Optimal Control of Heteroscedastic Macroeconomic Models

Online Appendix

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A Derivation of the general solution

Start with the Bellman equation (5) and substitute the optimizer (6) - (8) into the right side. Then rearrange to get:

\[ V(x, w, \Sigma) = \min_i \left[ x'Rx + \beta \{ tr(PQ)x'Sx + x'APA + x'FBPFBx - 2x'A'PBFx \} \right. \]
\[ + x'WFx - 2x'HFx + \beta c' \left( K + C'ww'C + G'\Sigma G + Lx's + Qx'Sx \right) c \]
\[ - 2x'Rx + 2x'W(f - i') + \beta g' \left( K + C'ww'C + G'\Sigma G + Lx's + Qx'Sx \right) g \]
\[ + 2x'H'x + 2x'H(f - i') - 2\beta x'A'p + 2\beta x'H'Bp - \beta x'H'BPF \]
\[ + 2\beta x'A'PBf + \beta tr(PL)x's + x'Rx + (f - i')'W(f - i') - 2x''H(f - i') \]
\[ + \beta (f'B'PBf + tr(PK) + tr(PC'ww'C) + tr(PG'\Sigma G) + k - 2f'B'p) \]  
\[ (A1) \]

Replace the trial solution (4) on the left side of (A1) and equate the coefficients of the quadratic forms in x to obtain the condition:

\[ P = R + F'WF - 2HF + \beta \left[ A'PA + F'B'PBF - 2A'PBF + tr(PQ)S + c'QcS + g'QgS \right]. \]

Substituting (7) and (8) in the above gives the matrix Riccati equation for P in (9). Equating the terms involving ww' in equations (4) and (A1) gives:

\[ c'ww'c = \beta \left( tr(PC'ww'C) + c'C'ww'Cc + g'C'ww'Cg \right). \]  
\[ (A2) \]

The Lyapunov equation (10) is obtained from this by substituting \( c'ww'c = w'cc'w, \) \( tr(PC'ww'C) = w'CPC'w, \) \( c'C'ww'Cc = w'Ccc'C'w \) and \( g'C'ww'Cg = w'Cgg'C'w \) into (A2) and equating the coefficients of the quadratic form in w. Collecting the terms in \( \Sigma \) on the right side of the Bellman equation (A1) and equating them to \( g'\Sigma g \) in equation (4) gives:

\[ g'\Sigma g = \beta tr(PG'\Sigma G) + \beta c'(G'\Sigma G)c + \beta g'G'\Sigma Gg. \]
Using the properties of the trace of a matrix, this can be written as:

\[ tr(\mathbf{gg}'\Sigma) = \beta tr([\mathbf{GPG}' + \mathbf{Gcc}'\mathbf{G'} + \mathbf{Ggg}'\mathbf{G'}]\Sigma) \].

Note that given three square matrices \(\mathbf{A}, \mathbf{B}\) and \(\mathbf{C}\), it follows that if \(tr(\mathbf{AB}) = tr(\mathbf{CB})\) then \(\mathbf{A} = \mathbf{C}\). Applying this result to the above equation, the coefficients of the matrix \(\Sigma\) on the left and right sides can be equated to get equation (11). It is readily verified that the solution for \(p\) in equation (12) is obtained by equating the coefficients of the linear terms in \(x\) of equations (4) and (A1), and then rearranging. Equating the remaining terms in (4) and (A1), and rearranging gives the solution for \(k\) in equation (13).

### B  Certainty-equivalent transformation

This appendix derives the objects \(\mathbf{\tilde{R}}, \mathbf{\tilde{x}^*}, \mathbf{\tilde{i}^*}\) and \(k\) such that the Bellman equation (5) can be written equivalently as (18). Start by expanding the first quadratic term in (5):

\[
V(\mathbf{x}, \mathbf{w}, \Sigma) = \min_{\mathbf{i}} \left[ \mathbf{x}'\mathbf{Rx} + \mathbf{x}'\mathbf{Rx}^* - 2\mathbf{x}'\mathbf{Rx}^* + (\mathbf{i} - \mathbf{i}^*)'\mathbf{W}(\mathbf{i} - \mathbf{i}^*) + 2(\mathbf{x} - \mathbf{x}^*)'\mathbf{H}(\mathbf{i} - \mathbf{i}^*) + \beta \{ k + tr\left(\mathbf{PC'ww'C} + \mathbf{PG'SG} + \mathbf{Lx's + Qx'Sx}\right)\mathbf{c} \} + \beta \{ tr\left(\mathbf{PL}\right)\mathbf{x's + trPQ}\mathbf{x'Sx} + \mathbf{g'}\left(\mathbf{K} + \mathbf{C'ww'C} + \mathbf{G'SG} + \mathbf{Lx's + Qx'Sx}\right)\mathbf{g} \} + I \right]
\]

Use \(\mathbf{\tilde{R}}\) to rearrange this as:

\[
V(\mathbf{x}, \mathbf{w}, \Sigma) = \min_{\mathbf{i}} \left[ \mathbf{x}'\mathbf{\tilde{R}x} + \mathbf{x}'\mathbf{Rx}^* - 2\mathbf{x}'\left[\mathbf{Rx}^* - \frac{1}{2} \beta tr\left(\mathbf{PL}\right)\mathbf{s} - \frac{1}{2} \beta \mathbf{sc'}\left(\mathbf{L}\right)\mathbf{c} - \frac{1}{2} \beta \mathbf{sc'}\left(\mathbf{L}\right)\mathbf{Lg} \right] \right] \\
+ (\mathbf{i} - \mathbf{i}^*)'\mathbf{W}(\mathbf{i} - \mathbf{i}^*) + 2(\mathbf{x} - \mathbf{x}^*)'\mathbf{H}(\mathbf{i} - \mathbf{i}^*) + \beta k + I + \beta tr\left(\mathbf{PC'ww'C}\right) + \beta tr\left(\mathbf{PG'SG}\right) \\
+ \beta \mathbf{g'}\left(\mathbf{K} + \mathbf{C'ww'C} + \mathbf{G'SG}\right)\mathbf{g} + \beta \mathbf{c'}\left(\mathbf{K} + \mathbf{C'ww'C} + \mathbf{G'SG}\right)\mathbf{c}.
\] (A3)
Note that the last four terms in this equation arise only because there is a GARCH component in the variance structure, which is certainty equivalent. We can write (18) as:

\[
\tilde{V}(x, w, \Sigma) = \min_{i} [x'\tilde{R}x + \tilde{x}'\tilde{R}\tilde{x} + 2x'\tilde{R}\tilde{x} + (i - i')' W (i - i') + (i' - \bar{i}')] W (i' - \bar{i}')
\]

\[
+ \beta \bar{k} + I + 2 (i - i')' W (i' - \bar{i}') + 2 (x - x')' H (i - i') + \beta \text{tr} \left( P C' w' C \right)
\]

\[
+ \beta \text{tr} \left( P G' \Sigma G \right) + 2 (x - x')' H \left( i' - \bar{i}' \right) + 2 (x' - \bar{x}')' H \left( i - \bar{i}' \right)
\]

\[
+ \beta c' \left( K + C' w w' C + G' \Sigma G \right) c + \beta g' \left( K + C' w w' C + G' \Sigma G \right) g \]

(A4)

After equating (A3) and (A4), and eliminating the common terms we get:

\[
x''Rx'' - 2x' \left[ Rx'' - \frac{1}{2} \beta \text{tr} (PL) s - \beta \frac{1}{2} sc' (L) c - \beta \frac{1}{2} sg' L g \right] + \beta k =
\]

\[
\tilde{x}''\tilde{R}\tilde{x}'' - 2x'\tilde{R}\tilde{x}'' + \beta \bar{k} + \left( i' - \bar{i}' \right)' W \left( i' - \bar{i}' \right)
\]

\[
+ 2 (i - i')' W \left( i' - \bar{i}' \right) + 2 (x - x')' H \left( i' - \bar{i}' \right) + 2 (x' - \bar{x}')' H \left( i - \bar{i}' \right)
\]

The definition of \( \tilde{i}^* \) can be used to re-write the last three terms in the above as:

\[
-2 (x' - \bar{x}')' H W^{-1} H' (x - \bar{x}').
\]

Thus the equality between the two Bellman equations becomes:

\[
x''Rx'' - 2x' \left[ Rx'' - \frac{1}{2} \beta \text{tr} (PL) s - \beta \frac{1}{2} sc' (L) c - \beta \frac{1}{2} sg' L g \right] + \beta k
\]

\[
= \tilde{x}''\tilde{R}\tilde{x}'' - 2x'\tilde{R}\tilde{x}'' + \beta \bar{k} + \left( i' - \bar{i}' \right)' W \left( i' - \bar{i}' \right) - 2x' H W^{-1} H' (x' - \bar{x}')
\]

\[
+ 2 \tilde{x}'' H W^{-1} H' (x' - \bar{x}').
\]

Finally \( \tilde{x}^* \) is found by equating the linear coefficients; while \( \bar{k} \) by equating the constant coefficients.

The intuition behind these results is quite simple. When a decision maker has a concave utility, volatility in the goal variables reduces expected utility in the same way that volatility of asset prices reduces the utility of an investor with a concave utility function. If the process
describing volatility is independent of the state vector, as in the homoscedastic optimal regulator problem and under GARCH, the decision rule exhibits CE. Thus volatility affects only welfare. This logic extends to any stochastic linear regulator problem with exogenous time-varying volatility.

If however volatility is state dependent and the decision maker can influence the state of the system, this should influence his behavior. The CE principle does not hold, since the coefficients of the optimal feedback rule as well as the minimum value of the loss depend on the variance structure. Sections 2.2.3 and 2.2.4 formalize this observation. When the conditional volatility terms are linear and quadratic they have an effect which is mathematically equivalent to the linear and quadratic terms describing the welfare loss. This isomorphism means that any linear-quadratic state dependent volatility optimal control problem can be re-written as a CE control problem by a suitable re-parameterization of the targets and welfare weights. Thus researchers can draw upon standard optimal control algorithms to solve heteroscedastic optimal control problems and interpreting their results.

C State-space representation

C.1 Unconditional covariance matrix VAR residuals

The unconditional covariance matrix of the residuals $\Omega$ is determined by evaluating the unconditional expectations on the right side of (24), setting $\Omega_t = \Omega$, $E[z_t]'s = \bar{\pi}$ and $E[z_t'Sz_t] = \bar{z}'S\bar{z} + E[(\pi_t - \bar{\pi})^2] = \bar{\pi}^2 + tr(S^2[(I - \Phi_1-\Phi_2)(I - \Phi_1-\Phi_2)])$ using (23), to obtain the discrete Lyapunov equation:

$$\Omega = \Omega_0 + \Omega_1 \bar{\pi} + \Omega_2 (\bar{\pi}^2 + tr(S^2[(I - \Phi_1-\Phi_2)(I - \Phi_1-\Phi_2)])$$
$$+ M\Omega M' + N\Omega N'. \quad (A5)$$

This can be solved numerically.
C.2 Mapping from VAR to state-space model

In the main text the VAR in equation (23) is written as:

\[ z_{t+1} = \Phi_1 z_t + \Phi_2 z_{t-1} + e_t. \]

with \( \Phi_1 = \begin{bmatrix} \Phi_1^1 & \Phi_1^2 \end{bmatrix} \), \( \Phi_2 = \begin{bmatrix} \Phi_2^1 & \Phi_2^2 \end{bmatrix} \). These two matrices of parameters can also be partitioned as \( \Phi_1 = \begin{bmatrix} \phi_1 & \gamma_1 \\ \chi_{y1} & \chi_{r1} \end{bmatrix} \) and \( \Phi_2 = \begin{bmatrix} \phi_2 & \gamma_2 \\ \chi_{y2} & \chi_{r2} \end{bmatrix} \). This separates the blocks of non-policy (upper part) and policy (bottom part) variables, since only the former is required for the solution of the optimal feedback rule. The ordering of the three variables is used to identify the Impulse Response Function (IRF), which is derived from the Cholesky decomposition of the estimated covariance matrix of the shocks. This ordering is standard in the small-scale monetary VAR literature and alternative orderings do not produce significant changes in the shape of the IRF.

The non-policy block of the VAR model (23) can be written in a companion form consistent with (2) using:

\[
\begin{align*}
    x_t &= \begin{bmatrix} y_t' & y_{t-1}' & r_{t-1} \end{bmatrix}' , \\
    i_t &= r_t , \\
    w_t &= \begin{bmatrix} e'_{y,t} & 0_3 \end{bmatrix}' , \\
    A &= \begin{bmatrix} \phi_1 & \phi_2 & \gamma_2 \\ I_2 & 0_{2,2} & 0_2 \\ 0_2' & 0_2' & 0 \end{bmatrix} , \\
    B &= \begin{bmatrix} \gamma_1' & 0_2' & 1 \end{bmatrix}'.
\end{align*}
\]

The 2 × 2 portion of the covariance matrix in equation (24) pertinent to the unemployment and inflation equations alone, \( \Omega_{1,t+1} \), can be written as

\[
\Omega_{1,t+1} = E_t \left[ e_{y,t+1}e'_{y,t+1} \right] = K_1 + C_1'e_y t e'_{y,t}C_1 + C_1'\Omega_{1,t}G_1 + L_1y_t s_y + Q_1y_t' s_1y_t
\]

with \( K_1, C_1, G_1, L_1, Q_1 \) and \( S_1 \) being 2 × 2 matrices, \( s_y = \begin{bmatrix} 0 & 1 \end{bmatrix}' \) and \( S_1 = s_y s_y' \).

Given the definition of \( x_t \) provided above, the covariance matrix can be written in a form
consistent with (3) using:

\[ K = \begin{bmatrix} K_{1} & 0_{2,3} \\ 0_{3,2} & 0_{3,3} \end{bmatrix}, \quad C = \begin{bmatrix} C_{1} & 0_{2,3} \\ 0_{3,2} & 0_{3,3} \end{bmatrix}, \quad G = \begin{bmatrix} G_{1} & 0_{2,3} \\ 0_{3,2} & 0_{3,3} \end{bmatrix}, \]

\[ L = \begin{bmatrix} L_{1} & 0_{2,3} \\ 0_{3,2} & 0_{3,3} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{1} & 0_{2,3} \\ 0_{3,2} & 0_{3,3} \end{bmatrix}, \quad s = \begin{bmatrix} s'_{y} & 0'_{3} \end{bmatrix}. \]

Using the above definitions of \( x_t \) and \( i_t \), the transformed welfare loss can be written in a form equivalent to (1) using:

\[ R = \begin{bmatrix} \lambda_u & 0 & 0'_{2} & 0 \\ 0 & \lambda_{\pi} & 0'_{2} & 0 \\ 0_{2} & 0_{2} & 0_{2,2} & 0 \\ 0 & 0 & 0'_{2} & \lambda_{\Delta r} \end{bmatrix} \]

\[ x^* = 0, \ i^* = 0, \ W = \lambda_{\Delta r} \text{ and } H = \begin{bmatrix} 0'_{4} & -\lambda_{\Delta r} \end{bmatrix}'. \]

The VAR model under control consists of the non-policy block in equation (23) and the policy rule (6). Following Polito and Wickens (2012), we partition the vector \( F \) conformably with \( x_t \), i.e. \( F = \begin{bmatrix} F'_{y} & F'_{y-1} & f_{r-1} \end{bmatrix}' \), and append this to the non-policy block to obtain:

\[ \begin{bmatrix} I_{2} & 0 \\ F'_{y} & 1 \end{bmatrix} \begin{bmatrix} y_{t} \\ r_{t}^{*} \end{bmatrix} = \begin{bmatrix} 0_{2} \\ f \end{bmatrix} + \begin{bmatrix} \phi_{1} & \gamma_{1} \\ -F'_{y-1} & -f_{r-1} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{2} & \gamma_{2} \\ 0'_{2} & 0 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} e_{yt} \\ 0 \end{bmatrix}. \]

This can be solved to obtain a reduced-form system congruent with (23):

\[ \begin{bmatrix} y_{t} \\ r_{t}^{*} \end{bmatrix} = \begin{bmatrix} 0_{2} \\ f \end{bmatrix} + \begin{bmatrix} \phi_{1} & \gamma_{1} \\ \lambda'_{y1} & \lambda'_{r1} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{2} & \gamma_{2} \\ \lambda'_{y2} & \lambda'_{r2} \end{bmatrix} \begin{bmatrix} y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} e_{yt} \\ e_{rt} \end{bmatrix}; \]
with $\chi_{y1}^* = -F'_y \phi_1 - F'_{y-1}; \chi_{r1}^* = -F'_y \gamma_1 + f'_1; \chi_{y2}^* = -F'_y \phi_2; \chi_{r2}^* = -F'_y \gamma_2$ and $e_{rt}^* = -F'_y e_{yt}$. For the purpose of dynamic simulation, note that the residuals required for the simulation of the VAR model (A6) are

$$e_{yt} = \Xi_{1,t+1} v_{y,t+1}$$

where $\Xi_{1,t+1}$ is the top-left $2 \times 2$ block of $\Xi_{t+1}$ and $v_{y,t+1}$ is the top $2 \times 1$ part of $v_{t+1}$.

D Maximum Likelihood Estimation

D.1 The likelihood function

Let $(z_{t+1}^+)' = \begin{bmatrix} z_{t+1}' & z_t' \end{bmatrix}, F_1 = \begin{bmatrix} \phi_1 & \gamma_1 \\ \chi_{y1} & \chi_{r1} \end{bmatrix}, F_2 = \begin{bmatrix} \phi_2 & \gamma_2 \\ \chi_{y2} & \chi_{r2} \end{bmatrix}$ and $v_{t+1}' = \begin{bmatrix} e_{t+1}' & 0_3' \end{bmatrix}$. Then the VAR equations in (23) can be written in the companion form as

$$z_{t+1}^+ = F^+ z_t^+ + v_{t+1}, \text{ with } F^+ = \begin{bmatrix} F_1 & F_2 \\ I_3 & 0_{3,3} \end{bmatrix}.$$ 

The covariance matrix in equation (24) can be written in the equivalent companion form as

$$\Omega_{t+1}^+ = E[v_{t+1}v_{t+1}'|z_{t+1}^+]$$

$$= \Omega_0^+ + \Omega_1^+ z_t^+ s^+ + \Omega_2^+ z_t^+ s^+ z_t^+ + M^+ v_t v_t' M^+ + N^+ \Omega_t^+ N^+$$

where

$$\Omega_0^+ = \begin{bmatrix} \Omega_0 & 0_{3,3} \\ 0_{3,3} & 0_{3,3} \end{bmatrix}, \quad \Omega_1^+ = \begin{bmatrix} \Omega_1 & 0_{3,3} \\ 0_{3,3} & 0_{3,3} \end{bmatrix}, \quad \Omega_2^+ = \begin{bmatrix} \Omega_2 & 0_{3,3} \\ 0_{3,3} & 0_{3,3} \end{bmatrix},$$

$$M^+ = \begin{bmatrix} M & 0_{3,3} \\ 0_{3,3} & 0_{3,3} \end{bmatrix}, \quad N^+ = \begin{bmatrix} N & 0_{3,3} \\ 0_{3,3} & 0_{3,3} \end{bmatrix},$$

and $s^+ = \begin{bmatrix} 0 & 1 & 0_4' \end{bmatrix}$ so that $z_t^+ s^+ = \pi_t$ and $z_t^+ S^+ s_t^+ = \pi_t^2$ (since $S^+ = s^+ s^+'$). Therefore, the log-likelihood for period $t$ of the heteroscedastic VAR model in equations (23) and (24)
can be written as:

\[
L_t = -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln(|\Omega_t^+|) - \frac{1}{2} v_t' \left( \Omega_t^+ \right)^{-1} v_t
\]

where:

\[
v_{t+1} = (\Omega_{t+1}^+)^{-1/2} (z_{t+1}^+ - F^+ z_t^+)
\]

Summing this over \(T\) periods gives the log-likelihood for the estimation period:

\[
L = -2T \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \ln(|\Omega_t^+|) - \frac{1}{2} \sum_{\tau=1}^{k} v_{\tau}^t \left( \Omega_{\tau}^+ \right)^{-1} v_{t}.
\]

The likelihood-based estimation of the parameters in equations (23) and (24) is straightforward: given an initial set of parameters values (typically this employs the OLS estimates of the homoscedastic model), a vector of shocks and a sample of the observations, we compute the likelihood function and update the original parameter values using simplex methods (we use the \textit{FindMinimum} numerical optimization package on MATLAB).

We require the variance structure to remain positive definite for all \(t\). The BEKK specification for the GARCH process of the variance structure in equation (24) guarantees that positive definiteness of the variance under the \(GH\) model. Admissibility is more problematic in discrete-time square-root volatility models (i.e. linear-dependent variance structures) because these use a Gaussian approximation, due originally to Sun (1992), that allows the driving variable to turn negative during a discrete-time interval. However this is not a problem under the linear-quadratic specification, since admissibility requires that the eigenvalues of the variance structure remain positive for all possible values of the driving factor. These are given by (24) requiring \(\sigma_{k,t+1}^2 \geq 0\), with \(k = u, \pi, i:\)

\[
\sigma_{k,t+1}^2 = \delta_{k,0} + \delta_{k,1} \pi_{t-1} + \delta_{k,2} \pi_{t-1}^2 + a_{k,T}^2 \varepsilon_{k,t}^2 + g_{k,T}^2 \sigma_{k,T}^2 \geq 0,
\]

which is ensured provided that \(4\delta_{k,2} \left( \delta_{k,0} a_{k,T}^2 \varepsilon_{k,T}^2 \sigma_{k,T}^2 \right) \geq \delta_{k,1}\), so that the roots of the associated quadratic equation are complex. As before, the simultaneous presence of GARCH and state-dependent processes in the variance structure imply a time-dependent condition
for admissibility of an heteroscedastic model, like the model EN. Our Matlab code automatically checks that this restriction is satisfied.

D.2 Impulse Response Function

Figure 1 plots the orthogonalized IRF for the HO, LN, LQ, GH and EN models. These responses are computed assuming that the inflation rate in the period before the shock hits the economy is zero. The shape of the IRFs is similar across models and in line with conventional findings in the VAR literature for the U.S. (see Stock and Watson 2001, for example): both inflation and the rate of interest fall in response to an unemployment shock; an interest rate shock increases unemployment and reduces inflation; the rate of interest increases following an inflation shock, and so does unemployment.

![Figure 1: Orthogonalized impulse-response function of the HO (solid-thin black line), LN (dotted black line), LQ (dashed black line), GH (solid black line) and EN (solid-thick grey line).](image)

E Stochastic simulation

We start by creating a flat benchmark path setting the starting values for the vector of non-policy variables to zero, to keep the steady-state values in line with the zero mean of
the empirical sample. Reflecting the change in the steady-state rate of inflation implied by linear dependence, we reduce the benchmark path for inflation in $LNX$, $LQX$, $ENX$ and $EN$ by the respective value of $\pi$ shown in Table 2. We then use the Matlab function \texttt{randn}, which produces standardized random normal variates, to generate 500 random paths for the standardized innovations that simulate the innovations in the vector $v$. Changing the sign on these then gives a total of 1000 antithetical normal variates.\footnote{The method of antithetic variables, first suggested by Hendry (1984), improves efficiency in Monte Carlo simulation. We trim 10\% of the simulated series to eliminate the impact of extreme draws.} This set is then used to perturb the benchmark path in each model/weight combination using equations (23) and $e_{t+1} = \Xi_{t+1}v_{t+1}$ for the analysis of the dynamics under the empirical rule; and equations (A6) and (A7) for the model dynamics under the optimal feedback rule.

Although these sets of shocks are the same for each such combination, it is important to note that the final disturbances (the $e$'s) generated by either $e_{t+1} = \Xi_{t+1}v_{t+1}$ or (A7) can be very different, particularly for sequences of large shocks with the same sign. The use of antithetical variable shocks makes the simulated path for unemployment, inflation and the rate of interest symmetric for positive and negative residual tracks in models $HO$ and $GHX$. However, recall that under linear dependence (models $LN$ and $LNX$) a series of negative unemployment and inflation shocks has the cumulative effect of lowering the simulated value of inflation, thus attenuating the effect of future shocks. This linear-dependence effect is also a feature of models $LQX$ and $LQX^*$, though it is partially offset by the quadratic dependence of the variance structure that amplifies volatility even in the presence of negative shocks. It is also a feature of models $ENX$ and $EN$. For a more detailed discussion on the dynamic responses of macroeconomic variables to shocks in models with state-dependent volatility, see Polito and Spencer (2011b). State dependent volatility is not the only source of amplification and asymmetric effects on the dynamic responses. These can also emerge when the central bank has asymmetric preferences towards macroeconomic stabilization, Ruge-Murcia (2004). They also are a typical feature of models with time-varying parameters, as in Primiceri (2005).
Further applications

There are several fields in economics, finance and econometrics where our framework could be used. Many of the models used in these fields still routinely assume that the error structure is homoscedastic, despite the evidence of heteroscedasticity frequently revealed by Breusch-Pagan and similar misspecification tests. Ljungqvist and Sargent (2004) show how to solve a wide range of optimal control problems under the assumption of homoscedasticity. Hansen and Sargent (2008) show how the solutions can be modified to allow for dynamic misspecification. Our framework deals with the effect of stochastic misspecification.

We have used this framework to revisit the monetary control problem given the extensive literature in this field and its relevance to policy, but the estimation and simulation techniques are also of wider relevance. They are also capable of development. For example, we have assumed that the policy targets are fixed, while the work of Ireland (2005), Dewachter and Lyrio (2004) and others suggests that these tend to move over time. These authors use the Kalman filter to model a moving inflation target as a latent variable, but this technique also assumes that the disturbances are homoscedastic. When models exhibit GARCH (Harvey, Ruiz and Santana (1992), Kim and Nelson (1999)) and state dependent volatility (Spencer (2008)) we need to employ the extended Kalman filter (Harvey (1989)). This technique could also be used to examine the monetary control problem.

Our illustrative model is based on the finding of GARCH and inflation-conditional volatility in a small-scale VAR model of the U.S. economy. In view of the Okun-Friedman conjecture that high inflation increases macroeconomic uncertainty, we felt confident in using inflation to drive volatility. However, this is by no means a general result. Further empirical research may find evidence of state dependent volatility in data for other variables or countries. For example, Benigno et al. (2010) report a significant correlation between the unemployment rate and the volatility of productivity shocks in the U.S.

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2 This tests the homoscedasticity assumption by regressing squared residuals from an econometric model upon lagged explanatory variables and squared error terms. The popular textbook of Heij et al (2004) provide interesting examples including unemployment & wage data, interest rates & bond yields, exchange rates and the money supply.

3 They formalize this evidence using a structural model in which an increase in the volatility of productivity increases the level of unemployment. The analysis of state-dependent volatility would involve the investigation of the reverse type of causality: how changes in the level of a variable affect its volatility or
Our framework could also be used for the estimation and optimal control of heteroscedastic rational expectation models. Recently, economists have looked at the effects of time-varying volatility in Dynamic Stochastic General Equilibrium (DSGE) models (see Justiniano and Primiceri (2008) for a specific example, and Fernández-Villaverde and Rubio-Ramírez (2010) for a broader survey). This literature focuses on the estimation of a DSGE model where changes in the variance structure are assumed to be driven by exogenous volatility processes that, like GARCH, have no direct feedback on the state of the economy. Both the inclusion of state-dependent components in the variance structure and the analysis of optimal policy represent feasible, though non-trivial, extensions of this literature.

The term-structure literature already provides a range of Gaussian and non-Gaussian state-dependent volatility models. Importantly, besides the Gaussian distribution used extensively in macroeconomics, our framework handles any probability distribution with mean and variance that are linear or quadratic in the state variables. That is because, as we note in the introduction to section 2, expectations of quadratic forms in the state variables include both mean and variance terms and the structure of the optimal control problem is preserved if these are linear-quadratic. This opens the way to monetary control problems in which the volatility of the interest rate has a lower bound as in the model of Cox, Ingersoll and Ross (1985) for example.\(^4\)

There is now a large literature, pioneered by the work of Primiceri (2005), on the estimation of macroeconomic models where exogenous time variation is allowed in the dynamic as well as the stochastic parameters. This relates to the debate over the respective roles of good shocks versus good policy during the Great Moderation. The evidence on exogenous time variation suggests that this can be ascribed mainly to changes in the volatility parameters, at least for small-scale VAR models of the U.S. economy (Primiceri (2005), Giannone, Lenza and Reichlin (2008)). Any time variation in the dynamic parameters will nevertheless affect the policy rule and welfare.\(^5\) Moreover, state dependence in these parameters would

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\(^4\) They show that the interest rate in their model has a non-central \(\chi^2\) distribution. The conditional mean and variance of any variable with this distribution is linear in its current value. The autoregressive gamma model of Gourieroux and Jasiak (2001), the Wishart model of Gourieroux and Sufana (2003) and the quadratic term-structure model of Ahn, Dittmar and Gallant (2002) share this property. These models are all ‘exponential-affine’ in the sense of Duffie et al (2003).

\(^5\) The solution is an optimal policy rule which updates in line with the time-varying parameters, as Chow
allow policymakers a further degree of control over the system. However, this sort of model is not linear quadratic, thus not immediately amenable to the solution techniques proposed in this paper.

G  Further References


— (1975) shows. We would conjecture that the sort of random walk drift in the parameters of Bayesian VAR models would take a similar form.


