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# Extreme Nash Equilibrium of Polymatrix Games in Electricity Market

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**Abstract**—Game theoretical approaches are widely used for the analysis of oligopolistic electricity markets. Nash equilibrium is a solution concept of game theoretical approaches. Due to existence of mixed strategy equilibrium and large number of multiple players, finding Nash equilibrium for problems in electricity market is a difficult task. To resolve these difficulties, this paper proposes a simplified approach for finding extreme Nash equilibrium, based on payoff matrix approach and mixed integer linear programming (MILP). To illustrate the proposed approach, a practical case study of Cournot poly-matrix game is considered. Eliminating constraints are appended on the proposed approach to find a global optimal solution. Obtained results show the strength of proposed approach, in terms of simplicity and computational time.

**Index Terms**—Electricity market, poly-matrix game, Nash equilibrium, payoff matrix approach, mixed-integer linear programming.

## I. INTRODUCTION

Electric power sector is being restructured for improving operational efficiency and enhancing consumer benefits. This introduces competition between generation companies and retailers. In a fully competitive market, market clearing price is always close to generator's marginal cost and there is no scope for market power. However, electricity markets are oligopolistic in nature, due to limited number of market participants and other operational constraints. In oligopolistic electricity markets, strategic behavior of dominating generation companies increases their expected profit. Therefore, market price is higher than expected, and is undesirable. To prevent dominating generation companies to abuse market power, forecasting of future market outcomes is essential [1].

Game theory is an appropriate tool to model strategic behavior of dominating participants in the market. Each strategic firm in the game acts as a player, and has a set of strategies. Players select the strategy to be offered in the market, which gives them maximum payoff, depending on other players' strategies. Among the variety of game

theoretical models, Cournot and Supply function equilibrium are the most popular for modeling oligopolistic electricity markets [2]. Solution of any game theoretical model is Nash equilibrium is a stable point of game where none of the player is interested to change its strategy unilaterally. This paper aims to find Nash equilibrium for a game that represents electricity market in an efficient and simplified manner.

Due to transmission constraints, and nonlinear and non-convex payoff, finding Nash equilibrium for games related to electricity markets are difficult. Various approaches proposed to find Nash equilibrium are broadly classified into three categories: mathematical programming approaches [3-5], co-evolutionary approaches [6-7] and payoff matrix approaches [8-10]. In mathematical programming approaches, equilibrium problem is formulated as an optimization problem, and then solved using iterative and optimization techniques. Cournot Nash equilibrium in both bilateral and pool based electricity market is obtained by linear complementarity approach [3]. Individual strategic bidding problem formulated as a mathematical program with equilibrium constraints can be solved by interior point algorithm [4]. Equilibrium problem with equilibrium constraints transform into mixed integer linear programming problem using binary expansion approach [5]. This type of approach can be used for finding multiplayer game Nash equilibrium. However, iterative techniques have convergence difficulties and optimization techniques may provide local optimal solution that is not a true Nash equilibrium. In addition, these types of approaches are suitable only for finding pure strategy equilibrium.

The co-evolutionary programming approaches employ artificial intelligence based optimization techniques for finding Nash equilibrium [6-7]. These approaches effectively deal with transmission constrained non-convex payoff. However, co-evolutionary programming approaches are designed for finding only pure strategy equilibrium and not for mixed strategy equilibrium.

Payoff matrix approaches can find global solution and are suitable for both pure and mixed strategy equilibrium. To reduce the size of payoff matrix, continuous strategies must be

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discretized. Several heuristic based on payoff matrix approach have been used to solve three-player Cournot and Bertrand game [9]. Nash equilibrium condition of payoff matrix is characterized by the polynomial equations that are solved by homotopy continuation algorithm [10]. Approaches proposed in [9-10] are complicated, intuitional and do not provide optimal Nash equilibrium.

This paper presents a solution method to find extreme Nash equilibrium of poly-matrix games using payoff matrix approach and mixed integer linear programming (MILP). Nash equilibrium is extreme because it is expressed as a solution of optimization problem. Linearization technique is employed to transform the equilibrium problem into MILP problem. Two numerical examples, one involving bi-matrix game and other poly-matrix game, are considered to test the proposed approach. Results obtained from the proposed method are verified with approaches proposed in [9] and GAMBIT [15].

Rest of the paper is organized as follows: In Section II, the basic poly-matrix game formulation and Nash equilibrium are described. Section III provides MILP formulation of payoff matrix approach and the simulation procedure. Section IV includes numerical and graphical results of testing the proposed approach through a comprehensive analysis on bi-matrix and poly-matrix game. In Section V, relevant conclusions are drawn.

## II. PROBLEM FORMULATION

### A. Polymatrix Cournot Game

A poly-matrix game is a normal form multi-player non-cooperative game. If players compete in game based on their production quantity, games become poly-matrix Cournot game. In this type of game, each player (Firm) selects its production level assuming known and constant behavior of rival players. Let  $N = \{1, 2, \dots, n\}$  be a set of players that participate strategically in electricity market. The cost function of each firm is expressed as:

$$C_i(P_{gi}) = \frac{1}{2} a_i P_{gi}^2 + b_i P_{gi} + c_i \quad (a_i > 0; i \in N) \quad (1)$$

where,  $P_{gi}$  is the output power of  $i^{th}$  power producer in MW, and  $a_i, b_i, c_i$  are the cost coefficients. Players select their offered power between minimum  $\underline{P}_{gi}$  and maximum  $\bar{P}_{gi}$  generation limit in MW defined as:

$$\underline{P}_{gi} \leq P_{gi} \leq \bar{P}_{gi} \quad i \in N \quad (2)$$

The sum of offered power of each firm is assumed equal to the total market demand, neglecting transmission losses such that

$$P_d = \sum_{i=1}^P P_{gi} \quad (3)$$

Market price  $\lambda$  in \$/MWh depends on total demand and is uniform for each node of the system. The relationship

between price and demand is represented by inverse linear demand curve, and is expressed as:

$$\lambda_n = \theta - \rho P_d \quad (4)$$

where,  $\theta$  (\$/MWh) and  $\rho$  (\$/MW<sup>2</sup>h) are demand benefit coefficients. Marginal cost of generators can be defined as:

$$MC_i(P_{gi}) = a_i P_{gi} + b_i \quad (5)$$

Marginal cost of generators is equal to the market clearing price, when no generation capacity constraints and transmission constraints are considered.

### B. Equilibrium Problem Formulation

Let us consider that each player  $i$  has a finite set of pure strategies  $S_i = \{s_i^1, s_i^2, \dots, s_i^{k_i}\}$ , where,  $k_i$  is the total number of strategies. Payoff of player  $i$  for each strategy is calculated using the following expression

$$a_i(s_i^1, s_i^2, \dots, s_i^{k_i}) = \lambda_n P_{gi} - C_i(P_{gi}) \quad (6)$$

Payoff matrix  $A_i$  consists of payoff for player  $i$  for each strategic combination. If player  $i$  chooses his strategy  $s_i^k$  and player  $j$  chooses his strategy  $s_j^l$ , a partial payoff  $a_{ij}(s_i^k, s_j^l)$  is assigned for player  $i$ . Therefore, for any pure strategic choice  $(s_1^k, \dots, s_n^l)$  of the  $n$  players, overall payoff of player  $i$  at the end of game is

$$A_i(s_1^k, \dots, s_n^l) = \sum_{j \neq i} a_{ij}(s_i^k, s_j^l) \quad (7)$$

For a mixed strategy game, let  $X_i$  be a probability vector over the set of pure strategy  $S_i$  having following condition:

$$\tilde{S}_i = \{X_i : e^T X_i = 1, X_i \geq 0\} \quad (8)$$

At the end of poly-matrix game, overall payoff of player  $i$  is

$$R_i(X) = (X_i)^T \sum_{j \neq i} A_{ij} X_j \quad (9)$$

In a poly-matrix game,  $n$ -tuple  $X^* = (X_1^*, \dots, X_n^*)$  of the mixed strategy is called Nash equilibrium if and only if, for any other  $n$ -tuple  $X^* = (X_1^*, \dots, X_{i-1}^*, X_i, X_{i+1}^*, \dots, X_n^*)$ , satisfy following condition:

$$(X_i^*)^T \sum_{j \neq i} A_{ij} X_j^* \geq (X_i)^T \sum_{j \neq i} A_{ij} X_j^* \quad \forall i \in N \quad (10)$$

The proof of pure and mixed strategy Nash equilibrium existence in finite multi-player games can be found in [11-12].

### III. PROPOSED SOLUTION APPROACH

#### A. MILP Formulation

The problem of poly-matrix Nash equilibrium is expressed in (10). This is transformed to MILP problem using linearization techniques [13-14]. The payoff maximization objective of player  $i$  shown in (8) and (9) can be transformed into linear minimization objective using duality theory.

$$\min_{\gamma_i} \left\{ \gamma_i : e^i \gamma_i \geq A_{ij} X_j \right\} \quad (11)$$

where,  $\gamma_i$  is a dual objective variable of player  $i$ , and equal to its primal payoff  $\gamma_i = (X_i)^T \sum_{j \neq i} A_{ij} X_j$ . The final optimization problem to find Nash equilibrium can be formulated as follows:

$$\min_{\gamma_i, X_i} \sum_{i \in N} \gamma_i \quad (12)$$

Subject to

$$\sum_{i \in N} e_i^T X_i = 1 \quad (13)$$

$$\gamma_i e_i - \sum_{j \in N: j \neq i} A_{ij} X_j - M_i U_i \leq 0 \quad (14)$$

$$X_i + U_i \leq e_i \quad (15)$$

$$U_i \in \{0, 1\} \quad (16)$$

$$X_i \geq 0 \quad (17)$$

$$X_j \geq 0 \quad (18)$$

$$\gamma_i \in \mathbb{R} \quad (19)$$

Objective function (12) is equal to the sum of players expected payoff. Constraint (13) ensures that sum of probability of mixed strategy is always equal to one. Equation (14) constitutes the complementarity slackness conditions of objective. In this condition,  $M_i$  is constant, and should be large enough, as obtained by the following expression:

$$M_i = \left( \max_{i, j \in I: i \neq j} a_{ij} \right) - \left( \min_{i, j \in I: i \neq j} a_{ij} \right) \quad (20)$$

Inequality constraint (16) shows that the sum of pure mixed strategy variable and binary variable is always less than unity. The solution of formulated MILP problem in (12)-(19) may be local optima. To find global optimal solution, eliminate constraints added on formulated MILP problem.

$$\sum_{i=1}^N 1 + U_i \{1\} - U_i \{0\} \leq L \quad \forall i \in N \quad (21)$$

$$L = \frac{\sum_{i=1}^N M_i}{N+1} \quad (22)$$

This constraint eliminates the binary variable combinations found in initial equilibrium solution.

#### B. Simulation Procedure

This section describes the simulation procedure used to obtain extreme Nash equilibrium by the proposed approach.

- Step 1: Model initialization: Define the number of generation companies that participate as a strategic player in the electricity market poly-matrix game. Collect their cost parameters and respective demand benefit coefficients.
- Step 2: Define strategy set: Continuous pure strategy set of players is defined between their maximum and minimum capacity. Using discretization continuous strategy set transform into discrete strategy set.
- Step 3: Payoff matrix construction: For each strategy combination, market-clearing price and players' profit is calculated using (1)-(6). Then construct a payoff matrix of each player to have their profit for all strategy of opponents.
- Step 4: MILP formulation: When all players' payoff matrix is constructed, then transpose equilibrium problem in MILP using (12)-(21).
- Step 5: Nash equilibrium: Solution of MILP using any commercial MILP solver provides Nash equilibrium.
- Step 6: Add elimination constraints: To check whether obtained Nash equilibrium is local or global optimal, add elimination constraints (21)-(22) on formulated MILP in (12)-(19). Again, run MILP solver to obtain new Nash equilibrium.
- Step 7: Check global optimal: If new Nash equilibrium obtained in Step 6 is equal to old Nash equilibrium obtained in Step 5, this means global optimal equilibrium is achieved, then go to next Step. If new Nash equilibrium differs from old Nash equilibrium, it means solution is local optimal, so go to Step 6.
- Step 8: End

### IV. CASE STUDY

To illustrate the proposed solution approach, two cases are considered. Case I is a bi-matrix Cournot game and Case II is a poly-matrix Cournot game.

#### A. Case I: Bimatrix Cournot Game

In this case, two generators  $G_1$  and  $G_2$  compete in oligopolistic market to meet demand. The strategic behavior of generators is modeled as a strategic player in a bi-matrix Cournot game. Fig. 1 shows system configuration and provides information about generators' location and network constraints.

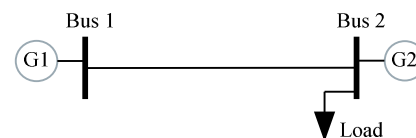


Figure 1. Two bus system.

In this case, generators' minimum and maximum generation capacity is considered as 0 MW and 100 MW respectively. The marginal cost of generators is assumed, as \$10/MWh for generator  $G_1$  and \$20/MWh for generator  $G_2$ .

Players' discrete strategy set are formulated by an increment of 10 MW from generators' minimum to maximum limit. Therefore, each generator has 10 strategies to offer in the market. Demand is equal to the sum of generators offered power.

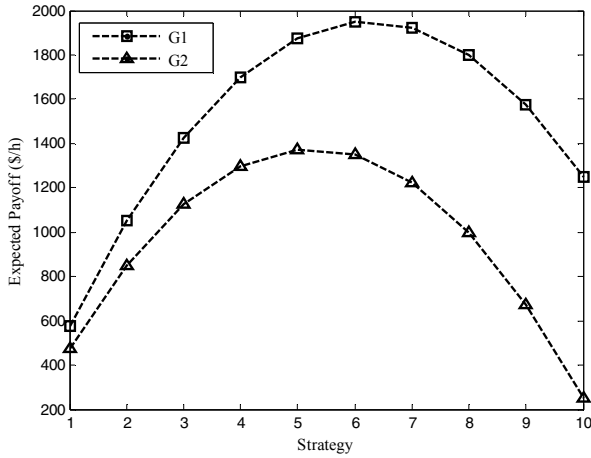


Figure 2. Generators expected payoff.

Demand benefit coefficients  $\theta$  and  $\rho$  are equal to 100 and 0.5 respectively. The size of payoff matrix for this game is  $10 \times 10$ . The player's expected payoff corresponds to their pure strategies, as shown in Fig. 2. Large coefficients  $M_1$  and  $M_2$  values are equal to 4600 and 4800 for player  $G_1$  and  $G_2$  payoff, respectively. Nash equilibrium obtained by the proposed approach results in an offer of 60 MW for  $G_1$  and 50 MW for  $G_2$ , in the absence of network limits.

**B. Case II: Polymatrix Cournot Game**

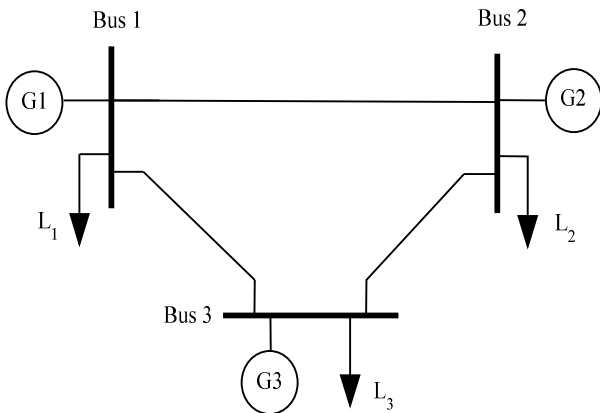


Figure 3. Three bus system.

In this case, three generators  $G_1$ ,  $G_2$  and  $G_3$  strategically interact in oligopolistic electricity market. Their strategic interaction represents a poly-matrix Cournot game. For this case, system configuration is shown in Fig. 3. The transmission lines are assumed to be lossless and have equal reactance. Line flow is obtained using DC power flow solution. Details about generators cost coefficients are provided in Table I.

TABLE I. GENERATORS COST COEFFICIENTS (SOURCE: TABLE 1, [8])

Generator	$G_1$	$G_2$	$G_3$
Cost coefficient $a$ (\$/MW <sup>2</sup> h)	0.015718	0.021052	0.012956
Cost coefficient $b$ (\$/MWh)	1.360575	-2.07807	8.105354
Cost coefficient $c$ (\$/h)	9490.366	11128.95	6821.482

Demand benefit coefficients  $\theta$  and  $\rho$  are 106.1176 and 0.0206, respectively. Minimum and maximum limit of each generator is considered as 600 MW and 1500 MW, respectively. The discrete strategy set of each player is formulated by discretization of 100 MW in the range 600~1500 MW. The size of payoff matrix for this game is  $10 \times 10 \times 10$ . The expected player's payoff corresponding to their pure strategies is shown in Fig. 4. Large coefficients  $M_1$ ,  $M_2$  and  $M_3$  are equal to 59263.356, 61520.718 and 61073.012 for player  $G_1$ ,  $G_2$  and  $G_3$  payoff, respectively. Nash equilibrium obtained by the proposed approach results in  $G_1$  offering 1100 MW,  $G_2$  offering 1000 MW and  $G_3$  offering 1000 MW in the absence of network limits. The obtained results are similar to unconstrained case of [9].

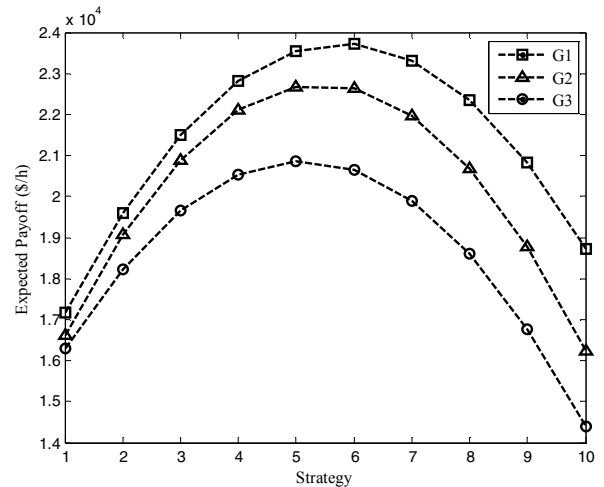


Figure 4. Generators expected payoff.

To minimize computational effort, payoff matrix construction and power flow operation is performed in MATLAB platform. MILP problem have been solved using CPLEX 12.0 under GAMS. All simulations are performed on a Windows based Personal Computer, 1.73 GHz processor and 2.0 GB RAM.



TABLE II. COMPUTATIONAL TIME REQUIREMENTS

Approach	Dimension	Computation Time (Sec.)
Proposed Approach	10×10×10	0.109
GAMBIT [15]	10×10×10	2250

Comparison of computational time taken by the proposed approach with others is shown in Table II. Visualization of this table shows that proposed approach takes only 0.109 seconds without elimination of dominated strategies. Solution of Case I bi-matrix game is obtained in 0.016 seconds.

#### V. CONCLUSION

The competition between strategic power producers in an oligopolistic electricity markets is generally modeled through game theoretical or equilibrium models. Due to presence of multiple strategic power producers in electricity markets finding Nash equilibrium is difficult task. This paper proposes a payoff matrix and MILP based solution approach for finding extreme Nash equilibrium of a poly-matrix game. Proposed approach has been tested on case studies of bi-matrix and poly-matrix game. Simplicity, solvability and less computational time are the advantages of proposed approach. Implementation on larger systems, with size reduction techniques, is a major aspect to be investigated in future.

#### REFERENCES

- [1] X. Guan, Y. Ho and D. L. Pepyne, "Gaming and price spikes in electric power markets," *IEEE Trans. Power Syst.*, vol. 16, no. 3, pp. 402–408, Aug. 2001.
- [2] R. Baldick and W. Hogan, Capacity Constrained Supply Function Equilibrium Models of Electricity Markets: Stability, Non-Decreasing Constraints, and Function Space Iterations, Univ. Calif. Energy Inst., Dec. 2001. [Online]. Available: <http://www.ucei.berkeley.edu/PDF/pwp089.pdf>.
- [3] B. F. Hobbs, "Linear complementarity models of Nash-Cournot competition in bilateral and POOLCO power market," *IEEE Trans. Power Syst.*, vol. 16, no. 2, pp. 194–202, May 2001.
- [4] B. F. Hobbs, C. B. Metzler, and J. S. Pang, "Strategic gaming analysis for electric power systems: An MPEC approach," *IEEE Trans. Power Syst.*, vol. 57, no. 2, pp. 638–645, May 2000.
- [5] M. V. Pereira, S. Granville, M. H. C. Fampa, R. Dix and L. A. Barroso, "Nash equilibrium in strategic bidding: a binary expansion approach," *IEEE Trans. Power Syst.*, vol. 21, no. 2, pp. 629–638, May 2006.
- [6] H. Chen, K. P. Wong, D. H. M. Nguyen, and C. Y. Chung, "Analyzing oligopolistic electricity market using coevolutionary computation," *IEEE Trans. Power Syst.*, vol. 21, no. 1, pp. 143–152, Feb. 2006.
- [7] T. C. Price, "Using co-evolutionary programming to simulate strategic behavior in markets," *J. Evol. Econ.*, vol. 7, pp. 219–254, 1997.
- [8] L. B. Cunningham, R. Baldick, and M. L. Baughman, "An empirical study of applied game theory: Transmission constrained Cournot behavior," *IEEE Trans. Power Syst.*, vol. 17, no. 1, pp. 166–172, Feb. 2002.
- [9] K. H. Lee and R. Baldick, "Solving three-player games by the matrix approach with application to an electric power market," *IEEE Trans. Power Syst.*, vol. 18, no. 4, pp. 1573–1580, Nov. 2003.
- [10] Y. Yang, Y. Zhang, F. Li and H. Chen, "Computing all nash equilibria of multiplayer games in electricity markets by solving polynomial equations," *IEEE Trans. Power Syst.*, vol. 27, no. 1, pp. 81–91, Feb. 2012.
- [11] J. Nash, "Non-cooperative games," *Ann. Math.*, vol. 54, no. 2, pp. 286–295, 1951.
- [12] D. Fudenberg and J. Tirole, *Game Theory*. Cambridge, MA: MIT Press, 1991.
- [13] C. Audit, S. Belhaiza and P. Hansen, "Enumeration of all extreme equilibrium in game theory: bimatrix and polymatrix games," *J. Optimiz. Theory App.*, vol. 129, no. 3, pp. 349–372, Jun. 2006.
- [14] M. R. Hesamzadeh and D. R. Biggar, "Computation of extremal-nash equilibria in a wholesale power market using a single-stage milp," *IEEE Trans. Power Syst.*, vol. 27, no. 1, pp. 1706–1707, Aug. 2012.
- [15] GAMBIT [Online]. Available: <http://www.hss.caltech.edu/gambit>