Left 3-Engel elements in groups of exponent 5

Gunnar Traustason
Department of Mathematical Sciences,
University of Bath,
Bath BA2 7AY, UK

It is still an open question whether a left 3-Engel element of a group G is always contained in the Hirsch-Plotkin radical of G. In this paper we begin a systematic study of this problem. The problem is first rephrased as saying that a certain type of groups are locally nilpotent. We refer to these groups as sandwich groups as they can be seen as the analogs of sandwich algebras in the context of Lie algebras. We show that any 3-generator sandwich group is nilpotent and obtain a power-conjugation presentation for the free 3-generator sandwich group. As an application we show that the left 3-Engel elements in any group G of exponent 5 are in the Hirsch-Plotkin radical of G.

1 Introduction

Let G be a group. An element $a \in G$ is a left Engel element in G, if for each $x \in G$ there exists a non-negative integer $n(x)$ such that

$$[[[x, \underbrace{a], a], \ldots, a]}_{n(x)} = 1.$$

If $n(x)$ is bounded above by n then we say that a is a left n-Engel element in G. It is straightforward to see that any element of the Hirsch-Plotkin radical $HP(G)$ of G is a left Engel element and the converse is known to be true for some classes of groups, including solvable groups and finite groups (more generally groups satisfying the maximal condition on subgroups) [3,6]. The converse is however not true in general and this is the case even for bounded left Engel elements. In fact whereas one sees readily that a left 2-Engel element is always in the Hirsch-Plotkin radical this is still an open question for left 3-Engel elements. There is some substantial progress by A. Abdollahi in [1] where he proves in particular that for any left 3-Engel p-element a in a group G one has that a^p is in $HP(G)$ (in fact he proves the stronger result that a^p is in the Baer radical), and that the subgroup generated by two left 3-Engel elements is nilpotent of class at most 4. See also [2] for some results about left 4-Engel elements.

Groups of prime power exponent are known to satisfy some Engel type conditions and the solution to the restricted Burnside problem in particular makes use of the fact that the associated Lie ring satisfies certain Engel type identities [12,13]. Considering left Engel elements, it was observed by William Burnside [4] that every element in a group of exponent 3, is a left 2-Engel element and so the fact that every left 2-Engel element lies in the Hirsch-Plotkin radical can be seen as the underlying reason why groups of exponent 3 are locally finite. For groups of 2-power exponent there is a close link with left Engel elements. Let G be a finitely generated group of exponent 2^n and a an element in G of order 2, then

$$[[[x, \underbrace{a], a], \ldots, a}_{n+1}] = [x, a]^{(-2)^n} = 1.$$

Thus a is a left $(n+1)$-Engel element of G. It follows from this that if $G/G^{2^{n-1}}$ is finite and the left $(n+1)$-Engel elements of G are in the Hirsch-Plotkin radical, then G is finite. As we know that for sufficiently large n the variety of groups of exponent 2^n is not locally finite [8,9], it follows that for sufficently large n there are left n-Engel elements that are not contained in the Hirsch-Plotkin radical. Notice also that if all left 4-Engel elements of a group G of exponent 8 are in $HP(G)$, then G is locally finite.

In this paper we focus on left 3-Engel elements. We first make the observation that an element $a \in G$ is a left 3-Engel element if and only if $\langle a, a^x \rangle$ is nilpotent of class at most 2 for all $x \in G$ [1]. We next introduce a related class of groups.

Definition. A *sandwich* group is a group G generated by a set X of elements such that $\langle x, y^g \rangle$ is nilpotent of class at most 2 for all $x, y \in X$ and all $g \in G$.

If $a \in G$ is a left 3-Engel element then $H = \langle a \rangle^G$ is a *sandwich* group and it is clear that the following statements are equivalent:

(1) For every pair (G, a) where a is a left 3-Engel element in the group G we have that a is in the locally nilpotent radical of G.

(2) Every sandwich group is locally nilpotent.

It is also clear that to prove (2), it suffices to show that every finitely generated sandwich group is nilpotent. We will show in next section that every 3-generator sandwich group is nilpotent and obtain a power-conjugation presentation for the free 3-generator sandwich group.

2 3-generator sandwich groups

The follwing definition will be useful [5].

Definition We say that a 3-generator group $\langle a, b, c \rangle$ is of type (r, s, t) if $\langle a, b \rangle$, $\langle a, c \rangle$ and $\langle b, c \rangle$ are nilpotent of class r, s and t respectively.

Notice that any 3-generator sandwich group is of type $(2, 2, 2)$.

2.1 Groups of type $(1, 2, 2)$

In this section we will work with groups $\langle a, b, c \rangle$ of type $(1, 2, 2)$. In [5] it was shown that these groups are solvable. Here we will provide more detailed analysis. We show that these groups are polycyclic and obtain a poly-cyclic presentation for the largest such group. We will also deal with the specical case when c is furthermore a left 3-Engel element and when the group is a sandwich group.

2.1.1 The free group of type $(1, 2, 2)$

Let $F = \langle a, b, c \rangle$ be the largest group of type $(1, 2, 2)$. As $\langle c \rangle^{\langle a \rangle}$ is abelian and $1 = [c, a, a]$, we have
$$c^{a^2} = c^{2a} c^{-1}.$$
Conjugating on both side by a^{-1} it follows as well that
$$c^{a^{-1}} = c^2 c^{-a}.$$

Notice also that as c commutes with c^a, c^b, it follows that c^{ab} commutes with c^a, c^b. Using $[c, c^b] = 1$, it follows that

$$\begin{aligned} 1 &= [c^{a^2 b}, c^{a^2}] \\ &= [c^{2ab} c^{-b}, c^{2a} c^{-1}] \\ &= [c^{2ab}, c^{2a} c^{-1}]^{c^{-b}} [c^{-b}, c^{2a} c^{-1}] \\ &= [c^{2ab}, c^{-1}]^{c^{-b}} [c^{-b}, c^{2a}]^{c^{-1}} \\ &= [c^{2ab}, c^{-1}][c^{-b}, c^{2a}] \end{aligned}$$

and hence
$$[c^{-1}, c^{2ab}] = [c^{-b}, c^{2a}].$$
The right hand side commutes with c, c^{ab} and the left hand side commutes with c^a, c^b. Thus the common element is in the center of

$$\langle c \rangle^F = \langle c, c^a, c^b, c^{ab} \rangle.$$

It follows that

$$[c^{2ab}, c] = [c^{-1}, c^{2ab}]^c = [c^{-b}, c^{2a}]^{c^b} = [c^{2a}, c^b].$$

Conjugating on both sides with a^{-1} gives $[c^{2b}, c^{a^{-1}}] = [c^2, c^{a^{-1}b}]$ or $[c^{2b}, c^2 c^{-a}] = [c^2, c^{2b} c^{-ab}]$ which implies that $[c^{2b}, c^a] = [c^2, c^{ab}]$. Notice also that as $c^{2ab} = c^{2ba}$ we have by symmetry that $[c^{2a}, c^b] = [c^{2b}, c^a]$. The conclusion is that

$$[c^2, c^{ab}] = [c^{2b}, c^a] = [c^{2a}, c^b] = [c^{2ab}, c]. \tag{1}$$

From this one sees as well that

$$[c^2, c^{2ab}] = [c^2, c^{ab}]^2 = [c^{2ab}, c]^2 = [c^{2ab}, c^2] = [c^2, c^{2ab}]^{-1}$$

and therefore $[c^2, c^{ab}]^4 = [c^2, c^{2ab}]^2 = 1$. We will see later that the order of $[c^2, c^{ab}]$ is in fact 2. Notice first that

$$[c^2, c^{ab}]^b = [c^{2b}, c^{-a}c^{2ab}] = [c^{2b}, c^a]^{-1} = [c^2, c^{ab}]^{-1}.$$

By symmetry we also have that $[c^2, c^{ab}]^a = [c^2, c^{ab}]^{-1}$. It follows from this and the previous relations that $\langle c^2, c^{2a}, c^{2b}, c^{2ab}, [c^2, c^{ab}]\rangle$ is normalised by $a, b, c, a^{-1}, b^{-1}, c^{-1}$ and thus

$$\langle c^2\rangle^F = \langle c^2, c^{2a}, c^{2b}, c^{2ab}, [c^2, c^{ab}]\rangle.$$

The relations above also imply that

$$\langle c^4\rangle^F = \langle c^4, c^{4a}, c^{4b}, c^{4ab}, [c^2, c^{ab}]^2\rangle.$$

Notice that we also have that all the elements in $\langle c^4\rangle^F$ commute with the elments of $\langle c^2\rangle^F$. The next step is to show that $[c^2, c^{ab}]$ commutes with b and this will imply that $[c^2, c^{ab}]^2 = 1$. Notice first that

$$\begin{aligned} a^{2c} &= a^2[a^2, c] = a^2[a, c^2] = a^2c^2c^{-2a} \\ a^{2c^{-1}} &= a^2[a^2, c^{-1}] = a^2[a, c^{-2}] = a^2c^{-2}c^{2a}. \end{aligned}$$

Thus

$$\begin{aligned} (a^2)^{ab^c} &= a^{2c^{-1}bc} \\ &= (a^2c^{-2}c^{2a})^{bc} \\ &= (a^2c^{2ab}c^{-2b})^c \\ &= a^2c^2c^{-2a}c^{2ab}c^{-2b}[c^{2ab}, c] \\ &= a^2c^2c^{2ab}c^{-2a}c^{-2b}[c^2, c^{ab}]. \end{aligned}$$

Both a and b^c commute with b and in order to show that $[c^2, c^{ab}]$ commutes with b it thus suffices to show that $c^2c^{2ab}c^{-2a}c^{-2b}$ commutes with b. But this follows from the following calculations. We will there be using the fact that $\langle c^4\rangle^F \leq Z(\langle c^2\rangle^F)$. We have

$$\begin{aligned} (c^2c^{2ab}c^{-2a}c^{-2b})^b &= c^{2b}c^{-2a}c^{4ab}c^{-2ab}c^2c^{-4b} = c^2c^{2ab}c^{-2a}c^{-2b}[c^{-2ab}, c^2][c^{2b}, c^{-2a}] \\ &= c^2c^{2ab}c^{-2a}c^{-2b}[c^2, c^{2ab}]^2 = c^2c^{2ab}c^{-2a}c^{-2b}. \end{aligned}$$

By symmetry the element $u = [c^2, c^{ab}]$ commutes also with a and is therefore in $Z(F)$. As $u^2 = 1$ it also follows that $\langle c^2\rangle^F$ is abelian.

Now calculating modulo $\langle c^2\rangle^F$. We have seen that a^2 commutes with ab^c and of course a^2 commutes with b^2 and a^cb. It is also clear that ab^c and a^cb commute. Notice that $(b^2)^c = b^2[b^2,c] = b^2[b,c^2]$ and thus modulo $(c^2)^F$ we have $(b^2)^{a^cb} = b^{2c^{-1}acb} = b^2$. Thus the group

$$\langle b^2, a^2, a^cb, ab^c\rangle \, \langle c^2\rangle^F / \langle c^2\rangle^F$$

is abelian. We next show that this is normal in $F/\langle c^2\rangle^F$. As b^2, a^2, a^cb, ab^c commute with a^2, b^2, c^2 modulo $\langle c^2\rangle^F$. It suffices to show that $u^a, u^b, u^c \in \langle b^2, a^2, a^cb, ab^c\rangle \, \langle c^2\rangle^F$ for all $u \in \{b^2, a^2, a^cb, ab^c\}$. The only such conjugates that remain to be checked are (calulating modulo $(c^2)^F$)

$$(a^cb)^b = b^{-1}a^cb^2 = b^{-1}a^{-c}a^{2c}b^2 = (a^cb)^{-1}a^2b^2,$$

$$(a^cb)^c = a^{c^2}b^c = ab^c,$$

$$(ab^c)^a = b^ca = b^{2c}b^{-c}a^{-1}a^2 = b^2(ab^c)^{-1}a^2$$

and $(ab^c)^c = a^cb$.

Let $H = \langle a^2, b^2, a^cb, ab^c\rangle \, \langle c^2\rangle^F$. Modulo H we have $a^c = (a^cb)b^{-1} = b^{-1}$ and $b^c = a^{-1}(ab^c) = a^{-1}$ and thus if $K = \langle a,b\rangle H$ then K/H is an abelian normal subgroup of F/H. Finally $F/K = \langle cK\rangle$ and thus we have seen that F is a polycyclic group.

Remark. Our way of writing polycyclic presentations in this paper follows [7]. It reflects a polycyclic series

$$\langle x_1\rangle \trianglelefteq \langle x_1, x_2\rangle \trianglelefteq \cdots \trianglelefteq \langle x_1, \ldots, x_m\rangle = G.$$

We also partion the set of generators into subsets $X_1, \ldots, X_r$ where $\langle X_1\rangle \leq \langle X_1 \cup X_2\rangle \leq \cdots \leq \langle X_1 \cup \cdots \cup X_r\rangle = G$ is a normal series with abelian factors.

We will be obtaining such a presentation for a number of groups in this paper and the main difficulty is to prove that these are nilpotent or polycyclic. When this has been achieved it is a routine matter to obtain a polycyclic presentation for these. This can be done with the aid of a computer although this has been done by hand here. Although the presentations given in this paper are confluent, this fact is not needed for the main results. We only need to know that the groups satisfy the set of relations as indicated.

Routine calculations show that we get the following polycyclic presentation of F. We only write down the conjugation relations that are non-trivial.

Generators

$X_1: \; x_1 = [c^2, c^{ab}],\ x_2 = c^{-2}c^{-2ab}c^{2a}c^{2b},\ x_3 = c^{-2}c^{2a},\ x_4 = c^{-2}c^{2b},\ x_5 = c^2$
$X_2: \; x_6 = b^2,\ x_7 = a^2,\ x_8 = a^c b,\ x_9 = ab^c$
$X_3: \; x_{10} = b,\ x_{11} = a$
$X_4: \; x_{12} = c.$

Relations

$x_1^2 = 1,\ x_{10}^2 = x_6,\ x_{11}^2 = x_7,\ x_{12}^2 = x_5$

$x_2^{x_{12}} = x_2x_1,$
$x_3^{x_6} = x_3x_2^{-2},\ x_3^{x_8} = x_3x_2^{-1},\ x_3^{x_9} = x_3x_2^{-1}x_1,\ x_3^{x_{10}} = x_3x_2^{-1},$
$x_4^{x_7} = x_4x_2^{-2},\ x_4^{x_8} = x_4x_2^{-1}x_1,\ x_4^{x_9} = x_4x_2^{-1},\ x_4^{x_{11}} = x_4x_2^{-1},$
$x_5^{x_6} = x_5x_4^2,\ x_5^{x_7} = x_5x_3^2,\ x_5^{x_8} = x_5x_4x_3x_2^{-1},\ x_5^{x_9} = x_5x_4x_3x_2^{-1}x_1,\ x_5^{x_{10}} = x_5x_4,$
$x_5^{x_{11}} = x_5x_3,$

$x_6^{x_8} = x_6x_2^{-1}x_1,\ x_6^{x_{12}} = x_6x_4^{-1},\ x_7^{x_9} = x_7x_2^{-1}x_1,\ x_7^{x_{12}} = x_7x_3^{-1}$

$x_8^{x_{10}} = x_8^{-1}x_7x_6x_3^{-1}x_2^2,\ x_8^{x_{12}} = x_9x_3^{-1}x_2x_1,\ x_9^{x_{11}} = x_9^{-1}x_7x_6x_4^{-1}x_2,\ x_9^{x_{12}} = x_8x_4^{-1}$

$x_{10}^{x_{12}} = x_{11}x_9x_7^{-1}x_2x_1,\ x_{11}^{x_{12}} = x_{10}x_8^{-1}x_7x_3^{-1}.$

2.1.2 Groups $\langle a, b, c\rangle$ of type $(1, 2, 2)$ where c is a left 3-Engel element

In this section, we will first determine the structure of the free 3-generator group $G = \langle a, b, c\rangle$ of type $(1, 2, 2)$ of which c is a left 3-Engel element. We know that G is a quotient of F.

Lemma 2.1 *$\langle c\rangle^G$ is nilpotent of class at most* 2.

Proof As c is a left 3-Engel element we have that $[c^{ab}, c]$ commutes with c, c^{ab} and as $[c, c^{ab}]$ commutes also with c^a, c^b it follows that $[c, c^{ab}] \in Z(\langle c\rangle^G)$. Conjugating with a gives that $[c^a, c^{2ab}c^{-b}] = [c^a, c^{-b}]$ is in $Z(\langle c\rangle^G)$ as well. Hence the generators c, c^a, c^b, c^{ab} of $\langle c\rangle^G$ commute modulo the center and $\langle c\rangle^G$ is nilpotent of class at most 2. $\Box$.

Remark. The proof above reveals that it suffices that $\langle c, c^{ab}\rangle$ is nilpotent of class at most 2. In fact it suffices that $[c^{ab}, c]$ commutes with c. The reason is that then $1 = [c^{ab}, c, c]^{a^{-1}b^{-1}} = [c, c^{ab}, c^{ab}]$.

It turns out that this extra condition is sufficient to obtain the free 3-generator group of type $(1, 2, 2)$ with c left 3-Engel. Previously we have seen that $[c^2, c^{ab}] = [c^{2a}, c^b]$ and $[c^2, c^{ab}]^2 = 1$. Hence

$$[c, c^{ab}]^2 = [c^a, c^b]^2, \quad [c, c^{ab}]^4 = 1.$$

We also know that c, c^{ab} commute with c^a, c^b. Armed with this information one derives the following polycyclic presentation of G.

Generators

$X_1: \quad x_1 = [c, c^{ab}][c^a, c^b],\ x_2 = [c, c^{ab}],\ x_3 = c^{-1}c^{-ab}c^a c^b,$
$X_2: \quad x_4 = c^{-1}c^a,\ x_5 = c^{-1}c^b,\ x_6 = c,$
$X_3: \quad x_7 = a,\ x_8 = b.$

Relations

$x_1^2 = 1,\ x_2^4 = 1$

$x_2^{x_7} = x_2x_1,\ x_2^{x_8} = x_2^3x_1,$
$x_3^{x_4} = x_3x_1,\ x_3^{x_5} = x_3x_2^2x_1,\ x_3^{x_6} = x_3x_2,\ x_3^{x_8} = x_3x_2^2x_1,$
$x_4^{x_5} = x_4x_2^3x_1,\ x_4^{x_8} = x_4x_3^{-1}x_1,$
$x_5^{x_7} = x_5x_3^{-1}x_2,$
$x_6^{x_7} = x_6x_4,\ x_6^{x_8} = x_6x_5.$

One can read from the presentation that $\gamma_3(\langle c\rangle^G) = \{1\}$. Thus c is indeed a left 3-Engel element. From the presentation one can also read that

$$\begin{aligned}
[G,G] &= \langle x_1, x_2, x_3, x_4, x_5\rangle,\\
\gamma_3(G) &= \langle x_1, x_2, x_3\rangle,\\
\gamma_4(G) &= \langle x_1, x_2\rangle,\\
\gamma_5(G) &= \langle x_1, x_2^2\rangle.
\end{aligned}$$

So the group G is nilpotent of class 5.

Remark. If G has no element of order 2 then G is nilpotent of class at most 3.

2.1.3 The free 3-generator sandwich group of type $(1,2,2)$

Consider the free sandwich group $H = \langle a, b, c\rangle$ of type $(1,2,2)$. This group is a quotient of G. Let us use the same notation above where $x_1, \dots, x_8$ are defined as before but that now we are working with the stronger assumption that the group is a sandwich group. We will obtain some new relations. Notice first that

$$\begin{aligned}
1 &= [a, b^c, a]\\
&= [x_7, x_8[x_8, x_6], x_7]\\
&= [x_7, x_8x_5^{-1}, x_7]\\
&= [x_7, x_5^{-1}, x_7]\\
&= [x_7, x_5, x_7]^{-1}\\
&= [x_3x_2^{-1}, x_7]\\
&= [x_2, x_7]^{-1}\\
&= x_1.
\end{aligned}$$

Also

$$\begin{aligned}
1 &= [b, a^c, b]\\
&= [x_8, x_7[x_7, x_6], x_8]\\
&= [x_8, x_7x_4^{-1}, x_8]\\
&= [x_8, x_4, x_8]^{-1}\\
&= [x_3x_1, x_8]\\
&= x_2^2x_1
\end{aligned}$$

It follows that $x_1 = x_2^2 = 1$. That is we have $[c, c^{ab}]^2 = 1$ and $[c^a, c^b] = [c, c^{ab}]$. We thus obtain a group H with the following presentation.

Generators

$X_1:\ \ x_1 = [c, c^{ab}],\ x_2 = c^{-1}c^{-ab}c^a c^b,$
$X_2:\ \ x_3 = c^{-1}c^a,\ x_4 = c^{-1}c^b,\ x_5 = c,$
$X_3:\ \ x_6 = a,\ x_7 = b.$

Relations

$x_1^2 = 1,$

$x_2^{x_5} = x_2 x_1,\ x_3^{x_4} = x_3 x_1,\ x_3^{x_7} = x_3 x_2^{-1},$
$x_4^{x_6} = x_4 x_2^{-1} x_1,\ x_5^{x_6} = x_5 x_3,\ x_5^{x_7} = x_5 x_4,$

From the presentation we read that H is nilpotent of class 4. One can also check that the group is a sandwich group.

2.2 Certain groups $\langle a, b, c\rangle$ of type $(1, 2, 3)$ where c is a left 3-Engel element

In this section $G = \langle a, b, c\rangle$ is a group of type $(1, 2, 3)$ with the further property that $[b, c, c] = 1$ and that c is a left 3-Engel element. We show that $\langle a, b, c\rangle$ is polycyclic and obtain a presentation for the group. Groups of similar kind were studied in [10]. There it was however assumed that there were no elements of order of order 2 or 3.

Notice that $1 = [c, a, a]$ gives $c^{a^2} = c^{2a-1}$ and $c^{a^{-1}} = c^2 c^{-a}$. Also $1 = [c, b, b, b] = 1$ gives $c^{b^3} = c^{3b^2-3b+1}$ and $c^{b^{-1}} = c^{b^2} c^{-3b} c^3$. Thus

$$\langle c\rangle^G = \langle c, c^a, c^b, c^{b^2}, c^{ab}, c^{ab^2}\rangle.$$

As c is left 3-Engel we have that $\langle c, c^g\rangle$ is nilpotent of class at most 2 for all $g \in G$. Furthermore $\langle c, c^a\rangle$ and $\langle c, c^b, c^{b^2}\rangle$ are abelian. It follows then as well that $c^{a^r b^s}$ commutes with $c^{a^r b^t}$ and $c^{a^t b^s}$ for all integers r, s, t. In particular we have

$$\begin{array}{llll} [c, c^{ab}] & \text{and} & [c^b, c^a] & \text{commute with} & c, c^a, c^b, c^{ab} \\ [c, c^{ab^2}] & \text{and} & [c^{b^2}, c^a] & \text{commute with} & c, c^a, c^{b^2}, c^{ab^2} \\ [c^b, c^{ab^2}] & \text{and} & [c^{b^2}, c^{ab}] & \text{commute with} & c^b, c^{b^2}, c^{ab}, c^{ab^2} \end{array}$$

Now

$$\begin{aligned} 1 &= [c^{a^2b^r}, c^{a^2}] \\ &= [c^{2ab^r} c^{-b^r}, c^{2a}c^{-1}] \\ &= [c^{2ab^r}, c^{2a}c^{-1}]^{c^{-b^r}} [c^{-b^r}, c^{2a}c^{-1}] \\ &= [c^{2ab^r}, c^{-1}][c^{-b^r}, c^{2a}]. \end{aligned}$$

It follows that $[c, c^{ab^r}]^2 = [c^{b^r}, c^a]^2$ and in particular

$$\begin{aligned} [c, c^{ab}]^2 &= [c^b, c^a]^2 \\ [c, c^{ab^2}]^2 &= [c^{b^2}, c^a]^2 \\ [c^b, c^{ab^2}]^2 &= [c^{b^2}, c^{ab}]^2. \end{aligned} \tag{2}$$

Proposition 2.2 *G is solvable.*

We establish that G is solvable in few steps.

Step 1. $c^2 \in Z_3(\langle c \rangle^G)$

Let $\tilde{a} = c^{ab}, \tilde{b} = c^{ab^2}$ and $\tilde{c} = c$. We then have that $\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ is a sandwich group of type $(1, 2, 2)$ and thus we can use the presentation from section 2.1.3 to see that

$$[c^2, c^{ab}, c^{ab^2}] = [\tilde{c}^2, \tilde{a}, \tilde{b}] = x_2(\tilde{a}, \tilde{b}, \tilde{c})^{-2} \in Z(\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle)$$

and

$$[c^2, c^{ab^2}, c^{ab}] = [\tilde{c}^2, \tilde{b}, \tilde{a}] = x_2(\tilde{a}, \tilde{b}, \tilde{c})^{-2} \in Z(\langle \tilde{a}, \tilde{b}, \tilde{c}).$$

Thus $[c^2, c^{ab}, c^{ab^2}] = [c^2, c^{ab^2}, c^{ab}]$ commutes with c, c^{ab}, c^{ab^2} and obviously also with c^a. This argument shows that

$$\begin{array}{lll}
[c^2, c^{ab}, c^{ab^2}] = [c^2, c^{ab^2}, c^{ab}] & \text{commutes with} & c, c^{ab}, c^{ab^2}, c^a \\
[c^{2ab^2}, c^b, c] = [c^{2ab^2}, c, c^b] & \text{commutes with} & c^{ab^2}, c^b, c, c^{b^2} \\
[c^{2ab}, c^{b^2}, c] = [c^{2ab}, c, c^{b^2}] & \text{commutes with} & c^{ab}, c^{b^2}, c, c^b \\
[c^{2b^2}, c^a, c^{ab}] = [c^{2b^2}, c^{ab}, c^a] & \text{commutes with} & c^{b^2}, c^a, c^{ab}, c^{ab^2} \\
[c^{2b}, c^{ab^2}, c^a] = [c^{2b}, c^a, c^{ab^2}] & \text{commutes with} & c^b, c^{ab^2}, c^a, c^{ab} \\
[c^{2a}, c^b, c^{b^2}] = [c^{2a}, c^{b^2}, c^b] & \text{commutes with} & c^a, c^b, c^{b^2}, c.
\end{array}$$

But we can do better than this. Using (2) we have

$$[c^2, c^{ab}, c^{ab^2}] = [c^{2b}, c^a, c^{ab^2}]$$

and

$$[c^2, c^{ab^2}, c^{ab}] = [c^{2b^2}, c^a, c^{ab}].$$

Thus

$$\begin{aligned}
[c^2, c^{ab}, c^{ab^2}] &= [c^2, c^{ab^2}, c^{ab}] \\
&= [c^{2b}, c^a, c^{ab^2}] \\
&= [c^{2b}, c^{ab^2}, c^a] \\
&= [c^{2b^2}, c^a, c^{ab}] \\
&= [c^{2b^2}, c^{ab}, c^a]
\end{aligned} \tag{3}$$

and as this common value commutes with $c, c^{ab}, c^{ab^2}, c^b, c^a, c^{b^2}$, we have that it lies in $Z(\langle c\rangle^G)$. Similarly

$$\begin{aligned}
[c^{2ab^2}, c^b, c] &= [c^{2ab^2}, c, c^b] \\
&= [c^{2ab}, c^{b^2}, c] \\
&= [c^{2ab}, c, c^{b^2}] \\
&= [c^{2a}, c^b, c^{b^2}] \\
&= [c^{2a}, c^{b^2}, c^b]
\end{aligned} \tag{4}$$

and the common value is again in $Z(\langle c\rangle^G)$. Now consider the group $\langle \tilde{a}, \tilde{b}, \tilde{c}\rangle$ where $\tilde{a} = c$, $\tilde{b} = c^{b^2}$ and $\tilde{c} = c^{ab}$. This is a sandwich group of type $(1, 2, 2)$ and the presentation in section 2.1.3 gives

$$[c^2, c^{ab}, c^{b^2}] = [\tilde{a}^2, \tilde{c}, \tilde{b}] = [\tilde{c}^2, \tilde{a}, \tilde{b}]^{-1} = [c^{2ab}, c, c^{b^2}]^{-1}.$$

Thus $[c^2, c^{ab}, c^{ab^2}]$ and $[c^2, c^{ab}, c^{b^2}]$ are in $Z(\langle c\rangle^G)$ and thus $[c^2, c^{ab}] \in Z_2(\langle c\rangle^G)$. Similarly it follows from (3) and (4) that $[c^2, c^{ab^2}] \in Z_2(\langle c\rangle^G)$. Hence $c^2 \in Z_3(\langle c\rangle^G)$ and thus in particular $\langle c^2\rangle^G$ is nilpotent.

Step 2. $\langle c\rangle^G$ is nilpotent of class at most 3.

Let $u = [c, b^2, a, [c, b^2]]$. First notice that

$$\begin{aligned} u &= [[c, b^2]^a, [c, b^2]] \\ &= [c^{-a}c^{ab^2}, c^{-1}c^{b^2}] \\ &= [c^{-a}, c^{-1}c^{b^2}]^{c^{ab^2}}[c^{ab^2}, c^{b^2}c^{-1}] \\ &= [c^{-a}, c^{b^2}]^{c^{ab^2}}[c^{ab^2}, c^{-1}] \\ &= [c^{b^2}, c^a] \cdot [c, c^{ab^2}]. \end{aligned}$$

We next calculate the action of b on u. Let

$$v = [c, b^2, b] = [c^2, b, b] = c^2c^{-4b}c^{2b^2}.$$

We have,

$$\begin{aligned} u^b &= [v[c, b^2], a, v[c, b^2]] \\ &= [[v, a]^{[c,b^2]}[c, b^2, a], v[c, b^2]] \\ &= [v^{-1}v^a, v[c, b^2]]^{[c,b^2][c,b^2,a]}[c, b^2, a, v[c, b^2]] \\ &= [v^a, [c, b^2]^b]^{[c,b^2]^a}[c, b^2, a, [c, b^2]] \cdot [c, b^2, a, v]^{[c,b^2]} \\ &\overset{(*)}{=} u \cdot [v^a, [c, b^2]^b]^{[c,b^2]^a}[[c, b^2]^a, v]^{[c,b^2]} \\ &= u \cdot [v^a, c^{-b}c^{b^3}]^{c^{-a}c^{ab^2}} \cdot [c^{-a}c^{ab^2}, v]^{c^{-1}c^{b^2}} \\ &= u \cdot [c^{2a}c^{-4ab}c^{2ab^2}, cc^{-4b}c^{3b^2}]^{c^{-a}c^{ab^2}} \cdot [c^{-a}c^{ab^2}, c^2c^{-4b}c^{2b^2}]^{c^{-1}c^{b^2}} \end{aligned}$$

where in (*) we have used the fact that $c^2 \in Z_3(\langle c\rangle^G)$. This shows in particular that u commutes with b modulo $\langle c^2\rangle^G$. As we are seeking a polycyclic presentation for G, we will work out the right hand side. This is equal to uw where

$$\begin{aligned} w \;=\; & [c^{2a}, c^{-4b}c^{3b^2}]^{c^{-a}c^{-4ab}c^{3ab^2}}[c^{-4ab}, cc^{3b^2}]^{c^{-a}c^{3ab^2}}[c^{2ab^2}, cc^{-4b}]^{c^{-a}c^{ab^2}} \\ & [c^{-a}, c^{-4b}c^{2b^2}]^{c^{-1}c^{b^2}c^{ab^2}}[c^{ab^2}, c^2c^{-4b}]^{c^{-1}c^{b^2}} \end{aligned}$$

$$\begin{aligned}
&= [c^{2a},c^{b^2}]^{3c^{-4ab}}[c^{2a},c^{b}]^{-4c^{3b^2}c^{3ab^2}}[c^{2ab},c^{b^2}]^{-6c^{-a}}[c^{2ab},c]^{-2c^{3b^2}c^{3ab^2}}\\
&\quad [c^{2ab^2},c^{b}]^{-4c^{-a}}[c^{2ab^2},c]^{c^{-4b}}[c^{a},c^{2b^2}]^{-1}[c^{a},c^{2b}]^{2c^{3b^2}c^{ab^2}}\\
&\quad [c^{ab^2},c^{2b}]^{-2c^{-1}}[c^{ab^2},c^{2}]^{c^{-4b}}\\
&\stackrel{(2)}{=} [c^{2},c^{ab^2}]^{-3c^{-4ab}}[c^{2},c^{ab}]^{4c^{3b^2}c^{3ab^2}}[c^{2b},c^{ab^2}]^{6c^{-a}}[c^{2},c^{ab}]^{2c^{3b^2}c^{3ab^2}}\\
&\quad [c^{2b},c^{ab^2}]^{4c^{-a}}[c^{2},c^{ab^2}]^{-c^{-4b}}[c^{2},c^{ab^2}][c^{2},c^{ab}]^{-2c^{3b^2}c^{ab^2}}\\
&\quad [c^{2b},c^{ab^2}]^{2c^{-1}}[c^{2},c^{ab^2}]^{-c^{-4b}}\\
&= [c^{2},c^{ab}]^{4c^{3b^2}c^{3ab^2}}[c^{2},c^{ab}]^{2c^{3b^2}c^{3ab^2}}[c^{2},c^{ab}]^{-2c^{3b^2}c^{ab^2}}\\
&\quad [c^{2},c^{ab^2}]^{-3c^{-4ab}}[c^{2},c^{ab^2}]^{-c^{-4b}}[c^{2},c^{ab^2}][c^{2},c^{ab^2}]^{-c^{-4b}}\\
&\quad [c^{2b},c^{ab^2}]^{6c^{-a}}[c^{2b},c^{ab^2}]^{4c^{-a}}[c^{2b},c^{ab^2}]^{2c^{-1}}\\
&= [c^{2},c^{ab}]^{4}[c^{2},c^{ab^2}]^{-4}[c^{2b},c^{ab^2}]^{12}\\
&\quad [c^{2},c^{ab},c^{b^2}]^{12}[c^{2},c^{ab},c^{ab^2}]^{16}[c^{2},c^{ab^2},c^{ab}]^{12}\\
&\quad [c^{2},c^{ab^2},c^{b}]^{8}[c^{2b},c^{ab^2},c^{a}]^{-10}[c^{2b},c^{ab^2},c]^{-2}\\
&= [c^{2},c^{ab}]^{4}[c^{2},c^{ab^2}]^{-4}[c^{2b},c^{ab^2}]^{12}\\
&\quad [c^{2ab},c,c^{b^2}]^{-12}[c^{2ab^2},c,c^{b}]^{-8}[c^{2ab^2},c^{b},c]^{2}\\
&\quad [c^{2},c^{ab},c^{ab^2}]^{16}[c^{2},c^{ab^2},c^{ab}]^{12}[c^{2b},c^{ab^2},c^{a}]^{-10}\\
&= [c^{2},c^{ab}]^{4}[c^{2},c^{ab^2}]^{-4}[c^{2b},c^{ab^2}]^{12}[c^{2},c^{ab},c^{ab^2}]^{18}[c^{2ab^2},c,c^{b}]^{-18}.
\end{aligned}$$

Now let

$$x = [c,c^{ab^2}][c^{a},c^{b^2}] = u[c^{2a},c^{b^2}] = u[c^{2},c^{ab^2}]^{-1}.$$

We have by (2) that $x^2 = 1$. We also have that x commutes with c and a. Then

$$\begin{aligned}
x^{b} &= u^{b}[c^{2ab},c^{b^3}]\\
&= u[c^{2ab},c^{b^3}]w\\
&= x[c^{2},c^{ab^2}][c^{2ab},c^{b^3}]w\\
&= x[c^{2},c^{ab^2}][c^{2ab},c^{3b^2}c]w \qquad (5)\\
&= x[c^{2},c^{ab^2}][c^{2},c^{ab}]^{-1}[c^{2b},c^{ab^2}]^{-3}[c^{2ab},c^{b^2},c]^{3}w\\
&= x[c^{2},c^{ab}]^{3}[c^{2},c^{ab^2}]^{-3}[c^{2b},c^{ab^2}]^{9}[c^{2},c^{ab},c^{ab^2}]^{18}[c^{2ab^2},c,c^{b}]^{-15}.
\end{aligned}$$

Now x commutes with $c, c^{b^2}, c^a, c^{ab^2}$ and thus x^b commutes with c^b and c^{ab}. The formula above for x^b gives then

$$\begin{aligned} [x, c^b] &= [c^2, c^{ab^2}, c^b]^3 = [c^{2ab^2}, c, c^b]^{-3} \\ [x, c^{ab}] &= [c^2, c^{ab^2}, c^{ab}]^3. \end{aligned} \tag{6}$$

In particular $x \in Z_2(\langle c\rangle^G)$. Next we use

$$\begin{aligned} 1 &= [c^{b^3}, c^{ab^3}] \\ &= [c^{3b^2}c^{-3b}c, c^{3ab^2}c^{-3ab}c^a] \\ &= [c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}c^a]^c[c, c^{3ab^2}c^{-3ab}c^a] \\ &= [c^{3b^2}c^{-3b}, c^a]^c[c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}]^{c^a c}[c, c^{3ab^2}c^{-3a}]. \end{aligned}$$

This gives

$$[c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}]^{c^a c} = [c^a, c^{3b^2}c^{-3b}][c, c^{3ab^2}c^{-3a}]^{-1} = [c^a, c^{b^3}][c, c^{ab^3}]^{-1}.$$

Notice that the right hand side commutes with c^a, c and thus we get

$$[c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}] = [c^a, c^{b^3}][c^{ab^3}, c].$$

Now we work with the left hand side. We have

$$\begin{aligned} [c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}] &= [c^{3b^2}, c^{3ab^2}c^{-3ab}]^{c^{-3b}}[c^{-3b}, c^{-3ab}c^{3ab^2}] \\ &= [c^{3b^2}, c^{-3ab}][c^{-3b}, c^{3ab^2}]. \end{aligned}$$

Hence

$$[c^{b^2}, c^{ab}]^{-9}[c^b, c^{ab^2}]^{-9} = [c^a, c^{b^3}][c^{ab^3}, c]. \tag{7}$$

The left hand side commutes with $c^b, c^{ab}, c^{b^2}, c^{ab^2}$, whereas the right hand side commutes with c, c^a. Hence, the common value is in $Z(\langle c\rangle^G)$. Let

$$y = [c^{b^2}, c^{ab}][c^{ab^2}, c^b].$$

Then $y^2 = 1$ and y commutes with $c^b, c^{b^2}, c^{ab}, c^{ab^2}$. Furthermore (7) gives us that

$$\begin{aligned} y &= [c^{b^2}, c^{ab}]^{10}[c^b, c^{ab^2}]^8[c^a, c^{b^3}][c^{ab^3}, c] \\ &= [c^b, c^{ab^2}]^{18}[c^a, c^{b^3}][c^{ab^3}, c]. \end{aligned}$$

It follows from this that

$$\begin{aligned} [y,c] &= [c^{2ab^2},c^b,c]^{-9} \\ [y,c^a] &= [c^{2b},c^{ab^2},c^a]^9 \overset{(3)}{=} [c^2,c^{ab^2},c^{ab}]^9. \end{aligned} \tag{8}$$

In particular $y \in Z_2(\langle c\rangle^G)$.

Conjugating equation (7) by b^{-1} gives

$$[c^b,c^a]^{-9}[c,c^{ab}]^{-9} = [c^{ab^{-1}},c^{b^2}][c^{ab^2},c^{b^{-1}}].$$

The common value is again in the centre of $\langle c\rangle^G$. Now let

$$z = [c,c^{ab}][c^a,c^b].$$

By (2), $z^2 = 1$ and z commutes clearly with c, c^b, c^a, c^{ab}. Furthermore

$$z = [c,c^{ab}]^{18}[c^{ab^{-1}},c^{b^2}][c^{ab^2},c^{b^{-1}}]$$

which gives

$$\begin{aligned} [z,c^{b^2}] &= [c^{2ab},c,c^{b^2}]^{-9} \overset{(4)}{=} [c^{2ab^2},c,c^b]^{-9} \\ [z,c^{ab^2}] &= [c^2,c^{ab},c^{ab^2}]^9. \end{aligned} \tag{9}$$

In particular $z \in Z_2(\langle c\rangle^G)$.

We have seen that $[c^a,c^{b^3}][c^{ab^3},c] \in Z(\langle c\rangle^G)$. Expanding this gives

$$\begin{aligned} [c^a,c^{b^3}][c^{ab^3},c] &= [c^a,c^{3b^2}c^{-3b}][c^{3ab^2}c^{-3ab},c] \\ &= [c^a,c^b]^{-3}[c^a,c^{b^2}]^{3c^{-3b}}[c^{ab^2},c]^{3c^{-3ab}}[c^{ab},c]^{-3} \\ &= [c^a,c^b]^{-3}[[c^a,c^{b^2}]^3[[c^a,c^{b^2}]^3,c^{-3b}] \\ &\quad\ [c^{ab^2},c]^3[[c^{ab^2},c]^3,c^{-3ab}][c^{ab},c]^{-3}. \end{aligned}$$

Letting $\tilde{c} = c, \tilde{a} = c^{ab}$ and $\tilde{b} = c^{ab^2}$, we have a sandwich group $\langle \tilde{a},\tilde{b},\tilde{c}\rangle$ of type $(1,2,2)$. From the presentation of these in 2.1.3, we read that

$$[[c^{ab^2},c]^3,c^{-3ab}] = [c,c^{ab^2},c^{ab}]^9$$

that commutes with $[c^{ab^2}, c]$. Similarly

$$[[c^a, c^{b^2}]^3, c^{-3b}] = [c^a, c^{b^2}, c^b]^{-9}.$$

We thus get that

$$[c^{ab^2}, c]^3 [c^{ab}, c]^{-3} [c^a, c^b]^{-3} [c^a, c^{b^2}]^3 [c^a, c^{b^2}, c^b]^{-9} [c, c^{ab^2}, c^{ab}]^9$$

is in $Z(\langle c \rangle^G)$.

It follows from the presentation in 2.1.3, applied on $\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ with $\tilde{a} = c^{ab}, \tilde{b} = c^{ab^2}$ and $\tilde{c} = c$, that $1 = [[c, c^{ab^2}, c^{ab}]^9, c] = [c, c^{ab^2}, c^{ab}, c]^9 = [c, c^{ab^2}, c^{ab}, c]$. Similarly $[c^a, c^{b^2}, c^b, c^a] = 1$. Thus

$$\begin{aligned} 1 &= [c, c^{ab^2}, c^{ab}, c] \\ 1 &= [c^a, c^{b^2}, c^b, c^a]. \end{aligned}$$

Conjugating this by b and b^2 gives then also

$$\begin{aligned} [c^b, c^a, c^{ab^2}, c^b] &= 1 \\ [c^{b^2}, c^{ab}, c^a, c^{b^2}] &= 1 \\ [c^{ab}, c, c^{b^2}, c^{ab}] &= 1 \\ [c^{ab^2}, c^b, c, c^{ab^2}] &= 1. \end{aligned}$$

Now consider again the sandwich group $\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ where $\tilde{a} = c^{ab}, \tilde{b} = c^{ab^2}$ and $\tilde{c} = c$. The presentation for $\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ then gives, using $1 = [\tilde{c}, \tilde{b}, \tilde{a}, \tilde{c}] = x_1(\tilde{a}, \tilde{b}, \tilde{c})$, that

$$[c, c^{ab^2}, c^{ab}] = [c, c^{ab}, c^{ab^2}]$$

and similarly one sees that

$$\begin{aligned} [c, c^{ab^2}, c^{ab}] &= [c, c^{ab}, c^{ab^2}] \\ [c^a, c^{b^2}, c^b] &= [c^a, c^b, c^{b^2}] \\ [c^b, c^a, c^{ab^2}] &= [c^b, c^{ab^2}, c^a] \\ [c^{b^2}, c^a, c^{ab}] &= [c^{b^2}, c^{ab}, c^a] \\ [c^{ab}, c, c^{b^2}] &= [c^{ab}, c^{b^2}, c] \\ [c^{ab^2}, c, c^b] &= [c^{ab^2}, c^b, c]. \end{aligned} \tag{10}$$

We have now enough material to show that $\langle c\rangle^G$ is nilpotent of class at most 3. We first calculate modulo $Z(\langle c\rangle^G)$. We see that

$$[c, c^{ab}, c^{ab^2}] = [[c^b, c^a]z, c^{ab^2}] = [c^b, c^a, c^{ab^2}]$$

and

$$[c, c^{ab^2}, c^{ab}] = [[c^{b^2}, c^a]x, c^{ab}] = [c^{b^2}, c^a, c^{ab}].$$

Thus modulo $Z(\langle c\rangle^G)$ we have

$$\begin{aligned}
[c, c^{ab}, c^{ab^2}] &= [c, c^{ab^2}, c^{ab}] \\
&= [c^b, c^a, c^{ab^2}] \\
&= [c^b, c^{ab^2}, c^a] \\
&= [c^{b^2}, c^a, c^{ab}] \\
&= [c^{b^2}, c^{ab}, c^a].
\end{aligned}$$

But $[c, c^{ab}, c^{ab^2}]$ commutes with c, c^{ab}, c^{ab^2} whereas $[c^b, c^a, c^{ab^2}]$ commutes with c^b, c^a. As $[c^{b^2}, c^a, c^{ab}]$ commutes with c^{b^2}, it is now clear that all these elements that are equal modulo $Z(\langle c\rangle^G)$ are in fact all in $Z(\langle c\rangle^G)$.

Similarly we have that $[c^{ab^2}, c, c^b] = [c^{ab^2}, c^b, c]$, $[c^{ab}, c^{b^2}, c] = [c^{ab}, c, c^{b^2}]$ and $[c^a, c^b, c^{b^2}] = [c^a, c^{b^2}, c^b]$ are all in $Z(\langle c\rangle^G)$.

Now $[c, c^{ab}]$ commutes with c, c^a, c^b, c^{ab} and as we have just seen that $[c, c^{ab}, c^{ab^2}]$ and $[c, c^{ab}, c^{b^2}]$ are in $Z(\langle c\rangle^G)$, it follows that $[c, c^{ab}] \in Z_2(\langle c\rangle^G)$. Similarly one sees that $[c, c^{ab^2}] \in Z_2(\langle c\rangle^G)$. As c commutes with c, c^b, c^{b^2}, c^a, it now follows that $c \in Z_3(\langle c\rangle^G)$. Hence $\langle c\rangle^G$ is nilpotent of class at most 3.

Step 3. G is solvable.

As $G/\langle c\rangle^G$ is abelian it is now clear that G is solvable. $\square$

The group G is in fact poly-cyclic. The next aim is to establish this and to obtain a poly-cyclic presentation for G. First we look into the structure of $\gamma_3(G)$. We have seen that $x^2 = 1$, where $x = [c, c^{ab^2}][c^a, c^{b^2}]$. Using this and (6), it follows that

$$1 = [x^2, c^{ab}] = [x, c^{ab}]^2 = [c, c^{ab^2}, c^{ab}]^{12}$$

and

$$1 = [x^2, c^b] = [x, c^b]^2 = [c^{ab^2}, c, c^b]^{-12}.$$

In fact we shall now see that 12 can be replaced by 6. To see this notice first that by our previous work, (6),(8) and (9), we have

$$\begin{aligned}
[x, c^b] &= [c^{ab^2}, c, c^b]^6 \\
[y, c] &= [c^{ab^2}, c, c^b]^6 \\
[z, c^{b^2}] &= [c^{ab^2}, c, c^b]^6 \\
[x, c^{ab}] &= [c, c^{ab}, c^{ab^2}]^6 \\
[y, c^a] &= [c, c^{ab}, c^{ab^2}]^6 \\
[z, c^{ab^2}] &= [c, c^{ab}, c^{ab^2}]^6.
\end{aligned}$$

Using these we see that

$$[c^a, c^{b^2}, c^b] = [[c^{ab^2}, c]x, c^b] = [c^{ab^2}, c, c^b]^7,$$

$$[c^{ab}, c^{b^2}, c] = [[c^{ab^2}, c^b]y, c] = [c^{ab^2}, c^b, c]^7,$$

$$[c^{b^2}, c^a, c^{ab}] = [[c, c^{ab^2}]x, c^{ab}] = [c, c^{ab^2}, c^{ab}]^7,$$

and

$$[c^b, c^a, c^{ab^2}] = [[c, c^{ab}]z, c^{ab^2}] = [c, c^{ab}, c^{ab^2}]^7.$$

Thus

$$\begin{aligned}
[c^{ab^2}, c, c^b]^6 &= [z, c^{b^2}] \\
&= [c, c^{ab}, c^{b^2}][c^a, c^b, c^{b^2}] \\
&= [c^{ab}, c, c^{b^2}]^{-1}[c^a, c^b, c^{b^2}] \\
&= [c^{ab^2}, c, c^b]^{-7}[c^{ab^2}, c, c^b]^7 \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
[c, c^{ab}, c^{ab^2}]^6 &= [y, c^a] \\
&= [c^{b^2}, c^{ab}, c^a][c^{ab^2}, c^b, c^a] \\
&= [c^{b^2}, c^{ab}, c^a][c^b, c^{ab^2}, c^a]^{-1} \\
&= [c, c^{ab}, c^{ab^2}]^7[c, c^{ab}, c^{ab^2}]^{-7} \\
&= 1.
\end{aligned}$$

Hence, using (10) and the calculations above, we have

$$\begin{aligned}
([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^6 &= 1 \\
[c, c^{ab}, c^{ab^2}]^6 &= 1 \\
[c, c^{ab^2}, c^{ab}] &= [c, c^{ab}, c^{ab^2}] \\
[c^b, c^a, c^{ab^2}] &= [c, c^{ab}, c^{ab^2}] \\
[c^b, c^{ab^2}, c^a] &= [c, c^{ab}, c^{ab^2}] \\
[c^{b^2}, c^a, c^{ab}] &= [c, c^{ab}, c^{ab^2}] \\
[c^{b^2}, c^{ab}, c^a] &= [c, c^{ab}, c^{ab^2}] \\
[c^{ab^2}, c^b, c] &= [c^{ab^2}, c, c^b] \\
[c^{ab}, c^{b^2}, c] &= [c^{ab^2}, c, c^b] \\
[c^{ab}, c, c^{b^2}] &= [c^{ab^2}, c, c^b] \\
[c^a, c^{b^2}, c^b] &= [c^{ab^2}, c, c^b] \\
[c^a, c^b, c^{b^2}] &= [c^{ab^2}, c, c^b].
\end{aligned}$$

Next we sort out the action of a, b on $\gamma_3(\langle c\rangle^G)$. First we have

$$\begin{aligned}
([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^a &= [c^a, c^{-b}, c^{2ab^2}c^{-b^2}][c^{-b^2}, c^a, c^{ab}] \\
&= [c^a, c^b, c^{b^2}][c^b, c^a, c^{ab^2}]^2[c^{b^2}, c^a, c^{ab}]^{-1} \\
&= [c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b].
\end{aligned}$$

Then

$$\begin{aligned}
([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b]^b &= [c^b, c^{ab^2}, c^{ab^3}][c^{ab^3}, c^b, c^{b^2}] \\
&= [c^b, c^{ab^2}, c^a][c^a, c^b, c^{b^2}] \\
&= [c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b]
\end{aligned}$$

and thus $[c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b]$ is in $Z(G)$. Next

$$\begin{aligned}
[c, c^{ab}, c^{ab^2}]^a &= [c^a, c^{-b}, c^{2ab^2}c^{-b^2}] \\
&= [c^a, c^b, c^{b^2}][c^b, c^a, c^{ab^2}]^2 \\
&= [c, c^{ab}, c^{ab^2}]([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b]),
\end{aligned}$$

and

$$\begin{aligned}[c, c^{ab}, c^{ab^2}]^b &= [c^b, c^{ab^2}, c^a] \\ &= [c, c^{ab}, c^{ab^2}]\end{aligned}$$

We next consider the action of a and b on $\gamma_2(\langle c\rangle^G)$. As before we let $x = [c, c^{ab^2}][c^a, c^{b^2}]$, $y = [c^b, c^{ab^2}][c^{ab}, c^{b^2}]$ and $z = [c, c^{ab}][c^a, c^b]$. From (2) we know that that $x^2 = y^2 = z^2 = 1$ and one can easily check that they commute with a. Then

$$\begin{aligned}x^b &= [c^b, c^{3ab^2}c^a][c^{ab}, c^{3b^2}c] \\ &= [c^b, c^a][c^{ab}, c] \cdot [c^b, c^{ab^2}]^3[c^{ab}, c^{b^2}]^3 \cdot [c^b, c^{ab^2}, c^a]^3[c^{ab}, c^{b^2}, c]^3 \\ &= zy([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^3,\end{aligned}$$

$$\begin{aligned}y^b &= [c^{b^2}, c^{-3ab}c^a][c^{ab^2}, c^{-3b}c] \\ &= [c^{b^2}, c^a][c^{ab^2}, c] \cdot [c^{b^2}, c^{ab}]^{-3}[c^{ab^2}, c^b]^{-3}[c^{b^2}, c^{ab}, c^a]^{-3}[c^{ab^2}, c^b, c]^{-3} \\ &= xy([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^3,\end{aligned}$$

and

$$z^b = [c^b, c^{ab^2}][c^{ab}, c^{b^2}] = y.$$

This together with the fact that $x, y, z \in Z(\langle c\rangle^G)$ sorts out the action of G on $\langle x, y, z\rangle$. We next consider the action of a and b on the remaining generators of $\gamma_2(\langle c\rangle^G)$, $[c, c^{ab}], [c, c^{ab^2}]$ and $[c^b, c^{ab^2}]$. We have

$$[c, c^{ab}]^a = [c^a, c^b]^{-1} = [c, c^{ab}]z^{-1} = [c, c^{ab}]z,$$

$$[c, c^{ab^2}]^a = [c^a, c^{b^2}]^{-1} = [c, c^{ab^2}]x$$

and

$$[c^b, c^{ab^2}]^a = [c^{ab}, c^{b^2}]^{-1} = [c^b, c^{ab^2}]y.$$

Also

$$[c, c^{ab}]^b = [c^b, c^{ab^2}],$$

$$\begin{aligned}[c, c^{ab^2}]^b &= [c^b, c^{3ab^2}c^a] \\ &= [c^b, c^a][c^b, c^{ab^2}]^3[c^b, c^{ab^2}, c^a]^3 \\ &= [c, c^{ab}]z[c^b, c^{ab^2}]^3[c, c^{ab}, c^{ab^2}]^3 \\ &= [c, c^{ab}][c^b, c^{ab^2}]^3z[c, c^{ab}, c^{ab^2}]^3,\end{aligned}$$

and

$$\begin{aligned}
[c^b, c^{ab^2}]^b &= [c^{b^2}, c^{-3ab}c^a] \\
&= [c^{b^2}, c^a][c^{b^2}, c^{ab}]^{-3}[c^{b^2}, c^{ab}, c^a]^{-3} \\
&= [c, c^{ab^2}]x[c^b, c^{ab^2}]^{-3}y^3[c, c^{ab}, c^{ab^2}]^3 \\
&= [c, c^{ab^2}][c^b, c^{ab^2}]^{-3}xy[c, c^{ab}, c^{ab^2}]^3.
\end{aligned}$$

We will next find some relations that hold in these generators. In the following we will use the fact that $\gamma_3(\langle c\rangle^G)^6 = 1$. Notice that

$$c^{a^2b^3} = \left(c^{2a}c^{-1}\right)^{b^3} = c^{2ab^3}c^{-b^3} = c^{6ab^2}c^{-6ab}c^{2a}c^{-3b^2}c^{3b}c^{-1}. \tag{11}$$

This is the same as

$$\begin{aligned}
c^{b^3a^2} &= \left(c^{3b^2}c^{-3b}c\right)^{a^2} \\
&= c^{3a^2b^2}c^{-3a^2b}c^{a^2} \\
&= c^{6ab^2}c^{-3b^2}c^{-6ab}c^{3b}c^{2a}c^{-1} \\
&= c^{6ab^2}c^{-6ab}c^{-3b^2}[c^{b^2}, c^{ab}]^{18}c^{2a}c^{3b}[c^b, c^a]^6c^{-1} \qquad (12) \\
&= c^{6ab^2}c^{-6ab}c^{-3b^2}c^{2a}c^{3b}c^{-1}[c^{b^2}, c^{ab}]^{18}[c^b, c^a]^6 \\
&= c^{6ab^2}c^{-6ab}c^{2a}c^{-3b^2}[c^{b^2}, c^a]^{-6}c^{3b}c^{-1}[c^{b^2}, c^{ab}]^{18}[c^b, c^a]^6 \\
&= c^{6ab^2}c^{-6ab}c^{2a}c^{-3b^2}c^{3b}c^{-1}[c^b, c^a]^6[c^{b^2}, c^a]^{-6}[c^{b^2}, c^{ab}]^{18}
\end{aligned}$$

Comparing (11) and (12) and using also (2), we get

$$1 = [c^b, c^a]^6[c^{b^2}, c^a]^{-6}[c^{b^2}, c^{ab}]^{18} = [c, c^{ab}]^6[c, c^{ab^2}]^{-6}[c^b, c^{ab^2}]^{18}. \tag{13}$$

Conjugating this by b, gives

$$\begin{aligned}
1 &= [c^b, c^{ab^2}]^6 \cdot [c, c^{ab}]^{-6}[c^b, c^{ab^2}]^{-18}[c, c^{ab^2}]^{18}[c^b, c^{ab^2}]^{-54} \\
&= [c, c^{ab}]^{-6}[c, c^{ab^2}]^{18}[c^b, c^{ab^2}]^{-66}.
\end{aligned}$$

Multiplying this with (13) gives

$$[c, c^{ab^2}]^{12}[c^b, c^{ab^2}]^{-48} = 1. \tag{14}$$

We saw earlier that

$$x^b = zy([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^3.$$

But from (5) and (13) we also have

$$x^b = x[c, c^{ab}]^6[c, c^{ab^2}]^{-6}[c^b, c^{ab^2}]^{18} = x.$$

These two equations give

$$y = zx([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^3 \tag{15}$$

Thinking about the Sylow structure of $\gamma_2(\langle c \rangle)^G)$. We choose our generating set for the poly-cyclic presentation to be

$$\begin{array}{ll} X_1: & \begin{array}{l} x_1 = [(c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^2, \quad x_2 = ([c, c^{ab}, c^{ab^2}][c^{ab^2}, c, c^b])^3, \\ x_3 = [c, c^{ab}, c^{ab^2}]^2 \quad x_4 = [c, c^{ab}, c^{ab^2}]^3, \\ x_5 = [c, c^{ab}][c^a, c^b], \quad x_6 = [c, c^{ab^2}][c^a, c^{b^2}], \quad x_7 = [c, c^{ab^2}]^4[c^b, c^{ab^2}]^{-16}, \\ x_8 = [c, c^{ab^2}]^3[c^b, c^{ab^2}]^{-12}, \quad x_9 = [c, c^{ab}]^4[c, c^{ab^2}]^{-4}[c^b, c^{ab^2}]^{12}, \\ x_{10} = [c, c^{ab}]^3[c, c^{ab^2}]^{-3}[c^b, c^{ab^2}]^9, \quad x_{11} = [c^b, c^{ab^2}] \end{array} \\ & \\ X_2: & x_{12} = c, \quad x_{13} = c^b, \quad x_{14} = c^{b^2}, \quad x_{15} = c^a, \quad x_{16} = c^{ab}, \quad x_{17} = c^{ab^2} \\ & \\ X_3: & x_{18} = a, \quad x_{19} = b \end{array}$$

We have seen that the following power relations hold for these.

$x_1^3 = 1,\ x_2^2 = 1,\ x_3^3 = 1,\ x_4^2 = 1,\ x_5^2 = 1,\ x_6^2 = 1,\ x_7^3 = 1,\ x_8^4 = 1,\ x_9^3 = 1,\ x_{10}^2 = 1.$

Then we have the following (non-trivial) conjugation relations:

$x_3^{x_{18}} = x_3x_1,\ x_4^{x_{18}} = x_4x_2,\ x_5^{x_{19}} = x_6x_5x_2,$
$x_7^{x_{12}} = x_7x_3x_1^2,\ x_7^{x_{13}} = x_7x_3^2x_1,\ x_7^{x_{15}} = x_7x_3,\ x_7^{x_{16}} = x_7x_3^2,\ x_7^{x_{19}} = x_9$
$x_8^{x_{13}} = x_8x_4x_2,\ x_8^{x_{16}} = x_8x_4,\ x_8^{x_{18}} = x_8x_6,\ x_8^{x_{19}} = x_{10}x_8x_5x_4,$
$x_9^{x_{13}} = x_9x_3x_1^2,\ x_9^{x_{14}} = x_9x_3^2x_1,\ x_9^{x_{16}} = x_9x_3,\ x_9^{x_{17}} = x_9x_3^2,\ x_9^{x_{19}} = x_9^2x_7^2,$
$x_{10}^{x_{12}} = x_{10}x_4x_2,\ x_{10}^{x_{13}} = x_{10}x_4x_2,\ x_{10}^{x_{14}} = x_{10}x_4x_2,\ x_{10}^{x_{15}} = x_{10}x_4,\ x_{10}^{x_{16}} = x_{10}x_4,$
$x_{10}^{x_{17}} = x_{10}x_4,\ x_{10}^{x_{18}} = x_{10}x_2,\ x_{10}^{x_{19}} = x_{10}x_8^2x_2,$
$x_{11}^{x_{12}} = x_{11}x_4x_3^2x_2x_1,\ x_{11}^{x_{15}} = x_{11}x_4x_3^2,\ x_{11}^{x_{18}} = x_{11}x_6x_5x_2,\ x_{11}^{x_{19}} = x_{11}x_8^3x_7x_5x_4x_2,$
$x_{12}^{x_{16}} = x_{12}x_{11}x_{10}x_9x_8^3x_7,\ x_{12}^{x_{17}} = x_{12}x_{11}^4x_8^3x_7,\ x_{12}^{x_{18}} = x_{15},\ x_{12}^{x_{19}} = x_{13},$
$x_{13}^{x_{15}} = x_{13}x_{11}x_{10}x_9x_8^3x_7x_5,\ x_{13}^{x_{17}} = x_{13}x_{11},\ x_{13}^{x_{18}} = x_{16},\ x_{13}^{x_{19}} = x_{14},$
$x_{14}^{x_{15}} = x_{14}x_{11}^4x_8^3x_7x_6,\ x_{14}^{x_{16}} = x_{14}x_{11}x_6x_5x_2,\ x_{14}^{x_{18}} = x_{17},\ x_{14}^{x_{19}} = x_{14}^3x_{13}^{-3}x_{12},$
$x_{15}^{x_{18}} = x_{15}^2x_{12}^{-1},\ x_{15}^{x_{19}} = x_{16},\ x_{16}^{x_{18}} = x_{16}^2x_{13}^{-1},\ x_{16}^{x_{19}} = x_{17},\ x_{17}^{x_{18}} = x_{17}^2x_{14}^{-1},\ x_{17}^{x_{19}} =$

$x_{17}^3 x_{16}^{-3} x_{15}$.

A confluence check was carried out to see that this presentation is consistent. For the next result we look at the special case when the group is of exponent 5.

Proposition 2.3 *Let $G = \langle a, b, c\rangle$ be a group of exponent 5 that is of type $(1, 2, 3)$, where $[b, c, c] = 1$ and where c is a left 3-Engel element in G. We have that G is nilpotent of class at most 4.*

Proof We have seen above that G is nilpotent. To see that the class is at most 4 we can without loss of generality assume that $\gamma_6(G) = \{1\}$. We know that

$$\gamma_2(\langle a, b\rangle) = \gamma_3(\langle a, c\rangle) = \gamma_4(\langle b, c\rangle) = \{1\}. \tag{16}$$

We need to show that any commutator of weight 5 in a, b, c is trivial. By (16) it is clear that we only need to deal with the multiweights $(1, 1, 3)$, $(1, 2, 2)$, $(1, 3, 1)$, $(3, 1, 1)$, $(2, 1, 2)$ and $(2, 2, 1)$ in a, b, c. The only commutators of these multiweights that we need to consider are

$$\begin{array}{ll}
\text{Weight}\,(1,1,3): & [a,c,b,c,c] \\
\text{Weight}\,(1,3,1): & [a,c,b,b,b] \\
\text{Weight}\,(1,2,2): & [a,c,b,b,c],\ [a,c,b,c,b] \\
\text{Weight}\,(3,1,1): & [b,c,a,a,a] \\
\text{Weight}\,(2,1,2): & [b,c,c,a,a],\ [b,c,a,c,a],\ [b,c,a,a,c] \\
\text{Weight}\,(2,2,1): & [c,a,a,b,b]
\end{array}$$

The following calculations, show that these are all trivial. Firstly it follows from (16) that

$$1 = [a, [b, c, c, c]] = [a, c, b, c, c]^{-3}$$

and thus $[a, c, b, c, c] = 1$. Next using (16) again

$$1 = [a, [c, b, b, b]] = [a, c, b, b, b].$$

Having dealt with weights $(1, 1, 3)$ and $(1, 3, 1)$, we turn to weight $(1, 2, 2)$. Firstly

$$1 = [a, [b, c, c, b]] = [a, [b, c, c], b] = [a, c, b, c, b]^{-2},$$

that gives $[a, c, b, c, b] = 1$. We next use the well known fact that in groups of exponent 5 we have that $\prod_{\sigma \in S_4}[y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}]$ is in $\gamma_6(G)$ for all $y, x_1, x_2, x_3, x_4 \in G$. This gives us the 'linearlised 4-Engel identity'

$$1 = \prod_{\sigma \in S_4} [y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}].$$

We will be making some use of this. Firstly let $y = a$, $x_1 = x_2 = b$ and $x_3 = x_4 = c$ to obtain $1 = [a, c, b, b, c]^4$. Hence $[a, c, b, b, c] = 1$. For weight $(3, 1, 1)$ simply notice that

$$1 = [b, [c, a, a], a] = [b, c, a, a, a].$$

For weight $(2, 1, 2)$, we need to deal with three commutators. First notice that

$$1 = [b, [c, a, a], c] = [b, c, a, a, c]$$

and using this we furthermore have

$$1 = [b, c, [c, a, a]] = [b, c, c, a, a][b, c, a, c, a]^{-2}.$$

Using again the linearlised 4-Engel identity for $y = b$, $x_1 = x_2 = a$ and $x_3 = x_4 = c$, we get

$$1 = [b, c, c, a, a]^4 [b, c, a, c, a]^4$$

From this and the last identity it follows that $[b, c, a, c, a]^3 = 1$ that gives us $[b, c, a, c, a] = 1$ and then also $[b, c, c, a, a] = 1$. It now only remains to deal with weight $(2, 2, 1)$. We apply once again the lineralised 4-Engel identity, this time with $y = c$, $x_1 = x_2 = a$ and $x_3 = x_4 = b$. This gives

$$1 = [c, a, a, b, b]^{4!} = [c, a, a, b, b]^{3 \cdot 8}.$$

It follows that $[c, a, a, b, b] = 1$. We have thus shown that G is nilpotent of class at most 4. $\Box$

2.3 The free 3-generator sandwich group

In this section we will show that the any 3-generator sandwich group is nilpotent and we will determine a presentation for the free 3-generator sandwich group.

Notice that any such group is of type (2,2,2). Let $R = \langle x, y, z\rangle$ be the free sandwich group of rank 3.

Theorem 2.4 *R is nilpotent of class at most* 5.

Proof We proceed in few steps.

Step 1. $\langle [z,x], y\rangle$ is nilpotent of class at most 3.

Let $a = z$, $b = z^x$ and $c = y$. We have that $\langle a, b, c\rangle$ is a sandwich group of type $(1,2,2)$ and thus a homomorphic image of H from Section 2.1.3. So it satisfies the presentation given for H and we will use the letters $x_1, \ldots, x_7$ as in there. Notice in particular that $\langle a, b, c\rangle$ is nilpotent of class at most 4. Now

$$\begin{aligned}
[z,x,y] &= [a^{-1}b, c] \\
&= [x_6^{-1}x_7, x_5] \\
&= (x_5^{-1})^{x_6^{-1}x_7}x_5 \\
&= (x_5^{-1}x_3)^{x_7}x_5 \\
&= x_5^{-1}x_4^{-1}x_3x_2^{-1}x_5 \\
&= x_4^{-1}x_3x_2^{-1}x_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
[z,x,y,[z,x]] &= [a^{-1}b, c, a^{-1}b] \\
&= [x_4^{-1}x_3x_2^{-1}x_1, x_6^{-1}x_7] \\
&= x_3^{-1}x_2x_1x_4(x_4^{-1}x_3x_2^{-1}x_1)^{x_6^{-1}x_7} \\
&= x_3^{-1}x_2x_1x_4(x_4^{-1}x_2^{-1}x_1x_3x_2^{-1}x_1)^{x_7} \\
&= x_3^{-1}x_2x_1x_4(x_4^{-1}x_3x_2^{-2})^{x_7} \\
&= x_3^{-1}x_2x_1x_4x_4^{-1}x_3x_2^{-1}x_2^{-2} \\
&= x_2^{-2}x_1.
\end{aligned}$$

But as x_1 and x_2^2 are in the center of $\langle z, z^x, y\rangle$, it follows that $[z,x,y,[z,x]]$ commutes with y and $[z,x]$. As y is a left 3-Engel element we also have

$[z, x, y, y, y] = 1$ and thus the class of $\langle [z, x], y\rangle$ is nilpotent of class at most 3.

<u>Step 2</u>. $\gamma_3(\langle [z, x], y\rangle) \leq Z(R)$ and $\gamma_4(\langle z, z^x, y\rangle) \leq Z(R)$.

We let $x_1, \ldots, x_7$ be as before. In Step 1 we saw that $[z, x, y, [z, x]] = x_2^{-2}x_1$. We also have that the element $[z, x, y, y]$ equals

$$[x_4^{-1}x_3x_2^{-1}x_1, x_5] = x_3^{-1}x_2x_1x_4(x_4^{-1}x_3x_2^{-1}x_1)^{x_5} = x_3^{-1}x_2x_1x_4x_4^{-1}x_3x_2^{-1}x_1^2 = x_1.$$

Thus $\gamma_3(\langle [z, x], y\rangle) = \langle x_1, x_2^2\rangle \leq Z(\langle z, z^x, y\rangle)$. In partcular all the elments in $\gamma_3(\langle [z, x], y\rangle)$ commute with z and y. But as $\gamma_3(\langle [z, x], y\rangle) = \gamma_3(\langle [x, z], y\rangle)$ we have that these elements also commute with x. Hence we get the first part of Step 2. The latter part follows from the fact that $\gamma_4(\langle z, z^x, y\rangle) = \langle x_1\rangle \leq \gamma_3(\langle [z, x], y\rangle)$.

From now on we calculate modulo $Z(R)$. We have seen that $x_1, x_2^2 \in Z(R)$. Let $a = z$, $b = z^x$ and $c = y$ and let $z_2, \ldots, z_7$ be the images of $x_2, \ldots, x_7$ in $\langle z, z^x, y\rangle\, Z(R)/Z(R)$. We have that $\langle z, z^x, y\rangle$ satisfies the following presentation (not necessarily confluent).

<u>Generators</u>

$X_1 :\quad z_2 = c^{-1}c^{-ab}c^ac^b,\ z_3 = c^{-1}c^a,$
$X_2 :\quad z_4 = c^{-1}c^b,\ z_5 = c,$
$X_3 :\quad z_6 = a,\ z_7 = b.$

<u>Relations</u>

$z_2^2 = 1,\ z_3^{z_7} = z_3z_2,\ z_4^{z_6} = z_4z_2,\ z_5^{z_6} = z_5z_3,\ z_5^{z_7} = z_5z_4.$

<u>Step 3</u>. $[z^x, z^y] \in Z_2(R)$.

We have,

$$[z^x, z^y] = [a^{-1}b, [a, c]] = [z_6^{-1}z_7, [z_6, z_5]] = [z_6^{-1}z_7, z_3^{-1}] = z_2$$

which is in the center of $\langle z, z^x, y\rangle Z(R)/Z(R)$. By symmetry $[z^y, z^x] = [z^x, z^y]^{-1}$ is in the centre of $\langle z, z^y, x\rangle Z(R)/Z(R)$. Hence $[z^x, z^y]$ commutes with x, y and z modulo $Z(R)$. This finishes the proof of Step 3.

Step 4. $[z, x, y] \in Z_3(R)$.

We can now calculate modulo $Z_2(R)$. Using the presentation for $\langle z_2, \ldots, z_7\rangle$ above, using the fact established above that $z_2 Z(R) \in Z_2(R)/Z(R)$, one can read that $\langle z, z^x, y\rangle Z_2(R)/Z_2(R)$ is nilpotent of class at most 2. In particular

$$[z, x, y] = [z^{-1}z^x, y]$$

commutes with z and y modulo $Z_2(R)$. But as modulo $Z_2(R)$ we have $[z, x, y] = [x, z, y]^{-1}$ (using step 2) we have by symmetry that $[z, x, y]$ also commutes with x modulo $Z_2(R)$ Hence $[z, x, y] \in Z_3(G)$.

We now finish the proof by showing that the R is nilpotent of class at most 5. As all the subgroups generated by two of the generators x, y, z are nilpotent of class at most 2. It suffices to show that any commutator of weight 3 involving all the generators is in $Z_3(R)$. But this was done in Step 4. $\Box$

We want to obtain a presentation for R. For this we need more detailed analysis of R. We first see that the normal closure of each of z, x and y is nilpotent of class at most 2. By symmetry it suffices to deal with z. As any subgroup generated by two of z, x, y is nilpotent of class at most 2 it suffices to show that any commutator of weight 5 with three entries of z, one of x and one of y is trivial. But these are all generated by $[z, x, y, z, z]$ and $[z, y, x, z, z]$ and it suffices to show that $[z, x, y, z, z] = 1$. We calculate in $\langle z, z^x, y\rangle$ and we let $e_1, \ldots, e_7$ be the images of $x_1, \ldots, x_7$ of H in $\langle z, z^x, y\rangle$. we see that

$$[z, x, y, z, z] = [e_6^{-1}e_7, e_5, e_6, e_6] = [e_4^{-1}e_3e_2^{-1}e_1, e_6, e_6] = 1.$$

Before going further we introduce some notation. For each ordered triple (r, s, t) such that $\{r, s, t\} = \{x, y, z\}$ we consider the group $\langle r, r^s, t\rangle$. Let $a = r$, $b = r^s$ and $c = t$, we let $e_1 = e_1(r, r^s, t), \ldots, e_7 = e_7(r, r^s, t)$ be the images of $x_1, \ldots, x_7$ in $\langle r, r^s, t\rangle$. First notice that by the calculations above we have $[z, x, y, y] = e_1(z, z^x, y)$. But as $\langle [z, x], y\rangle$ is nilpotent of class at most 3 it follows that

$$e_1(x, x^z, y) = [x, z, y, y] = [z, x, y, y]^{-1} = e_1(z, z^x, y).$$

Hence

$$\begin{array}{rcl} e_1(x, x^z, y) & = & e_1(z, z^x, y) \\ e_1(y, y^x, z) & = & e_1(x, x^y, z) \\ e_1(z, z^y, x) & = & e_1(y, y^z, x). \end{array} \tag{17}$$

We next deal with the e_2's. Calculating in $\langle z, z^x, y\rangle$ we see that

$$[z, x, y, [z, x]] = e_2(z, z^x, y)^{-2} e_1(z, z^x, y).$$

Now notice that $[z, x, y, z, x] = [z, x, [y, z], x] \in \langle z, z^y, x\rangle$ and calculations show that this is $e_1(z, z^y, x)$. By symmetry $[x, z, y, x, z] = e_1(x, x^y, z)$. As $[z, x, y, [z, x]] = [z, x, y, z, x][x, z, y, x, z]$, it follows from this and (16) that

$$e_2(z, z^x, y)^2 = e_1(z, z^x, y) \cdot e_1(y, y^z, x) \cdot e_1(x, x^y, z). \tag{18}$$

By symmetry and (16), we see that $e_2(r, r^s, t)^2$ takes the same value for all ordered triples (r, s, t) with $\{r, s, t\} = \{x, y, z\}$. In particular it follows that $e_2(r, r^s, t)^4 = 1$.

By the calculations above we have that any non-trivial commutator of weight 5 in x, y, z must involve one of the generators once and the other two twice. If the commutator that occurs once is z then any such non-trivial commutator of weight 5 is generated by $[z, x, y, x, y]$ and $[z, y, x, y, x]$ that are in $\gamma_4(\langle x, x^z, y\rangle) = \langle e_1(x, x^z, y)\rangle$ and $\gamma_4(\langle y, y^z, x\rangle) = \langle e_1(y, y^z, x)\rangle$ respectively. Hence

$$\gamma_5(R) = \langle e_1(x, x^y, z),\ e_1(y, y^z, x),\ e_1(z, z^x, y)\rangle.$$

Next we turn to commutators of weight 4. Every such commutator is generated modulo $\gamma_5(R)$ by commutators of the form $[z, y, x, y]$ and $[z, x, y, y]$. We have that the latter one is in $\gamma_5(R)$ and, modulo $\gamma_5(R)$, the former is equal to $[z, y, [x, y]] = [y^z, y^x] = e_2(y, y^x, x)^{-1}$. Thus

$$\gamma_4(R) = \langle [z, x, [z, y]],\ [x, y, [x, z]],\ [y, z, [y, x]]\rangle \cdot \gamma_5(R).$$

Then of course $\gamma_3(R) = \langle [z, x, y],\ [z, y, x]\rangle \cdot \gamma_4(R)$, $\gamma_2(R) = \langle [z, x],\ [z, y],\ [y, x]\rangle \cdot \gamma_3(R)$ and $R = \langle z, x, y\rangle \cdot \gamma_2(G)$. We can now easily come up with the following presentation using the generators above.

<u>Generators</u>

X_1 : $x_1 = e_1(z, z^x, y),\ x_2 = e_1(x, x^y, z),\ x_3 = e_1(y, y^z, x)$

X_2 : $x_4 = [z, x, [z, y]],\ x_5 = [x, y, [x, z]],\ x_6 = [y, z, [y, x]],$
$x_7 = [z, x, y],\ x_8 = [z, y, x]$

X_3 : $x_9 = [z, x],\ x_{10} = [z, y],\ x_{11} = [x, y]$

X_4 : $x_{12} = x,\ x_{13} = y,\ x_{14} = z.$

<u>Relations</u>

$x_1^2 = x_2^2 = x_3^2 = 1,\ x_4^2 = x_5^2 = x_6^2 = x_3x_2x_1$

$x_4^{x_{12}} = x_4x_3,\ x_4^{x_{13}} = x_4x_1,\ x_5^{x_{13}} = x_5x_1,\ x_5^{x_{14}} = x_5x_2,$
$x_6^{x_{12}} = x_6x_3,\ x_6^{x_{14}} = x_6x_2,$
$x_7^{x_9} = x_7x_2x_3,\ x_7^{x_{10}} = x_7x_1,\ x_7^{x_{11}} = x_7x_1,\ x_7^{x_{12}} = x_7x_5x_3x_2,$
$x_7^{x_{13}} = x_7x_1,\ x_7^{x_{14}} = x_7x_4x_3x_2,$
$x_8^{x_9} = x_8x_3,\ x_8^{x_{10}} = x_8x_2x_1,\ x_8^{x_{11}} = x_8x_3,\ x_8^{x_{12}} = x_8x_3,$
$x_8^{x_{13}} = x_8x_6x_3,\ x_8^{x_{14}} = x_8x_4x_3,$
$x_9^{x_{10}} = x_9x_4,\ x_9^{x_{11}} = x_9x_5,\ x_9^{x_{13}} = x_9x_7,\ x_{10}^{x_{11}} = x_{10}x_6,\ x_{10}^{x_{12}} = x_{10}x_8,$
$x_{11}^{x_{14}} = x_{11}x_8x_7^{-1}x_6x_5x_4x_3,\ x_{12}^{x_{13}} = x_{12}x_{11},\ x_{12}^{x_{14}} = x_{12}x_9^{-1},\ x_{13}^{x_{14}} = x_{13}x_{10}^{-1}.$

Remark. If there are no elements of order 2, the class is at most 3.

3 Left 3-Engel elements in groups of exponent 5

In this section we show that the left 3-Engel groups of any group G of exponent 5 are contained in the Hirsch-Plotkin radical of G. The proof uses the main results of sections 2.2 and 2.3 and follows in outline the proof of the corresponding result on 4-Engel groups [11].

Theorem 3.1 *Let G be a group of exponent 5 and let $a \in G$ be a left 3-Engel element in G. We then have that a is in the Hirsch-Plotkin radical of G.*

Proof Let $a_1, \ldots, a_k$ be conjugates of a, and let $H = \langle a_1, \ldots, a_k \rangle$. We will show by induction that H is nilpotent of class at most k and that the normal closure of a_i in H is abelian for $i = 1, \ldots, k$.

The case $k = 2$ holds by the asummption that a is a left 3-Engel element and the case $k = 3$ follows from Theorem 2.4 (see the remark after the proof). Now suppose that $k \geq 3$. Let $u = [a_1, a_2, \ldots, a_{k-2}]$ then the subgroup $\langle a_{k-1}, a_{k-1}^u, a_k \rangle$ is generated by three conjugates of a and is thus nilpotent of class at most 3. By this and the fact that any two conjugates generate a subgroup of class at most 2, it follows that

$$[a_1, a_2, \ldots, a_{k-1}, a_k, a_k] = [a_{k-1}^{-u} a_{k-1}, a_k, a_k] = 1$$

and

$$[a_{k,3} [a_1, a_2, \ldots, a_{k-1}]] = [a_{k,3}\, a_{k-1}^{-u} a_{k-1}] = 1.$$

We thus have the following identities which hold for any conjugates $a_1, a_2, \ldots, a_k$ of a and for any $k \geq 3$.

$$\begin{aligned} [a_1, a_2, \ldots, a_{k-1}, a_k, a_k] &= 1, \\ [a_k, [a_1, a_2, \ldots, a_{k-1}], [a_1, a_2, \ldots, a_{k-1}], [a_1, a_2, \ldots, a_{k-1}]] &= 1. \end{aligned} \tag{19}$$

We now proceed with the induction step. Let $k \geq 4$ and suppose that the result is true for all smaller values of k. We first show that if $1 \leq r \leq k$, then

$$[[a_1, a_2, \ldots, a_r], [a_1, a_k, a_{k-1}, \ldots, a_{r+1}]] = [a_1, a_2, \ldots, a_k, a_1]^{(-1)^{k-r}}. \tag{20}$$

This is obvious when $r = k$. Now consider the case $r = k - 1$. Let $u = [a_1, \ldots, a_{k-1}]$. By the induction hypothesis and (19) we have that $\langle a_1, u, a_k \rangle$ is of type $(1, 2, 3)$ and satisfies all the criteria for the group in Section 2.2. It is thus nilpotent of class at most 4 by Proposition 2.3. Using the fact that u commutes with a_1 and the first identity in (19) one sees easily that all commutators of weight $(2, 1, 1)$ and $(1, 1, 2)$ in a_1, u, a_k are trivial. The only commutators that one needs to consider are $[u, a_k, a_1, a_1]$ and $[u, a_k, a_1, a_k]$ but as $[u, [a_k, a_1, a_1]] = [u, [a_1, a_k, a_k]] = 1$ we get by expanding these that

$$\begin{aligned} 1 &= [u, a_k, a_1, a_1], \\ 1 &= [u, a_k, a_1, a_k]. \end{aligned}$$

From this one sees that $[[u,[a_1,a_k]] = [u,a_k,a_1]^{-1}$ that gives us the identity (20) when $r = k-1$. This argument also tells us that

$$[[a_1,a_k,\ldots,a_3],[a_1,a_2]] = [a_1,a_k,\ldots,a_2,a_1]^{-1}$$

and thus

$$[a_1,[a_1,a_k,\ldots,a_2]] = [[a_1,a_2],[a_1,a_k,\ldots,a_3]]^{-1}$$

that shows that the case $r = 1$ follows it is holds for $r = 2$. To establish (20) it is thus sufficient to show that if $2 \leq r \leq k-2$, then

$$[[a_1,a_2,\ldots,a_r],[a_1,a_k,\ldots,a_{r+1}]] = [[a_1,a_2,\ldots,a_{r+1}],[a_1,a_k,\ldots,a_{r+2}]]^{-1}.$$

Let $u = [a_1,a_2,\ldots,a_r]$ and $v = [a_1,a_k,\ldots,a_{r+2}]$. By the induction hypothesis we have that u and v commute and that $\langle u,a_{r+1}\rangle$, $\langle v,a_{r+1}\rangle$ are nilpotent of class at most 2. Thus $\langle u,v,a_{r+1}\rangle$ is of type $(1,2,2)$ and thus nilpotent of class at most 3 by section 2.1.2. Thus $[u,[v,a_{r+1}]] = [u,a_{r+1},v]^{-1}$ as was required. This establishes (20).

We want to show that H is nilpotent of class at most k. We arrive at this in two steps. First we show that H is nilpotent (of class at most $k+1$) and then that $\gamma_k(H) \leq \gamma_{k+1}(H)$. We turn to the first step. Consider a commutator $c = [b_1,b_2,\ldots,b_{k+1}]$ where $b_1,\ldots,b_{k+1}$ lie in $\{a_1,\ldots,a_k\}$. We want to show that $c \in Z(H)$. By induction $c = 1$ unless $\{b_1,\ldots,b_k\} = \{a_1,\ldots,a_k\}$. Also by (19) we have that $c = 1$ if $b_k = b_{k+1}$. So there is no loss of generality in assuming that $b_{k+1} = a_1, b_k = a_k$ and that $\{b_1,\ldots,b_{k-1}\} = \{a_1,\ldots,a_{k-1}\}$. Then, using the inductive hypothesis, we see that $[b_1,b_2,\ldots,b_{k-1}]$ can be expressed as a product $u_1u_2\cdots u_r$ where each u_i is a commutator of the form $[a_1,a_{\sigma(2)},a_{\sigma(3)},\cdots,a_{\sigma(k-1)}]$ for some permutation σ of $\{2,3,\ldots,k-1\}$. So

$$c = [b_1,\ldots,b_{k+1}] = [u_1\cdots u_r,a_k,a_1] = [\prod_{i=1}^{r}[u_i,a_k]^{u_{i+1}u_{i+2}\cdots u_r},a_1].$$

Now the inductive hypothesis implies that $u_1,u_2,\ldots,u_r$ commute with a_1. So c is the product of conjugates of the commutators $[u_1,a_k,a_1],\ldots,[u_r,a_k,a_1]$. To show that $c \in Z(H)$ it thus clearly suffices to show that $[a_1,a_2,\ldots,a_k,a_1] \in Z(H)$.

So consider $d = [a_1,a_2,\ldots,a_k,a_1,a_i]$, where $1 \leq i \leq k$. If $i = 1$ then

$d = 1$ by (19). If $i = k$, let $u = [a_1, a_2, \ldots, a_{k-1}]$. Then, using the induction hypothesis, $\langle a_1, u, a_k \rangle$ is of type $(1, 2, 3)$ and satisfies the criteria for the group in Section 2.2. It is thus nilpotent of class at most 4. Thus

$$1 = [u, [a_1, a_k, a_k]] = [u, a_k, a_1, a_k]^{-2}.$$

This implies that $[u, a_k, a_1, a_k] = 1$ and thus $d = 1$ when $i = k$.

Now let $1 < i < k$. To show that $d = 1$, it suffices by (20) to show that $[u, a_i, v, a_i] = 1$ where $u = [a_1, a_2, \ldots, a_{i-1}]$ and $v = [a_1, a_k, a_{k-1}, \ldots, a_{i+1}]$. Now by the induction hypothesis $\langle u, v, a_i \rangle$ is of type $(1, 2, 3)$ and satisfies the criteria from Section 2.2. Thus it is nilpotent of class at most 4. Hence again

$$1 = [u, [v, a_i, a_i]] = [u, a_i, v, a_i]^{-2}$$

that implies that $[a_1, a_2, \ldots, a_k, a_1]$ commutes with a_i. This finishes the proof that H is nilpotent of class at most $k + 1$. To show that the class is actually k it suffices to show that $[a_1, a_2, \ldots, a_k, a_1] = 1$ since by the argument above, this will imply that $[b_1, b_2, \ldots, b_{k+1}] = 1$ for all $b_1, \ldots, b_{k+1} \in \{a_1, \ldots, a_k\}$.

In order to achieve this we will first show that

$$[a_1, a_2, \ldots, a_{k-3}, a_{\sigma(k-2)}, a_{\sigma(k-1)}, a_{\sigma(k)}, a_1] = [a_1, a_2, \ldots, a_k, a_1]$$

for all permutations σ of $\{k-2, k-1, k\}$. By the induction hypothesis we have that $[a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{-1}, a_{k-2}^{a_{k-1}}] = 1$ and that the elements $[a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{-1}]$ and $[a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{a_{k-1}}]$ commute with $[a_k, a_1]$. Thus

$$[a_1, a_2, \ldots, a_{k-3}, [a_{k-2}, a_{k-1}], a_k, a_1] = [a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{-1} a_{k-2}^{a_{k-1}}, [a_k, a_1]] = 1.$$

Similarly, we have that $[a_1, a_2, \ldots, a_{k-3}, a_k]$, $[a_1, a_2, \ldots, a_{k-3}, a_k^{-[a_{k-2}, a_{k-2}]}]$ and $[a_1, a_2, \ldots, a_{k-3}, a_k^{-[a_{k-2}, a_{k-1}]}, a_k]$ commute with a_1 by the induction hypothesis, and thus that

$$\begin{aligned} 1 &= [a_1, a_2, \ldots, a_{k-3}, [a_{k-2}, a_{k-1}, a_k], a_1] \\ &= [a_1, \ldots, a_{k-3}, [a_{k-2}, a_{k-1}], a_k, a_1][a_2, \ldots, a_{k-3}, a_k, [a_{k-1}, a_{k-2}], a_1]^{-1}. \end{aligned}$$

By the previous part this implies that $[a_1, \ldots, a_{k-3}, a_k, [a_{k-1}, a_{k-2}], a_1] = 1$. This gives us that $[a_1, \ldots, a_{k-3}, a_{\sigma(k-2)}, a_{\sigma(k-1)}, a_{\sigma(k)}, a_1]$ is equal to the element $[a_1, a_2, \ldots, a_k, a_1]$ for all the permutations σ of $\{k-2, k-1, k\}$ as we wished to show.

Now using the linearised 4-Engel identity we have

$$\prod_{\sigma}[a_1, a_2, \ldots, a_{k-3}, a_{\sigma(k-2)}, a_{\sigma(k-1)}, a_{\sigma(k)}, a_{\sigma(1)}] = 1,$$

where the product ranges over all permutations of $\{k-2, k-1, k, 1\}$. By the induction hypothesis all the factors where $\sigma(1) \neq 1$ are trival and by the analysis above we know that all the remaining six factors are equal to $[a_1, \ldots, a_k, a_1]$. Hence $1 = [a_1, \ldots, a_k, a_1]^6 = [a_1, \ldots, a_k, a_1]$ as required. This finishes the proof of the inductive hypothesis and thus of the Theorem. $\Box$.

Theorem 3.2 *Let G be a group of exponent 5. Then G is locally finite if and only if it satisfies the law*

$$[z, [y, x, x, x], [y, x, x, x], [y, x, x, x]]$$

Proof. It is known (see for example Lemma 15 in [11]) that if G is a finite group of exponent 5 and $x \in G$, then $\langle x\rangle^G$ is nilpotent of class at most 6. Thus every finite group of exponent 5 satisfies the identity

$$[z, [y, x, x, x], [y, x, x, x], [y, x, x, x]] = 1.$$

Conversely suppose G is any group of exponent 5 and let let $N = HP(G)$ be the Hirsch-Plotkin radical of G (that is the locally finite radical of G). Then the Hirsch-Plotkin radical of G/N is trivial. We want to show that $N = G$, that is $G/N = \{1\}$. Without loss of generality we can thus assume that $HP(G) = \{1\}$ and we wish to show that $G = \{1\}$. Let $a \in G$. As the identity holds we have that $[b, a, a, a]$ is a left 3-Engel element in G for all $b \in G$ and hence $[b, a, a, a] \in HP(G) = \{1\}$ for all $b \in G$. Thus a is a left 3-Engel element of G and thus trival. This shows that $G = \{1\}$ and we have finished the proof. $\Box$

Remark. This gives us a new proof of a fact that was originally proved in [11] that a group of exponent 5 is locally finite if and only if all the 3-generator subgroups are finite.

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