Global Existence for a Nonlinear System in Thermoviscoelasticity with Nonconvex Energy

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A three-dimensional thermoviscoelastic system derived from the balance laws of momentum and energy is considered. To describe structural phase transitions in solids, the stored energy function is not assumed to be convex as a function of the deformation gradient. A novel feature for multi-dimensional, nonconvex, and non-isothermal problems is that no regularizing higher order terms are introduced. The mechanical dissipation is not linearized. We prove existence global in time. The approach is based on a fixed-point argument using an implicit time discretization and the theory of renormalized solutions for parabolic equations with L^1 data.

1. INTRODUCTION

This article is concerned with global solvability of an initial-boundary value problem in three-dimensional thermoviscoelasticity arising from the theory of solid-solid phase transitions. We consider a three-dimensional body, identified with its reference configuration Ω in $\mathbb{R}^3$. Here, Ω is assumed to be a bounded, nonempty domain with Lipschitz boundary. Let $T > 0$ be an arbitrary, but fixed time. The thermomechanical evolution of the body will be described in terms of the deformation field $u\colon \Omega \times [0,T[\to \mathbb{R}^n$ and the absolute temperature field $\theta\colon \Omega \times [0,T[\to \mathbb{R}$. The evolution of the body also depends on the stored energy function $\Phi(F,\theta)\colon \mathrm{Mat}\,(n \times n) \times \mathbb{R} \to \mathbb{R}$ which enters the equation through the stress tensor $\sigma(F,\theta) = \frac{\partial \Phi(F,\theta)}{\partial F}$.

The balance laws of momentum and energy ultimately lead to the nonlinear coupled system

$$u_{tt} = \mathrm{Div}\,(\sigma(\nabla u, \theta) + \nabla u_t) \quad \text{in } \Omega \times]0,T[, \tag{1a}$$

$$\theta_t = \Delta\theta + \theta\sigma_\theta(\nabla u, \theta) : \nabla u_t + \nabla u_t : \nabla u_t \quad \text{in } \Omega \times]0,T[. \tag{1b}$$

Initial and boundary conditions are specified in Section 2. Remarks about the derivation of this system follow at the end of this section.

To model phase transitions, we do *not* assume σ to be monotone in F for temperatures below the critical temperature (i.e., the temperature at which the phase transition occurs). Hence, the stored energy Φ will be nonconvex as a function of F below the critical temperature.

Some remarks about the peculiarities of this model and related systems are in order. In this vein, we will focus on systems with nonconvex energies. Equations of thermoviscoelasticity with convex energy have been considered in [2], where renormalized solutions are used to show existence of a solution. We rely heavily on this machinery and comment later on differences between convex and nonconvex energies. The novelty of the existence result presented in this paper (Theorem 3.1) is that, to our knowledge, all previous results for similar systems with nonconvex energies either

(i) study the one-dimensional case, i.e., $\Omega = [0, 1]$,

or

(ii) in the multi-dimensional case, concentrate on the isothermal problem or include capillarity-like higher-order terms that have a regularizing effect.

Even given these assumptions, it is nonetheless remarkably difficult to prove global existence. Let us highlight a few salient results.

– The one-dimensional case has been a focus of study for a long time. See, e.g., [6, 7, 17, 5]. Recent advances have been made in [22], where rather weak solutions are considered. In the noteworthy paper [26], Watson studies solids as well as gaseous materials. It is important to note that all these results do not hold for the multi-dimensional case: though very different techniques are used throughout these papers, the crucial step is always a bound on u_x in $L^\infty(\Omega; \mathbb{R})$. In higher space dimensions, even in the isothermal case, this estimate is *wrong* unless one imposes rotational invariance of Ω or physically unrealistic smoothness assumptions on the given data [24].

– The three-dimensional isothermal case has been studied by Rybka [23]. Among others, he proves existence and uniqueness of a solution in the multi-dimensional isothermal case for Lipschitz continuous σ. Later, this result was generalized by Friesecke and Dolzmann [14] to allow a more general kind of nonlinearity in σ. Our proof uses the ideas developed in [14]. The main difficulty of generalizing their results to the non-isothermal case is the mechanical dissipation $\nabla u_t : \nabla u_t$ in the heat equation.

– The three-dimensional, non-isothermal case including capillarity-like higher-order terms was an open problem for some years until recently when Pawłow and Żochowski [19] proved global existence. The proof relies on the observation that the higher-order term allows a parabolic decomposition of the equation of motion. Preliminary results were, for example, obtained in [15].

System (1a)–(1b) is derived from the the balance laws of momentum and energy,

$$
\begin{aligned}
u_{tt} &= \operatorname{Div}\left(\sigma(\nabla u, \theta) + \nabla u_t\right) \quad \text{in } \Omega\times]0,T[, && \text{(2a)}\\
-\theta\Phi_{\theta\theta}\left(\nabla u, \theta\right)\theta_t &= \Delta\theta + \theta\sigma_\theta(\nabla u, \theta) : \nabla u_t + \nabla u_t : \nabla u_t && \text{(2b)}\\
&\quad \text{in } \Omega\times]0,T[.
\end{aligned}
$$

In Section 2, the assumptions on Φ are listed. In particular, by (E2), one has

$$-\theta\Phi_{\theta\theta}\left(\nabla u, \theta\right) = 1 - \theta\phi''(\theta)\Phi_1(F).$$

For physical reasons, one expects $\phi(\theta)$ to be linear in θ. In this case, $1 - \theta\phi''(\theta)\Phi_1(F) = 1$, which means that (2a)–(2b) reduces to the system (1a)–(1b) under consideration. Yet, the growth conditions we must impose allow Φ to be linear only in an arbitrarily large, but fixed range of θ, so (1a)–(1b) represent the balance of energy within this restricted range. This explains why we replace the nonlinear term $1 - \theta\Phi_{\theta\theta}\left(\nabla u, \theta\right)$ with 1 and consequently study the system (1a)–(1b) as an approximation of the full system (2a)–(2b).

It should be noted that the viscous part ∇u_t of the stress tensor is not frame-indifferent. This is a typical weakness of multidimensional models of thermoviscoelasticity that are analytically tractable.

We will use a self-explanatory notation for Lebesgue and Sobolev space. E.g., $L^1\left(\Omega; \mathbb{R}^n\right)$ denotes the class of Lebesgue integrable functions defined on Ω with values in $\mathbb{R}^n$. Time-dependent function spaces will be denoted, e.g., $W^{1,2}\left(0, T; W^{1,p}\left(\Omega; \mathbb{R}\right)\right)$.

To prove global existence of a solution to the system (1a)–(1b), we combine methods developed in [14] (to deal with the nonconvexity) with the theory of renormalized solutions in classical thermoviscoelasticity (to deal with the mechanical dissipation; see in particular the paper by Blanchard and Guibé [2]).

2. THE INITIAL-BOUNDARY VALUE PROBLEM

The system (1a)–(1b) has to be furnished with appropriate initial and boundary conditions. We study the initial-boundary value problem

$$u_{tt} = \operatorname{Div}\left(\sigma(\nabla u, \theta) + \nabla u_t\right) \quad \text{in } \Omega\times]0,T[, \tag{3a}$$

$$
\begin{aligned}
\theta_t &= \Delta\theta + \theta\sigma_\theta(\nabla u, \theta) : \nabla u_t + \nabla u_t : \nabla u_t \qquad (3b)\\
&\quad \text{in } \Omega\times]0, T[,\\
u &= g \quad \text{on } \partial\Omega \times [0, T[, \qquad (3c)\\
u &= u_0 \quad \text{in } \Omega \times \{0\}, \qquad (3d)\\
u_t &= v_0 \quad \text{in } \Omega \times \{0\}, \qquad (3e)\\
\theta &= 0 \quad \text{on } \partial\Omega \times [0, T[, \qquad (3f)\\
\theta &= \theta_0 \quad \text{in } \Omega \times \{0\}. \qquad (3g)
\end{aligned}
$$

Here, g, u_0, v_0 and θ_0 are given functions. Their regularity is specified in Section 2.1. The boundary condition $\theta = 0$ on $\partial\Omega$ seems to be inappropriate, since this corresponds to zero (absolute) temperature on the boundary. However, after reformulating the above equations for the incremental temperature field with respect to a fixed temperature $\overline{\Theta}$, rather than for the absolute temperature θ, one can see that the same proof holds for (positive) boundary data $\overline{\Theta}$. We refrain from spelling this out, to keep the notation simple.

On the stored energy $\Phi(F, \theta)$, the following conditions will be imposed:

1. Φ is sufficiently smooth:

$$
\Phi \in C^2(\mathrm{Mat}\,(n \times n) \times \mathbb{R}; \mathbb{R}). \qquad (E1)
$$

2. The stored energy is of the form

$$
\Phi(F, \theta) = \alpha + \theta - \theta \ln(\theta) + \phi(\theta)\Phi_1(F) + \Phi_2(F), \qquad (E2)
$$

where α is constant. The other quantities will be specified in the next paragraphs.

To simplify the notation, let us define

$$
\phi_j(F) := \frac{\partial \Phi_j(F)}{\partial F} \qquad (j = 1, 2). \qquad (4)
$$

3. Growth condition on $\phi\colon \mathbb{R} \to \mathbb{R}$:

$$
|\phi(\theta)|, |\theta\phi'(\theta)|, |\phi'(\theta)| \leq C. \qquad (E3)
$$

4. Growth condition on Φ_1:

Φ_1 has linear growth near infinity. That is to say, ϕ_1 is bounded.(E4)

Additionally, we require that

ϕ_1 is a globally Lipschitz continuous function. (E5)

5. Growth condition on Φ_2:

$$\phi_2 = \frac{\partial \Phi_2}{\partial F} \text{ is a globally Lipschitz continuous function,} \tag{E6}$$

and $c, c' > 0$ exist such that

$$c|F|^2 - c' \leq \Phi_2(F) \leq c' \left(|F|^2 + 1\right) \text{ and } |\phi_2(F)| \leq c' \left(|F| + 1\right). \tag{E7}$$

Remark 2. 1. We need to explain that the growth conditions (E1)–(E7) are consistent with experimental observations of martensitic phase transitions. First, it is an inherent property of martensitic phase transitions that the energy is convex for high temperatures (beyond the so-called $\mathrm{M_d}$ temperature, about 300°C for NiTi). And as there are no nonconvexities at high temperature, there are no nonconvexities for large strains. The reason is that only the parent (austenitic) and the martensitic phases are stable, and their strains differ only by a few percent. Phases with large strain are always unstable, which means that they are in the convex region of the energy landscape, away from the minimizers. Therefore, for high temperatures or strains, the problem reduces to one which is very similar to the one studied by Blanchard and Guibé [2]. The challenge is exactly to address the nonconvexity in small strains and below the transition temperature. Indeed, it is important to note that even the modeling implicitly relies on the small strain assumption. It follows that the difficult mathematical question of growth conditions at Infinity seems largely irrelevant from a viewpoint of applications. Specifically, the modeling of phase transitions on the continuum level relies on the existence of Ericksen-Pitteri neighborhoods [12, 20]. This approach is only valid if the different stable phases can be confined to a neighborhood which does not, on the crystalline level, include shifts by one atomic layer (the cut-off function ξ in the example below singles out an analogue neighborhood on the continuum level). The fact that martensitic phase transitions are a small strain phenomenon is illustrated in [1]. Here, it is demonstrated that for specific phase transitions like the fcc-to-bcc transition occurring in iron, the crystalline energy necessarily has to be in L^∞. Finally, we remark that some of the conditions stated above are familiar from related problems: the conditions imposed on the absolute temperature in (E3) for $\phi(\theta)$ can also be found, for example, in [25] (however, the restrictive assumption (E4) is not imposed there).

Remark 2. 2. A priori, it is not even clear that the strong growth conditions to be imposed on the energy to obtain existence results of multi-dimensional thermoviscoelastic models of phase transitions (even with higher order terms [19]) can be met for a frame-indifferent function. However, a general method to derive energy functions meeting all physical require-

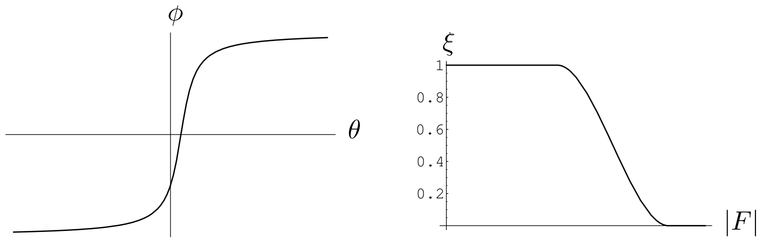

FIG. 1. Left: Prototypical shape of $\phi(\theta)$, satisfying (E3). Right: The cut-off function ξ used in the example in Remark 2.2. It takes the value 1 in a region surrounding possibly stable phases, and decreases to 0 for large strains.

ments (and arbitrary growth conditions) is presented in [27, 28]. For illustrational purposes, we give an example of the tetragonal-orthorhombic (orthoI) symmetry breaking, as it occurs in Zirconia (ZrO_2). As explained in [11], this problem can be studied in two space dimensions. To ensure frame-indifference of the energy (that is, to satisfy $\Phi(QF) = \Phi(F)$ for every $Q \in \mathrm{SO}(3)$), we use the polar decomposition and write the energy in $C := F^T F$. We will write both $\Phi = \Phi(F)$ and $\Phi = \Phi(C)$ if no confusion can arise. It is convenient to introduce the Voigt notation for the components of C,

$$C = \begin{pmatrix} c_1 & \frac{1}{2}c_6 \\ \frac{1}{2}c_6 & c_2 \end{pmatrix}.$$

Using the ideas of [27, 28, 11], it is clear to see that every function in C with tetragonal symmetry can be written as

$$\Phi(c_1, c_2, c_6) := \overline{\Phi}\left(c_1 + c_2, c_1^2 + c_2^2, c_6^2\right).$$

We derive an energy function for a first order phase transition. Let us introduce $\rho_1(c_1, c_2, c_6) := c_1 + c_2$ and $\rho_2(c_1, c_2, c_6) := c_1^2 + c_2^2$. Then, if $\xi = \xi(c_1, c_2, c_6)$ is a smooth cut-off function (see Figure 1 on the right) and $\gamma > 0$ is a constant, a possible energy is given by

$$\begin{aligned} \Phi(c_1, c_2, c_3) \;:=\; & \alpha + \theta - \theta \ln(\theta) \\ & + \sum_{j=1}^{2} \xi \Bigg[\arctan\left(\theta - \theta_c + \frac{1}{4}\right) \cdot \rho_j(c_1, c_2, c_6) \\ & - \frac{1}{2}\rho_j(c_1, c_2, c_6)^2 + \frac{1}{3}(c_1, c_2, c_6)^3 \Bigg] \\ & + \xi \cdot c_6^2 + \gamma(c_1 + c_2). \end{aligned} \tag{5}$$

We first motivate this structure of Φ. The right-hand side of the first line of (5) gives the caloric term. The second and third line of (5) describe the behavior for 'small' strains, including the strain region in which the phase transition occurs. The last line models the growth rate for 'large' strains in c_1, c_2 and the behavior in the off-diagonal c_6, which is independent of the phase transition. The function in square brackets is the usual Landau-Ginzburg energy for first order transitions [13], where the planar symmetry has been factored out and the commonly used term $\theta - \theta_c + \frac{1}{4}$ has been replaced by $\arctan\left(\theta - \theta_c + \frac{1}{4}\right)$. The elimination of the the planar symmetry is a physical necessity; one avoids unwanted minimizers. The growth conditions (E3) on θ are introduced for mathematical reasons. Obviously, Φ is of the type (E2) and satisfies the smoothness assumption (E1). It is plain to see that $\phi(\theta) := \arctan\left(\theta - \theta_c + \frac{1}{4}\right)$ satisfies (E3), and $\Phi_2(F) := \xi c_6^2 + \gamma(c_1 + c_2)$ satisfies for large strains the equality $\Phi_2(F) = \gamma(c_1 + c_2) = \gamma \left|F\right|^2$. Consequently, the growth conditions (E6), (E7) are met. The example shows that these growth conditions are physically reasonable (an experimental determination of the growth condition for large strains is not possible; and in engineering literature, the use of piecewise quadratic functions is common [16]). Finally, the two remaining growth conditions (E4) and (E5) hold since we use the cut-off function ξ in the definition of Φ. Though these growth conditions are introduced for mathematical convenience, the discussion in the preceding remark provides a physical justification. Namely, (E4) and (E5) model the phase transition, which happens only in a bounded set of strains. The cut-off function ξ singles out such a set. Since we merely aim to give a prototypical example for an energy function, we refrain from introducing appropriate constants and parameters. In practice, it is compelling to choose γ small enough to make sure Φ_2 does not hide the potential wells. Similarly, one has to select ξ in such a way that its decrease does not destroy the convexity away from the minimizers. Both choices can easily be accomplished. The details are explained in [11], where it is also shown how to fit the location of minimizers, the energy barriers, and the elastic moduli. Since the result is necessarily more technical and does not shed deeper light on our goals, we refrain from repeating it here.

2.1. Initial and Boundary Conditions

The given data should satisfy the following smoothness assumptions:

$$\begin{aligned}
g &\in W^{1,2}\left(\Omega; \mathbb{R}^n\right), \\
u_0 &\in W_g^{1,2}(\Omega; \mathbb{R}^n) := \{u \in W^{1,2}\left(\Omega; \mathbb{R}^n\right) \mid u - g \in W_0^{1,2}\left(\Omega; \mathbb{R}^n\right)\}, \\
v_0 &\in L^2\left(\Omega; \mathbb{R}^n\right), \\
\theta_0 &\in L^1\left(\Omega; \mathbb{R}\right).
\end{aligned}$$

2.2. Resulting System of Equations

Using (E2), one obtains

$$\sigma(F,\theta) = \frac{\partial \Phi(F,\theta)}{\partial F} = \phi(\theta)\phi_1(F) + \phi_2(F)$$

and $\theta\sigma_\theta(\nabla u,\theta) : \nabla u_t = \theta\phi'(\theta)\phi_1(\nabla u) : \nabla u_t$. To simplify the notation, let us write $f(\theta) := \theta\phi'(\theta)$. Then, system (3a)–(3g) becomes

$$\begin{aligned}
u_{tt} &= \operatorname{Div}\left(\phi(\theta)\phi_1(\nabla u) + \phi_2(\nabla u) + \nabla u_t\right) \quad \text{in } \Omega\times]0,T[, && (6a)\\
u &= g \quad \text{on } \partial\Omega\times[0,T[, && (6b)\\
u &= u_0 \quad \text{in } \Omega\times\{0\}, && (6c)\\
u_t &= v_0 \quad \text{in } \Omega\times\{0\}, && (6d)\\
\theta_t &= \Delta\theta + f(\theta)\phi_1(\nabla u) : \nabla u_t + \nabla u_t : \nabla u_t \quad \text{in } \Omega\times]0,T[, && (7a)\\
\theta &= 0 \quad \text{on } \partial\Omega\times[0,T[, && (7b)\\
\theta &= \theta_0 \quad \text{in } \Omega\times\{0\}. && (7c)
\end{aligned}$$

3. EXISTENCE OF A WEAK-RENORMALIZED SOLUTION GLOBAL IN TIME

In order to prove existence of a weak solution, we have to overcome two main difficulties: the nonconvexity of the energy density and the mechanical dissipation, i.e., the term $\nabla u_t : \nabla u_t$ in the heat equation.

The latter will lead to a parabolic equation with initial data in $L^1(\Omega;\mathbb{R})$ and a right-hand side in $L^1\left(0,T;L^1(\Omega;\mathbb{R})\right)$. We will use the concept of renormalized solutions, introduced by Lions and DiPerna in their investigation of the Boltzmann equation [10, 9]. Further references for renormalized solutions for parabolic equations in L^1 are, e.g., [18, 4, 3]. Other frameworks are, for example, SOLA [8], and entropy solutions [21].

In this paper, we combine ideas of [14] and [2]. In [2], renormalized solutions are applied to a thermoviscoelastic system with a convex stored energy. Apart from the free energy, the coupling of the equations studied here differs from [2]: for phase transitions, the essential parameter is the temperature θ, not its gradient $\nabla\theta$.

3.1. Quick Review of Renormalized Solutions

For the reader's convenience, the basic properties of renormalized solutions are briefly summarized. With our application in mind, we concentrate on parabolic equations.

This section deals with equations of the type

$$\theta_t - \Delta\theta = H \text{ in } \Omega\times]0,T[, \tag{8a}$$
$$\theta = 0 \text{ on } \partial\Omega\times]0,T[, \tag{8b}$$
$$\theta = \theta_0 \text{ in } \Omega\times\{0\}, \tag{8c}$$

with $H \in L^1\left(0,T;L^1(\Omega;\mathbb{R})\right)$ and $\theta_0 \in L^1(\Omega;\mathbb{R})$.

$T_K(r) := \max(\min(r,K),-K)$ denotes the truncation function at height $K \geq 0$.

Definition 3.1. A measurable function $\theta\colon \Omega\times]0,T[\to \mathbb{R}$ is a *renormalized solution* of problem (8a)–(8c) if it satisfies the following properties:

(i) $\theta \in L^\infty\left(0,T;L^1(\Omega;\mathbb{R})\right)$,

(ii) $T_K(\theta) \in L^2\left(0,T;W_0^{1,2}(\Omega;\mathbb{R})\right)$ for every $K \geq 0$,

(iii) $\lim_{n\to+\infty} \iint_{\{(x,t)\ \mid\ n\leq|\theta(x,t)|\leq n+1\}} |\nabla\theta|^2 \,\mathrm{d}y\,\mathrm{d}s = 0$,

and, for every $S \in C^\infty(\mathbb{R};\mathbb{R})$ with $S' \in C_0^\infty(\mathbb{R};\mathbb{R})$,

(iv) $S(\theta)_t - \operatorname{div}\left[S'(\theta)\nabla\theta\right] + S''(\theta)\left|\nabla\theta\right|^2 = HS'(\theta)$ in $\mathcal{D}'(\Omega\times]0,T[)$,

(v) $S(\theta) = S(\theta_0)$ in $\Omega\times\{0\}$.

Remark 3. 1. In our application, the right hand side H of the heat equation will depend on an absolute temperature $\hat\theta$ and u:

$$\theta_t - \Delta\theta = H(\hat\theta,u) \text{ in } \Omega\times]0,T[,$$

where $H(\hat\theta,u) := f(\hat\theta)\phi_1(\nabla u) : \nabla u_t + \nabla u_t : \nabla u_t$. To obtain the regularity $H \in L^1\left(0,T;L^1(\Omega;\mathbb{R})\right)$, aiming for $u \in W^{1,2}\left(0,T;W^{1,2}(\Omega;\mathbb{R}^n)\right)$ as regularity of the deformation looks reasonable. This is a physically reasonable solution space: Since solids undergoing a phase transformation are likely to form microstructures, the regularity of the deformation will be low. It is not necessarily the case that the second spatial derivatives are to be in $L^p(\Omega;\mathbb{R})$. Accepting this point, we arrive at a heat equation in $L^1(\Omega;\mathbb{R})$. Since the heat capacity is constant, temperature is proportional to energy density, which is naturally measured in an $L^1(\Omega;\mathbb{R})$ norm. Therefore, the treatment of the heat equation in $L^1(\Omega;\mathbb{R})$ seems to be physically reasonable.

3.2. Statement of the Theorems

Now we are in a position to define a weak-renormalized solution of (6a)–(7c).

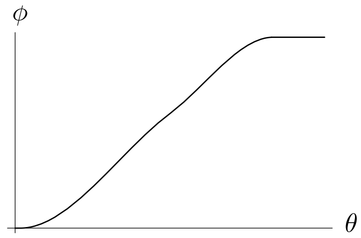

FIG. 2. Prototypical shape of $\phi(\theta)$, satisfying $\phi'(r_0) = 0$ for some point $r_0 > 0$ in addition to (E3). Here, ϕ is chosen to be constant close to the absolute temperature $\theta = 0$, which is plotted at the origin.

Definition 3.2. A pair (u, θ) with $u\colon \Omega\times]0, T[\to \mathbb{R}^n$ and $\theta\colon \Omega\times]0, T[\to \mathbb{R}$ is said to be a *weak renormalized solution of the system (6a)–(7c)* if it satisfies the following conditions:

(i) Regularity of u:

$$\begin{aligned} u \ &\in \ L^\infty\left(W_g^{1,2}(\Omega;\mathbb{R}^n)\right) \cap W^{1,\infty}\left(0, T; L^2(\Omega;\mathbb{R}^n)\right) \\ &\cap \ W^{1,2}\left(0, T; W^{1,2}(\Omega;\mathbb{R}^n)\right) \cap W^{2,2}\left(0, T; W^{-1,2}(\Omega;\mathbb{R}^n)\right), \end{aligned}$$

(ii) u is a weak solution of (6a)–(6d): for every $\zeta \in C_0^\infty(\Omega\times]0, T[;\mathbb{R}^n)$,

$$\int_0^T \int_\Omega [(\sigma(\nabla u, \theta) + \nabla u_t) : \nabla\zeta - u_t \cdot \zeta_t] \,\mathrm{d}x \,\mathrm{d}t = 0,$$

$$\begin{aligned} u(\cdot, 0) &= u_0(\cdot) \quad \text{in } \Omega \times \{0\}, \\ u_t(\cdot, 0) &= v_0(\cdot) \quad \text{in } \Omega \times \{0\}, \end{aligned}$$

(iii) θ is a renormalized solution of (7a)–(7c).

The main theorem of this paper can be formulated as follows.

Theorem 3.1. *Let Ω be a nonempty, bounded domain in $\mathbb{R}^n$ ($n = 2$ or $n = 3$) with a Lipschitz boundary. Assume the initial and boundary conditions satisfy the conditions stated in Section 2.1 and Φ satisfies hypotheses (E1)–(E7). Then a weak renormalized solution of system (6a)–(7c) exists.*

One also has the following result concerning the positivity of the absolute temperature.

THEOREM 3.2. *Suppose in addition to the assumptions of Theorem 3.1 that* $f(r_0) = 0$ *for some* $r_0 \geq 0$ *and* $\theta_0 \geq r_0$ *almost everywhere in* $\Omega\times]0,T[$. *Then the absolute temperature* θ *of the weak renormalized solution of system (6a)–(7c) satisfies* $\theta \geq r_0$ *almost everywhere in* $\Omega\times]0,T[$.

We note that Theorem 3.2 specifically gives for the prototypical energy of Remark 2.2 the estimate $\theta \geq 0$ almost everywhere. To obtain $\theta > 0$, one has to modify the toy model for the energy to fulfill $\phi'(r_0) = 0$ for some point $r_0 > 0$. This can easily be achieved, an example of such a function is plotted in Figure 3.2.

4. PROOF OF THEOREMS 3.1 AND 3.2

The main part of this section will be devoted to the proof of Theorem 3.1. The proof of Theorem 3.2 follows at the end of the section; and it is a straightforward application of ideas of Blanchard and Guibé [2].

We will use the Schauder Fixed Point Theorem to prove Theorem 3.1. Let $\hat\theta = \hat\theta(x,t)$ be an arbitrary element in $L^1\left(0,T;L^1(\Omega;\mathbb{R})\right)$. First, we consider the problem

$$\begin{aligned} u_{tt} &= \operatorname{Div}\Big(\phi(\hat\theta)\phi_1(\nabla u) + \phi_2(\nabla u) + \nabla u_t\Big) \quad \text{in } \Omega\times]0,T[, && \text{(9a)}\\ u &= g \quad \text{on } \partial\Omega\times[0,T[, && \text{(9b)}\\ u &= u_0 \quad \text{in } \Omega\times\{0\}, && \text{(9c)}\\ u_t &= v_0 \quad \text{in } \Omega\times\{0\}. && \text{(9d)} \end{aligned}$$

We will show that this system has a unique solution $\hat u$. Substitution of $\hat\theta$ and $\hat u$ in the nonlinear term on the right-hand side of Equation (7a) will give a solution $\hat\theta$ of (7a)–(7c). We will investigate continuity and compactness of the map $\Psi\colon \hat\theta \to \theta$.

THEOREM 4.1. *The system (9a)–(9d) admits a unique weak solution* $\hat u = \hat u(x,t)$ *with the regularity* $\hat u \in L^\infty\left(W^{1,2}_g(\Omega;\mathbb{R}^n)\right) \cap W^{1,\infty}\left(0,T;L^2(\Omega;\mathbb{R}^n)\right)$ $\cap\, W^{1,2}\left(0,T;W^{1,2}(\Omega;\mathbb{R}^n)\right) \cap W^{2,2}\left(0,T;W^{-1,2}(\Omega;\mathbb{R}^n)\right)$.

Proof. Since the temperature is fixed, this is a purely viscoelastic problem. Existence and uniqueness of a solution follows on from the work of Friesecke and Dolzmann [14, Theorem 4.1]: a semi-implicit discretization in time leads to a variational problem. The integrand is convex, due to the discretized viscosity. Since the functional can be shown to be coercive (here, we need the assumption that $\partial\Omega$ is Lipschitz), a solution exists. Using different approximations in time, one can easily obtain most of the necessary weak convergences (to obtain strong convergence of the velocity, we use again that $\partial\Omega$ is Lipschitz to apply an Aubin-type argument). The crucial step is to show strong convergence of ∇u; this is [14, Proposition 3.1]. $\square$

A multiplication of equation (9a) by $\hat{u}_t$ and integration over space and time yields

$$\frac{1}{2}\int_\Omega |\hat{u}_t(t)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\nabla \hat{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \Phi_2(\nabla \hat{u})(t)\,\mathrm{d}x$$
$$= -\int_0^t \int_\Omega \phi(\hat{\theta})\phi_1(\nabla\hat{u}) : \nabla\hat{u}_t \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\hat{u}_t(0)|^2 \,\mathrm{d}x + \int_\Omega \Phi_2(\nabla\hat{u})(0)\,\mathrm{d}x$$

for almost every t in $]0,T[$.

By (E7), there are positive constants c and c' such that

$$c\,|\nabla\hat{u}(t)|^2 - c' < \Phi_2(\nabla\hat{u})(t) \text{ and } \Phi_2(\nabla\hat{u})(0) < c'\left(|\nabla\hat{u}(0)|^2 + 1\right).$$

Hence, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\hat{u}_t(t)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\nabla \hat{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}s + c\int_\Omega |\nabla\hat{u}(t)|^2 \,\mathrm{d}x \\
&\leq\ -\int_0^t \int_\Omega \phi(\hat{\theta})\phi_1(\nabla\hat{u}) : \nabla\hat{u}_t \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} \\
&\quad + c'\|u_0\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} + C
\end{aligned} \tag{10}$$

for almost every t in $]0,T[$.

By (E4), $\phi_1(\nabla\hat{u})$ is bounded. Using this and Young's inequality, the first term on the right-hand side can be estimated as follows:

$$\begin{aligned}
&-\int_0^t \int_\Omega \phi(\hat{\theta})\phi_1(\nabla\hat{u}) : \nabla\hat{u}_t \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad\leq C\int_0^t \int_\Omega \left|\phi(\hat{\theta})\right|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_\Omega |\nabla\hat{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{11}$$

Combining (10) and (11), we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\hat{u}_t(t)|^2 \,\mathrm{d}x + \frac{1}{2}\int_0^t \int_\Omega |\nabla \hat{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}s + c\int_\Omega |\nabla\hat{u}(t)|^2 \,\mathrm{d}x \\
&\leq\ C\int_0^t \int_\Omega \left|\phi(\hat{\theta})\right|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + c'\|u_0\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} + C
\end{aligned}$$

for almost every t in $]0,T[$.

Since the terms on the left hand side are nonnegative, we can take the supremum over t. This yields

$$\begin{aligned}
\frac{1}{2}\,\|\hat{u}_t\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} &+ \frac{1}{2}\,\|\nabla\hat{u}_t\|^2_{L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{n\times n}\right)\right)} \\
&+ c\,\|\nabla\hat{u}\|^2_{L^\infty\left(0,T;L^2\left(\Omega;\mathbb{R}^{n\times n}\right)\right)} \\
\leq\ C&\left[\left\|\phi(\hat{\theta})\right\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}))} + \|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|u_0\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} + 1\right] \leq C,
\end{aligned}$$

where C is independent of $\hat{\theta}$ by (E3).
Poincaré's inequality

$$\|\hat{u} - g\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq C\left(\|\nabla\hat{u}\|^2_{L^2\left(\Omega;\mathbb{R}^{n\times n}\right)} + 1\right)$$

finally gives the bound

$$\begin{aligned}
&\frac{1}{2}\,\|\hat{u}_t\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} \\
&\qquad\qquad + \frac{1}{2}\,\|\nabla\hat{u}_t\|^2_{L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{n\times n}\right)\right)} + \|\hat{u}\|^2_{L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^n))} \leq C.
\end{aligned}$$

Next, we prove continuous dependence of $\hat{u}$ on $\hat{\theta}$. Let $\hat{\theta}_1, \hat{\theta}_2$ be two absolute temperatures. Denote the corresponding solutions of (9a)–(9d) by $\hat{u}_1$ and $\hat{u}_2$. A multiplication of the differences of the two equations by $\hat{u}_1 - \hat{u}_2$ and integration over space and time yields

$$\begin{aligned}
&\partial_t \frac{1}{2}\left(\int_\Omega |\hat{u}_1(t) - \hat{u}_2(t)|^2 \,\mathrm{d}x + \int_0^t\int_\Omega |\nabla\hat{u}_1 - \nabla\hat{u}_2|^2 \,\mathrm{d}x\,\mathrm{d}s\right) \qquad (12)\\
&= -\int_0^t\int_\Omega \left[\left(\phi(\hat{\theta}_1)\phi_1(\nabla\hat{u}_1) - \phi(\hat{\theta}_2)\phi_1(\nabla\hat{u}_2)\right) : (\nabla\hat{u}_1 - \nabla\hat{u}_2)\right] \mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t\left[\int_\Omega (\phi_2(\nabla\hat{u}_1) - \phi_2(\nabla\hat{u}_2)) : (\nabla\hat{u}_1 - \nabla\hat{u}_2)\right] \mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega |\partial_t\hat{u}_1 - \partial_t\hat{u}_2|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leq \Bigg|\int_0^t\int_\Omega \left(\phi(\hat{\theta}_1) - \phi(\hat{\theta}_2)\right)\phi_1(\nabla\hat{u}_1) : (\nabla\hat{u}_1 - \nabla\hat{u}_2) \\
&\quad + \phi(\hat{\theta}_2)\left(\phi_1(\nabla\hat{u}_1) - \phi_1(\nabla\hat{u}_2)\right) : (\nabla\hat{u}_1 - \nabla\hat{u}_2)\,\mathrm{d}x\,\mathrm{d}s\Bigg| \\
&\quad + \mathrm{Lip}\,(\phi_2)\int_0^t\int_\Omega |\nabla\hat{u}_1 - \nabla\hat{u}_2|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega |\partial_t\hat{u}_1 - \partial_t\hat{u}_2|^2 \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le\ C\int_0^t\int_\Omega\left|\phi(\hat\theta_1)-\phi(\hat\theta_2)\right|^2\,\mathrm{d}x\,\mathrm{d}s+\frac12\int_0^t\int_\Omega|\nabla\hat u_1-\nabla\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+C\,\mathrm{Lip}\,(\phi_1)\int_0^t\int_\Omega|\nabla\hat u_1-\nabla\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\mathrm{Lip}\,(\phi_2)\int_0^t\int_\Omega|\nabla\hat u_1-\nabla\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_\Omega|\partial_t\hat u_1-\partial_t\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le\ C\int_0^t\int_\Omega\left|\phi(\hat\theta_1)-\phi(\hat\theta_2)\right|^2\,\mathrm{d}x\,\mathrm{d}s+C\int_0^t\int_\Omega|\nabla\hat u_1-\nabla\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t\int_\Omega|\partial_t\hat u_1-\partial_t\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

for almost every t in $]0,T[$ (the first inequality uses (E6), the second one Young's inequality, (E3), (E4) and (E5)). Similarly, using $\partial_t\hat u_1-\partial_t\hat u_2$ as test function and invoking (E3), (E4), (E5), (E6) and Young's inequality, one obtains

$$
\begin{aligned}
&\partial_t\frac12\int_0^t\int_\Omega|\partial_t\hat u_1-\partial_t\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s \qquad (13)\\
&\le\ \partial_t\frac12\int_0^t\int_\Omega|\partial_t\hat u_1-\partial_t\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s+\frac12\int_0^t\int_\Omega|\nabla\partial_t\hat u_1-\nabla\partial_t\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s\\
&=\ -\int_0^t\int_\Omega\Big(\phi(\hat\theta_1)\phi_1(\nabla\hat u_1)-\phi(\hat\theta_2)\phi_1(\nabla\hat u_2)\Big):(\nabla\partial_t\hat u_1-\nabla\partial_t\hat u_2)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\int_0^t\int_\Omega(\phi_2(\nabla\hat u_1)-\phi_2(\nabla\hat u_2)):(\nabla\partial_t\hat u_1-\nabla\partial_t\hat u_2)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\frac12\int_0^t\int_\Omega|\nabla\partial_t\hat u_1-\nabla\partial_t\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le\ C\int_0^t\int_\Omega\left|\phi(\hat\theta_1)-\phi(\hat\theta_2)\right|^2\,\mathrm{d}x\,\mathrm{d}s+C\int_0^t\int_\Omega|\nabla\hat u_1-\nabla\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

for almost every t in $]0,T[$.

Let us add (12) and (13). The second inequality uses $|\phi|\le C$, the third one $|\phi'|\le C$.

$$
\begin{aligned}
&\partial_t\frac12\Big(\int_\Omega|\hat u_1(t)-\hat u_2(t)|^2\,\mathrm{d}x \qquad (14)\\
&\qquad+\int_0^t\int_\Omega|\nabla\hat u_1-\nabla\hat u_1|^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_\Omega|\partial_t\hat u_1-\partial_t\hat u_2|^2\,\mathrm{d}x\,\mathrm{d}s\Big)\\
&\le\ C\int_0^t\int_\Omega\left|\phi(\hat\theta_1)-\phi(\hat\theta_2)\right|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C \int_0^t \int_\Omega |\nabla \hat{u}_1 - \nabla \hat{u}_2|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega |\partial_t \hat{u}_1 - \partial_t \hat{u}_2|^2 \, \mathrm{d}x \, \mathrm{d}s \\
\leq\ & C \int_0^t \int_\Omega \left| \phi(\hat{\theta}_1) - \phi(\hat{\theta}_2) \right| \mathrm{d}x \, \mathrm{d}s \\
&\quad + C \int_0^t \int_\Omega |\nabla \hat{u}_1 - \nabla \hat{u}_2|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega |\partial_t \hat{u}_1 - \partial_t \hat{u}_2|^2 \, \mathrm{d}x \, \mathrm{d}s \\
\leq\ & C \int_0^t \int_\Omega \left| \hat{\theta}_1 - \hat{\theta}_2 \right| \mathrm{d}x \, \mathrm{d}s \\
&\quad + C \int_0^t \int_\Omega |\nabla \hat{u}_1 - \nabla \hat{u}_2|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega |\partial_t \hat{u}_1 - \partial_t \hat{u}_2|^2 \, \mathrm{d}x \, \mathrm{d}s
\end{aligned}
$$

for almost every t in $]0,T[$.

Hence, by Gronwall's inequality, $\hat{u}$ depends continuously on $\hat{\theta}$. In particular, the map

$$\hat{\theta} \mapsto \nabla \hat{u}$$

is continuous from $L^1\left(0,T;L^1\left(\Omega;\mathbb{R}\right)\right)$ to $L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{n\times n}\right)\right)$; the second line of (13) shows that

$$\hat{\theta} \mapsto \nabla \hat{u}_t$$

is continuous from $L^1\left(0,T;L^1\left(\Omega;\mathbb{R}\right)\right)$ to $L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{n}\right)\right)$.

Using the estimates derived so far and the boundedness of $f(\hat{\theta}) = \hat{\theta}\phi'(\hat{\theta})$ required in (E3), one easily obtains a bound in L^1 on the nonlinear term on the right-hand side of the heat equation:

$$
\begin{aligned}
&\left\| f(\hat{\theta})\phi_1(\nabla \hat{u}) : \nabla \hat{u}_t + \nabla \hat{u}_t : \nabla \hat{u}_t \right\|_{L^1(0,T;L^1(\Omega;\mathbb{R}))} \\
&\qquad \leq\ C \left[\|u_0\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} + \|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + 1 \right] \leq C,
\end{aligned}
\tag{15}
$$

C being a constant independent of $\left\|\hat{\theta}\right\|_{L^1(0,T;L^1(\Omega;\mathbb{R}))}$.

Substituting $\hat{u}$ and $\hat{\theta}$ in the right-hand side of the heat equation (7a), we get a unique solution θ:

$$
\begin{aligned}
\theta_t &= \Delta\theta + f(\hat{\theta})\phi_1(\nabla \hat{u}) : \nabla \hat{u}_t + \nabla \hat{u}_t : \nabla \hat{u}_t \quad \text{in } \Omega\times]0,T[, \\
\theta &= 0 \quad \text{on } \partial\Omega\times]0,T[, \\
\theta &= \theta_0 \quad \text{in } \Omega\times\{0\}
\end{aligned}
\tag{16}
$$

(since the right hand side is in $L^1\left(0,T;L^1\left(\Omega;\mathbb{R}\right)\right)$, it has a unique renormalized solution, see [2, Proposition 1 & 2]).

Recall that the map $\hat{\theta} \to \theta$ is denoted Ψ. The two maps $\hat{\theta} \mapsto \nabla \hat{u}_t$ and $\hat{\theta} \mapsto \nabla \hat{u}$ are continuous from $L^1\left(0,T;L^1\left(\Omega;\mathbb{R}\right)\right)$ to $L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{n\times n}\right)\right)$. Hence, the nonlinear term $f(\hat{\theta})\phi_1(\nabla \hat{u}) : \nabla \hat{u}_t + \nabla \hat{u}_t : \nabla \hat{u}_t$ of the right hand side of the heat equation is a continuous mapping of $L^1\left(0,T;L^1\left(\Omega;\mathbb{R}\right)\right)$ to itself. Hence, by [2, Proposition 1], Ψ is continuous. Furthermore, by (15) and [2, Proposition 3], Ψ is compact for $1 \le p < 1 + \frac{2}{n}$ as a map from $L^1\left(0,T;L^1\left(\Omega;\mathbb{R}\right)\right)$ to $L^p\left(0,T;L^p\left(\Omega;\mathbb{R}\right)\right)$.

The existence of a bounded set $B \subseteq L^1\left(0,T;L^1\left(\Omega;\mathbb{R}\right)\right)$ with $\Psi(B) \subseteq B$ is now straightforward: inequality (15) gives a bound on the right-hand side of the heat equation by

$$C\left[\|u_0\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} + \|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + 1\right].$$

According to the theory of renormalized solutions (see, e.g., [2, Proposition 1], there exists a constant C independent of $\left\|\hat{\theta}\right\|_{L^1(0,T;L^1(\Omega;\mathbb{R}))}$ such that

$$\|\theta\|_{L^1(0,T;L^1(\Omega;\mathbb{R}))} < R := C\left[\|u_0\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} \right. \\ \left. + \|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|\theta_0\|_{L^1(\Omega;\mathbb{R})} + 1\right].$$

One can choose $B := B(0,R)$. An application of Schauder's Fixed Point Theorem finishes the proof. $\square$

We now sketch the proof of Theorem 3.2. It follows the proof of Theorem 4 in [2]. We denote by $\tilde{f}(r)$ the function

$$\tilde{f}(r) := \begin{cases} 0 & \text{if } r \le r_0 \\ f(r) & \text{else} \end{cases}.$$

It is easy to verify that the proof of Theorem 3.1 also applies to the system (6a)–(7c) with $f(\theta)$ replaced by $\tilde{f}(\theta)$. Considering the special test function $S(r) := \min(T_K(r-r_0), 0)$ for $K > 0$, one obtains from Definition 3.1 (iv) (χ_A is the characteristic function of a set A and $\tilde{S}(r) := \int_0^r S(x)\,\mathrm{d}x$ the primitive of S)

$$\int_\Omega \tilde{S}(\theta)(t)\,\mathrm{d}x + \int_0^t \int_\Omega \chi_{]-K+r_0,r_0[}\nabla\theta^2\,\mathrm{d}x\,\mathrm{d}s \\ = \int_0 \int_\Omega \left[\nabla u_t : \nabla u_t + \tilde{f}(\theta)\phi_1(\nabla u) : \nabla u_t\right] S(\theta)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \tilde{S}(\theta_0)\,\mathrm{d}x.$$

By considering the signs of $\tilde{f}$ and S, one obtains

$$\int_\Omega \tilde{S}(\theta)(t)\,\mathrm{d}x \le 0$$

for almost every t in $\Omega\times]0,T[$. This implies $\theta \geq r_0$ almost everywhere in $\Omega\times]0,T[$. The nature of the definition of $\tilde{f}$ finally implies then

$$\tilde{f}(\theta) = f(\theta) \text{ almost everywhere in } \Omega\times]0,T[,$$

and Theorem 3.2 is established. □

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