NEW INVARIANTS OF G_2-STRUCTURES

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Abstract. We define a $\mathbb{Z}_{48}$-valued homotopy invariant $\nu(\varphi)$ of a G_2-structure φ on the tangent bundle of a closed 7-manifold in terms of the signature and Euler characteristic of a coboundary with a $Spin(7)$-structure. For manifolds of holonomy G_2 obtained by the twisted connected sum construction, the associated torsion-free G_2-structure always has $\nu(\varphi) = 24$. Some holonomy G_2 examples constructed by Joyce by desingularising orbifolds have odd ν.

We define a further homotopy invariant $\xi(\varphi)$ such that if M is 2-connected then the pair (ν, ξ) determines a G_2-structure up to homotopy and diffeomorphism. The class of a G_2-structure is determined by ν on its own when the greatest divisor of $p_1(M)$ modulo torsion divides 224; this sufficient condition holds for many twisted connected sum G_2-manifolds.

We also prove that the parametric h-principle holds for coclosed G_2-structures.

1. Introduction

In this paper we develop methods to determine when two G_2-structures on a closed 7-manifold are deformation-equivalent, by which we mean related by homotopies and diffeomorphisms. The main motivation is to study the problem of deformation-equivalence of metrics with holonomy G_2. Such metrics can be defined in terms of torsion-free G_2-structures. The torsion-free condition is a complicated PDE, but we ignore that and consider only the G_2-structure as a topological residue of the holonomy G_2 metric: for a pair of G_2 metrics to be deformation-equivalent, it is certainly necessary that the associated G_2-structures are. One would not expect this necessary condition to be sufficient since the torsion-free constraint is quite rigid. A much weaker constraint on a G_2-structure is for it to be coclosed, and we find that the h-principle holds in this case: if two coclosed G_2-structures can be connected by a path of G_2-structures then they can also be connected by a path of coclosed G_2-structures.

1.1. The ν-invariant.

A G_2-structure on a 7-manifold M is a reduction of the structure group of the frame bundle of M to the exceptional Lie group G_2. As we review in §2.1, a G_2-structure on M is equivalent to a 3-form $\varphi \in \Omega^3(M)$ of a certain type and we will therefore refer to such 'positive' 3-forms as G_2-structures. A G_2-structure induces a Riemannian metric and spin structure on M. Throughout this introduction M shall be a closed connected spin 7-manifold and all G_2-structures φ will be compatible with the chosen spin structure. We denote the space of all such G_2-structures by $\mathcal{G}_2(M)$.

We say that two G_2-structures are homotopic if they can be connected by a continuous path of G_2-structures, so the set of homotopy classes of G_2-structures on M is $\pi_0\mathcal{G}_2(M)$. The following observation is not new, but the closest statement we have found in the literature is Witt [34, Proposition 3.3]. The proof is simple and provided in §3.1.

Lemma 1.1. *The group $H^7(M; \pi_7(S^7)) \cong \mathbb{Z}$ acts freely and transitively on $\pi_0\mathcal{G}_2(M) \equiv \mathbb{Z}$.*

The group of spin diffeomorphisms of M, $\mathrm{Diff}(M)$, acts by pull-back on $\mathcal{G}_2(M)$ with quotient $\bar{\mathcal{G}}_2(M) := \mathcal{G}_2(M)/\mathrm{Diff}(M)$. Since $\mathcal{G}_2(M)$ is locally path connected

$$\pi_0\bar{\mathcal{G}}_2(M) = \pi_0\mathcal{G}_2(M)/\pi_0\mathrm{Diff}(M),$$

and we call $\pi_0\bar{\mathcal{G}}_2(M)$ the set of deformation classes of G_2-structures on M. Up until now neither invariants of $\pi_0\bar{\mathcal{G}}_2(M)$ nor results about its cardinality have appeared in the literature.

Our starting point for studying both of these problems is the following characteristic class formula, valid for any closed spin 8-manifold X (see Corollary 2.5):

$$e_+(X) = 24\widehat{A}(X) + \frac{\chi(X) - 3\sigma(X)}{2}. \tag{1}$$

Here the terms are the integral of the Euler class of the positive spinor bundle, and the $\widehat{A}$-genus, ordinary Euler characteristic and signature of X ($\widehat{A}(X)$ is an integer because X is spin, and $\sigma(X) \equiv \chi(X) \bmod 2$ for any closed oriented X). Moving from $Spin(8)$ to $Spin(7)$, if we use the (real dimension 8) spin representation of $Spin(7)$ to regard $Spin(7)$ as a subgroup of $GL(8, \mathbb{R})$, then a $Spin(7)$-structure on an 8-manifold X can be characterised by a certain kind of 4-form $\psi \in \Omega^4(X)$. A $Spin(7)$-structure defines a spin structure and Riemannian metric on X, and (up to a sign) a unit spinor field of positive chirality. In particular, if a closed 8-manifold X has a $Spin(7)$-structure then $e_+(X) = 0$, and (1) implies

$$48\widehat{A}(X) + \chi(X) - 3\sigma(X) = 0. \tag{2}$$

If W is a compact 8-manifold with boundary M then a $Spin(7)$-structure on W induces a G_2-structure on M. From (2) one deduces that the "$\widehat{A}$ defect" $\chi(W) - 3\sigma(W) \bmod 48$ depends only on the induced G_2-structure on M. It turns out, see Lemma 3.4, that any G_2-structure φ on M bounds a $Spin(7)$-structure on some compact 8-manifold and this allows us to define an invariant $\nu(\varphi)$.

Definition 1.2. Let (M, φ) be a closed spin 7-manifold with G_2-structure and $Spin(7)$-coboundary (W, ψ). The ν-invariant of φ is the residue

$$\nu(\varphi) := \chi(W) - 3\sigma(W) \mod 48 \ \in \mathbb{Z}_{48}.$$

This definition makes sense even if M is not connected, and is additive under disjoint unions. Among the many analogous invariants in differential topology, perhaps the one best known to non-topologists is Milnor's $\mathbb{Z}_7$-valued λ-invariant of homotopy 7-spheres, defined as a "p_2 defect" of a spin coboundary [28]. To distinguish all 28 smooth structures on a homotopy sphere one can use the Eells-Kuiper invariant μ [14], which is another $\widehat{A}$ defect (see (9)).

In §1.2 we describe how ν is related to Lemma 1.1 by interpreting G_2-structures in terms of spinor fields, and we develop most of the theory in those terms. However, the definition above is sometimes useful when dealing with examples. It lets us compute ν from a coboundary with the right type of 4-form, and finding such 4-forms can be easier than describing spinor fields directly, *e.g.* in the proof of Theorem 1.7 and Examples 1.14 and 1.15.

Theorem 1.3 below summarises the basic properties of ν. Note that if φ is a G_2-structure on M, then the 3-form $-\varphi$ is also a G_2-structure, but compatible with the *opposite* orientation; $-\varphi$ is a G_2-structure on $-M$. In addition, if X is a closed $(2n+1)$–manifold, we define its rational semi-characteristic by $\chi_{\mathbb{Q}}(X) := \sum_{i=0}^{n} b_i(X) \bmod 2$.

Theorem 1.3. *For all G_2-structures φ on M, $\nu(\varphi) \in \mathbb{Z}_{48}$ is well-defined, and invariant under homotopies and diffeomorphisms. Hence ν defines a function*

$$\nu : \pi_0\bar{\mathcal{G}}_2(M) \to \mathbb{Z}_{48}. \tag{3}$$

Moreover $\nu(-\varphi) = -\nu(\varphi)$, and ν takes exactly the 24 values allowed by the parity constraint

$$\nu(\varphi) \equiv \chi_{\mathbb{Q}}(M) \mod 2. \tag{4}$$

Theorem 1.3 entails that $\pi_0\bar{\mathcal{G}}_2(M)$ has at least 24 elements. Here are some related questions that motivate our investigations:

- What are the values of ν for torsion-free G_2-structures, *i.e.* ones arising from G_2 holonomy metrics? Are there G_2 metrics on the same manifold that can be distinguished by ν?
- Do there exist G_2 metrics that are not deformation-equivalent, but whose associated torsion-free G_2-structures belong to the same class in $\pi_0\bar{\mathcal{G}}_2(M)$?
- What is the cardinality of $\pi_0\bar{\mathcal{G}}_2(M)$? For example, for which closed spin manifolds M is ν a complete invariant of $\pi_0\bar{\mathcal{G}}_2(M)$?

We give partial answers to the first and third of these questions below, and discuss directions for further research in §1.7.

1.2. **The affine difference D, spinors and the ν-invariant.** An important feature of homotopy classes of G_2-structures is that the identification $\pi_0\mathcal{G}_2(M) \equiv \mathbb{Z}$ from Lemma 1.1 should be regarded as affine, or as a $\mathbb{Z}$-torsor: there is no preferred base point, but Lemma 1.1 has the following consequence.

Lemma 1.4. *For any pair of G_2-structures φ, φ' on M there is a difference $D(\varphi, \varphi') \in \mathbb{Z}$ such that $(\pi_0\mathcal{G}_2(M),\ D) \cong (\mathbb{Z},\ \text{subtraction})$, i.e. $D(\varphi, \varphi') = 0$ if and only if φ is homotopic to φ', and*

$$D(\varphi, \varphi') + D(\varphi', \varphi'') = D(\varphi, \varphi''). \tag{5}$$

To understand the relationship between D and ν, we first explain the reasoning which goes into the proof of Lemma 1.1. As we describe in §2.2, a choice of Riemannian metric and unit spinor field on the spin manifold M defines a G_2-structure. Because any two Riemannian metrics are homotopic, this sets up a bijection between $\pi_0\mathcal{G}_2(M)$ and homotopy classes of sections of the unit spinor bundle. This is an S^7-bundle, and Lemma 1.1 follows from obstruction theory for sections of sphere bundles.

We can both describe D in concrete terms and prove Lemma 1.4 by counting zeros of homotopies of spinor fields (see §3.1). With this understanding of D, the next lemma is elementary. The intuitive notion of a $Spin(7)$-bordism is spelt out in §3.3.

Lemma 1.5. *Let φ, φ' be G_2-structures on M. Suppose (W, ψ) is a $Spin(7)$-bordism from (M, φ) to (M, φ'), and let $\overline{W}$ be the closed spin 8-manifold formed by identifying the two boundary components (*cf. (20)*). Then*

$$D(\varphi, \varphi') = -e_+(\overline{W}). \tag{6}$$

Combining Lemma 1.5 with the characteristic class formula (1), the mod 24 residue of $D(\varphi, \varphi')$ can be computed from just the signature and Euler characteristic of $\overline{W}$, which equal those of W. So while D only makes sense as an "affine" invariant, its mod 24 residue is related to the "absolute" invariant ν (in particular, ν is affine linear).

Proposition 1.6. *Let φ and φ' be G_2-structures on M. Then*

$$\nu(\varphi') - \nu(\varphi) \equiv 2D(\varphi, \varphi') \mod 48. \tag{7}$$

1.3. **The ν-invariant for manifolds with G_2 holonomy.** The exceptional Lie group G_2 also occurs as an exceptional case in the classification of Riemannian holonomy groups due to Berger [3]. It is immediate from the definitions that a metric on a 7-manifold M has holonomy contained in G_2 if and only if it is induced by a G_2-structure $\varphi \in \Omega^3(M)$ that is parallel. The covariant derivative $\nabla\varphi$ of φ with respect to the Levi–Civita connection ∇ of its induced metric can be identified with the intrinsic torsion of the G_2-structure, so metrics with holonomy in G_2 correspond to torsion-free G_2-structures [31, Corollary 2.2, §11].

One can define a moduli space of torsion-free G_2-structures on a fixed closed G_2-manifold M, which is an orbifold locally homeomorphic to finite quotients of $H^3_{dR}(M)$. But while the local structure is well understood, little is known about the global structure. One basic question is whether the moduli space is connected, *i.e.* whether any pair of torsion-free G_2-structures are equivalent up to homotopies through torsion-free G_2-structures and diffeomorphism. If one could find examples of diffeomorphic G_2-manifolds where the associated G_2-structures have different values of ν, this would prove that the moduli space is disconnected.

Finding compact manifolds with holonomy G_2 is a hard problem. The known constructions solve the non-linear PDE $\nabla\varphi = 0$ using gluing methods. Joyce [22] found the first examples by desingularising flat orbifolds, and later Kovalev [24] implemented a 'twisted connected sum' construction. In [10], the classification theory of closed 2-connected 7-manifolds is used to find examples of twisted connected sum G_2-manifolds that are diffeomorphic, but without any evidence either way as to whether the torsion-free G_2-structures are in the same component of the moduli space.

The twisted connected sum G_2-manifolds are constructed by gluing a pair of pieces of the form $S^1 \times V$, where V are asymptotically cylindrical Calabi–Yau 3-folds with asymptotic ends $\mathbb{R} \times S^1 \times K3$. We review this construction in §4.3 and then compute ν for all such G_2-structures.

Theorem 1.7. *If (M, φ) is a twisted connected sum then $\nu(\varphi) = 24$.*

We carry out this calculation by finding an explicit $Spin(7)$-bordism from a twisted connected sum G_2-structure φ to a G_2-structure that is a product of structures on lower-dimensional manifolds, for which ν is easier to evaluate.

For all the explicit examples of pairs of diffeomorphic G_2-manifolds found in [10], Corollary 1.13 below implies that ν classifies the homotopy classes of G_2-structures up to diffeomorphism. Thus diffeomorphisms between these G_2-manifolds can always be chosen so that the corresponding torsion-free G_2-structures are homotopic. Theorem 1.8 implies that they are then also homotopic as coclosed G_2-structures, but the question whether they can be connected by a path of torsion-free G_2-structures, so that they are in the same component of the moduli space of G_2 metrics, remains open.

Theorem 1.7 does not necessarily apply to more general gluings of asymptotically cylindrical G_2-manifolds. For example, a small number of the G_2-manifolds M constructed by Joyce [23, §12.8.4] have $\chi_{\mathbb{Q}}(M) = 1$, so those torsion-free G_2-structures have odd $\nu \neq 24$; yet they can be regarded at least topologically as a gluing of asymptotically cylindrical manifolds.

1.4. **The h-principle for coclosed G_2-structures.** We call a G_2-structure with defining 3-form φ *closed* if $d\varphi = 0$ and *coclosed* if $d^*\varphi = 0$, where d^* is defined in terms of the metric induced by the G_2-structure. For φ to be torsion-free is equivalent to it being both closed and coclosed (Fernández–Gray [16]). Individually, the conditions of being closed or coclosed are much more flexible than the torsion-free condition, and we show that coclosed G_2-structures satisfy the h-principle. Let $\mathcal{G}_2^{cc}(M) \subset \mathcal{G}_2(M)$ be the subspace of coclosed G_2-structures.

Theorem 1.8. *The inclusion $\mathcal{G}_2^{cc}(M) \hookrightarrow \mathcal{G}_2(M)$ is a homotopy equivalence.*

If M is an open manifold then Theorem 1.8 is a straight-forward application of Theorem 10.2.1 from Eliashberg–Mishachev [15] (*cf.* Lê [27, Theorem-Remark 3.17]). h-principles are generally much harder to prove on closed manifolds, but for coclosed G_2-structures we can use a micro-extension trick to reduce the problem to an application of [15, Theorem 10.2.1] on $M \times (-\epsilon, \epsilon)$. (There is no apparent way to apply the same trick to closed G_2-structures, which seem closer to symplectic structures in this sense.)

One motivation for considering coclosed G_2-structures is that they are the structures induced on 7-manifolds immersed in 8-manifolds with holonomy $Spin(7)$. One can attempt to construct $Spin(7)$ metrics on $M \times (-\epsilon, \epsilon)$ using the 'Hitchin flow' of coclosed G_2-structures [21]. Bryant [5, Theorem 7] shows that this can be solved provided that the initial coclosed G_2-structure is real analytic.

Theorem 1.8 implies that any spin 7-manifold M admits smooth coclosed G_2-structures. When M is closed, Grigorian [19] proves short-time existence of solutions φ_t for a version of the 'Laplacian coflow' of coclosed G_2-structures. Even if the initial G_2-structure φ_0 is merely smooth, the coclosed G_2-structures φ_t will be real analytic for $t > 0$ (sufficiently small so that the solution exists). As a consequence, we deduce the following

Corollary 1.9. *For every closed spin 7-manifold M, $M \times (-\epsilon, \epsilon)$ admits torsion-free $Spin(7)$-structures.*

1.5. **Counting deformation classes of G_2-structures.** We can think of the set of deformation-equivalence classes of G_2-structures as the quotient (isomorphic to $\pi_0\bar{\mathcal{G}}_2(M)$) of $\pi_0\mathcal{G}_2(M)$ under the action

$$\pi_0\mathcal{G}_2(M) \times \mathrm{Diff}(M) \to \pi_0\mathcal{G}_2(M), \quad ([\varphi], f) \mapsto [f^*\varphi].$$

The deformation invariance of ν implies that this action on $\pi_0\mathcal{G}_2(M) \cong \mathbb{Z}$ is by translation by multiples of 24, so that $\pi_0\bar{\mathcal{G}}_2(M)$ has at least 24 elements. To determine to what extent ν classifies elements of $\pi_0\bar{\mathcal{G}}_2(M)$ we need to understand precisely which multiples of 24 are realised as translations. Combining the characteristic class formula (1) with Lemma 1.5 we arrive at

Proposition 1.10. *Let $f\colon M \cong M$ be a spin diffeomorphism with mapping torus T_f. Then*

$$D(\varphi, f^*\varphi) = 24\widehat{A}(T_f) \in \mathbb{Z}.$$

The possible values of $\widehat{A}(T_f)$ are closely related to the spin characteristic class $p_M := \frac{p_1}{2}(M)$ (see §6.1). More precisely, the theory developed in [11] identifies the following two key quantities:

$$d_o(M) := \begin{cases} 0 & \text{if } p_M \text{ is torsion,} \\ \operatorname{Max}\{s \mid s, m \in \mathbb{Z},\ m^2 s \text{ divides } m p_M\} & \text{otherwise,} \end{cases}$$

and a certain value $r \in \{0, 1, 2\}$ that depends on the properties of the automorphisms of $H^4(M)$ preserving p_M and the torsion linking form. If $H^4(M)$ is torsion-free then $d_o(M)$ is simply the greatest integer dividing p_M, and $r = 1$ whenever $H^4(M)$ is 2-torsion-free. $d_o(M)$ is always even by Lemma 6.1.

The following theorem gives lower bounds on $|\pi_0\bar{\mathcal{G}}_2(M)|$. For $\frac{a}{b}$ a fraction without common factors, denote $\operatorname{Num}\left(\frac{a}{b}\right) = a$.

Theorem 1.11. *If $p_M = 0 \in H^4(M;\mathbb{Q})$ then $\pi_0\bar{\mathcal{G}}_2(M) \equiv \pi_0\mathcal{G}_2(M) \equiv \mathbb{Z}$. In general*

$$|\pi_0\bar{\mathcal{G}}_2(M)| \geq 24 \cdot \operatorname{Num}\left(\frac{2^r d_o(M)}{224}\right).$$

So, in particular, if $H^4(M)$ has no 2-torsion then $|\pi_0\bar{\mathcal{G}}_2(M)| \geq 24 \cdot \operatorname{Num}\left(\frac{d_o(M)}{112}\right)$.

For upper bounds on $|\pi_0\bar{\mathcal{G}}_2(M)|$ we need spin diffeomorphisms $f\colon M \cong M$ with $D(\varphi, f^*\varphi) \neq 0$. When M is 2-connected and p_M is not torsion, these are provided by [11].

Theorem 1.12. *If M is 2-connected and $p_M \neq 0 \in H^4(M;\mathbb{Q})$ then*

$$|\pi_0\bar{\mathcal{G}}_2(M)| = 24 \cdot \operatorname{Num}\left(\frac{2^r d_o(M)}{224}\right);$$

then also $|\pi_0\bar{\mathcal{G}}_2(N\sharp M)| \leq 24 \cdot \operatorname{Num}\left(\frac{2^r d_o(M)}{224}\right)$ for any connected spin 7-manifold N.

Theorem 1.12 helps identify certain manifolds M for which ν is a complete invariant of $\pi_0\bar{\mathcal{G}}_2(M)$.

Corollary 1.13. *If $2^r d_o(M_0)$ divides 224 for some 2-connected M_0 such that $M \cong N\sharp M_0$, then $|\pi_0\bar{\mathcal{G}}_2(M)| = 24$. In this case two G_2-structures φ and φ' on M are deformation-equivalent if and only if $\nu(\varphi) = \nu(\varphi')$.*

1.6. **The ξ-invariant.** We now describe a further invariant that, depending on the topology of M, can distinguish more classes of $\pi_0\bar{\mathcal{G}}_2(M)$. For the moment we restrict to the special case when p_M is rationally trivial, and postpone the full definition to §6.4.

In dimension 7, the Eells-Kuiper invariant μ arises from considering the following characteristic class formula [14, §6]: if X is a closed spin 8-manifold then

$$224\widehat{A}(X) = p_X^2 - \sigma(X). \tag{8}$$

If M is closed spin with p_M a torsion class and W is a spin coboundary, then $p_W \in H^4(W;\mathbb{Q})$ is in the image of $H^4(W, M;\mathbb{Q})$, the cohomology relative to the boundary, and $p_W^2 \in \mathbb{Q}$ is well-defined. Then (8) implies that the $\widehat{A}$ defect,

$$\mu(M) := \frac{p_W^2 - \sigma(W)}{8} \in \mathbb{Q}/28\mathbb{Z}, \tag{9}$$

is independent of the choice of W. (This differs from the definition in [14] by a factor of 28. The mod $\mathbb{Z}$ residue of $\mu(M)$ is determined by the almost-smooth structure of M because p_W is a characteristic element for the intersection form; therefore $\mu(M)$ can take 28 different values if the underlying almost-smooth manifold is fixed.)

If we consider a G_2-structure φ on a spin manifold M such that p_M is torsion, then we can in a sense cancel the ambiguities in the definitions of the $\widehat{A}$ defects ν and μ to obtain a stronger invariant. A linear combination of (2) and (8) gives that

$$7\chi(X) + \frac{3p_X^2 - 45\sigma(X)}{2} = 0$$

for any closed X^8 with $Spin(7)$-structure. Hence setting

$$\xi(\varphi) := 7\chi(W) + \frac{3p_W^2 - 45\sigma(W)}{2} \in \mathbb{Q} \tag{10}$$

is independent of choice of $Spin(7)$-coboundary W. If we consider G_2-structures on a fixed smooth M with p_M torsion then the relation

$$\xi(\varphi) = 7\nu(\varphi) + 12\mu(M) \mod 336\mathbb{Z}$$

means that $\nu(\varphi)$ can be determined from $\xi(\varphi)$ and $\mu(M)$. The ξ-invariant takes precisely the values allowed by the constraint

$$\xi(\varphi) = 7\chi_{\mathbb{Q}}(M) + 12\mu(M) \mod 14\mathbb{Z}.$$

Similarly to Proposition 1.6, $14D(\varphi, \varphi') = \xi(\varphi') - \xi(\varphi)$, so ξ distinguishes between all elements of $\pi_0\mathcal{G}_2(M)$. Since ξ is patently invariant under diffeomorphisms, this entails the claim from Theorem 1.12 that $\pi_0\mathcal{G}_2(M) = \pi_0\bar{\mathcal{G}}_2(M)$ when p_M is torsion.

Example 1.14. S^7 has a standard G_2-structure φ_{rd}, induced as the boundary of B^8 with a flat $Spin(7)$-structure. Clearly $\nu(\varphi_{rd}) \equiv \chi(B^8) - 3\sigma(B^8) \equiv 1$. Meanwhile $p_{B^8} = 0$, so $\xi(\varphi_{rd}) = 7$.

On the other hand, the flat $Spin(7)$-structure on the complement of $B^8 \subset \mathbb{R}^8$ induces the G_2-structure $-\varphi_{rd}$ on S^7 (with the orientation reversed). If r is a reflection of S^7 then $\widehat{\varphi}_{rd} = r^*(-\varphi_{rd})$ is a different G_2-structure on S^7 inducing the same orientation as φ_{rd}. Since $\nu(\widehat{\varphi}_{rd}) = \nu(-\varphi_{rd}) = -\nu(\varphi_{rd}) = -1$ (and $\xi(\widehat{\varphi}_{rd}) = \xi(-\varphi_{rd}) = -\xi(\varphi_{rd}) = -7$) there can be no homotopy between φ_{rd} and $\widehat{\varphi}_{rd}$.

Example 1.15. S^7 has a 'squashed' G_2-structure φ_{sq} that is invariant under $Sp(2)Sp(1)$ and nearly parallel (*i.e.* the corresponding cone metric on $\mathbb{R} \times S^7$ has exceptional holonomy $Spin(7)$). This G_2-structure is the asymptotic link of the asymptotically conical $Spin(7)$-manifold constructed by Bryant and Salamon [6] on the total space W of the positive spinor bundle of S^4. This bundle is $\mathcal{O}(-1)$ over $\mathbb{H}P^1$ with the orientation reversed. Since this space has $\sigma = 1$ and $\chi = 2$, it follows that $\nu(\varphi_{sq}) = 2 - 3 = -1$.

Further $p_W^2 = 1$, so $\xi(\varphi_{sq}) = -7$. In particular, φ_{sq} is homotopic to $\widehat{\varphi}_{rd}$; if we glue W and B^8 to form $\mathbb{H}P^2$ then we can interpolate to define a $Spin(7)$-structure on $\mathbb{H}P^2$.

The definition of ξ becomes more involved when p_M is rationally non-trivial. In general, let d_π denote the greatest integer dividing p_M modulo torsion (which is even by Lemma 6.1), and $\tilde{d}_\pi := \gcd(d_\pi, 4)$. One can then replace the $p_W^2 \in \mathbb{Q}$ that appears in (10) with a $\mathbb{Q}/2\tilde{d}_\pi\mathbb{Z}$-valued function on $S_{d_\pi} := \{k \in H^4(M) : p_M - d_\pi k \text{ is torsion}\}$. Hence one can define $\xi(\varphi)$ as a function $S_{d_\pi} \to \mathbb{Q}/3\tilde{d}_\pi\mathbb{Z}$, see Definition 6.8. It is invariant in the sense that if $f : M' \to M$ is a diffeomorphism, then $f^* : H^4(M) \to H^4(M')$ restricts to a bijection $S_{d_\pi} \to S'_{d_\pi}$, and $\xi(f^*\varphi) \circ f^* = \xi(\varphi)$ for any G_2-structure φ on M.

Lemma 1.16.

$$\xi(\varphi') - \xi(\varphi) = 14D(\varphi, \varphi') \mod 3\tilde{d}_\pi \tag{11}$$

Together with Proposition 1.6, this means that the values of (ν, ξ) determine $D(\varphi, \varphi')$ modulo $\mathrm{lcm}(24, \mathrm{Num}(\frac{3\tilde{d}_\pi}{14})) = 24\mathrm{Num}\left(\frac{d_\pi}{112}\right)$. However, this does not mean that the pair (ν, ξ) distinguishes between $24\mathrm{Num}\left(\frac{d_\pi}{112}\right)$ classes in $\pi_0\bar{\mathcal{G}}_2(M)$, but only that it distinguishes that many classes modulo homotopies and diffeomorphisms *acting trivially on cohomology*. The reason is that for a general diffeomorphism f of M, $\xi(\varphi) \circ f^* - \xi(\varphi)$ can be a non-zero constant in $\mathbb{Q}/3\tilde{d}_\pi\mathbb{Z}$. Understanding the action of f on ξ reduces to the same technical problem as for the action on $\pi_0\mathcal{G}_2(M)$, and we find that in general (ν, ξ) can distinguish between $24\mathrm{Num}\left(\frac{2^r d_o(M)}{224}\right)$ elements of $\pi_0\bar{\mathcal{G}}_2(M)$, which in a sense is a more precise version of Theorem 1.11. In particular, combining with Theorem 1.12 we find

Theorem 1.17. *If M is 2-connected then (ν, ξ) is a complete invariant of $\pi_0\bar{\mathcal{G}}_2(M)$.*

In combination with the diffeomorphism classification of closed 2-connected 7-manifolds from [11], we obtain a classification result for 2-connected 7-manifolds with G_2-structures, stated in Theorem 6.9.

1.7. **Further problems.** The main motivation for this work is to help distinguish between connected components of the moduli space of G_2 metrics on a fixed M. One supply of candidates comes from 2-connected twisted connected sums, but Theorem 1.7 shows that ν is not enough to distinguish between those. All twisted connected sum G_2-manifolds M have $d_o(M)$ a divisor of $d_o(K3) = 24$, so when M is 2-connected, the only remaining chance of using the homotopy theory to distinguish between different twisted connected sums G_2 metrics is when d_o is divisible by 3: by Theorem 1.11 there are in this case 3 different homotopy classes of G_2-structures with $\nu = 24$, and they can be distinguished by ξ. A number of examples with $d_o(M) = d_\pi(M) = 6$ are exhibited in [10], and it seems likely that a more exhaustive search will provide diffeomorphic pairs of such twisted connected sums, but we do not currently have any way to compute ξ in this situation.

The examples of Joyce with odd ν mentioned above can be viewed as a kind of twisted connected sums, gluing asymptotically cylindrical manifolds with holonomy a proper subgroup of G_2 and cross-section $K3 \times T^2$, but where the torus factor is not rectangular (as for usual twisted connected sums) but hexagonal. Such "extra-twisted connected sums" provide candidates of 2-connected G_2-manifolds with fewer restrictions on the possible values of ν, and we will return to this elsewhere.

The definition of ν in terms of a coboundary is not always amenable to explicit computations. A common theme in differential topology is to find ways to express 'extrinsic' invariants (defined in terms of a coboundary) intrinsically, *e.g.* the classical Eells-Kuiper invariant can be expressed in terms of eta invariants [13]. Sebastian Goette informs us that it is possible to express ν analytically, and we plan to study this and applications to extra-twisted connected sums further in future work.

Some necessary conditions are known for a closed spin 7-manifold M to admit a metric with holonomy G_2 (see *e.g.* [23, §10.2]), but there is currently no conjecture as to what the right sufficient conditions would be. A refinement of this already very hard problem would be to ask: which deformation classes of G_2-structures on M contain torsion-free G_2-structures? This is of course related to the problem of whether there is any M with torsion-free G_2-structures that are not deformation-equivalent, which was one of our motivations for introducing ν. If one attempts to find torsion-free G_2-structures as limits of a flow of G_2-structures as in [7, 19, 33, 35], does the homotopy class of the initial G_2-structures affect the long-term behaviour of the flow?

Organisation. The rest of the paper is organised as follows. In Section 2 we establish preliminary results needed to define and compute ν. In Section 3 we define the affine difference $D(\varphi, \varphi')$ and the ν-invariant, establish the existence of $Spin(7)$-coboundaries for G_2-structures and hence prove Theorem 1.3. We also describe examples of G_2-structures on S^7 in more detail. In Section 4 we compute the ν-invariant for twisted connected sum G_2-manifolds, proving Theorem 1.7. Section 5 establishes the h-principle for coclosed G_2-structures stated in Theorem 1.8. In Section 6 we describe the action of spin diffeomorphisms on $\pi_0\mathcal{G}_2(M)$, give the general definition of the ξ-invariant and prove the results from §1.5-1.6.

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2. Preliminaries

In this section we describe G_2-structures and $Spin(7)$-structures on 7 and 8-manifolds, and their relationships to spinors. We also establish some basic facts about the characteristic classes of spin manifolds in dimensions 7 and 8.

2.1. **The Lie groups $Spin(7)$ and G_2.** We give a brief review of how $Spin(7)$ and G_2-structures can be characterised in terms of forms. For more detail on the differential geometry of such structures, and how they can be used in the study metrics with exceptional holonomy, see *e.g.*

Salamon [31] or Joyce [23]. We defer the analogous discussion of $SU(3)$ and $SU(2)$-structures until we use it in §4.

The stabiliser in $GL(8,\mathbb{R})$ of the 4-form

$$\begin{aligned}\psi_0 = dx^{1234} + dx^{1256} + dx^{1278} + dx^{1357} - dx^{1368} - dx^{1458} - dx^{1467} - \\ dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678} \in \Lambda^4(\mathbb{R}^8)^*\end{aligned} \tag{12}$$

is $Spin(7)$ (identified with a subgroup of $SO(8)$ by the spin representation). Here and elsewhere, dx^{1234} abbreviates $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ etc. On an 8-dimensional manifold X, a 4-form $\psi \in \Omega^4(X)$ which is pointwise equivalent to ψ_0 defines a $Spin(7)$-structure, and induces a metric and orientation (the orientation form is ψ^2).

The exceptional Lie group G_2 can be defined as the automorphism group of $\mathbb{O}$, the normed division algebra of octonions. Equivalently, G_2 is the stabiliser in $GL(7,\mathbb{R})$ of the 3-form

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*. \tag{13}$$

On a 7-dimensional manifold M, a 3-form $\varphi \in \Omega^3(M)$ that is pointwise equivalent to φ_0 defines a G_2-structure, which induces a Riemannian metric and orientation. Note that

$$dt \wedge \varphi_0 + {*\varphi_0} \cong \psi_0 \tag{14}$$

on $\mathbb{R} \oplus \mathbb{R}^7$, so the stabiliser in $Spin(7)$ of a non-zero vector in $\mathbb{R}^8$ is exactly G_2. Therefore the product of a 7-manifold with a G_2-structure and S^1 or $\mathbb{R}$ has a natural product $Spin(7)$-structure, while a $Spin(7)$-structure ψ on W^8 induces a G_2-structure on ∂W by contracting ψ with an outward pointing normal vector field.

Remark 2.1. If φ is G_2-structure on M^7, then $-\varphi$ is a G_2-structure too, inducing the same metric and opposite orientation (because φ_0 is equivalent to $-\varphi_0$ under the orientation-reversing isomorphism $-1 \in O(7)$). The product $Spin(7)$-structure $dt \wedge \varphi + {*\varphi}$ on $M \times [0,1]$ induces φ on the boundary component $M \times \{1\} \cong M$, and $-\varphi$ on $M \times \{0\} \cong -M$.

2.2. **G_2-structures and spinors.** In this paper we are concerned with G_2-structures on a manifold M^7 up to homotopy. Since there is an obvious way to reverse the orientation of a G_2-structure, while any two Riemannian metrics are homotopic, we may as well consider G_2-structures compatible with a fixed orientation and metric. Because G_2 is simply-connected, the inclusion $G_2 \hookrightarrow SO(7)$ lifts to $G_2 \hookrightarrow Spin(7)$. Therefore a G_2-structure on M also induces a spin structure, and we focus on studying G_2-structures compatible also with a fixed spin structure. As in the introduction, we let $\pi_0\mathcal{G}_2(M)$ denote the homotopy classes of G_2-structures on M with a choice of spin structure.

As we already saw, G_2 is exactly the stabiliser of a non-zero vector in the spin representation Δ of $Spin(7)$; as a representation of G_2, Δ splits as the sum of a 1-dimensional trivial part and the standard 7-dimensional representation. $Spin(7)$ acts transitively on the unit sphere in Δ with stabiliser G_2, so $Spin(7)/G_2 \cong S^7$.

From the above, we deduce that given a spin structure on M, a compatible G_2-structure φ induces an isomorphism $SM \cong \underline{\mathbb{R}} \oplus TM$ for the spinor bundle SM: here $\underline{\mathbb{R}}$ denotes the trivial line bundle. Hence we can associate to φ a unit section of SM, well-defined up to sign. Conversely, any unit section of SM defines a compatible G_2-structure. A transverse section s of the spinor bundle SM of a spin 7-manifold has no zeros, so defines a G_2-structure; thus a 7-manifold admits G_2-structures if and only if it is spin (*cf.* Gray [17], Lawson–Michelsohn [26, Theorem IV.10.6]).

Note that s and $-s$ are always homotopic, because they correspond to sections of the trivial part in a splitting $SM \cong \underline{\mathbb{R}} \oplus TM$ and the Euler class of an oriented 7-manifold vanishes. It follows that SM contains a trivial 2-plane field $K \supset \underline{\mathbb{R}}$ which accommodates a homotopy from s to $-s$. Therefore $\pi_0\mathcal{G}_2(M)$ can be identified with homotopy classes of unit sections of the spinor bundle. As stated in the introduction, Lemma 1.1 now follows by a standard application of obstruction theory, but we will describe the bijection $\pi_0\mathcal{G}_2(M) \cong \mathbb{Z}$ in elementary terms in §3.1.

Remark 2.2. Let us make some further comments on the signs of the spinors. Given a principal $Spin(7)$ lift $\widetilde{F}$ of the frame bundle F of M, the principal G_2-subbundles of $\widetilde{F}$ are in bijective correspondence with sections of the associated unit spinor bundle. The G_2-subbundles corresponding to spinors s and $-s$ have the same image in F, hence they define the same G_2-structure on M (they have the same 3-form φ).

While $SO(7)$ does not itself act on Δ, the action of $Spin(7)$ on $(\Delta - \{0\})/\mathbb{R}^* \cong \mathbb{R}P^7$ does descend to an action of $SO(7)$. Therefore the orbit $SO(7)\varphi_0$, the set of G_2-structures on $\mathbb{R}^7$ defining the same orientation and metric as φ_0, is $SO(7)/G_2 \cong \mathbb{R}P^7$. G_2-structures compatible with a fixed orientation and metric on M but without any constraint on the spin structure therefore correspond to sections of an $\mathbb{R}P^7$ bundle. If M is not spin then this bundle has no sections. Given a spin structure, the unit sphere bundle in the associated spinor bundle is an S^7 lift of the $\mathbb{R}P^7$-bundle, and two G_2-structures induce the same spin structure if they can both be lifted to the same S^7 bundle.

2.3. **$Spin(7)$-structures and characteristic classes of $Spin(8)$-bundles.** The spin representation of $Spin(7)$ is faithful, so defines an inclusion homomorphism $Spin(7) \hookrightarrow SO(8)$, which has a lift $i_\Delta : Spin(7) \hookrightarrow Spin(8)$. The restriction of the positive half-spin representation Δ^+ of $Spin(8)$ to $Spin(7)$ is a sum of a trivial rank 1 part and the 7-dimensional vector representation (factoring through $Spin(7) \to SO(7)$). Therefore $i_\Delta(Spin(7)) \subset Spin(8)$ can be characterised as the stabiliser of a unit positive spinor $s_0 \in \Delta^+$, and $Spin(7)$-structures on a spin 8-manifold are equivalent to unit positive spinor fields (up to sign, in the same sense as G_2-structures). Hence there is an obvious obstruction to the existence of $Spin(7)$-structures on an 8-manifold X: it must be spin, and the Euler class in $H^8(X)$ of the positive half-spinor bundle on X must vanish.

Remark 2.3. One can of course also define an embedding $i_0 : Spin(7) \hookrightarrow Spin(8)$ as the stabiliser of the coordinate vector e_8 in the vector representation $\mathbb{R}^8$ of $Spin(8)$. The restrictions to this copy of $Spin(7)$ of the half-spin representation $\Delta^\pm$ of $Spin(8)$ are both isomorphic to the spin representation of $Spin(7)$. Therefore, if W^8 is a spin manifold then the restrictions of the half-spinor bundles $S^\pm W$ to ∂W are naturally isomorphic to the spinor bundle $S(\partial W)$.

In particular, a positive spinor field on W^8 can be restricted to a spinor field on ∂W, so the restriction of a $Spin(7)$-structure on W to a G_2-structure on ∂W can be described in terms of the spinorial picture. Of course, this gives exactly the same result as if we describe the restriction in terms of differential forms. This is because the image of the composition of the inclusions $G_2 \hookrightarrow Spin(7) \overset{i_0}{\hookrightarrow} Spin(8)$ is equally well described as the stabiliser in $Spin(8)$ of $(s_0, e^8) \in \Delta^+ \times \mathbb{R}^8$ and as the lift of the stabiliser in $GL(\mathbb{R}, 8)$ of $(\psi_0, e_8) \in \Lambda^4\mathbb{R}^8 \times \mathbb{R}^8$.

Let us describe briefly our conventions for orientations on the half-spin representations of $Spin(8)$. For each fixed non-zero $v \in \mathbb{R}^8$, the Clifford multiplication $\mathbb{R}^8 \times \Delta^\pm \to \Delta^\mp$ defines orientation-preserving isomorphisms $c_v^\pm : \Delta^\pm \to \Delta^\mp$. A feature of the 'triality' in dimension 8 is that the map $\widehat{c}_{s_\pm} : \mathbb{R}^8 \to \Delta^\mp$ induced by Clifford multiplication with a fixed non-zero spinor $s_\pm \in \Delta^\pm$ is an isomorphism too. The Clifford relations imply that, for $s_+ = vs_-$,

$$c_v^+ \circ \widehat{c}_{s_-} = \widehat{c}_{s_+} \circ r_v \; : \; \mathbb{R}^8 \to \Delta^-,$$

where $r_v : \mathbb{R}^8 \to \mathbb{R}^8$ is reflection in the hyperplane orthogonal to v. Thus $\widehat{c}_{s_+}$ and $\widehat{c}_{s_-}$ have opposite orientability. Our convention is that $\widehat{c}_{s_-}$ is orientation-preserving, while $\widehat{c}_{s_+}$ is not.

More explicitly, $\mathbb{R}^8$, Δ^+ and Δ^- can each be identified with the octonions $\mathbb{O}$ so that the Clifford multiplication $\mathbb{R}^8 \times \Delta^- \to \Delta^+$ corresponds to the octonionic multiplication $(x, y) \mapsto xy$, *cf.* Baez [2, p.162 above (5)]. Then, to satisfy the Clifford relations, $\mathbb{R}^8 \times \Delta^+ \to \Delta^-$ must correspond to $(x, y) \mapsto -\bar{x}y$, where $\bar{x}$ is the octonion conjugate of x. This map is orientation-reversing on the first factor.

Let X be a spin 8-manifold, $e \in H^8(X)$ the Euler class of TX, and $e_\pm \in H^8(X)$ the Euler classes of the half-spinor bundles $S^\pm X$. More generally, for any principal $Spin(8)$-bundle on any X, let $e, e_\pm$ denote the Euler classes of the vector bundles associated to the vector and half-spin representations of $Spin(8)$. With our orientation conventions, the non-degeneracy of the Clifford product implies

$$e_+ = e + e_-. \tag{15}$$

The following statement can be found for instance in Gray–Green [18, p.89].

Proposition 2.4. *For any principal $Spin(8)$-bundle*

$$e_{\pm} = \frac{1}{16}\left(p_1^2 - 4p_2 \pm 8e\right).$$

In degree 8, the $\widehat{A}$ and L genera are given by

$$\begin{aligned} 45 \cdot 2^7 \widehat{A} &= 7p_1^2 - 4p_2, \\ 45L &= 7p_2 - p_1^2, \end{aligned} \tag{16}$$

so Proposition 2.4 can be rewritten as $e_{\pm} = 24\widehat{A} + \dfrac{\pm e - 3L}{2}$. If X is closed and orientable then the integral of the L genus of TX is the signature of X by the Hirzebruch signature theorem, while the integral of the Euler class is just the ordinary Euler characteristic.

Corollary 2.5. *If X is a closed spin 8-manifold then*

$$e_{\pm}(X) = 24\widehat{A}(X) + \frac{\pm\chi(X) - 3\sigma(X)}{2}.$$

Remark 2.6. Modulo torsion, the group of integral characteristic classes of a principal $Spin(8)$-bundle in dimension 8 is generated by p_1^2, p_2 and e, so we could prove Corollary 2.5 (and hence Proposition 2.4) by checking that the formula holds for the following spin 8-manifolds.

- S^8: $\chi = 2$, $\widehat{A} = \sigma = 0$, $e_{\pm} = \pm 1$.
- $K3 \times K3$: $\chi = 24^2$, $\sigma = (-16)^2$. $\widehat{A} = 4$ because the holonomy is $SU(2) \times SU(2)$. Because this also defines a $Spin(7)$-structure (*cf.* (22)), $e_+ = 0$ and $e_- = -\chi$.
- $\mathbb{H}P^2$: $\chi = 3$, $\sigma = 1$. $\widehat{A} = 0$ by the Lichnerowicz formula since there is a metric with positive scalar curvature. $e_- = -\chi$ because $S^-X \cong -TX$ for any spin 8-manifold X with $Sp(2)Sp(1)$-structure. This structure also splits S^+X into a sum of a rank 5 and a rank 3 part, so $e_+ = 0$. (Alternatively, we can identify a quaternionic line subbundle of $T\mathbb{H}P^2$, like that spanned by the projection of the vector field $(q_1, q_2, q_3) \mapsto (0, q_1, q_2)$ on $\mathbb{H}^3$, with a non-vanishing section of the rank 5 part of S^+X.)

3. The ν-invariant

In this section we study the set $\pi_0\mathcal{G}_2(M)$ of homotopy classes of G_2-structures on a closed spin 7-manifold M, and prove the basic properties of the invariants D and ν. We conclude the section with some concrete examples.

3.1. **The affine difference.** Let M be a closed connected spin 7-manifold, and φ, φ' a pair of G_2-structures on M. We describe how to define the difference $D(\varphi, \varphi') \in \mathbb{Z}$ from Lemma 1.4.

A homotopy of G_2-structures is equivalent to a path of non-vanishing spinor fields. Any path of spinor fields on M can be identified with a positive spinor field s on $M \times [0,1]$. We can always find s such that the restrictions to $M \times \{1\}$ and $M \times \{0\}$ are the non-vanishing spinor fields corresponding to φ and $-\varphi'$, respectively. Then the pull-back by s of the Thom class of the positive spinor bundle defines a relative Euler class in $H^8(W, M)$, independent of the choice of s, and we define $D(\varphi, \varphi')$ to be its integral $n_+(M \times [0,1], \varphi, \varphi')$. If we take s to have transverse zeroes then we can interpret this geometrically as the intersection number of the graph of s with the zero section.

It is obvious from this definition that the affine relation (5) holds. If $n_+(M \times [0,1], \varphi, \varphi') = 0$ then s can be chosen to be non-vanishing, so φ and φ' are homotopic if and only if $D(\varphi, \varphi') = 0$. Given φ we can construct φ' such that $D(\varphi, \varphi') = 1$ by modifying the defining spinor of φ in a 7-disc B^7: in a local trivialisation we change it from a constant map $B^7 \to S^7$ to a degree 1 map. Thus D can take any integer value, so D really corresponds to the difference function under a bijection $\mathbb{Z} \cong \pi_0\mathcal{G}_2(M)$, completing the proof of Lemma 1.4.

To compute $D(\varphi, \varphi')$, we can consider more general spin 8-manifolds W with boundary $M \sqcup -M$. Generalising the above, let $n_+(W, \varphi, \varphi')$ be the intersection number with the zero section of a positive spinor whose restriction to the two boundary components correspond to φ and $-\varphi'$. Form a closed spin 8-manifold $\overline{W}$ by gluing the M piece of the boundary of W to the $-M$ piece. We can define a continuous positive spinor field on $\overline{W}$ by modifying the spinor field from W in an

$M \times [0,1]$ neighbourhood of the former boundary, to interpolate between φ' on $M \times \{1\}$ and $-\varphi$ on $M \times \{0\}$. Its intersection number with the zero section is $n_+(W,\varphi,\varphi') - D(\varphi,\varphi')$, so we can compute D as

$$D(\varphi,\varphi') = n_+(W,\varphi,\varphi') - e_+(\overline{W}). \tag{17}$$

3.2. **The definition of ν.** Let M be a closed spin 7-manifold (not necessarily connected) with G_2-structure φ, and W a compact spin 8-manifold with $\partial W = M$. Such W always exist since the bordism group Ω_7^{Spin} is trivial [29]. The restrictions of the half-spinor bundles $S^\pm W$ of W to M are isomorphic to the spinor bundle on M (Remark 2.3), and the composition $S^+W_{|M} \to S^-W_{|M}$ of these isomorphisms is Clifford multiplication by a unit normal vector field to the boundary. Let $n_\pm(W,\varphi)$ be the intersection number with the zero section of a section of $S^\pm W$ whose restriction to M is the non-vanishing spinor field defining φ. Let

$$\bar\nu(W,\varphi) := -2n_+(W,\varphi) + \chi(W) - 3\sigma(W) \in \mathbb{Z}. \tag{18}$$

Reversing the orientations, $-W$ is a spin 8-manifold whose boundary $-M$ is equipped with a G_2-structure $-\varphi$.

Lemma 3.1. *Let W be a compact spin 8-manifold, and φ a G_2-structure on $M = \partial W$.*

(i) *If φ' is another G_2-structure on M then $\bar\nu(W,\varphi') - \bar\nu(W,\varphi) = 2D(\varphi,\varphi')$*

(ii) $\bar\nu(W,\varphi) \equiv \chi_{\mathbb{Q}}(M) \mod 2$

(iii) $\bar\nu(-W,-\varphi) = -\bar\nu(W,\varphi)$

(iv) *If W' is another compact spin 8-manifold with $\partial W' = M$ then the closed spin 8-manifold $X = W \cup_{\mathrm{Id}_M} (-W')$ has*

$$48\widehat{A}(X) = \bar\nu(W',\varphi) - \bar\nu(W,\varphi).$$

Proof. (i) Clearly $n_+(W,\varphi) = n_+(M \times I, \varphi, \varphi') + n_+(W,\varphi')$.

(ii) For W^{4n} any compact oriented manifold with boundary, $\sigma(W)$ is by definition the signature of a non-degenerate symmetric form on the image $H_0^{2n}(W)$ of $H^{2n}(W,M) \to H^{2n}(W)$. In particular, $\sigma(W) \equiv \dim H_0^{2n}(W) \bmod 2$. Writing $\chi(W) = \sum_{i=0}^{2n-1} b_i(W) + \sum_{i=0}^{2n} b_{4n-i}(W)$ and using $b_{4n-i}(W) = b_i(W,M)$ and the definition that $\chi_{\mathbb{Q}}(W) = \sum_{i=0}^{2n-1} b_i(\partial W) \bmod 2$, the exactness of the sequence

$$0 \to H^0(W,M) \to H^0(W) \to \cdots \to H^{2n-1}(\partial W) \to H^{2n}(W,M) \to H_0^{2n}(W) \to 0$$

implies

$$\sigma(W) + \chi(W) \equiv \chi_{\mathbb{Q}}(\partial W) \mod 2. \tag{19}$$

(iii) Let v be a vector field on W that is a unit outward-pointing normal field along M, and $s \in \Gamma(S^+W)$ a spinor field whose restriction to M induces φ. Then the restriction of the Clifford product $v \cdot s \in \Gamma(S^-W)$ also induces φ. By the Poincare–Hopf index theorem, the number of zeros of v is $\chi(W)$, so $n_-(W,\varphi) = n_+(W,\varphi) - \chi(W)$ (these signs are compatible with (15)).

Reversing the orientations swaps sections of S^+W and S^-W, and reverses the signs assigned to the zeros, so $n_+(-W,-\varphi) = -n_-(W,\varphi)$. It also reverses the signature, but preserves the Euler characteristic. Thus

$$\bar\nu(-W,-\varphi) = 2n_-(W,\varphi) + \chi(W) + 3\sigma(W) = 2n_+(W,\varphi) - 2\chi(W) + \chi(W) + 3\sigma(W) = -\bar\nu(W,\varphi).$$

(iv) $\sigma(W) + \sigma(-W') = \sigma(X)$ by Novikov additivity [1, 7.1], $\chi(W) + \chi(-W') = \chi(X)$ because $\chi(M) = 0$, and X has a transverse positive spinor field whose intersection number with the zero section is $n_+(W,\varphi) + n_+(-W',-\varphi)$. Hence

$$\bar\nu(W',\varphi) - \bar\nu(W,\varphi) = -\bar\nu(-W',-\varphi) - \bar\nu(W,\varphi) = 2e_+(X) - \chi(X) + 3\sigma(X) = 48\widehat{A}(X)$$

by Corollary 2.5. □

Corollary 3.2. $\nu(\varphi) := \bar\nu(W,\varphi) \bmod 48 \in \mathbb{Z}_{48}$ *is independent of the choice of W, and*

$$\nu(\varphi') - \nu(\varphi) \equiv 2D(\varphi,\varphi') \mod 48.$$

This gives the majority of the proofs of Theorem 1.3 and Proposition 1.6. To complete the proofs it remains only to show the existence of *Spin*(7)-coboundaries, since Definition 1.2 is phrased in terms of those. We show the existence of the required *Spin*(7)-coboundaries in the following subsection.

3.3. *Spin*(7)**-bordisms.** Let φ, φ' be G_2-structures on closed 7-manifolds M, M'. A *Spin*(7)-bordism from (M, φ) to (M', φ') is a compact 8-manifold with boundary $M \sqcup -M'$ and a *Spin*(7)-structure ψ inducing the respective G_2-structures on the boundary. More formally, we require that $\partial W = f(M) \ \sqcup \ f'(M')$ for embeddings $f : M \hookrightarrow \partial W$, $f' : M' \hookrightarrow \partial W$ that pull back the contraction of ψ with the outward normal field to φ and $-\varphi'$, respectively. If $M = M'$ then we can form a closed spin 8-manifold by identifying the boundary components,

$$\overline{W} := W/(f' \circ f^{-1}). \tag{20}$$

Clearly, there is a topologically trivial *Spin*(7)-bordism W (*i.e.* there is a diffeomorphism $W \cong M \times [0,1]$, but it does not have to preserve the *Spin*(7)-structure) from φ to φ' if and only if they are deformation-equivalent, *i.e.* $f^*\varphi'$ is homotopic to φ for some diffeomorphism $f \colon M \cong M$.

Remark 3.3. If (W, ψ, f, f') is a *Spin*(7)-bordism from (M, φ) to (M', φ') then (W, ψ, f', f) is *Spin*(7)-bordism from $(-M', -\varphi')$ to $(-M, -\varphi)$. However, it does not follow in general that $-W$ has a *Spin*(7)-structure making it a *Spin*(7)-bordism from (M', φ') to (M, φ) (because the orientation of a *Spin*(7)-structure cannot be reversed). In particular, if W is a *Spin*(7)-coboundary for (M, φ) then $-W$ is not necessarily a *Spin*(7)-coboundary for $(-M, -\varphi)$, unless $\chi(W) = 0$, *cf.* proof of Lemma 3.1(iii).

The *Spin*(7)-structure ψ induces a non-vanishing positive spinor field s on W. By Remark 2.3 the restriction of s to ∂W is the spinor defining the G_2-structures φ and $-\varphi'$, so $n_+(W, \varphi, \varphi') = 0$. In particular, when φ and φ' are G_2-structures on the same manifold $M = M'$, Lemma 1.5 follows from (17). Similarly, if W is a *Spin*(7)-coboundary for (M, φ) then $\bar{\nu}(W, \varphi) = \chi(W) - 3\sigma(W)$, so Corollary 3.2 together with Lemma 3.4(ii) imply Theorem 1.3.

Lemma 3.4.
(i) *For a connected compact spin 8-manifold W with connected boundary M, there is a unique homotopy class of G_2-structures on M that bound Spin(7)-structures on W.*
(ii) *Any G_2-structure has a Spin(7) coboundary (any two G_2-structures are Spin(7)-bordant).*

Proof. If W is connected with non-empty boundary then there is no obstruction to defining a non-vanishing positive spinor field on W, so there is some G_2-structure φ on M that bounds a *Spin*(7)-structure on W. If φ' is another G_2-structure bounding a *Spin*(7)-structure on W, consider an arbitrary spin filling W' of $-M$, and let $-\varphi''$ be a G_2-structure on $-M$ that bounds a *Spin*(7)-structure on W'. Then $W \sqcup W'$ admits two *Spin*(7)-structures that define bordisms from φ and φ', respectively, to φ''. Hence

$$D(\varphi, \varphi') = D(\varphi, \varphi'') - D(\varphi', \varphi'') = 0,$$

and φ and φ' must be homotopic.

For (ii), take any spin filling W of M, and let φ be a G_2-structure on M that bounds a *Spin*(7)-structure. In order to find a *Spin*(7)-coboundary for some other φ' with $D(\varphi, \varphi') = \pm k$, we use that if X and X' are closed spin 8-manifolds then, since $\widehat{A}$ and σ are bordism-invariants, and in particular additive under connected sums, Corollary 2.5 implies that

$$e_+(X \sharp X') = e_+(X) + e_+(X') - 1.$$

(We could also see that for any pair of positive spinor fields s, s' on X, X' one can define a spinor field on $X \sharp X'$ that equals s and s' outside the connecting neck, and with a single zero on the neck.) Therefore φ' will bound a *Spin*(7)-structure on W' the connected sum of W with k copies of a manifold with $e_+ = 2$ or 0, *e.g.* $S^4 \times S^4$ or T^8. ☐

3.4. **Examples of G_2-structures on S^7.** To make the discussion more concrete, we elaborate on some examples on S^7, where D can be described in the following direct way. The spinor bundle of S^7 can be trivialised by identifying it with the restriction of the positive half-spinor bundle on B^8, thus up to homotopy, a G_2-structure φ on S^7 can be identified with a map f from S^7 to the unit sphere in Δ^+. The difference D between two G_2-structures on S^7 equals the difference of the degrees of the corresponding maps $S^7 \to S^7$: $D(\varphi, \varphi') = \deg f - \deg f'$.

Example 3.5. We first illustrate how this description works for the standard round G_2-structure φ_{rd} and its reverse $\widehat{\varphi}_{rd}$, which we already understand from Example 1.14. By definition, φ_{rd} corresponds to a constant map $f_{rd} : x \mapsto s_0$. The G_2-structure φ_{rd} is invariant under the action of $Spin(7)$, and so is f_{rd}, in the sense that $f_{rd}(gx) = s_0 = gs_0 = gf_{rd}(x)$ for any $g \in Spin(7)$.

Let r be a reflection of S^7, and $\widehat{\varphi}_{rd} = r^*(-\varphi_{rd})$ as before. Then $\widehat{\varphi}_{rd}$ is invariant under the action of the conjugate subgroup $rSpin(7)r \subset Spin(8)$. If $x_0 \in S^7$ is a vector orthogonal to the hyperplane of the reflection, then φ_{rd} and $\widehat{\varphi}_{rd}$ take the same value at x_0. Thus $\hat{f}_{rd}(x_0) = s_0$, and $\hat{f}_{rd}(rgrx_0) = (rgr)s_0$ for any $g \in Spin(7)$. The outer automorphism on $Spin(8)$ of conjugating by r swaps the the positive and negative spin representations via Clifford multiplication by x_0, so $(rgr)s_0 = x_0 \cdot (g(x_0 \cdot s_0)) = x_0 \cdot (g(x_0) \cdot s_0)$ for $g \in Spin(7)$. Hence $\hat{f}_{rd} : S^7 \to S^7$ equals the orientation-preserving diffeomorphism $c^-_{x_0} \circ \widehat{c}_{s_0} \circ (-r)$, and $D(\widehat{\varphi}_{rd}, \varphi_{rd}) = \deg \hat{f}_{rd} - \deg f_{rd} = 1$.

Example 3.6. Consider the octonionic left-multiplication parallelism on S^7, *i.e.* the trivialisation of TS^7 obtained by considering $u \in S^7$ as a unit octonion and defining $L_u : \operatorname{Im}\mathbb{O} \cong T_uS^7$ as left multiplication by u. Its associated G_2-structure $\varphi_{\mathbb{O}}$ has $\varphi_{\mathbb{O}}(u) = L_u\varphi_0$ for a fixed G_2-structure φ_0. The associated map $f_{\mathbb{O}} : S^7 \to S^7$ is $u \mapsto \tilde{L}_u s_0$ where $S^7 \to Spin(8)$, $u \mapsto \tilde{L}_u$ is the continuous lift of $S^7 \to SO(8)$, $u \mapsto L_u$ (with $\tilde{L}_1 = \mathrm{Id}$) which acts on $s_0 \in \Delta^+$.

Here is one way to understand $\tilde{L}_u$. The Moufang identity $u(xy)u = (ux)(yu)$ holds for any $u, x, y \in \mathbb{O}$ [20, Lemma A.16(c)], so $(L_u, R_u, L_u \circ R_u) \in SO(8)^3$ preserves the Cayley multiplication. As mentioned before, the Cayley multiplication on $\mathbb{O}$ can be identified with Clifford multiplication $\mathbb{R}^8 \times \Delta^- \to \Delta^+$, whose stabiliser in the group $SO(\mathbb{R}^8) \times SO(\Delta^-) \times SO(\Delta^+)$ is precisely $Spin(8)$ [2, (5)]. Hence a copy of S^7 in $Spin(8)$ whose action on $\mathbb{R}^8$ is by L_u must act on Δ^+ by $L_u \circ R_u$. If we choose the identification $\Delta^+ \cong \mathbb{O}$ so that s_0 corresponds to 1, then $f_{\mathbb{O}}(u) = \tilde{L}_u s_0$ corresponds to u^2, so $\deg f_{\mathbb{O}} = 2$. Hence $D(\varphi_{\mathbb{O}}, \varphi_{rd}) = 2$, and $\nu(\varphi_{\mathbb{O}}) = -3$.

Example 3.7. The G_2-structure φ_{rd} is invariant under the order 4 diffeomorphism given by scalar multiplication by i on $S^7 \subset \mathbb{C}^4$ (since $i\,\mathrm{Id} \in SU(4) \subset Spin(7)$) so descends to a G_2-structure $\varphi_{rd}/\mathbb{Z}_4$ on the quotient $S^7/\mathbb{Z}_4$. This is the boundary of the unit disc bundle of $\mathcal{O}(-4)$ on $\mathbb{C}P^3$ (the canonical bundle of $\mathbb{C}P^3$), which has an $SU(4)$-structure restricting to $\varphi_{rd}/\mathbb{Z}_4$ (indeed, the total space admits a Calabi–Yau metric asymptotic to $\mathbb{C}^4/\mathbb{Z}_4$, *cf.* Calabi [9, §4]). The self-intersection number of a hyperplane in the zero-section is -4, so $\sigma = -1$, and $\nu(\varphi_{rd}/\mathbb{Z}_4) = 4 + 3 = 7$.

Remark 3.8. While Example 3.7 illustrates that ν itself is not multiplicative under covers, if φ and φ' are G_2-structures on the same closed spin 7-manifold M and $p : \widetilde{M} \to M$ is a degree k covering map then $D(p^*\varphi, p^*\varphi') = kD(\varphi, \varphi')$.

Remark 3.9. The fact that φ_{rd} and $\widehat{\varphi}_{rd}$ are both invariant under the antipodal map on S^7 is not incompatible with $D(\varphi_{rd}, \widehat{\varphi}_{rd})$ being odd, because the G_2-structures they define on $\mathbb{R}P^7 = S^7/\pm 1$ induce different spin structures. The actions of $Spin(7)$ and the conjugate $rSpin(7)r$ on $\mathbb{R}P^7$ can both be lifted to the spinor bundle. Since -1 acts trivially on $\mathbb{R}P^7$, its image under either lift will be $\pm\mathrm{Id}$, and the two spin structures can be distinguished by which of the two lifts acts as $+\mathrm{Id}$.

Similarly, φ_{rd} defines the same spin structure on $\mathbb{R}P^7$ as the octonionic left-multiplication parallelism of $\mathbb{R}P^7$, but not the right-multiplication one. This is related to the fact that $Spin(7)$ can be described as the subgroup of $SO(8)$ generated by left multiplication by unit imaginary octonions, while the subgroup generated by right multiplications is a conjugate of $Spin(7)$ by a reflection.

4. ν of twisted connected sum G_2-manifolds

Our motivation for introducing the invariant ν is to give a tool for studying the homotopy classes of G_2-structures. We now show how the definition of ν in terms of $Spin(7)$-bordisms allows us to compute it for the large class of 'twisted connected sum' manifolds with holonomy G_2. Before describing the twisted connected sums, we explain how to compute ν of G_2-structures defined as products of structures on lower-dimensional manifolds. This is then used in the proof of Theorem 1.7, that the torsion-free G_2-structures of twisted connected sum G_2-manifolds always have $\nu = 24$.

4.1. **$SU(3)$ and $SU(2)$-structures.** Let us first describe $SU(3)$ and $SU(2)$-structures in terms of forms, along the lines of §2.1.

Let $z^k = x^{2k-1} + ix^{2k}$ be complex coordinates on $\mathbb{R}^6$. Then the stabiliser in $GL(6,\mathbb{R})$ of the pair of forms

$$\Omega_0 = dz^1 \wedge dz^2 \wedge dz^3 \in \Lambda^3(\mathbb{R}^6)^* \otimes \mathbb{C},$$
$$\omega_0 = \tfrac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) \in \Lambda^2(\mathbb{R}^6)^*,$$

is $SU(3)$. An $SU(3)$-structure (Ω, ω) on a 6-manifold induces a Riemannian metric, almost complex structure and orientation (the volume form is $-\frac{i}{8}\Omega \wedge \overline{\Omega} = \frac{1}{6}\omega^3$). On $\mathbb{R} \oplus \mathbb{R}^6$

$$dt \wedge \omega_0 + \operatorname{Re}\Omega_0 \cong \varphi_0, \tag{21}$$

and $SU(3)$ is exactly the stabiliser in G_2 of a non-zero vector in $\mathbb{R}^7$. The product of a 6-manifold with $SU(3)$-structure and S^1 or $\mathbb{R}$ has a product G_2-structure, while the boundary of a 7-manifold with G_2-structure has an induced $SU(3)$-structure.

The stabiliser in $GL(4,\mathbb{R})$ of the triple of forms

$$\omega_0^I = dx^{12} + dx^{34},\ \omega_0^J = dx^{13} - dx^{24},\ \omega_0^K = dx^{14} + dx^{23} \in \Lambda^2(\mathbb{R}^4)^*$$

is $SU(2)$. The stabiliser in $SU(2)$ of a non-zero vector is clearly trivial, and the boundary of a 4-manifold W with $SU(2)$-structure $(\omega^I, \omega^J, \omega^K)$ has a natural coframe defined by contracting each of the three 2-forms with an outward pointing normal vector field.

If e^1, e^2, e^3 is a coframe on $\mathbb{R}^3$ then

$$e^{123} + e^1 \wedge \omega_0^I + e^2 \wedge \omega_0^J + e^3 \wedge \omega_0^K \cong \varphi_0$$

on $\mathbb{R}^3 \oplus \mathbb{R}^4$. Therefore the product of a parallelised 3-manifold and a 4-manifold with $SU(2)$-structure has a natural product G_2-structure. Similarly, if we let $\omega_1^I, \omega_1^J, \omega_1^K$ denote an equivalent triple of 2-forms on a second copy of $\mathbb{R}^4$, and $\mathrm{vol}_0 = \frac{1}{2}(\omega_0^I)^2$ etc, then

$$\mathrm{vol}_0 + \omega_0^I \wedge \omega_1^I + \omega_0^J \wedge \omega_1^J + \omega_0^K \wedge \omega_1^K + \mathrm{vol}_1 \cong \psi_0 \tag{22}$$

on $\mathbb{R}^4 \oplus \mathbb{R}^4$, so the product of two 4-manifolds W_0, W_1 with $SU(2)$-structures has a natural product $Spin(7)$-structure. If W_0 is closed while ∂W_1 is non-empty, clearly the G_2-structure induced on $\partial(W_0 \times W_1)$ by this $Spin(7)$-structure equals the product of $\omega_0^\bullet$ with the coframe on ∂W_1 induced by $\omega_1^\bullet$.

4.2. **Product G_2-structures and spinors.** Above we described two types of product G_2-structures. In order to compute ν of such products, we shall need to describe $SU(3)$ and $SU(2)$ in terms of spinors.

The half-spin representations $\Delta^\pm$ of $Spin(6) \cong SU(4)$ are the standard 4-dimensional representation of $SU(4)$ and its dual. The inclusion $SU(3) \hookrightarrow SO(6)$ lifts to the obvious inclusion $SU(3) \hookrightarrow SU(4)$, so the stabiliser of a non-zero element in Δ^+ is exactly $SU(3)$. Hence, analogously to §2.2, $SU(3)$-structures on a 6-manifold N compatible with a fixed spin structure and metric can be defined by positive unit spinor fields (which always exist and any two are homotopic since the real rank of S^+N is 8).

If N is the boundary of a spin 7-manifold M, then the half-spinor bundles on N are both isomorphic, as real vector bundles, to the restriction of the spinor bundle from M. Analogously to Remark 2.3, the restrictions of G_2-structures on M to $SU(3)$-structures on N can be described equivalently in terms of differential forms or spinors. As there is no obstruction to extending a non-vanishing section of a rank 8 bundle on M from the boundary to the interior, it follows that any $SU(3)$-structure on N is induced as the boundary of a G_2-structure on M.

Lemma 4.1. *If N is a 6-manifold with an $SU(3)$-structure (Ω, ω), then the product G_2-structure $\varphi = d\theta \wedge \omega + \operatorname{Re}\Omega$ on $S^1 \times N$ has $\nu(\varphi) = 0$.*

Proof. Any spin 6-manifold N bounds some spin 7-manifold M, as the bordism group Ω_6^{Spin} is trivial [29]. Then any product G_2-structure φ on $S^1 \times N$ bounds a product $Spin(7)$-structure on $S^1 \times M$. The S^1 factor makes $\sigma(S^1 \times M) = \chi(S^1 \times M) = 0$, so $\nu(\varphi) = 0$. □

Now we consider dimensions 3 and 4. Before looking at the spinors we prove a topological lemma.

Lemma 4.2. *For any compact spin 4-manifold W with boundary Y,*

$$\chi(W) \equiv \chi_2(Y) \mod 2,$$

where $\chi_2(Y)$ is the mod 2 semi-characteristic $\sum_{i=0}^{1} \dim H^i(Y; \mathbb{Z}_2)$.

Proof. Repeating the argument in the proof of (19) with $\mathbb{Z}_2$-coefficients instead of $\mathbb{Q}$-coefficients shows that there is a mod 2 identity

$$\chi(W) \equiv \dim H_0^2(W; \mathbb{Z}_2) + \chi_2(Y) \mod 2,$$

where $H_0^2(W; \mathbb{Z}_2)$ is the image of $H^2(W, Y; \mathbb{Z}_2) \to H^2(W; \mathbb{Z}_2)$. The intersection form of W defines a non-singular bilinear form over $\mathbb{Z}_2$ on $H_0^2(W; \mathbb{Z}_2)$. This injects as an orthogonal summand into the mod 2 intersection form of the manifold $X := W \cup_{\mathrm{Id}_Y} -W$. Since X is a closed spin 4-manifold, its intersection form is even, and hence the form on $H_0^2(W; \mathbb{Z}_2)$ is too. By [30, Ch. III Lemma 1.1] the rank of every non-singular even bilinear form over $\mathbb{Z}_2$ is even, which completes the proof. □

The spin representations of $Spin(4) \cong SU(2) \times SU(2)$ are the standard 2-dimensional complex representations of the two factors. Therefore the stabiliser of a non-zero positive spinor is one of the $SU(2)$ factors, and a unit spinor field on a spin 4-manifold defines an $SU(2)$-structure.

The spin representation of $Spin(3) \cong SU(2)$ is again the standard representation of $SU(2)$. The stabiliser of a non-zero spinor is trivial, so a unit spinor field defines a parallelism, *i.e.* a trivialisation of the tangent bundle. For a spin 4-manifold with boundary Y, the restriction of either the positive or negative spinor bundle to Y is isomorphic to the spinor bundle of Y. The analogue in dimension 4 of Corollary 2.5 is that

$$e_\pm(X) = \tfrac{3}{4}\sigma(X) \pm \tfrac{1}{2}\chi(X) \tag{23}$$

for any closed spin 4-manifold X (it suffices to check for $X = S^4$ and $K3$). Recall Rokhlin's theorem that $\sigma(X)$ is divisible by 16.

Lemma 4.3. *Let X be a closed 4-manifold with an $SU(2)$-structure $(\omega^I, \omega^J, \omega^K)$ and Y a closed 3-manifold with a coframe field (e^1, e^2, e^3). Then*

$$\nu(\varphi) = 24\chi_2(Y)\frac{\sigma(X)}{16} \mod 48$$

for the product G_2-structure $\varphi = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge \omega^I + e^2 \wedge \omega^J + e^3 \wedge \omega^K$ on $Y \times X$.

Proof. Pick a spin coboundary W of Y. Let $n_+(W, \pi)$ be the intersection number with the zero section of a positive spinor field on W whose restriction to Y is the defining spinor field of the parallelism π equivalent to the coframe field. We can apply connected sums with T^4 or $S^2 \times S^2$ to make $n_+(W, \pi) = 0$ (this is the same argument as in Lemma 3.4), so we can assume that π bounds an $SU(2)$-structure on W.

If X has an $SU(2)$-structure then $e_+(X) = 0$, so (23) implies $\chi(X) = -\frac{3}{2}\sigma(X)$. $W \times X$ is a $Spin(7)$-coboundary for φ so, applying Lemma 4.2 in the final step,

$$\nu(\varphi) = \chi(W \times X) - 3\sigma(W \times X) = \big(-24\chi(W) - 48\sigma(W)\big)\frac{\sigma(X)}{16} = 24\chi_2(Y)\frac{\sigma(X)}{16} \mod 48. \quad \square$$

4.3. **Twisted connected sums.** Now we sketch the basics of the twisted connected sum construction, ignoring many details that are required to justify that the resulting G_2-structures are torsion-free (see [24, 10]). The construction starts from a pair of asymptotically cylindrical Calabi–Yau 3-folds $V_\pm$. We can think of these as a pair of (usually simply connected) 6-manifolds with boundary $S^1 \times \Sigma_\pm$, for $\Sigma_\pm$ a K3 surface. They are equipped with $SU(3)$-structures $(\omega_\pm, \Omega_\pm)$ such that on a collar neighbourhood $C_\pm \cong [0,1) \times \partial V_\pm$ of the boundary

$$\begin{aligned} \omega_\pm &= dt \wedge du + \omega^I_\pm, \\ \Omega_\pm &= (du - idt) \wedge (\omega^J_\pm + i\omega^K_\pm), \end{aligned} \tag{24}$$

where u is the S^1-coordinate, t is the collar coordinate and $(\omega^I_\pm, \omega^J_\pm, \omega^K_\pm)$ is an $SU(2)$-structure on $\Sigma_\pm$. The construction assumes that there is a diffeomorphism $f : \Sigma_+ \to \Sigma_-$ such that

$$f^*\omega^I_- = \omega^J_+, \ f^*\omega^J_- = \omega^I_+ \text{ and } f^*\omega^K_- = -\omega^K_+.$$

Now define G_2-structures on $S^1 \times V_\pm$ by

$$\varphi_\pm = dv \wedge \omega_\pm + \operatorname{Re}\Omega_\pm,$$

where v denotes the S^1-coordinate, and a diffeomorphism

$$\begin{aligned} F :\ \partial(S^1 \times V_+) \cong S^1 \times S^1 \times \Sigma_+ &\longrightarrow S^1 \times S^1 \times \Sigma_- \cong \partial(S^1 \times V_-), \\ (v, u, x) &\longmapsto (u, v, f(x)). \end{aligned}$$

In the collar neighbourhoods $C_\pm$

$$\varphi_\pm = dv \wedge dt \wedge du + dv \wedge \omega^I_\pm + du \wedge \omega^J_\pm + dt \wedge \omega^K_\pm,$$

so φ_+ and φ_- patch up to a well-defined G_2-structure φ on the closed manifold

$$M = (S^1 \times V_+) \cup_F (S^1 \times V_-). \tag{25}$$

One arranges that this G_2-structure can be perturbed to a torsion-free one. Because F swaps the circle factors at the boundary, M is simply-connected if V_+ and V_- are.

4.4. **A** ***Spin*****(7)-bordism.** We now proceed with the proof of Theorem 1.7, that the twisted connected sum G_2-structures defined above always have $\nu = 24$. Consider the diffeomorphism

$$\widetilde{F} = \mathrm{Id} \times -\mathrm{Id} \times f \ :\ S^1 \times S^1 \times \Sigma_+ \to S^1 \times S^1 \times \Sigma_-,$$

and the "untwisted connected sum" $\widetilde{M} = (S^1 \times V_+) \cup_{\widetilde{F}} (S^1 \times V_-)$. Then $\widetilde{M} = S^1 \times N$, where $N = V_+ \cup_{-\mathrm{Id}\times f} V_-$. Let r denote the right angle rotation $(v,u) \mapsto (u,-v)$ of $S^1 \times S^1$ and $g := F \circ \widetilde{F}^{-1}$, and let T_r and T_g denote their mapping tori. Then $g = r \times \mathrm{Id}_\Sigma$, so $T_g \cong T_r \times \Sigma$.

To compute $\nu(\varphi)$ of the twisted connected sum G_2-structure φ on M and prove Theorem 1.7 we will construct a $Spin(7)$-bordism W to product G_2-structures on $\widetilde{M} \sqcup T_g$. Let

$$\begin{aligned} B_\pm &= \left\{(y - \tfrac{1}{2})^2 + t^2 < \tfrac{1}{16}\right\} \subset I \times S^1 \times C_\pm, \\ W_\pm &= I \times S^1 \times V_\pm \setminus B_\pm, \end{aligned}$$

where y denotes the I-coordinate, and t the collar coordinate on $C_\pm \subset V_\pm$ as before. $\partial W_\pm$ is a union of five pieces, meeting in edges at $\{y\} \times S^1 \times S^1 \times \Sigma$ for $y = 0, \frac{1}{4}, \frac{3}{4}$ and 1: a 'top' and 'bottom' piece each diffeomorphic to $S^1 \times V_\pm$, $[0, \frac{1}{4}] \times S^1 \times S^1 \times \Sigma_\pm$ and $[\frac{3}{4}, 1] \times S^1 \times S^1 \times \Sigma_\pm$, and $E_\pm := \left\{(y - \frac{1}{2})^2 + t^2 = \frac{1}{4}\right\} \subset I \times S^1 \times C_\pm$.

We form a 'keyhole' bordism W by gluing some of these pieces: identify $[0, \frac{1}{4}] \times S^1 \times S^1 \times \Sigma_\pm$ via $\mathrm{Id} \times \widetilde{F}$, and $[\frac{3}{4}, 1] \times S^1 \times S^1 \times \Sigma_\pm$ via $\mathrm{Id} \times F$. Then ∂W is a disjoint union $M \sqcup \widetilde{M} \sqcup T_g$, where M is formed by gluing the top pieces of ∂W_+ and ∂W_- and $\widetilde{M}$ by gluing the bottom pieces, while the keyhole boundary component $E_+ \cup E_-$ can be identified with the mapping torus T_g.

It is easy to compute that $H_1(T_r) \cong \mathbb{Z} \times \mathbb{Z}_2$, so $\chi_2(T_r) \equiv 1$. Since $\sigma(\Sigma) = -16$, Lemma 4.3 implies that any product G_2-structure on $T_r \times \Sigma$ has $\nu = 24$, while a product G_2-structure on $\widetilde{M}$ has $\nu = 0$. To complete the calculation of $\nu(\varphi)$ it remains to show that W does indeed admit a suitable $Spin(7)$-structure, and to compute the topological invariants of the $Spin(7)$-bordism W.

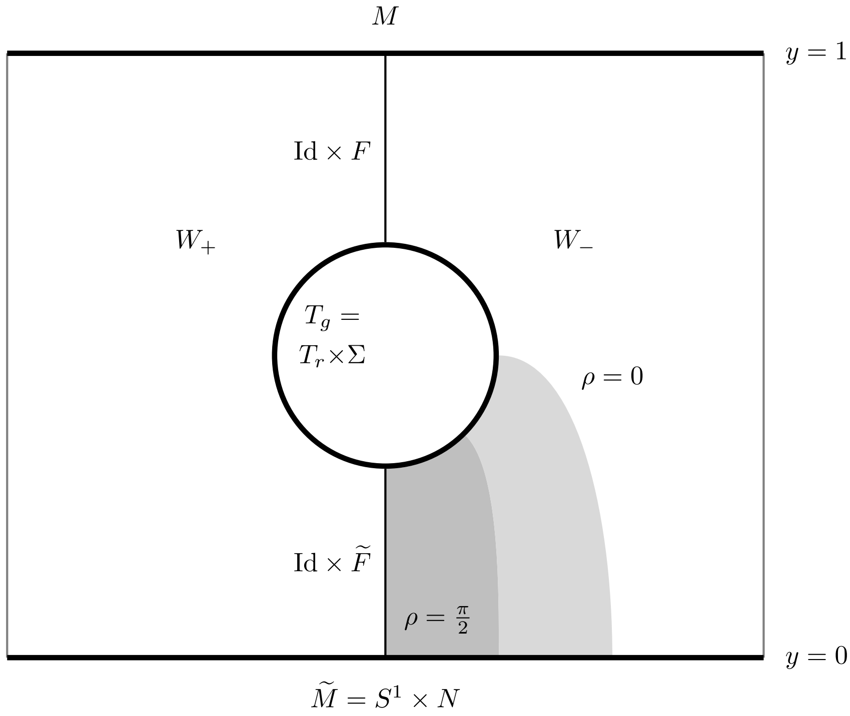

FIGURE 1. The 'keyhole' bordism W

Lemma 4.4. *$\chi(W) = 0$ and $\sigma(W) = -16$.*

Proof. For the Euler characteristic, we use the usual inclusion-exclusion formula. The spaces W_+, W_- and $W_+ \cap W_-$ all contain S^1 factors, so $\chi(W) = \chi(W_+) + \chi(W_-) - \chi(W_+ \cap W_-) = 0$.

For the signature, we must apply Wall's signature formula [32] because W is formed by gluing W_+ and W_- along only parts of boundary components. The piece of the boundaries of W_+ and W_- that we glue is $X_0 = \big([0, \frac{1}{4}] \sqcup [\frac{3}{4}, 1]\big) \times T^2 \times \Sigma$. Let $Z = \partial X_0 = \big\{0, \frac{1}{4}, \frac{3}{4}, 1\big\} \times T^2 \times \Sigma$ (the edges of $\partial W_\pm$), and

$$X_\pm := \partial(W_\pm) \setminus X_0 = \big(\{0,1\} \times S^1 \times V_\pm\big) \sqcup E_\pm,$$

where $E_\pm$ are the keyhole pieces as defined above.

Throughout this proof we will use real coefficients for all cohomology groups. We need to identify the images A, B and C in $H^3(Z)$ of $H^3(X_0)$, $H^3(X_+)$ and $H^3(X_-)$, respectively; each is a Lagrangian subspace with respect to the intersection form $(\ ,\ )$ on $H^3(Z)$. The vector space $K = \frac{A\cap(B+C)}{(A\cap B)+(A\cap C)}$ admits the following natural non-degenerate symmetric bilinear form q: if $a, a' \in A \cap (B + C)$ (representing $[a], [a'] \in K$) and $a' = b' + c'$, $b' \in B$, $c' \in C$, then we set

$$q([a], [a']) := -(a, b').$$

Since $W_\pm$ both have signature 0, the signature formula [32, Theorem p. 271] implies that the signature of W equals the signature of (K, q).

We can identify $Z_y := \{y\} \times T^2 \times \Sigma$ with $S^1 \times \partial V_+$. On Z_y, let v denote the coordinate on the S^1 factor from $S^1 \times V_+$, and u the coordinate on the S^1 factor in ∂V_+. Let $\theta_+ = [dv]$ and $\theta_- = [du] \in H^1(Z_y)$. If $w \in H^4(\Sigma)$ is positive then $\theta_+ \wedge \theta_- \wedge w \in H^6(Z_y)$ is positive with respect to the orientation on Z_y given by the identification with $S^1 \times \partial V_+$. The orientation on Z that we should use to define its intersection form in the application of the signature formula is that induced as the boundary of X_+, *i.e.*

$$Z = Z_1 \sqcup -Z_{\frac{3}{4}} \sqcup\ Z_{\frac{1}{4}} \sqcup -Z_0.$$

Since the K3 surface Σ has no cohomology in odd degrees, the vector space $H^3(Z)$ decomposes as the sum of 8 copies of $L := H^2(\Sigma)$: we let $L_{y\pm}$ denote the image of $L \to H^3(Z_y)$, $\ell \mapsto \theta_\pm \wedge \ell$. (This means for example that if $\alpha_\pm \in H^2(V_\pm)$ then the restriction of $[dv] \wedge \alpha_\pm \in H^3(W_\pm)$ to Z_y lies in L_{y+} for $y = 0, \frac{1}{4}$, and in $L_{y\pm}$ for $y = \frac{3}{4}, 1$.) For $h \in H^3(Z)$, let $h_{y\pm} \in L$ denote the $L_{y\pm}$ component under this isomorphism. Then the intersection form on $H^3(Z)$ is given in terms of the inner product $\langle\ ,\ \rangle$ on L by

$$\begin{aligned}(h,h') = \langle h_{1+}, h'_{1-}\rangle - \langle h_{1-}, h'_{1+}\rangle - \langle h_{\frac{3}{4}+}, h'_{\frac{3}{4}-}\rangle + \langle h_{\frac{3}{4}-}, h'_{\frac{3}{4}+}\rangle \\ + \langle h_{\frac{1}{4}+}, h'_{\frac{1}{4}-}\rangle - \langle h_{\frac{1}{4}-}, h'_{\frac{1}{4}+}\rangle - \langle h_{0+}, h'_{0-}\rangle + \langle h_{0-}, h'_{0+}\rangle.\end{aligned} \tag{26}$$

Let $N_\pm$ denote the image of $H^2(V_\pm)$ in $H^2(\Sigma) \cong L$, and $T_\pm \subset L$ the orthogonal complement. By the long exact sequence of the pair $(V_+, S^1 \times \Sigma_+)$ and Poincaré-Lefschetz duality, the image of $H^3(V_+)$ in $H^3(S^1 \times \Sigma)$ is the annihilator of the image of $H^2(V_+)$ under the intersection pairing, which equals $[du] \wedge T_+$. We find that

$$\begin{aligned}A &= \{h \in H^3(Z) : h_{0\pm} = h_{\frac{1}{4}\pm},\ h_{\frac{3}{4}\pm} = h_{1\pm}\}, \\ B &= \{h \in H^3(Z) : h_{0+}, h_{1+} \in N_+,\ h_{0-}, h_{1-} \in T_+,\ h_{\frac{1}{4}\pm} = h_{\frac{3}{4}\pm}\}, \\ C &= \{h \in H^3(Z) : h_{0+}, h_{1-} \in N_-,\ h_{0-}, h_{1+} \in T_-,\ h_{\frac{1}{4}\pm} = \pm h_{\frac{3}{4}\mp}\}.\end{aligned} \tag{27}$$

Given an element of K represented by $a = b + c$, we can certainly find some $h \in A \cap B$ with $h_{1\pm} = b_{1\pm}$. Replacing a by $a - h$, we may assume without loss of generality that $b_{1\pm} = 0$. Similarly we can assume $c_{1\pm} = 0$, and then $a_{1\pm} = 0$ too. Setting

$$n := a_{0+},\ t := a_{0-},\ n_+ := b_{0+},\ t_+ := b_{0-},\ n_- := c_{0+}, \quad \text{and } t_- := c_{0-},$$

the remaining components are determined by (27). Thus we find that any element of K can be represented by $a = b + c$ such that

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ n & t \\ n & t \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ \frac{n+t}{2} & \frac{-n+t}{2} \\ \frac{n+t}{2} & \frac{-n+t}{2} \\ n_+ & t_+ \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 \\ \frac{-n-t}{2} & \frac{n-t}{2} \\ \frac{n-t}{2} & \frac{n+t}{2} \\ n_- & t_- \end{pmatrix}$$

(where the top left matrix entry corresponds to the 1+ component etc), and

$$n_\pm \in N_\pm, \quad t_\pm \in T_\pm, \quad n = n_+ + n_-, \quad t = t_+ + t_-.$$

Representing a pair of classes $[a], [a'] \in K$ by elements of that form, applying (26) and rearranging gives

$$\begin{aligned}2q([a],[a']) = -2(a,b') &= -\langle n, -n'+t'\rangle + \langle t, n'+t'\rangle + \langle n, 2t'_+\rangle - \langle t, 2n'_+\rangle \\ &= \langle n, n'\rangle + \langle t, t'\rangle + \langle n, t'_+-t'_-\rangle + \langle t, -n'_++n'_-\rangle.\end{aligned} \tag{28}$$

Now consider

$$\begin{aligned}K_0 &= \{[a] \in K : n \in N_+ \cap N_-,\ t \in T_+ + T_-\}, \\ K_\pm &= \{[a] \in K : n = t \in N_\pm \cap (T_+ + T_-)\}.\end{aligned}$$

If we use (28) to evaluate the product of two elements of K_0 then the cross terms $\langle n, t'\rangle$ etc vanish, and $q([a],[a']) = \langle n, n'\rangle + \langle t, t'\rangle = \langle n+t, n'+t'\rangle$. Hence K_0 is isometric to L, so has signature -16.

If $[a] \in K_+$, then the RHS of (28) reduces to $2\langle t, n'_-\rangle$, which vanishes if $[a'] \in K_0 + K_+$. Similarly K_- is orthogonal to $K_0 + K_-$. This implies in particular that K_+ and K_- are transverse, and since $K_+ \oplus K_-$ is a sum of isotropic spaces it has signature 0.

Finally, note that $K_+ \oplus K_-$ is a complement to K_0 in K: given $(n,t) \in (N_+ + N_-) \times (T_+ + T_-)$ we can certainly subtract an element of $N_+ \cap N_-$ from n to ensure that $n \in T_+ + T_-$, and then an element of $T_+ + T_-$ from t to ensure $n = t$. Hence the orthogonal complement to K_0 is precisely $K_+ \oplus K_-$, and

$$\sigma(W) = \sigma(K) = \sigma(K_0) + \sigma(K_+ \oplus K_-) = -16. \qquad \square$$

To finish the proof of Theorem 1.7, we need to exhibit a $Spin(7)$-structure on W with the right restrictions to the boundary components: the restriction to M should be the twisted connected sum G_2-structure φ, while the restrictions to $\widetilde{M} = S^1 \times N$ and $T_g = T_r \times \Sigma$ should be product G_2-structures. We can define an $SU(3)$-structure on N as follows. Let V'_- be the complement of the collar neighbourhood $C_- \subset V_-$. On C_- set

$$\omega' = dt \wedge du + c_\rho \omega_-^I + s_\rho \omega_-^J,$$
$$\Omega' = (du - idt) \wedge (c_\rho \omega_-^J - s_\rho \omega_-^I + i\omega_-^K),$$

where $c_\rho = \cos\rho,\ s_\rho = \sin\rho$ for a smooth function ρ supported on C_-, such that $\rho = \frac{\pi}{2}$ on ∂V_-. Take $\tilde\omega$ to be ω_+ on V_+, ω' on C_-, and ω_- on V'_-, and define $\tilde\Omega$ analogously. Then $(\tilde\omega, \tilde\Omega)$ is a well-defined $SU(3)$-structure on N, and $\tilde\varphi = d\theta \wedge \tilde\omega + \operatorname{Re}\tilde\Omega$ is a product G_2-structure on $\widetilde{M}$.

Next we define the $Spin(7)$-structure ψ on W. Let y be the I coordinate on each half. First, define ρ on $I \times C_-$ to be $\frac{\pi}{2}$ on a neighbourhood of $[0, \frac14] \times \partial V_-$ and have compact support in $\big[0, \frac12\big) \times C_-$ (see Figure 1), and use this to define forms $\tilde\omega$ and $\tilde\Omega$ on $I \times V_-$. Since dy is a global covector field on W_0, defining a $Spin(7)$-structure is equivalent to defining a G_2-structure on each slice $y = $ const. Take this to be $\varphi_+ = d\theta \wedge \omega_+ + \operatorname{Re}\Omega_+$ on $\{y\} \times S^1 \times V_+$, and $d\theta \wedge \tilde\omega + \operatorname{Re}\tilde\Omega$ on $\{y\} \times S^1 \times V_-$. Then the restriction of ψ to the boundary components M and $\widetilde{M}$ are φ and $-\tilde\varphi$ respectively, as desired.

Finally we show that the restriction of ψ to the 'keyhole' boundary component $T_g = E_+ \cup E_-$ is a product G_2-structure too. We first outline the argument, starting from $E_\pm \cong [0, \pm\pi] \times S^1 \times S^1 \times \Sigma_\pm$ (the first factor corresponding to one half of the circle $\{(y - \frac12)^2 + t^2 = \frac{1}{16}\}$) being embedded as a product inside $I \times C_\pm$. The restriction of ψ to $I \times C_\pm$ is a product of two $SU(2)$-structures, so the induced G_2-structure on $E_\pm$ is a product of a coframe field on $[0, \pm\pi] \times S^1 \times S^1$ and an $SU(2)$-structure on Σ. The coframes on the two copies of $[0, \pm\pi] \times S^1 \times S^1$ patch up to a coframe on their union T_r, and the G_2-structure on T_g is the product of that with an $SU(2)$-structure on Σ.

In order to fill in the details of this sketch we need to write down the structures explicitly, which is rather cumbersome. To make the notation slightly more manageable we will use a complex form as a shorthand for an ordered pair of real forms, so that an $SU(2)$-structure can be defined by one complex and one real 2-form, or a coframe field on a 3-manifold by one complex and one real 1-form. Also, we identify both Σ_+ and Σ_- with a standard K3 surface Σ, so that f corresponds to Id_Σ. Setting $y = -\frac12 c_\alpha + \frac12,\ t = \frac12 s_\alpha$ for $\alpha \in [0, \pi]$ lets us identify $E_+ \subset I \times C_+$ with $[0, \pi] \times S^1 \times S^1 \times \Sigma$. On $I \times C_+$, ψ is the product of the $SU(2)$-structure

$$\big((dy - idt) \wedge (dv + idu),\ dy \wedge dt - dv \wedge du\big) \tag{29}$$

on $I \times [0,1) \times S^1 \times S^1$ and $(\omega_+^I + i\omega_+^J,\ \omega_+^K)$ on Σ. The induced G_2-structure on E_+ is given by contraction with the normal vector field $c_\alpha \frac{\partial}{\partial y} - s_\alpha \frac{\partial}{\partial t}$. The result is the product of the same $SU(2)$-structure on Σ with the coframe field $(e^{i\alpha}(dv + idu),\ \frac12 d\alpha)$ on $[0, \pi] \times S^1 \times S^1$.

Similarly, for $\alpha \in [\pi, 2\pi]$ we set $y = -\frac12 c_\alpha + \frac12,\ t = -\frac12 s_\alpha$ to identify $[\pi, 2\pi] \times S^1 \times S^1 \times \Sigma_- \cong E_-$. On $I \times C_-$, the restriction of ψ is given by the product of (29) on $I \times [0,1) \times S^1 \times S^1$ and $\big(e^{-i\rho}(\omega_-^I + i\omega_-^J),\ \omega_-^K\big)$ on the tangent space to the Σ factor. Contracting with the normal vector field $c_\alpha \frac{\partial}{\partial y} + s_\alpha \frac{\partial}{\partial t}$ gives the coframe $\big(e^{-i\alpha}(dv + idu),\ -\frac12 d\alpha\big)$ on $[\pi, 2\pi] \times S^1 \times S^1$. Now, as product G_2-structures

$$\big(e^{-i\alpha}(dv + idu),\ -\tfrac12 d\alpha\big) \cdot \big(e^{-i\rho}(\omega_-^I + i\omega_-^J),\ \omega_-^K\big) =$$
$$\Big(e^{i(\rho-\alpha)}(dv + idu),\ -\tfrac12 d\alpha\Big) \cdot (\omega_-^I + i\omega_-^J,\ \omega_-^K) = \Big(e^{i(\alpha-\rho)}(du + idv),\ \tfrac12 d\alpha\Big) \cdot (\omega_+^I + i\omega_+^J,\ \omega_+^K).$$

T_g is formed by gluing the boundaries of $[0, \pi] \times S^1 \times S^1 \times \Sigma$ and $[\pi, 2\pi] \times S^1 \times S^1 \times \Sigma$ using $(\pi, v, u, x) \mapsto (\pi, u, v, x)$ and $(0, v, u, x) \mapsto (2\pi, v, -u, x)$. These maps preserve the $SU(2)$-structure on the Σ factor, and match up the coframes $(e^{i\alpha}(dv + idu),\ \frac12 d\alpha)$ and $\big(e^{i(\alpha-\rho)}(du + idv),\ \frac12 d\alpha\big)$ to a well-defined coframe on T_r (since $\rho = 0$ at $\alpha = \pi$ and $\rho = \frac{\pi}{2}$ at $\alpha = 0, 2\pi$). Thus the G_2-structure on $T_g = T_r \times \Sigma$ is a product, completing the proof of Theorem 1.7.

4.5. **Orbifold resolutions.** For some of Joyce's examples of compact G_2-manifolds constructed by resolving flat orbifolds, the torsion-free G_2-structures are homotopic to twisted connected sum G_2-structures, and thus have $\nu = 24$. It is proved in [25] that in some cases there is even a connecting path of torsion-free G_2-structures, but that is irrelevant for the calculation of ν.

We have no general technique for computing ν of orbifold resolution G_2-manifolds. We note, however, that a small number of examples have $b_2(M) + b_3(M)$ even, *e.g.* [23, §12.8.4]. Those G_2-manifolds have $\chi_{\mathbb{Q}}(M)$—and hence ν—odd.

5. The h-principle for coclosed G_2-structures

We now prove Theorem 1.8, that coclosed G_2-structures satisfy the h-principle. We first set up some notation, continuing from §2.1.

5.1. **Positive 4-forms.** For a vector space V of dimension 7, let $\Lambda^3_+V^*$ and $\Lambda^4_+V^*$ denote the space of forms equivalent to φ_0 (as defined in (13)) and $*\varphi_0$ respectively. These are *open* subsets of the spaces of forms. Any $\varphi \in \Lambda^3_+V^*$ defines a G_2-structure, and thus an inner product and orientation, and a Hodge star operator. This gives a non-linear map $\Lambda^3_+V^* \to \Lambda^4_+V^*, \varphi \mapsto *\varphi$, which is 2-to-1. The stabiliser of a $\sigma \in \Lambda^4_+V^*$ is isomorphic to $G_2 \times \{\pm 1\}$, so σ *together with* a choice of orientation on V determines a G_2-structure [4, §2.8.3].

We say that a G_2-structure on a 7-manifold M, defined by a positive 3-form $\varphi \in \operatorname{Sec}\Lambda^3_+(M)$, is coclosed if the associated 4-form $\sigma = *\varphi \in \operatorname{Sec}\Lambda^4_+(M)$ is closed. The set of coclosed G_2-structures on an oriented manifold M is therefore the same as the space of closed positive 4-forms $\operatorname{Clo}\Lambda^4_+(M) \subset \operatorname{Sec}\Lambda^4_+(M)$. (Each section induces a spin structure, and the space $\mathcal{G}_2^{cc}(M)$ appearing in the statement of Theorem 1.8 is a subset of $\operatorname{Clo}\Lambda^4_+(M)$ compatible with a fixed spin structure on M.)

5.2. **Microextension.** It is generally easier to prove h-principles for partial differential relations on open manifolds than on closed manifolds. The Hirsch microextension trick is the strategy to prove h-principles on closed manifolds by reducing the problem to an h-principle on an open manifold of higher dimension.

In order to apply the microextension trick, we consider 4-forms on 8-manifolds such that the restriction to every hypersurface is a positive 4-form. The key point that makes the argument work is that not only is the set of such forms open, but moreover any positive 4-form from a hypersurface can be extended this way. This is the feature that enables us to prove the h-principle for coclosed G_2-structures on closed manifolds, but not for, say, symplectic structures or closed G_2-structures.

Definition 5.1. For a vector space W of dimension 8, let

$$\mathcal{R}(W) = \{\chi \in \Lambda^4W^* \,|\, \chi_{|V} \in \Lambda^4_+V^* \text{ for every hyperplane } V \subset W\}.$$

If $W = V \oplus \mathbb{R}$ and $\varphi \in \Lambda^3_+V^*$ then the invariance of $\psi = dt \wedge \varphi + *\varphi$ under $Spin(7)$ (*cf.* (14)), which acts transitively on the hyperplanes, shows that $\psi \in \mathcal{R}(W)$.

Lemma 5.2. *$\mathcal{R}(W)$ is open in Λ^4W^*.*

Proof. Let $G \cong \mathbb{R}P^7$ denote the Grassmannian of hyperplanes in W, and $\pi : \mathcal{V} \to G$ the tautological bundle. If $f : \pi^{-1}(U) \to U \times \mathbb{R}^7$ is a local trivialisation, then $\Lambda^4W^* \times U \to \Lambda^4(\mathbb{R}^7)^*$, $(\chi, V) \mapsto f_{V*}(\chi_{|V})$ is continuous, so the pre-image of $\Lambda^4_+(\mathbb{R}^7)^*$ is open. Hence if $\chi \in \mathcal{R}(W)$ then for each $V \in G$ there are open neighbourhoods $B_V \subset \Lambda^4W^*$ of χ and $C_V \subset G$ of V such that $\chi'_{|V'} \in \Lambda^4_+V'^*$ for each $\chi' \in B_V$ and $V' \in C_V$. Since G is compact it can be covered by $C_{V_1}, \dots, C_{V_k}$ for finitely many $V_1, \dots, V_k \in G$. Then $B_{V_1} \cap \dots \cap B_{V_k}$ is an open neighbourhood of χ in Λ^4W^* and contained in $\mathcal{R}(W)$. □

For an 8-manifold N, let $\mathcal{R}(N) \subset \Lambda^4(N)$ be the subbundle with fibres $\mathcal{R}(T_xN) \subset \Lambda^4T^*_xN$. Let $\operatorname{Clo}\mathcal{R}(N) \subset \operatorname{Sec}\mathcal{R}(N)$ denote the subspace of closed 4-forms, and $\operatorname{Clo}_a\mathcal{R}(N)$ the subspace of forms representing a fixed cohomology class $a \in H^4_{dR}(N)$. Because the subbundle $\mathcal{R}(N) \subset \Lambda^4(N)$ is open and invariant under the natural action of Diff(N), [15, Theorem 10.2.1] immediately implies that $\operatorname{Clo}_a\mathcal{R}(N) \hookrightarrow \operatorname{Sec}\mathcal{R}(N)$ is a homotopy equivalence if N is an open manifold.

5.3. **The proof of Theorem 1.8.** We prove the following stronger version of Theorem 1.8.

Theorem 5.3. *Let M be a closed 7-manifold. Let $I^k \to \operatorname{Sec}\Lambda^4_+(M)$, $s \mapsto \sigma_s$ and $I^k \to H^4_{dR}(M)$, $s \mapsto a_s$ be families such that $\sigma_s \in \operatorname{Clo}_{a_s}\Lambda^4_+(M)$ for all $s \in \partial I^k$. Then the family σ_s is homotopic in $\operatorname{Sec}\Lambda^4_+(M)$, relative to ∂I^k, to a family σ'_s such that $\sigma'_s \in \operatorname{Clo}_{a_s}\Lambda^4_+(M)$ for all $s \in I^k$.*

In particular

- $\operatorname{Clo}\Lambda^4_+(M) \hookrightarrow \operatorname{Sec}\Lambda^4_+(M)$ *is a homotopy equivalence;*
- $\operatorname{Clo}_a\Lambda^4_+(M) \hookrightarrow \operatorname{Sec}\Lambda^4_+(M)$ *is a homotopy equivalence for each fixed $a \in H^4_{dR}(M)$.*

Proof. Identify σ_s with its pull-back to $M\times\mathbb{R}$, and let $\chi_s = \sigma_s + dt\wedge *\sigma_s - td(*\sigma_s) \in \operatorname{Sec}\Lambda^4(M\times\mathbb{R})$. Then there is $\epsilon > 0$ such that χ_s takes values in $\mathcal{R}$ over $N := M \times (-\epsilon, \epsilon)$ for all $s \in I^k$, and $\chi_s \in \operatorname{Clo}_{a_s}\mathcal{R}(N)$ for $s \in \partial I^k$. If $a_s \equiv a$ is constant in s then it follows immediately from [15, Theorem 10.2.1] that the family χ_s is homotopic in $\operatorname{Sec}\mathcal{R}(N)$, relative to ∂I^k, to a family $\chi'_s \in \operatorname{Clo}_a\mathcal{R}(N)$. If we set $\sigma'_s = \chi'_{s|M}$ then $\sigma'_s \in \operatorname{Clo}_a\Lambda^4_+(M)$ for all $s \in I^k$, and the restriction to M of the homotopy from χ to χ' gives a homotopy from σ to σ' in $\operatorname{Sec}\Lambda^4_+(M)$.

The proof of [15, Theorem 10.2.1] builds on [15, Proposition 4.7.4], which is stated for the case when a_s is constant. However, the proof still works if a_s is allowed to depend on s (*cf.* [15, Exercise in §10.2]). □

6. The action of spin diffeomorphisms on $\pi_0\mathcal{G}_2(M)$

Let (M, φ) be a closed connected spin 7-manifold with G_2-structure. In this section we investigate the action of the group of spin diffeomorphisms of M on the set of homotopy classes of G_2-structures on M:

$$\pi_0\mathcal{G}_2(M) \times \operatorname{Diff}(M) \to \pi_0\mathcal{G}_2(M), \quad ([\varphi], f) \mapsto [f^*\varphi].$$

The quotient is the set $\pi_0\bar{\mathcal{G}}_2(M)$ of deformation classes of G_2-structures. To determine the action for a specific spin diffeomorphism $f\colon M \cong M$ amounts to computing the difference class $D(\varphi, f^*\varphi) \in \mathbb{Z}$. The existence of the ν-invariant ensures that $D(\varphi, f^*\varphi) = 24k$ for some integer k. In this section we relate the possible values of k to the topology of M and in particular $p_M \in H^4(M)$. At the end we provide the general definition of the ξ-invariant.

6.1. **The spin characteristic class p_M.** Recall that the classifying space $BSpin$ is 3-connected and $\pi_4(BSpin) \cong \mathbb{Z}$. It follows that $H^4(BSpin) \cong \mathbb{Z}$ is infinite cyclic. A generator is denoted $\pm\frac{p_1}{2}$ and the notation is justified since for the canonical map $\pi\colon BSpin \to BSO$ we have $\pi^*p_1 = 2\frac{p_1}{2}$ where p_1 is the first Pontrjagin class. Given a spin manifold X we write

$$p_X := \frac{p_1}{2}(X) \in H^4(X).$$

The following lemma is well known to experts.

Lemma 6.1 ([11, Lemma 2.2(i)])**.** *For a closed spin 7-manifold M, $p_M \in 2H^4(M)$.*

For later use, we recall from the introduction that d_π denotes the greatest divisor of p_M modulo torsion, while $d_o := \operatorname{Max}\{s \mid s, m \in \mathbb{Z},\ m^2 s \text{ divides } mp_M\}$; we set $d_\pi = d_o = 0$ if p_M is torsion. Both are even by Lemma 6.1.

Example 6.2. If $H^4(M) \cong \mathbb{Z} \oplus \mathbb{Z}_4$ and $p_M \mapsto (8, 2)$ then $d_\pi = 8$ while $d_o = 4$.

6.2. **Translations of G_2-structures and mapping tori.** Given (M, φ) and a spin diffeomorphism $f\colon M \cong M$, we wish to calculate the difference element $D(\varphi, f^*\varphi) \in \mathbb{Z}$. Note that (given φ) the homotopy class $[f^*\varphi] \in \pi_0\mathcal{G}_2(M)$ depends only on the pseudo-isotopy class of f. For suppose that F is pseudo-isotopy between diffeomorphisms f_0 and f_1, *i.e.* a diffeomorphism $F\colon M \times I \cong M \times I$ such that $F|_{M\times\{i\}} = f_i$ for $i = 0, 1$. Then contracting the pull-back $F^*\psi$ of the product $Spin(7)$-structure $\psi = dt \wedge \varphi + *\varphi$ with $\frac{\partial}{\partial t}$ and restricting to the slices $M \times \{t\}$ defines a homotopy between $f_0^*\varphi$ and $f_1^*\varphi$. On the other hand, Proposition 6.3 shows that $D(\varphi, f^*\varphi)$ does not depend upon the G_2-structure φ. Hence we obtain a well-defined function

$$D_M\colon \widetilde{\pi}_0\operatorname{Diff}(M) \to \mathbb{Z}, \quad [f] \mapsto D_M(f) := D(\varphi, f^*\varphi),$$

where $\widetilde{\pi}_0\mathrm{Diff}(M)$ denotes the group of pseudo-isotopy classes of spin diffeomorphisms of M.

The integer $D_M(f)$ measures the translation action of f on the set of homotopy classes of G_2-structures. Next we show how to calculate $D_M(f)$ using the mapping torus of f:

$$T_f := (M \times [0,1])/(x,0) \sim (f(x),1).$$

Since f is a spin diffeomorphism the closed 8-manifold T_f admits a spin structure. We choose a spin structure and let T_f to denote the corresponding 8-dimensional spin manifold: no confusion shall arise since we are interested only in the characteristic number

$$p^2(f) := \langle p^2_{T_f}, [T_f]\rangle \in \mathbb{Z},$$

which depends only on the oriented diffeomorphism type of T_f since $2p_{T_f} = p_1(T_f)$ and $H^8(T_f) \cong \mathbb{Z}$ (in fact p_{T_f} is independent of the choice of spin structure by [8, p. 170]). Therefore $p^2(f)$ is an invariant of the pseudo-isotopy class of f and we define the function

$$p^2 \colon \widetilde{\pi}_0\mathrm{Diff}(M) \to \mathbb{Z}, \quad [f] \mapsto p^2(f).$$

The following proposition proves Proposition 1.10 and shows how the mapping torus T_f can be used to compute the difference class $D(\varphi, f^*\varphi)$.

Proposition 6.3. *The function $D_M \colon \widetilde{\pi}_0\mathrm{Diff}(M) \to \mathbb{Z}$ is a homomorphism given by*

$$D(\varphi, f^*\varphi) \;=\; \frac{3 \cdot p^2(f)}{28} \;=\; 24\widehat{A}(T_f).$$

Proof. From the definition of $D(\varphi,\varphi')$ in §3 it is clear that $D(f^*\varphi, f^*\varphi') = D(\varphi,\varphi')$ for any spin diffeomorphism f and any pair of G_2-structures φ and φ' on M. Now for two spin diffeomorphisms $f_0, f_1 \colon M \cong M$, the affine property (5) of D gives

$$D(\varphi, (f_1 \circ f_0)^*\varphi) \;=\; D(\varphi, f_0^*\varphi) + D(f_0^*\varphi, f_0^*(f_1^*\varphi)) \;=\; D(\varphi, f_0^*\varphi) + D(\varphi, f_1^*\varphi).$$

This shows that D_M is a homomorphism.

Turning to the mapping torus, we can use Lemma 1.5 to compute $D(\varphi, f^*\varphi)$ by treating the product $M \times [0,1]$ together with the embeddings $(\mathrm{Id}, 0)$ and $(f,1) : M \hookrightarrow M \times [0,1]$ as a $Spin(7)$-bordism W_f from $(M, f^*\varphi)$ to (M,φ). Clearly the manifold $\overline{W}_f$ obtained by closing up the bordism as in (20) is nothing other than the mapping torus T_f, so (6) gives

$$D(f^*\varphi, \varphi) \;=\; -e_+(\overline{W}_f) \;=\; -e_+(T_f).$$

By Proposition 2.4, $e_+(T_f) = \frac{1}{16}(4p^2_{T_f} - 4p_2 + 8e)$ and using the signature theorem to eliminate p_2 from this equation we have

$$D(\varphi, f^*\varphi) \;=\; e_+(T_f) \;=\; \frac{3p^2_{T_f}}{28} - \frac{45\sigma(T_f)}{28} + \frac{\chi(T_f)}{2}.$$

Since T_f is a mapping torus both $\sigma(T_f)$ and $\chi(T_f)$ vanish which proves the first equality of the proposition. Now the second equality follows from Corollary 2.5. □

Since Proposition 6.3 determines D_M in terms of p^2, the proofs of Theorems 1.11 and 1.12 are completed by quoting the following result. Here b_M denotes the torsion linking form on $\mathrm{Tor}\, H^4(M)$.

Theorem 6.4 ([11, Definition 4.4 and Corollary 4.17(iv)])**.** *For any spin 7-manifold M, there is an $r \in \{0,1,2\}$ depending only on $(H^4(M), b_M, p_M)$ such that*

$$p^2(\mathrm{Diff}(M)) \subseteq \mathrm{lcm}(224, 2^r d_o(M))\mathbb{Z}, \tag{30}$$

with equality if M is 2-connected.

The next subsection summarises some ingredients of the proof of this theorem. However, before we do so let us prove an elementary special case of (30) in order to make the appearance of $d_o(M)$ less mysterious.

Lemma 6.5. *Let M be a closed spin 7-manifold and f a spin diffeomorphism of M. Then*

$$p^2(f) \in \mathrm{lcm}(224, d_o(M))\mathbb{Z}. \tag{31}$$

Proof. First recall (8): for a closed 8-dimensional spin manifold X, combining the definitions (16) of the L-genus and the $\widehat{A}$-genus gives

$$p_X^2 - \sigma(X) = 8 \cdot 28\widehat{A}(X).$$

Since the mapping torus T_f is a closed 8-dimensional spin manifold with $\sigma(T_f) = 0$ we deduce that

$$p_{T_f}^2 \in 8 \cdot 28 \cdot \mathbb{Z}. \tag{32}$$

From the definition of d_o there is a positive integer m such that $m^2 d_o$ divides mp_M. Applying Lemma 6.6 below with $x = mp_{T_f}$ and $s = m^2 d_o(M)$ gives that $m^2 d_o(M)$ divides $m^2 p_{T_f}^2$ and hence

$$p_{T_f}^2 \in d_o(M) \cdot \mathbb{Z}.$$

□

Lemma 6.6. *Let T_f be the mapping torus of $f \colon M \cong M$ and $i \colon M \to T_f$ the inclusion. If $x \in H^4(T_f)$ and $s \in \mathbb{Z}$ divides i^*x then s divides $x^2 \in H^8(T_f) \cong \mathbb{Z}$.*

Proof. Consider the following fragment of the long exact cohomology sequence for the mapping torus T_f with $\mathbb{Z}_s$ coefficients:

$$H^3(M;\mathbb{Z}_s) \xrightarrow{\mathrm{Id}-f^*} H^3(M;\mathbb{Z}_s) \xrightarrow{\partial} H^4(T_f;\mathbb{Z}_s) \xrightarrow{i^*} H^4(M;\mathbb{Z}_s) \xrightarrow{\mathrm{Id}-f^*} H^4(M;\mathbb{Z}_s).$$

For a space X, let $\rho_s \colon H^*(X) \to H^*(X;\mathbb{Z}_s)$ denote reduction mod s. By assumption $i^*\rho_s(x) = 0$ and so $\rho_s(x)$ lies in the image of ∂. But the cup-product

$$H^4(T_f;\mathbb{Z}_s) \times H^4(T_f;\mathbb{Z}_s) \to \mathbb{Z}_s$$

vanishes on $\mathrm{Im}(\partial)$. Hence $\rho_s(x)^2 = \rho_s(x^2) = 0 \in H^8(T_f;\mathbb{Z}_s)$ and so s divides x^2. □

6.3. **Diffeomorphisms of spin 7-manifolds.** We shall now summarise the main ideas of the proof of Theorem 6.4 from [11]. For this recall that an almost-diffeomorphism is a homeomorphism that is smooth away from a finite set of points, and we denote the group of almost-diffeomorphisms of a spin manifold M that preserve the spin structure by $\mathrm{ADiff}(M)$. Below we recall the technical notion of a Gauss refinement from [11, §2.5] and how it detects aspects of the action of diffeomorphisms and almost-diffeomorphisms. This leads to a generalisation of the Eells-Kuiper invariant [11, §2.6], and in the next subsection we use these ideas in the general definition of the ξ-invariant of a G_2-structure.

Let M be a closed spin 7-manifold as usual. We can associate to it the invariants p_M, b_M and q_M°, where b_M is the torsion linking form on $\mathrm{Tor}\, H^4(M)$, and q_M° is a "family of quadratic refinements" of b_M [11, §2.4]. Group isomorphisms F act naturally on these objects by pull-backs, *e.g.* $F^\# p_M$ is simply $F^{-1}(p_M)$. For any spin diffeomorphism f of M, the induced action f^* on $H^4(M)$ preserves these invariants, *i.e.*

$$(f^*)^\# p_M = p_M, \quad (f^*)^\# b_M = b_M, \quad (f^*)^\# q_M^\circ = q_M^\circ;$$

in fact, this remains true even if f is merely an almost-diffeomorphism or even a homeomorphism [11, Theorem 1.2]. We define a function

$$P : \mathrm{Aut}(H^4(M), b_M, p_M) \to \mathbb{Z}/2d_\pi(M)\mathbb{Z}$$

as follows [11, (39)]. Let

$$S_{d_\pi} := \{k \in H^4(M) : p_M - d_\pi k \text{ is torsion}\}. \tag{33}$$

For $F \in \mathrm{Aut}(H^4(M), b_M, p_M)$, pick $k \in S_{d_\pi}$, let $t := F(k) - k$, and

$$P(F) := d_\pi^2 b_M(t,t) - 2d_\pi\, b_M(p_M - d_\pi k,\, t) \mod 2d_\pi(M)\mathbb{Z}.$$

Then [11, Prop 4.16] states that

$$p^2(f) = P(f^*) \mod 2d_\pi(M). \tag{34}$$

Meanwhile [11, (42)] states that

$$\mathrm{Im}\, P = 2^r d_o(M)\mathbb{Z}/2d_\pi(M)\mathbb{Z}, \tag{35}$$

for some $r(H^4(M), b_M, p_M) \in \{0,1,2\}$, and $r = 1$ unless $H^4(M)$ has 2-torsion. Combined with (32) this implies (30), and hence Theorem 1.11.

Further, if M is 2-connected then [11, Proposition 3.10] states that there exist $f \in \mathrm{ADiff}(M)$ with $f^* = \mathrm{Id}$ on $H^4(M)$ and $p^2(f) = 2\tilde{d}_\pi n$ for any $n \in \mathbb{Z}$; as in the introduction, $\tilde{d}_\pi := \mathrm{lcm}(4, d_\pi)$. It is well known that f is pseudo-isotopic to a diffeomorphism if $p^2(f)$ is divisible by 224 [11, Lemma 3.7(iii)], so one can find $f \in \mathrm{Diff}(M)$ such that $p^2(f) = \mathrm{lcm}(224, 2\tilde{d}_\pi)$. Hence equality holds in (30), completing the proof of Theorem 6.4 (and hence Theorem 1.12).

A key step in the above argument is that $p^2(f) \bmod 2d_\pi$ can be determined purely algebraically, from the action f^* on $H^4(M)$. A related fact is that $p^2(f) \bmod 2\tilde{d}_\pi$ can be determined by the action of f^* on *Gauss refinements* associated to spin coboundaries of M. A Gauss refinement on M is a function

$$g : S_{d_\pi} \to \mathbb{Q}/\tfrac{\tilde{d}_\pi}{4}\mathbb{Z}$$

whose mod $\mathbb{Z}$ reduction is determined by the quadratic linking family q°_M and which satisfies

$$g(k+t) - g(k) = \frac{d_\pi^2 b_M(t,t) - 2d_\pi\, b_M(p_M - d_\pi k,\, t)}{8} \mod \tfrac{d_\pi}{4}\mathbb{Z}.$$

For our present purposes the significance of these conditions is that the difference between two Gauss refinements of M is just a constant in $\mathbb{Z}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$, and if $f \in \mathrm{ADiff}(M)$ then

$$g - (f^*)^\# g = \frac{p^2(f)}{8} \mod \frac{\tilde{d}_\pi}{4}, \tag{36}$$

where $F^\# g := g \circ F$ for any isomorphism F of $H^4(M)$.

Let W be a 3-connected coboundary of M, and $j : H^4(W) \to H^4(M)$ the restriction map. We can associate a Gauss refinement g_W to W by setting

$$g_W(jn) := \frac{(p_W - d_\pi n)^2 - \sigma(W)}{8} \mod \tfrac{\tilde{d}_\pi}{4}\mathbb{Z}$$

for any $n \in H^4(W)$ such that $jn \in S_{d_\pi}$ (then the image of $p_W - d_\pi n \in H^4(W;\mathbb{Q})$ has compact support, and its cup-square is well-defined in $H^8(W, M; \mathbb{Q}) \cong \mathbb{Q}$) [11, (18)]. The key property of g_W is that if $f : M \to M'$ is a diffeomorphism and W' is another 3-connected coboundary of M' then

$$g_{W'} - (f^*)^\# g_W = \frac{p_X^2 - \sigma(X)}{8} = 28\widehat{A}(X) \mod \tfrac{\tilde{d}_\pi}{4}, \tag{37}$$

where X is the closed spin manifold $(-M) \cup_f M'$ [11, (24)]. In particular, the mod 28 reduction of g_W is independent of the choice of spin coboundary W. This defines a generalisation of the Eells-Kuiper invariant,

$$\mu_M : S_{d_\pi} \to \mathbb{Q}/\gcd(28, \tfrac{\tilde{d}_\pi}{4})\mathbb{Z},$$

which distinguishes between $\gcd\left(28, \mathrm{Num}\left(\frac{2^r d_o}{8}\right)\right)$ different smooth structures on the topological manifold underlying M [11, Corollary 4.14]. Together with the homeomorphism invariants $(H^4(M), q^\circ_M, p_M)$, it classifies 2-connected 7-manifolds up to diffeomorphism.

Theorem 6.7 ([11, Theorem 1.3])**.** *For a pair of closed* 2*-connected* 7*-manifolds* M_0 *and* M_1 *and an isomorphism* $F\colon H^4(M_1) \to H^4(M_0)$*, there is a diffeomorphism* $f\colon M_0 \cong M_1$ *such that* $F = f^*$ *if and only if* $(q^\circ_{M_1}, \mu_{M_1}, p_{M_1}) = F^\#(q^\circ_{M_0}, \mu_{M_0}, p_{M_0})$.

6.4. **The ξ-invariant.** We now give the definition of the ξ-invariant of a G_2-structure φ, which is a function $\xi(\varphi) : S_{d_\pi} \to \mathbb{Q}/3\tilde{d}_\pi\mathbb{Z}$ (with S_{d_π} as in (33)). We also explain how the pair (ν, ξ) distinguishes between $24\,\mathrm{Num}\left(\frac{2^r d_o(M)}{224}\right)$ deformation-equivalence classes of G_2-structures on a spin 7-manifold M. This entails Theorem 1.11, and when M is 2-connected combining with Theorem 1.12 implies Theorem 1.17, that (ν, ξ) is a complete invariant of $\pi_0\bar{\mathcal{G}}_2(M)$.

Definition 6.8. Let φ be a G_2-structure on a closed 7-manifold with $Spin(7)$-coboundary W. The ξ-invariant of φ is the function

$$\xi(\varphi) := 7(\chi(W) - 3\sigma(W)) + 12 g_W \ : \ S_{d_\pi} \to \mathbb{Q}/3\tilde{d}_\pi\mathbb{Z}.$$

Combining (2) and (37) shows that $\xi(\varphi)$ is diffeomorphism-invariant (and in particular independent of the choice of W): if $f : M' \to M$ is a diffeomorphism then

$$(f^*)^\#(\xi(f^*\varphi)) = \xi(\varphi).$$

The relations

$$\begin{aligned} 2D(\varphi,\varphi') &= \nu(\varphi') - \nu(\varphi) \mod 48, \\ 14D(\varphi,\varphi') &= \xi(\varphi') - \xi(\varphi) \mod 3\tilde{d}_\pi, \end{aligned}$$

for G_2-structures φ and φ' on the same M mean that (ν,ξ) determine D mod $\operatorname{lcm}\left(24, \operatorname{Num}\left(\frac{3\tilde{d}_\pi}{14}\right)\right)$. Moreover, these relations help us see that precisely $\operatorname{lcm}\left(24, \operatorname{Num}\left(\frac{3\tilde{d}_\pi}{14}\right)\right) = 24\operatorname{Num}\left(\frac{d_\pi}{112}\right)$ pairs (ν,ξ) are realised, namely the ones satisfying

$$\nu = \chi_{\mathbb{Q}}(M) \mod 2, \tag{38a}$$

$$\frac{\xi - 7\nu}{12} = \mu_M \mod \gcd\left(28, \tfrac{\tilde{d}_\pi}{4}\right). \tag{38b}$$

However, this does not mean that there are $24\operatorname{Num}\left(\frac{d_\pi}{112}\right)$ different deformation-equivalence classes, as one has to take into account that $f \in \operatorname{Diff}(M)$ acts non-trivially on Gauss refinements and hence on ξ: (36) implies

$$\xi(f^*\varphi) - \xi(\varphi) = \frac{3}{2}p^2(f) \mod 3\tilde{d}_\pi.$$

Using Theorem 6.4, the mod $2^{r-1}3d_o(M)$ reductions of ξ of deformation-equivalent G_2-structures on M must still be equal. Hence we can use (ν,ξ) to distinguish between at least

$$\operatorname{lcm}\left(24, \operatorname{Num}\left(\frac{2^{r-1}3d_o(M)}{14}\right)\right) = 24\operatorname{Num}\left(\frac{2^r d_o(M)}{224}\right)$$

deformation-equivalence classes. For 2-connected M this is precisely the number of deformation-equivalence classes on M according to Theorem 1.12, so (ν,ξ) distinguishes between all the classes, completing the proof of Theorem 1.17.

Given a G_2-structure, we can use (38b) to recover the Eells-Kuiper invariant of the underlying smooth manifold from (ν,ξ). Hence Theorem 6.7 implies that we can classify closed 2-connected manifolds with homotopy classes of G_2-structures using the quintuple $(H^4(M), q^\circ_M, p_M, \nu, \xi)$.

Theorem 6.9. *Let M_i be closed 2-connected 7-manifolds, and φ_i G_2-structures on M_i. Given an isomorphism $F : H^4(M_1) \to H^4(M_0)$, there is a diffeomorphism $f : M_0 \cong M_1$ such that $F = f^*$ and $f^*\varphi_1$ is homotopic to $f^*\varphi_0$ if and only if $\nu(\varphi_0) = \nu(\varphi_1)$ and $F^\#(p_{M_0}, q^\circ_{M_0}, \xi(\varphi_0)) = (p_{M_1}, q^\circ_{M_1}, \xi(\varphi_1))$.*

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