Phase transition for finite-speed detection among moving particles

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Abstract

Consider the model where particles are initially distributed on $\mathbb{Z}^d$, $d \geq 2$, according to a Poisson point process of intensity $\lambda > 0$, and are moving in continuous time as independent simple symmetric random walks. We study the escape versus detection problem, in which the target, initially placed at the origin of $\mathbb{Z}^d$, $d \geq 2$, and changing its location on the lattice in time according to some rule, is said to be detected if at some finite time its position coincides with the position of a particle. For any given $S > 0$, we consider the case where the target can move with speed at most S, according to any continuous function and can adapt its motion based on the location of the particles. We show that, for any $S > 0$, there exists a sufficiently small $\lambda_* > 0$, so that if the initial density of particles $\lambda < \lambda_*$, then the target can avoid detection forever.

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1. Introduction

Let Π be a Poisson point process of intensity $\lambda > 0$ on $\mathbb{Z}^d$, $d \geq 2$. We label all points of this process by positive integers in some arbitrary way, *i.e.* $\Pi = \{p_j\}_{j\geq 1}$, and interpret the points of Π as *particles*. We denote by $\eta_j(0)$, $j \geq 1$, the initial position of the j^{th} particle, and we will assume that each particle p_i, $i \geq 1$, moves as an independent continuous-time random walk on $\mathbb{Z}^d$. More formally, for each $k \geq 1$, let $(\zeta_k(t))_{t\geq 0}$ be an independent continuous-time random walk on $\mathbb{Z}^d$ starting from the origin. Then $\eta_k(t) := \eta_k(0)+\zeta_k(t)$ denotes the location of the k-th particle at time t.

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In addition, we consider an extra particle, called *target*, which at time 0 is positioned at the origin, and is moving on $\mathbb{Z}^d$, $d \geq 2$ in time, according to a certain prescribed rule. We say that the target is *detected* at time t, if there exists a particle p_j located at time t at the same vertex as the target. We will assume that the target particle wants to evade detection and can do so by moving in continuous time by means of nearest-neighbor jumps on $\mathbb{Z}^d$, which can depend on the past, present and future positions of the particles.

More precisely, let $\mathcal{P}$ be the set of functions $g\colon \mathbb{R}_+ \to \mathbb{Z}^d$ for which $g(0) = 0$ and such that the following holds:

$$\text{for any } g \in \mathcal{P}, \text{ any } t \geq 0 \text{ and any } \xi > 0, \text{ if } \|g(t+\xi) - g(t)\| > 1 \text{ then there exists } \xi' \in (0, \xi) \text{ for which } 0 < \|g(t+\xi') - g(t)\| < \|g(t+\xi) - g(t)\|. \quad (1)$$

We view $\mathcal{P}$ as the set of all permitted trajectories for the target, and $g(t),\ g \in \mathcal{P}$, denotes the position of the target at time t. Condition (1) in the definition of $\mathcal{P}$ prevents the target to make long range jumps, i.e. for any trajectory $g \in \mathcal{P}$, the target is allowed to jump only between nearest neighbor vertices of $\mathbb{Z}^d$.

We say that $g \in \mathcal{P}$ is *detected* at time t if there exists a particle $p_j \in \Pi$, for some $j \geq 1$, such that $\eta_j(t) = g(t)$, and define the detection time of g as follows:

$$T_{det}(g) = \inf\left\{t \geq 0\colon g(t) \in \bigcup\nolimits_{k\geq 1} \eta_k(t)\right\}.$$

In [9, Theorem 1.1] it was shown that there exists a phase transition in λ so that, if λ is large enough, $\mathbf{P}\left(T_{det}(g) < \infty \text{ for all } g \in \mathcal{P}\right) = 1$. Hence, the target cannot avoid detection forever even if it knew the past, present and *future* positions of the particles at all times, and could move at any time at any arbitrarily large speed.

Here we consider a parameter $0 < S < +\infty$ and let $\mathcal{P}_S \subset \mathcal{P}$ be the set of all trajectories $g \in \mathcal{P}$ with maximum *speed* S, *i.e.*,

$$\mathcal{P}_S := \{\ g \in \mathcal{P}\colon \forall t \geq 0\ \forall \xi > 0,\ \|g(t+\xi) - g(t)\| \leq \xi S \vee 1\}.$$

Then define

$$\lambda_{det}(S) = \inf\left\{\lambda \geq 0\colon\ \mathbf{P}\left(T_{det}(g) < \infty \text{ for all } g \in \mathcal{P}_S\right) = 1\right\}$$

and

$$\lambda_{det}(\infty) = \inf\left\{\lambda \geq 0\colon\ \mathbf{P}\left(T_{det}(g) < \infty \text{ for all } g \in \mathcal{P}\right) = 1\right\}.$$

The main result in [9, Theorem 1.1], mentioned above, gives that $\lambda_{det}(\infty) \in (0, \infty)$. Since for any $S \leq S'$ we have $\mathcal{P}_S \subseteq \mathcal{P}_{S'}$, then

$$\lambda_{det}(S) \leq \lambda_{det}(S') \leq \lambda_{det}(\infty) < \infty.$$

It was also observed in [9], that for sufficiently small $\lambda > 0$, there is a strictly positive probability for the target, starting from the origin, to avoid detection forever, provided it can move at any time at any arbitrarily large speed, *i.e.* $\lambda_{det}(\infty) > 0$.

The main contribution of this work is to establish an analogous result for any bounded speed, *i.e.* to show the existence of a non-trivial phase transition for all finite speeds $0 < S < +\infty$. In other words, for any $S > 0$, if the density λ of particles is small enough, with positive probability a target moving with maximum speed S can avoid detection forever.

Theorem 1.1. *For any $S > 0$, we have $\lambda_{det}(S) > 0$.*

Continuous-space model. In this variant, particles are given by a Poisson point process of intensity λ on $\mathbb{R}^d$, and move independently as Brownian motions. Then, we say that the target is detected at time t if there exists a particle within distance 1 from the target at that time. This variant is an extension of the widely studied *Boolean model* (also called *random geometric graph* or *continuum percolation*) to a mobile setting. We remark that, with little change in the proof, Theorem 1.1 can also be shown to hold in this continuous-space version. We discuss how to adapt our proof to this setting in Section 4.

Related work. The problem of detecting a target by moving particles has been studied in other settings. For example, [4, 6] considered the continuous-space variant mentioned above, and studied the case where the target is non-mobile and stays put at the origin (using our notation, this corresponds to $g \equiv 0$). Using arguments from stochastic geometry, they derived the precise distribution of the detection time; in particular, they showed that

$$\mathbf{P}\left(T_{det}(g) > t\right) = \exp\left(-\lambda \operatorname{vol}\left(W_d(t)\right)\right) \qquad \text{when } g \equiv 0, \tag{2}$$

where $W_d(t)$ is the d-dimensional Wiener sausage up to time t. The volume of the Wiener sausage is known to be of order $\sqrt{t}$ in $d = 1$, $\frac{t}{\log t}$ in $d = 2$, and t in $d \geq 3$.

For the case of a mobile target, if the target has to move *independently* of the particles (i.e., g is a deterministic function), in [7] it was shown that, for any given g, a similar expression as in (2) holds with $W_d(t)$ replaced by a Wiener sausage with drift $-g$. Also, [7], and in particular [8], showed that, among all deterministic functions g, the one that maximizes $\mathbf{P}\left(T_{det}(g) > t\right)$ is $g \equiv 0$. In other words, if the target has to move independently of the particles, the best strategy for the target to avoid detection is to stay put. See also the corresponding result for random walks on $\mathbb{Z}^d$ in [2]. For the case where the motion of the target may depend on the positions of the particles, it is shown in [9, Theorem 1.1] via a multi-scale analysis that, for sufficiently large λ, the target cannot avoid detection almost surely even if it knows beforehand the position of all particles at all times. A result of similar flavor was established in [5, Proposition 8] for the study of the rate at which an infection spreads among moving particles. The result in [9, Theorem 1.1] establishes also that, provided λ is large enough, $\mathbf{P}\left(\exists g \in \mathcal{P} \colon T_{det}(g) > t\right)$ decays at least as quickly as $\exp\left(-\frac{C\sqrt{t}}{\log^c t}\right)$ in $d = 1$, $\exp\left(-\frac{Ct}{\log^c t}\right)$ in $d = 2$, $\exp\left(-Ct\right)$ in $d \geq 3$. This bound is tight (up to the constant factor C) and matches up with the case $g \equiv 0$ for $d \geq 3$. Intuitively, this gives that a target that knows the positions of all nodes at all times cannot evade detection much longer than a non-mobile target.

2. Proof of Theorem 1.1

The hardest case is to prove Theorem 1.1 in two dimensions. In higher dimensions, we simply show that the target can avoid detection by moving only in the first two dimensions; i.e., we define the hyperplane

$$\mathbb{H}_d = \mathbb{Z}^2 \times O_{d-2}, \tag{3}$$

where O_{d-2} stands for the origin of $\mathbb{Z}^{d-2}$, and show that the target can avoid detection by only moving within $\mathbb{H}_d$. (In the case $d = 2$, we simply define $\mathbb{H}_2 = \mathbb{Z}^2$.)

For any $i \in \mathbb{H}_d$, consider the time interval

$$T_i = \left[\frac{\|i\|_1}{S}, \frac{\|i\|_1 + 1}{S}\right], \tag{4}$$

and the space-time line segment

$$K_i = i \times T_i.$$

We will show that for λ small enough, there exists a trajectory g for the target that is contained in the space-time region $\bigcup_{i\in\mathbb{H}_d} K_i$ and is never detected. Note that, for such a trajectory g, we have $g \in \mathcal{P}_S$. We say that K_i is *vacant* if there is no particle of Π inside K_i, and E_i will denote the indicator random variable that K_i is vacant, i.e. $E_i := \mathbb{I}_{\{K_i \text{ is vacant}\}}$. A key step in the proof is the proposition below, which establishes that the process induced by $\{E_i\}_{i\in\mathbb{H}_d}$ stochastically dominates an independent Bernoulli percolation process on the square lattice.

Proposition 2.1. *For any $\lambda > 0$ and $S > 0$, there exists $p = p(\lambda, S) > 0$, so that if $\{X_i\}_{i\in\mathbb{H}_d}$ are i.i.d. Bernoulli random variables taking values 0 or 1 with mean p, then $\{E_i\}_{i\in\mathbb{H}_d}$ stochastically dominates $\{X_i\}_{i\in\mathbb{H}_d}$. Moreover, for any $S > 0$, we have*

$$\liminf_{\lambda\downarrow 0} p(\lambda, S) = 1. \tag{5}$$

The proof of Theorem 1.1 is a straightforward application of Proposition 2.1.

Proof of Theorem 1.1. Eq. (5) of Proposition 2.1 implies that, given $S > 0$, there exists $\lambda_c(S) > 0$, such that for $0 < \lambda < \lambda_c(S)$, we have $p(\lambda, S) > p_c$. Here, p_c is the critical probability for *oriented* site percolation on $\mathbb{Z}^2$; i.e., for $p > p_c$ with positive probability there exists an infinite oriented path starting from the origin which is composed entirely of open sites of $\mathbb{Z}^2$ and such that each jump of the path is either to the east or to the north direction (that is, for any two consecutive sites j_1, j_2 on the path we have that j_2 is either $j_1 + (1, 0)$ or $j_1 + (0, 1)$). Hence, with positive probability, there exists an infinite oriented path of adjacent sites of $\mathbb{H}_d$, say $i_0 = 0, i_1, i_2, \ldots$, such that for all $j \geq 0$ we have $\|i_j\|_1 = j$ and K_{i_j} is vacant. Thus, the target can avoid detection if for all $j \geq 0$ it stays at site i_j during the time interval $\left[\frac{\|i_j\|_1}{S}, \frac{\|i_j\|_1+1}{S}\right)$, and jumps to i_{j+1} at time $\frac{\|i_j\|_1+1}{S}$; i.e., the function $g \in \mathcal{P}_S$ given by $g(s) = i_j$ for all $j \geq 0$ and all $s \in \left[\frac{\|i_j\|_1}{S}, \frac{\|i_j\|_1+1}{S}\right)$ is the function for which $T_{det}(g) = \infty$. Thus $\lambda_{det}(S) \geq \lambda_c(S) > 0$, and the proof is completed. □

3. Proof of Proposition 2.1

For any $k \geq 1$, let $J_k := \{x \in \mathbb{H}_d : ||x||_1 = k\}$, and $\mathcal{G}_k$ be the σ-algebra generated by $\{E_i\}_{i \in \mathbb{H}_d \colon \|i\|_1 \leq k}$. The goal of this section is to show that, for $k \geq 1$, the following holds:

$$\text{conditioned on any } G \in \mathcal{G}_{k-1},\ \{E_i\}_{i \in J_k} \text{ stochastically dominates } \{X_i\}_{i \in J_k}. \tag{6}$$

We will analyze the states of sites of J_k inductively on $k = 0, 1, \ldots$ Once (6) is established, Proposition 2.1 follows directly. The proof of (6) will be split in several steps and lemmas. We start with an informal description of the proof, discussing the main ingredients used to establish (6), and then proceed to the rigorous arguments.

The main idea of the proof is the following: by definition, the space-time region $\bigcup_{i \in \mathbb{H}_d} K_i$ grows linearly in time and moves away from the origin at linear speed. In particular, for any time t, the site i, such that $t \in T_i$, has ℓ_1 norm of order t. Since by time t a particle, performing simple symmetric random walk, typically moves a distance of order $\sqrt{t}$, it implies that each individual particle can spend only a limited amount of time inside the region $\bigcup_{i \in \mathbb{H}_d} K_i$. Thus, if the intensity of the Poisson point process is sufficiently small, we will show that the union of all vacant K_i's contains an infinite connected component; i.e., the region of $\bigcup_{i \in \mathbb{H}_d} K_i$ that is not visited by particle "percolates" in space-time.

To make the above argument rigorous, fix $\lambda > 0$, small enough, such that there exists $1 \leq k_0 < +\infty$, so that, with sufficiently large probability, there is no particle in the space-time region $\bigcup_{i \in J_k} K_i$ for all $k \leq k_0$. Let $k = k_0 + 1$, and select all particles that visit the space-time region $\bigcup_{i \in J_k} K_i$. Let u be one such particle. We observe the motion of u from the time it first visits $\bigcup_{i \in J_k} K_i$ onwards. In order to do this, we introduce the *region of influence* of u, which is a random region given by a ball centered at the space point which is the canonical space-coordinate projection of the space-time point where u first visits $\bigcup_{i \in J_k} K_i$, and which has a random radius that depends on the motion of u from that time onwards. This region of influence will intersect all sites i' of $\mathbb{H}_d$ for which u can enter $K_{i'}$. As discussed above, u can only spend a finite time inside $\{K_i\}_{i \in \mathbb{H}_d}$, so the region of influence of u is bounded. We show that the region of influence of u has a radius with an exponentially decaying tail.

For a general level k, we repeat the argument above: among all particles that enter the space-time region $\bigcup_{i \in J_k} K_i$ select only those which have not entered the space-time region $\bigcup_{j=0}^{k-1} \bigcup_{i \in J_j} K_i$, and then define their region of influence in a similar way. The goal is to show that the sites of $\mathbb{H}_d$ that do *not* belong to the region of influence of any particle stochastically dominates an independent percolation process that is known to be supercritical.

Now we begin the rigorous proof of Proposition 2.1. First we establish (6). For $k = 0$ the set J_k has only one element and (6) holds in a trivial manner. Now fix $k \geq 1$ and let $\Psi_0 = \Pi$. Consider the particles that did not enter the space-time region $\bigcup_{j=0}^{k-1} \bigcup_{i \in J_j} K_i$, and let Ψ_k be the point process determined by the location of these particles at time $\frac{k}{S}$.

Lemma 3.1. *For any $k \geq 1$, Ψ_k is a non-homogenenous Poisson point process of intensity uniformly bounded above by λ.*

Proof. Let Υ be the point process determined by the location of the particles of Ψ_0 at time k/S, which is a Poisson point process of intensity λ. For any x, let $p(x)$ be the probability that a random walk that at time k/S is located at x does not visit $\bigcup_{j=0}^{k-1} \bigcup_{i \in J_j} K_i$ during $[0, k/S)$. Then, Ψ_k is a Poisson point process obtained by *thinning* Υ in such a way that its intensity measure at position x is $\lambda p(x) \leq \lambda$. $\square$

For each $i \in J_k$, let

$$N_i := \text{number of particles of } \Psi_k \text{ that visit the set } J_k \text{ during the interval } [k/S, (k+1)/S] \text{ and enter } J_k \text{ through } i.$$

Lemma 3.2. *There exists a positive constant $c = c(d, S)$ so that the set $\{N_i\}_{i \in \mathbb{H}_d}$ is stochastically dominated by $\{M_i\}_{i \in \mathbb{H}_d}$, where M_i are i.i.d. Poisson random variables of mean $c\lambda$.*

Proof. We define a set of random variables $\{N_i'\}_{i \in J_k}$ which are distributed independently across different values of k. For any given k, consider an independent configuration of particles distributed as a Poisson point process of intensity λ over $\mathbb{Z}^d$. Let each particle perform a continuous-time random walk for a time interval of length $1/S$. Then, for each $i \in J_k$, let N_i' be the number of particles that visit i during $(0, 1/S)$ and visit i before visiting any other site of J_k. By Lemma 3.1 and by independence across different values of k, we have that $\{N_i'\}_{i \in \mathbb{H}_d}$ stochastically dominates $\{N_i\}_{i \in \mathbb{H}_d}$. It then suffices to show that, for any given k, $\{N_i'\}_{i \in J_k}$ is stochastically dominated by $\{M_i\}_{i \in J_k}$.

By thinning of Poisson point processes we have that $\{N_i'\}_{i \in J_k}$ are independent Poisson random variables. It remains to show that there exists a constant $c = c(d, S)$ so that, uniformly for all i, we have $\mathbf{E}[N_i'] \leq c\lambda$. Fix $i \in J_k$ and let $\tilde{p}(x)$ be the probability that a particle starting from $x \in \mathbb{Z}^d$ visits i during $[0, 1/S)$ and does so before visiting any other site of J_k. Then, we have that

$$\mathbf{E}[N_i'] = \lambda \sum_{x \in \mathbb{Z}^d} \tilde{p}(x).$$

Since the number of jumps of a particle during $[0, 1/S)$ is a Poisson random variable of mean $1/S$, we can apply a Chernoff bound for Poisson random variables (e.g., using [1, Theorem A.1.15] with $\epsilon \geq 1$) to conclude that there is a constant $c_1 > 0$ such that, for any x so that $\|x - i\|_1 \geq 2/S$, we have $\tilde{p}(x) \leq e^{-c_1 \|x - i\|_1}$.

Then, using that the number of sites at distance z from i is at most $c_2 z^{d-1}$ for some constant $c_2 > 0$, we have

$$\mathbf{E}\left[N_i'\right] \leq \lambda \left(\sum_{x:\ \|x-i\|_1 < 2/S} \tilde{p}(x) + \sum_{z=2/S}^{\infty} c_2 z^{d-1} e^{-c_1 z} \right) \leq c\lambda,$$

for $c = c(d, S)$ sufficiently large. □

We now introduce some notations that we will use to define the region of influence of a site. Fix $\delta = \frac{S}{4\sqrt{d}}$ and let $C_{0,0}^{\delta} \equiv C^{\delta} \subset \mathbb{Z}^d \times \mathbb{R}_+$ be the space-time cone

$$C^{\delta} = \{(y,t) \colon y \in \mathbb{Z}^d, t \geq 0 \text{ and } \|y\|_2 < \delta t\}.$$

We claim that for any $x \in \mathbb{H}_d$ and any $t \in T_x$, the shifted cone $C_{x,t}^{\delta} = (x,t) + C^{\delta}$ does not intersect K_j for any $j \neq x$. In order to see this, let $j \in \mathbb{H}_d$ be such that $\|j\|_1 \geq \|x\|_1$. Then, for any s for which $(j,s) \in K_j$ we have

$$\begin{aligned} s - t \leq \frac{1}{S} + \frac{\|j\|_1 - \|x\|_1}{S} &\leq \frac{1 + \|j-x\|_1}{S} \\ &\leq \frac{1 + \sqrt{d}\|j-x\|_2}{S} \leq \frac{1 + \|j-x\|_2}{4\delta} \leq \frac{\|j-x\|_2}{2\delta}. \end{aligned}$$

On the other hand, by the definition of C^{δ}, for any $(j, s') \in C_{x,t}^{\delta}$ we have $s' - t > \frac{\|j-x\|_2}{\delta}$.

For a random walk $(\xi(t))_t$ that starts from the origin define τ as the last time that $(\xi(t))_t$ is outside C^{δ}; i.e.,

$$\tau = \inf\{t \geq 0 \colon (\xi(s), s) \in C^{\delta} \text{ for all } s \geq t\}.$$

Since C^{δ} grows linearly with time, we have that τ is finite almost surely. Now define the random variable

$$\chi = \sup\{\|\xi(t)\|_2 \colon t \in [0, \tau]\}. \tag{7}$$

The definition of χ is illustrated in Figure 1(a).

We are now ready to define the region of influence of a site. From now on we fix k and $i \in J_k$, and we denote by B_i the region of influence of site i. We couple M_i and N_i so that $M_i \geq N_i$. If $M_i = 0$, we set $B_i = \emptyset$. Otherwise we proceed as follows. We construct a region for each of the N_i particles that visit K_i, where we say that a particle visits K_i if the particle visits i during T_i. Consider the jth such particle and let χ_j be an independent random variable distributed as χ, and define t_j as the first time the particle visits K_i. With this, define the space-time cylinder

$$S_j = (B(i, \chi_j) \cap \mathbb{H}_d) \times [t_j, t_j + \chi_j/\delta],$$

where $B(x,r) \subset \mathbb{Z}^d$ stands for the ball of radius r centered at x. Note that, for any time $s \geq t_j + \chi_j/\delta$, the particle is inside the space-time cone C_{i,t_j}^{δ}.

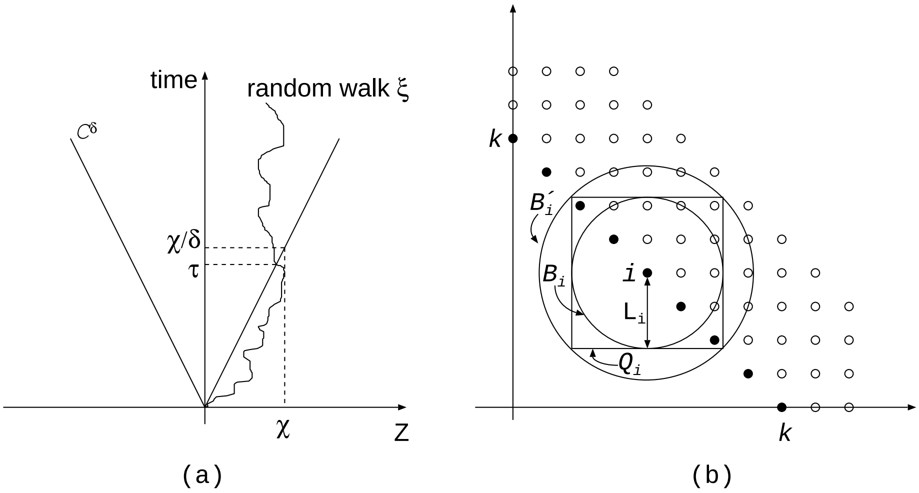

Figure 1: (a) Illustration of the space-time cone C^δ for $d = 1$ and the definition of τ and χ. (b) Illustration of the region of influence of site i for $d = 2$.

Consequently, at any time $s \geq t_j + \chi_j/\delta$, the jth particle cannot intersect $\bigcup_{z\in\mathbb{H}_d} K_z$; hence the sites $\iota \in \mathbb{H}_d$ for which j can intersect K_ι are contained in S_j. Define χ_j in the same way as above for all $N_i < j \leq M_i$, and take $L_i = \max_{j=1}^{M_i} \chi_j$ and $B_i = B(i, L_i) \cap \mathbb{H}_d$. Note that B_i contains all sites that intersect $\bigcup_{j=1}^{N_i} S_j$. Since the $\{M_i\}_{i\in J_k}$ are i.i.d. random variables, the regions $\{B_i\}_{i\in J_k}$ are also i.i.d.

We have the following lemma bounding the size of B_i.

Lemma 3.3. *There exist constants $c, c' > 0$ independent of λ such that, for all $x \geq 1$ and $i \in \mathbb{H}_d$,*

$$\mathbf{P}(L_i > x) \leq c\lambda \exp(-c'x).$$

Proof. First we derive an upper bound for $\mathbf{P}(\chi \geq x)$. The probability that a random walk performs at least x jumps in a time interval of length $x/2$ is $e^{-c_1 x}$ for some positive constant c_1. If this does not happen, then χ can only be at least x if at some time after $x/2$ the random walk is outside the cone C^δ. For any integer $a \geq 0$, let I_a be the time interval $[x/2 + a, x/2 + a + 1]$. We show that, during I_a, the probability that the distance between the random walk and the origin exceeds $\delta(x/2 + a)$ is at most $e^{-c_2(x+a)}$ for some positive constant $c_2 = c_2(d, S)$. This follows since, with probability $1 - e^{-c_3(x+a)}$, the random walk is within distance $\frac{\delta(x/2+a)}{2}$ from the origin at time $x/2 + a$ and, with probability $1 - e^{-c_4(x+a)}$, the random walk performs less than $\frac{\delta(x/2+a)}{2}$ jumps during a time interval of length 1. Then, summing over a we obtain

$$\mathbf{P}(\chi \geq x) \leq c_5 e^{-c_6 x},$$

for some positive constants $c_5 = c_5(d, S)$ and $c_6 = c_6(d, S)$. From this, we obtain

$$\mathbf{P}(L_i > x) \leq \mathbf{E}[M_i]\, \mathbf{P}(\chi > x) \leq cc_5\lambda e^{-c_6 x},$$

where c comes from Lemma 3.2. $\square$

Now we refer to Figure 1(b). If $B_i = \emptyset$, set $Q_i = \emptyset$. Otherwise, let Q_i be the square $i + [-L_i, L_i]^d \cap \mathbb{H}_d$ of side length $2L_i$; note that B_i is inscribed inside Q_i. Consider the 2-dimensional circle B_i' that circumscribe Q_i; the radius of B_i' is $\sqrt{2}L_i$. Now consider any site $\iota \in B_i$ so that $\iota \in J_{k'}$ for some $k' \geq k$, and take any oriented path from the origin to ι. By construction, this path must contain a site in $Q_i \cap J_k$.

Now, for any $i \in \mathbb{H}_d$, we define $Y_i = 0$ if there exists a $j \in \mathbb{H}_d$ with $\|j\|_1 = \|i\|_1$ for which $i \in Q_j$. Otherwise, we set $Y_i = 1$. From the argument above we have that we can couple Y_i and E_i so that $Y_i \leq E_i$. Therefore, if $\{Y_i\}_{i \in J_k}$ stochastically dominates $\{X_i\}_{i \in J_k}$ we establish (6). This last statement holds since the radius of B_i' has an exponential tail by Lemma 3.3 (thus, it is stochastically dominated by a Geometric random variable). Also, the sites $i \in \mathbb{H}_d$ for which $\|i\|_1 = k$ form a one-dimensional line segment. These two properties allow us to apply a result by Holroyd and Martin [3, Theorem 1.3], which establishes that $\{Y_i\}_{i \in J_k}$ stochastically dominates $\{X_i\}_{i \in J_k}$, where $\{X_i\}_{i \in J_k}$ are i.i.d. Bernoulli random variables with mean approaching 1 as $\lambda \to 0$. This establishes (6) and completes the proof of Proposition 2.1.

4. Brownian motions on $\mathbb{R}^d$

In this section we discuss how the proof of Theorem 1.1 can be adapted to the setting where particles perform independent Brownian motions on $\mathbb{R}^d$, $d \geq 2$, and the target is detected as soon as it is within distance 1 from any particle.

The main changes needed in the proof regard the definition of the space-time region K_i and the definition of the region of influence B_i. We start with K_i. For all $i \in \mathbb{H}_d$, define $K_i = B(i, 4/3) \times T_i$, where $B(i, r)$ is the d-dimensional closed ball on $\mathbb{R}^d$ of radius r centered at i, and T_i is defined as in (4). Then, the proof of Theorem 1.1 (assuming Proposition 2.1) carries through with no further changes, and it remains to show how the proof of Proposition 2.1 needs to be changed to this setting.

The proof of Proposition 2.1 is composed of three lemmas. Lemma 3.1 holds without any changes. For Lemma 3.2, the only change we need is to define N_i as the number of particles of Ψ_k that visit $B(i, 4/3)$ during the interval $[k/S, (k+1)/S]$, and first visit $B(i, 4/3)$ not after visiting $B(j, 4/3)$ for every $j \in J_k \setminus \{i\}$. (Note that we allow that the particle visits $B(i, 4/3)$ concurrently to visiting $B(j, 4/3)$ for some $j \in J_k \setminus \{i\}$; in this case, this particle counts to N_i and an independent copy of the particle counts to N_j.) Then Lemma 3.2 follows in the same way.

For Lemma 3.3, we need to do more changes since we need to define B_i and L_i differently. From now on, fix k and $i \in J_k$. Then let $x \in B(i, 4/3)$ and $t \in T_i$ be arbitrary. We regard x as the location and t the time that the particle first visits $B(i, 4/3)$. Consider the cone $C^\delta_{x,t} = (x, t) + C^\delta$. Then, for any $j \notin B(i, 5)$

and $s \in T_j$ we have

$$s - t \le \frac{1 + \|j\|_1 - \|i\|_1}{S} \le \frac{1 + \|j-i\|_2}{4\delta} \le \frac{1 + 4/3 + \|j-x\|_2}{4\delta} \le \frac{\|j-x\|_2}{2\delta},$$

where in the second to last step we apply the triangle inequality, and in the last step we used that $\|j - x\|_2 \ge 3$ since $j \notin B(i, 5)$. Since, for any $(j, s') \in C^\delta$, it holds that $s' - t > \frac{\|j-x\|_2}{\delta}$, we obtain that C^δ does not intersect any T_j for which $j \notin B(i, 5)$. Now let ℓ be a particle from the set of the N_i particles that visit $B(i, 4/3)$ during the interval $[k/S, (k+1)/S]$, and do so before visiting $B(i', 4/3)$ for every $i' \in J_k \setminus \{i\}$. Let χ_ℓ be a random variable distributed as χ (cf (7)), and let L_i be the maximum of χ_ℓ over all ℓ. Then, we set $B_i = B(i, 10+L_i)$ if $M_i \ge 1$. With these definitions, Lemma 3.3 holds without further changes and we obtain that the random variable L_i has an exponential tail. Then, the remaining of the proof of Proposition 2.1 hold by setting $Q_i = i + [-10 - L_i, 10 + L_i]^d \cap \mathbb{H}_d$ and B'_i as the ball that circumscribe Q_i. No further change is needed.

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