible to embed $N_+ \perp N_-$ primitively in L because that would violate the necessary condition (6.10) (recall that $\ell(N_+) = 5$ for the polarising lattice of the Burkhardt block). This illustrates what we mean in (i').

General terminal degenerations and smaller G_2*–manifolds.* In the discussion above for simplicity (and because all the examples we presented were of this type) we referred only to degenerations of smooth Fanos to singular Fanos with only ordinary nodes. However, we could consider more general degenerations, especially to Fano 3-folds with terminal Gorenstein singularities and seek projective small resolutions of these also. This will lead to a wider variety of semi-Fanos related to a single deformation family of smooth Fano 3-folds. For example, one could look for projective small resolutions of defect one terminal quartics with worse than ODPs, thereby generalising Examples 7.3–7.6.

As part of the partial classification scheme (summarised in [21]) for smooth weak Fano 3-folds of rank two, Cutrone-Marshburn [26, Nos. 54–76, Table 2] present a list of 19 potential candidates for rank two weak Fano 3-folds Y with small anticanonical morphism which (if they exist) can be obtained as projective small resolutions of terminal quartics. They all arise as the blowup of a smooth rank 1 Fano 3-fold along a smooth curve of known degree and genus (their numerical link types are all E1-E1); this makes it straightforward to compute $b^3(Y)$ (and hence also $b^3(Z)$ for the associated block) and the polarising lattices for these putative examples. The possible values for $b^3(Z)$ which arise from their list are

$$b^3(Z) \in \{6, 8, 10, 12, 14, 16, 18, 20, 26, 28, 34\},$$

whereas for our previously discussed quartic-related blocks we had

$$b^3(Z) \in \{6, 34, 36, 44, 50, 66\},$$

where 6 and 66 in the latter list are realised by the Burkhardt block and the smooth quartic block respectively.

Assuming all these examples could be realised as rank two semi-Fano 3-folds (which admittedly may not be the case) then because they have rank two polarising lattices we can certainly achieve primitive perpendicular gluing of any such pair of semi-Fano blocks. This would yield 2-connected 7-manifolds with torsion-free cohomology with the following 46 values of b^3

$$b^3 \in \{35, 37, \ldots, 111, 115, 117, 123, 125, 133, 139, 155\},$$

compared to the 14 values of b^3 (all of which satisfied $b^3 \geq 91$) obtained in (8.6) from the smooth and nodal quartic blocks we already discussed.

The main thing we learn from this discussion is that by allowing ourselves to degenerate to worse than ordinary nodes we may be able to obtain semi-Fano 3-folds with small b^3 without having to dramatically increase b^2—as happens for example in the Burkhardt example where we achieve $b^3(Y) = 0$ but the price is that $b^2(Y) = 16$. As we have seen the large Picard rank of the Burkhardt example makes it very problematic to match; by contrast we could perpendicularly glue these resolutions of terminal defect one quartics to most blocks of semi-Fano type. The price we pay for allowing semi-Fano 3-folds constructed by resolving more general terminal degenerations is that we can no longer be guaranteed to be able to produce rigid associative 3-folds in the resulting G_2–manifolds.

Remark. It would be particularly interesting to know the existence (or not) of Nos. 57 and 71 in [26, Table 2] since both would give rise to rank two semi-Fanos with $b^3(Y) = 0$; for this one needs to study rational curves of degree 8 in the rank one Fano V_{22} or in the smooth quadric Q respectively.

Metric G_2*–transitions.* We indicate the technical problems that currently stop us from using the twisted connected sum construction to produce families of G_2–metrics degenerating to a compact singular G_2–manifold. One basic problem is that given a sequence of ACyl Calabi–Yau metrics on V_+ that degenerate in a satisfactory way, we would need to be able to match the asymptotic limits of this whole family to ACyl Calabi–Yau metrics on V_-, in a continuous manner. One situation where this problem is simplified is when Z_+ is a building block like Example 7.8. There is a sequence of Kähler classes on Z_+ that shrink the $(-1,-1)$-curves in Z_+, but whose restriction to S_+ is constant. We should therefore find a sequence of degenerating ACyl Calabi–Yau structures on V_+ with a fixed asymptotic limit, that can be matched to a fixed ACyl Calabi–Yau 3-fold V_-.

This easy case of the matching-in-families problem is different from the sort of transitions we were discussing earlier in the subsection, in that the degeneration of the ACyl Calabi–Yau structures on V_+ comes from changing the resolution of a non-generic pencil rather than from a degeneration of the underlying semi-Fano to a nodal Fano X. In the latter case, suppose that the sequence of ACyl Calabi–Yau structures on V_+ can be matched with ones on V_-. Then, because $H^2(V_+) \to H^2(S_+)$ is injective for semi-Fano type blocks, the sequence of matching data must degenerate too in some sense. The problem of finding the limiting matching data therefore turns out to be too constrained to use the argument of Proposition 6.18 (the number of missing degrees of freedom is exactly the defect of X). This is the same kind of problem as in handcrafted gluing, and requires the same remedy: more detailed information about the deformation theory.

Given a matching in families, another problem is to control the neck length parameter in the gluing. If we can find a 1-parameter family of compatible pairs of ACyl metrics on V_+ and V_-, then Theorem 3.13 says that for each pair there is a parameter T such that we can form a twisted connected sum with neck length T. Joyce's perturbation results give bounds on T in terms of the geometry of V_+ and V_-. If the family of metrics on V_- degenerates to a metric with conical singularities, then the upper bound for T goes to infinity, so there is no guarantee that we can find a 1-parameter family of G_2–metrics whose Gromov-Hausdorff limit is compact. It may be difficult to get around this without resolving the conjecture about existence of conically singular Calabi–Yau metrics.

References

[1] B. S. Acharya, *On mirror symmetry for manifolds of exceptional holonomy*, Nuclear Phys. B **524** (1998), no. 1-2, 269–282.

[2] B. S. Acharya and S. Gukov, *M theory and singularities of exceptional holonomy manifolds*, Phys. Rep. **392** (2004), no. 3, 121–189.

[3] B. S. Acharya, G. Kane, P. Kumar, R. Lu, and B. Zheng, *Mixed Wino-Axion Dark Matter in String/ M Theory and the 130 GeV Gamma-line "Signal"*, arXiv:1205.5789, 2012.

[4] B. S. Acharya, G. Kane, and E. Kuflik, *String Theories with Moduli Stabilization Imply Non-Thermal Cosmological History, and Particular Dark Matter*, arXiv:1006.3272, 2010.

[5] B. S. Acharya, G. Kane, and P. Kumar, *Compactified String Theories – Generic Predictions for Particle Physics*, Int. J. Mod. Phys. **A27** (2012), 1230012.

[6] B. S. Acharya, M. O'Loughlin, and B. Spence, *Higher-dimensional analogues of Donaldson-Witten theory*, Nuclear Physics B **503** (1997), no. 3, 657 – 674.

[7] B. S. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, arXiv:hep-th/0109152, 2001.

[8] M. Arap, J. W. Cutrone, and N. A. Marshburn, *On the existence of certain weak Fano threefolds of Picard number two*, arXiv:1112.2611, 2011.

[9] M. Atiyah and E. Witten, *M theory dynamics on a manifold of G_2 holonomy*, Adv. Theor. Math. Phys. **6** (2003), 1–106.

[10] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, Second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3., vol. 4, Springer-Verlag, Berlin, 2004.

[11] L. Baulieu, H. Kanno, and I. M. Singer, *Special quantum field theories in eight and other dimensions*, Comm. Math. Phys. **194** (1998), no. 1, 149–175.

[12] A. Beauville, *Fano threefolds and $K3$ surfaces*, The Fano Conference, Univ. Torino, Turin, 2004, pp. 175–184.

[13] J. Bolton, L. Vrancken, and L. M. Woodward, *On almost complex curves in the nearly Kähler* 6*-sphere*, Q. J. Math. **45** (1994), no. 180, 407–427.

[14] R. L. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. (2) **126** (1987), no. 3, 525–576.

[15] M. Čadek, M. Crabb, and J. Vanžura, *Obstruction theory on 8-manifolds*, Manuscripta Math. **127** (2008), no. 2, 167–186.

[16] G. Cavalcanti, *Formality of k-connected spaces in $4k+3$ and $4k+4$ dimensions*, Math. Proc. Camb. Phil. Soc. **141** (2006), 101–112.

[17] Y.-M. Chan, *Desingularizations of Calabi–Yau* 3*-folds with a conical singularity*, Q. J. Math. **57** (2006), 151–181.

[18] ———, *Desingularizations of Calabi–Yau 3-folds with conical singularities. II. The obstructed case*, Q. J. Math. **60** (2009), 1–44.

[19] I. Cheltsov and J. Park, *Sextic double solids*, Cohomological and Geometric Approaches to Rationality Problems, Progress in Mathematics, vol. 282, Birkhäuser, 2010, pp. 75–132.

[20] T. Coates, M. Haskins, A. M. Kasprzyk, and J. Nordström, In preparation, 2013.

[21] A. Corti, M. Haskins, J. Nordström, and T. Pacini, *Asymptotically Cylindrical Calabi-Yau 3-folds from weak Fano 3-folds*, Geom. Topol. **17** (2013), 1955–2059.

[22] D. Crowley, *The classification of highly connected manifolds in dimensions 7 and 15*, Ph.D. thesis, Indiana University, 2001, arXiv:math/0203253.

[23] D. Crowley and C. Escher, *The classification of S^3-bundles over S^4*, Differential Geom. Appl. **18** (2003), no. 3, 363–380.

[24] D. Crowley and J. Nordström, *New invariants of G_2-structures*, arXiv:1211.0269v3, 2012.

[25] ———, *The classification of 2-connected 7-manifolds*, arXiv:1406.2226, 2014.

[26] J. W. Cutrone and N. A. Marshburn, *Towards the classification of weak Fano threefolds with* $\rho = 2$, Cent. Eur. J. Math. **11** (2013), no. 9, 1552–1576.

[27] J. De Boer, A. Naqvi, and A. Shomer, *Towards a topological G_2 string*, Tech. Report hep-th/0502140. IFTA-2005-03. SCIPP-2004-22, Calif. Univ. Santa Cruz. Inst. Part. Phys., Santa Cruz, CA, Feb 2005.

[28] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.

[29] I. Dolgachev, *Integral quadratic forms: applications to algebraic geometry (after V. Nikulin)*, Bourbaki seminar, Vol. 1982/83, Astérisque, vol. 105, Soc. Math. France, Paris, 1983, pp. 251–278.

[30] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), Oxford Univ. Press, Oxford, 1998, pp. 31–47.

[31] S. K. Donaldson and E. Segal, *Gauge theory in higher dimensions, II*, Geometry of special holonomy and related topics, Surv. Differ. Geom., vol. 16, Int. Press, Somerville, MA, 2011, pp. 1–41.

[32] J. Eells and N. Kuiper, *An invariant for certain smooth manifolds*, Annali di Math. **60** (1962), 93–110.

[33] H. Finkelnberg, *On the Geometry of the Burkhardt Quartic*, Ph.D. thesis, Leiden, February 1989.

[34] D. Gayet, *Smooth moduli spaces of associative submanifolds*, arXiv:1011.1744, 2010.

[35] A. Gray, *Vector cross products on manifolds*, Trans. Amer. Math. Soc. **141** (1969), 465–504.

[36] P. A. Griffiths, *On the periods of certain rational integrals, I*, Ann. Math. **90** (1969), 460–495.

[37] S. Gukov, S.-T. Yau, and E. Zaslow, *Duality and fibrations on G_2 manifolds*, Turkish J. Math. **27** (2003), no. 1, 61–97.

[38] J. A. Harvey and G. Moore, *Superpotentials and Membrane Instantons*, arXiv:hep-th/9907026, 1999.

[39] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[40] M. Haskins, H.-J. Hein, and J. Nordström, *Asymptotically cylindrical Calabi-Yau manifolds*, arXiv:1212.6929, 2012.

[41] R. Hepworth, *Generalized Kreck-Stolz invariants and the topology of certain 3-Sasakian 7-manifolds*, Ph.D. thesis, University of Edinburgh, 2005.

[42] N. J. Hitchin, *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Sup. Norm. Pisa Sci. Fis. Mat. **25** (1997), 503–515.

[43] ———, *The geometry of three-forms in six and seven dimensions*, arXiv:math/0010054, 2000.

[44] V. A. Iskovskih and Y. G. Prokhorov, *Fano varieties*, Algebraic geometry, V, Encyclopaedia Math. Sci., vol. 47, Springer, Berlin, 1999, pp. 1–247.

[45] D. D. Joyce, *Compact Riemannian* 7*-manifolds with holonomy G_2. I, II*, J. Differential Geom. **43** (1996), no. 2, 291–328, 329–375.

[46] ———, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[47] S. Karigiannis, *Desingularization of G_2 manifolds with isolated conical singularities*, Geom. Topol. **13** (2009), no. 3, 1583–1655.
[48] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160.
[49] A. Kovalev and N.-H. Lee, *$K3$ surfaces with non-symplectic involution and compact irreducible G_2-manifolds*, Math. Proc. Cambridge Philos. Soc. **151** (2011), no. 2, 193–218.
[50] M. Kreck and S. Stolz, *A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry*, Ann. of Math. (2) **127** (1988), no. 2, 373–388.
[51] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Univ. Press, 1989.
[52] R. Lockhart, *Fredholm, Hodge and Liouville theorems on noncompact manifolds*, Trans. Amer. Math. Soc. **301** (1987), 1–35.
[53] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), no. 4, 705–747.
[54] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 76.
[55] S. Mori and S. Mukai, *Classification of Fano 3-folds with $B_2 \geq 2$*, Manuscripta Math. **36** (1981/82), no. 2, 147–162.
[56] ———, *On Fano 3-folds with $B_2 \geq 2$*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 101–129.
[57] ———, *Classification of Fano 3-folds with $B_2 \geq 2$. I*, Algebraic and topological theories (Kinosaki, 1984), Kinokuniya, Tokyo, 1986, pp. 496–545.
[58] ———, *Erratum: "Classification of Fano 3-folds with $B_2 \geq 2$" [Manuscripta Math.* **36** *(1981/82), no. 2, 147–162; MR0641971 (83f:14032)]*, Manuscripta Math. **110** (2003), no. 3, 407.
[59] ———, *Extremal rays and Fano 3-folds*, The Fano Conference, Univ. Torino, Turin, 2004, pp. 37–50.
[60] Y. Namikawa, *Smoothing Fano 3-folds*, J. Algebraic Geom. **6** (1997), no. 2, 307–324.
[61] V. Nikulin, *Integer symmetric bilinear forms and some of their applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 111–177,238, English translation: *Math. USSR Izvestia* **14** (1980), 103–167.
[62] J. Nordström, *Deformations of asymptotically cylindrical G_2-manifolds*, Math. Proc. Camb. Phil. Soc. **145** (2008), 311–348.
[63] ———, *Deformations of glued G_2-manifolds*, Comm. Anal. Geom. **17** (2009), no. 3, 481–503.
[64] S. P. Novikov, *Homotopy equivalent smooth manifolds I*, Izv. Akad. Nauk. SSSR **28** (1965), 365–474, English Translation: A.M.S. Transl (2) 48 (1965) 271-396.
[65] T. Pantev and M. Wijnholt, *Hitchin's equations and M-theory phenomenology*, J. Geom. Phys. **61** (2011), no. 7, 1223–1247.
[66] M. Reid, *Chapters on algebraic surfaces*, Complex algebraic geometry (Park City, UT, 1993), IAS/Park City Math. Ser., vol. 3, Amer. Math. Soc., Providence, RI, 1997, pp. 3–159.
[67] F. Rohsiepe, *Lattice polarized toric K3 surfaces*, arXiv:hep-th/0409290, 2004; see also `http://www.th.physik.uni-bonn.de/People/rohsiepe/k3/`.
[68] R. Roiban, C. Römelsberger, and J. Walcher, *Discrete torsion in singular G_2-manifolds and real LG*, Adv. Theor. Math. Phys. **6** (2002), no. 2, 207–278.
[69] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics Series, vol. 201, Longman Scientific & Technical, Harlow, 1989.
[70] Y. T. Siu, *Every $K3$ surface is Kähler*, Invent. Math. **73** (1983), no. 1, 139–150.
[71] E. Thomas, *On the cohomology groups of the classifying space for the stable spinor groups*, Bol. Soc. Mat. Mexicana **7** (1962), 57–69.
[72] G. Tian, *Gauge theory and calibrated geometry. I*, Ann. of Math. (2) **151** (2000), no. 1, 193–268.
[73] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. I*, J. Amer. Math. Soc. **3** (1990), no. 3, 579–609.
[74] M. Verbitsky, *Manifolds with parallel differential forms and Kähler identities for G_2-manifolds*, J. Geom. Phys. **61** (2011), no. 6, 1001–1016.
[75] T. Walpuski, *G_2-instantons, associative submanifolds and Fueter sections*, arXiv:1205.5350, 2012.
[76] M. Y. Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989), 59–68.
[77] D. L. Wilkens, *Closed $(s-1)$-connected $(2s+1)$-manifolds, $s = 3, 7$*, Bull. London Math. Soc. **4** (1972), 27–31.
[78] ———, *On the inertia group of certain manifolds*, J. London Math. Soc. **9** (1975), no. 2, 537–548.
[79] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.