Combinatorial Reid’s Recipe for Consistent Dimer Models

submitted by

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Signed on behalf of the Faculty of Science ............................................
The aim of this thesis is to generalise Reid’s recipe as first defined by Reid [Rei97] for $G$-$\text{Hilb}(\mathbb{C}^3)$ ($G$ a finite abelian subgroup of $\text{SL}(3, \mathbb{C})$) to the setting of consistent dimer models. We study the $\theta$-stable representations of a quiver $Q$ with relations $R$ dual to a consistent dimer model $\Gamma$ in order to introduce a well-defined recipe that marks interior lattice points and interior line segments of a cross-section of the toric fan $\Sigma$ of the moduli space $M_A(\theta)$ with vertices of $Q$, where $A = \mathbb{C}Q/\langle R \rangle$. After analysing the behaviour of ‘meandering walks’ on a consistent dimer model $\Gamma$ and assuming two technical conjectures, we introduce an algorithm - the arrow contraction algorithm - that allows us to produce new consistent dimer models from old. This algorithm could be used in the future to show that in doing combinatorial Reid’s recipe, every vertex of $Q$ appears ‘once’ and that combinatorial Reid’s recipe encodes the relations of the tautological line bundles of $M_A(\theta)$ in $\text{Pic}(M_A(\theta))$. 
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INTRODUCTION

In the late nineties, Reid [Rei97] described a recipe to mark the interior line segments and interior lattice points of a cross section of the toric fan of the $G$-Hilbert scheme, a crepant resolution of the quotient $\mathbb{C}^3/G$, by irreducible representations of $G$, for $G$ a finite abelian subgroup of $\text{SL}(3, \mathbb{C})$. One of the main features of this recipe is that it encodes the relations between the tautological line bundles of $G$-Hilb($\mathbb{C}^3$) in Pic($G$-Hilb($\mathbb{C}^3$)). Some years later, Craw [Cra05] proved that the recipe can be done for any finite abelian subgroup $G$ of $\text{SL}(3, \mathbb{C})$.

The quotient $\mathbb{C}^3/G$ and crepant resolution $G$-Hilb($\mathbb{C}^3$) are examples of affine Gorenstein toric threefolds and crepant resolutions arising from the study of dimer models and moduli spaces of quiver representations. Therefore, it is natural to consider a generalisation of Reid’s recipe for dimer models. Dimer models are combinatorial objects which encode quivers with relations. To any dimer model $\Gamma$, we associate a quiver $Q$ with relations $\mathcal{R}$ and a complex algebra $A$. When $\Gamma$ satisfies some ‘consistency’ conditions, the centre of the algebra $A$ is the coordinate ring of an affine Gorenstein toric threefold $X$. For some stability parameter $\theta$, the fine moduli space $\mathcal{M}_A(\theta)$ of $\theta$-stable representations of $A$ with dimension vector $(1,1,\ldots,1)$ satisfying the relations $\mathcal{R}$ is a crepant resolution of $X$. In the case when the dimer model is hexagonal, there is a special choice of parameter $\theta$ for which $\mathcal{M}_A(\theta)$ is $G$-Hilb($\mathbb{C}^3$) for some $G \subset \text{SL}(3, \mathbb{C})$.

Reid’s recipe for consistent dimer models marks each interior lattice point and interior line segment in a cross-section of the fan $\Sigma$ of $\mathcal{M}_A(\theta)$ by a vertex or vertices of the quiver $Q$. Some progress has been done towards generalising Reid’s recipe to consistent dimer models. The work of Cautis–Logvinenko [CL09] and Logvinenko [Log10] for the $G$-Hilbert scheme inspired Bocklandt–Craw–Quintero-Vélez [BCQV15, Theorem 1.4] to introduce Geometric Reid’s Recipe for consistent dimer models. In their paper, Bocklandt–Craw–Quintero-Vélez study the images of vertex simples $S_v$ under one of the functors

$$\Psi : D^b(\text{mod } -A) \longrightarrow D^b(\text{coh}(\mathcal{M}_A(\theta)))$$

which gives the equivalence between the bounded derived category of finitely-generated $A$-modules and the bounded derived category of coherent sheaves on $\mathcal{M}_A(\theta)$. The insight of geometric Reid’s recipe, due to Cautis–Logvinenko [CL09], and expanded upon by Logvinenko [Log10] and Craw–Cautis–Logvinenko [CCL12], is that the support of the object $\Psi(S_v)$ is closely related to classical Reid’s recipe for the $G$-Hilbert scheme. The results of Bocklandt–Craw–Quintero-Vélez [BCQV15, Theorem 1.4] suggest how interior lattice points and certain interior line segments of a cross-section of $\Sigma$ might
be marked by vertices \( v \) of \( Q \). This thesis presents a more combinatorial approach to the problem.

In order to define combinatorial Reid’s recipe for consistent dimer models, we study the \( \theta \)-stable representations of \( A \) with dimension vector \((1,1,\ldots,1)\) satisfying the relations \( \mathcal{R} \), for a special choice of stability parameter \( \theta \). This enables us to introduce a well-defined recipe which marks every interior lattice point and interior line segment of the fan \( \Sigma \) by a vertex of \( Q \). Combinatorial Reid’s recipe is compatible with both classical Reid’s recipe from Reid [Rei97] and Craw [Cra05], and with geometric Reid’s recipe. With the aim to show that every vertex appears ‘once’ and that the marking encodes the relations of the line bundles of \( \mathcal{M}_A(\theta) \) in \( \text{Pic}(\mathcal{M}_A(\theta)) \), we introduce a ‘localisation’ algorithm, called the arrow contraction algorithm.

As a word of warning to the reader, at times throughout this thesis we will need to simultaneously refer to dimer models, quivers and toric fans of crepant resolutions, which we identify with their cross-sections at height one. All three of these have zero-dimensional and one-dimensional components. In order to avoid confusion we give to each of them different names. Table 1 summarises this information and introduces the symbol we normally use to denote each of them.

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<th>One-dimensional</th>
<th>Two-dimensional</th>
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<td>Nodes (n)</td>
<td>Edges (e)</td>
</tr>
<tr>
<td>Quiver (( Q ))</td>
<td>Vertices (v)</td>
<td>Arrows (a)</td>
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<tr>
<td>Fan (( \Sigma ))</td>
<td>Lattice points (( \rho ))</td>
<td>Line segments (( \tau ))</td>
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Table 1: Convention of names and symbols used for dimer models, quivers and toric fans

Structure of the thesis

This thesis comes in four parts. The first two chapters present some background material and define some important notions, such as fundamental hexagons and meandering walks. Chapter 3 uses some of the results from Chapter 2 to introduce a well-defined generalisation of Reid’s recipe to the setting of consistent dimer models. The third part of this thesis (Chapters 4, 5 and 6) defines an algorithm, called the arrow contraction algorithm, which enables us to produce new consistent dimer models from old. In this part of the thesis, we assume two conjectures which we make based on numerous computational evidence. Finally, Chapter 7 aims to bring the latter two parts of the thesis together, suggesting future work and a link between combinatorial Reid’s recipe for consistent dimer models and the arrow contraction algorithm.
Chapter 1 provides a basic introduction to dimer models and moduli spaces of quiver representations. We start by defining dimer models, their quivers with relations, perfect matchings and zigzag and characteristic polygons. Throughout this thesis we will only be interested in dimer models in the two-torus $T$ which satisfy a ‘nondegeneracy’ condition. This is weaker than the consistency condition assumed in Chapters 3–7. Section 1.3 gives some background material on toric varieties. We recall King’s construction of moduli spaces of quiver representations and introduce two equivalent notions of consistency for dimer models. The following theorem due to Ishii–Ueda [IU08], Craw–Quintero-Vélez [CQV12] and Broomhead [Bro12], brings most of the background material together:

**Theorem 1.48.** Let $\Gamma$ be a consistent dimer model with $\theta$ generic. The moduli space $M_A(\theta)$ is a toric variety that gives a crepant resolution $f : M_A(\theta) \to X$ of the Gorenstein affine toric threefold $X := \text{Spec}(\mathbb{Z}(A))$.

At the end of Chapter 1 we remind the reader of classical Reid’s recipe, as first described by Reid [Rei97] and proved by Craw [Cra05], for $G$ a finite abelian subgroup of $\text{SL}(3, \mathbb{C})$.

In Chapter 2, we introduce the notion of a fundamental hexagon. Fundamental hexagons were first studied by Ishii–Ueda [IU08] to define toric coordinates on each of the charts $U_\sigma$ of $M_A(\theta)$, for any three-dimensional cone $\sigma$ of the fan $\Sigma$ of $M_A(\theta)$, $\Gamma$ a nondegenerate dimer model and $\theta$ generic. They are fundamental domains of the universal cover of $\Gamma$ associated to each torus-invariant $\theta$-stable representation of $M_A(\theta)$. Section 2.2 is expository and summarises results from Ishii–Ueda [IU08]. Section 2.3 introduces the notion of meandering walks on a nondegenerate dimer model $\Gamma$. Meandering walks are subsets of edges of $\Gamma$ which lie on the boundary of fundamental hexagons. They generalise the notion of zigzag paths and are defined in terms of symmetric differences of perfect matchings. Meandering walks will play an important role throughout the thesis, especially when defining the arrow contraction algorithm.

As an example, consider the dimer model $\Gamma$ in Figure 1 depicted next to its associated quiver $Q$. The fan of $M_A(\theta)$ for $\theta = (-7, 1, 1, \ldots, 1)$ is presented in Figure 2. It has eight lattice points $\rho_1, \rho_2, \ldots, \rho_8$, marked by $1, 2, \ldots, 8$ in the picture.

![Figure 1: A dimer model $\Gamma$ with its dual quiver $Q$](image-url)
Fundamental hexagons are defined uniquely up to translations by \( \mathbb{Z}^2 \). Figure 3 shows a tiling of \( \mathbb{R}^2 \) by a fundamental hexagon. Any choice of vertex 0 in the universal cover of \( \Gamma \) determines a connected subset of \( \mathbb{R}^2 \) and thus a fundamental hexagon. Note that each of the eight tiles of \( \Gamma \) appears once in the fundamental hexagon. Moreover, the fundamental hexagon has six trivalent nodes along its boundary, three white and three black. The quiver depicted in the fundamental hexagon supports a \( \theta \)-stable representation ‘associated’ to the cone \( \sigma \) with lattice points \( \rho_2, \rho_7 \) and \( \rho_8 \).

**Figure 2:** Fan \( \Sigma \) of \( \mathcal{M}_A(\theta) \)

We conclude Chapter 2 by generalising Nakamura’s \( G \)-igssaw transformations to the setting of dimer models. Nakamura [Nak01] introduced an algorithm to produce the crepant resolution \( G \)-Hilb(\( \mathbb{C}^3 \)) of \( \mathbb{C}^3/G \) for a finite abelian subgroup \( G \subset \text{SL}(3, \mathbb{C}) \). This algorithm is generalised to the setting of consistent dimer models by using our understanding of fundamental hexagons and the meandering walks on their boundaries. For two adjacent cones \( \sigma_+ \) and \( \sigma_- \) with common line segment \( \tau \), we can consider their \( \theta \)-stable representations \( M_+ \) and \( M_- \). We cut the corresponding fundamental hexagon

**Figure 3:** A tiling of \( \mathbb{R}^2 \) by a fundamental hexagon
of $M_+$, denoted $\text{Hex}(\sigma_+)$, using a subset of edges $c_-$ in $\Gamma$ which lie along the boundary of $\text{Hex}(\sigma_-)$. We call the set of edges $c_-$ the cut of $\text{Hex}(\sigma_+)$ by $\text{Hex}(\sigma_-)$. This cut splits $\text{Hex}(\sigma_+)$ into connected components called jigsaw pieces.

**Theorem 2.21** (Generalised jigsaw transformation). Given the $\theta$-stable $A$-module $M_+$, we may cut the corresponding fundamental hexagon $\text{Hex}(\sigma_+)$ along the edges of $c_-$ and rearrange the resulting jigsaw pieces (i.e., translate each piece by a carefully chosen element of $\mathbb{Z}^2$) in order to obtain $\text{Hex}(\sigma_-)$ and hence produce the corresponding $\theta$-stable $A$-module $M_-$. 

Figure 4 shows a tiling of $\mathbb{R}^2$ by translates of the fundamental hexagon associated to the cone with ray generators $\rho_2, \rho_3$ and $\rho_7$. This cone is adjacent to the cone which defines the fundamental hexagon from Figure 3.

![Figure 4: A tiling of $\mathbb{R}^2$ by the fundamental hexagon of an adjacent cone](image)

Note that we obtain the fundamental hexagon from Figure 4 by splitting the fundamental hexagon from Figure 3 into two connected components, one containing the tile dual to the vertex 7 and the complement of this tile. By rearranging these tiles, i.e. choosing a different lift of the tile dual to the vertex 7 in the universal cover, we can obtain the fundamental hexagon from Figure 4.

In Chapter 3, we continue our study of jigsaw pieces. From this chapter onwards we assume $\Gamma$ is a consistent dimer model and $\theta$ is a ‘0-generated’ stability parameter. Every 0-generated stability parameter is generic. Among all jigsaw pieces of $\text{Hex}(\sigma_-)$ and $\text{Hex}(\sigma_-)$ we distinguish the zero jigsaw piece $\text{Jig}_+(0)$ and the one jigsaw piece
Jig\(\tau\)\(1\). Much of Section 3.2 is dedicated to showing that the one jigsaw piece Jig\(\tau\)\(+\)(1) of Hex\((\sigma+)\) coincides with the one jigsaw piece Jig\(\tau\)\(−\)(1) of Hex\((\sigma−)\) up to translation in the universal cover of \(\Gamma\). More importantly, this implies that the ‘sources’ of the two subquivers \(Q_+\)\(\tau\)\(\) and \(Q_-\)\(\tau\)\(\) of the support of \(M_+\) and \(M_-\), as introduced in Definition 3.19, coincide. We use this last result to define Reid’s recipe for interior line segments of \(\Sigma\) (see Definition 3.21). Reid’s recipe for interior lattice points of \(\Sigma\) is defined using ‘socles’ in a way reminiscent of Craw–Ishii [CI04, Proposition 9.1] (see Definition 3.23). We show compatibility of combinatorial Reid’s recipe with classical Reid’s recipe:

**Theorem 3.27 (Reid’s recipe for the G-Hilbert scheme).** Let \(G\) be a finite abelian subgroup of \(\text{SL}(3, \mathbb{C})\) and let \(\Sigma\) be the fan of the G-Hilbert scheme. Reid’s recipe for \(\Sigma\) as in Definitions 3.21 and 3.23 agrees with the classical recipe from Reid [Rei97] and Craw [Cra05].

We also show that the marking of interior lattice points by vertices of \(Q\) according to Definition 3.23 is compatible with the geometric Reid’s recipe for consistent dimer models as stated by Bocklandt–Craw–Quintero-Véliz [BCQV15, Theorem 1.4]:

**Theorem 3.32 (Geometric Reid’s recipe for lattice points).** Let \(v \in Q_0\) be a nonzero vertex and let \(\rho \in \Sigma(1)\) be an interior lattice point of the triangulation \(\Sigma\). Then \(v\) marks \(\rho\) according to Reid’s recipe if and only if the divisor \(D_\rho\) is contained in the support of the sheaf \(\Psi(S_v)\).

New phenomena can be observed in combinatorial Reid’s recipe for consistent dimer models which did not appear in classical Reid’s recipe: two vertices of \(Q\) can mark a single line segment; a vertex can mark more than one lattice point; and the recipe for lattice points is not determined purely by the shape of the incoming line segments and their markings. Section 3.6 computes geometric and combinatorial Reid’s recipe for three examples depicting some of these interesting features.

In the example above, the one jigsaw piece consists of a single tile, the tile dual to the vertex 7, while the zero jigsaw piece contains every other tile of \(\Gamma\). This is one of the simplest examples of a jigsaw transformation. The two subquivers \(Q_+\)\(\tau\)\(\) and \(Q_-\)\(\tau\)\(\) coincide in this case, they are the quiver with a single vertex, 7, and no arrows. Reid’s recipe marks the common line segment to both cones by the ‘sources’ of the quiver \(Q_+\)\(\tau\)\(\), in this case 7.

The aim of Chapters 4, 5 and 6 is to introduce an algorithm - the arrow contraction algorithm - which takes a consistent dimer model \(\Gamma\) to a new consistent dimer model \(\Gamma'\) such that for any 0-generated stability parameters \(\theta\) and \(\theta'\), the moduli space of \(\theta'\)-stable \(A'\)-modules \(\mathcal{M}_{A'}(\theta')\), for \(A'\) the Jacobian algebra of \(Q'\), is an open subvariety of \(\mathcal{M}_{A}(\theta)\).

Chapter 4 motivates the arrow contraction algorithm. Section 4.1 introduces an example which illustrates many of the results presented in the remaining of the thesis. We then recall the tautological \(\mathbb{C}\)-algebra isomorphism \(\phi: A \rightarrow \text{End}_{\mathcal{M}_{A}(\theta)}(T)\) between the Jacobian algebra \(A\) and the endomorphism algebra of the tautological bundle \(T\) of \(\mathcal{M}_{A}(\theta)\). This isomorphism associates to each path \(p\) in \(Q\) a torus-invariant section \(\phi(p) = x^{\text{div}(p)}\) of the line bundle \(L_{h(p)} \otimes L_{e(p)}^{-1}\), where the label on \(p\) is the torus-invariant
divisor \( \text{div}(p) = \sum_{a \in \text{supp}(p)} \text{div}(a) \in \mathbb{N}^{\Sigma(1)} \) on \( \mathcal{M}_A(\theta) \). We prove that for any arrow \( \alpha \) in \( Q \) with tail at vertex 0, there exists consecutive boundary divisors \( D_1, D_2, \ldots, D_r \) such that \( \text{div}(\alpha) = D_1 + D_2 + \cdots + D_r \). We then associate to \( \alpha \) a convex polygon \( \Delta' \) and a fan \( \Sigma' := \{ \sigma \in \Sigma \mid \rho_i \not\subseteq \sigma \text{ for all } 1 \leq i \leq r \} \), where \( \Sigma \) denotes the toric fan of \( \mathcal{M}_A(\theta) \).

Finally, we show that for a consistent dimer model \( \Gamma \) dual to \( Q \) and \( \theta \) 0-generated, the toric variety defined by \( \Sigma' \) is an open subvariety of \( \mathcal{M}_A(\theta) \):

**Theorem 4.8.** Suppose \( \Delta' \) is a nondegenerate polygon. Let \( Y' \) denote the toric variety defined by the fan \( \Sigma' \) and let \( \sigma' := \left\{ \sum_{\lambda_p \geq 0} \lambda_p u_p \in \mathbb{N}_R \mid u_p \in \Delta' \right\} \) denote the cone over \( \Delta' \). Then:

1. \( Y' \) is the open subvariety of \( \mathcal{M}_A(\theta) \) satisfying \( Y' = \mathcal{M}_A(\theta) \setminus \text{supp}(\text{div}(\alpha)) \);
2. \( Y' \) is a crepant resolution of the affine Gorenstein toric variety \( U_{\sigma'} \).

In the arrow contraction algorithm, \( \Delta' \) will be the characteristic polygon of \( \Gamma' \), while \( Y' \) will be the moduli space \( \mathcal{M}_{A'}(\theta') \), for \( \theta' \) a 0-generated stability parameter. Lemma 4.9 gives the first link between combinatorial Reid’s recipe and the toric fan of \( Y' \).

In order to define \( \Gamma' \) we need to determine a subset of edges \( S_{\alpha} \) of \( \Gamma \) that will be removed. Chapter 5 defines the set \( S_{\alpha} \) in terms of the meandering walks of line segments in a strip of triangles \( \text{strip}(\alpha) \). Most of the results in this chapter assume a technical conjecture (Conjecture 5.7) about some of the edges in the meandering walks associated to the line segments between the triangles in the strip. Section 5.2 studies the behaviour of the meandering walks of \( \Gamma \) associated to all the line segments in \( \text{strip}(\alpha) \). It partitions two of them into subsets of edges linking trivalent nodes of the fundamental hexagons of the triangles in \( \text{strip}(\alpha) \). Each segment corresponds to a jump between two adjacent triangles in \( \text{strip}(\alpha) \), and the behaviour of the walks depends on whether the two triangles in \( \text{strip}(\alpha) \) are ‘lower’ or ‘upper’. This understanding allows us to give two alternative characterisations of the edges in \( S_{\alpha} \) (see Proposition 5.15 and Theorem 5.16).

For any \( \alpha \in Q_1 \) with tail at vertex 0, Theorem 5.16 associates to some of the line segments in \( \text{strip}(\alpha) \) one or three of the edges in \( S_{\alpha} \). This result is key in proving that the dimer model obtained from \( \Gamma \) by removing the edges in \( S_{\alpha} \) is consistent. At the end of the chapter, we summarise all the information we have about the meandering walks of line segments in \( \text{strip}(\alpha) \) in a single diagram, called the circuit diagram of \( \alpha \). The circuit diagram also contains all the edges in \( S_{\alpha} \).

Chapter 6 makes use of the results from Chapters 4 and 5 to finally introduce the arrow contraction algorithm. We define a bicoloured graph \( \Gamma' \) associated to a consistent dimer model \( \Gamma \) and an arrow \( \alpha \) in its dual quiver with tail at vertex 0. The bicoloured graph \( \Gamma' \) is obtained from \( \Gamma \) by removing the set of edges \( S_{\alpha} \). We need to assume another technical conjecture (Conjecture 6.4) about the cycles made by the arrows dual to the edges in \( S_{\alpha} \), in order to prove that \( \Gamma' \) is indeed a dimer model. Once this fact is established, we recall the definition of the zigzag polygon of a dimer model \( \Gamma \), defined as the convex hull of points obtained using the homology classes of the zigzag paths of \( \Gamma \). We make further use of the results from Chapter 5 to show that the zigzag polygon of \( \Gamma' \) is the convex polygon \( \Delta' \) associated to the arrow \( \alpha \). Using a result by Gulotta [Gul08, Theorem 3.1], we prove the following:
Theorem 6.15. Assume Conjectures 5.7 and 6.4. The dimer model $\Gamma'$ is consistent.

In a manner similar to that of Ishii–Ueda [IU09], we show that for 0-generated stability parameters $\theta$ and $\theta'$ there is an open immersion $j : \mathcal{M}_A(\theta') \rightarrow \mathcal{M}_A(\theta)$ whose image is the complement of the support of $\text{div}(\alpha)$. This leads to a geometric characterisations of the arrows dual to the edges in $\mathcal{S}_\alpha$ (see Corollary 6.19).

We are then ready to state the arrow contraction algorithm:

**Algorithm 6.20 (The Arrow Contraction Algorithm).** Let $\Gamma$ be a consistent dimer model with dual quiver $Q$ and Jacobian algebra $A$. We assume $\Gamma$ has no bivalent nodes and Conjectures 5.7 and 6.4 hold for $\Gamma$.

1. Pick an arrow $\alpha \in Q_0$ with tail at vertex $0$. By Proposition 4.6, we can associate to $\alpha$ a convex polygon $\Delta'$. Assume $\alpha$ is such that $\Delta'$ is nondegenerate.

2. Find the set of edges $\mathcal{S}_\alpha$ using Definition 5.4 or either characterisations of the sets $\mathcal{S}_\alpha$ (see Proposition 5.15 and Theorem 5.16) or $\mathcal{S}_\alpha$ (see Corollary 6.19).

3. Remove the edges in $\mathcal{S}_\alpha$ and all nodes of $\Gamma$ connected only to edges in $\mathcal{S}_\alpha$ from $\Gamma$ to obtain a new dimer model $\Gamma'$ with characteristic polygon $\Delta(\Gamma') = \Delta'$.

4. The dimer model $\Gamma'$ is consistent and for 0-generated stability parameters $\theta$ and $\theta'$, there exists an open immersion $\mathcal{M}_A(\theta') \rightarrow \mathcal{M}_A(\theta)$, so that the toric fan of $\mathcal{M}_A(\theta')$ is $\Sigma' := \{\sigma \in \Sigma \mid \rho_i \not\subset \sigma \text{ for all } 1 \leq i \leq r\}$, where $\Sigma$ is the fan of $\mathcal{M}_A(\theta)$.

5. Since $\Gamma'$ is consistent, and therefore nondegenerate, we may remove all bivalent nodes of $\Gamma'$ without altering the number and direction of its zigzag paths or the Jacobian algebra of its dual quiver (see Remark 1.38).

One can now repeat using $\Gamma'$ and $\theta'$.

The arrow contraction algorithm coincides with Ishii–Ueda’s algorithm [IU09] in the special case when the ‘label’ of the arrow $\alpha$ is a torus-invariant prime divisor. We briefly discuss the connection between these two algorithms. In the last section of Chapter 5, we study the restrictions of the tautological bundle $T$ of $\mathcal{M}_A(\theta)$ to weak toric Fano surfaces. Craw, King and (independently) Logvinenko observed that for some compact surfaces $Z \subset Y = G\text{-Hilb}(\mathbb{C}^3)$, the restriction of the tautological bundle, after removing all redundant summands, is tilting on $Z$. The arrow contraction algorithm gives a partial explanation for this phenomenon.

**Theorem 6.22.** Let $\Gamma$ be a consistent dimer model with Jacobian algebra $A$. If we can obtain a dimer model $\Gamma'$ with Jacobian algebra $A'$ from $\Gamma$ by performing the arrow contraction algorithm such that $\mathcal{M}_{A'}(\theta')$ is the total space of the canonical bundle of a weak toric Fano surface $Z$, the restriction $\bigoplus_{v \in Q_0} L_v|_Z$ of the tautological bundle $T$ of $\mathcal{M}_A(\theta)$ to $Z$ is tilting.

It is not yet clear whether this is also a necessary condition. We give an example which suggests this might be the case.
Chapter 7 concludes this thesis by conjecturing a link between combinatorial Reid’s recipe and the arrow contraction algorithm. The head and tail of the arrow or arrows associated to a line segment $\tau$ in $\text{strip}(\alpha)$ appear to coincide with the vertices that mark either $\tau$ or an endpoint $\rho$ of $\tau$ or another line segment in $\text{strip}(\alpha)$ attached to $\rho$ by Reid’s recipe. We formulate a conjecture based on computational evidence (see Conjecture 7.1) and suggest how this conjecture could be used to show that in the labelling of $\Sigma$ by Reid’s recipe, every nonzero vertex of $Q$ appears ‘once’ and that combinatorial Reid’s recipe for consistent dimer models encodes the relations of the line bundles of $\mathcal{M}_A(\theta)$ in $\text{Pic}(\mathcal{M}_A(\theta))$. 

CHAPTER 1

PRELIMINARIES

The aim of this chapter is to provide a basic introduction to dimer models and moduli spaces of quiver representations. We start by defining dimer models as introduced by Hanany et al. [HK05] [FHK+06]. We associate to each dimer model a quiver $Q$ with relations $R$ and a characteristic polygon $\Delta(\Gamma)$. We then review some classical toric geometry and present King’s construction of the moduli space $M_A(\theta, d)$ of $\theta$-stable representations of $Q$ with relations $R$ of dimension vector $d$. After introducing the nondegeneracy and consistency conditions for dimer models, we describe the link between $\Gamma$ and $M_A(\theta, d)$ in the case when $\theta$ is generic and $d = 1$. Finally, we recall Reid’s recipe as introduced by Reid for a special kind of dimer models whose associated quiver is the McKay quiver of $G$ for some finite abelian subgroup $G$ of $\text{SL}(3, \mathbb{C})$.

1.1 Dimer models and their quivers with relations

We define a graph to be a triple $(\mathcal{N}, \mathcal{E}, r)$ consisting of a set of nodes $\mathcal{N}$, a set of edges $\mathcal{E}$ and a relation $r$ that associates to every edge a pair of nodes called its endnodes. A graph is said to be bipartite if

1. the set of nodes $\mathcal{N}$ is the union of two disjoint sets $\mathcal{B}$ and $\mathcal{W}$, and

2. every edge in $\mathcal{E}$ has an endnode in $\mathcal{B}$ and an endnode in $\mathcal{W}$.

We may choose a colour for each of the disjoint sets of nodes $\mathcal{B}$ and $\mathcal{W}$. We will refer to nodes in $\mathcal{B}$ as black nodes while nodes in $\mathcal{W}$ will be called white nodes. A bipartite graph with a choice of colouring is a bicoloured graph. A black node is said to be connected to a white node (and vice versa) if there exists an edge in $\mathcal{E}$ whose endnodes are the black and white nodes. Associate to any bicoloured graph a one-dimensional CW complex whose 0-cells are the nodes in $\mathcal{N} = \mathcal{B} \sqcup \mathcal{W}$ and 1-cells the edges in $\mathcal{E}$.

Let $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ be the real two-torus. The torus $\mathbb{T}$ inherits an orientation from the standard orientation of $\mathbb{R}^2$. An embedding of a bicoloured graph $(\mathcal{B}, \mathcal{W}, \mathcal{E})$ into the torus $\mathbb{T}$ is an injective continuous map from its associated one-dimensional CW
complex to the torus. When an embedding exists, we identify the nodes and edges of the bicoloured graph with their images under this map.

**Definition 1.1.** A dimer model is a bicoloured graph $\Gamma = (B, W, E)$ embedded in $T$ with the following two properties:

1. $\Gamma$ has no univalent nodes, and
2. every connected component of $T \setminus E$ is simply connected.

A tile of $\Gamma$ is the closure of a connected component in $T \setminus E$. In this way, $\Gamma$ is a CW complex whose 0-cells are the black and white nodes of $\Gamma$, 1-cells the edges in $E$, and 2-cells its tiles, such that $|\Gamma| = T$.

**Definition 1.2.** A quiver $Q$ is the quadruple $(Q_0, Q_1, h, t)$ consisting of a set of vertices $Q_0$, a set of arrows $Q_1$ and maps $h, t : Q_1 \to Q_0$ which give an orientation to its arrows. For $a \in Q_1$, we call $h(a)$ the head of $a$ and $t(a)$ the tail of $a$.

A path $p$ of length $k$ ($k \geq 1$) in $Q$ is a sequence of arrows $a_k a_{k-1} \cdots a_1$ with $h(a_j) = t(a_{j+1})$ for $1 \leq j \leq k - 1$. The path has head $h(p) := h(a_k)$ and tail $t(p) := t(a_0)$. Every vertex $v \in Q_0$ also determines a path $e_v$ called the trivial path of $v$ with $h(e_v) = t(e_v) = v$. Trivial paths are said to be paths of length 0. A path $p$ in $Q$ is said to be a cycle if the head and tail of $p$ coincide. A relation in $Q$ is a non-zero $C$-linear combination of paths of length at least two sharing a common head and a common tail. Let $R$ be a set of relations in $Q$. We call $Q := (Q_0, Q_1, h, t, R)$ a quiver with relations.

**Definition 1.3.** The path algebra $CQ$ of a quiver $Q = (Q_0, Q_1, h, t)$ is the associative algebra whose underlying $C$-vector space has basis the set of paths in $Q$. The trivial paths are orthogonal idempotent elements of $CQ$, i.e.

$$e_v \cdot e_{v'} = \begin{cases} e_v & \text{if } v = v' \\ 0 & \text{otherwise} \end{cases}$$

Moreover, for any path $p$ in $Q$, $e_{h(p)} \cdot p \cdot e_{t(p)} = p$. The product of any two paths $p$ and $p'$ is defined as concatenation when possible, and zero otherwise:

$$p \cdot p' = \begin{cases} pp' & \text{if } h(p') = t(p) \\ 0 & \text{otherwise} \end{cases}$$

Every dimer model $\Gamma$ encodes the information of a quiver with relations $Q$ in the following way. Consider the dual tiling of the bicoloured graph $\Gamma$ in $T$. Tiles of $\Gamma$ are dual to vertices which make the set $Q_0$. Edges in $\Gamma$ are dual to arrows connecting vertices in $Q_0$. The choice of orientation of the arrows comes from the colouring of the graph $\Gamma$. Fix the convention that black vertices are to the left of every arrow, while white vertices are to the right.

To define the set of relations $R$ of $Q$ we introduce the superpotential $W$ of a quiver $Q$. Let $Q$ be the quiver dual to a dimer model $\Gamma$. A face $F$ of $Q$ is the closure of any connected component of the complement $T \setminus Q_1$. We denote the set of faces of $Q$ by $Q_2$. Faces of $Q$ are dual to nodes of $\Gamma$. The fixed orientation means that arrows on
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the boundary of faces dual to white nodes are oriented clockwise, while those around a face dual to a black node have an anticlockwise orientation. Consider the quotient $\mathbb{C}Q_{cyc} := \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$, where $[\mathbb{C}Q, \mathbb{C}Q]$ is the complex vector space spanned by all commutators in $\mathbb{C}Q$. The quotient $\mathbb{C}Q_{cyc}$ has basis corresponding to all cycles in $Q$ up to cyclic shift. For any face $F \in Q_2$, the cycle of arrows around its boundary, denoted $\partial F$, is a basis element of $\mathbb{C}Q_{cyc}$. The superpotential $W$ of $Q$ is defined to be the element of $\mathbb{C}Q_{cyc}$ determined by the faces of $Q$ in the following way:

$$W := \sum_{F \in Q_2} (-1)^F \partial F,$$

where $(-1)^F$ takes the value 1 when $F$ is dual to a black node of $\Gamma$, and the value $-1$ when $F$ is dual to a white node of $\Gamma$. The set of relations $\mathcal{R}$ of the quiver $Q$ dual to $\Gamma$ is the set $\{ \frac{\partial}{\partial a} W \mid a \in Q_1 \}$, where the map $\frac{\partial}{\partial a} : \mathbb{C}Q_{cyc} \to \mathbb{C}Q$ is defined for every basis element $c$ of $\mathbb{C}Q_{cyc}$ to be

$$\frac{\partial}{\partial a} c = \begin{cases} p' & \text{if } \exists p \in \mathbb{C}Q \text{ such that } [p] = c \text{ and } p = p' \cdot a \\ 0 & \text{otherwise} \end{cases}.$$

Remark 1.4. Since every arrow $a \in Q_1$ appears only in two oppositely oriented faces $F_+$ and $F_-$ of $Q$, we can write every relation in $\mathcal{R}$ as $p_+ - p_-$, where $p_{\pm}$ is the path around $F_{\pm}$ starting at $h(a)$ and finishing at $t(a)$.

Throughout this thesis we will only consider quivers which come from a dimer model, and thus have a superpotential $W$ and a set of relations $\mathcal{R}$ associated to them.

Definition 1.5. The Jacobian algebra of a quiver $Q$ with superpotential $W$ and set of relations $\mathcal{R}$ is the quotient algebra

$$A := \mathbb{C}Q/I_{\mathcal{R}}$$

where $I_{\mathcal{R}}$ denotes the two-sided ideal of $\mathbb{C}Q$ generated by set of relations in $\mathcal{R}$.

Example 1.6. One of the simplest examples of a dimer model comes from the tiling of $\mathbb{R}^2$ by hexagons. The dashed lines in Figure 1.1 indicate a fundamental domain, and the induced CW complex on $T$ is a dimer model that has one black node, one white node, three edges and one tile. The dual quiver $Q$ of $\Gamma$ is depicted in Figure 1.2. It has a single vertex $v$, three arrows, one white face $F_+$ and a black face $F_-$. The path algebra $\mathbb{C}Q$ is the free algebra $\mathbb{C}\langle a_1, a_2, a_3 \rangle$ generated by the elements $a_1, a_2$ and $a_3$ each corresponding to the arrows marked by 1, 2 and 3, respectively. Note that the element $e_v \in \mathbb{C}Q$ is the identity. The superpotential of $Q$ is

$$W = \partial F_+ - \partial F_- = a_1 a_2 a_3 - a_1 a_3 a_2.$$

The set of relations $\mathcal{R}$ has three elements

$$\frac{\partial}{\partial a_1} W = a_2 a_3 - a_3 a_2, \quad \frac{\partial}{\partial a_2} W = a_3 a_1 - a_1 a_3, \quad \text{and } \frac{\partial}{\partial a_3} W = a_1 a_2 - a_2 a_1.$$

Thus, the Jacobian algebra $A$ of $Q$ is $\mathbb{C}\langle a_1, a_2, a_3 \rangle/I_{\mathcal{R}} \cong \mathbb{C}[a_1, a_2, a_3]$ the polynomial
ring in three variables.

Figure 1.1: A dimer model coming from the tiling of $\mathbb{R}^2$ by hexagons

Figure 1.2: The dual quiver to the dimer model coming from the tiling of $\mathbb{R}^2$ by hexagons

In general, the Jacobian algebra of a quiver dual to a dimer model is noncommutative.

**Example 1.7.** Let $G$ be a finite abelian subgroup of $\text{SL}(3, \mathbb{C})$ of order $r$ and let $\text{Irr}(G)$ denote the set of irreducible representations of $G$. Since $G$ is abelian, every irreducible representation is one-dimensional and the number of irreducible representations is the order of the group. The inclusion of $G$ in $\text{SL}(3, \mathbb{C})$ gives a three-dimensional representation which decomposes into irreducible representations $\varrho_1 \oplus \varrho_2 \oplus \varrho_3$. The *McKay quiver* $Q$ of the subgroup $G$ is the quiver with $r$ vertices, one for each element of $\text{Irr}(G)$. For every $\varrho \in \text{Irr}(G)$, there is an arrow $\alpha_i^\varrho \in Q_1$ with tail at vertex $\varrho \varrho_i$ ($1 \leq i \leq 3$) and head at vertex $\varrho$. Every such quiver is dual to a dimer model with $r$ hexagonal tiles [UY11]. The Jacobian algebra $A$ of $Q$ is isomorphic to the noncommutative skew group algebra $\mathbb{C}[x, y, z] \ast G$ [CMT07, Proposition 2.8].

**Notation 1.8.** Let $\Gamma$ be a dimer model with dual quiver $Q$ and let $a$ be an arrow of $Q$. We write $e_a$ for the edge in $\Gamma$ dual to $a$.

### 1.2 Perfect matchings and the characteristic polygon

A *perfect matching* of a bipartite graph $(\mathcal{B}, \mathcal{W}, \mathcal{E})$ is a subset of edges $\Pi \subset \mathcal{E}$ such that each node in $\mathcal{B} \cup \mathcal{W}$ is the endnode of precisely one edge in $\Pi$.

**Definition 1.9.** A dimer model $\Gamma = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ is *nondegenerate* if for every edge of $\mathcal{E}$ there is a perfect matching $\Pi$ of $\Gamma$ containing it.
Remark 1.10. It is easy to verify whether a dimer model is nondegenerate. Broomhead [Bro12, Remark 2.12] remarks that as a consequence of Hall’s theorem a dimer model is nondegenerate if and only if $|\mathcal{B}| = |\mathcal{W}|$ and every proper subset of black nodes of size $n$ is connected to at least $n + 1$ white nodes, and vice versa. Broomhead refers to this condition as the strong marriage condition.

Following Ishii–Ueda [IU09, Section 4.2], we use perfect matchings to define the characteristic polygon $\Delta(\Gamma)$ associated to a dimer model $\Gamma$. The universal cover $\pi: \mathbb{R}^2 \to \mathbb{T}$ enables one to pull back a dimer model $\Gamma$ to a bicoloured graph $\widetilde{\Gamma}$ on $\mathbb{R}^2$. We identify a perfect matching $\Pi$ on $\Gamma$ with its lift to a $\mathbb{Z}^2$-periodic perfect matching of $\widetilde{\Gamma}$. Fix a reference perfect matching $\Pi_1$. For any perfect matching $\Pi$, the union of $\Pi \cup \Pi_1$ splits $\mathbb{R}^2$ into connected components.

Definition 1.11. Define the height function $h_{\Pi_1}$ as a locally-constant function on $\mathbb{R}^2 \setminus (\Pi \cup \Pi_1)$ which increases by one when one crosses

(a) an edge $e \in \Pi_1$ with a black node on the right; or
(b) an edge $e \in \Pi$ with a white node on the right;
and decreases by one when one crosses

(a) an edge $e \in \Pi_1$ with a white node on the right; or
(b) an edge $e \in \Pi$ with a black node on the right.

The height function is determined up to the addition of a constant. We use the height function to define the height change $h(\Pi, \Pi_1) := (h_x(\Pi, \Pi_1), h_y(\Pi, \Pi_1)) \in \mathbb{Z}^2$ of the perfect matching $\Pi$ with respect to $\Pi_1$ as follows

$$h_x(\Pi, \Pi_1) = h_{\Pi, \Pi_1}(p + (1, 0)) - h_{\Pi, \Pi_1}(p)$$
$$h_y(\Pi, \Pi_1) = h_{\Pi, \Pi_1}(p + (0, 1)) - h_{\Pi, \Pi_1}(p).$$

The height change does not depend on the choice of point $p \in \mathbb{R}^2 \setminus (\Pi \cup \Pi_1)$. Moreover, for any three perfect matchings $\Pi, \Pi_1$ and $\Pi_2$

$$h(\Pi, \Pi_2) = h(\Pi, \Pi_1) - h(\Pi_2, \Pi_1). \quad (1.2.1)$$

Height changes are elements of $\mathbb{Z}^2$ and thus can be considered as elements of the cohomology group $H^1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$ of $\mathbb{T}$.

Definition 1.12. Define the characteristic polygon $\Delta(\Gamma)$ of a dimer model $\Gamma$ to be the convex hull

$$\Delta(\Gamma) = \text{Conv}\{h(\Pi, \Pi_1) \in \mathbb{Z}^2 \mid \Pi \text{ a perfect matching of } \Gamma\} \subset \mathbb{R}^2.$$ 

By equation (1.2.1), computing the characteristic polygon with a different choice of reference perfect matching results in a translation of the original characteristic polygon.

Example 1.13. As an example, consider the dimer model $\Gamma$ in Figure 1.3 drawn together with its dual quiver $Q$. 

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The dimer model has sixty perfect matchings, listed in Table A.1 in Appendix A. Choose $\Pi_1$ as the reference perfect matching. Figure 1.4 shows the unions $\Pi_i \cup \Pi_1$ for $i = 2, 10, 11$. Edges in $\Pi_1 \setminus \Pi_i$ are coloured in grey. Using Figure 1.4 we can compute the height changes $h(\Pi_2, \Pi_1), h(\Pi_{10}, \Pi_1)$ and $h(\Pi_{11}, \Pi_1)$ as follows:

$$
\begin{align*}
 h(\Pi_2, \Pi_1) &= (0 - 0, (0 + 1) - 0) = (0, 1) \\
 h(\Pi_{10}, \Pi_1) &= ((0 - 1) - 0, (0 + 2) - 0) = (-1, 2) \\
 h(\Pi_{11}, \Pi_1) &= ((0 - 1) - 0, (0 + 2) - 0) = (-1, 2)
\end{align*}
$$

for a special choice of point $p$.

Note that $h(\Pi_{10}, \Pi_1) = h(\Pi_{11}, \Pi_1)$. In general the height changes of any two perfect matchings may give the same lattice point $\rho \in \mathbb{Z}^2$. In fact, the sixty perfect matchings of $\Gamma$ only give ten distinct lattice points (see Table A.1).

Figure 1.5 shows ten perfect matchings $\Pi_1, \Pi_2, \ldots, \Pi_{10}$ chosen so that their height changes are all different. Plotting the lattice points $h(\Pi_i, \Pi_1)$ ($1 \leq i \leq 10$) and taking their convex hull gives the characteristic polygon $\Delta(\Gamma)$ of $\Gamma$, which in this case is the elongated hexagon depicted in Figure 1.6. For $1 \leq i \leq 10$, we mark the lattice point $h(\Pi_i, \Pi_1)$ in $\Delta(\Gamma)$ by $i$. 

Figure 1.3: A dimer model with its dual quiver

Figure 1.4: The unions $\Pi_i \cup \Pi_1$ for $i = 2, 10, 11$
Figure 1.5: Ten perfect matchings of the dimer model $\Gamma$ in Figure 1.3
1.3 Toric varieties

We now recall some classical toric geometry as presented in [CLS11]. The theory begins with a (complex) algebraic torus $T$, which is an affine variety isomorphic to $(\mathbb{C}^\times)^n$ for some $n \in \mathbb{N}$. The torus $T$ inherits a group structure from $(\mathbb{C}^\times)^n$. Write $M \cong \mathbb{Z}^n$ for the character lattice of $T$ and $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ for its dual lattice.

**Definition 1.14.** An affine toric variety $X$ is an irreducible normal affine variety that contains a torus $T$ as a dense open subset such that the action of $T$ on itself extends to an action on the whole of $X$.

Affine toric varieties can be encoded by means of cones in $N_R := N \otimes \mathbb{R}$. A polyhedral cone in $N_R$ is a subset $\sigma = \text{Cone}(S) = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\} \subset N_R$ for some $S \subset N$. For a polyhedral cone $\sigma$, the dual cone $\sigma^\vee := \{m \in M_R \mid \langle m, u \rangle \geq 0 \forall u \in \sigma\}$ is a polyhedral cone in $M_R := M \otimes \mathbb{R}$. A face of $\sigma$ is $\tau = H_m \cap \sigma \subset N_R$ where $H_m$ is the hyperplane $\{u \in N_R \mid \langle m, u \rangle = 0\}$ for some $m \in \sigma^\vee$. A face of a polyhedral cone is a polyhedral cone and the intersection of any two faces is a face. A polyhedral cone $\sigma = \text{Cone}(S)$ is said to be rational if $S$ is finite. It is strongly convex if the zero cone $\{0\} := \text{Cone}(\emptyset)$ is a face of $\sigma$. The latter is equivalent to $\sigma$ containing no positive-dimensional subspaces of $N_R$. Given a rational polyhedral cone $\sigma \subset N_R$, the subset $S_\sigma = \sigma^\vee \cap M \subset M$ forms a saturated semigroup.

**Theorem 1.15.** Let $\sigma \subset N_R$ be a rational polyhedral cone. Then, $U_\sigma := \text{Spec}(\mathbb{C}[S_\sigma])$ is an affine toric variety. Moreover, $\sigma$ is strongly convex if and only if $T = N \otimes_{\mathbb{Z}} \mathbb{C}^\times$ is the dense open torus of $U_\sigma$.

We say a cone $\sigma$ is smooth if and only if its minimal generators form part of a $\mathbb{Z}$-basis of $N$. When $\sigma$ is a strongly convex rational polyhedral cone, $U_\sigma$ is smooth if and only if $\sigma$ is smooth. Note that if $\tau$ is a face of $\sigma$, there is a natural embedding $U_\tau \hookrightarrow U_\sigma$.

We can extend the definition of an affine toric variety by introducing the notion of fans.

**Definition 1.16.** A fan $\Sigma$ in $N_R$ is a collection of strongly convex rational polyhedral cones in $N_R$ such that:

1. Every face of a cone in $\Sigma$ is also a cone in $\Sigma$; and
2. The intersection of any two cones is a face of each one of them.

Write $\Sigma(i)$ for the set of all $i$-dimensional cones of $\Sigma$. Cones $\rho \in \Sigma(1)$ are called rays.
Define a toric variety $X_\Sigma$ to be the variety covered by charts $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$ for all cones $\sigma \in \Sigma$, where $U_\sigma$ and $U_{\sigma'}$ are glued along the open subsets $U_\tau$ where $\tau$ is a common face of $\sigma$ and $\sigma'$. The variety $X_\Sigma$ is smooth if and only every cone $\sigma$ of $\Sigma$ is smooth. We will be particularly interested in toric varieties which are semiprojective. These are characterised as varieties of the form $\text{Proj}(\mathbb{C}[S])$ for some $\mathbb{Z}$-graded semigroup algebra $\mathbb{C}[S]$. A semiprojective toric variety is said to be projective if the 0-graded piece $\mathbb{C}[S]_0 = \mathbb{C}$.

Recall that for every $\sigma \in \Sigma$, there is a bijective correspondence between points of $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$ and semigroup homomorphisms $S_\sigma \rightarrow \mathbb{C}$ [CLS11, Proposition 1.3.1]. Define the distinguished point of $U_\sigma$ to be the semigroup homomorphism $\gamma_\sigma : S_\sigma \rightarrow \mathbb{C}$ which sends an element $m \in S_\sigma$ to 1 if $m \in \sigma^\perp \cap M$ and to 0 otherwise. Now, consider the action of $T$ on $X_\Sigma$. The point $\gamma_\sigma$ determines a torus orbit $O(\sigma) = T \cdot \gamma_\sigma \subseteq X_\Sigma$.

**Theorem 1.17** (The Orbit-Cone Correspondence). For a toric variety $X_\Sigma$ with fan $\Sigma$ in $\mathbb{N}_0^d$ there is a bijective correspondence between its set of cones and the set of $T$-orbits of $X_\Sigma$ given by $\sigma \rightarrow O(\sigma)$. If $n$ is the dimension of $T$, $\dim O(\sigma) = n - \dim \sigma$. Moreover, for any cone $\sigma$ of $\Sigma$, the affine open subset $U_\sigma$ is the union $\bigcup_{\tau \text{ a face of } \sigma} O(\tau)$.

Suppose $X_\Sigma$ is a smooth toric variety. By the Orbit-Cone Correspondence, one-dimensional cones of $\Sigma$ correspond to codimension one $T$-orbits in $X_\Sigma$. For every ray $\rho \in \Sigma(1)$, let $D_\rho$ denote the $T$-invariant prime divisor $\overline{O(\rho)}$ on $X_\Sigma$, where the overline denotes the closure of the orbit. Write $u_\rho \in N$ for the primitive lattice point of $\rho$, and $d = |\Sigma(1)|$ for the number of $T$-invariant divisors of $X_\Sigma$. Since $X_\Sigma$ is smooth, every Weil divisor is Cartier. Assume that the vectors $\{u_\rho \in N \mid \rho \in \Sigma(1)\}$ span $N_\mathbb{R}$, then we have the following short exact sequence:

$$0 \longrightarrow M \xrightarrow{\text{div}} \mathbb{Z}^d \xrightarrow{\text{deg}} \text{Pic}(X_\Sigma) \longrightarrow 0$$  \hspace{1cm} (1.3.1)

where $\mathbb{Z}^d$ is the lattice of all $T$-invariant Cartier divisors of $X_\Sigma$. The injective map $\text{div} : M \rightarrow \mathbb{Z}^d$ sends an element $m \in M$ to the divisor $\sum_{\rho \in \Sigma(1)} (m, u_\rho) D_\rho$. The degree map $\text{deg} : \mathbb{Z}^d \rightarrow \text{Pic}(X_\Sigma)$ sends a Cartier divisor $D$ to the invertible sheaf $\mathcal{O}_{X_\Sigma}(D)$.

The Cox ring $S := \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ of the toric variety $X_\Sigma$ is isomorphic to the semigroup algebra $\mathbb{N}^d$, the subsemigroup of $\mathbb{Z}^d$ consisting of effective torus-invariant Weil divisors on $X_\Sigma$. For any torus-invariant divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \in \mathbb{Z}^d$ we write $x^D := \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho}$ for the corresponding Laurent monomial. For $L \in \text{Pic}(X_\Sigma)$, the set of torus-invariant effective divisors on $X_\Sigma$ satisfying $\mathcal{O}_{X_\Sigma}(D) \cong \mathcal{O}_L$ is $\text{deg}^{-1}(L) \cap \mathbb{N}^d$, so

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{D \in \text{deg}^{-1}(L) \cap \mathbb{N}^d} \mathbb{C} x^D.$$

In particular, every torus-invariant section of $L$ is of the form $x^D$ for some divisor $D \in \text{deg}^{-1}(L) \cap \mathbb{N}^d$.

Recall the following definitions:

**Definition 1.18.** A normal toric variety is Gorenstein if its canonical divisor is Cartier.

**Definition 1.19.** Let $X$ be an irreducible variety. A resolution of singularities of $X$ is a proper morphism $f : Y \rightarrow X$ such that $Y$ is smooth and irreducible and
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\[ f^{-1}(X \setminus X_{\text{sing}}) \cong X \setminus X_{\text{sing}}. \]  
If the morphism is projective, we say the resolution is projective. When \( X \) is a normal Gorenstein variety we say a resolution of singularities \( f : Y \to X \) is crepant if \( K_Y = f^*(K_X) \).

For a fan \( \Sigma \) in \( \mathbb{N} \mathbb{R} \), we say \( \Sigma' \) is a refinement of \( \Sigma \) if every cone of \( \Sigma' \) is contained in a cone \( \Sigma \) and \( \bigcup_{\sigma \in \Sigma'} \sigma = \bigcup_{\sigma \in \Sigma} \sigma \).

**Theorem 1.20.** Every fan \( \Sigma \) has a refinement \( \Sigma' \) such that the morphism \( f : X_{\Sigma'} \to X_\Sigma \) is a resolution of singularities.

Let \( U_\sigma \) be a three-dimensional Gorenstein toric singularity, where \( \sigma \) has primitive ray generator \( u_1, u_2, \ldots, u_k \). The Gorenstein condition means that there exists a monomial \( m \in M \) such that \( \langle m, u_i \rangle = 1 \) for all \( 1 \leq i \leq k \), i.e. all \( u_i \) lie in a common hyperplane at height one. A smooth subdivision of \( \sigma \) into a fan \( \Sigma \) determines a crepant resolution \( X_\Sigma \) if the generators of all the subcones of \( \Sigma \) lie in the same hyperplane, i.e. if \( \Sigma \) is determined completely by a triangulation of the slice \( \Delta \) of \( \sigma \). We will abuse notation by identifying \( \Sigma \) with this triangulation of \( \Delta \). In particular, we identify:

1. rays in the fan \( \Sigma \) with lattice points in the triangulation of \( \Delta \), typically denoted \( \rho \in \Sigma(1) \);
2. two-dimensional cones in the fan \( \Sigma \) with line segments in the triangulation of \( \Delta \), typically denoted \( \tau \in \Sigma(2) \); and
3. three-dimensional cones in the fan \( \Sigma \) with triangles in the triangulation of \( \Delta \), typically denoted \( \sigma \in \Sigma(3) \).

Suppose \( X_\Sigma \) is a smooth three-dimensional toric variety. For any \( \sigma \in \Sigma(3) \), \( U_\sigma = \text{Spec} \left( \mathbb{C}[\sigma^\vee \cap M] \right) \cong \text{Spec} \mathbb{C}[t_1, t_2, t_3] \cong \mathbb{C}^3 \). We can express \( t_1, t_2, t_3 \) as Laurent monomials in the variables of the Cox ring \( S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)] \) of \( X_\Sigma \), as follows. Find the inner-pointing normal vectors to each of the three two-dimensional cones of \( \sigma \). Each vector gives a lattice point in \( M \). Using the injective map \( \text{div} : M \to \mathbb{Z}^d \) from the short exact sequence (1.3.1), we get a Cartier divisor \( D = \Sigma_{\rho \in \Sigma(1)} a_\rho D_\rho \) for each lattice point. Finally, the monomial \( x^D = \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho} \in S \) defines a coordinate of \( \sigma \) for \( t_i \).

**Example 1.21.** Let \( X_\Sigma \) be the smooth three-dimensional toric variety defined by the fan \( \Sigma \) in Figure 1.7. We compute the three coordinates for the cone \( \sigma \in \Sigma(3) \) with ray generators \( \rho_8, \rho_9 \) and \( \rho_{10} \). Choose coordinates for the lattice \( N \) so that

\[
\begin{align*}
u_{\rho_8} &= (-1, 1, 1), \quad \nu_{\rho_9} = (0, 0, 1) \quad \text{and} \quad \nu_{\rho_{10}} = (1, 0, 1).
\end{align*}
\]

The inner pointing normal vectors to the three two-dimensional cones of \( \sigma \) are:

\[
\begin{align*}
m_{8,9} &= \nu_{\rho_8} \times \nu_{\rho_9} = (1, 1, 0), \\
m_{9,10} &= \nu_{\rho_9} \times \nu_{\rho_{10}} = (0, 1, 0), \\
m_{10,8} &= \nu_{\rho_{10}} \times \nu_{\rho_8} = (-1, -2, 1).
\end{align*}
\]

Using the injective map \( \text{div} : M \to \mathbb{Z}^d \) from the short exact sequence (1.3.1), we
associate to each of these vectors a divisor of $X_{\Sigma}$ as follows:

$$D_{8,9} = \sum_{\rho \in \Sigma(1)} \langle (1, 1, 0), (1, -1, 1) \rangle D_{\rho_1} + \cdots + \langle (1, 1, 0), (0, 1, 1) \rangle D_{\rho_{10}}$$

$$= D_{\rho_2} + D_{\rho_3} - D_{\rho_5} - D_{\rho_6} + D_{\rho_{10}}$$

$$D_{8,10} = 2D_{\rho_1} + D_{\rho_2} - D_{\rho_3} - D_{\rho_4} + 2D_{\rho_6} + D_{\rho_7} + D_{\rho_9}$$

$$D_{9,10} = -D_{\rho_1} - D_{\rho_2} + D_{\rho_3} + 2D_{\rho_4} + 2D_{\rho_5} + D_{\rho_7} + D_{\rho_8}$$

The Laurent monomials

$$x^{D_{8,9}} = x_{\rho_2} x_{\rho_3} x_{\rho_{10}}, \quad x^{D_{8,10}} = \frac{x_{\rho_1}^2 x_{\rho_2} x_{\rho_6} x_{\rho_7} x_{\rho_9}}{x_{\rho_3} x_{\rho_4}}, \quad \text{and} \quad x^{D_{9,10}} = \frac{x_{\rho_3} x_{\rho_4}^2 x_{\rho_5}^2 x_{\rho_7} x_{\rho_8}}{x_{\rho_1} x_{\rho_2}}$$

give the three coordinates of the chart $U_\sigma \cong \mathbb{C}^3$ in terms of the variables of the Cox ring of $X_{\Sigma}$.

![Figure 1.7: The fan $\Sigma$ encoding a crepant resolution of $X$](image)

### 1.4 Quiver representations and their moduli spaces

Let $Q = (Q_0, Q_1, h, t)$ be a quiver. A **representation** $W$ of a quiver $Q$ over $\mathbb{C}$ consists of a collection of complex vector spaces $W_v$, one for each vertex $v \in Q_0$, and a $\mathbb{C}$-linear map $\phi_a : W_{t(a)} \to W_{h(a)}$ for each arrow $a \in Q_1$. More compactly, we write $W = \left( \bigoplus_{v \in Q_0} W_v, \{ \phi_a \}_{a \in Q_1} \right)$. A representation is said to be **finite-dimensional** if for all $v \in Q_0$, $W_v$ is a finite-dimensional vector space. The dimension vector of a finite-dimensional representation is the nonnegative tuple $d = (\dim W_v)_{v \in Q_0}$. For any two representations $W, W'$ of the quiver $Q$, a morphism $g : W \to W'$ consists of a $\mathbb{C}$-linear map $g_v : W_v \to W'_v$ for each vertex $v \in Q_0$, such that for all arrows $a \in Q_1$, the following diagram commutes:

$$
\begin{array}{ccc}
W_{t(a)} & \xrightarrow{\phi_a} & W_{h(a)} \\
\downarrow{g_{t(a)}} & & \downarrow{g_{h(a)}} \\
W'_{t(a)} & \xrightarrow{\phi'_a} & W'_{h(a)}
\end{array}
$$

The morphism $g : W \to W'$ is said to be an **isomorphism** of quiver representations if and only if for all $v \in Q_0$ the map $g_v$ is an isomorphism of vector spaces.
Define $\text{Rep}(Q)$ to be the category whose objects are representations of the quiver $Q$ and morphisms are the morphism between quiver representations. Let $\text{Mod} - \mathbb{C}Q$ denote the category of left $\mathbb{C}Q$-modules. We use lower case to denote the category $\text{rep}(Q)$ of finite-dimensional representations of $Q$ and the category $\text{mod} - \mathbb{C}Q$ of finite-dimensional left $\mathbb{C}Q$-modules. The following is a well-known result:

**Proposition 1.22.** [ASS06, Theorem III 1.6] There is an equivalence of categories $\text{Rep}(Q) \cong \text{Mod} - \mathbb{C}Q$ which restricts to an equivalence $\text{rep}(Q) \cong \text{mod} - \mathbb{C}Q$ between the categories of finite-dimensional representations of $Q$ and finite-dimensional left $\mathbb{C}Q$-modules.

**Proof.** We first define a functor from $\text{Mod} - \mathbb{C}Q$ to $\text{Rep}(Q)$ on objects. Given a left $\mathbb{C}Q$-module $M$ with module structure $\mu : \mathbb{C}Q \times M \to M$, we obtain $\mathbb{C}$-linear maps $\phi_v : M \to M$ satisfying $\phi_v(m) = \mu(e_v, m)$ for all $v \in Q_0$. Set $W_v := \phi_v(M)$. Note that $W_v \cap W'_v = \{0\}$ for $v \neq v'$ as $e_v \cdot e'_v = 0$. Similarly, each $a \in Q_1$ defines a $\mathbb{C}$-linear map $\phi_a : M \to M$ satisfying $\phi_v(m) = \mu(a, m)$, and the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\phi_a} & M \\
\phi_{\varepsilon(a)} & \downarrow & \phi_{\varepsilon'(a)} \\
M & \xrightarrow{\phi_v} & M
\end{array}
$$

commutes, so in fact $\phi_a : W_{\varepsilon(a)} \to W_{\varepsilon'(a)}$. Thus, $M$ determines a quiver representation $W = \left( \bigoplus_{v \in Q_0} W_v, \{ \phi_a \}_{a \in Q_1} \right)$ of $Q$. One can extend this assignment to morphisms, giving the required functor.

Now we define a functor on objects in the opposite direction. Given a representation $W = \left( \bigoplus_{v \in Q_0} W_v, \{ \phi_a \}_{a \in Q_1} \right)$, set $M := \bigoplus_{v \in Q_0} W_v$. For $v \in Q_0$ define first a $\mathbb{C}Q_0$-module structure $\mu : \mathbb{C}Q_0 \times M \to M$ by setting $\mu(e_v, m) = m$ for all $m \in W_v$ and $\mu(e_v, m) = 0$ otherwise. Next, for $a \in Q_1$ and $m \in M$, define $\mu(a, m) = \phi_a(m)$ if $m \in W_{\varepsilon(a)}$ and $\mu(a, m) = 0$ otherwise. Every nontrivial path is a composition of arrows, so we obtain the required module structure $\mu : \mathbb{C}Q \times M \to M$ that makes $M$ into a left $\mathbb{C}Q$-module. Again, this assignment extends to morphisms, and it is easy to see that these operations are inverse.

For the second statement, note that the functors above take finite-dimensional representations of $Q$ to finite-dimensional left $\mathbb{C}Q$-modules, and vice versa. Therefore, the restrictions of the functors to finite-dimensional object give the second equivalence.

A representation of a quiver $Q$ with relations $\mathcal{R}$ is defined to be a representation $W = \left( \bigoplus_{v \in Q_0} W_v, \{ \phi_a \}_{a \in Q_1} \right)$ with the added condition that the linear maps $\{ \phi_a \}_{a \in Q_1}$ satisfy the relations $\mathcal{R}$. Define $\text{Rep}(Q, \mathcal{R})$ to be the category of representations of the quiver $Q$ with relations $\mathcal{R}$. Let $\text{rep}(Q, \mathcal{R})$ denote the category of finite-dimensional representations of the quiver $Q$ with relations $\mathcal{R}$. The equivalences of categories in Proposition 1.22 translate into

$$
\text{Rep}(Q, \mathcal{R}) \cong \text{Mod} - A \quad \text{and} \quad \text{rep}(Q, \mathcal{R}) \cong \text{mod} - A
$$

for $A = \mathbb{C}Q/\mathbb{I}_\mathcal{R}$. 
Following King [Kin94], we now present the construction of the fine moduli space of quiver representations of $Q$ with dimension vector $d$. We first recall some standard definitions and results from Geometric Invariant Theory (GIT) [MFK94].

Let $G$ be a linearly reductive group acting algebraically on an affine variety $X = \text{Spec}(R)$. The set of $G$-orbits of $X$ is $X/G = \{G \cdot x \mid x \in X\}$. For $g \in G$, the morphism $\varphi_g : X \to X$ defining the action of $g$ on $X$ comes from a homomorphism $\varphi^*_g : R \to R$. Therefore, the action of $G$ on $X$ induces an action on $R$ defined for $f \in R$ by $g \cdot f = \varphi^*_{g^{-1}}(f)$. This action determines the ring of invariants $R^G = \{f \in R \mid g \cdot f = f \text{ for all } g \in G\}$. As $G$ is linearly reductive, $R^G$ is finitely generated as a $\mathbb{C}$-algebra and $\text{Spec}(R^G)$ is an affine variety.

**Definition 1.23.** The map $\pi : X \to Y$ is a good categorical quotient of $X$ by $G$ if:

(i) For any open set $V \subset Y$, the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(\pi^{-1}(V))$ induces an isomorphism $\mathcal{O}_Y(V) \cong \mathcal{O}_X(\pi^{-1}(V))^G$;

(ii) For any $G$-invariant closed set $U \subset X$, $\pi(U)$ is closed; and

(iii) For any two disjoint $G$-invariant closed sets $U, U' \subset X$, $\pi(U) \cap \pi(U') = \emptyset$.

We write $\pi : X \to X//G$ for a good categorical quotient of $X$ by $G$. Good categorical quotients satisfy the following properties:

**Proposition 1.24.** Let $\pi : X \to X//G$ be a good categorical quotient of $X$ by $G$. Then:

(i) The map $\pi$ is surjective;

(ii) For any two points $x, y \in X$

$$\pi(x) = \pi(y) \iff \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset,$$

where the overline denotes the closure of the orbits.

**Definition 1.25.** A geometric quotient of $X$ by $G$ is a good categorical quotient $\pi : X \to X//G$ that satisfies the following equivalent conditions:

(i) All $G$-orbits are closed in $X$;

(ii) $X//G$ is a $G$-orbit space, i.e. $\pi$ induces a bijection $\{G - \text{orbits of } X\} \cong X//G$; and

(iii) For any two points $x, y \in X$

$$\pi(x) = \pi(y) \iff x \text{ and } y \text{ lie in the same } G\text{-orbit}.$$

Let $\chi : G \to \mathbb{C}^\times$ be a character of $G$. We say $f \in R$ is a $\chi$-semi-invariant function if for all $x \in X$ and $g \in G$

$$f(g \cdot x) = \chi(g)f(x).$$

Let $R_{\chi}$ denote the complex vector space of all $\chi$-semi-invariant functions.

**Definition 1.26.** The GIT quotient of $X$ by $G$ corresponding to $\chi$ is the semiprojective variety $X//\chi G := \text{Proj}(\bigoplus_{n \geq 0} R_{\chi^n})$. 


Note that when \( \chi = 0 \), \( X/\chi G = \text{Spec}(R^G) \). The GIT quotient \( X/\chi G \) is projective over \( \text{Spec}(R^G) \), the degree zero part of the graded ring.

We define the semistable and stable loci \( X^{ss}(\chi) \) and \( X^s(\chi) \) of \( X \) with respect to \( \chi \) as follows:

\[
X^{ss}(\chi) := \{ x \in X \mid \exists n \geq 1 \text{ and } f \in R_\chi^n \text{ with } f(x) \neq 0 \}
\]
\[
X^s(\chi) := \{ x \in X^{ss}(\chi) \mid G \cdot x \text{ is closed in } X^{ss}(\chi) \text{ and } G_x \text{ is finite} \},
\]
where \( G_x := \{ g \in G \mid g \cdot x = x \} \) is the stabiliser of \( x \) by \( G \).

**Proposition 1.27.** Let \( X/\chi G \) be the GIT quotient of \( X \) by \( G \) corresponding to \( \chi \). Then:

(i) The map \( \pi : X^{ss}(\chi) \to X/\chi G \) is a good categorical quotient of \( X^{ss}(\chi) \) by \( G \); and

(ii) The open subset \( X^s(\chi) \subseteq X^{ss}(\chi) \) gives a geometric quotient \( \pi : X^s(\chi) \to X/G \subset X/\chi G \).

Now, let \( Q \) be a quiver, fix a dimension vector \( \underline{d} := (d_v)_{v \in Q_0} \) and let \( W_v = \mathbb{C}^{d_v} \). The set of all representations of \( Q \) with vector spaces \( W_v \) \((v \in Q_0)\)

\[
\text{Rep}(Q, \underline{d}) = \bigoplus_{a \in Q_1} \text{Hom}(W_{t(a)}, W_{h(a)})
\]

is an affine space. The group \( \mathfrak{G} := \prod_{v \in Q_0} \text{GL}(d_v; \mathbb{C}) \) acts on \( \text{Rep}(Q, \underline{d}) \) by conjugation:

\[
(g_v)_{v \in Q_0} \cdot (\phi_a)_{a \in Q_1} = (g_{h(a)}\phi_ag_{t(a)}^{-1})_a \in Q_1.
\]

\( \text{Rep}(Q, \underline{d}) \) contains an affine subscheme \( \mathbb{V}_R \) consisting of those representations of \( Q \) with dimension vector \( \underline{d} \) that satisfy the relations \( R \).

Consider the rational vector space

\[
\Theta := \left\{ \theta \in \text{Hom}(\mathbb{Z}^{Q_0}; \mathbb{Q}) \mid \sum_{v \in Q_0} \theta_v d_v = 0 \right\}.
\]

For any representation \( W \in \text{Rep}(Q, \underline{d}) \) define \( \theta(W) := \theta(\underline{d}) \). Any \( \theta \in \Theta \) determines a fractional character \( \chi_\theta \) of \( \mathfrak{G} \) by \( \chi_\theta(g) := \prod_{v \in Q_0} \det(g_v)^{\theta_v} \).

**Definition 1.28.** A representation \( W \in \text{Rep}(Q, \underline{d}) \) is said to be \( \theta \)-semistable (resp. \( \theta \)-stable) if and only if \( \theta(W) = 0 \) and \( \theta(W') \geq 0 \) (resp. \( \theta(W') > 0 \)) for every nonzero proper subrepresentation \( W' \subset W \). A parameter \( \theta \in \Theta \) is said to be generic if every \( \theta \)-semistable representation is also \( \theta \)-stable.

**Definition 1.29.** Two \( \theta \)-semistable representations \( W, W' \) of \( Q \) are said to be \( S \)-equivalent if their composition series of \( \theta \)-stable representations have the same composition factors.

King [Kin94] showed that the notion of \( \theta \)-semistability (resp. \( \theta \)-stability) for a representation \( W \) coincides with the notion of \( \chi_\theta \)-semistability (resp. \( \chi_\theta \)-stability) for the corresponding point \([W] \in \text{Rep}(Q, \underline{d})\). Moreover:
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Proposition 1.30. Let \( R \) denote the coordinate ring of the affine scheme \( \mathbb{V}_R \). Then for \( \theta \in \Theta \):

(i) The scheme

\[
\mathcal{M}_A(\theta, d) := \text{Proj} \left( \bigoplus_{n \geq 0} R_{\chi_n^\theta} \right)
\]

is a coarse moduli space of \( \theta \)-semistable representations of \( Q \) with dimension vector \( d \) up to \( S \)-equivalence;

(ii) The scheme \( \mathcal{M}_A(\theta, d) \) is projective over \( \mathbb{V}_R \); and

(iii) When \( \theta \) is generic and \( d \) is primitive, \( \mathcal{M}_A(\theta, d) \) is the fine moduli space of isomorphism classes of \( \theta \)-stable representations of \( Q \) with dimension vector \( d \).

Notation 1.31. We will be interested in the representations of a quiver \( Q \) with relations \( R \) with dimension vector \( 1 \). In light of Proposition 1.22, henceforth we use \( \theta \)-stable \( A \)-modules to refer to \( \theta \)-stable representations of \( Q \) with dimension vector \( 1 \) satisfying the relations \( R \) of \( Q \). When \( \theta \) is generic, we write \( \mathcal{M}_A(\theta) := \mathcal{M}_A(\theta, 1) \) for the fine moduli space of \( \theta \)-stable \( A \)-modules.

As \( \mathcal{M}_A(\theta) \) is the fine moduli space of \( \theta \)-stable \( A \)-modules, it carries a universal family

\[
T = \bigoplus_{v \in Q_0} L_v(\theta)
\]
called the tautological bundle of \( \mathcal{M}_A(\theta) \), and a tautological \( \mathbb{C} \)-algebra homomorphism

\[
\phi : A \to \text{End}_{\mathcal{O}_{\mathcal{M}_A(\theta)}}(T).
\]

Choose a vertex \( 0 \in Q_0 \) and set \( L_0 \cong \mathcal{O}_{\mathcal{M}_A(\theta)} \). For a point \( y \in \mathcal{M}_A(\theta) \), let \( \iota : \{y\} \to \mathcal{M}_A(\theta) \) be its closed immersion. The tautological bundle \( T \) on \( \mathcal{M}_A(\theta) \) pulls back to \( \bigoplus_{v \in Q_0} W_v \), and the maps \( \{L_t(a)(\theta) \to L_h(a)(\theta) \mid a \in Q_1\} \) restrict to maps \( \{\varphi_a : W_t(a) \to W_h(a) \mid a \in Q_1\} \) that determine a \( \theta \)-stable \( A \)-module \( M_y(\theta) \).

Definition 1.32. For any point \( y \in \mathcal{M}_A(\theta) \), define \( M_y(\theta) \) to be the \( \theta \)-stable \( A \)-module parametrised by \( y \), which is the fibre of the tautological bundle \( T \).

Ishii–Ueda studied the moduli space \( \mathcal{M}_A(\theta) \) for the case when \( Q \) is the dual quiver to a nondegenerate dimer model \( \Gamma \). Theorem 1.33 summarises some of their results (see [IU08, Proposition 5.1, Proposition 6.3, Theorem 6.4]).

Theorem 1.33. Let \( \Gamma \) be a nondegenerate dimer model and suppose \( \theta \) is generic. The moduli space \( \mathcal{M}_A(\theta) \) is a crepant resolution of \( X := \text{Spec}(\mathbb{C}[\sigma^\vee \cap M]) \) where \( \sigma \) the cone over the characteristic polygon \( \Delta(\Gamma) \).

Example 1.34. Let \( G \subset \text{SL}(3, \mathbb{C}) \) be a finite abelian subgroup and write \( \Gamma \) for its corresponding dimer model (see Example 1.7). Recall that \( Q_0 = \text{Irr}(G) \) and let \( v_0 \in Q_0 \) denote the trivial representation. For any stability parameter \( \theta = (\theta_v) \in \Theta \) satisfying \( \theta_v > 0 \) for \( v \neq v_0 \), the moduli space \( \mathcal{M}_A(\theta) \) coincides with the \( G \)-Hilbert scheme introduced by Nakamura [Nak01].
Example 1.35. Recall the dimer model $\Gamma$ introduced in Example 1.13. The parameter $\theta = (-9, 1, 1, \ldots, 1)$ is generic (see Lemma 3.2) and the fine moduli space of $\theta$-stable $A$-modules is the three-dimensional toric variety encoded by the fan $\Sigma$ in Figure 1.7.

1.5 Zigzag paths and the consistency condition

We now introduce the notion of a zigzag path which is key in the definition of a consistent dimer model.

Definition 1.36. A zigzag path on a bicoloured graph $(B, W, E)$ in an oriented surface is a path which turns maximally right at white nodes and maximally left at black nodes.

Zigzag paths are either periodic or infinite. We parametrise a zigzag path $z$ in an oriented surface by a map $\gamma_z : \mathbb{Z} \to \mathcal{E}$ satisfying the following two conditions:

(i) For all $i \in \mathbb{Z}$, $h(\gamma_z(i)) = t(\gamma_z(i + 1));$

(ii) The path $z$ turns maximally right at $h(\gamma_z(2i))$ and maximally left at $h(\gamma_z(2i + 1))$.

The following definition is independent of the choice of parametrisation $\gamma_z$ as defined above.

Definition 1.37. An edge $e$ of a zigzag path $z$ is called a zig (resp. zag) of $z$ if $e$ is the image of an even (resp. odd) integer under the map $\gamma_z$.

Any edge $e$ of a bicoloured graph in an oriented surface determines at most two zigzag paths, depending on whether $e$ is a zig or a zag edge. We denote these paths by $\text{zig}(e)$ and $\text{zag}(e)$, respectively.

Two infinite zigzag paths in a bicoloured graph $(B, W, E)$ are said to intersect if and only if they share an odd number of consecutive edges in $\mathcal{E}$. Suppose two infinite zigzag paths $z$ and $z'$ intersect each other twice and let $e$ and $e'$ be edges in the two different intersections. We say the pair $z$ and $z'$ intersect each other more than once in the same direction if and only if either $\gamma_z(e) < \gamma_z(e')$ and $\gamma_z'(e) < \gamma_z'(e')$ or $\gamma_z(e) > \gamma_z(e')$ and $\gamma_z'(e) > \gamma_z'(e')$. Two zigzag paths are said to share a common node $n$ if and only if they both pass through the node $n$.

Remark 1.38. If a node of a dimer model is bivalent and its edges have only it as a common node, it is possible to remove the bivalent node to produce a new dimer model. When $\Gamma$ is a nondegenerate dimer model, any bivalent node must be connected to two distinct nodes, as otherwise $\Gamma$ would not satisfy the strong marriage condition introduced in Remark 1.10. Moreover, if $\Gamma'$ is a dimer model obtained from a nondegenerate dimer model $\Gamma$ by removing its bivalent nodes, the Jacobian algebra of the quiver dual to $\Gamma$ is isomorphic to the Jacobian algebra of the quiver dual to $\Gamma'$. There also exists a natural bijection between the set of zigzag paths of $\Gamma$ and $\Gamma'$ (see [Bro12, Section 2.1.5] and [IU09, Section 5.1]). Note that if the bicoloured graph has no bivalent nodes, two zigzag paths intersect if and only if they share an edge.

Assumption 1.39. In light of Remark 1.38, we will assume throughout this thesis that every nondegenerate dimer model has no bivalent nodes.
Let \( \mathfrak{z} \) be a zigzag path in \( \Gamma \). Since \( \Gamma \) is a bicoloured graph embedded in the torus \( \mathbb{T} \), zigzag paths are periodic.

**Definition 1.40.** The *slope* of a zigzag path is its homology class \([\mathfrak{z}] \in H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2\).

The set of slopes \((u, v) \in \mathbb{Z}^2\) of zigzag paths of \( \Gamma \) which are not homologically trivial has a natural cyclic order when thought of as elements of the unit circle

\[
\frac{(u, v)}{\sqrt{u^2 + v^2}} \in S^1.
\]

Similarly, every node in \( \Gamma \) has a natural cyclic order given by the directions of the outgoing paths passing through the node. The following consistency condition for a dimer model, first introduced by Gulotta [Gul08], requires both these natural cyclic orders to coincide:

**Definition 1.41.** A dimer model \( \Gamma \) is said to be *properly-ordered* if

1. no zigzag path of \( \Gamma \) is homologically trivial;
2. no zigzag path of \( \Gamma \) intersects itself on the universal cover;
3. two zigzag paths of \( \Gamma \) with the same homology class do not share a common node; and
4. for every node of \( \Gamma \), the natural cyclic order of the zigzag paths passing through it coincides with the natural cyclic order determined by their slopes.

Ishii–Ueda [IU11b, Definition 3.5] introduced the following consistency condition for dimer models:

**Definition 1.42.** A dimer model \( \Gamma \) is said to be *consistent* if

1. no zigzag path of \( \Gamma \) is homologically trivial;
2. no zigzag path of \( \Gamma \) intersects itself on the universal cover; and
3. no pair of zigzag paths on the universal cover intersect each other more than once in the same direction.

Ishii–Ueda’s consistency condition is equivalent to Gulotta’s properly-ordered condition:

**Proposition 1.43.** [IU11b, Lemmata 4.2, 4.3] A dimer model \( \Gamma \) is properly-ordered if and only if \( \Gamma \) is consistent.

**Remark 1.44.** There are several notions of consistency for dimer models in the literature. Broomhead [Bro12, Definition 5.12] introduced a notion of algebraic consistency, which by results of Bocklandt [Boc12] and Ishii–Ueda [IU11b] is equivalent to Ishii–Ueda’s consistency condition. We shall not define Broomhead’s notion of algebraic consistency as we do not use it except in Proposition 1.47 below.
Let \( \Gamma \) be a dimer model and let \( \{z_1, z_2, \ldots, z_k\} \) be the set of all zigzag paths in \( \Gamma \). We can assume that the zigzag paths \( z_1, z_2, \ldots, z_k \) are ordered cyclically anticlockwise starting from a randomly chosen zigzag path \( z_1 \). We now define a sequence \( (b_i)_{i=1}^k \) of elements of \( \mathbb{Z}^2 \) by setting \( b_0 = (0, 0) \) and \( b_{i+1} = b_i + [z_{i+1}]' \) (\( 0 \leq i \leq k - 1 \)) where \([z_{i+1}]'\) denotes the vector normal to \([z_{i+1}]\) obtained by rotating \([z_{i+1}]\) by 90 degrees in the positive direction. If every edge in \( \Gamma \) appears in two distinct zigzag paths, once as a zig and once as a zag, we have \( b_k = (0, 0) \) and we can define the zigzag polygon of \( \Gamma \) as follows.

**Definition 1.45.** The zigzag polygon is the convex hull of the set of points \( \{b_1, b_2, \ldots, b_k\} \).

The following is a result of Gulotta [Gul08, Theorem 3.3] and Ishii–Ueda [IU09, Theorem 11.1]:

**Theorem 1.46.** For a consistent dimer model \( \Gamma \), the characteristic polygon of \( \Gamma \) coincides with the zigzag polygon of \( \Gamma \) up to translation.

Consistency is a stronger condition for a dimer model than nondegeneracy. Indeed, Broomhead [Bro12, Definition 3.2] defines consistency for a dimer model as a strengthening of the nondegeneracy condition. Ishii–Ueda [IU09, Proposition 7.1] also showed that a consistent dimer model is non-degenerate. Broomhead [Bro12] proved the following result for consistent dimer models:

**Proposition 1.47.** Let \( \Gamma \) be a consistent dimer model with dual quiver \( Q \). The centre \( Z(A) \) of the Jacobian algebra \( A \) of \( Q \) is isomorphic to the Gorenstein semigroup algebra \( \mathbb{C}[\sigma \cap \Delta] \), where \( \sigma \) is the cone over \( \Delta(\Gamma) \). In particular, \( X := \text{Spec}(Z(A)) \) is a Gorenstein affine toric threefold.

For \( \theta \) generic, Craw–Quintero-Vélez [CQV12] give a GIT construction of a toric subvariety \( Y_\theta \) of \( \mathcal{M}_A(\theta, \underline{1}) \) which they call the coherent component of \( \mathcal{M}_A(\theta, \underline{1}) \). The subvariety \( Y_\theta \) admits a birational morphism \( f : Y_\theta \to X := \text{Spec}(Z(A)) \) obtained by variation of GIT [CQV12, Proposition 2.14]. This result strengthens Theorem 1.33:

**Theorem 1.48.** Let \( \Gamma \) be a consistent dimer model with \( \theta \) generic. The moduli space \( \mathcal{M}_A(\theta) \) is a toric variety that gives a crepant resolution \( f : \mathcal{M}_A(\theta) \to X \) of the Gorenstein affine toric threefold \( X := \text{Spec}(Z(A)) \).

Consistency is also enough to show an equivalence between the bounded derived category \( D^b(\text{coh}(\mathcal{M}_A(\theta))) \) of coherent sheaves on \( \mathcal{M}_A(\theta) \) and the bounded derived category \( D^b(\text{mod} - A) \) of finite-dimensional \( A \)-modules:

**Theorem 1.49.** [IU09] Let \( \Gamma \) be a consistent dimer model with \( \theta \) generic and let \( T = \bigoplus_{v \in Q_0} L_v \) be the universal family of \( A \)-modules on \( \mathcal{M}_A(\theta) \). Then:

1. The tautological \( \mathbb{C} \)-algebra homomorphism \( \phi : A \to \text{End}_{\mathcal{O}_{\mathcal{M}_A(\theta)}}(T) \) is an isomorphism; and

2. The tautological bundle \( T \) is tilting. In particular, the functors

\[
\begin{align*}
R\text{Hom}_{\mathcal{O}_{\mathcal{M}_A(\theta)}}(T, -) : D^b(\text{coh}(\mathcal{M}_A(\theta))) &\to D^b(\text{mod} - A) \\
- \otimes_A T : D^b(\text{mod} - A) &\to D^b(\text{coh}(\mathcal{M}_A(\theta)))
\end{align*}
\]
Remark 1.50. Ishii–Ueda [IU13, Section 7] remark that the functor $R\text{Hom}_{\mathcal{M}\mathcal{A}(\theta)}(T, -)$ is an equivalence of categories if and only if the functor

$$\Phi(-) = R\Gamma(T \otimes -): D^b(\text{coh}(\mathcal{M}\mathcal{A}(\theta))) \to D^b(\text{mod} - A)$$

is an equivalence too. The functor $\Phi$ sends the structure sheaf $\mathcal{O}_y$ of a point $y \in \mathcal{M}\mathcal{A}(\theta)$ to the $A$-module $M_y$ introduced in Definition 1.32.

Definition 1.51. An $A$-module $M$ is nilpotent if there exists $n \in \mathbb{N}$ such that any path of length greater than $n$ acts on $M$ by 0. We write $\text{mod} - A$ for the full subcategory of $\text{mod} - A$ consisting of all nilpotent modules. Following Ishii–Ueda [IU13, Section 7], the functor $\Phi$ from Remark 1.50 induces a functor $\Phi_0$ from $\text{coh}(\mathcal{M}\mathcal{A}(\theta))$, the full subcategory of $\text{coh}(\mathcal{M}\mathcal{A}(\theta))$ consisting of coherent sheaves supported on the fibre over the unique-torus invariant point $x_0$ of $X$, to $\text{mod} - A$.

The tautological $\mathbb{C}$-algebra isomorphism $\phi: A \to \text{End}_{\mathcal{O}\mathcal{M}\mathcal{A}(\theta)}(T)$ from Theorem 1.49 associates to each path $p$ in $Q$ a torus-invariant section

$$\phi(p) = x^{\text{div}(p)}$$

of the line bundle $L_h(p) \otimes L_t(p)^{-1}$, where the label (or labelling divisor) on $p$ is the torus-invariant divisor $\text{div}(p) = \sum_{a \in \text{supp}(p)} \text{div}(a) \in \mathbb{N}\Sigma(1)$ on $\mathcal{M}\mathcal{A}(\theta)$. Since $L_0 \cong \mathcal{O}_{\mathcal{M}\mathcal{A}(\theta)}$ we have the following result:

Lemma 1.52. The tautological bundle $T = \bigoplus_{v \in Q_0} L_v$ satisfies $L_v \cong \mathcal{O}_{\mathcal{M}\mathcal{A}(\theta)}(\text{div}(p))$ for each vertex $v \in Q_0$, where $p$ is any path from the vertex 0 to the vertex $v$.

The main result of Bender–Mozgovoy [BM09, Theorem 4.2] presents the following formula for the labelling divisors in terms of the torus-invariant prime divisors in $\mathcal{M}\mathcal{A}(\theta)$:

$$x^{\text{div}(a)} = \prod_{\{\rho \in \Sigma(1) | e_\rho \in \Pi_i\}} x_\rho.$$  \hfill (1.5.1)

Example 1.53. Consider the dimer model $\Gamma$ introduced in Example 1.13. Figure 1.5 shows all $\theta$-stable perfect matchings $\Pi_i := \Pi_{\rho_i}$ ($0 \leq i \leq 10$) of $\Gamma$ for $\theta = (-9, 1, 1, \ldots, 1)$. The quiver $Q$ dual to $\Gamma$ has 28 arrows. By equation (1.5.1), a divisor $D_i := D_{\rho_i}$ appears in the label $\text{div}(a)$ of an arrow $a \in Q_1$ if its dual edge $e_\rho$ lies in its corresponding perfect matching $\Pi_i$. For example, the label of the arrow $a_1$ with tail at vertex 0 and head at vertex 4 is $\text{div}(a_1) = D_1 + D_2 + D_6$, as its dual edge appears in $\Pi_1, \Pi_2$ and $\Pi_6$. It is a simple exercise using Figure 1.5 to verify that the labels of the arrows of $Q$ are those presented in Figure 1.8, where an arrow marked $i_1i_2 \cdots i_j$ has labelling divisor $D_i_1 + D_i_2 + \cdots + D_i_j$.

1.6 Reid’s recipe for McKay quivers

Reid [Rei97] computed $G$-$\text{Hilb}(\mathbb{C}^3)$ for several examples of $G \subset \text{SL}(3, \mathbb{C})$ and introduced a recipe which marks the interior lattice points and interior line segments of the toric
fan $\Sigma$ of $G$-$\text{Hilb}(\mathbb{C}^3)$ with irreducible representations of $G$. This decoration encodes the relations between the line bundles in the Picard group of $G$-$\text{Hilb}(\mathbb{C}^3)$. Craw [Cra05] proved that Reid’s recipe can be done for any finite abelian subgroup $G \subset SL(3, \mathbb{C})$. The proof uses a deep understanding of the crepant resolution $G$-$\text{Hilb}(\mathbb{C}^3)$ of $X := \mathbb{C}^3/G$, in particular knowledge of the Craw–Reid algorithm [CR02] which gives an explicit construction of the fan $\Sigma$ of $G$-$\text{Hilb}(\mathbb{C}^3)$. We now describe the marking of $\Sigma$ by Reid’s recipe.

To every line segment $\tau$ of $\Sigma$ we can associate a $G$-invariant ratio of monomials in the Cox ring $S = \mathbb{C}[x, y, z]$ of $X$, in a similar way as in Example 1.21. Both monomials in each ratio lie in the same character space of the $G$-action on $S$. Mark every line segment $\tau$ with the character $\varrho$ if and only if the monomials associated to $\tau$ lie in the $\varrho$ character space. The marking of the interior lattice points is done case by case depending on the valency of $\rho \in \Sigma(1)$. Note that the valency of $\rho$ can only be 3, 4, 5 or 6 [CR02, Section 1.3]. The marking of lattice points is defined as follows:

(i) If the valency of $\rho \in \Sigma(1)$ is 3, a single character $\varrho$ marks all three line segments with endpoint $\rho$ and we mark the lattice point $\rho$ with the character $\chi = \varrho \varrho$;

(ii) If the valency of $\rho \in \Sigma(1)$ is 4, two distinct characters $\varrho_1$ and $\varrho_2$ mark a pair of line segments with endpoint $\rho$. Mark $\rho$ with the character $\chi = \varrho_1 \varrho_2$;

(iii) If the valency of $\rho \in \Sigma(1)$ is 5, two distinct characters $\varrho_1$ and $\varrho_2$ mark a pair of line segments with endpoint $\varrho$. The fifth line segments is marked with a third distinct character of $G$. Mark $\rho$ with $\chi = \varrho_1 \varrho_2$;

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{quiver.png}
\caption{The quiver $Q$ dual to the dimer model $\Gamma$ from Figure 1.3 with labels}
\end{figure}
(iv) If the valency of $\rho \in \Sigma(1)$ is 6 either

(a) Two distinct characters $\varrho_1$ and $\varrho_2$ mark a pair of line segments with endpoint $\rho$ and the remaining two line segments are marked by another two different characters of $G$. In this case, we mark $\rho$ with the character $\chi = \varrho_1 \varrho_2$; or

(b) Three distinct characters $\varrho_1, \varrho_2$ and $\varrho_3$ mark a pair of line segments each with endpoint $\rho$ and we mark $\rho$ with the characters $\chi_1$ and $\chi_2$ of $G$ such that $\chi_1 \chi_2 = \varrho_1 \varrho_2 \varrho_3$, as defined in [Cra05, Lemma 3.4].

**Theorem 1.54.** [Cra05, Corollary 4.6] Every nontrivial character of $G$ appears once on $\Sigma$ as:

(i) a character $\chi$ marking a lattice point $\rho \in \Sigma(1)$; or

(ii) a character $\varrho$ marking a line segment $\tau \in \Sigma(2)$, or possibly several line segments each sharing an endpoint with another line segment marked with $\varrho$.

Recall that the McKay quiver $Q$ of $G$ is dual to a hexagonal dimer model $\Gamma$ in $T$. The Jacobian algebra $A$ of $Q$ is the skew group algebra $\mathbb{C}[x,y,z] \ast G$. The quiver $Q$ has a vertex for each irreducible representation of $G$. Let $v_0 \in Q_0$ denote the trivial representation of $G$. For any stability parameter $\theta = (\theta_v) \in \Theta$ satisfying $\theta_v > 0$ for all $v \neq v_0$, the moduli space $\mathcal{M}_A(\theta)$ is the $G$-Hilbert scheme. The tautological bundle $T$ of $\mathcal{M}_A(\theta)$ decomposes into the direct sum of line bundles, one for each irreducible representation of $G$. Let $\Sigma$ be the toric fan of $\mathcal{M}_A(\theta)$. The set of relations in $\text{Pic}(\mathcal{M}_A(\theta))$ between the line bundles can be stated using Reid’s recipe:

**Theorem 1.55.** [Cra05, Theorem 6.1] The following relations hold in $\text{Pic}(X_\Sigma)$.

(i) $L_\chi = L_\varrho \otimes L_\varrho$ when $\chi = \varrho \varrho$ marks a trivalent lattice point $\rho \in \Sigma(1)$;

(ii) $L_\chi = L_{\varrho_1} \otimes L_{\varrho_2}$ when $\chi = \varrho_1 \varrho_2$ is the only character marking a lattice point $\rho \in \Sigma(1)$ of valency 4, 5 or 6;

(iii) $L_{\chi_1} \otimes L_{\chi_2} = L_{\varrho_1} \otimes L_{\varrho_2} \otimes L_{\varrho_3}$ when $\chi_1$ and $\chi_2$ satisfying $\chi_1 \chi_2 = \varrho_1 \varrho_2 \varrho_3$ mark a lattice point $\rho \in \Sigma(1)$ of valency 6.

Moreover, these relations generate all other relations between the tautological line bundles in $\text{Pic}(X_\Sigma)$.

Reid [Rei97] and Craw [Cra05] computed $G$-Hilb($\mathbb{C}^3$) and Reid’s recipe for several large examples of finite abelian subgroups $G \subset \text{SL}(3, \mathbb{C})$. We now present a small example to illustrate the recipe and theorems above.

**Example 1.56.** Let $G$ be the finite abelian subgroup of $\text{SL}(3, \mathbb{C})$ generated by the diagonal matrix $\text{diag}(\varepsilon, \varepsilon^2, \varepsilon^3)$, where $\varepsilon$ is a primitive sixth root of unity. Figure 1.9 shows the toric fan of $G$-Hilb($\mathbb{C}^3$) decorated using Reid’s recipe. Note that every nontrivial character of $G$ appears once as described in Theorem 1.54. The only nontrivial relation in $\text{Pic}(G$-Hilb($\mathbb{C}^3$)) between its tautological line bundles comes from the interior lattice point of its toric fan $L_5 = L_2 \otimes L_3$. 

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Figure 1.9: The fan of $G$-Hilb($\mathbb{C}^3$) decorated using Reid’s recipe
The aim of this chapter is to study fundamental hexagons for nondegenerate dimer models, and to provide a dimer model analogue of Nakamura’s ‘$G$-jigsaw transformations’ of monomial $G$-clusters for any finite abelian subgroup $G \subset \text{SL}(3, \mathbb{C})$.

For a nondegenerate dimer model $\Gamma$ with Jacobian algebra $A$, and for any generic stability parameter $\theta$, Ishii-Ueda [IU08] associated a fundamental domain - a ‘fundamental hexagon’ - of the universal cover of $\Gamma$ to every torus-invariant $\theta$-stable $A$-module, and hence computed the toric coordinate charts on the fine moduli space $M_A(\theta)$. For any pair of adjacent 3-dimensional cones $\sigma_{\pm} \in \Sigma(\theta)$ in the toric fan defining $M_A(\theta)$, we show how to pass between the fundamental hexagons for the torus-invariant $A$-modules $M_{\sigma_{\pm}}$ in a manner that generalises $G$-jigsaw transformations for the $G$-Hilbert scheme.

### 2.1 The fundamental hexagon

Let $\Gamma = \{\mathfrak{B}, \mathfrak{W}, \mathfrak{E}\}$ be a nondegenerate dimer model in $\mathbb{T}$ and let $Q = \{Q_0, Q_1, h, t, R\}$ be its dual quiver with relations. Let $A$ be the Jacobian algebra of $\Gamma$ and write $\Delta(\Gamma)$ for the characteristic polygon of $\Gamma$. Recall from Chapter 1 that:

(i) the cone $\sigma$ over the characteristic polygon $\Delta(\Gamma)$ defines an affine toric variety $X = \text{Spec}(\mathbb{C}[\sigma^{\vee} \cap M])$; and

(ii) for any generic $\theta \in \Theta$, the fine moduli space $M_A(\theta)$ of $\theta$-stable $A$-modules provides a crepant resolution $M_A(\theta) \rightarrow X$.

Throughout this chapter, let $\theta \in \Theta$ be generic and let $\Sigma := \Sigma(\theta) \subseteq \mathbb{N}_R$ denote the fan of the toric variety $M_A(\theta)$.

**Definition 2.1.** Let $M = \left( \bigoplus_{v \in Q_0} W_v, \{\phi_a\}_{a \in Q_1} \right)$ be a representation of $Q$. The co-support of $M$ is the subset

$$\text{cosupp}(M) := \{e_a \mid \phi_a = 0\} \subset \mathfrak{E}$$

of edges of $\Gamma$. 


Let $\rho \in \Sigma(1)$ be a ray and let $O_\rho \subset \mathcal{M}_A(\theta)$ denote the corresponding two-dimensional open orbit of the action by the dense torus. For any point $y \in O_\rho$, define

$$\Pi_\rho(\theta) := \text{cosupp}(M_y(\theta)) \subset \mathfrak{C},$$

where $M_y(\theta)$ is the $\theta$-stable $A$-module associated to the point $y \in \mathcal{M}_A(\theta)$ by Definition 1.32. The subset $\Pi_\rho(\theta) \subseteq \mathfrak{C}$ does not depend on the choice of point $y \in O_\rho$, and it is a perfect matching by Ishii–Ueda [IU08, Lemma 6.1]. We call any such perfect matching a $\theta$-stable perfect matching. We now use these perfect matchings to associate a $\theta$-stable $A$-module to each cone of $\Sigma$. For $\sigma \in \Sigma$, write $\sigma(1) := \{\rho \in \Sigma(1) \mid \rho \subseteq \sigma\}$, where the zero cone $\sigma = \{0\}$ satisfies $\sigma(1) = \emptyset$.

**Definition 2.2.** Let $\theta \in \Theta$ be generic and let $\Sigma$ denote the fan of $\mathcal{M}_A(\theta)$. For any cone $\sigma \in \Sigma$, the $\theta$-stable $A$-module associated to $\sigma$ is $M_\sigma(\theta) = (\oplus_{v \in Q_0} \mathbb{C}, \{\psi_a\}_{a \in Q_1})$, where

$$\psi_a := \begin{cases} 0 & \text{if } \mathfrak{e}_a \in \bigcup_{\rho \in \sigma(1)} \Pi_\rho(\theta) \\ 1 & \text{otherwise} \end{cases}.$$

The following result establishes in particular that $M_\sigma(\theta)$ is a $\theta$-stable $A$-module for every generic $\theta \in \Theta$ and for every cone $\sigma \in \Sigma$.

**Lemma 2.3.** Let $\theta \in \Theta$ be generic and let $\Sigma$ denote the fan of $\mathcal{M}_A(\theta)$. For all $\sigma \in \Sigma$ there exists a unique point $y \in O_\sigma \subseteq \mathcal{M}_A(\theta)$ such that $M_\sigma(\theta) = M_y(\theta)$.

**Proof.** This is a restatement of results from Ishii–Ueda [IU08] and Mozgovoy [Moz09], so we only sketch the proof. For $\sigma \in \Sigma$, we claim that the point $[(\psi_a)] \in \mathcal{M}_A(\theta)$ lies in $O_\sigma$. The proof is case-by-case according to the dimension of $\sigma$ as follows. For $0 \in \Sigma$, we have $\psi_a = 1$ for all $a \in Q_1$, so $(\psi_a) \in \mathcal{M}_A(\theta)$ in $O_\sigma$. Every point in $(\mathbb{C}^\times)^{Q_1}$ is $\theta$-stable for all $\theta \in \Theta$, so $y := [(\psi_a)]$ satisfies $y \in (\mathcal{M}_A(\theta))/\mathcal{M}_A(\theta) = O_\sigma \subset \mathcal{M}_A(\theta)$ as required. For $\rho \in \Sigma(1)$ and $\tau \in \Sigma(2)$, the result is due to Ishii–Ueda [IU09, Lemma 6.2] and Mozgovoy [Moz09, Corollary 4.19] respectively. For $\sigma \in \Sigma(3)$, the statement is [IU08, Lemma 4.5] (see also [Moz09, Corollary 4.18]).

**Remark 2.4.** If $\rho$ is an interior lattice point of $\Sigma$, the $A$-module $M_\rho$ is nilpotent. Indeed, as $\rho$ is interior, the point $y \in \mathcal{M}_A(\theta)$ such that $M_y = M_\rho$ lies in $f^{-1}(x_0)$, where $f : \mathcal{M}_A \to X$ is the crepant resolution of $X$ and $x_0$ is the unique torus-invariant point of $X$. Remark 1.50 introduces the functor $\Phi : D^b(\text{coh}(\mathcal{M}_A(\theta))) \to D^b(\text{mod} - A)$ that takes the structure sheaf $\mathcal{O}_y$ of $y$ to the $A$-module $M_y$. Since $y$ lies over $x_0$, by Definition 1.51 $\mathcal{O}_y \subset \text{coh}(\mathcal{M}_A(\theta))$ and the functor $\Phi$ induced by $\Phi$ takes $\mathcal{O}_y$ to $M_y \in \text{mod} - A$, i.e. $M_y$ is nilpotent.

Lemma 2.3 relies on the description of the coordinates on the affine toric chart $U_\sigma \subseteq \mathcal{M}_A(\theta)$ by Ishii–Ueda [IU08, Lemma 4.5]. We now recall some of the combinatorics from their construction. For $\sigma \in \Sigma(3)$, consider the $\theta$-stable $A$-module $M_\sigma(\theta)$ and let $Q_\sigma := Q_{M_\sigma}$ denote the subquiver that supports $M_\sigma(\theta)$. The universal cover $\pi : \mathbb{R}^2 \to \mathbb{T}$ of the two-dimensional torus lifts $Q$ to a $\mathbb{Z}^2$-periodic infinite quiver $\tilde{Q}$ in $\mathbb{R}^2$. Similarly, $\Gamma$ is lifted to regular cell decomposition $\tilde{\Gamma}$ of $\mathbb{R}^2$, in which the 2-cells are polygonal tiles,
each corresponding to a unique vertex \( v \) of \( \widetilde{Q} \). Fix a vertex \( v_0 \in Q_0 \) and lift it to a vertex \( \widetilde{v}_0 \in \widetilde{Q}_0 \) with \( \pi(\widetilde{v}_0) = v_0 \in Q_0 \). Since \( M_\sigma(\theta) \) is torus-invariant, for any path \( p \) in \( Q_\sigma \) from \( v_0 \) to a vertex \( v \) in \( Q_\sigma \), the end point of its lift \( \widetilde{p} \) does not depend on the choice of \( p \). Thus, the map \( \pi \) lifts \( Q_\sigma \) to a subquiver \( \widetilde{Q}_\sigma \) of \( \widetilde{Q} \) such that \( \pi: \widetilde{Q}_\sigma \to Q_\sigma \) is an isomorphism [IU08, Lemma 4.1].

**Definition 2.5.** For \( \sigma \in \Sigma(3) \), the fundamental hexagon \( \text{Hex}(\sigma) \) is the subset of \( \mathbb{R}^2 \) covered by the tiles corresponding to vertices of \( Q_\sigma \). The honeycomb tiling of \( \mathbb{R}^2 \) induced by \( \sigma \) is the tiling of \( \mathbb{R}^2 \) given by the union of the \( \mathbb{Z}^2 \)-translates of \( \text{Hex}(\sigma) \).

Consider the \( \mathbb{Z}^2 \)-periodic graph of \( \sigma \)

\[
\text{Graph}(\sigma) := \bigcup_{n \in \mathbb{Z}^2} (\partial \text{Hex}(\sigma) + n)
\]

obtained as the union of the \( \mathbb{Z}^2 \)-translates of the boundary of \( \text{Hex}(\sigma) \), and let \( \text{supp}(\text{Graph}(\sigma)) \) denote the set of edges of \( \Gamma \) that lie in \( \text{Graph}(\sigma) \).

**Remarks 2.6.**

1. Both \( \widetilde{Q}_\sigma \) and \( \text{Hex}(\sigma) \) are defined uniquely up to the choice of the lift \( \widetilde{v}_0 \in \widetilde{Q} \) of the vertex \( v_0 \), i.e., up to translations by \( \mathbb{Z}^2 \).

2. Since \( \widetilde{Q}_\sigma \) is isomorphic to the subquiver \( Q_\sigma \) that supports the \( \theta \)-stable \( A \)-module \( M_\sigma(\theta) \), the tiles in \( \text{Hex}(\sigma) \) correspond one-to-one with vertices of \( Q_0 \). In particular, \( \partial \text{Hex}(\sigma) \) bounds a fundamental domain of \( \tilde{\Gamma} \).

3. Since \( \text{Graph}(\sigma) \) is \( \mathbb{Z}^2 \)-periodic in \( \mathbb{R}^2 \), we may regard it as a subset of the torus \( T \), in which case \( \text{supp}(\text{Graph}(\sigma)) \) becomes a subset of the set of edges \( E \) of \( \Gamma \). This identification gives

\[
\text{supp}(\text{Graph}(\sigma)) \subseteq \text{cosupp}(M_\sigma(\theta)). \tag{2.1.1}
\]

Moreover, since \( \partial \text{Hex}(\sigma) \subseteq \text{Graph}(\sigma) \), we may regard the edges in the boundary of \( \text{Hex}(\sigma) \) as a subset of \( \text{cosupp}(M_\sigma) \) which is itself a subset of the torus \( T \). In particular, whenever we work with a set of edges in \( \tilde{\Gamma} \) that lie in \( \partial \text{Hex}(\sigma) \) for some \( \sigma \in \Sigma(3) \), we may regard the edges as a subset of the set of edges \( E \) of \( \Gamma \).

When combined with Remarks 2.6(2) above, the next key result of Ishii–Ueda [IU08, Lemma 4.4] and the discussion that follows it justifies the terminology of Definition 2.5.

**Lemma 2.7.** For \( \sigma \in \Sigma(3) \), there are six trivalent nodes of \( \text{Graph}(\sigma) \) lying on \( \partial \text{Hex}(\sigma) \). All other nodes of \( \text{Graph}(\sigma) \) have valency two.

Each trivalent node in \( \text{Graph}(\sigma) \) corresponds to a corner of three hexagons in the honeycomb tiling of \( \mathbb{R}^2 \), while the bivalent nodes arise along the sides of the hexagon. Ishii–Ueda [IU08, Lemma 4.5] showed that if \( n_1, n_2, \ldots, n_6 \) are the six trivalent nodes of \( \text{Graph}(\sigma) \) lying clockwise around \( \partial \text{Hex}(\sigma) \), then the nodes \( n_1, n_3, \) and \( n_5 \) lie in one \( \mathbb{Z}^2 \)-orbit in \( \mathbb{R}^2 \) and \( n_2, n_4 \) and \( n_6 \) lie in a different one. One of these orbits consists of white nodes of \( \tilde{\Gamma} \), and the other of black nodes. Hence, the boundary of \( \text{Hex}(\sigma) \) is made up of three distinct sides, each appearing twice; opposite sides of the hexagon are identified under \( \pi \). See Figure 2.1 for a schematic picture of a fundamental hexagon in part of the honeycomb tiling.
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**Definition 2.8.** An edge $e \in \text{cosupp}(M_\sigma(\theta))$ is a boundary edge of $\text{Hex}(\sigma)$ if $e \in \text{supp}(\text{Graph}(\sigma))$, and otherwise it is an interior edge of $\text{Hex}(\sigma)$.

**Example 2.9.** Consider the cone $\sigma \in \Sigma(3)$ generated by the rays $\rho_8, \rho_9, \rho_{10}$ in the toric diagram of the crepant resolution from Example 1.35. Figure 2.2 shows the lifts of both the quiver $Q_\sigma$ that supports $M_\sigma(\theta)$ and the cosupport of $M_\sigma(\theta)$ to the universal cover. The green dashed arrows in Figure 2.2 represent a choice of the quiver $\tilde{Q}_\sigma$, while the red dashed edges of $\tilde{\Gamma}$ show all edges in the boundary of a choice of $\text{Hex}(\sigma)$. Note that $\text{Graph}(\sigma)$ provides all boundary edges in the honeycomb tiling for $\sigma$, and it is characterised as the unique connected component of the subset of dashed and solid edges in Figure 2.2 consisting of more than a single edge.

Figure 2.1: Schematic of a fundamental hexagon

Figure 2.2: The honeycomb tiling and fundamental hexagon for $\sigma$
2.2 Ishii–Ueda’s local coordinates and perfect matchings

We now summarise results from Ishii-Ueda [IU08] to explain how the fundamental hexagon \( \text{Hex}(\sigma) \) can be used to describe the coordinates of the affine chart \( U_\sigma \cong \mathbb{C}^3 \).

For \( \sigma \in \Sigma(3) \), write \( M_\sigma(\theta) = \left( \bigoplus_{e \in Q_0} \mathbb{C}, \{ \psi_a \}_{a \in Q_1} \right) \). The affine chart \( U_\sigma \subseteq \mathcal{M}_A(\theta) \) parametrises the set of all \( \theta \)-stable \( A \)-modules \( \left( \bigoplus_{e \in Q_0} \mathbb{C}, \{ \phi_a \}_{a \in Q_1} \right) \) satisfying \( \phi_a = 1 \) for arrows \( a \in Q_1 \) with \( \psi_a = 1 \) [IU08, Lemma 4.3]. It follows that for \( y \in U_\sigma \), the \( \theta \)-stable \( A \)-module \( M_y(\theta) := \left( \bigoplus_{e \in Q_0} \mathbb{C}, \{ \varphi_a \}_{a \in Q_1} \right) \) is determined by the maps \( \varphi_a \) corresponding to arrows \( a \in Q_1 \) with \( \psi_a = 0 \).

To investigate the values of these maps, let \( n_1, n_2, \ldots, n_6 \) label the trivalent nodes around the boundary of \( \text{Hex}(\sigma) \) in clockwise order. Let \( e_0, e_1, \ldots, e_5 \) be the edges around \( n_1 \) and \( n_2 \), respectively, arranged also in an clockwise direction, so that \( e_1 \) and \( e_2 \) lie along the same side of the hexagon \( \text{Hex}(\sigma) \) (see Figure 2.1). For \( i = 0, 1, \ldots, 5 \), write \( t_i \) for the value of the map \( \varphi_{a_i} \), where \( a_i \) is the arrow in \( Q \) dual to \( e_i \). The relations \( \{ p_+ (a) - p_-(a) \in \mathbb{C}Q \mid a \in Q_1 \} \) determined by the dimer model \( \Gamma \) imply that the edges along the side starting with \( e_i \) must correspond to maps with alternating values \( t_i \) and \( t_i \) for \( i = 0, 1, 2 \), where the overline denotes addition modulo 3. Therefore, \( t_0 = t_3, t_1 = t_4 \) and \( t_2 = t_5 \). Since the maps in \( M_\sigma(\theta) \) are commutative and, for any arrow \( a \in Q_1 \), we have \( ap_+(a) = ap_-(a) \), the value of the path \( ap_+(a) \) does not depend on the arrow \( a \in Q_1 \). Thus, for any interior edge \( e \) of \( \text{Hex}(\sigma) \), \( \varphi_{ae} = t_0 t_1 t_2 \). Hence, any choice of values \( t_0, t_1, t_2 \in \mathbb{C} \) for the maps \( \varphi_{a_0}, \varphi_{a_1}, \varphi_{a_2} \) determines the values of the rest of the maps of \( M_y(\theta) \), giving an isomorphism \( U_\sigma \cong \mathbb{C}^3 \) that sends the torus-invariant point \( y = [M_\sigma(\theta)] \in U_\sigma \) to the origin of \( \mathbb{C}^3 \). Figure 2.3 records the value of \( \varphi_{a_1} \) for every edge \( e \in \text{cosupp}(M_\sigma) \).

![Diagram](https://via.placeholder.com/150)

**Figure 2.3:** A fundamental hexagon with the values of its maps

To restate this result in terms of perfect matchings, let \( \rho_0, \rho_1, \rho_2 \in \Sigma(1) \) denote the rays in \( \sigma \), and write \( \Pi_0(\theta), \Pi_1(\theta), \Pi_2(\theta) \) for the corresponding \( \theta \)-stable perfect matchings. Recall from Definition 2.2, that the cosupport of \( M_\sigma(\theta) \) is equal to the union of \( \Pi_0(\theta), \Pi_1(\theta) \) and \( \Pi_2(\theta) \). For \( 0 \leq i \leq 2 \), let \( (e_{i,1}, e_{i,2}, \ldots, e_{i,2k_i+1}) \) be the chains of edges making the three inequivalent sides of the boundary of \( \text{Hex}(\sigma) \); in light of Re-
marks 2.6(3), these edges may be chosen from $\mathcal{E} \subset \mathbb{T}$. Mozgovoy [Moz09, Corollary 4.18] noted that the set of edges

$$\Pi'_i := \cosupp(M_\sigma(\theta)) \setminus \{ \{e_j, 2k+1 \mid 0 \leq k \leq k_j, j \neq i\} \cup \{e_i, 2k \mid 1 \leq k \leq k_i\} \} \quad (2.2.1)$$

is a perfect matching of $\Gamma$ and, moreover, these are the only perfect matchings in $\cosupp(M_\sigma(\theta))$. They must therefore coincide with the $\theta$-stable perfect matchings $\Pi_0(\theta), \Pi_1(\theta)$ and $\Pi_2(\theta)$. Relabelling if necessary gives $\Pi'_i = \Pi_i(\theta)$ for $0 \leq i \leq 2$. Equation (2.2.1) implies the following:

**Corollary 2.10.** Let $\sigma \in \Sigma(3)$ and consider an edge $\mathcal{E} \in \cosupp(M_\sigma(\theta))$. For $0 \leq i \leq 2$, we have $\mathcal{E} \in \Pi_i(\theta)$ if and only if the edge $\mathcal{E}$ is labelled with $\Pi_i$ in Figure 2.2. That is, either:

1. $\mathcal{E}$ is an interior edge of $\text{Hex}(\sigma)$, in which case $\mathcal{E} \in \bigcap_{i=0}^2 \Pi_i(\theta)$; or
2. $\mathcal{E}$ is a boundary edge of $\text{Hex}(\sigma)$, in which case $\mathcal{E}$ is one of the edges in a chain $\{e_j, 1, e_j, 2, \ldots, e_j, 2k_j+1\}$ in $\text{Graph}(\sigma)$ linking adjacent trivalent points of $\text{Hex}(\sigma)$, for some $0 \leq j \leq 2$. The edges $e_j, 1, e_j, 2, \ldots, e_j, 2k_j+1$ belong alternately to either a single perfect matching $\Pi_j(\theta)$ or to two perfect matchings $\Pi_j(\theta) \cap \Pi_j(\theta)$, where the overline denotes addition modulo 3.

In particular, $\text{Graph}(\sigma) \subset \mathbb{T}$ is the unique connected component of the locus $\bigcup_{0 \leq i \leq 2} \Pi_i(\theta) \subset \mathbb{T}$ comprising more than a single edge.

**Example 2.11.** Consider again the cone $\sigma \in \Sigma(3)$ generated by the rays $\rho_8, \rho_9, \rho_{10}$ in the toric diagram of the crepant resolution introduced in Example 1.35. The perfect matchings $\Pi_8, \Pi_9, \Pi_{10}$ are shown in Figure 1.5, and it is a simple exercise to verify the statement of Corollary 2.10 in this case by comparing Figure 2.2.

### 2.3 Meandering walks

In this section we generalise the notion of a zig-zag path by associating a walk in a nondegenerate dimer model $\Gamma$ to any line segment $\tau \in \Sigma(2)$, where $\Sigma := \Sigma(\theta)$ is the toric fan of the moduli space $\mathcal{M}_A(\theta)$ defined by any generic stability parameter $\theta \in \Theta$.

**Definition 2.12.** Let $\Pi, \Pi'$ be $\theta$-stable perfect matchings of $\Gamma$. The symmetric difference of $\Pi$ and $\Pi'$ is the subset $\Pi \ominus \Pi' := (\Pi \cup \Pi') \setminus (\Pi \cap \Pi')$ of edges in $\Gamma$. For $\tau \in \Sigma(2)$, let $\rho, \rho'$ be the two ray generators of $\tau$. The meandering walk of $\tau$ is the set $m_\tau := \Pi_\rho \ominus \Pi_\rho'$ of edges in $\Gamma$. We justify the terminology later.

**Remark 2.13.** Definition 2.12 generalises the notion of a $\sigma$-strand introduced by Logvinenko [Log04, Definition 6.59] to the setting of dimer models.

By Corollary 2.10, if $\rho$ and $\rho'$ are the endpoints of a line segment $\tau$ in a triangle $\sigma \in \Sigma(3)$, the edges along two adjacent sides of the boundary of $\text{Hex}(\sigma)$ belong alternately to $\Pi_\rho(\theta)$ and $\Pi_{\rho'}(\theta)$. Therefore, the edges in a meandering walk $m_\tau$ make a cycle in $\Gamma$, and edges along any two adjacent sides of the fundamental hexagon $\text{Hex}(\sigma)$ belong to one of the meandering walks $m_\tau$ for $\tau$ a two-dimensional cone in $\sigma$. We illustrate this
in Figure 2.4, where \( \rho_0, \rho_1, \rho_2 \) are the ray generators in \( \sigma \) and \( \Pi_i := \Pi_{\rho_i}(\theta) \) \( (0 \leq i \leq 2) \) denotes the corresponding \( \theta \)-stable perfect matchings. Let \( m_{i,j} \) denote the meandering walk \( m_\tau \) where \( \tau \) is a line segment in \( \sigma \) with endpoints \( \rho_i \) and \( \rho_j \). Figure 2.4 shows \( \text{Hex}(\sigma) \) in part of the honeycomb tiling with edges labelled by the perfect matchings each belongs to. Edges along each side of \( \partial \text{Hex}(\sigma) \) are in the intersection of two meandering walks.

**Figure 2.4:** The meandering walks and perfect matchings associated to \( \text{Hex}(\sigma) \)

To this point, each meandering walk is just a set of edges whose support is a cycle in \( \Gamma \). We wish to trace the edges in such cycles, and to do this we have to choose a direction along each of them. For this, let \( \Sigma \) denote the set of all tiles of \( \Gamma \) and consider the chain complex

\[
0 \xrightarrow{\partial_1} \mathbb{Z}^\Sigma \xrightarrow{\partial_2} \mathbb{Z}^\mathcal{E} \xrightarrow{\partial_3} \mathbb{Z}^{3\mathbb{R}\Sigma} \xrightarrow{\partial_4} 0
\]

which computes the homology of \( \Sigma \). The boundary operator \( \partial_1 \) sends an edge \( \epsilon \in \mathcal{E} \) to \( \mathfrak{w}(\epsilon) - b(\epsilon) \), where \( \mathfrak{w}(\epsilon) \) and \( b(\epsilon) \) are the white and black endnodes of \( \epsilon \), respectively. A perfect matching \( \Pi \) of \( \Gamma \) gives a 1-chain \( \sum_{\epsilon \in \Pi} \epsilon \). The difference of 1-chains associated to a pair of perfect matchings \( \Pi \) and \( \Pi' \) is a 1-cycle, whose homology class we denote by \( [\Pi - \Pi'] \in H_1(\Sigma, \mathbb{Z}) \). This class is equivalent to the class of a 1-cycle supported on the symmetric difference \( \Pi \oplus \Pi' = (\Pi \cup \Pi') \setminus (\Pi \cap \Pi') \).

Given any \( \tau \in \Sigma(2) \), we may turn the meandering walk \( m_\tau \) into a directed meandering walk by setting it to be the cycle supported on \( m_\tau \) together with a choice of direction determined by the choice of either \( [\Pi_\rho(\theta) - \Pi_{\rho'}(\theta)] \) or \( [\Pi_{\rho'}(\theta) - \Pi_\rho(\theta)] \). For a meandering walk \( m_\tau \), we denote either choice of direction by \( [m_\tau] \). In general, the homology class of \( m_\tau \) is normal to \( \tau \).

**Lemma 2.14.** Let \( \Sigma \) be the fan of \( \mathcal{M}_A(\theta) \) for \( \theta \in \Theta \) generic. For either choice of direction on the meandering walk \( m_\tau \) of \( \tau \in \Sigma(2) \), its homology class \( [m_\tau] \in H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^2 \) is normal to the line segment \( \tau \).
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Proof. For every \( \tau \in \Sigma(2) \), let \( \rho, \rho' \in \Sigma(1) \) be its ray generators and let \( \sigma \in \Sigma(3) \) be any cone containing it. The meandering walk \( m_\tau \) passes through two of the three distinct sides of the fundamental hexagon \( \text{Hex}(\sigma) \). Choose coordinates so that the homology class of \( m_\tau \) is \((0, 1)\). Figure 2.5 below depicts a part of \( \text{Graph}(\sigma) \) in the universal cover of \( T \): the thicker edges of \( \text{Graph}(\sigma) \) indicate the edges in the lift of \( m_\tau \) to the universal cover \( \tilde{T} \). The dashed lines indicate a fundamental domain of \( T \). Computing the height change \( h(\Pi_\rho(\theta), \Pi_{\rho'}(\theta)) \), we get that \( h(\Pi_\rho(\theta), \Pi_{\rho'}(\theta)) = (1, 0) \in H^1(T, \mathbb{Z}) \), as there exists a path from \((0, 0)\) to \((1, 0)\) that crosses an edge \( e \in \Pi_{\rho'} \) with a black node on the right when moving from a connected component of \( \mathbb{R}^2 \setminus (\Pi_\rho \cup \Pi_{\rho'}) \) to another, and both \((0, 0)\) and \((0, 1)\) lie in the same connected component of \( \mathbb{R}^2 \setminus (\Pi_\rho \cup \Pi_{\rho'}) \) (see Definition 1.11). Since \( h(\Pi_\rho(\theta), \Pi_{\rho'}(\theta)) \) indicates the direction of the line segment \( \tau \) connecting the nodes \( \rho \) and \( \rho' \) and \( h(\Pi_\rho(\theta), \Pi_{\rho'}(\theta)) \cdot [m_\tau] = (1, 0) \cdot (0, 1) = 0 \), \( m_\tau \) is normal to the line segment \( \tau \).

Figure 2.5: The meandering walk \( m_\tau \) with \([m_\tau] = (0, 1)\)

Remark 2.15. Let \( \Gamma \) be a consistent dimer model. By Theorem 1.46, there is a one-to-one correspondence between the set of zigzag paths of \( \Gamma \) and the set of line segments on the boundary of \( \Delta(\Gamma) \). Moreover, Ishii–Ueda [IU09, Theorem 11.1] showed that for \( \theta \) generic, each zigzag path \( z \) in \( \Gamma \) is supported on the symmetric difference of the \( \theta \)-stable perfect matchings of \( \rho \) and \( \rho' \), where \( \rho \) and \( \rho' \) are the ray generators of the line segment in \( \partial \Delta \) corresponding to \( z \). Therefore, every zigzag path \( z \) in \( \Gamma \) is a directed meandering walk with the choice of direction \( [\Pi_{\rho'} - \Pi_\rho] \), where \( \rho' \) is adjacent to \( \rho \) in a clockwise direction. An edge \( e \in \text{supp}(z) \) is a zig (resp. zag) of \( z \) if \( e \) lies in \( \Pi_{\rho'} \) (resp. \( \Pi_\rho \)).

Remark 2.16. The name meandering walk was chosen as the cycle of \( T \) supported on the set of edges making a meandering walk does not always turn maximally right at white nodes and maximally left at black nodes, but rather, it meanders.

2.4 Generalised Nakamura jigsaw transformations

For any finite abelian subgroup \( G \subseteq \text{SL}(3, \mathbb{C}) \), Nakamura [Nak01] introduced an algorithm to construct a crepant resolution of \( \mathbb{C}^3/G \) where the key step in each iteration of the algorithm was a combinatorial procedure called a ‘G-igsaw transformation’. The crepant resolution produced by this algorithm is \( G\text{-Hilb}(\mathbb{C}^3) \) which, by Example 1.34, coincides with the fine moduli space of \( \theta \)-stable \( A \)-modules for \( A = \mathbb{C}[x, y, z]*G \) and for a special choice of stability parameter \( \theta \). We now generalise Nakamura’s algorithm to
construct the fine moduli space $\mathcal{M}_A(\theta)$ for any nondegenerate dimer model $\Gamma$ and any generic stability parameter $\theta$. Our construction differs from those introduced previously by Mozgovoy [Moz09] and Craw–Quintero-Vélez [CQV12].

Let $\sigma_+ \in \Sigma(3)$ be adjacent cones in $\Sigma$ with $\tau = \sigma_+ \cap \sigma_- \in \Sigma(2)$ the common two-dimensional cone. Let $\rho_0, \rho_1, \rho_2$ and $\rho_1, \rho_2, \rho_3$ denote the rays in $\sigma_+$ and $\sigma_-$, respectively. Write $M_+ := M_{\sigma_+}$ and $M_- := M_{\sigma_-}$ for the $\theta$-stable $A$-modules of $\sigma_+$ and $\sigma_-$, and write $\text{Hex}(\sigma_+)$ and $\text{Hex}(\sigma_-)$ for their fundamental hexagons (choose any lift, see Remark 2.6(1)).

**Figure 2.6:** Cones $\sigma_+$ and $\sigma_-$ in the triangulation $\Sigma$

**Definition 2.17.** Consider the subset $\bigcup_{i=0}^{3} \Pi_i(\theta)$ of edges of $\Gamma$. Define a jigsaw piece of $\tau$ to be the closure of any connected component of $T \setminus \bigcup_{i=0}^{3} \Pi_i(\theta)$.

**Remark 2.18.** Each jigsaw piece of $\tau$ coincides with a union of tiles of $\Gamma$. Since we may lift tiles to $\tilde{\Gamma}$, we often draw the jigsaw pieces in the universal cover. While the choice of lift is not unique, we often refer to any choice of lift of a given jigsaw piece as being a ‘jigsaw piece’ itself. The terminology is chosen to suggest that one might move jigsaw pieces around in $\tilde{\Gamma}$ by deliberately choosing different lifts. In fact, this procedure plays the essential role in the jigsaw transformations introduced in Theorem 2.21 below.

**Proposition 2.19.** Let $\sigma_+ \in \Sigma(3)$ satisfy $\tau = \sigma_+ \cap \sigma_-$ as in Figure 2.6 above, and regard

$$c_- := m_{1,3} \cap m_{2,3}$$

as a subset of the edges in the boundary of $\text{Hex}(\sigma_-)$. The closure of any connected component of $\text{Hex}(\sigma_+ \setminus c_-)$ is a jigsaw piece of $\tau$, and every jigsaw piece of $\tau$ arises in this way. In particular, if we cut $\text{Hex}(\sigma_+)$ along the edges of $c_-$ then we obtain precisely the jigsaw pieces of $\tau$.

**Proof.** The rays $\rho_1, \rho_2$ are contained in both $\sigma_+$ and $\sigma_-$, so the fundamental hexagons $\text{Hex}(\sigma_\pm)$ share two of their three distinct boundary sides, namely those along the meandering walk $m_{1,2}$. We also know the edges of $m_{0,1} \cap m_{0,2}$ traverse the third boundary side of $\text{Hex}(\sigma_+)$, and cutting $\text{Hex}(\sigma_+)$ along the set $c_-$ requires that we cut along the edges of $m_{1,3} \cap m_{2,3}$. All together then, the edges that cut out the connected components of $\text{Hex}(\sigma_+ \setminus c_-)$ are those in the set

$$\mathcal{C} := m_{1,2} \cup (m_{0,1} \cap m_{0,2}) \cup (m_{1,3} \cap m_{2,3}).$$
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This set contains every edge in the boundary of both $\text{Hex}(\sigma)$ and $\text{Hex}(\sigma_-)$. If we regard both $\text{Graph}(\sigma_\pm)$ as subsets of $T$ as in Remark 2.6(3), we have that

$$\pi(C) = \pi(\partial \text{Hex}(\sigma_+)) \cup \pi(\partial \text{Hex}(\sigma_-)) = \text{Graph}(\sigma_+) \cup \text{Graph}(\sigma_-).$$

Now Corollary 2.10 implies that this subset of $T$ is the unique connected component of the locus $\bigcup_{0 \leq i \leq 3} \Pi_i(\theta)$ comprising more than a single edge. Definition 2.17 implies that we may ignore isolated edges of $\bigcup_{0 \leq i \leq 3} \Pi_i(\theta)$ when computing the jigsaw pieces, so the jigsaw pieces of $\tau$ are precisely the (images under $\pi$ of the) closures of the connected components of $\text{Hex}(\sigma_+) \setminus c_-$. 

Remarks 2.20. 1. By symmetry, a similar statement holds for the cut of $\text{Hex}(\sigma_-)$ along $c_+ := m_{0,1} \cap m_{0,2}$ (2.4.3)

in $\partial \text{Hex}(\sigma_+)$. The proof goes through verbatim if we transpose the numbers 0 and 3 throughout.

2. Since $\text{Hex}(\sigma_+)$ and $\text{Hex}(\sigma_-)$ are different, there are necessarily at least two jigsaw pieces of $\tau$.

Proposition 2.19 and Remark 2.20(1) imply the following result:

**Theorem 2.21** (Generalised jigsaw transformation). Given the $\theta$-stable $A$-module $M_+$, we may cut the corresponding fundamental hexagon $\text{Hex}(\sigma_+)$ along the edges of $c_-$ and rearrange the resulting jigsaw pieces (i.e., translate each piece by a carefully chosen element of $\mathbb{Z}^2$) in order to obtain $\text{Hex}(\sigma_-)$ and hence produce the corresponding $\theta$-stable $A$-module $M_-$. 

**Example 2.22.** Let $G \subset \text{SL}(3, \mathbb{C})$ be a finite abelian subgroup, let $A = \mathbb{C}[x, y, z] * G$ denote the skew group algebra and write $\Gamma$ for the corresponding consistent dimer model (see Example 1.34). The jigsaw transformations described above recover the $G$-jigsaw transformations introduced by Nakamura [Nak01] in constructing the $G$-Hilbert scheme of $\mathbb{C}^3$.

Recall that jigsaw pieces are by definition closed subsets of $T$.

**Definition 2.23.** We say that an edge $e \in E$ is an interior edge of a jigsaw piece if $e$ is not contained in the boundary of the jigsaw piece.

Let $Q^+$ and $Q^-$ denote the quivers that support the $A$-modules $M_+$ and $M_-$, respectively. Write $Q^+_0$ and $Q^+_1$ for the sets of vertices and arrows of the quiver $Q^+$. Similarly, $Q^-_0$ and $Q^-_1$ denote the sets of vertices and arrows of the quiver $Q^-$. 

**Corollary 2.24.** Let $a \in Q_1^+$ have head and tail dual to tiles in different jigsaw pieces of $\tau$. If $a \in Q_1^+$ then $e_a \in \Pi_3$. If $a \in Q_1^-$ then $e_a \in \Pi_0$

Proof. Suppose first that $a \in Q_1^+$. Proposition 2.19 shows that $e_a \in c_- = \text{supp}(m_1,3 \cap m_{2,3})$, so either $e_a \in \Pi_3$ or $e_a \in \Pi_1 \cap \Pi_2$. However, $e_a \notin \text{cosupp}(M_+) = \bigcup_{i=0}^2 \Pi_i$ because $a \in Q_1^+$, giving $e_a \in \Pi_3$ as required. A similar proof works for the case when $a \in Q_1^-$. \qed
Example 2.25. Consider the dimer from Example 1.13. Let $\sigma_{\pm} \in \Sigma(3)$ be the cones such that $\rho_7, \rho_8, \rho_9 \in \sigma_{\pm}(1)$ and $\rho_8, \rho_9, \rho_{10} \in \sigma_{\pm}(1)$ as in Figure 1.7. Figure 2.7 illustrates that the edges from $\bigcup_{7 \leq i \leq 10} \Pi_i(\theta)$ decompose the fundamental domain of $\mathcal{T}$ into two jigsaw pieces. Edges in $\mathbf{m}_{1,2}$ (and its $\mathbb{Z}^2$-translates) are shown in blue, while edges in $c_+$ and $c_-$ are coloured green and red respectively. We illustrate several copies of the fundamental hexagons $\text{Hex}(\sigma_{\pm})$ for clarity.

The fundamental hexagon $\text{Hex}(\sigma_-)$ has boundary comprising edges in red and blue, and the green edges of $c_+$ split this hexagon into two pieces. Since the union of all edges in blue, red and green is precisely $\bigcup_{7 \leq i \leq 10} \Pi_i(\theta)$, we see that these two pieces are precisely the jigsaw pieces of $\tau$. If we keep one of these pieces in place, say that in Figure 2.7 containing the tile dual to the vertex of the quiver labelled 0, and translate the second jigsaw piece along the direction of the blue edges until it sits on the opposite side of the first jigsaw piece, then we cover precisely $\text{Hex}(\sigma_+)$ whose boundary edges are blue and green.

Finally, we discuss some properties of the cuts $c_{\pm}$ of the fundamental hexagons $\text{Hex}(\sigma_{\pm})$.

Lemma 2.26. The cuts $c_{\pm}$ of the fundamental hexagons $\text{Hex}(\sigma_{\pm})$ satisfy the following:

(a) Neither $c_-$ nor $c_+$ intersects itself;

(b) If the cuts $c_{\pm}$ intersect each other, they do so along an odd number of edges;

(c) If $c_{\pm}$ meets $\text{Graph}(\sigma_{\pm})$ at $n_1$ and $n_2$ consecutively, the nodes $n_1$ and $n_2$ have different colours.

Proof. For (a), if $c_-$ intersects itself at a node $n$ of $\bar{\Gamma}$, then at least three edges of $c_-$ touch $n$. Edges on $c_-$ belong alternately to $\Pi_3(\theta)$ and $\Pi_1(\theta) \cap \Pi_2(\theta)$, so two of the three edges touching $n$ belong to either set of edges. This is a contradiction because $\Pi_i(\theta)$...
are perfect matchings \((i = 1, 2, 3)\). The proof for \(c_+\) is similar. For (b), suppose \(c_-\) and \(c_+\) intersect. Edges along \(c_-\) and \(c_+\) belong alternately to \(\Pi_3(\theta)\) and \(\Pi_1(\theta) \cap \Pi_2(\theta)\), and to \(\Pi_0(\theta)\) and \(\Pi_1(\theta) \cap \Pi_2(\theta)\) respectively. Thus the chain of edges along which \(c_+\) and \(c_-\) intersect must begin and end with an edge in \(\Pi_1(\theta) \cap \Pi_2(\theta)\). Therefore there must be an odd number of them. Finally, for (c), suppose \(c_-\) intersects \(\text{Graph}(\sigma_+)\) at \(n_1\) and \(n_2\), consecutively. Let \(e_1\) and \(e_2\) be the first and last edge between \(n_1\) and \(n_2\). If \(e_1 \notin \text{supp}(\text{Graph}(\sigma_+))\), then \(e_2 \notin \text{supp}(\text{Graph}(\sigma_+))\) and \(e_1, e_2 \in \Pi_0\). Otherwise, \(e_1, e_2 \in \text{supp}(\text{Graph}(\sigma_+))\) and \(e_1\) coincides with \(e_2\). In both cases, \(e_1\) and \(e_2\) belong to the same set of perfect matchings, and thus \(n_1\) and \(n_2\) must have different colours. The same prove holds for \(c_+\) and \(\text{Graph}(\sigma_-)\).
Reid [Rei97] calculated the toric fan $\Sigma$ of the $A$-Hilbert scheme $A$-Hilb$(\mathbb{C}^3)$ for several examples of finite abelian subgroups $A \subset \text{SL}(3, \mathbb{C})$, and in each case introduced a recipe to mark the interior lattice points and line segments of $\Sigma$ by irreducible representations of $A$. Craw [Cra05] later proved that this recipe can be carried out for any finite abelian subgroup $A \subset \text{SL}(3, \mathbb{C})$.

In this chapter, we generalise Reid’s recipe so that it can be carried out for the toric fan $\Sigma$ of a particular variety $M_A$ associated to any consistent dimer model $\Gamma$. This recipe marks the interior lattice points and line segments of the fan $\Sigma$ by vertices of the quiver $Q$ dual to $\Gamma$. New phenomena can be observed in this generalisation: line segments in $\Sigma$ may be marked by more than a single vertex; and vertices may mark more than a single lattice point in $\Sigma$. We provide several examples to illustrate the construction, and we establish that this new marking is compatible (as far as can be verified) with ‘Geometric Reid’s recipe for dimer models’ in the sense of Bocklandt–Craw–Quintero-Vélez [BCQV15].

### 3.1 The $0$-generated stability parameter and the module $M(\tau)$

Let $\Gamma$ be a nondegenerate dimer model with dual quiver $Q$. Choose once and for all a vertex $0 \in Q_0$. We refer to the tile of $\Gamma$ dual to 0 as the zero tile.

**Definition 3.1.** A stability parameter $\theta := (\theta_v) \in \Theta$ is said to be 0-generated if $\theta_v > 0$ for all $v \neq 0$.

**Lemma 3.2.** If $\theta \in \Theta$ is 0-generated, then $\theta$ is generic.

**Proof.** Let $M$ be a $\theta$-semistable $A$-module of dimension vector 1. To show $\theta$ is generic we must show $M$ is $\theta$-stable. Let $N := (\oplus_{v \in Q_0} W_v, \{\phi_a\}_{a \in Q_1})$ be a nonzero proper submodule of $M$, so $\theta(N) \geq 0$. Since $\theta(M) = 0$, we have $\theta_0 = -\sum_{v \neq 0} \theta_v$. If $W_0 \neq 0$, then properness of $N$ implies $\theta(N) = \theta_0 + \sum_{v \neq 0} \theta(W_v) < 0$ which is a contradiction. Thus $W_0 = 0$ and hence $\theta(N) > 0$. \qed
Remark 3.3. It is well known that for any 0-generated stability parameter $\theta \in \Theta$, an $A$-module $M$ of dimension vector $\underline{1}$ is $\theta$-stable if and only if for all $v \in Q_0$ there exists a path in $Q_M$ from 0 to $v$ (see for example Bocklandt–Craw–Quintero-Vélez [BCQV15, Lemma 3.1(ii)]).

From now on, let $\theta \in \Theta$ be a 0-generated stability parameter.

Notation 3.4. To simplify notation, we write $\mathcal{M}_A$, $L_v$, $M_\sigma$ and $\Pi_\rho$ in place of $\mathcal{M}_A(\theta)$, $L_v(\theta)$, $M_\sigma(\theta)$ and $\Pi_\rho(\theta)$ respectively. We continue to let $\Sigma := \Sigma(\theta)$ denote the toric fan of the moduli space $\mathcal{M}_A$.

Choose once and for all a lift $\tilde{0} \in \tilde{Q}$ of the zero vertex. We refer to the tile of $\tilde{\Gamma}$ dual to $\tilde{0}$ as the zero tile in $\tilde{\Gamma}$. From now on, for every three-dimensional cone $\sigma \in \Sigma$ in the toric fan of $\mathcal{M}_A$ we choose the lift $\tilde{Q}_\sigma$ of the subquiver $Q_\sigma \subset Q$ where the vertex $\tilde{0}$ plays the role of the vertex $\tilde{v}_0$ as in Section 2.1. It follows that the fundamental hexagon $\text{Hex}(\sigma)$ of every three-dimensional cone $\sigma \in \Sigma(3)$ contains the zero tile in $\tilde{\Gamma}$.

We now study jigsaw transformations of torus-invariant $\theta$-stable $A$-modules for a 0-generated stability parameter $\theta \in \Theta$. Choose adjacent three-dimensional cones $\sigma_\pm \in \Sigma(3)$ and let $\tau := \sigma_+ \cap \sigma_-$ be the common face of dimension two as in Figure 2.6. As before, let $M_\pm := M_{\sigma_\pm}$ denote the $\theta$-stable $A$-modules associated to the three-dimensional cones $\sigma_\pm$, and write $Q^\pm := Q_{M_\pm}$ for the quivers that support these modules. Write $Q^0_\pm$ and $Q^1_\pm$ for the sets of vertices and arrows of $Q^\pm$. It is often convenient to think of $Q^\pm$ as subquivers of $\tilde{Q}$ in the universal cover as in Example 2.9, where each vertex of $Q^0_\pm$ is drawn inside the dual tile in the corresponding fundamental hexagon $\text{Hex}(\sigma_\pm)$.

Definition 3.5. The zero jigsaw piece of any interior line segment $\tau \in \Sigma(2)$ in the triangulation $\Sigma$ is the jigsaw piece in $\Gamma$ containing the zero tile. Let $\text{Jig}_\tau(0)$ denote this jigsaw piece.

Recall that both $\text{Hex}(\sigma_+)$ and $\text{Hex}(\sigma_-)$ can be obtained by glueing together the jigsaw pieces of $\tau$. One may ask how many jigsaw pieces of $\tau$ lie adjacent to $\text{Jig}_\tau(0)$ in $\text{Hex}(\sigma_+)$ or $\text{Hex}(\sigma_-)$.

Lemma 3.6. The zero jigsaw piece in $\text{Hex}(\sigma_\pm)$ is adjacent to only one jigsaw piece in $\text{Hex}(\sigma_\pm)$ that we denote $\text{Jig}_\tau^\pm(1)$, and the common boundary of $\text{Jig}_\tau(0)$ and $\text{Jig}_\tau^\pm(1)$ in $\text{Hex}(\sigma_\pm)$ is a single chain of edges from $c_\pm$.

Proof. We give the argument for $\text{Hex}(\sigma_+)$; the argument for $\text{Hex}(\sigma_-)$ is identical except that we swap $+$ and $-$ throughout. Proposition 2.19 implies that every edge in the boundary of each jigsaw piece in $\text{Hex}(\sigma_+)$ lies in either $\partial \text{Hex}(\sigma_+) \setminus c_-$ or in $c_-$. The boundary edges of $\text{Jig}_\tau(0)$ do not all lie in $\partial \text{Hex}(\sigma_+) \setminus c_-$ because $\text{Hex}(\sigma_+)$ and $\text{Hex}(\sigma_-)$ are different. Equally, the boundary edges of $\text{Jig}_\tau(0)$ do not all lie in $c_-$, because Lemma 2.26(a) asserts that $c_-$ does not intersect itself. We claim that the boundary of $\text{Jig}_\tau(0)$ is made up precisely of a chain of edges in $c_-$ and a chain of edges in $\partial \text{Hex}(\sigma_+) \setminus c_-$. To see this, note that each chain of edges from $c_-$ in $\partial \text{Jig}_\tau(0)$ starts and ends at a node on $\partial \text{Hex}(\sigma_+)$. Suppose $\text{Jig}_\tau(0)$ has boundary made of more than one chain of edges from $c_-$. Figure 3.1 shows the fundamental hexagon $\text{Hex}(\sigma_+)$. The dashed lines indicate all edges in $c_-$. The shaded area in the fundamental hexagon is
Chapter 3. Definition of Reid’s recipe

the zero jigsaw piece Jig_τ(0) with boundary made of two disjoint chains of edges from c_. The interior edges of Hex(σ+) which are not labelled belong to all four perfect matchings Π_0, Π_1, Π_2, and Π_3, thus arrows dual to those edges are not in Q^+ or Q^−. Without loss of generality, suppose that the first chain of edges from c_− in ∂Jig_τ(0) starts at a black node on ∂Hex(σ+). Lemma 2.26 part (c) implies that the chain of edges will end at a white node on ∂Hex(σ+). By Lemma 2.26 part (b), c_− and c_+ will intersect along an odd number of edges, so the second chain of edges from c_− in ∂Jig_τ(0) starts at a white node on ∂Hex(σ+). It ends at a black node on ∂Hex(σ+) by Lemma 2.26 part (c). Therefore, the edges in Π_3 \ (Π_0 ∪ Π_1 ∪ Π_2) which make up one of the disjoint chains of edges of c_− along ∂Jig_τ(0) are dual to arrows in Q^+ with tail at vertices dual to tiles in a non-zero jigsaw piece and head at vertices dual to tiles in Jig_τ(0). Let v ∈ Q_0 be one of the vertices dual to a tile in a nonzero jigsaw piece such that there exists an arrow in Q^+ with tail at v and head at a vertex dual to a tile in Jig_τ(0). Then, there is no path in Q^+ from any vertex dual to a tile in Jig_τ(0) to v. In particular, there is no path in Q^+ from 0 to v, contradicting our choice of stability parameter θ (see Remark 3.3). Hence, there cannot be more than one chain of edges from c_− in ∂Jig_τ(0), and the result follows.

Our next goal is to show that the modules M_± share a common quotient module M(τ), and to do this we first define the subquiver Q(τ) of Q that supports M(τ).

**Lemma 3.7.** Let e_a ∈ E be an interior edge of a jigsaw piece of τ. If the dual arrow satisfies a ∈ Q^+ ⇒ ∪ Q^−, then in fact a ∈ Q^+ ∩ Q^−. In particular,

{a ∈ Q^+ | e_a is an interior edge of Jig_τ(0)} = {a ∈ Q^− | e_a is an interior edge of Jig_τ(0)}. 

**Proof.** Since e_a ∈ E is an interior edge of a jigsaw piece of τ, Definition 2.17 and Corollary 2.10 imply that either e_a ∉ ∪_{0 ≤ i ≤ 3} Π_i, or e_a ∈ ∩_{0 ≤ i ≤ 3} Π_i. The assumption a ∈ Q^+ ∪ Q^− is equivalent to e_a ∉ ∪_{0 ≤ i ≤ 3} Π_i. It follows that e_a ∉ ∪_{0 ≤ i ≤ 3} Π_i, which is equivalent to a ∈ Q^+ ∩ Q^− as required. The second statement is immediate.
Definition 3.8. Let $Q(\tau)$ be the subquiver of $Q$ with vertex set
\[ Q_0(\tau) := \{ v \in Q_0 \mid v \text{ is dual to a tile in } Jig_\tau(0) \} \]
and arrow set
\[ Q_1(\tau) := \{ a \in Q_1^+ \mid \epsilon_a \text{ is an interior edge of } Jig_\tau(0) \} \]
\[ = \{ a \in Q_1^+ \mid \epsilon_a \text{ is an interior edge of } Jig_\tau(0) \}, \]
where the equality follows from Lemma 3.7. Define $M(\tau) := (\oplus_{v \in Q_0(\tau)} C, \{ \phi_a \}_{a \in Q_1})$ by setting
\[ \phi_a = \begin{cases} 1 & \text{if } a \in Q_1(\tau), \\ 0 & \text{else} \end{cases} . \]

Remark 3.9. We regard $M(\tau)$ as a representation of $Q$ (that is, a $\mathbb{C}Q$-module) with dimension vector $d := (d_v) \in \mathbb{N}^{Q_0}$ given by $d_v = 1$ for $v \in Q_0(\tau)$ and $d_v = 0$ otherwise. Notice that

1. at least one entry of $d$ equals 1, because $Jig_\tau(0)$ is non-empty; and
2. at least one entry of $d$ equals 0, because there are at least two jigsaw pieces of $\tau$.

The next result establishes in particular that $M(\tau)$ is an $A$-module (note that $M(\tau)$ is not equal to the $\theta$-stable $A$-module $M_\tau$ associated to the cone $\tau \in \Sigma(2)$ in Definition 2.2).

Proposition 3.10. There exist nonzero proper $A$-submodules $N_\pm(\tau) \subset M_\pm$ such that
\[ M_+/N_+(\tau) \cong M(\tau) \cong M_-/N_-(\tau) . \]

Proof. Recall that $M_+ = (\oplus_{v \in Q_0} C, \{ \psi_a \}_{a \in Q_1})$, where $\psi_a = 1$ for $a \in Q_1^+$ and $\psi_a = 0$ otherwise. Define a $\mathbb{C}Q$-module $N_+(\tau) := (\oplus_{v \in Q_0} Q_0(\tau) C, \{ \phi_a \}_{a \in Q_1})$ by setting
\[ \phi_a = \begin{cases} 1 & \text{if } a \in Q_1^+ \text{ and } t(a), h(a) \in Q_0 \setminus Q_0(\tau), \\ 0 & \text{else} \end{cases} . \]

To show that $N_+(\tau)$ is an $A$-submodule of $M_+$, we must show that there does not exist $a \in Q_1^+$ with $t(a) \in Q_0 \setminus Q_0(\tau)$ and $h(a) \in Q_0(\tau)$. Suppose for a contradiction that $a \in Q_1^+$ is one such arrow. The head of $a$ is dual to a tile in $Jig_\tau(0)$ and the tail is not, so Lemma 3.6 implies that $t(a)$ is dual to a tile in $Jig_\tau^+(1)$ and, moreover, that the dual edge $\epsilon_a \in c_-$ lies in the unique chain of edges from $c_-$ that forms part of the boundary of $Jig_\tau(0)$. The edges of $c_-$ alternate between those in $\Pi_3$ and those in $\Pi_1 \cap \Pi_2$, and we know $\epsilon_a \in \Pi_3$ by Corollary 2.24. Since $t(a)$ is dual to a tile in $Jig_\tau^+(1)$ and $h(a)$ is dual to a tile in $Jig_\tau(0)$, it follows that every arrow $b \in Q_1^+$ in the opposite direction (that is, $t(b) \in Q_0(\tau)$ and $h(b) \in Q_0(\tau)$) satisfies $\epsilon_b \in \Pi_1 \cap \Pi_2$ and $c_1 \cap \Pi_2$. However, $\text{cosupp}(M_+) = \bigcup_{0 \leq i \leq 2} \Pi_i$, so no such arrow $b \in Q_1^+$ can exist. This is a contradiction, because a vertex $v_0 \in Q_0 \setminus Q_0(\tau)$ exists by Remark 3.9 and there must exist at least one path in $Q^+$ from 0 to $v_0$ by Remark 3.9. This contradiction proves the claim, so $N_+(\tau)$ is a $\mathbb{C}Q$-submodule of $M_+$. Regarding $M_+$ as a representation of the quiver $Q$,
we have that $N_+(\tau)$ is a subrepresentation of $M_+$. Since $M_+$ satisfies the relations $\mathcal{R}$ in $Q$, it follows that $N_+(\tau)$ does as well, so $N_+(\tau)$ is an $A$-submodule of $M_+$.

The quotient $A$-module $M_+/N_+(\tau)$ is isomorphic to $(\oplus_{v \in Q_{0}(\tau)} \mathbb{C}, \{\varphi_a\}_{a \in Q_1})$ where

$$
\varphi_a = \begin{cases} 
1 & \text{if } a \in Q_1^+ \text{ and } t(a), h(a) \in Q_0(\tau) \\
0 & \text{else}
\end{cases}
$$

If an arrow $a \in Q_1^+$ has both head and tail in $Q_0(\tau)$, then the dual edge $e_a$ is an interior edge of $\text{Jig}_\tau(0)$, so this quotient module is $M(\tau)$ as required. The proof of the analogous result for $N_-(\tau)$ is identical, except in that we swap $+$ with $-$ and $0$ with $3$ throughout.

**Remark 3.11.** As in the proof of Lemma 3.6, one can show that the only arrows in $Q^\pm$ with head dual to a tile in $\text{Jig}_\tau^\pm(1)$ have tail dual to a tile in $\text{Jig}_\tau(0) \cup \text{Jig}_\tau^\pm(1)$. Using a similar argument to the one in the proof of Lemma 3.10, suppose there exists an arrow in $Q^\pm$ from a tile in a jigsaw piece $\text{Jig}_\tau(i)$ ($i \neq 0, 1$) to a tile in $\text{Jig}_\tau(1)$. Then, there are no arrows in $Q^\pm$ from tiles in $\text{Jig}_\tau(1)$ to tiles in $\text{Jig}_\tau(i)$. Hence, for any tile $v$ in $\text{Jig}_\tau(i)$, there does not exist a path in $Q^\pm$ from $0$ to it, a contradiction as $M_\pm$ is $\theta$-stable.

**Example 3.12.** Consider the dimer model $\Gamma$ with dual quiver $Q$ presented in Figure 3.2. The fan $\Sigma$ of $M_\mathcal{A}$ for $\theta = (-7, 1, 1, \ldots, 1)$ is depicted in Figure 3.3.

**Figure 3.2:** The dimer model $\Gamma$ with dual quiver $Q$

**Figure 3.3:** The fan $\Sigma$ of $M_\mathcal{A}$

Let $\sigma_+$ and $\sigma_-$ denote the cones of $\Sigma$ with ray generators $\rho_5, \rho_7, \rho_8$ and $\rho_2, \rho_7, \rho_8$, respectively. The arrows and vertices in Figure 3.4 show the lift of the quiver $Q_{\sigma_+}$ that supports $M_{\sigma_+}$ to the universal cover of $\Gamma$.  

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Figure 3.4: The honeycomb tiling of $\mathbb{R}^2$ by $\text{Hex}(\sigma_+)$ and the lift of edges in $\text{c}_{-}$

Figure 3.5: The honeycomb tiling of $\mathbb{R}^2$ by $\text{Hex}(\sigma_-)$
The zero jigsaw piece has two tiles, while the one jigsaw piece is made out of the tiles dual to the vertices 2, 3, 6 and 7. The jigsaw pieces can be rearranged to give the fundamental hexagon Hex(σ⁻) depicted in Figure 3.5.

### 3.2 On consistency and sections of tautological bundles

The modules $N_{±}(τ)$ from Proposition 3.10 are not isomorphic in general. Nevertheless, our goal in this section is to prove that the set of minimal $A$-module generators of $N_{±}$ coincide. For this we must recall some geometric properties of the tautological bundle $T = \bigoplus_{v \in Q_0} L_v$ on the fine moduli space $M_A$, where $A$ is the Jacobian algebra of a consistent dimer model $Γ$.

**Assumption 3.13.** *From now on, assume that $Γ$ is a consistent dimer model.*

Recall from Chapter 1 that every torus-invariant section of a line bundle $L$ on the toric variety $M_A$ is a multiple of a monomial $x^D$ in the Cox ring $S := \mathbb{C}[x_ρ | ρ ∈ \Sigma(1)]$ of $M_A$. Since $Γ$ is a consistent dimer model, the tautological $\mathbb{C}$-algebra isomorphism $ϕ: A → \text{End}_{O_{M_A}}(T)$ from Theorem 1.49 associates to each path $p$ in $Q$ the torus-invariant section

$$x^{\text{div}(p)} = \prod_{a ∈ \text{supp}(p)} x^{\text{div}(a)}$$

$$= \prod_{\{ρ ∈ \Sigma(1) | a_ρ ∈ Π_ρ\}} x_ρ$$

by (1.5.1)

of the line bundle $L_{h(ψ)} \otimes L_{t(ψ)}^{-1}$. For any $v ∈ Q_0$, Lemma 1.52 shows that $L_v = O_{M_A}(\text{div}(p))$, where $p$ is any path from vertex 0 to vertex $v$. Hence by restricting to either of the affine open subsets $U_{σ_{±}} := U_{±} \subseteq M_A$, we have that

$$L_v|_{U_{±}} \cong O_{U_{±}}(\text{div}(p_{±})),$$

where $p_{±}$ is any path in $Q_{±}$ from 0 to $v$, so $x^{\text{div}(p_{±})}$ generates $L_v|_{U_{±}}$ as an $O_{U_{±}}$-module.

**Remark 3.14.** Since $L_v = O_{M_A}(\text{div}(p_{±}))$, we have $h(p_{+}) = h(p_{-})$ as vertices of $Q$. However, the lifts to the universal cover of the paths $p_{±}$ differ in general (unless $v = 0$, in which case $L_0 \cong O_{M_A}$ when $p_{±}$ both equal the trivial path at vertex 0).

Let $σ_{±} ∈ Σ(3)$ satisfy $τ = σ_{+} ∩ σ_{−}$ as in Figure 2.6. The cones $σ_{±}$ and $σ_{−}$ have ray generators $ρ_0, ρ_1, ρ_2$ and $ρ_1, ρ_2, ρ_3$, respectively.

**Lemma 3.15.** *For $v ∈ Q_0^+$, let $p$ be a path in $Q^+$ from vertex 0 to vertex $v$. Then $v$ is dual to a tile in $\text{Jig}^+(1)$ if and only if $x_{ρ_3}$ appears with multiplicity one in the labelling section $x^{\text{div}(p)}$.*

**Proof.** The path $p$ must traverse one arrow $a$ with tail in $\text{Jig}_τ(0)$ and head in $\text{Jig}_τ^+(1)$, so $a ∈ Π_3$ by Corollary 2.24 and hence $x_{ρ_3} | x^{\text{div}(a)}$. By the proof of Lemma 3.6, since $p$ is a path in $Q^+$, the path $p$ cannot travel from $\text{Jig}_τ^+(1)$ to $\text{Jig}_τ(0)$. Therefore, every other arrow $a'$ in the path $p$ has head and tail in the same jigsaw piece, either $\text{Jig}_τ^+(0)$ or $\text{Jig}_τ^+(1)$, so the dual edge $e_{a'}$ is an interior edge in a jigsaw piece of $τ$. Since $a' ∈ Q_1^+$,
Lemma 3.7 implies that $a' \in Q_1^-$. In particular, $x_{p_3}$ does not divide $x^{\div(a')}$ because $\cosupp(M_-) = \bigcup_{1 \leq i \leq 3} \Pi_i$. Thus, $a$ is the only arrow in the path $p$ whose label is divisible by $x_{p_3}$, so the multiplicity of $x_{p_3}$ in $x^{\div(a)}$ equals one as required.

\[\square\]

Remark 3.16. A vertex $v \in Q_0^\pm$ is dual to a tile in $\text{Jig}_\tau^+(1)$ if and only if $x_{p_0}$ appears with multiplicity one in the section $x^{\div(p)}$ on any path $p$ in $Q_1^\tau$ from vertex 0 to vertex $v$. The proof is identical to that of Lemma 3.15 except in that we replace $+$ by $-$ and 0 by 3 throughout.

Proposition 3.17. There exists a jigsaw piece $\text{Jig}_\tau(1) \subset \mathbb{T}$ that lifts to the jigsaw piece $\text{Jig}_\tau^+(1)$ in $\text{Hex}(\sigma_+)$ and to $\text{Jig}_\tau(1)$ in $\text{Hex}(\sigma_-)$.

Proof. We first compute the toric coordinate function on the chart $U_+$ corresponding to the face $\tau$ of $\sigma_+$ satisfying $\tau = \sigma_+ \cap \sigma_-$. Let $m \in M$ be the primitive vector that lies perpendicular to the cone $\tau$ such that $\langle m, n \rangle \geq 0$ for all $n \in \sigma_+$. Since $p_1, p_2 \subset \tau$, we have $\langle m, v_{p_1} \rangle = \langle m, v_{p_2} \rangle = 0$. Also, since $p_0$ and $p_3$ lie on opposite sides of $\tau$, our choice of $m$ gives $\langle m, v_{p_0} \rangle > 0$ and $\langle m, v_{p_3} \rangle < 0$, and since $m$ is a primitive generator and both cones $\sigma_\pm$ are basic, we have $\langle m, v_{p_0} \rangle = 1$ and $\langle m, v_{p_3} \rangle = -1$. Thus, the toric coordinate function on $U_+$ corresponding to the face $\tau$ of $\sigma_+$ satisfies $x^m = (x_{p_0}/x_{p_3}) \cdot x^{m'}$, where $x^{m'} \in \mathbb{C}[M]$ is independent of $x_{p_0}, x_{p_1}, x_{p_2}, x_{p_3}$.

Let $v \in Q_0^\pm$ be dual to a tile of $\text{Jig}_\tau^+(1)$ in $\text{Hex}(\sigma_+)$ and let $p_\pm$ be any path in $Q_1^\pm$ from 0 to $v$. We now compute the generating section $x^{\div(p_-)}$ of $L_v|_{U_+}$ directly from $x^{\div(p_+)}$ using the transition function for $L_v$ from $U_+$ to $U_-$. Indeed, given the section $x^{\div(p_+)}$ that generates $L_v|_{U_+}$, the section $x^{\div(p_-)}$ that generates $L_v|_{U_-}$ is then obtained by multiplying $x^{\div(p_+)}$ by the minimal power of the above Laurent monomial $x^m$ such that the resulting Laurent monomial lies in $\mathbb{C}[\sigma_- \cap M]$, as $m \in M$ is the primitive vector perpendicular to the common face $\tau$ of $\sigma_+$ and $\sigma_-$. Putting this together gives

$$x^{\div(p_-)} = x^{\div(p_+)} \cdot \left( \frac{x_{p_0}}{x_{p_3}} \cdot x^{m'} \right)^d \in \mathbb{C}[\sigma_- \cap M]$$  \hspace{1cm} (3.2.1)

for some Laurent monomial $x^{m'} \in \mathbb{C}[M]$ that is independent of $x_{p_0}, x_{p_1}, x_{p_2}, x_{p_3}$, where $d \in \mathbb{N}$ is the minimal non-negative integer such that (3.2.1) holds. Now, $p_-$ is a path in the quiver $Q^-$ and $\Pi_3 \subseteq \cosupp(Q^-)$, so the left hand side of (3.2.1) is independent of $x_{p_3}$. The same is true of $x^{m'}$, and yet $x_{p_3}$ appears in $x^{\div(p_+)}$ with multiplicity one by Lemma 3.15, so we have $d = 1$.

We now claim that the vertex $v = h(p_-) \in Q_0^-$ is dual to a tile in $\text{Jig}_\tau^-(1)$. To see this, note that $p_-$ is a path in $Q^+$ and $\Pi_0 \subseteq \cosupp(Q^+)$, so $x^{\div(p_+)}$ is independent of $x_{p_0}$. The monomial $x^{m'}$ is also independent of $x_{p_0}$, and since $d = 1$ we conclude from equation (3.2.1) that $x_{p_0}$ appears in $x^{\div(p_-)}$ with multiplicity one. The claim follows directly from Remark 3.16.

Thus far, we know that for a vertex $v \in Q_0^\pm$ such that $v = h(p_-) \in Q_0^+$ is dual to a tile in $\text{Jig}_\tau^+(1)$, we have that the vertex $v = h(p_-) \in Q_0^-$ is dual to a tile in $\text{Jig}_\tau^-(1)$. This is true for the vertex $v$ dual to some tile in $\text{Jig}_\tau^+(1)$, and hence for the vertices dual to every tile in $\text{Jig}_\tau^+(1)$. The statement is symmetric, so $\text{Jig}_\tau^+(1)$ are two lifts of the same jigsaw piece of $\tau$ as required.

\[\square\]
Remark 3.18. The non-negative integer $d$ from equation (3.2.1) equals the degree of the line bundle $L_v$ on the toric curve $C_\tau \cong \mathbb{P}^1$ in $\mathcal{M}_A$ defined by $\tau \in \Sigma(2)$.

Definition 3.19. Let $Q_{\pm}(\tau)$ denote the subquiver of $Q_{\pm}$ that supports the $A$-module $N_{\pm}(\tau)$. Explicitly, $Q_{\pm}(\tau)$ has vertex set $Q_0 \setminus Q_0(\tau)$ and arrow set $\{a \in Q_1^\pm \mid t(a), h(a) \in Q_0 \setminus Q_0(\tau)\}$.

Corollary 3.20. The set of minimal $A$-module generators of $N_{\pm}(\tau)$ coincide. Equivalentlly, the source vertices of the quivers $Q_{\pm}(\tau)$ coincide.

Proof. We claim that the source vertices of the quivers $Q_{\pm}(\tau)$ are dual to tiles in $\text{Jig}^\pm_\tau(1)$. To see this, let $v \in Q_0 \setminus Q_0(\tau)$ be any vertex. By our choice of stability parameter $\theta \in \Theta$, there exist paths $p^\pm$ in the quivers $Q^\pm$, each with tail at 0 and head at $v$. Lemma 3.6 implies that each path $p^\pm$ travels along precisely one arrow $a^\pm$ with tail in $Q_0(\tau)$ and head in $Q_0 \setminus Q_0(\tau)$. Thus, there is a path in $Q_{\pm}(\tau)$ from the vertex $h(a^\pm)$ to the vertex $v$, and the claim follows because $h(a^\pm)$ is dual to a tile in $\text{Jig}^\pm_\tau(1)$.

To complete the proof, Remark 3.11 implies that the only arrows of $Q_{\pm}(\tau)$ with head dual to a tile of $\text{Jig}^\pm_\tau(1)$ have tail dual to a tile of $\text{Jig}^\pm_\tau(1)$. It follows from the above claim that the source vertices of $Q_{\pm}(\tau)$ are precisely the source vertices of the quivers with vertex set $\{v \in Q_0^\pm \mid v \text{ is dual to a tile in } \text{Jig}^\pm_\tau(1)\}$ and arrow set $\{a \in Q_1^\pm \mid e_a \text{ is an interior edge of } \text{Jig}^\pm_\tau(1)\}$. These latter two quivers coincide by Proposition 3.17, and hence so do the source vertices of the quivers $Q_{\pm}(\tau)$. \qed

3.3 Combinatorial Reid’s recipe

We are now in a position to introduce Reid’s recipe for a consistent dimer model $\Gamma$. We first introduce the recipe for interior line segments of $\Sigma$.

Definition 3.21 (Reid’s recipe for interior line segments). Let $\tau \in \Sigma(2)$ define an interior line segment of $\Sigma$. We say a vertex $v \in Q_0$ marks the line segment $\tau$ if $v$ is one of the common source vertices of the quivers $Q_{\pm}(\tau)$ as in Corollary 3.20.

Remarks 3.22. 1. Reid’s recipe never marks an interior line segment with the vertex $0 \in Q_0$, because the zero vertex cannot be a vertex of $Q_{\pm}(\tau)$.

2. Reid’s recipe does not mark boundary line segments of $\Sigma$ with any vertices, because any such line segment $\tau \in \Sigma(2)$ is not of the form $\tau = \sigma_+ \cap \sigma_-$ for cones $\sigma_{\pm} \in \Sigma(3)$.

We now introduce Reid’s recipe for marking interior lattice points of $\Sigma$. For $v \in Q_0$, write $S_v := \mathcal{C}e_v$ for the corresponding vertex simple $A$-module. For any $\theta$-stable $A$-module $M$, the module $S_v$ is contained in the socle of $M$ if and only if $S_v$ is a submodule of $M$, in which case we write $S_v \subseteq \text{soc}(M)$.

Definition 3.23 (Reid’s recipe for interior lattice points). Let $\rho \in \Sigma(1)$ define an interior lattice point of $\Sigma$. We say that a vertex $v \in Q_0$ marks the lattice point $\rho$ if for every torus-invariant point $y \in D_\rho$, $S_v \subseteq \text{soc}(M_y)$ where $M_y$ is the $\theta$-stable $A$-module associated to the point $y \in \mathcal{M}_A$. 

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Remarks 3.24. 1. Reid’s recipe never marks a lattice point with the vertex 0 ∈ Q₀ because our choice of stability parameter θ ensures that for any dimer model Γ, S₀ is never in soc(M_y) for every torus-invariant point y ∈ M_A.

2. Note that for ρ ∈ Σ(1) and v ∈ Q₀, S_v ⊆ soc(M_ρ) implies that S_v ⊆ soc(M_y) for every torus-invariant point y ∈ D_ρ. Indeed, if S_v ⊆ soc(M_ρ) every arrow in M_y with tail at vertex v is zero. As the zero arrows of M_ρ are zero in M_y, S_v ⊆ soc(M_y).

3. Reid’s recipe marks every interior lattice point of Σ with at least one vertex of Q. Indeed, if ρ ∈ Σ(1) defines an interior lattice point, Remark 2.4 shows that the A-module M_ρ is nilpotent. It follows that there exists a nonzero vertex v ∈ Q₀ such that M_ρ contains S_v as a submodule, in which case vertex v marks ρ by the previous remark.

Example 3.25. Consider the dimer model from Example 1.13. The fan of M_A for the 0-generated stability parameter θ = (−9, 1, 1, ..., 1) is presented in Example 1.35. By looking at the jigsaw transformation in Example 2.25, it is clear from Figure 2.7 that the primitive generators of the degree 1 jigsaw piece are 3 and 9. Hence, the vertices 3 and 9 mark the unique line segment τ ∈ Σ with ρ₈,ρ₉ ⊂ τ. Figure 3.6 presents the perfect matchings associated to the rays ρ₈ and ρ₉ and their underlying quiver. Both S₂ and S₅ lie in the socle of M_ρ₈, while only S₂ lies in the socle of M_ρ₉. Hence, vertices 2 and 5 mark the lattice point ρ₈, while the vertex 2 marks the lattice point ρ₉. By drawing the underlying quivers of the θ-stable A-modules M_y for all torus-invariant points y in D_ρ₈ and D_ρ₉, one can verify that these are the only vertices marking the lattice points ρ₈ and ρ₉.

Figure 3.6: The θ-stable perfect matchings Π₈ and Π₉ for Example 1.35

Remark 3.26. The combinatorial input required for computing Reid’s recipe is the set of θ-stable perfect matchings of Γ. Raf Bocklandt has written a code in GAP which produces for any choice of convex polygon ∆ a quiver Q dual to a dimer model Γ with characteristic polygon ∆. The output of the code also includes the set of θ-stable perfect matchings of Γ, for whatever choice of generic stability parameter θ.
3.4 Compatibility with classical Reid’s recipe

Now that we have defined Reid’s recipe for a consistent dimer model, we must verify that our description is compatible with that in the literature. We begin with the classical Reid’s recipe from Reid [Rei97] and Craw [Cra05].

**Theorem 3.27 (Reid’s recipe for the \(G\)-Hilbert scheme).** Let \(G\) be a finite abelian subgroup of \(\text{SL}(3, \mathbb{C})\) and let \(\Sigma\) be the fan of the \(G\)-Hilbert scheme. Reid’s recipe for \(\Sigma\) as in Definitions 3.21 and 3.23 agrees with the classical recipe from Reid [Rei97] and Craw [Cra05].

**Proof.** In the classical version of Reid’s recipe for the \(G\)-Hilbert scheme, the vertex set of the McKay quiver \(Q\) is the set \(\text{Irr}(G)\) of isomorphism classes of irreducible representation of \(G\).

Consider first the marking of an interior lattice point \(\rho \in \Sigma(1)\). Craw–Ishii [CI04, Proposition 9.1] proved that a vertex \(v \in Q_0\) of the McKay quiver marks \(\rho\) in the classical recipe if and only if \(S_v\) is a submodule of every \(G\)-cluster defined by a torus-invariant point of \(D_\rho\). This agrees with the marking from Definition 3.23.

Now let \(\tau \in \Sigma(2)\) be an interior line segment in \(\Sigma\). To each \(\tau\) we associated a \(G\)-invariant ratio \(m_\tau^+ : m_\tau^-\) of monomials in the Cox ring \(S = \mathbb{C}[x, y, z]\) of \(X := \mathbb{C}^3 / G\). A vertex \(v\) marks the line segment \(\tau\) if \(v\) is the common character space of the monomials \(m_\tau^+\). We now show that this marking is compatible with Definition 3.21. In \(G\)-Hilb(\(\mathbb{C}^3\)), a \(G\)-igigsaw transformation along \(\tau\) from \(\sigma_+\) to \(\sigma_-\) moves every tile \(v\) such that the label of the path from 0 to \(v\) is divisible by \(m_\tau^+\). Let \(v\) be the unique vertex such that \(\text{div}(p) = m_\tau^-\). The vertex \(v\) is the unique irredundant generator of the degree one jigsaw piece \(\text{Jig}_\tau^+(1)\), and by Proposition 3.17 and Corollary 3.20 the marking of interior line segments in \(\Delta\) by the classical recipe of Reid and Craw agrees with that by Definition 3.21.

**Remark 3.28.** It follows from [Cra05, Corollary 4.6] that at least in the case of the \(G\)-Hilbert scheme, every vertex of the quiver \(Q\) appears ‘once’ in the marking of interior lattice points and line segments of \(\Sigma\).

3.5 Compatibility with geometric Reid’s recipe

We now consider geometric Reid’s recipe in the sense of Cautis-Logvinenko [CL09], Logvinenko [Log10], Cautis–Craw–Logvinenko [CCL12] and Bocklandt–Craw–Quintero-Vélez [BCQV15]. We will show that the marking of interior lattice points by vertices of \(Q\) according to Definition 3.23 is compatible with the generalisation by Bocklandt–Craw–Quintero-Vélez [BCQV15, Theorem 1.4] of the work of Logvinenko [Log10, Theorem 1.1]. We can only give a partial link here, namely, for the marking of interior lattice points and certain line segments that are marked with a single vertex. For the general statements we need first to prove that every vertex of \(Q\) appears ‘once’ on the fan \(\Sigma\).

Rather than using the functor \(\mathcal{L}_T^\text{L} : D^b(\text{mod} -A) \to D^b(\text{coh}(\mathcal{M}_A))\) from Theo-
rem 1.49 we consider
\[ \Psi(-) := T^\vee \otimes_A \mod A : D^b(M_A) \to D^b(\coh(M_A)) \]
where \( T^\vee = \bigoplus_v L_v^{-1} \). This equivalence is chosen to be compatible with the equivalence considered by [CL09]. Recall that for \( v \in Q_0 \), we write \( S_v \) for the vertex simple \( A \)-module \( C_{e_v} \). Bocklandt–Craw–Quintero-Vélez [BCQV15, Theorem 1.4] studied the image \( \Psi(S_v) \) for \( v \) a vertex of a quiver \( Q \) dual to a consistent dimer model generalising [Log10, Theorem 1.1]:

**Theorem 3.29.** [BCQV15, Theorem 1.4] Let \( \Gamma \) be a consistent dimer model with dual quiver \( Q \). Then for any \( v \in Q_0 \), \( \Psi(S_v) \) is quasi-isomorphic to one of the following:

(i) \( L_v^{-1} \) restricted to a connected union of compact irreducible torus-invariant divisors;

(ii) \( L_v^{-1} \) restricted to a compact torus-invariant curve;

(iii) \( F[1] \) for a sheaf \( F \) supported on a connected union of compact torus-invariant divisors;

(iv) The dualising complex of the compact exceptional locus \( f^{-1}(x_0) \), where \( f : M_A \to X \) is a crepant resolution and \( x_0 \) is the unique torus-invariant point of \( X \).

The insight of geometric Reid’s recipe, due to Cautis–Logvinenko [CL09], and expanded upon by Logvinenko [Log10] and Cautis–Craw–Logvinenko [CCL12], is that the support of the object \( \Psi(S_v) \) is closely related to classical Reid’s recipe. Theorem 3.29 parts (i) and (ii) suggest a marking of the interior lattice points and some of the interior line segments of \( \Sigma \) by vertices of \( Q \). For the purposes of this thesis, geometric Reid’s recipe marks a lattice point \( \rho \) with the vertex \( v \in Q_0 \) if \( \Psi(S_v) \) is quasi-isomorphic to \( L_v^{-1} \) restricted to the compact irreducible torus-invariant divisor \( D_\rho \). It marks a line segment \( \tau \) of \( \Sigma \) with the vertex \( v \in Q_0 \) if \( \Psi(S_v) \) is quasi-isomorphic to \( L_v^{-1} \) restricted to the compact torus-invariant curve \( C_\tau \). The next result provides the appropriate generalisation to consistent dimer models of a result of Cautis–Craw–Logvinenko [CCL12, Proposition 4.8].

**Lemma 3.30.** Let \( \sigma \in \Sigma(3) \) and suppose \( S_v \) lies in the socle of the corresponding torus-invariant \( \theta \)-stable \( A \)-module \( M_\sigma \). Then the vertex \( v \) marks either:

1. an edge of the triangle \( \sigma \), i.e., there exists \( \tau \in \Sigma(2) \) with \( \tau \subset \sigma \) such that \( v \) marks \( \tau \). In this case, \( S_v \) lies in the socle of every \( \theta \)-stable \( A \)-module in the curve \( C_\tau \) or

2. a node of the triangle \( \sigma \), i.e., there exists \( \rho \in \Sigma(1) \) with \( \rho \subset \sigma \) such that \( v \) marks \( \rho \). In this case, \( S_v \) lies in the socle of every \( \theta \)-stable \( A \)-module in the divisor \( D_\rho \).

**Proof.** Let \( y \in M_A \) denote the torus-invariant point defined by the cone \( \sigma \in \Sigma(3) \), so \( M_y = M_\sigma \) by Lemma 2.3. Since \( S_v \) lies in the socle of \( M_\sigma \), Remark 3.24(1) implies that \( v \) is not the zero vertex. Bocklandt–Craw–Quintero-Vélez [BCQV15, Propositions 3.4, 4.7, Lemma 4.8] shows

\[
Z_v = \{ y \in M_A \mid S_v \subseteq \soc(M_y) \}
\]
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is either a single \((-1, -1)-\)curve or a connected union of compact torus-invariant divisors in \(\mathcal{M}_A\). We have \(S_v \subseteq \text{soc}(M_y)\) by assumption, hence \(y \in Z_v\). We consider two cases.

If \(Z_v\) is a \((-1, -1)-\)curve in \(\mathcal{M}_A\) then there exists \(\tau \in \Sigma(2)\) such that \(Z_v = C_\tau\), so \(S_v\) lies in the socle of every \(\theta\)-stable \(A\)-module in \(C_\tau\). Also, \(y \in Z_v = C_\tau\) so the inclusion-reversing correspondence between orbit-closures in \(\mathcal{M}_A\) and cones in \(\Sigma\) gives \(\tau \subset \sigma\). By the proof of [BCQV15, Lemma 4.10], the tile dual to \(v\) is the only tile in \(\Gamma\) not contained in the zero jigsaw piece. Hence, \(v\) marks the line segment \(\tau\) by Reid’s recipe.

Otherwise, \(Z_v\) is a torus-invariant divisor. Since \(y \in Z_v\), there exists \(\rho \in \Sigma(1)\) such that \(y \in D_\rho \subseteq Z_v\), so \(S_v\) lies in the socle of every \(\theta\)-stable \(A\)-module in \(D_\rho\). Also, \(y \in Z_v = C_\tau\) so the inclusion-reversing correspondence between orbit-closures in \(\mathcal{M}_A\) and cones in \(\Sigma\) gives \(\tau \subset \sigma\) as required.

We now strengthen Lemma 3.30 slightly in order to present a geometric result:

**Proposition 3.31.** Let \(v \in Q_0\) be a nonzero vertex. If \(H^0(\Psi(S_v)) \neq 0\), then either:

1. there exists a line segment \(\tau \in \Sigma(2)\) such that vertex \(v\) marks \(\tau\), and the corresponding torus-invariant curve satisfies \(C_\tau = \text{supp}(\Psi(S_v))\); or
2. there exists a lattice point \(\rho \in \Sigma(1)\) such that vertex \(v\) marks \(\rho\), and the corresponding torus-invariant divisor satisfies \(D_\rho \subseteq \text{supp}(\Psi(S_v))\).

**Proof.** Since \(v\) is a nonzero vertex and \(H^0(\Psi(S_v)) \neq 0\), Bocklandt–Craw–Quintero-Vélez [BCQV15, Proposition 4.7] gives \(\sigma \in \Sigma(3)\) such that the corresponding torus-invariant \(\theta\)-stable \(A\)-module \(M_\sigma\) contains \(S_v\) in its socle. Now Lemma 3.30 produces either:

1. an interior line segment \(\tau \in \Sigma(2)\) such that \(v\) marks \(\tau\), and \(S_v\) lies in the socle of every \(\theta\)-stable \(A\)-module in the curve \(C_\tau\); or
2. an interior lattice point \(\rho \in \Sigma(1)\) such that \(v\) marks \(\rho\), and \(S_v\) lies in the socle of every \(\theta\)-stable \(A\)-module in the divisor \(D_\rho\).

It remains to note [BCQV15, Proposition 3.4] that since \(H^0(\Psi(S_v)) \neq 0\), the support of the object \(\Psi(S_v)\) is precisely the locus \(\{y \in \mathcal{M}_A \mid S_v \subseteq \text{soc}(M_y)\}\). □

The next result provides a partial link between combinatorial Reid’s recipe and geometric Reid’s recipe.

**Theorem 3.32 (Geometric Reid’s recipe for lattice points).** Let \(v \in Q_0\) be a nonzero vertex and let \(\rho \in \Sigma(1)\) be an interior lattice point of the triangulation \(\Sigma\). Then \(v\) marks \(\rho\) according to Reid’s recipe if and only if the divisor \(D_\rho\) is contained in the support of the sheaf \(\Psi(S_v)\).

**Proof.** Lemma 3.30 implies that \(v\) marks \(\rho\) if and only if \(S_v\) lies in the socle of some (equivalently, every) \(\theta\)-stable \(A\)-module in the divisor \(D_\rho\). This is equivalent to having \(D_\rho\) in the support of \(\Psi(S_v)\) by Bocklandt–Craw–Quintero-Vélez [BCQV15, Proposition 1.3]. □
3.6 Conjectures regarding combinatorial Reid’s recipe

We expect that in the marking of the fan $\Sigma$ by combinatorial Reid’s recipe, every nonzero vertex of $Q$ appears ‘once’ in the following sense:

**Conjecture 3.33.** In the marking of $\Sigma$ by combinatorial Reid’s recipe, every nonzero vertex $v$ of $Q$ appears either:

1. marking a collection of lattice points, connected by line segments of $\Sigma$;
2. marking a single line segment $\tau$;
3. marking a collection of line segments in $\Sigma$, such that any $\tau$ marked with $v$ shares an endpoint with another line segment $\tau'$ marked with $v$.

Moreover, a vertex $v \in Q_1$ marks a single line segment of $\Sigma$ by combinatorial Reid’s recipe if and only if $S_v$ lies in the socle of every $\theta$-stable $A$-module in the curve $C_\tau$, as in Proposition 3.31.

We also expect combinatorial Reid’s recipe to encode the relations between the line bundles in Pic($\mathcal{M}_A$):

**Conjecture 3.34.** Let $\rho$ be an interior line segment. Then the following relation holds in Pic($\mathcal{M}_A$):

\[
\bigotimes_{v \text{ marks } \rho} L_v = \bigotimes_{v \text{ marks at least 2 line segments attached to } \rho} L_v = \bigotimes_{v \text{ marks more than 2 line segments attached to } \rho} L_v.
\]

The search for a proof of these two conjectures has led us to the arrow contraction algorithm, which is motivated in the following chapter. The connection with the arrow contraction algorithm is explained in Chapter 7 (see Corollary 7.1).

3.7 Examples

**Example 3.35.** Let $G$ be a finite abelian subgroup of SL(3, $\mathbb{C}$). By Corollary 3.27, the examples calculated in Reid [Rei97], Craw [Cra05] and Cautis–Logvinenko [CL09] illustrate combinatorial Reid’s recipe for consistent dimer models. Example 1.56 illustrates combinatorial Reid’s recipe for a dimer model $\Gamma$ dual to the McKay quiver $Q$ of $G$, where $G \subset$ SL(3, $\mathbb{C}$) is generated by the diagonal matrix $\text{diag}(\varepsilon, \varepsilon^2, \varepsilon^3)$, $\varepsilon$ a primitive sixth root of unity.

**Example 3.36.** Consider the dimer model from Example 1.35. The stability parameter $\theta = (-9, 1, 1, \ldots, 1)$ is 0-generated and the fan $\Sigma$ of $\mathcal{M}_A$ is presented in Figure 1.7. Geometric Reid’s recipe suggests the marking of $\Sigma$ presented in Figure 3.7(a). Combinatorial Reid’s recipe gives a complete marking of the interior lattice points and interior line segments of the toric fan. Notice that unlike classical Reid’s recipe, two vertices, 3 and 9, mark an interior line segment of $\Sigma$. Also, the vertex 2 marks two interior lattice points of $\Sigma$. Moreover, the recipe for lattice points is not determined by the shape of the incoming line segments and their marking, as one of the interior lattice points of $\Sigma$ is marked by two vertices of $Q$ while the other by a single one.
Example 3.37. Figure 3.8 shows a consistent dimer model with its dual quiver. The stability parameter $\theta = (-21, 1, 1, \ldots, 1)$ is 0-generated. The fan $\Sigma$ of $\mathcal{M}_A$ is depicted in Figure 3.9. The interior line segments and nodes of $\Sigma$ are decorated using geometric and combinatorial Reid’s recipe. While geometric Reid’s recipe only marks a single line segment in $\Sigma$, combinatorial Reid’s recipe marks every interior line segment with a vertex of $Q$.

Figure 3.7: Marking of $\Sigma$ by geometric and combinatorial Reid’s recipe

Figure 3.8: A dimer model with its dual quiver
Example 3.38. Consider the dimer model in Figure 3.10. The fan $\Sigma$ of $\mathcal{M}_A$ for $\theta = (-25, 1, 1, \ldots, 1)$ is depicted in Figure 3.11. The interior line segments and interior lattice points of $\Sigma$ are decorated using geometric and combinatorial Reid’s recipe.

The first thing to note in this example is that there exists an interior lattice point of valency 7. This is unlike the fan of the $G$-Hilbert scheme, for $G$ a finite abelian subgroup of $\text{SL}(3, \mathbb{C})$, which Craw [CR02, Section 1.3] showed only has interior lattice points of valency 3, 4, 5 and 6. Combinatorial Reid’s recipe for $\Sigma$ also shows two vertices
7 and 21 marking a single interior line segment and one vertex 8 marking two interior lattice points of Σ.

![Diagram](image)

(a) Geometric

(b) Combinatorial

**Figure 3.11:** Marking of Σ by geometric and combinatorial Reid’s recipe
CHAPTER 4

THE CONVEX POLYGON $\Delta'$ AND THE OPEN SUBVARIETY $Y'$

This chapter motivates the arrow contraction algorithm. We begin by introducing an example which illustrates many of the results presented in the remaining of this thesis. We then associate to any arrow $\alpha$ in $Q$ with tail at vertex 0 a convex polygon $\Delta'$ and an open subvariety $Y'$ of $\mathcal{M}_A(\theta)$, for $\theta$ 0-generated and $\Gamma$ a consistent dimer model dual to $Q$. The aim of the arrow contraction algorithm will be to produce a new consistent dimer model $\Gamma'$ from our original consistent dimer model $\Gamma$, such that the characteristic polygon of $\Gamma'$ is $\Delta'$ and the moduli space $\mathcal{M}_A'(\theta')$, for $\theta'$ a 0-generated stability parameter, is $Y'$. Lemma 4.9 gives the first link between combinatorial Reid’s recipe and the toric fan of $Y'$.

4.1 Motivating example

Consider the consistent dimer model $\Gamma$ from Example 3.38. Figure 4.1 shows the dual quiver $Q$ of $\Gamma$. The characteristic polygon $\Delta(\Gamma)$ of $\Gamma$ is an irregular heptagon. The toric fan $\Sigma$ of the moduli space $\mathcal{M}_A$ of $\theta$-stable $A$-modules for the 0-generated stability parameter $\theta = (-25, 1, 1, \ldots, 1)$ is presented in Figure 4.2. The fan has 19 rays $\rho_1, \rho_2, \ldots, \rho_{19}$, marked 1, 2, $\ldots$, 19 in the toric picture, each corresponding to a torus-invariant prime divisor $D_i := D_{\rho_i}$ in $\mathcal{M}_A$ with $\theta$-stable perfect matching $\Pi_i$ ($1 \leq i \leq 19$). The isomorphism $\phi : A \rightarrow \text{End}(T)$ between the Jacobian algebra of $Q$ and the endomorphism algebra of the tautological bundle of $\mathcal{M}_A$ gives a label $\text{div}(a)$ to each arrow $a \in Q_1$. Figure 4.3 shows the quiver $Q$ with a selection of the arrows labelled. An arrow $a$ in Figure 4.3 labelled by $i_1, i_2, \ldots, i_j$ has labelling divisor $\text{div}(a) = D_{i_1} + D_{i_2} + \ldots + D_{i_j}$. For a list containing all of the labels for the arrows of $Q$ see Appendix B.
Pick the arrow $a_3$ in $Q$ with tail at vertex 0 and head at vertex 11. The arrow has labelling divisor $\text{div}(a_3) = D_4 + D_5 + D_8$. Note that the torus-invariant prime divisors $D_4$, $D_5$ and $D_8$ correspond to adjacent lattice points $\rho_4, \rho_5, \rho_8$ on the boundary of the characteristic polygon $\Delta(\Gamma)$. Moreover, the complement of all the triangles in $\Sigma$ containing one of the boundary lattice points $\rho_4$, $\rho_5$ or $\rho_8$ defines a convex polygon $\Delta'$. It is easy to verify that this is also the case for the arrows $a_1$ and $a_2$ in $Q$. In fact, both statements hold for any arrow $\alpha$ with tail at vertex 0 (see Lemma 4.3 and Proposition 4.6). Therefore, any choice of arrow $\alpha \in Q_1$ with tail at vertex 0 determines a convex polygon $\Delta'$ and strip of triangles $\text{strip}(\alpha)$. For $\alpha = a_3 \in Q_1$, Figure 4.2 shows its convex polygon $\Delta'$. Its strip of triangles $\text{strip}(a_3)$ is depicted in Figure 4.4 together with the names we use to denote the triangles and line segments contained in it.

![Figure 4.1: The quiver $Q$ dual to the dimer model from Example 3.38](image)

![Figure 4.2: The toric fan of $\mathcal{M}_A$ and convex polygon $\Delta'$ for $a_3 \in Q_1$](image)
Chapter 4. The convex polygon $\Delta'$ and the open subvariety $Y'$

Figure 4.3: The quiver $Q$ with a selection of arrows labelled

Figure 4.4: Strip of triangles strip($a_3$)
We would like to construct a consistent dimer model $\Gamma'$ whose characteristic polygon is $\Delta'$. For this, we define a subset of edges $S_{a_3} \subset \mathcal{E}$ that will be removed from $\Gamma$ in order to produce $\Gamma'$. Recall from Definition 2.12 that for any line segment $\tau$ in $\Sigma$ with endnodes $\rho$ and $\rho'$, we define the meandering walk $m_{\tau}$ of $\tau$ to be the set of of edges in the symmetric difference $\Pi_{\rho} \ominus \Pi_{\rho'}$. By Corollary 2.10, every meandering walk is the support of a cycle in $\Gamma$. Note that the line segments $\tau_0, \tau_1', \tau_3'$ and $\tau_6$ are all boundary line segments on $\Delta(\Gamma)$, and Remark 2.15 shows that their meandering walks support zigzag paths. We will use the meandering walks of the line segments in strip($a_3$) to define the set of edges $S_{a_3}$.

For $0 \leq i \leq 6$ and $0 \leq j \leq 5$, write $m_i$ and $m'_j$ for the meandering walks of $\tau_i$ and $\tau'_j$, respectively. In order to create a dimer model $\Gamma'$ that is consistent, we consider what its zig-zag paths must look like. By Theorem 1.46, there should be one zigzag path $z$ for each line segment $\tau$ in the boundary of $\Delta'$, such that sup$(j) = m_{\tau}$ and the homology class of $z$ is the inward-pointing normal vector to $\tau$. As $\Gamma$ is consistent, $\Gamma'$ already has a zigzag path for every boundary line segment of $\Sigma$ except for $\tau'_0, \tau'_2, \tau'_4$ and $\tau'_5$. By Lemma 2.14, the homology classes of the meandering walks $m'_0, m'_2, m'_4, m'_6$ are normal to the line segment $\tau'_0, \tau'_2, \tau'_4$ and $\tau'_5$, respectively, making them candidates for the missing zigzag paths. Thus, every edge we remove from $\Gamma$ should not be an edge in any of these meandering walks. Edges in $S_{a_3}$ would ideally prevent the zigzag paths $m_0, m'_1, m'_3, m_6$ from appearing in $\Gamma'$ and would make the meandering walks $m'_0, m'_2, m'_4, m'_5$ into zigzag paths. Table 4.1 lists all meandering walks $m_\tau$ for $\tau$ a line segment in strip($a_3$). To simplify notation $e_i$ denotes the edges in $\Gamma$ dual to the arrow $a_i$ in $Q$.

<table>
<thead>
<tr>
<th>Walk</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_0$</td>
<td>$e_3, e_{29}, e_{31}, e_{42}, e_{47}, e_{63}, e_{64}, e_7, e_{11}, e_5, e_{10}, e_{16}, e_{20}, e_{24}, e_{27}, e_{37}$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$e_3, e_{31}, e_{10}, e_{16}, e_{20}, e_{24}, e_{27}, e_{37}$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$e_3, e_{31}, e_{17}, e_{32}, e_{36}, e_{46}, e_{50}, e_{66}, e_{67}, e_{14}, e_{21}, e_{13}, e_{18}, e_{16}, e_{20}, e_{24}, e_{27}, e_{37}$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$e_3, e_{31}, e_{17}, e_{34}, e_{20}, e_{24}, e_{27}, e_{37}$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>$e_3, e_{31}, e_{17}, e_{34}, e_{25}, e_{40}, e_{49}, e_{53}, e_{69}, e_{70}, e_{22}, e_{26}, e_{24}, e_{27}, e_{37}$</td>
</tr>
<tr>
<td>$m_5$</td>
<td>$e_3, e_{31}, e_{17}, e_{34}, e_{25}, e_{38}, e_{27}, e_{37}$</td>
</tr>
<tr>
<td>$m_6$</td>
<td>$e_3, e_{31}, e_{17}, e_{34}, e_{25}, e_{38}, e_{27}, e_{2}, e_{56}, e_{59}, e_{72}, e_{55}, e_{52}, e_{43}, e_{39}, e_{39}$</td>
</tr>
<tr>
<td>$m'_0$</td>
<td>$e_{29}, e_{33}, e_{42}, e_{47}, e_{63}, e_{64}, e_7, e_{11}, e_5, e_{31}$</td>
</tr>
<tr>
<td>$m'_1$</td>
<td>$e_{10}, e_{18}, e_{13}, e_{21}, e_{14}, e_{67}, e_{66}, e_{50}, e_{46}, e_{36}, e_{32}, e_{17}$</td>
</tr>
<tr>
<td>$m'_2$</td>
<td>$e_{16}, e_{18}, e_{13}, e_{21}, e_{14}, e_{67}, e_{66}, e_{50}, e_{46}, e_{36}, e_{32}, e_{34}$</td>
</tr>
<tr>
<td>$m'_3$</td>
<td>$e_{20}, e_{26}, e_{22}, e_{70}, e_{69}, e_{53}, e_{49}, e_{40}, e_{35}, e_{25}$</td>
</tr>
<tr>
<td>$m'_4$</td>
<td>$e_{24}, e_{26}, e_{22}, e_{70}, e_{69}, e_{53}, e_{49}, e_{40}, e_{35}, e_{38}$</td>
</tr>
<tr>
<td>$m'_5$</td>
<td>$e_{37}, e_{2}, e_{56}, e_{59}, e_{72}, e_{55}, e_{52}, e_{43}, e_{39}, e_{30}$</td>
</tr>
</tbody>
</table>

Table 4.1: Edges of the meandering walks coming from line segments in strip($a_3$)

The first thing to note from Table 4.1 is that $e_3$, the arrow dual to our chosen arrow $a_3 \in Q_1$, appears in all the meandering walks $m_0, m_1, \ldots, m_6$. This is always the case for any $\alpha \in Q_1$ with tail at vertex 0 (see proof of Lemma 5.5). Also note that the meandering walks $m'_0, m'_2, m'_4, m'_5$ share no edges and each of their edges belongs to one of the zigzag paths $m_0, m'_1, m'_3, m_6$ of $\Gamma$. For example, $m'_0$ consists of nine edges from $m_0$ and one edge from $m_6$. The meandering walk $m'_2$ shares an edge with $m_0$,
ten edges with $m'_1$ and an edge with $m_6$. Therefore, $m'_0, m'_2, m'_4, m'_5$ have each a natural direction inherited from the zigzag paths they share edges with. The edges at which these meandering walks with their natural direction fail to be zigzag are zigs and zags of $m_0, m'_1, m'_3, m_6:

$$s_0 = e_3 \in m_0 \cap m_6, \quad s_1 = e_{10} \in m_0 \cap m'_1, \quad s_2 = e_{17} \in m_6 \cap m'_1,$$

$$s_3 = e_{20} \in m_0 \cap m'_3, \quad s_4 = e_{25} \in m_6 \cap m'_3, \quad s_5 = e_{27} \in m_0 \cap m_6.$$ 

These edges make the set $S_{a_3} = (m_0 \cup m_6) \setminus (m'_0 \cup m'_2 \cup m'_4 \cup m'_5)$ as introduced in Definition 5.4.

Figure 4.5: The meandering walks $m'_0, m'_2, m'_4, m'_5$ and the edges at which they fail to be zigzag

Figure 4.5 shows the meandering walks $m'_0, m'_2, m'_4, m'_5$ with their natural direction and the edges in $S_{a_3}$. Notice that the edges in the set $S_{a_3}$ are those that prevent the lower meandering walks $m'_0, m'_2, m'_4, m'_5$ from turning maximally right at a white node and maximally left at a black node, i.e., they prevent these walks from being zigzags (see Lemma 6.10 and Remark 6.11). Observe that the endnodes of each of the edges in $S_{a_3}$ is a trivalent node of a fundamental hexagon $\text{Hex}(\sigma)$ for $\sigma$ a triangle in strip($a_3$). Indeed, the black endnode of $e_3$ is attached to the edges $e_3 \in m_0 \cap m_1, e_{25} \in m_6 \cap m'_4$ and $e_{31} \in m_1 \cap m'_0$, making it a trivalent node of $\text{Hex}(\sigma_0)$. Write $b_i$ and $w_i$ for the black and white trivalent nodes of $\text{Hex}(\sigma_i)$ ($0 \leq i \leq 5$), respectively. It is an easy exercise to verify that for $0 \leq i \leq 5$, the edge $s_i$ has endnodes $w'_{i-1}$ and $b_i$, where the overline denotes addition modulo 6.

Figure 4.6 shows a schematic depicting all edges in $S_{a_3}$ and all meandering walks $m_\tau$ for $\tau$ a line segment in strip($a_3$), called the circuit diagram for $a_3$. The circuit diagram encodes much of the structure described earlier. Although in this example $w_0 = w_1$ and $w_2 = w_3$, we intentionally draw the circuit diagram in its most general form assuming
there is an even number of edges (possibly zero) connecting two trivalent nodes of the same colour in $\Gamma$. Notice that the walk $m_1$ splits from $m_0$ at $b_0$ and travels up to $w_1$ with each of $m_i$ (for $2 \leq i \leq 6$). At $w_1$ it splits from these, travelling along $m'_0$ until it rejoins $m_0$ at $w_0$, and then travels with $m_0$ until $b_0$, when it completes a cycle around $T$. The meandering walks $m_2, m_3, m_4$ and $m_5$ exhibit a similar behaviour. All of them split from $m_0$ at $b_0$ and travel with $m_6$ until a trivalent node of a triangle in strip($a_3$), when they split from $m_6$ to eventually rejoin $m_0$ at another trivalent node. For $1 \leq i \leq 5$, the walk $m_i$ splits from $m_6$ before the walks $m_{i+1}, m_{i+2}, \ldots, m_5$ do, and rejoins $m_0$ after $m_1, m_2, \ldots, m_{i-1}$ do. Because of this orderly behaviour, the walks $m_1, m_2, m_3, m_4$ and $m_5$ appear to make a Mexican wave in the circuit diagram.

![Circuit diagram for $a_3$](image)

### Figure 4.6: Circuit diagram for $a_3$

#### 4.2 The convex polygon $\Delta'$ and the open subvariety $Y'$

Let $\Gamma = \{\mathfrak{B}, \mathfrak{W}, \mathfrak{E}\}$ be a consistent dimer model and let $Q = \{Q_0, Q_1, h, t, R\}$ be its dual quiver with relations. Write $A$ for the Jacobian algebra of $Q$. Recall that for a 0-generated stability parameter $\theta$ (see Definition 3.1):

(i) The moduli space $M_A$ of $\theta$-stable $A$-modules provides a projective crepant resolution $f : M_A \to X$ of the affine Gorenstein toric threefold $X := \text{Spec}(Z(A))$; and

(ii) The tautological $\mathbb{C}$-algebra morphism $\phi : A \to \text{End}_{O_{M_A}}(T)$ between the algebra $A$ and the endomorphism algebra of the tautological bundle $\bigoplus_{v \in Q_0} L_v$ on $M_A$ is an isomorphism.

The tautological $\mathbb{C}$-algebra isomorphism $\phi : A \to \text{End}_{O_{M_A}}(T)$ associates to each path $p$ a label $\text{div}(p)$. A divisor $D$ is said to be a summand of $\text{div}(p)$ if and only if $\text{div}(p) - D \geq 0$. Every lattice point $\rho$ in $\Sigma$ corresponds to a torus-invariant prime divisor $D_\rho$. It follows from equation (1.5.1) that for $a \in Q_1$, $D_\rho$ is a summand of $\text{div}(a)$ if and only if its dual edge $e_\alpha$ lies in the $\theta$-stable perfect matching $\Pi_\rho$.

We distinguish between two different types of prime divisors:
Definition 4.1. Every lattice point \( \rho \in \Sigma \) lies either:

(i) on the boundary of \( \Delta \), in which case we call \( D_\rho \) a boundary divisor of \( \mathcal{M}_A \); or

(ii) in the interior of \( \Delta \), in which case we call \( D_\rho \) an interior divisor of \( \mathcal{M}_A \).

Lemma 4.2. Let \( a \) be an arrow in \( Q_1 \) with head at vertex 0. Then every interior divisor of \( \mathcal{M}_A \) is a summand of \( \text{div}(a) \).

Proof. Let \( x_0 \) denote the unique-torus invariant point of \( X \). Pick any point \( y \in \mathcal{M}_A \) in an interior divisor \( D_\rho \), so that \( y \in f^{-1}(x_0) \), where \( f : \mathcal{M}_A \to X \) is a crepant resolution of \( X \). By [BCQV15, Lemma 3.1 (ii)], all maps in \( M_y \) with head at vertex 0 are zero. Therefore, \( e_\alpha \) is an edge of the perfect matching \( \Pi_\rho \) and \( D_\rho \) is a summand of \( \text{div}(a) \). As this holds for every interior divisor \( D_\rho \) of \( \mathcal{M}_A \), the result follows.

Suppose the characteristic polygon \( \Delta(\Gamma) \) of \( \Gamma \) has \( n \) lattice points \( \rho_0, \rho_1, \ldots, \rho_{n-1} \) arranged clockwise around its boundary. For \( 0 \leq i \leq n - 1 \), let \( D_i := D_{\rho_i} \) and \( \Pi_i := \Pi_{\rho_i} \) denote the boundary divisor and the \( \theta \)-stable perfect matching associated to \( \rho_i \). Recall from Remark 2.15 that any zigzag path in \( \Gamma \) is supported on the symmetric difference of two \( \theta \)-stable perfect matchings coming from adjacent boundary divisors. For \( 0 \leq i \leq n - 1 \), write \( z_i \) for the zigzag path supported on the symmetric difference \( \Pi_i \cap \Pi_{i+1} \), where the overline denotes addition modulo \( n \).

Pick once and for all an arrow \( a \in Q_1 \) with tail at vertex 0.

Lemma 4.3. After relabelling the boundary divisors, there exist consecutive boundary divisors \( D_1, D_2, \ldots, D_r \) in \( \Delta(\Gamma) \) such that \( \text{div}(a) = D_1 + D_2 + \cdots + D_r \) for some \( 1 \leq r \leq n - 1 \).

Proof. Since \( \Gamma \) is consistent, the edge \( e_\alpha \) appears in exactly two distinct zigzag paths, once as a zig and once as a zag. Without loss of generality, assume \( e_\alpha \) lies in the support of \( z_0 \) and \( z_r \) for some \( 1 \leq r \leq n - 1 \). Recall that for \( 0 \leq i \leq n - 1 \), an edge \( e \in \text{supp}(z_i) = \Pi_i \cap \Pi_{i+1} \) is a zig (resp. zag) of \( z_i \) if and only if \( e \) lies in \( \Pi_{i+1} \) (resp. \( \Pi_i \)). Hence either \( \text{zig}(e_\alpha) = z_0 \) and \( \text{zag}(e_\alpha) = z_r \), in which case \( e_\alpha \in \Pi_1 \cap \Pi_r \), or \( \text{zig}(e_\alpha) = z_0 \) and \( \text{zag}(e_\alpha) = z_r \), which implies that \( e_\alpha \in \Pi_0 \cap \Pi_{r+1} \).

Suppose the first holds and \( e_\alpha \in \Pi_1 \cap \Pi_r \). The fact that \( e_\alpha \) is not in the support of \( z_i \) for \( i \neq 1, r \) means both that \( e_\alpha \in \Pi_2, \Pi_3, \ldots, \Pi_{r-1} \) and \( e_\alpha \notin \Pi_{r+1}, \Pi_{r+2}, \ldots, \Pi_n, \Pi_0 \).

Therefore, the only boundary divisors that appear as summands of the label of \( \alpha \) are \( D_1, D_2, \ldots, D_r \), whose lattice points \( \rho_1, \rho_1, \ldots, \rho_r \) lie in a segment of the boundary of \( \Delta(\Gamma) \). When \( \text{zag}(e_\alpha) = z_0 \) and \( \text{zig}(e_\alpha) = z_r \), the same argument shows that the boundary divisors in the label of \( \alpha \) are \( D_{r+1}, D_{r+2}, \ldots, D_0 \). Their corresponding lattice points also lie in a segment of the boundary of \( \Delta(\Gamma) \). After relabelling, \( \text{div}(a) = D_1 + D_2 + \cdots + D_r \) for some \( 1 \leq r \leq n - 1 \).

It remains to show that every summand of \( \text{div}(a) \) is a boundary divisor. The edge \( e_\alpha \) dual to the arrow \( \alpha \) shares a node with an edge \( e_\alpha \) dual to an arrow \( a \in Q_1 \) with head at vertex 0. By Lemma 4.2, all interior divisors of \( \mathcal{M}_A \) are summands of \( \text{div}(a) \). Therefore, \( e_\alpha \in \Pi_\rho \) for all interior divisors \( D_\rho \). It follows that \( e_\alpha \notin \Pi_\rho \) for all interior divisors \( D_\rho \) of \( \mathcal{M}_A \), as \( \Pi_\rho \) is a perfect matching. Hence, no interior divisor of \( \mathcal{M}_A \) is a summand of \( \text{div}(a) \).
Remark 4.4. The proof of Lemma 4.3 also shows that the segment of the boundary of $\Delta(\Gamma)$ determined by lattice points of the summands of $\text{div}(\alpha)$ is bounded by the line segments corresponding to the zigzag paths $\text{zig}(\epsilon_a)$ and $\text{zag}(\epsilon_a)$. Raf Bocklandt proved this result and Lemma 4.3 independently in unpublished work.

Definition 4.5. By Lemma 4.3, after relabelling the boundary divisors, there exist consecutive boundary divisors $D_1, D_2, \ldots, D_r$ in $\Delta(\Gamma)$ such that $\text{div}(\alpha) = D_1 + D_2 + \ldots + D_r$ for some $1 \leq r \leq n - 1$. Define

$$\Sigma' := \{ \sigma \in \Sigma \mid \rho_i \not\in \sigma \text{ for all } 1 \leq i \leq r \}$$

(4.2.1)

to be the fan made of all cones $\sigma \in \Sigma$ such that $\rho_i$ is not a ray of $\sigma$ ($1 \leq i \leq r$). Write $\Delta' := \Sigma' \cap \Delta(\Gamma)$ for the cross-section of $\Sigma'$ at height one.

Proposition 4.6. The subset $\Delta'$ of $\Delta$ is a convex polygon.

Proof. Our choice of 0-generated stability parameter $\theta$ implies that the line bundle $L_{h(\alpha)}$ is globally generated and that the subset of $\Delta(\Gamma)$

$$P(\alpha) := \left\{ p \in \Delta(\Gamma) \mid \exists \sigma \in \Sigma(3) \text{ such that } p \in \sigma \cap \Delta(\Gamma) \text{ and } x^{\text{div}(\alpha)} \text{ generates } L_{h(\alpha)}|_{U_{\sigma}} \text{ as an O}_{U_{\sigma}}\text{-module} \right\}$$

is convex (see proof of [Cra05, Lemma 5.3(i)]). Hence, it suffices to show that $\Delta' = P(\alpha)$. Let $\sigma \in \Sigma(3)$ be a three-dimensional cone and let $y \in U_{\sigma} \subseteq M_{\alpha}$ denote the origin in $U_{\alpha} \cong \mathbb{C}^3$. Suppose $p \in \sigma \cap \Delta(\Gamma)$. Then, $p \in P(\alpha)$ if and only if the corresponding torus-invariant $\theta$-stable representation $M_\sigma$ has a path from 0 to $v \in Q_0$ that begins along the arrow $\alpha$. Recall that the cosupport of $M_\sigma$ is the union of the three perfect matchings corresponding to the rays of $\sigma$. Therefore, $\alpha \in Q_1$ is in $M_\sigma$ if and only if $\rho \not\in \sigma$ for all $\rho \in \Sigma_\alpha(1) := \{ \rho \in \Sigma(1) \mid \text{div}(\alpha) - D_\rho \geq 0 \}$. This is true if and only if $\sigma$ is a cone in $\Sigma'$. Hence, the equality holds, and the result follows.

Definition 4.7. A polygon is said to be nondegenerate if not all of its vertices lie on a straight line.

Theorem 4.8. Suppose $\Delta'$ is a nondegenerate polygon. Let $Y'$ denote the toric variety defined by the fan $\Sigma'$ from (4.2.1) and let $\sigma' := \left\{ \sum_{\lambda_\rho \geq 0} \lambda_\rho u_\rho \in \mathbb{N}_\mathbb{R} \mid u_\rho \in \Delta' \right\}$ denote the cone over $\Delta'$. Then:

1. $Y'$ is the open subvariety of $M_{\alpha}$ satisfying $Y' = M_{\alpha} \setminus \text{supp(div(\alpha))}$;

2. $Y'$ is a crepant resolution of the affine Gorenstein toric variety $U_{\sigma'}$.

Proof. This follows from Proposition 4.6. As $\Delta'$ is a nondegenerate convex polygon, the cone $\sigma'$ defines a three-dimensional affine toric variety $U_{\sigma'}$. The variety $U_{\sigma'}$ is Gorenstein as all the primitive ray generators of $\sigma'$ lie in a hyperplane. The nondegeneracy and convexity of $\Delta'$ also imply that the fan $\Sigma'$ defines a three-dimensional toric variety $Y'$. The variety $Y'$ is smooth as each of its cones is a cone of $M_{\alpha}$, which is itself a smooth variety. As $\Sigma'$ is a subfan of $\Sigma$, $Y'$ is a subvariety of $M_{\alpha}$, and by construction $Y' = M_{\alpha} \setminus \text{supp(div(\alpha))}$. Finally, since $\Sigma'$ is determined completely by a triangulation of the slice $\Delta'$ of $\sigma'$, $Y'$ is a crepant resolution of $U_{\sigma'}$.

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The following lemma gives the first connection between combinatorial Reid’s recipe, the arrow $\alpha$ and certain line segments of $\Sigma'$:

**Lemma 4.9.** The line segments along $\partial \Sigma' \setminus \partial \Sigma$ have label $h(\alpha)$ by Reid’s recipe, as line segments of $\Sigma$.

**Proof.** Consider a jigsaw transformation along any of the line segments $\tau$ in $\partial \Sigma' \setminus \partial \Sigma$. We need to show that $h(\alpha)$ is the generator of the quiver $Q_{\pm}(\tau)$. First, note that the tile dual to $h(\alpha)$ is not in $\text{Jig}_\tau(0)$, as the arrow $\alpha$ is contained in the the perfect matchings of $D_1, \ldots, D_r$. Therefore, $h(\alpha)$ is a vertex of $Q_{\pm}(\tau)$. Suppose $h(\alpha)$ is not a generator of $Q_{\pm}(\tau)$, then there exists a path $p$ in $Q$ from 0 to a vertex $v \in Q_0$ such that $\text{div}(p)$ is a summand of $\text{div}(\alpha)$. In particular, there exists an arrow $\alpha'$ with tail at zero, such that $\text{div}(\alpha')$ is a summand of $\text{div}(\alpha)$. The divisor $D = \text{div}(\alpha) - \text{div}(\alpha')$ defines a section of $L_{h(\alpha)} \otimes L_{\text{div}(\alpha')}^{-1}$ and hence a path in $Q$. So, there is a relation in which $\alpha$ appears as a linear term, a contradiction, as $\Gamma$ has no bivalent nodes. \qed
CHAPTER 5

THE SUBSET OF EDGES $S_\alpha$

In Chapter 4, we associated to each arrow $\alpha \in Q_1$ with tail at vertex 0 a convex polygon $\Delta'$ and an open subvariety $Y' \hookrightarrow \mathcal{M}_A$. The aim of the arrow contraction algorithm is to generate a new dimer model $\Gamma'$ from $\Gamma$, whose associated affine Gorenstein toric variety is $U_{\sigma'}$, for $\sigma'$ the cone over $\Delta'$, and its moduli space of $\theta'$-stable $A'$-modules, for $\theta'$ a 0-generated stability parameter, is $Y'$. In order to define $\Gamma'$ we need to determine a subset of edges $S_\alpha \subseteq \mathcal{E}$ that will be removed from $\Gamma$.

In this chapter, we define the set $S_\alpha$ in terms of the meandering walks of line segments in a strip of triangles $\text{strip}(\alpha)$. Assuming Conjecture 5.7, we give two alternative characterisations of $S_\alpha$ using the triangles in $\text{strip}(\alpha)$ and the trivalent nodes of their fundamental hexagons (see Proposition 5.15 and Theorem 5.16). These characterisations will be useful in proving that the dimer model obtained from $\Gamma$ by removing the edges in $S_\alpha$ is consistent (see Theorem 6.15). We conclude the chapter by introducing the circuit diagram for $\alpha$, which gives a schematic depiction of the edges in $S_\alpha$ and all meandering walks $m_\tau$, where $\tau$ is a line segment in $\text{strip}(\alpha)$.

5.1 The strip of triangles $\text{strip}(\alpha)$

Let $\Gamma$ be a consistent dimer model with dual quiver $Q$. For a 0-generated stability parameter $\theta$, let $\Sigma$ denote the fan of $\mathcal{M}_A$. Fix $\alpha \in Q_1$ with tail at vertex 0.

**Definition 5.1.** Define

$$\text{strip}(\alpha) := \Sigma \setminus \Sigma'$$

to be the set of triangles in the complement of $\Sigma'$.

The geometry of $\text{strip}(\alpha)$ forces it to be a strip of triangles, which justifies the notation. Let $\sigma_0, \ldots, \sigma_k$ be the $k + 1$ triangles along the strip $\text{strip}(\alpha)$ arranged in a clockwise direction. For $1 \leq i \leq k$, let $\tau_i$ denote the line segment $\sigma_{i-1} \cap \sigma_i$. Write $\tau_0$ and $\tau_{k+1}$ to be the line segments of $\sigma_0$ and $\sigma_k$ which form part of the boundary of $\Sigma$. For $0 \leq i \leq k$, write $\tau'_i$ for the line segment of $\sigma_i$ that does not coincide with either $\tau_{i-1}$ or $\tau_i$. 

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We distinguish between two different types of triangles in \(\text{strip}(\alpha)\):

**Definition 5.2.** For \(0 \leq i \leq k\), each \(\tau'_i\) is either:

(i) on the boundary of \(\Sigma'\), in which case we call \(\sigma_i\) a *lower triangle*; or

(ii) on the boundary \(\Sigma\), in which case we call \(\sigma_i\) an *upper triangle*.

Recall that to any two-dimensional cone \(\tau\) in \(\Sigma\) we associate a subset of edges of \(\Gamma\) called the meandering walk \(m_{\tau}\) of \(\tau\) (see Definition 2.12).

**Definition 5.3.** For \(\tau\) a line segment in \(\text{strip}(\alpha)\), the meandering walk \(m_{\tau}\) is called:

(i) a *lower meandering walk* of \(\text{strip}(\alpha)\) if \(\tau = \tau'_i\) for a lower triangle \(\sigma_i\) \((0 \leq i \leq k)\);

(ii) a *middle meandering walk* of \(\text{strip}(\alpha)\) if \(\tau = \tau_i\) for \(1 \leq i \leq k\);

(iii) an *upper meandering walk* of \(\text{strip}(\alpha)\) if \(\tau = \tau'_i\) for an upper triangle \(\sigma_i\) \((0 \leq i \leq k)\) or \(\tau = \tau_0\) or \(\tau_k\).

To simplify notation, we write \(m_i\) and \(m'_i\) for the meandering walks \(m_{\tau_i}\) and \(m_{\tau'_i}\) \((0 \leq i \leq k)\) and \(m_{k+1}\) for \(m_{\tau_{k+1}}\). Both upper and lower meandering walks carry a natural direction as follows:

(i) Every upper meandering walk supports a zigzag path in \(\Gamma\) (see Remark 2.15) which has a direction determined by the fact that it turns maximally right at white nodes and maximally left at black nodes;

(ii) To each lower meandering walk \(m_{\tau}\) we assign the direction \([\Pi_{\rho'} - \Pi_{\rho}]\), where \(\rho\) and \(\rho'\) are the ray generators of \(\tau\) and \(\rho'\) is adjacent to \(\rho\) in a clockwise direction along the boundary of \(\Sigma'\). With this convention, the homology class of a lower meandering walk is the inward-pointing normal vector to the boundary edge of \(\Delta'\).

We use the meandering walks of \(\text{strip}(\alpha)\) to define a subset of edges \(S_\alpha \subset E\) as follows:

**Definition 5.4.** Define the subset of \(E\) given by

\[
S_\alpha := (m_0 \cup m_{k+1}) \setminus \bigcup_{m \text{ a lower meandering walk}} m.
\]
Lemma 5.5. The edge \( e_\alpha \) is an element of \( S_\alpha \).

Proof. The edge \( e_\alpha \) lies in \( \text{supp}(j_0) = m_0 \) and \( \text{supp}(j_r) = m_{k+1} \). For all \( 1 \leq i \leq k \), write \( \rho'_i \) and \( \rho''_i \) for the ray generators of \( \tau_i \). Since by Lemma 4.3 the label \( \text{div}(\alpha) = D_1 + D_2 + \cdots + D_r \) contains no interior divisors, \( e_\alpha \in \Pi \rho'_i \cap \Pi \rho''_i = m_i \) for all \( 1 \leq i \leq k \). Moreover, as \( e_\alpha \) is an edge in \( m_0, m_1, \ldots, m_{k+1} \), \( e_\alpha \notin m_i \) for all \( 0 \leq i \leq k \). In particular, \( e_\alpha \notin m \) for every lower meandering walk \( m \) and hence \( e_\alpha \) is an element of \( S_\alpha \). \( \square \)

Lemma 5.6. The black and white endnodes of \( e_\alpha \) are the black and white trivalent nodes of \( \text{Hex}(\sigma_0) \) and \( \text{Hex}(\sigma_k) \), respectively.

Proof. By the proof of Lemma 5.5, \( e_\alpha \) is an edge in \( m_0, m_1, \ldots, m_{k+1} \). Moreover, since \( m_0 \) and \( m_{k+1} \) support the zigzag paths \( j_0 \) and \( j_r \) and the edge \( e_\alpha \) is a zig of \( j_0 \) and a zag of \( j_r \), the edges \( e \in m_0 \) and \( e' \in m_{k+1} \) following \( e_\alpha \) are dual to arrows with head at vertex 0. Therefore, by Lemma 4.2 the labels \( \text{div}(\alpha) \) and \( \text{div}(\alpha') \) have all the interior divisors of \( M_A \) as their summands. It follows that \( e \in m_0 \cap m_1 \cap \cdots \cap m_k \cap m'_k \) and \( e' \in m'_0 \cap m_1 \cap m_2 \cap \cdots \cap m_{k+1} \). Therefore the white node common to both \( e \) and \( e_\alpha \) and the black node common to both \( e' \) and \( e_\alpha \) are the white and black trivalent nodes of \( \text{Hex}(\sigma_k) \) and \( \text{Hex}(\sigma_0) \), respectively. \( \square \)

5.2 Walking along \( m_0 \) and \( m_{k+1} \)

In this section, we describe the behaviour of the middle meandering walks \( m_1, m_2, \ldots, m_k \) as we travel along \( m_0 \) and \( m_{k+1} \). This understanding will enable us to characterise the edges in \( S_\alpha \) in terms of the triangles in \( \text{strip}(\alpha) \) and the trivalent nodes of their fundamental hexagons (see Theorem 5.16).

![Figure 5.2: Edges \( e_i, e'_i, e''_i \) and \( f_i, f'_i, f''_i \) in \( \text{Hex}(\sigma_i) \)](image)

For the purposes of this section, we assign to \( m_0 \) and \( m_{k+1} \) the direction opposite to that of the zigzag paths supported on them. We say two meandering walks travel together between two nodes \( n \) and \( n' \) if they share a sequence of edges \( \{e_1, e_2, \ldots, e_j\} \) in
\(\mathcal{E}\) and \(n\) and \(n'\) are endpoints of \(e_1\) and \(e_j\), respectively. We say two meandering walks travelling together along a set of edges \(\{e_1, e_2, \ldots, e_j\}\) split at the endnode \(n\) of \(e_j\) if the edges following \(e_j\) are different for both meandering walks. For \(0 \leq i \leq k\), write \(m_i\) and \(b_i\) for the white and black trivalent nodes of \(\text{Hex}(\sigma_i)\) in \(\Gamma\) and let \(e_i, e'_i\) and \(f_i, f'_i\) denote the edges in \(\partial \text{Hex}(\sigma_i)\) around \(w_i\) and \(b_i\), respectively so that \(e_i, f_i \in m_i \cap m_{i+1}\), \(e'_i, f'_i \in m_i \cap m'_i\) and \(e''_i, f''_i \in m_{i+1} \cap m'_i\), as depicted in Figure 5.2.

### 5.2.1 Walking along \(m_0\)

We assume the following conjecture:

**Conjecture 5.7.** Let \(\sigma_i\) and \(\sigma_{i+1}\) be triangles in the strip \(\text{strip}(\alpha)\), then:

1. If \(\sigma_i\) is an lower triangle and \(\sigma_{i+1}\) is a upper triangle, \(e_i \in m_{i+1}'\);
2. If \(\sigma_i\) is an upper triangle and \(\sigma_{i+1}\) is a lower triangle, \(f_i \in m_{i+1}'\);
3. If \(\sigma_i\) and \(\sigma_{i+1}\) are lower triangles, \(e_i \in m_{i+2}\) and \(f_{i+1} \in m_0\);
4. If \(\sigma_i\) and \(\sigma_{i+1}\) are upper triangles, \(f_i \in m_{i+2}\) and \(e_{i+1} \in m_0\);
5. If \(e \in m_0 \cap m_1 \cap \cdots \cap m_i \cap m'_i\) for an upper meandering walk \(m_i', e\) is not an edge in any lower meandering walk of \(\text{strip}(\alpha)\).

Lemmata 5.8 – 5.11 enable us to describe the walk along \(m_0\) inductively as we jump between adjacent triangles in the strip \(\text{strip}(\alpha)\) in a clockwise direction.

![Figure 5.3: Depiction of results in Lemma 5.8](image)

**Lemma 5.8.** Suppose \(\sigma_i\) is a lower triangle and \(\sigma_{i+1}\) is an upper triangle and suppose:

1. The edge \(e_i\) lies in \(m_0, m_1, \ldots, m_{i+1}\); and
2. \(m_0, m_1, \ldots, m_{i+1}\) travel together from \(w_i\) until \(b_0\).

Then the white and black nodes of \(e_i\) are the white trivalent node \(w_i\) of \(\text{Hex}(\sigma_i)\) and the black trivalent node \(b_{i+1}\) of \(\text{Hex}(\sigma_{i+1})\). Moreover:

1. The edge \(f_{i+1}\) lies in \(m_0, m_1, \ldots, m_{i+2}\); and
Lemma 5.9. Suppose $\sigma_i$ is an upper triangle and $\sigma_{i+1}$ is a lower triangle and suppose:

(i) The edge $f_i$ lies in $m_0, m_1, \ldots, m_{i+1}$; and

(ii) $m_0, m_1, \ldots, m_{i+1}$ travel together from $b_i$ until $b_0$.

Then, the walks $m_0, \ldots, m_{i+1}$ meet $m_{i+1}$ and moreover:

(i') The edge $e_{i+1}$ lies in $m_0, m_1, \ldots, m_{i+2}$; and

(ii') $m_0, m_1, \ldots, m_{i+2}$ travel together from $w_{i+1}$ until $b_0$.

Proof. By Conjecture 5.7 (ii), $f_i \in m'_{i+1}$. Travel along $m_{i+1}, m'_{i+1}$ until the next trivalent node of $\text{Hex}(\sigma_{i+1})$. Since $f_i \in \Pi_{\rho_{i+1}}'$ and edges from $b_i$ onwards belong alternatively to $\Pi_{\rho_{i+1}}'$ and $\Pi_{\rho_i} \cap \Pi_{\rho_{i+2}}'$, the trivalent node is white (see Figure 5.4(a)). As none of the edges between $b_i$ and $m_{i+1}$ belongs to $m_{i+1}$, we do not meet $b_0$ before $m_{i+1}$. Therefore, $e_{i+1}$ lies in $m_0, m_1, \ldots, m_{i+2}$. Moreover, by the time we reach $b_0$ again, we must be travelling along $m_0, m_1, \ldots, m_k$. So, we cannot meet $b_{i+1}$ along $m_0$, as this would imply that $m_i$ and $m_{i+1}$ stop travelling together. Hence, $m_0, \ldots, m_{i+2}$ travel together from $w_{i+1}$ until $b_0$. □

Lemma 5.10. Suppose $\sigma_i$ and $\sigma_{i+1}$ are lower triangles and suppose:

(ii') $m_0, m_1, \ldots, m_{i+2}$ travel together from $b_{i+1}$ until $b_0$.

Proof. By Conjecture 5.7 (i), $e_i \in m'_{i+1}$. Since $\sigma_i$ is an upper triangle, $m'_{i+1}$ is a zigzag path. Thus $e_i \in m_0 \cap m'_{i+1}$, both zigzag paths. By assumption, $\Gamma$ has no bivalent nodes. Therefore $m_0$ and $m'_{i+1}$ split at the black node of $e_i$. Note that the black node of $e_i$ is not $b_0$, as $0 \notin m_{i+2}$. Since $m_0, \ldots, m_{i+1}$ travel together from $w_i$ until $b_0$, the black node of $e_i$ is the black trivalent node $b_{i+1}$ of $\text{Hex}(\sigma_{i+1})$. As we have not met $b_0$, the edge $f_{i+1}$ lies in $m_0, m_1, \ldots, m_{i+2}$. Moreover, by the time we reach $b_0$ again, we must be travelling along $m_0, m_1, \ldots, m_k$. So, we cannot meet $w_{i+1}$ along $m_0$, as this would imply $m_{i+1}$ and $m_{i+2}$ stop travelling together and we cannot meet $b_{i+1}$ again before $b_0$ as $m_0$ does intersect itself. Hence, $m_0, \ldots, m_{i+2}$ travel together from $b_{i+1}$ until $b_0$. □

Figure 5.4: Depiction of results in Lemma 5.9

Lemma 5.10. Suppose $\sigma_i$ and $\sigma_{i+1}$ are lower triangles and suppose:

(ii') $m_0, m_1, \ldots, m_{i+2}$ travel together from $w_{i+1}$ until $b_0$.

Proof. By Conjecture 5.7 (ii), $f_i \in m'_{i+1}$. Travel along $m_{i+1}, m'_{i+1}$ until the next trivalent node of $\text{Hex}(\sigma_{i+1})$. Since $f_i \in \Pi_{\rho_{i+1}}'$ and edges from $b_i$ onwards belong alternatively to $\Pi_{\rho_{i+1}}'$ and $\Pi_{\rho_i} \cap \Pi_{\rho_{i+2}}'$, the trivalent node is white (see Figure 5.4(a)). As none of the edges between $b_i$ and $m_{i+1}$ belongs to $m_{i+1}$, we do not meet $b_0$ before $m_{i+1}$. Therefore, $e_{i+1}$ lies in $m_0, m_1, \ldots, m_{i+2}$. Moreover, by the time we reach $b_0$ again, we must be travelling along $m_0, m_1, \ldots, m_k$. So, we cannot meet $b_{i+1}$ along $m_0$, as this would imply that $m_i$ and $m_{i+1}$ stop travelling together. Hence, $m_0, \ldots, m_{i+2}$ travel together from $w_{i+1}$ until $b_0$. □
Lemma 5.11. Suppose $\text{Hex}(\sigma_1) \in \Pi$ the next trivalent node of $\text{Hex}(\sigma)$ Then $\text{Hex}(\sigma_1)$ and black trivalent nodes of $\sigma_1$ travel together from $w$ Now, travel along $m$ we could had not passed through $b$ By Conjecture 5.7 (iii),
Proof. By Conjecture 5.7 (iii), $e_1 \in m_{i+1}$ and $f_{i+1} \in m_0$. Travel along $m_i, m_{i+1}$ until the next trivalent node of $\text{Hex}(\sigma_{i+1})$. Since $e_1 \in \Pi_{i+1}$ and edges from $w_i$ onwards belong alternatively to $\Pi_{i+1}$, $\text{Hex}(\sigma_{i+1})$ cannot lie in $m_{i+1}, m'_{i+1}$ and $m_{i+2}$, we do not meet $b_0$ before $m_{i+1}$. Thus, $e_{i+1}$ lies in $m_0, m_1, \ldots, m_{i+2}$. Moreover, we cannot meet another trivalent node of $\text{Hex}(\sigma_{i+1})$ before meeting $b_0$, as $m_0$ does not intersect itself. Hence, $m_0, \ldots, m_{i+2}$ travel together from $w_{i+1}$ until $b_0$

Figure 5.5: Depiction of results in Lemma 5.10

(i) The edge $e_i$ lies in $m_0, m_1, \ldots, m_{i+1}$; and
(ii) $m_0, m_1, \ldots, m_{i+1}$ travel together from $m_i$ until $b_0$.

Then the walks $m_0, \ldots, m_{i+1}$ meet $b_{i+1}$ and $w_{i+1}$ and moreover:
(i') The edge $e_{i+1}$ lies in $m_0, m_1, \ldots, m_{i+2}$; and
(ii') $m_0, m_1, \ldots, m_{i+2}$ travel together from $w_{i+1}$ until $b_0$.

Lemma 5.11. Suppose $\sigma_i$ and $\sigma_{i+1}$ are upper triangles and suppose:

(i) The edge $f_i$ lies in $m_0, m_1, \ldots, m_{i+1}$; and
(ii) $m_0, m_1, \ldots, m_{i+1}$ travel together from $b_i$ until $b_0$.

Then $m_0, \ldots, m_{i+1}$ meet $w_{i+1}$ and $b_{i+1}$. The white and black nodes of $e_i$ are the white and black trivalent nodes of $\text{Hex}(\sigma_{i+1})$. Moreover:
(i') The edge $f_{i+1}$ lies in $m_0, m_1, \ldots, m_{i+2}$; and
(ii') $m_0, m_1, \ldots, m_{i+2}$ travel together from $b_{i+1}$ until $b_0$.
**Proof.** By Conjecture 5.7 (ii), \( f_i \in m_{i+2} \) and \( \epsilon_{i+1} \in m_0 \). Travel along \( m_{i+1}, m_{i+2} \) until the next trivalent node of \( \text{Hex}(\sigma_{i+1}) \). Since \( f_i \in \Pi_{\rho_{i+1}}^* \) and edges from \( b_i \) onwards belong alternatively to \( \Pi_{\rho_{i+1}}^* \) and \( \Pi_{\rho_{i+2}}^* \cap \Pi_{\rho_{i+3}}^* \), the trivalent node is white (see Figure 5.6(a)). By conjecture, \( \epsilon_{i+1} \in m_0 \) and we know \( \epsilon_{i+1} \in m_i, m_{i+1} \). Thus, we could not have passed through \( b_0 \) before \( m_{i+1} \) and \( \epsilon_{i+1} \) lies in \( m_0, m_1, \ldots, m_i, m'_{i+1} \). Since \( \sigma_{i+1} \) is an upper triangle, \( m'_{i+1} \) is a zigzag path. As \( \Gamma \) has no bivalent nodes, \( m_0 \) and \( m'_{i+1} \) split at the black node of \( f'_{i+1} \). The edge \( f'_{i+1} \) does not lie in \( m_{i+1} \) and hence its black node is not \( b_0 \). Therefore, after the black node of \( f'_{i+1} \), \( m_0, m_1, \ldots, m_i \) travel together, which implies that the black node is in fact \( b_{i+1} \). It follows that \( \epsilon_{i+1} \) lies in \( m_0, m_1, \ldots, m_{i+2} \). Since \( m_0 \) does not intersect itself, \( m_{i+1} \) and \( m_{i+2} \) travel together at least up to \( b_0 \) and thus \( m_0, m_1, \ldots, m_{i+2} \) travel together from \( b_{i+1} \) until \( b_0 \). \( \square \)

We now describe an inductive mechanism that traces a cycle of the walk \( m_0 \) around \( T \) starting and finishing at \( \epsilon_\alpha \), as we jump between adjacent triangles in \( \text{strip}(\alpha) \) in a clockwise direction.

**Algorithm 5.12.** Algorithm to trace a cycle of the walk \( m_0 \) around \( T \) starting and finishing at \( \epsilon_\alpha \):

1. **Start at** \( \epsilon_\alpha \). By Lemma 5.6, the black node of \( \epsilon_\alpha \) is the black trivalent node of \( \text{Hex}(\sigma_0) \). Travel from \( b_0 \) to \( m_0 \) along edges in \( m_0 \cap m'_0 \). The edge \( \epsilon_0 \) attached to \( m_0 \) satisfies the initial conditions of Lemma 5.8 and Lemma 5.10.

2. **Use either** Lemma 5.8 or Lemma 5.10, **depending on whether** \( \sigma_1 \) **is an upper or lower triangle**, to trace \( m_0 \) from \( m_0 \) to a trivalent node of \( \text{Hex}(\sigma_1) \).

3. **For** \( 1 \leq i \leq k-1 \), **use the appropriate Lemma to trace** \( m_0 \) **from a trivalent node of** \( \text{Hex}(\sigma_i) \) **to a trivalent node of** \( \text{Hex}(\sigma_{i+1}) \), **depending on whether** \( \sigma_i \) **and** \( \sigma_{i+1} \) **are upper or lower triangles**.

4. **The last triangle** \( \sigma_k \) **in** \( \text{strip}(\alpha) \) **is a lower triangle**, so we finish at the white trivalent node \( w_k \) of \( \text{Hex}(\sigma_k) \) with \( \epsilon_k \) in \( m_0, \ldots, m_k \). By Lemma 5.6, \( \epsilon_k = \epsilon_\alpha \) and the black node of \( \epsilon_k \) is \( b_0 \). Therefore, we have completed tracing a cycle of \( m_0 \) around \( T \).
5.2.2 Walking along $m_{k+1}$

A similar inductive mechanism can be used to trace a cycle of the walk $m_{k+1}$ around \( T \), starting and finishing at \( e_0 \), as we jump between adjacent triangles in strip(\( \alpha \)) in an anticlockwise direction. To obtain the equivalent statements of the Lemmata 5.8 – 5.11 for the case $m_{k+1}$:

(i) Replace $b_0$ by $w_k$;

(ii) Replace $m_0, m_1, \ldots, m_{i-1}$ by $m_i, m_{i+1}, \ldots, m_{k+1}$ and replace $m_0, m_1, \ldots, m_{i+2}$ by $m_{i-1}, m_i, \ldots, m_{k+1}$;

(iii) For the fixed index $0 \leq i \leq k - 1$, replace the black nodes $b_i$ and $b_{i+1}$ by white nodes $w_i$ and $w_{i+1}$, and vice versa;

(iv) For the fixed index $0 \leq i \leq k - 1$, replace the edges $e_i$ and $e_{i+1}$ by $f_i$ an $f_{i+1}$, and vice versa;

(v) Replace the indices $i + 1$ from nodes, edges and cones by $i - 1$; and

(vi) In the case of Lemma 5.11, replace $e'_{i+1}$ by $f''_{i+1}$.

For example, Lemma 5.8 becomes:

**Lemma 5.13.** Suppose $\sigma_i$ is a lower triangle and $\sigma_{i-1}$ is an upper triangle and suppose:

(i) The edge $f_i$ lies in $m_i, m_{i+1}, \ldots, m_{k+1}$; and

(ii) $m_i, m_{i+1}, \ldots, m_{k+1}$ travel together from $b_i$ until $w_k$.

Then the black and white nodes of $e_i$ are the black trivalent node $b_i$ of Hex($\sigma_i$) and the white trivalent node $w_{i-1}$ of Hex($\sigma_{i-1}$). Moreover:

(i') The edge $e_{i-1}$ lies in $m_{i-1}, m_i, \ldots, m_{k+1}$; and

(ii') $m_{i-1}, m_i, \ldots, m_{k+1}$ travel together from $w_{i-1}$ until $w_k$.

A similar conjecture to Conjecture 5.7 is needed in order to prove the equivalent statements of the Lemmata 5.8 – 5.11 for the case $m_{k+1}$. It can be obtained from Conjecture 5.7 in a manner similar to how the equivalent statements were obtained.

We can deduce more information about the edges in the intersection of the meandering walks $m_0, m_1, \ldots, m_{i+2}$ from Lemma 5.8 and Lemma 5.11 by using their equivalent statements for the meandering walk $m_{k+1}$.

**Lemma 5.14.** If $\sigma_i$ and $\sigma_{i+1}$ are both lower (resp. upper) triangles, there exists an edge in $m_0, m_1, \ldots, m_{k+1}$ whose nodes are $w_i$ and $b_{i+1}$ ($b_i$ and $w_{i+1}$, respectively).

*Proof.* Suppose $\sigma_i$ and $\sigma_{i+1}$ are both lower triangles. Consider the edges $e_i$ and $f_{i+1}$ attached to $w_i$ and $b_{i+1}$, respectively. By Lemma 5.10, $e_i$ lies in $m_0, m_1, \ldots, m_{i+2}$.

The equivalent Lemma for $m_{k+1}$ shows that $f_{i+1}$ lies in $m_i, m_{i+1}, \ldots, m_{k+1}$. Since $m_0, m_1, \ldots, m_{i+2}$ travel together from $w_i$ to $b_{i+1}$, and $m_i, m_{i+1}, \ldots, m_{k+1}$ travel together from $b_{i+1}$ to $w_i$, both $e_i$ and $f_{i+1}$ lie in $m_0, m_1, \ldots, m_{k+1}$. As $m_0$ and $m_{k+1}$ are zigzag paths and $\Gamma$ has no bivalent nodes, $e_i = f_{i+1}$ is an edge in $m_0, m_1, \ldots, m_{k+1}$ whose nodes are $w_i$ and $b_{i+1}$.

A similar proof works for the case when $\sigma_i$ and $\sigma_{i+1}$ are both upper triangles. \( \square \)
### 5.2.3 Summary of results

Lemmas 5.8 – 5.11 and their equivalent statement for $m_{k+1}$ imply that the meandering walks $m_0$ and $m_{k+1}$ can be partitioned into segments each corresponding to either:

1. The starting point of the inductive mechanism; or
2. A jump between adjacent triangles in $\text{strip}(\alpha)$.

Table 5.1 gives a summary of the results in this section. To simplify notation, we write $m_{[i,j]}$ for the intersection $\bigcap_{t=i}^{j} m_t$.

<table>
<thead>
<tr>
<th>Starting Point</th>
<th>Segment of $m_0$</th>
<th>Segment of $m_{k+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
</tbody>
</table>

| | $m'_0$ is a lower meandering walk | $m'_1$ is a lower meandering walk |
| | $m'_{i+1}$ is an upper meandering walk | $m'_{i+1}$ is a lower meandering walk |
| | $m'_{i+1}$ is a lower meandering walk | $m'_{i+1}$ is an upper meandering walk |
| | $m'_{i+1}$ is an upper meandering walk | $m'_{i+1}$ is an upper meandering walk |

Table 5.1: The walks $m_0$ and $m_{k+1}$ partitioned into segments

### 5.3 New characterisations of the edges in $S_\alpha$

Recall that for $0 \leq i \leq k + 1$, $b_i$ and $w_i$ denote the black and white trivalent nodes of $\text{Hex}(\sigma_i) \in \Gamma$ and we write $e_i, e'_i$ and $e''_i$, and $f_i, f'_i$ and $f''_i$ for the edges in $\partial \text{Hex}(\sigma_i)$ around $b_i$ and $w_i$, respectively so that

\[ e_i, f_i \in m_i \cap m_{i+1}, \quad e'_i, f'_i \in m_i \cap m'_i \quad \text{and} \quad e''_i, f''_i \in m_{i+1} \cap m'_i, \]

as in Figure 5.2. Throughout this section and for the remainder of this thesis, overline denotes addition modulo $k + 1$. 
We now use the results from the previous section, summarised in Table 5.1, to give two new characterisations of the edges in $S_\alpha$ in terms of the triangles in strip($\alpha$) and the trivalent nodes of their fundamental hexagons:

**Proposition 5.15.** Assume Conjecture 5.7 and let $\epsilon$ be an edge of $\Gamma$. Then $\epsilon \in S_\alpha$ if and only if

1. $\epsilon \in m_0 \setminus m_{k+1}$ and there exists an upper triangle $\sigma_i$ ($1 \leq i \leq k - 1$) in strip($\alpha$) such that either
   
   (i) $\sigma_{i-1}$ is a lower triangle, and the endnodes of $\epsilon$ are $w_{i-1}$ and $b_i$; or
   
   (ii) $\sigma_{i-1}$ is an upper triangle, and the endnodes of $\epsilon$ are $w_i$ and $b_i$.

2. $\epsilon \in m_{k+1} \setminus m_0$ and there exists an upper triangle $\sigma_i$ ($1 \leq i \leq k - 1$) in strip($\alpha$) such that either
   
   (i) $\sigma_{i+1}$ is a lower triangle, and the endnodes of $\epsilon$ are $w_i$ and $b_{i+1}$; or
   
   (ii) $\sigma_{i+1}$ is an upper triangle, and the endnodes of $\epsilon$ are $w_i$ and $b_i$.

3. $\epsilon \in m_0 \cap m_{k+1}$ and there exists $0 \leq i \leq k$ such that either
   
   (i) $\sigma_i$ and $\sigma_{i+1}$ are both lower triangles, and the endnodes of $\epsilon$ are $w_i$ and $b_{i+1}$; or
   
   (ii) $\sigma_i$ and $\sigma_{i+1}$ are both upper triangles, and the endnodes of $\epsilon$ are $w_i$ and $b_{i+1}$.

**Proof.** Note that if $\epsilon$ is an element of $S_\alpha \subseteq m_0 \cap m_{k+1}$, then $\epsilon$ is an edge of one of the segments listed in Table 5.1. Suppose $\epsilon$ is an edge in the segment of $m_0$ (resp. $m_{k+1}$) corresponding to the starting point of the induction mechanism. The only edge of $S_\alpha$ in that segment is $e_\alpha$, as the edges in $m_0$ (resp. $m_{k+1}$) between $b_0$ and $w_0$ (resp. $w_k$ and $b_k$) are elements of the lower meandering walk $m'_0$ (resp. $m'_k$, respectively). Moreover, $e_\alpha \in m_0 \cap m_k$ satisfies 3(i) when $i = k$. Similarly, the only edge in the segment of $m_0$ (resp. $m_{k+1}$) corresponding to a jump between two lower triangles $\sigma_i$ and $\sigma_{i+1}$ is the edge $\epsilon \in m_0 \cap m_1 \cap \cdots \cap m_{k+1}$ with endnodes $w_i$ and $b_{i+1}$ satisfying 3(i).

Now, suppose $\epsilon$ is the edge in $m_0$ (resp. $m_{k+1}$) corresponding to the jump between a lower (resp. an upper) triangle $\sigma_i$ and an upper (resp. a lower) triangle $\sigma_{i+1}$. Then, by Conjecture 5.7(v) $\epsilon \in S_\alpha$ and $\epsilon$ satisfies 1(i) (2(i), respectively). Note that no edges in the segment of $m_0$ (resp. $m_{k+1}$) corresponding to a jump between an upper (resp. a lower) triangle $\sigma_i$ and a lower (resp. an upper) triangle $\sigma_{i+1}$ belong to $S_\alpha$, as they are edges of the lower meandering walk $m'_0$ (resp. $m'_k$, respectively). Finally, in the case when $\sigma_i$ and $\sigma_{i+1}$ are upper triangles, the corresponding segments of $m_0$ and $m_{k+1}$ share an edge $\epsilon \in m_{[0,k+1]}$ with nodes $b_i$ and $w_{i+1}$, satisfying 3(ii). Also, the segment of $m_0$ (resp. $m_{k+1}$) has another edge in $S_\alpha$ as described by 1(ii) (2(ii), respectively). Since every edge of $S_\alpha$ corresponds to one of the listed edges and every listed edge is an element of $S_\alpha$, this completes the proof. \qed

For $0 \leq i \leq k + 1$, we associate one or three edges of $S_\alpha$ to every line segment $\tau_i$ in strip($\alpha$) in the following way:
Theorem 5.16. Assume Conjecture 5.7 and let \( e \) be an edge of \( \Gamma \). Then \( e \in S_\alpha \) if and only if for \( 0 \leq i \leq k \):

1. The edge \( e = e_i = e_{i-1} \) with endnodes \( b_i \) and \( w_{i-1} \) lies in \( m_0 \cap m_{k+1} \), in which case \( \sigma_i \) and \( \sigma_{i-1} \) are both lower triangles. We associate to \( \tau_i \) the edge \( s_i := e \in S_\alpha \);

2. The edge \( e = e_i = e_{i-1} \) with endnodes \( b_i \) and \( w_{i-1} \) lies in \( m_0 \setminus m_{k+1} \), in which \( \sigma_i \) and \( \sigma_{i-1} \) are an upper and a lower triangle, respectively. We associate to \( \tau_i \) the edge \( s_i := e \in S_\alpha \);

3. The edge \( e = e_i = e_{i-1} \) with endnodes \( b_i \) and \( w_{i-1} \) lies in \( m_{k+1} \setminus m_0 \), in which case \( \sigma_i \) and \( \sigma_{i-1} \) are a lower and an upper triangle, respectively. We associate to \( \tau_i \) the edge \( s_i := e \in S_\alpha \);

4. The edge \( e \) is one of \( s_i := e_i = e_{i-1} \), \( s_i' := e_i = e_{i-1} \) or \( s_i'' := e_i = e_{i-1} \) with endnodes \( b_i \) and \( w_{i-1} \) or \( s_i \) and \( w_i \) and \( b_i \) and \( w_i \) and \( b_i-1 \) and \( w_i \) and \( w_{i-1} \), respectively. In this case \( \sigma_i \) and \( \sigma_{i-1} \) are both upper triangles and we associate to \( \tau_i \) the edges \( s_i, s_i', s_i'' \in S_\alpha \).

Proof. This is essentially a reworking of the Proposition 5.15. Case 1, 2 and 3 correspond to 3(i), 1(i) and 2(ii) in Theorem 5.16, respectively. Case 4 merges 3(ii), 1(ii) and 2(ii). Note that as Proposition 5.15 makes use of Lemmata 5.8-5.11, all of which assume Conjecture 5.7, this result depends on the same conjecture. \( \square \)

Corollary 5.17. A node \( n \) is a trivalent node of \( \text{Hex}(\sigma) \), for a cone \( \sigma \) in \( \text{strip}(\alpha) \) if and only if \( n \) is an endnode of an edge in \( S_\alpha \).

Proof. By Proposition 5.15, if \( e \) is an edge in \( S_\alpha \) either there exists a pair of triangles \( \sigma_i, \sigma_{i+1} \) (\( 0 \leq i \leq k \)) in \( \text{strip}(\alpha) \) such that the endnodes of \( e \) are trivalent nodes of \( \text{Hex}(\sigma_i) \) and \( \text{Hex}(\sigma_{i+1}) \); or there exists a triangle \( \sigma \) in \( \text{strip}(\alpha) \) such the endnodes of \( e \) are the white and black trivalent nodes of \( \text{Hex}(\sigma) \). Thus, if \( n \) is an endnode of an edge in \( S_\alpha \), \( n \) is a trivalent node of \( \text{Hex}(\sigma) \), for a cone in \( \text{strip}(\alpha) \). For the converse, let \( \sigma_i \) be a cone in \( \text{strip}(\alpha) \) (\( 0 \leq i \leq k \)). By Theorem 5.16, if \( \sigma_i \) is a lower triangle, the edges \( s_i \) and \( s_{i+1} \) have the white and black trivalent nodes of \( \text{Hex}(\sigma_i) \) as their endnodes, respectively. Else \( \sigma_i \) is an upper triangle and either \( \sigma_{i-1} \) and \( \sigma_{i+1} \) are both lower triangles, in which case the black endnode of \( s_i \) is \( b_i \) and the white endnode of \( s_{i+1} \) is \( w_i \); or \( \sigma_{i-1} \) or \( \sigma_{i+1} \) is an upper triangle and the endnodes of \( s_i \) or \( s_{i+1} \) are \( b_i \) and \( w_i \). \( \square \)

5.4 Circuit diagrams

Using our understanding of the meandering walks \( m_0 \) and \( m_{k+1} \), we can deduce information about \( m_0', m_1', \ldots, m_k' \).

Lemma 5.18. Assume Conjecture 5.7. For \( 0 \leq i \leq k \), we can trace the meandering walk \( m_i' \) by travelling along the zigzag paths of \( \Gamma \) supported on \( m_0 \), \( m_{k+1} \) and the upper meandering walks of \( \text{strip}(\alpha) \). Moreover, if the meandering walk \( m_i' \) splits from one such zigzag path \( \gamma \) at a node \( n \) and starts travelling along a new zigzag path \( \gamma' \), then there exists an edge \( s \in S_\alpha \) with endnode \( n \) such that \( s \in \text{supp}(\gamma) \cap \text{supp}(\gamma') \).
Proof. Both statements are trivially true for all upper meandering walks of strip$(\alpha)$, as they are zigzag paths of $\Gamma$ (see Definition 5.3). To show both statements hold for all lower meandering walks, we use Lemmata 5.8-5.11 to trace a cycle of $m'_i$ ($0 \leq i \leq k$) around $T$ by travelling between $b_i$ and $w_i$ twice, first along $m_i \setminus m_{i+1}$ and then along $m_{i+1} \setminus m'_i$. For $0 \leq i \leq k$, the way the cycle of $m'_i$ is completed depends on whether $\sigma_l$ and $\sigma_l'$ are upper or lower triangles. We complete the cycle of a lower meandering walk $m'_i$ in the cases when $\sigma_l$ is between two lower and two upper triangles. All eight cases are summarised in Table 5.2 and their proofs are similar to the ones here presented.

Let $0 \leq i \leq k$. Suppose $\sigma_l = \sigma_l'$ and $\sigma_l''$ are all lower triangles. If $i \neq 0$, we know by Lemma 5.10 that the walks $m_0, m_1, \ldots, m_s, m'_i$ travel together from $b_i$ to $w_i$. The equivalent statement of Lemma 5.10 for $m_{k+1}$ shows that for $i \neq k$ the meandering walks $m'_i, m_{i+1}, m_{i+2}, \ldots, m_{k+1}$ travel together from $w_i$ to $b_i$. When $i = 0$ ($i = k$, resp.) $m_0$ ($m_{k+1}$, resp.) travels from $w_0$ ($w_k$, resp.) to $b_0$ ($b_k$, resp.) as $\sigma_0$ ($\sigma_k$, resp.) is a line segment in $\sigma_0$ ($\sigma_k$, resp.). Therefore, every edge of $m'_i$ is an edge in the support of a zigzag path supported in either $m_0$ or $m_{k+1}$. The only nodes where $m'_i$ and $m_0$, and $m'_i$ and $m_{i+1}$ split are $b_i$ and $w_i$. For $i \neq k$, since $\sigma_l = \sigma_l'$ and $\sigma_l''$ are all lower triangles, the edges $s_i \in m_0 \cap m_{k+1}$ and $s_{i+1} \in m_0 \cap m_{k+1}$ of $S_\alpha$ have endnodes $b_i$ and $w_i$, respectively (see Theorem 5.16). When $i = k$, the edges $s_k, s_0 \in m_0 \cap m_{k+1}$ have $b_k$ and $w_k$ as endnodes, respectively.

Now suppose $\sigma_{l-1} = \sigma_l'$ and $\sigma_{l+1}$ are an upper, a lower and an upper triangle, respectively. By Lemma 5.8 and its equivalent for $m_{k+1}$, the nodes $b_{l-1}$ and $w_l$, and $b_l$ and $w_{l+1}$ are connected by edges in $m_0, m_1, \ldots, m_s, m'_i, m_{i+1}, m_{i+2}, \ldots, m_{k+1}, \ldots$, respectively. Moreover, the meandering walks $m'_i$ and $m_{i+1}$ travel together from $w_i$ to $b_i$ and therefore must travel together from $w_i$ to $w_{l+1}$. Conjecture 5.7(i) ensures that $m'_{l+1}$ also travels with $m'_i$ and $m_{i+1}$ between $w_i$ and $w_{l+1}$. By the same argument, the nodes $b_{l-1}$ and $b_l$ are connected by edges in $m'_i, m_i$ and $m'_{i-1}$. So the lower meandering walk $m'_i$ can be traced by travelling along the zigzag paths supported on $m_0, m'_{i+1}, m_{k+1}$ and $m'_{i-1}$. The meandering walks supported on $m'_i$, $m_0$, $m'_i$ and $m'_{i+1}$, $m'_i$ and $m_{k+1}$, and $m'_i$ and $m'_{i-1}$ split at the nodes $w_i, w_{i+1}, b_i$ and $b_{l-1}$, respectively. The edges $s_{i+1}, s_{i+2}, s_i$ and $s_{i+1}$ have $m_i$, $m_{i+1}, b_i$ and $b_{l-1}$ as an endnode. By Lemma 5.9 and Lemma 5.8, $s_{i+1} \in m_0 \cap m'_{i+1}, s_{i+2} \in m_{k+1} \cap m'_{i+1}, s_i \in m_{k+1} \cap m'_{i-1}$ and $s_{i-1} \in m_0 \cap m'_{i-1}$.

Corollary 5.19. Every edge in a zigzag path of $\Gamma$ supported on the meandering walk of a line segment in strip$(\alpha)$ is an edge in a lower meandering walk of strip$(\alpha)$ or an edge in $S_\alpha$.

Using Lemmata 5.8-5.11 and Lemma 5.18, we draw a schematic, called a circuit diagram, depicting all meandering walks $m'_i$ for $\tau$ a line segment of strip$(\alpha)$ and edges in $S_\alpha$. For $\alpha \in Q_1$, use Table 5.2 to draw the cycle of $m'_i$ for each $0 \leq i \leq k$. The walks $m_0$ and $m_{k+1}$ complete the circuit diagram of $\alpha$, linking every cycle of $m'_i$. The following example presents the circuit diagram for a different choice of arrow $\alpha$ in the quiver $Q$ from Section 4.1.

Example 5.20. Consider the dimer model $\Gamma$ from Section 4.1. Its dual quiver $Q$ is depicted in Figure 4.1. Figure 5.7(a) shows the fan $\Sigma$ of its moduli space of $\theta$-stable $A$-modules for the $0$-generated stability parameter $\theta = (-25, 1, 1, \ldots, 1)$. Choose $\alpha$ to
### Table 5.2: Cycles $m'_i$

<table>
<thead>
<tr>
<th>$\sigma_{i-1}$</th>
<th>$\sigma_i$</th>
<th>$\sigma_{i+1}$</th>
<th>Cycle $m'_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower triangle</td>
<td>Lower triangle</td>
<td>Lower triangle</td>
<td>$b_i \quad \cdots \quad m'<em>i \quad \cdots \quad m</em>{[0,i]}$</td>
</tr>
<tr>
<td>Upper triangle</td>
<td>Lower triangle</td>
<td>Lower triangle</td>
<td>$b_i \quad \cdots \quad m'<em>i \quad \cdots \quad m</em>{[0,i]}$</td>
</tr>
<tr>
<td>Lower triangle</td>
<td>Lower triangle</td>
<td>Upper triangle</td>
<td>$b_i \quad \cdots \quad m'<em>{i+1} \quad \cdots \quad m</em>{[0,i]}$</td>
</tr>
<tr>
<td>Upper triangle</td>
<td>Lower triangle</td>
<td>Upper triangle</td>
<td>$b_i \quad \cdots \quad m'<em>{i+1} \quad \cdots \quad m</em>{[0,i]}$</td>
</tr>
<tr>
<td>Upper triangle</td>
<td>Upper triangle</td>
<td>Upper triangle</td>
<td>$w_i \quad \cdots \quad m'<em>i \quad \cdots \quad m</em>{[0,i]}$</td>
</tr>
<tr>
<td>Lower triangle</td>
<td>Upper triangle</td>
<td>Upper triangle</td>
<td>$w_i \quad \cdots \quad m'<em>{i+1} \quad \cdots \quad m</em>{[0,i]}$</td>
</tr>
<tr>
<td>Upper triangle</td>
<td>Upper triangle</td>
<td>Lower triangle</td>
<td>$m'<em>{i+1} \quad \cdots \quad m</em>{[0,i]} \quad b_i$</td>
</tr>
<tr>
<td>Lower triangle</td>
<td>Upper triangle</td>
<td>Lower triangle</td>
<td>$w_i \quad \cdots \quad m'<em>{i+1} \quad \cdots \quad m</em>{[0,i]} \quad b_i$</td>
</tr>
</tbody>
</table>
be the arrow $a_2$ with tail at vertex 0 and head at vertex 10. The arrow $a_2$ has labelling divisor $D_2 + D_3 + D_{16}$ (see Figure 4.3). Its respective convex polygon $\Delta'$ and strip of triangles strip$(a_2)$ are shown in Figures 5.7(b) and 5.8.

![Figure 5.7: The toric fan $\Sigma$ of $M_A$ and convex polygon $\Delta'$ for $a_2 \in Q_1$](image)

Figure 5.7: The toric fan $\Sigma$ of $M_A$ and convex polygon $\Delta'$ for $a_2 \in Q_1$

![Figure 5.8: Strip of triangles strip$(a_2)$](image)

Figure 5.8: Strip of triangles strip$(a_2)$

The subset $S_{a_2}$ has eight edges all depicted in its circuit diagram in Figure 5.9. As in Figure 4.6, the meandering walks $m_i$ ($0 \leq i \leq 6$) make a ‘Mexican wave’ as they orderly move from $m_6$ to $m_0$. Figure 5.10 shows all lower meandering walks of strip$(a_2)$ with their natural direction. Note that no lower meandering walk fails to be zigzag at $s_4$.

![Figure 5.9: The circuit diagram of $a_2$](image)

Figure 5.9: The circuit diagram of $a_2$
Figure 5.10: The meandering walks $m'_0$, $m'_1$, $m'_2$, $m'_5$ and the edges in $S_{a_2}$
CHAPTER 6

THE ARROW CONTRACTION ALGORITHM

In this chapter, we describe an algorithm - the arrow contraction algorithm - that allows us to produce new consistent dimer models from old. Any choice of arrow $\alpha$ in the dual quiver of a consistent dimer model $\Gamma$ with tail at vertex 0, produces a new consistent dimer model $\Gamma'$ whose corresponding characteristic polygon coincides with $\Delta'$. Ishii–Ueda [IU09] introduced an algorithm which takes a consistent dimer model $\Gamma$ with characteristic polygon $\Delta(\Gamma)$ to a new consistent dimer model $\Gamma'$ whose characteristic polygon is the convex hull of $\Delta(\Gamma) \setminus \{\rho\}$, for any choice of corner $\rho$ in $\Delta(\Gamma)$. The arrow contraction algorithm coincides with Ishii–Ueda’s in the special case when $\text{div}(\alpha) = D_\rho$ (see Section 6.6). However, the two algorithms are different in general.

6.1 The dimer model $\Gamma'$

We start by defining the bicoloured graph $\Gamma'$ associated to a consistent dimer model $\Gamma := (E, B, W)$ and arrow $\alpha \in Q_1$ with tail at vertex 0.

**Definition 6.1.** Define $\Gamma'$ to be the bicoloured graph whose set of nodes is the set of nodes $n$ of $\Gamma$ such that $n$ is an endnode of an edge in $E \setminus S_\alpha$. Its set of edges is

$$E' := E \setminus S_\alpha.$$ 

By definition, every node of $\Gamma'$ has an edge of $E'$ attached to it. In fact, every node of $\Gamma'$ is at least bivalent:

**Lemma 6.2.** Assume Conjecture 5.7. Then for every node $n$ of $\Gamma'$, there exists at least two edges $e, e' \in E'$, such that $n$ is an endnode of both $e$ and $e'$.

**Proof.** Since every node of $\Gamma$ is at least trivalent, we only need to check those nodes of $\Gamma'$ attached to edges in $S_\alpha$. To do this, we use the characterisation of the set $S_\alpha$ introduced in Theorem 5.16, which assumes Conjecture 5.7. Let $\sigma_i$ be a triangle in
Therefore, every endnode of an edge in \( S \) is simply-connected. This follows from Conjecture 6.4.

**Proof.** We know that \( \Gamma \) is a dimer model. Proposition 6.5. Assume Conjectures 5.7 and 6.4. Then the bicoloured graph \( \Gamma' \) is a dimer model.

**Proof.** We know that \( \Gamma' \) is a bicoloured graph embedded in the torus \( T \). By Lemma 6.2, assuming Conjecture 5.7, \( \Gamma' \) has no univalent nodes. It remains to show that every tile of \( \Gamma \) is simply-connected. This follows from Conjecture 6.4.
One can think of every tile in $\Gamma'$ as a union of tiles of $\Gamma$. As $\Gamma'$ is a dimer model, it is dual to a quiver $Q'$ with relations $\mathcal{R}'$. Note that the quiver $Q'$ inherits a choice of vertex $0 \in Q'_0$, namely the vertex $v' \in Q'_0$ dual to the tile containing the zero tile of $\Gamma$.

**Definition 6.6.** Let $\overline{Q} := (\overline{Q}_0, \overline{Q}_1, \overline{h}, \overline{t}, \overline{\mathcal{R}})$ be a quiver with

(i) set of vertices $\overline{Q}_0 = Q_0$;

(ii) set of arrows $\overline{Q}_1$ containing the set $Q_1 \setminus \hat{S}_\alpha$ and an arrow $a^{-1}$ for every arrow $a \in S_\alpha \setminus \hat{S}_\alpha$ with $\overline{t}(a^{-1}) = \overline{h}(a)$ and $\overline{h}(a^{-1}) = \overline{t}(a)$; and

(iii) set of relations $\overline{\mathcal{R}}$ containing the set $\mathcal{R}$ and two additional relations $aa^{-1} - e_{h(a)}$ and $a^{-1}a - e_{t(a)}$ for each arrow $a \in S_\alpha \setminus \hat{S}_\alpha$.

**Proposition 6.7.** Assume Conjectures 5.7 and 6.4. The Jacobian algebra $\overline{A} = \mathbb{C}\overline{Q}/I_{\overline{\mathcal{R}}}$ of $\overline{Q}$ is Morita equivalent to the Jacobian algebra $A' = \mathbb{C}\overline{Q}'/I_{\overline{\mathcal{R}}'}$ of $Q'$ dual to the dimer model $\Gamma'$.

**Proof.** Recall that the equivalence of categories from Proposition 1.22 translates into an equivalence between the category $\text{rep}(Q, \mathcal{R})$ of finite-dimensional representation of a quiver $Q$ with relations $\mathcal{R}$ and the category $\text{mod} - A$ of finite-dimensional left $A$-modules for $A = \mathbb{C}Q/I_{\mathcal{R}}$. To show Morita equivalence, we define functors between the categories $\text{rep}(Q', \mathcal{R}')$ and $\text{rep}(\overline{Q}, \overline{\mathcal{R}})$.

We start by defining a functor from $\text{rep}(\overline{Q}, \overline{\mathcal{R}})$ to $\text{rep}(Q', \mathcal{R}')$ on objects. Let $\overline{W} = (\bigoplus_{\pi \in \overline{Q}_0} \overline{W}_\pi, \{\overline{\phi}_\pi\}_{\pi \in \overline{Q}_1})$ be a representation of $\overline{Q}$. The set of vertices of $\overline{Q}$ coincides with the set of vertices of $Q$. The relations $\mathcal{R}$ of $\overline{Q}$ ensure that if the tiles dual to two vertices $\pi$ and $\pi'$ in $\overline{Q}_0 = Q_0$ lie in the same tile of $\Gamma'$, then $\overline{W}_\pi \cong \overline{W}_{\pi'}$. For $v \in Q'_0$, let $\overline{W}_v := \overline{W}_\pi$ for any $\pi$ dual to a tile of $\Gamma$ in the tile of $\Gamma'$ dual to $v$. By definition, the set of arrows of $\overline{Q}$ contains the set $Q_1 \setminus S_\alpha$. For $a \in Q'_1 = Q_1 \setminus S_\alpha$, let $\phi_a = \overline{\phi}_\pi$ where $a$ and $\pi$ are the same arrow, thought of as elements of $Q_1$. Thus, a representation $\overline{W}$ of $\overline{Q}$ determines a representation $W = (\bigoplus_{v \in Q_0} W_v, \{\phi_a\}_{a \in Q_1})$ of $Q$, and one can extend this assignment to morphisms between representations.

Now we define a functor on objects between the categories in the opposite direction. Let $W = (\bigoplus_{v \in Q'_0} W_v, \{\phi_a\}_{a \in Q'_1})$ be a representation of $Q'$. The functor sends $W$ to $\overline{W} = (\bigoplus_{\pi \in \overline{Q}_0} \overline{W}_\pi, \{\overline{\phi}_\pi\}_{\pi \in \overline{Q}_1})$, where $\overline{W}_\pi := W_v$ for every vertex $\pi \in \overline{Q}_0 = Q_0$ dual to a tile of $\Gamma'$ contained in the tile of $\Gamma$ dual to $v$. Every arrow in $\overline{Q}$ is either in $Q_1 \setminus \hat{S}_\alpha$, in $S_\alpha \setminus \hat{S}_\alpha$ or is an arrow $a^{-1}$ for $a \in S_\alpha \setminus \hat{S}_\alpha$. If $a \in Q_1 \setminus \hat{S}_\alpha$, set $\overline{\phi}_\pi = \phi_a$, where both $\pi$ and $a$ are the same arrow, thought of as elements of $Q_1$. For $a \in S_\alpha \setminus \hat{S}_\alpha$, choose any nonzero value for the maps $\overline{\phi}_a$. This choice determines the value of the maps $\overline{\phi}_{a^{-1}}$. The relations in $\mathcal{R}$ ensure that any other choice of nonzero values for the maps $\overline{\phi}_a$ $(a \in S_\alpha \setminus \hat{S}_\alpha)$ gives a new representation of $\overline{Q}$ isomorphic to $\overline{W}$. So a representation of $Q'$ defines a unique representation of $\overline{Q}$ up to isomorphism. This assignment also extends to morphisms. Moreover, the composition of these functors gives isomorphic representations and the categories are equivalent. $\square$

**Remarks 6.8.** 1. Introducing inverted arrows in a quiver $Q$ is a special case of ‘universal localisation’ in the sense of A. H. Schofield [Sch85, Chapter 4].
2. Note that in the proof of Proposition 6.7 not all the arrows in \( S_\alpha \) are inverted. The arrows in \( \tilde{S}_\alpha \) are deleted, since the relations in \( \tilde{A} \) determine the arrows in \( \tilde{S}_\alpha \), as follows. Suppose \( a \in \tilde{S}_\alpha \). Then there exists arrows \( a_1, a_1', a_2, a_2' \in S_\alpha \setminus \tilde{S}_\alpha \) as in Figure 6.1 below. For \( i = 1, 2 \), the following relations hold in \( A \):

\[
a_i^{-1}a_i = e_{t(a_i)} = e_{h(a)}
\]

\[
(a_i')^{-1}a_i' = e_{t(a_i')} = e_{h(a_i)}
\]

\[a_i'a_i = q,
\]

where \( q \) is any small cycle starting at vertex \( e_{t(a)} \). Hence, \( a = a_i^{-1}(a_i')^{-1}q \in \tilde{A} \).

![Figure 6.1: An arrow a in \( \tilde{S}_\alpha \)](image)

6.2 The zigzag polygon of \( \Gamma' \)

In order to show that \( \Gamma' \) is a consistent dimer model, we study its set of zigzag paths.

**Definition 6.9.** We say a directed meandering walk \( m = \{e_i\}_{i \in \mathbb{Z}} \) in \( \Gamma \) fails to be zigzag at an edge \( e \in \Gamma \) if the presence of \( e \) prevents the directed meandering walk \( m \) from turning maximally right at a white node or maximally left at a black node, i.e. if there exists an edge \( e \) such that for for some \( i \in \mathbb{Z} \) the edges \( e_i, e_i+1 \) and \( e \) share either

(i) a black node and \( e \) is between \( e_i \) and \( e_{i+1} \) in a clockwise direction; or

(ii) a white node and \( e \) is between \( e_{i+1} \) and \( e_i \) in a clockwise direction.

Recall that every lower meandering walk of \( \text{strip}(\alpha) \) comes with a natural direction introduced in Definition 5.3. The homology class of a lower meandering walk with its natural direction is the inward-pointing normal vector to the boundary edge of \( \Delta' \).

**Lemma 6.10.** Assume Conjecture 5.7. The set \( S_\alpha \) contains all edges \( e \in \mathcal{E} \) such that \( m \) fails to be zigzag at \( e \), for \( m \) a lower meandering walk with its natural direction.

**Proof.** Note that a lower meandering walk \( m \) only fails to be zigzag when it stops following a zigzag path \( \mathfrak{z} \) and starts following a new zigzag path \( \mathfrak{z}' \). By Lemma 5.18, which assumes Conjecture 5.7, this only happens at nodes attached to edges in \( S_\alpha \). Moreover, since the edge at which \( m \) fails to be zigzag is an element of both \( \mathfrak{z} \) and \( \mathfrak{z}' \), \( m \) is a zigzag path after removing all edges of \( S_\alpha \). Thus, \( S_\alpha \) contains all edges at which \( m \) fails to be a zigzag. \( \square \)
Remark 6.11. In general, the set $S_\alpha$ is not equal to the set of edges where the lower meandering walks fail to be zigzag. For an example where the latter set is a proper subset of $S_\alpha$, see Example 5.20.

Suppose $m$ is a meandering walk in $\Gamma$ that does not traverse any edges of $S_\alpha$, i.e. $m \subseteq E \setminus S_\alpha = E'$. Then, $m$ induces uniquely a path $m'$ in $\Gamma'$.

Lemma 6.12. Assume Conjectures 5.7 and 6.4. Every zigzag path in $\Gamma'$ is either:

1. a path in $\Gamma'$ induced by a zigzag path of $\Gamma$ that traverses no edges of $S_\alpha$, in which case it is the zigzag to an edge of $\Delta$ that is also an edge of $\Delta'$; or

2. a path in $\Gamma'$ induced by a lower meandering walk of $\text{strip}(\alpha)$ with its natural direction.

Proof. Suppose $m$ is a zigzag path of $\Gamma$ with homology class normal to a line segment in $\Delta'$. By Lemma 5.18, every edge of $S_\alpha$ is contained in two zigzag paths with homology classes normal to line segments in $\text{strip}(\alpha)$. Therefore, $m$ contains no edges in $S_\alpha$. Thus, $m$ induces a path in $\Gamma'$ which is a zigzag path. Suppose $m$ is a lower meandering walk of $\text{strip}(\alpha)$. By Lemma 6.10, the path of $\Gamma'$ induced by the lower meandering walk $m$ with its natural direction is a zigzag path. As every edge of $\Gamma'$ belongs to two zigzag paths of either type, $\Gamma'$ has no other zigzag paths.

Corollary 6.13. Assume Conjectures 5.7 and 6.4. The zigzag polygon of $\Gamma'$ coincides with $\Delta'$, up to translation.

Proof. Recall from Definition 1.45 that the zigzag polygon of $\Gamma'$ is constructed by taking the homology classes of all its zigzag paths $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_\ell$, cyclically ordered, and defining a sequence of lattice points $b_0 = (0, 0)$ and $b_{i+1} = b_i + \bar{z}_{i+1}(0 \leq i \leq \ell - 1)$, where $\bar{z}_{i+1}$ denotes the vector normal to $\bar{z}_{i+1}$ obtained by rotating $\bar{z}_{i+1}$ by 90 degrees in the positive direction. As every edge of $\Gamma'$ appears in two distinct zigzag paths, $b_\ell = 0$. The zigzag polygon is the convex hull of the set of points $b_1, b_2, \ldots, b_\ell$. By construction, it is the convex polygon that has a boundary line segment for each zigzag path of $\Gamma$. The boundary line segment is normal to the homology class of its zigzag path. Therefore, it suffices to show that for every boundary line segment of $\Delta'$, there exists a zigzag path in $\Gamma'$ whose homology class is the primitive inward-pointing normal vector to the line segment.

By Theorem 1.46, we know that the homology classes of the zigzag paths of $\Gamma'$ induced by zigzag paths of $\Gamma$ are in one-to-one correspondence with the primitive inward-pointing normal vectors to line segments in $\partial\Delta'$ which form part of the boundary of $\Delta(\Gamma)$. Moreover, there is a lower meandering walk for each of the remaining boundary line segments of $\Delta'$. By Lemma 6.12, each of these induces a zigzag path in $\Gamma$ and by Lemma 2.14, the homology classes of these zigzag paths are normal to the line segments in the boundary of $\Delta'$ they are associated to. The choice of natural direction ensures that their homology classes are in fact the inward-pointing normal vectors to their associated line segment.
6.3 Γ′ is a consistent dimer model

We now use the fact that the zigzag polygon of Γ′ is Δ′ to prove that Γ′ is a consistent dimer model. Recall the following theorem from Gulotta:

**Theorem 6.14.** [Gul08, Theorem 3.1] A dimer model Γ is properly-ordered if and only if its number of tiles is equal to twice the area of its zigzag polygon.

**Theorem 6.15.** Assume Conjectures 5.7 and 6.4. The dimer model Γ′ is consistent.

**Proof.** Since Γ is a consistent dimer model its zigzag polygon coincides with its characteristic polygon ∆(Γ) [IU09, Theorem 11.1]. Recall that every consistent dimer model is properly-ordered [IU11b, Lemmata 4.2, 4.3]. Now, by definition

\[
\text{Area of } Δ′ = \text{Area of } ∆(Γ) - \frac{1}{2} \# \text{Triangles in strip}(α). \tag{6.3.1}
\]

Therefore,

\[
2 \times \text{Area of } Δ′ = 2 \times \text{Area of } ∆(Γ) - \# \text{Triangles in strip}(α) \nonumber
\]

\[
= \# \text{Tiles of } Γ - \# \text{Triangles in strip}(α) \quad \text{as } Γ \text{ properly-ordered.} \tag{6.3.2}
\]

By Theorem 6.14, the dimer model Γ′ is properly-ordered if and only if

\[
2 \times \text{Area of } Δ′ = \# \text{Tiles of } Γ′. \tag{6.3.3}
\]

Putting equations (6.3.2) and (6.3.3) together, Γ′ is properly-ordered if and only if

\[
\# \text{Tiles of } Γ′ = \# \text{Tiles of } Γ - \# \text{Triangles in strip}(α). \tag{6.3.4}
\]

Theorem 5.16, which assumes Conjecture 5.7, associates to every line segment τ′ in strip(α) one or three edges of S′. By Conjecture 6.4, if s′ ∈ S′ is associated to a line segment τ′ (0 ≤ i ≤ k) between two triangles σ′_{i−1} and σ′_i and at least one of σ′_{i−1} and σ_1 is a lower triangle, the effect of removing s′ from Γ is merging two different tiles of Γ into one tile of Γ′. If s′, s″_i and s‴_i are the three edges associated to a line segment τ′ in strip(α) (0 ≤ i ≤ k) between two upper triangles σ_1−1 and σ_1, s′_i merges two different tiles t_0 and t_2 of Γ into a single tile of Γ′. The edges s′_i and s″_i merge two other tiles t_1 with t_2 and t_3 with t_0, respectively, into a tile of Γ′. However, the edges associated to the line segments τ_{i−1} and τ_{i+1} make a bow tie shape with the edges of τ′ merging t_2 with t_3 and t_2 with t_0, respectively. Therefore, the effect of removing all edges associated to the line segments τ_{i−1}, τ′_i and τ_{i+1} is merging four different tiles of Γ into a single tile of Γ′. In summary, for 0 ≤ i ≤ k the effect of removing the edge or edges associated to a line segment τ′ is decreasing the number of tiles of Γ by 1. Since there are k + 1 triangles in strip(α) and k + 1 line segments in strip(α) to which we associate one or three edges of S′, equation (6.3.4) holds and the dimer model Γ′ is properly-ordered and hence consistent [IU11b, Lemmata 4.2, 4.3].

**Remark 6.16.** It follows from the proof of Theorem 6.15, that number |Q′_0| of vertices of the quiver Q′, which is equal to the number of tiles of Γ′, is the number of vertices of Q minus k + 1, the number of triangles in strip(α).

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6.4 Geometric consequences of consistency

Fix $\theta': = (|Q_0|, 1, 1, \ldots, 1)$. The stability parameter $\theta'$ is 0-generated and thus generic (see Definition 3.1 and Remark 3.3). Since $\Gamma'$ is a consistent, the following corollary follows:

**Corollary 6.17.** Assume Conjectures 5.7 and 6.4 and suppose $\Delta'$ is a nondegenerate polygon. Let $\sigma' := \{ \sum_{\lambda \geq 0} \lambda u_\rho \in \mathbb{N}_R \mid u_\rho \in \Delta' \}$ denote the cone over $\Delta'$ and let $\theta'$ be generic. Then:

(i) The zigzag polygon $\Delta'$ of $\Gamma'$ coincides with its characteristic polygon $\Delta(\Gamma')$, up to translation;

(ii) The moduli space of $\theta'$-stable $A'$-modules $M_A$ is a projective crepant resolution of the Gorenstein toric threefold $X = \text{Spec}(\mathbb{C}[[\sigma']^\vee \cap M])$;

(iii) The tautological algebra morphism $\phi' : T' \to \text{End}_{M_A} A'$ from the tautological bundle $T' = \bigoplus_{v \in Q_0} L'_v$ of $M_A$ to the endomorphism algebra of $A'$ is an isomorphism; and

(iv) There is a derived equivalence between the bounded derived category $D^b(\text{coh}(M_A))$ of coherent sheaves of $M_A$ and the bounded derived category $D^b(\text{mod} - A')$ of left $A'$-modules.

Following Ishii-Ueda [IU09, Section 4.9], we define for every quiver $Q$ with relations $\mathcal{R}$ a $\mathbb{C}$-linear category $Q$ whose set of objects is the set $Q_0$ of vertices of $Q$. For any two objects $v, v'$ of $Q$, the space of morphisms between $v$ and $v'$ is the vector space $e_{v'} \cdot \mathcal{R} \cdot e_v$ of paths with tail at vertex $v$ and head at vertex $v'$. Composition of morphisms is defined to be the product of elements in the path algebra $\mathbb{C} Q$. Note that a representation of $Q$ is a linear functor from the category $Q$ of $Q$ to the category of complex vector spaces.

**Proposition 6.18.** Assume Conjectures 5.7 and 6.4 and suppose $\Delta'$ is a nondegenerate polygon. Then, there is an open immersion $j : M_A' \to M_A$ whose image is the complement of the support of $\text{div} (\alpha)$. Moreover, for $v' \in Q'_0$ the tautological line bundle $L'_v$ of $M_A'$ is the restriction of a tautological line bundle $L_v$ of $M_A$ to $M_A'$, for some $v \in Q_0$. In particular, the tautological bundle $T'$ of $M_A'$ is obtained by restricting a subbundle of the tautological bundle $T$ of $M_A$ to $M_A'$.

**Proof.** There is a functor from the category $Q$ of the quiver $Q$ dual to $\Gamma$ to the category $Q'$ of the quiver $Q'$ dual to $\Gamma'$. The functor $F : Q \to Q'$ takes a vertex $v \in Q$ to the vertex $v' \in Q'$ dual to the tile of $\Gamma'$ containing the tile of $\Gamma$ dual to $v$. An arrow $a \in Q_1$ is sent to itself if $a \notin S_a$, the identity if $a \notin S_a \setminus \mathring{S}_a$, and zero otherwise. Since representations of $Q$ are linear functors from the category $Q$ to the category of complex vector spaces, the pullback of the functor $F : Q \to Q'$ gives a functor $G : \text{mod} A' \to \text{mod} A$, whose image consists of representations $(\bigoplus_{v \in Q_0} W_{v'} \{ \phi_a \}_{a \in Q_1})$ satisfying $W_{v_1} = W_{v_2}$ if the functor $F$ sends $v_1$ and $v_2$ to the same vertex $v' \in Q'_0$ and $\phi_a = \text{id}_{W(v_a)}$ for every arrow $a \in S_a \setminus \mathring{S}_a$. This functor induces the open immersion $j : M_A' \to M_A$. 

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For the second statement, the tautological bundle $T'$ of $\mathcal{M}_{A'}$ is a family of $\theta'$-stable $A'$-modules and by the Morita equivalence of Proposition 6.7, we get a family $\overline{T}$ of $\theta$-stable $A$-modules which defines a family of $\theta$-stable $A$-modules. By the universal property of moduli spaces over the image $j(\mathcal{M}_{A'})$, this family is obtained by restriction. \qed

The following result gives a geometric characterisations of the arrows in $S_\alpha$:

**Corollary 6.19.** Assume Conjectures 5.7 and 6.4 and suppose $\Delta'$ is a nondegenerate polygon. Let $a$ be an arrow in $S_\alpha$. Then, either:

1. The arrow $a$ lies in $S_\alpha \setminus \hat{S}_\alpha$ and $\text{div}(a)$ is a summand of $\text{div}(\alpha)$; or

2. The arrow $a$ lies in $\hat{S}_\alpha$ and $\text{div}(a)$ is the complement $\sum_{\rho \in \Sigma(1)} D_\rho - \text{div}(\alpha) = \sum_{\rho \in \Sigma(1)} D_\rho$.

**Proof.** Let $a$ be an arrow in $S_\alpha \setminus \hat{S}_\alpha$. The relations $aa^{-1} - e_{h(a)}$ and $a^{-1}a - e_{t(a)}$ defined for the quiver $\overline{Q}$ and Proposition 6.18 imply that the maps between the line bundles $L_{t(a)}$ and $L_{h(a)}$ are isomorphisms on the complement of $\text{div}(\alpha)$, hence the labelling divisor of $a$ is a summand of $\text{div}(\alpha)$. Let $a = s_i$ be an arrow in $\hat{S}_\alpha$. By Lemma 5.18, the dual edge $s_i$ of $s_i$ is a zag of $\delta_0$ and a zig of $\delta_{k+1}$, where $\delta_0$ and $\delta_{k+1}$ are the zigzag paths of $\Gamma$ supported on $m_0$ and $m_{k+1}$, respectively. Thus, every boundary divisor which is not a summand of $\text{div}(\alpha)$ is a summand of $\text{div}(s_i)$. Moreover, there exists arrows $s_{i+1}$ and $s'_i$ in $S_\alpha \setminus \hat{S}_\alpha$ such that $s_{i+1} s'_i s_i$ is a small cycle of $\overline{Q}$. Since $s_{i+1}, s'_i \in S_\alpha \setminus \hat{S}_\alpha$ their labels are summands of $\text{div}(\alpha)$. Hence, every interior divisor is a summand of $\text{div}(s_i)$ and $\text{div}(s_i) = \sum_{\rho \in \Sigma(1)} D_\rho$. \qed

**6.5 The arrow contraction algorithm**

We are now ready to state the arrow contraction algorithm.

**Algorithm 6.20 (The Arrow Contraction Algorithm).** Let $\Gamma$ be a consistent dimer model with dual quiver $Q$ and Jacobian algebra $A$. We assume $\Gamma$ has no bivalent nodes and Conjectures 5.7 and 6.4 hold for $\Gamma$.

1. Pick an arrow $\alpha \in Q_0$ with tail at vertex $0$. By Proposition 4.6, we can associate to $\alpha$ a convex polygon $\Delta'$. Assume $\alpha$ is such that $\Delta'$ is nondegenerate.

2. Find the set of edges $S_\alpha$ using Definition 5.4 or either characterisations of the sets $S_\alpha$ (see Proposition 5.15 and Theorem 5.16) or $S_\alpha$ (see Corollary 6.19).

3. Remove the edges in $S_\alpha$ and all nodes of $\Gamma$ connected only to edges in $S_\alpha$ from $\Gamma$ to obtain a new dimer model $\Gamma'$ with characteristic polygon $\Delta(\Gamma') = \Delta'$.

4. The dimer model $\Gamma'$ is consistent and for 0-generated stability parameters $\theta$ and $\theta'$, there exists an open immersion $\mathcal{M}_{A'} \hookrightarrow \mathcal{M}_A$, so that the toric fan of $\mathcal{M}_{A'}$ is $\Sigma' := \{ \sigma \in \Sigma \mid \rho_i \nsubseteq \sigma \text{ for all } 1 \leq i \leq r \}$, where $\Sigma'$ is the fan of $\mathcal{M}_A$.

5. Since $\Gamma'$ is consistent, and therefore nondegenerate, we may remove all bivalent nodes of $\Gamma'$ without altering the number and direction of its zigzag paths or the Jacobian algebra of its dual quiver (see Remark 1.38).
One can now repeat using $\Gamma'$ and $\theta'$.

### 6.6 Recovering the Ishii–Ueda algorithm

For a consistent dimer model $\Gamma$ with dual quiver $Q$ and 0-generated stability parameter $\theta$, suppose there exists $\alpha \in Q_1$ with tail at vertex 0 and label $\text{div}(\alpha)$ equal to a torus-invariant primer divisor $D_\rho$ of $\mathcal{M}_A$. Assume without loss of generality that $\text{div}(\alpha) = D_{\rho_1}$. Then, $\epsilon_\alpha$ is a zig of $\mathcal{I}_1$ and a zag of $\mathcal{I}_2$. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ intersect and $\Gamma$ is consistent, the ray $\rho_1$ is a corner of the characteristic polygon $\Delta(\Gamma)$ of $\Gamma$. The polygon $\Delta'$ is the convex hull of $\Delta(\Gamma) \setminus \{\rho_1\}$ and the arrow contraction algorithm coincides with Ishii–Ueda’s operation [IU09]. Every triangle in $\text{strip}(\alpha)$ is a lower triangle and by Theorem 5.16, the number of edges in $\mathcal{S}_\alpha$ equals the number of triangles in $\text{strip}(\alpha)$. Lemma 5.18 gives an alternative description of the ‘new’ zigzag paths of $\Gamma$, those induced by the lower meandering walks of $\text{strip}(\alpha)$, to that introduced in the proof of [IU09, Proposition 10.1].

**Remark 6.21.** The fact that $\Delta'$ is the characteristic polygon of $\Gamma'$ [IU09, Proposition 11.2], left largely as an exercise to the reader, requires an understanding of the combinatorics of the Wunram’s $i$-series (see Lemma 10.2 and proof of Proposition 10.1 in [IU09]). Corollary 6.13 provides a simple geometric argument to establish this fact in the special case when the divisor $D_\rho$ equal the label of an arrow $\alpha$ with tail at vertex 0.

Unlike Ishii-Ueda’s algorithm, the arrow contraction algorithm is constrained by the labels of the arrows of $Q$ with tail at vertex 0. However, the arrow contraction algorithm preserves the triangulation of $\Sigma$, i.e. the toric fan of $\mathcal{M}_{A'}$, for $\theta'$ a 0-generated stability parameter, is $\Sigma'$.

### 6.7 Restrictions of the tautological bundle

Given any compact surface $Z \subset \mathcal{M}_A$, we may restrict the tautological bundle to it. The restriction $\bigoplus_{v \in Q_0} L_v|_Z$ contains redundant summands, as some of the line bundles are identified. Write $\bigoplus_{v \in Q_0} L_v|_Z$ for the restriction of the tautological bundle $T$ to $Z$ after removing every redundant summand. Craw, King and (independently) Logvinenko observed that for some compact surfaces $Z \subset G\text{-Hilb}(\mathbb{C}^3)$, the restriction of the tautological bundle $\bigoplus_{v \in Q_0} L_v|_Z$ is tilting on $Z$. The arrow contraction algorithm gives a partial explanation for this phenomenon.

**Theorem 6.22.** Let $\Gamma$ be a consistent dimer model with Jacobian algebra $A$. If we can obtain a dimer model $\Gamma'$ with Jacobian algebra $A'$ from $\Gamma$ by performing the arrow contraction algorithm such that $\mathcal{M}_{A'}$ is the total space of the canonical bundle of a weak toric Fano surface $Z$, the restriction $\bigoplus_{v \in Q_0} L_v|_Z$ of the tautological bundle $T$ of $\mathcal{M}_A$ to $Z$ is tilting.

**Proof.** By assumption, there is a sequence of contractions which take the dimer model $\Gamma$ associated to $\mathcal{M}_A$ to a dimer model $\Gamma'$ such that $\mathcal{M}_{A'}$ is the total space of the canonical bundle of a weak toric Fano surface $Z$. Since $\Gamma'$ was obtained by performing
the arrow contraction algorithm, the dimer model $\Gamma'$ is consistent. Therefore, the tautological bundle $T'$ of $\mathcal{M}_A'$ is tilting. Moreover, by Proposition 6.18, we know $T'$ is obtained by restricting a subbundle of the tautological bundle $T$ of $\mathcal{M}_A$ to $\mathcal{M}_A'$. The fact that the bundle $T'$ restricted to $Z$ is tilting is a result of Carqueville–Quintero-Vélez [CQV10, Proposition C.1] and Ishii–Ueda [IU11a, Theorem 7.1].

Therefore, the arrow contraction algorithm gives a sufficient condition for which the restriction of the tautological bundle to a weak toric Fano surface $Z$ is tilting. It is not yet clear whether this condition is also a necessary one. The following example suggests that this could be the case.

**Example 6.23.** Consider the dimer model $\Gamma$ depicted in Figure 6.2 with its dual quiver. An arrow of the quiver $Q$ marked by $i_1, i_2, \ldots, i_j$ has labelling divisor $D_{i_1} + D_{i_2} + \ldots + D_{i_j}$. The dimer model $\Gamma$ can be obtained from the dimer model in Example 3.37 by applying the arrow contraction algorithm five times. The toric fan $\Sigma$ of the moduli space $\mathcal{M}_A$ of $\theta$-stable $A$-modules, for the 0-generated stability parameter $\theta = (-7, 1, 1, \ldots, 1)$, is depicted in Figure 6.3(a). Figure 6.3(b) shows the labelling of the fan by combinatorial Reid’s recipe.

![Figure 6.2: The quiver Q dual to a dimer model $\Gamma$ with arrows labelled](image)

The fan contains two compact surfaces $Z_1$, $Z_2$ associated to the rays $\rho_6$ and $\rho_7$, respectively. Note that since the labels of the arrows with tail at 0 are $D_1$, $D_2 + D_3 + D_8$ and $D_5$, it is not possible to use the arrow contraction algorithm to find a new consistent dimer model $\Gamma'$ whose characteristic polygon has corners $\rho_1, \rho_2, \rho_5, \rho_8$. Moreover, after removing redundant summands, the restriction of the tautological bundle $T$ of $\mathcal{M}_A$ to $Z_1$ decomposes into six line bundles, and thus it is not one of the tilting bundles of $Z_1$. 

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Figure 6.3: The fan $\Sigma$ of $M_A$ for $\theta = (-7, 1, 1, \ldots, 1)$
We conclude this thesis by suggesting a link between the arrow contraction algorithm and combinatorial Reid’s recipe. Numerous computational evidence suggests that the marking of the line segments and lattice points in \( \text{strip}(\alpha) \) is closely related to the head and tails of the arrows in \( S_\alpha \). We give an explicit conjecture as to the relation of these two and suggest how this conjecture could be used to prove that in a fan \( \Sigma \) marked by Reid’s recipe, every vertex of \( Q \) appears ‘once’ and that combinatorial Reid’s recipe encodes the relations of the line bundles of \( \mathcal{M}_A \) in \( \text{Pic}(\mathcal{M}_A) \).

### 7.1 Example 3.38 revisited

Recall the dimer model \( \Gamma \) introduced in Example 3.38.

![Dimer model from Example 3.38](image)
When \( \alpha = a_3 \), \( S_{a_3} \) has six edges \( s_0, s_1, \ldots, s_5 \) one for each line segments \( \tau_i \) \((0 \leq i \leq 5)\) depicted in Figure 7.3 (see Theorem 5.16).

Table 7.1 lists the arrows in \( S_{a_3} \) dual to the edges in \( S_{a_3} \) along with their corresponding labelling divisor, tail and head. Notice the relation between Reid’s recipe and the tails and heads of the arrows in \( S_{a_3} \). The arrow \( s_0 \) has head at vertex 11, which marks all line segments in the boundary of \( \text{strip}(a_3) \) which are not in the boundary of \( \Sigma \). As the line segments in \( \partial \text{strip}(\alpha) \setminus \partial \Sigma \) coincide with the line segments in \( \partial \Sigma' \setminus \partial \Sigma \), this is Lemma 4.9. For \( 1 \leq i \leq 5 \), the arrow \( s_i \) has tail at the vertex marking \( \tau_i \) by Reid’s recipe. The arrows \( s_2, s_4 \) and \( s_5 \) have head at the vertex marking the unique interior lattice point of \( \Sigma \) in \( \tau_2, \tau_4 \) and \( \tau_5 \), respectively. The arrows \( s_1 \) and \( s_3 \) have head at the vertices marking \( \tau_2 \) and \( \tau_4 \), respectively.

<table>
<thead>
<tr>
<th>Arrow</th>
<th>Label</th>
<th>Tail</th>
<th>Head</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_0 )</td>
<td>( a_3 )</td>
<td>( D_4 + D_5 + D_8 )</td>
<td>0</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( a_{10} )</td>
<td>( D_5 )</td>
<td>3</td>
</tr>
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<td>( s_2 )</td>
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<td>6</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>( a_{20} )</td>
<td>( D_5 + D_8 )</td>
<td>7</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>( a_{25} )</td>
<td>( D_4 )</td>
<td>9</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>( a_{27} )</td>
<td>( D_4 + D_5 + D_8 )</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 7.1: Labelling divisors, heads and tails for the arrows in \( S_{a_3} \)
A similar behaviour appears in the case when \( \alpha = a_2 \). Figure 7.4 shows the strip of triangles \( \text{strip}(a_2) \). From Example 5.20, we know that there are a total of 8 arrows in \( S_{a_2} \). Table 7.2 lists all arrows in \( S_{a_2} \) with their labelling divisor, head and tail. As expected, the head of \( s_0 \) marks the lines segments in the boundary of \( \text{strip}(a_2) \) that are not in the boundary of \( \Sigma \). For \( i = 1, 2, 3, 5 \), the arrow \( s_i \in S_{a_2} \) has tail at the vertex marking the line segment \( \tau_i \) by Reid’s recipe and head at the vertex marking the unique interior lattice point of \( \Sigma \) in \( \tau_i \). The arrow \( s_4 \) exhibits an opposite behaviour. It has tail at vertex \( 25 \), which marks the line segment \( \tau_4 \) and head at vertex \( 21 \), which marks the interior lattice point connected to \( \tau_4 \). The arrows \( s'_4 \) and \( s''_4 \) both have tail at the vertex marking \( \tau_4 \) and head at the vertex making \( \tau_5 \) and \( \tau_3 \), respectively.

![Figure 7.4: Strip of triangles \( \text{strip}(a_2) \)](image)

<table>
<thead>
<tr>
<th>Arrow</th>
<th>Label</th>
<th>Tail</th>
<th>Head</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_0 )</td>
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<td>0</td>
<td>10</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( a_{30} )</td>
<td>11</td>
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<tr>
<td>( s_2 )</td>
<td>( a_{43} )</td>
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<td>18</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>( a_{55} )</td>
<td>19</td>
<td>25</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>( a_{72} )</td>
<td>( \sum_{j=1}^{19} D_j - (D_2 + D_3 + D_{16}) )</td>
<td>25</td>
</tr>
<tr>
<td>( s'_4 )</td>
<td>( a_{50} )</td>
<td>( D_3 + D_{16} )</td>
<td>21</td>
</tr>
<tr>
<td>( s''_4 )</td>
<td>( a_{58} )</td>
<td>( D_2 + D_{16} )</td>
<td>21</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>( a_{57} )</td>
<td>( D_2 )</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 7.2: Labelling divisors, heads and tails for the arrows in \( S_{a_2} \)

Finally, when \( \alpha = a_1 \), Theorem 5.16 tells us that the only arrows in \( S_{a_1} \) are \( s_0 = a_1 \) and \( s_1 = a_{19} \). The head of the arrow \( s_0 \) is the vertex 1, which marks the boundary line segments of \( \text{strip}(a_1) \) not in the boundary of \( \Sigma \), as expected. The arrow \( s_1 \) has tail at vertex 7, which marks the unique interior line segment in \( \text{strip}(a_1) \) and head at vertex 8, which marks the unique interior lattice point of \( \Sigma \) in \( \text{strip}(a_1) \).

### 7.2 The conjecture

It is clear from the examples in the previous section that Lemma 4.9 provides the first link between the arrow contraction algorithm and combinatorial Reid’s recipe for consistent dimer models. The following conjecture suggests a stronger link between these two:
Conjecture 7.1. Let $\Gamma$ be a consistent dimer model and let $\alpha$ be an arrow in its dual quiver $Q$ with tail at vertex 0. The arrows $s_i$, $s'_i$ and $s''_i$ denote the arrow or arrows in $S_\alpha$ dual to the edge or edges $s_i$, $s'_i$ and $s''_i$ associated to $\alpha$ (see Theorem 5.16). Let $\rho$ be a lattice point on $\partial\text{strip}(\alpha) \setminus \partial\Sigma$ corresponding to an interior divisor $D_\rho$ of $\mathcal{M}_A$. Then, one of the following holds:

1. Two lower triangles $\sigma_{i-1}$ and $\sigma_i$ on $\text{strip}(\alpha)$ contain the lattice point $\rho$. Reid’s recipe marks the lattice point $\rho$ and the line segment $\tau_i$ with distinct vertices $v_\tau$ and $v_{\tau_i}$, respectively, and there exists a line segment $\tau'$ in $\Sigma'$ containing $\rho$ marked by $v_{\tau_i}$. Then

$$t(s_i) = v_{\tau_i} \quad \text{and} \quad h(s_i) = v_{\rho}.$$ 

2. Two lower triangles $\sigma_{i-1}$ and $\sigma_{i+1}$ and an upper triangle $\sigma_i$ contain the lattice point $\rho$. Reid’s recipe marks the line segments $\tau_i$ and $\tau_{i+1}$ with distinct vertices $v_{\tau_i}, v_{\tau_{i+1}}$ and either:

   (a) Reid’s recipe marks the lattice point $\rho$ with a unique vertex $v_\rho \in Q_0$ and there exists a line segment $\tau'$ in $\Sigma'$ containing $\rho$ marked with either $v_{\tau_i}$ or $v_{\tau_{i+1}}$. Then the arrows $s_i$ and $s_{i+1}$ have tail at $v_{\tau_i}$ and $v_{\tau_{i+1}}$, respectively. If $v_{\tau_i}$ marks $\tau'$

$$t(s_i) = v_{\tau_i} \quad h(s_i) = v_{\tau_{i+1}} \quad t(s_{i+1}) = v_{\tau_{i+1}} \quad h(s_{i+1}) = v_{\rho}.$$ 

Else, $v_{\tau_{i+1}}$ marks $\tau'$ and

$$t(s_i) = v_{\tau_i} \quad h(s_i) = v_\rho \quad t(s_{i+1}) = v_{\tau_{i+1}} \quad h(s_{i+1}) = v_{\tau_i}.$$ 

(b) Reid’s recipe marks the lattice point $\rho$ with two vertices $v_\rho$ and $v'_\rho$ and there exists line segments $\tau'_1$ and $\tau'_2$ in $\Sigma'$ containing $\rho$ marked with $v_{\tau_i}$ and $v_{\tau_{i+1}}$, respectively. Then

$$t(s_i) = v_{\tau_i} \quad h(s_i) = v_\rho \quad t(s_{i+1}) = v_{\tau_{i+1}} \quad h(s_{i+1}) = v'_\rho.$$ 

3. Two lower triangles $\sigma_{i-2}$ and $\sigma_{i+1}$ and two upper triangles $\sigma_{i-1}$ and $\sigma_i$ all contain the lattice point $\rho$. Reid’s recipe marks the lattice point $\rho$ and line segments $\tau_{i-1}, \tau_i$ and $\tau_{i+1}$ with distinct vertices $v_\rho, v_{\tau_{i-1}}, v_{\tau_i}$ and $v_{\tau_{i+1}}$ and there exists a line segment $\tau'$ in $\Sigma'$ containing $\rho$ marked with $v_{\tau_i}$. Then

$$t(s_{i-1}) = v_{\tau_{i-1}} \quad h(s_{i-1}) = v_\rho \quad t(s'_i) = v_{\tau_i} \quad h(s'_i) = v_{\tau_{i+1}}$$
$$t(s_{i+1}) = v_{\tau_{i+1}} \quad h(s_{i+1}) = v_\rho \quad t(s''_i) = v_{\tau_i} \quad h(s''_i) = v_{\tau_{i-1}}$$
$$t(s_i) = v_\rho \quad h(s_i) = v_{\tau_i}.$$ 

Assuming Conjecture 7.1, we would like to use an inductive argument to show that in the marking of $\Sigma$ with vertices of $Q$ by Reid’s recipe, every nonzero vertex appears ‘once’, in the sense of Conjecture 3.33. To do this, we need to verify that Reid’s recipe is recursive with respect to the arrow contraction algorithm. First note that Conjecture 7.1 suggests a labelling of the vertices of $Q'$ by the labels of some of the vertices of $Q$. Either:
1. The vertex $v'$ of $Q'$ is dual to a tile made up of a single tile $t$ of $\Gamma$, in which case the label of $v'$ is the label of the vertex $v \in Q_1$ dual to the tile $t$; or

2. The vertex $v'$ of $Q'$ is dual to a tile made up by merging two or more tiles of $\Gamma$, in which case either:

   (a) Its dual tile contains the zero tile of $\Gamma$, and we set $v' = \text{zero vertex of } Q'$; or

   (b) Its dual tile contains a tile of $\Gamma$ dual to a vertex $v \in Q_1$ that labels an interior lattice point $\rho$ in $\Sigma \setminus \text{strip}(\alpha)$, and we set $v' = v$; or

   (c) Its dual tile contains no tile of $\Gamma$ dual to a vertex marking an interior lattice point $\rho$ in $\Sigma \setminus \text{strip}(\alpha)$, but contains a tile dual to a vertex marking an interior line segment $\tau$ in $\Sigma \setminus \text{strip}(\alpha)$, and we set $v' = v$.

**Conjecture 7.2.** After performing Reid’s recipe for $\Gamma'$, a vertex $v$ marks an interior lattice point or an interior line segment of $\Sigma'$ by Reid’s recipe if and only if $v$ labels a vertex of $Q'$ and $v$ marks the corresponding interior lattice point or interior line segment of $\Sigma \setminus \text{strip}(\alpha)$.

The proof of Conjecture 3.33 would be an inductive argument which performs the arrow contraction algorithm as many times as needed to produce the dimer model from Example 1.6. At each stage, the number of vertices will be reduced as tiles of $\Gamma$ merge to give new tiles of $\Gamma'$ until we are left with a single vertex, the zero vertex. It is not yet clear that for any consistent dimer model $\Gamma$ one can always repeatedly perform the arrow contraction algorithm to produce the dimer model from Example 1.6, as the algorithm is only well-defined for when $\Delta'$ is a nondegenerate polygon.

**Example 7.3.** Consider the dimer model $\Gamma$ with dual quiver $Q$ in Figure 7.5(a). For $\theta = (-9,1,1,\ldots,1)$, the fan $\Sigma$ of $M_A$ is depicted in Figure 7.6(a).

![Image](image.png)

(a) The dimer model $\Gamma$ and its dual quiver $Q$  
(b) The quiver $Q$ with labelled arrows

**Figure 7.5:** The dimer model $\Gamma$ and its dual quiver $Q$
Let $\alpha = a_1$, with label $\text{div}(a_1) = D_1 + D_2 + D_6$ (see Figure 7.5(b)). The set $\mathcal{S}_{a_1}$ consists of six edges corresponding to the arrows:

<table>
<thead>
<tr>
<th>Arrow</th>
<th>Label</th>
<th>Tail</th>
<th>Head</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$D_1 + D_2 + D_6$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$a_{28}$</td>
<td>$D_1 + D_6$</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>$a_{23}$</td>
<td>$D_2$</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>$a_8$</td>
<td>$\sum_{j=1}^{10} D_j - (D_1 + D_2 + D_6)$</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>$a_{26}$</td>
<td>$D_1 + D_2$</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$D_6$</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 7.3: Labelling divisors, heads and tails for the arrows in $\mathcal{S}_{a_1}$

To obtain $\Gamma'$ from $\Gamma$, the zero tile of $\Gamma$ merges with the tile dual to vertex 4 in $Q$ to make a tile of $\Gamma'$ dual to the vertex labelled 0 in $Q'$. The tiles dual to the vertices 1, 2, 7 and 9 of $Q$ merge into a single tile of $\Gamma'$. As the vertex 2 appears labelling an interior lattice point of $\Sigma \setminus \text{strip}(a_1)$, we assign the label 2 to the vertex dual to this tile of $\Gamma'$.

To obtain $\Gamma'$ from $\Gamma$, the zero tile of $\Gamma$ merges with the tile dual to vertex 4 in $Q$ to make a tile of $\Gamma'$ dual to the vertex labelled 0 in $Q'$. The tiles dual to the vertices 1, 2, 7 and 9 of $Q$ merge into a single tile of $\Gamma'$. As the vertex 2 appears labelling an interior lattice point of $\Sigma \setminus \text{strip}(a_1)$, we assign the label 2 to the vertex dual to this tile of $\Gamma'$.

To obtain $\Gamma'$ from $\Gamma$, the zero tile of $\Gamma$ merges with the tile dual to vertex 4 in $Q$ to make a tile of $\Gamma'$ dual to the vertex labelled 0 in $Q'$. The tiles dual to the vertices 1, 2, 7 and 9 of $Q$ merge into a single tile of $\Gamma'$. As the vertex 2 appears labelling an interior lattice point of $\Sigma \setminus \text{strip}(a_1)$, we assign the label 2 to the vertex dual to this tile of $\Gamma'$.

Figure 7.6: The fan $\Sigma$ of $\mathcal{M}_A$ and Reid’s recipe on $\Sigma$ and $\Sigma'$

Conjecture 7.1 could also be used to show that the relations of the tautological line bundles of $\mathcal{M}_A$ in $\text{Pic}(\mathcal{M}_A)$ are encoded by combinatorial Reid’s recipe. Proposition 7.4 shows that the relations from Conjecture 3.34 hold when restricted to some local charts. The proof assumes Conjecture 7.1 and considers each its four cases individually.

**Proposition 7.4.** Let $\Gamma$ be a consistent dimer model and let $\alpha$ be an arrow in its dual quiver $Q$ with tail at vertex 0. Assume Conjecture 7.1. For all $\sigma \not\subseteq \text{strip}(\alpha)$, the following relations hold in $\text{Pic}(\mathcal{M}_A)$:

1. \(L_{v_{\tau_1}|U_\sigma} = L_{v_{\tau_1}|U_\sigma} \otimes L_{h(\alpha)|U_\sigma}\) in Case 1 of Conjecture 7.1;

2. \(L_{v_{\tau_1}|U_\sigma} = L_{v_{\tau_1}|U_\sigma} \otimes L_{h(\alpha)|U_\sigma}\) or \(L_{v_{\tau_{i+1}}|U_\sigma} = L_{v_{\tau_{i+1}}|U_\sigma} \otimes L_{h(\alpha)|U_\sigma}\) in Case 2(a) of Conjecture 7.1, depending on whether \(v_{\tau_1}\) or \(v_{\tau_{i+1}}\) mark \(\tau';\)
3. \[ L_{v_p}|_{U_\sigma} \otimes L_{v'_p}|_{U_\sigma} = L_{v_\tau_i}|_{U_\sigma} \otimes L_{v_{\tau_{i+1}}}|_{U_\sigma} \otimes L_{h(\alpha)}|_{U_\sigma} \] in Case 2(b) of Conjecture 7.1.

4. \[ L_{v_p}|_{U_\sigma} = L_{v_\tau_i}|_{U_\sigma} \otimes L_{h(\alpha)}|_{U_\sigma} \] in Case 3 of Conjecture 7.1.

Proof. Let \( s_i, s'_i \) and \( s''_i \) denote the arrow or arrows in \( S_\alpha \) dual to the edge or edges \( s_i, s'_i \) and \( s''_i \) associated to \( \tau_i \) (see Theorem 5.16).

For Case 1, first observe that by Lemma 5.18, the label of the arrow \( s_i \) coincides with \( \text{div}(\alpha) \). Moreover, the arrow \( s_i \) has tail at \( v_{\tau_i} \) and head at \( v_p \). Therefore for all \( \sigma \nsubseteq \text{strip}(\alpha) \), there exists a path from 0 to \( v_p \) in \( Q_\sigma \), the quiver supporting \( M_\sigma \), made up of a path from 0 to \( v_{\tau_i} \) followed by the arrow \( s_i \). This shows that the relation \( L_{v_p}|_{U_\sigma} = L_{v_\tau_i}|_{U_\sigma} \otimes L_{h(\alpha)}|_{U_\sigma} \) holds, as \( L_v|_{U_\sigma} = O_{U_\sigma}(\text{div}(p)) \) where \( p \) is a path in a path in \( Q_\sigma \) from 0 to \( v \).

For Case 2(a), Lemma 5.18 says that \( \text{div}(s_i) + \text{div}(s_{i+1}) = \text{div}(\alpha) \). Therefore for all \( \sigma \nsubseteq \text{strip}(\alpha) \), there exists a path from 0 to \( v_p \) in \( Q_\sigma \) made up of a path from 0 to either \( v_{\tau_i} \) or \( v_{\tau_{i+1}} \), depending on whether \( v_{\tau_i} \) or \( v_{\tau_{i+1}} \) marks \( \tau' \), followed by one of \( s_{i+1} \) or \( s_i \) and then the other. Thus, either \( L_{v_p}|_{U_\sigma} = L_{v_\tau_i}|_{U_\sigma} \otimes L_{h(\alpha)}|_{U_\sigma} \) or \( L_{v_p}|_{U_\sigma} = L_{v_{\tau_{i+1}}}|_{U_\sigma} \otimes L_{h(\alpha)}|_{U_\sigma} \).

For Case 2(b), again Lemma 5.18 tells us that \( \text{div}(s_i) + \text{div}(s_{i+1}) = \text{div}(\alpha) \). Thus for \( \sigma \nsubseteq \text{strip}(\alpha) \), there exist paths in \( Q_\sigma \) from 0 to \( v_p \) and from 0 to \( v'_p \) made up of a path from 0 to \( v_{\tau_{i+1}} \) followed by the arrow \( s_i \) in one case, and a path from 0 to \( v_{\tau_{i+1}} \) followed by the arrow \( s_{i+1} \) in the other. Hence, \( L_{v_p}|_{U_\sigma} \otimes L_{v'_p}|_{U_\sigma} = L_{v_{\tau_i}}|_{U_\sigma} \otimes L_{v_{\tau_{i+1}}}|_{U_\sigma} \otimes L_{h(\alpha)}|_{U_\sigma} \).

Finally, for Case 3 we know that \( \text{div}(s_{i-1}) + \text{div}(s''_i) = \text{div}(s_{i+1}) + \text{div}(s'_i) = \text{div}(\alpha) \). Moreover for \( \sigma \nsubseteq \text{strip}(\alpha) \), there exist a minimum of two paths from 0 to \( v_p \). Both start by travelling from 0 to \( v_{\tau_i} \), then one goes through \( s''_i \) followed by \( s_{i-1} \) while the other travels through \( s'_i \) and then \( s_{i+1} \). It follows that \( L_{v_p}|_{U_\sigma} = L_{v_{\tau_i}}|_{U_\sigma} \otimes L_{h(\alpha)}|_{U_\sigma} \), as required.

Therefore, to prove Conjecture 3.34, it suffices to show that Conjecture 7.1 holds and that the relations of Proposition 7.4 hold for all \( \sigma \subseteq \text{strip}(\alpha) \).
APPENDIX A

PERFECT MATCHINGS OF THE DIMER MODEL IN FIGURE 1.3

Table A.1 lists all perfect matchings of the dimer model in Figure 1.3. We write $e_i$ for the edge in $\Gamma$ dual to the arrow marked by $i$ in Figure 1.3. The last column of the table shows the value of the height change for each perfect matching with respect to $\Pi_1$.

<table>
<thead>
<tr>
<th>Perfect Matching</th>
<th>Edges</th>
<th>$h(\cdot, \Pi_1)$</th>
</tr>
</thead>
<tbody>
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<td>$e_1$ $e_6$ $e_7$ $e_{10}$ $e_{14}$ $e_{16}$ $e_{17}$ $e_{26}$ $e_{28}$</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$\Pi_2$</td>
<td>$e_1$ $e_7$ $e_{13}$ $e_{16}$ $e_{17}$ $e_{19}$ $e_{21}$ $e_{23}$ $e_{26}$</td>
<td>(0,1)</td>
</tr>
<tr>
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</tr>
<tr>
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<td>(-3.3)</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
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</tr>
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<tr>
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</tr>
<tr>
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<tr>
<td>$\Pi_{19}$</td>
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<td>(-2.2)</td>
</tr>
</tbody>
</table>

Continued on next page
Table A.1 – Continued from previous page

<table>
<thead>
<tr>
<th>Perfect Matching</th>
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<th>$h(-, \Pi_2)$</th>
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Table A.1: Perfect matchings for the dimer model in Figure 1.3
Table B.1 lists all the labels for the arrows of the quiver in Figure B.1 below. We write $a_i$ for the arrow marked by $i$ in Figure B.1. An arrow $a$ in Table B.1 with label $i_1, i_2, \ldots, i_j$ has labelling divisor $\text{div}(a) = D_{i_1} + D_{i_2} + \ldots + D_{i_j}$.

![Quiver Diagram](image)

**Figure B.1:** The quiver $Q$ from Figure 4.1
### Appendix B. Labels for the arrows of the quiver in Figure 4.1

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### Table B.1: Labels for the arrows of the quiver in Figure B.1

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