Coercivity of Combined Boundary Integral Equations
in High-Frequency Scattering

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Dedicated to Cathleen S. Morawetz on the occasion of her 90th birthday

Abstract
We prove that the standard second-kind integral equation formulation of the exterior Dirichlet problem for the Helmholtz equation is coercive (i.e., sign-definite) for all smooth convex domains when the wavenumber $k$ is sufficiently large. (This integral equation involves the so-called combined potential, or combined field, operator.) This coercivity result yields $k$-explicit error estimates when the integral equation is solved using the Galerkin method, regardless of the particular approximation space used (and thus these error estimates apply to several hybrid numerical-asymptotic methods developed recently). Coercivity also gives $k$-explicit bounds on the number of GMRES iterations needed to achieve a prescribed accuracy when the integral equation is solved using the Galerkin method with standard piecewise-polynomial subspaces. The coercivity result is obtained by using identities for the Helmholtz equation originally introduced by Morawetz in her work on the local energy decay of solutions to the wave equation. © 2015 Wiley Periodicals, Inc.

1 Introduction
The Helmholtz equation,

$$\Delta u + k^2 u = 0,$$

with wavenumber $k > 0$, posed in the domain exterior to a bounded obstacle, is arguably the simplest possible model of wave scattering and thus has been the subject of vast amounts of research.

On the one hand, much effort has gone into constructing the asymptotics as $k \to \infty$ of solutions of the Helmholtz equation in exterior domains using geometrical optics and Keller’s geometrical theory of diffraction, and then proving error

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bounds that justify these asymptotics (often via proving bounds on the inverse of the Helmholtz operator).

On the other hand, much research effort has gone into solving the Helmholtz equation numerically. For example, one popular method is the finite element method, which is based on the weak form of the PDE. Alternatively, if the wavenumber $k$ is constant, then an explicit expression for the fundamental solution of the Helmholtz equation is available, and this allows the problem of finding $u$ in the exterior domain to be reduced to solving an integral equation on the boundary of the obstacle (the so-called boundary integral method). The resulting integral equation can then be solved numerically in a variety of ways (e.g., using Galerkin, collocation, or Nyström methods).

Over the last two decades, there has been a lot of interest in

(a) determining how conventional numerical methods for solving the Helmholtz equation behave as $k$ increases, and
(b) designing new methods that perform better as $k$ increases than the conventional ones.

Regarding (a): the standard numerical analysis approach to numerical methods for the Helmholtz equation is to prove results about the convergence and conditioning of the methods as the number of degrees of freedom $N$ increases with $k$ fixed. In particular, the constants in the classical error estimates as $N \to \infty$ are not explicit in $k$. More recent work has sought to determine the dependence of these constants on $k$ and, more generally, determine how these methods perform as $k$ increases with $N$ either fixed or a prescribed function of $k$ (see, e.g., the recent review articles [22] and [10, §5, §6]).

Regarding (b): the engineering rule of thumb is that conventional numerical methods for the Helmholtz equation need a fixed number of degrees of freedom per wavelength to maintain accuracy as $k$ increases (see, e.g., [38]); therefore, if the problem is $d$-dimensional, $N$ must grow like $k^d$ as $k \to \infty$ for volume discretizations and like $k^{d-1}$ as $k \to \infty$ for discretizations on the boundary of the domain. (The investigations addressing (a) discussed above actually show that in some cases $N$ must increase faster than $k^d$ to maintain accuracy due to the so-called pollution effect; see, e.g., [1, 22].) This growth of $N$ with $k$ puts many high-frequency problems out of the range of standard numerical methods, and thus there has been much recent interest in designing methods that reduce this growth (see, e.g., [10] §1 and the references therein).

The classical results about the asymptotics of solutions to the Helmholtz equation and the associated bounds on the inverse of the Helmholtz operator have played an essential role in many of the attempts at tackling one or the other of the tasks (a) and (b) above. Indeed, very roughly speaking, the knowledge of the large-$k$ asymptotics of the Helmholtz equation can be used for task (b) (leading to so-called hybrid numerical-asymptotic methods), and knowledge of the bounds on
the inverse of the Helmholtz operator can be used for task (a) (for more details, see [10, §3] and [10, §5, §6], respectively).

This paper considers a standard integral operator associated with the Helmholtz equation posed in the exterior of a bounded obstacle with Dirichlet boundary conditions (physically this corresponds to sound-soft acoustic scattering), and seeks to address a question that is relevant to both tasks (a) and (b) above. This oscillatory integral operator is often called the “combined potential operator” or “combined field operator,” and we denote it by $A'_{k,\eta}$, where $k$ is the wavenumber and $\eta$ is a parameter that is usually chosen to be proportional to $k$ (we define $A'_{k,\eta}$ and derive the associated integral equation in Section 1.1 below). We seek to prove that $A'_{k,\eta}$ is coercive as an operator on $L^2(\Gamma)$, where $\Gamma$ is the boundary of the obstacle, i.e., that there exists an $\alpha_{k,\eta} > 0$ such that

$$(1.1) \quad |(A'_{k,\eta} \phi, \phi)_{L^2(\Gamma)}| \geq \alpha_{k,\eta} \|\phi\|_{L^2(\Gamma)}^2 \quad \text{for all } \phi \in L^2(\Gamma),$$

at least for $k$ sufficiently large. The $k$- and $\eta$-subscripts on the coercivity constant $\alpha_{k,\eta}$ indicate that, if this constant exists, it might depend on $k$ and $\eta$; we see below that in some cases it can be independent of both. (Note that if $A'_{k,\eta}$ is coercive, then the Lax-Milgram theorem implies that $A'_{k,\eta}$ is invertible with $\|(A'_{k,\eta})^{-1}\| \leq 1/\alpha_{k,\eta}$, but the converse is not true; i.e., $A'_{k,\eta}$ can be invertible but not coercive.)

Establishing coercivity has two important consequences:

(i) It allows one to prove $k$-explicit error estimates when the integral equation involving $A'_{k,\eta}$ is solved numerically using the Galerkin method both for conventional methods (which use approximation spaces consisting of piecewise polynomials) and for hybrid numerical-asymptotic methods (in which the approximation space is designed using knowledge of the large-$k$ asymptotics). Note that establishing coercivity is currently the only known way to prove these error estimates for the hybrid methods.

(ii) It proves that the numerical range (also known as the field of values) of the operator is bounded away from zero. This fact, along with a $k$-explicit bound on $\|A'_{k,\eta}\|$, then gives a $k$-explicit bound on the number of GMRES iterations needed to achieve a prescribed accuracy when the integral equation is solved using the Galerkin method with piecewise-polynomial approximation spaces (and where GMRES is the generalized minimal residual method). Note that no such bounds are currently available in the literature.

(Both of these consequences of coercivity are discussed in more detail in Section 1.3 below.)

Although it is well-known that $A'_{k,\eta}$ is invertible for every $k > 0$, it is not a priori clear that $A'_{k,\eta}$ will be coercive. Indeed, the usual numerical analysis of Helmholtz problems (posed either in the domain or on the boundary) seeks to prove that the
relevant operator is coercive up to a compact perturbation (i.e., satisfies a Gårding inequality). However, in [19] $A'_{k,\eta}$ was proved to be coercive, with $\alpha_{k,\eta}$ independent of $k$, when $\Gamma$ is the circle (in two dimensions) or the sphere (in three dimensions), $\eta = k$, and $k$ is sufficiently large. Furthermore, numerical experiments conducted in [5] suggest that $A'_{k,\eta}$ is coercive for a wide variety of two-dimensional domains, with $\alpha_{k,\eta}$ independent of $k$, when $\eta = k$ and $k$ is sufficiently large (in particular, domains that are nontrapping).

When considering the question of whether or not $A'_{k,\eta}$ is coercive, it is instructive to also consider two other questions about $A'_{k,\eta}$, namely, how do $\|A'_{k,\eta}\|$ and $\|(A'_{k,\eta})^{-1}\|$ depend on $k$ (where $\|\cdot\|$ denotes the operator norm on $L^2(\Gamma)$)? Bounds on $\|A'_{k,\eta}\|$ that are sharp in their $k$-dependence for a wide variety of domains can be obtained just by using general techniques for bounding the norms of oscillatory integral operators; see [9], [10, §5.5], [56, §1.2]. In contrast, to obtain $k$-explicit bounds on $\|(A'_{k,\eta})^{-1}\|$ it is necessary to use the fact that $A'_{k,\eta}$ arises from solving boundary value problems (BVPs) for the Helmholtz equation, and convert the problem of finding a bound on $\|(A'_{k,\eta})^{-1}\|$ into bounding the exterior Dirichlet-to-Neumann map and the interior impedance-to-Dirichlet map for the Helmholtz equation. (Although it might seem strange that $\|(A'_{k,\eta})^{-1}\|$ depends on the solution operator to the interior impedance problem, it turns out that this interior problem can also be formulated as an integral equation involving $A'_{k,\eta}$, thus this dependence is natural.) Appropriate bounds on these interior and exterior Helmholtz problems can then be used to bound $\|(A'_{k,\eta})^{-1}\|$; see [10, theorem 2.33 and §5.6.1], [11], and [57, §1.3].

In contrast to the task of bounding $\|(A'_{k,\eta})^{-1}\|$, the task of proving that $A'_{k,\eta}$ is coercive apparently cannot be reformulated in terms of bounding the solutions of Helmholtz BVPs. In this paper, however, we show that this task can be tackled using identities for solutions of the Helmholtz equation originally introduced by Morawetz. (This builds on the earlier work of two of the authors and their collaborators in [58].) Recall that Morawetz showed in [46] that bounding the solution of the exterior Dirichlet problem could be converted (via her identities) into constructing an appropriate vector field in the exterior of the obstacle, and then such a vector field was constructed by Morawetz, Ralston, and Strauss for two-dimensional nontrapping domains in [48, §4]. (This bound on the solution is equivalent to bounding the exterior Dirichlet-to-Neumann map, and can also be used to show local energy decay of solutions of the wave equation.)

Here we convert the problem of proving that $A'_{k,\eta}$ is coercive into that of constructing a suitable vector field in both the exterior and the interior of the obstacle. In addition to needing a vector field in the interior as well as the exterior, the conditions that the vector field must satisfy for coercivity are stronger than those in [46] and [48]. Indeed, we prove that the conditions for coercivity cannot be satisfied if the obstacle is nonconvex, and then we construct a vector field satisfying these
conditions for smooth, convex obstacles with strictly positive curvature in both two and three dimensions.

1.1 Formulation of the Problem

Let \( \Omega_- \subset \mathbb{R}^d \), \( d = 2 \) or \( 3 \), be a bounded, connected Lipschitz open set such that the open complement \( \Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-} \) is connected. In what follows we use domain to mean a connected open set, and thus \( \overline{\Omega} \) is a Lipschitz domain.

Let \( \Gamma := \partial \Omega_- \) (so \( \Gamma = \partial \Omega_+ \) too). Let \( H_{\text{loc}}^1(\Omega_+) \) denote the set of functions, \( v \), such that \( v \) is locally integrable on \( \Omega_+ \) and \( \psi v \in H^1(\Omega_+) \) for every compactly supported \( \psi \in C^\infty(\overline{\Omega}_+) := \{ \psi|_{\overline{\Omega}_+} : \psi \in C^\infty(\mathbb{R}^d) \} \).

**Definition 1.1 (Sound-soft scattering problem).** Given \( k > 0 \) and an incident plane wave \( u^I(x) = \exp(ik\mathbf{x} \cdot \mathbf{a}) \) for some \( \mathbf{a} \in \mathbb{R}^d \) with \( |\mathbf{a}| = 1 \), find \( u^S \in C^2(\Omega_+) \cap H_{\text{loc}}^1(\Omega_+) \) such that the total field \( u := u^I + u^S \) satisfies

\[
Lu := \Delta u + k^2 u = 0 \quad \text{in} \ \Omega_+,
\]

\[
u = 0 \quad \text{on} \ \Gamma,
\]

and \( u^S \) satisfies the Sommerfeld radiation condition,

\[
\frac{\partial u^S}{\partial r}(x) - ik u^S(x) = o\left(\frac{1}{r^{(d-1)/2}}\right)
\]

as \( r := |x| \to \infty \), uniformly in \( \tilde{x} := x/r \).

It is well-known that the solution to this problem exists and is unique; see, e.g., [10, theorem 2.12].

Note that, although we are restricting our attention to the case where the incident field is a plane wave, the results of this paper also apply to scattering by other incident fields, for example those satisfying [10, def. 2.11], and also to the general exterior Dirichlet problem, i.e., given a function \( g_D \) on \( \Gamma \) (with suitable regularity), find \( u^S \) satisfying both the Helmholtz equation in \( \Omega_+ \) and the Sommerfeld radiation condition, and also such that \( u^S = g_D \) on \( \Gamma \).

The BVP in Definition 1.1 can be reformulated as an integral equation on \( \Gamma \) in two different ways. The first, the so-called direct method, uses Green’s integral representation for the solution \( u \), i.e.,

\[
u(x) = u^I(x) - \int_{\Gamma} \Phi_k(x, y) \frac{\partial u^S(y)}{\partial n(y)} \, ds(y), \quad x \in \Omega_+,
\]

where \( \partial/\partial n \) is the derivative in the normal direction, with the unit normal \( \mathbf{n} \) directed into \( \Omega_+ \), and \( \Phi_k(x, y) \) is the fundamental solution of the Helmholtz equation given by

\[
\Phi_k(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \quad d = 2, \quad \Phi_k(x, y) = \frac{e^{ik|x - y|}}{4\pi|x - y|}, \quad d = 3
\]
(note that to obtain (1.3) from the usual form of Green’s integral representation one must use the fact that $u^I$ is a solution of the Helmholtz equation in $\Omega_-$; see, e.g., [10, theorem 2.43]).

Taking the Dirichlet and Neumann traces of (1.3) on $\Gamma$ one obtains two integral equations for the unknown Neumann boundary value $\frac{\partial u}{\partial n}$:

\begin{align}
(1.4) & \quad S_k \frac{\partial u}{\partial n} = u^I, \\
(1.5) & \quad \left( \frac{1}{2} I + D'_k \right) \frac{\partial u}{\partial n} = \frac{\partial u^I}{\partial n},
\end{align}

where the integral operators $S_k$ and $D'_k$, the single-layer operator and the adjoint-double-layer operator, respectively, are defined for $\psi \in L^2(\Gamma)$ by

\begin{align}
(1.6) & \quad S_k \psi(x) = \int_{\Gamma} \Phi_k(x, y) \psi(y) \, ds(y), \\
(1.7) & \quad D'_k \psi(x) = \int_{\Gamma} \frac{\partial \Phi_k(x, y)}{\partial n(x)} \psi(y) \, ds(y), \quad x \in \Gamma.
\end{align}

Both integral equations (1.4) and (1.5) fail to be uniquely solvable for certain values of $k$ (for (1.4) these are the $k$ such that $k^2$ is a Dirichlet eigenvalue of the Laplacian in $\Omega_-$, and for (1.5) these are the $k$ such that $k^2$ is a Neumann eigenvalue). The standard way to resolve this difficulty is to take a linear combination of the two equations, which yields the integral equation

\begin{equation}
(1.8) \quad A'_{k, \eta} \frac{\partial u}{\partial n} = f,
\end{equation}

where

\begin{equation}
(1.9) \quad A'_{k, \eta} := \frac{1}{2} I + D'_k - i\eta S_k
\end{equation}

is the combined potential or combined field operator, with $\eta \in \mathbb{R} \setminus \{0\}$ the so-called coupling parameter, and

\[ f(x) = \frac{\partial u^I}{\partial n}(x) - i\eta u^I(x), \quad x \in \Gamma. \]

When $\Omega_-$ is Lipschitz, standard trace results imply that the unknown Neumann boundary value $\frac{\partial u}{\partial n}$ is in $H^{-1/2}(\Gamma)$. When $\Omega_-$ is $C^2$, elliptic regularity implies that $\frac{\partial u}{\partial n} \in L^2(\Gamma)$ (see, e.g., [23 §6.3.2, theorem 4]), but this is true even when $\Omega_-$ is Lipschitz via a regularity result of Nečas [49 §5.1.2], [40 theorem 4.24(ii)]. Therefore, even for Lipschitz $\Omega_-$ we can consider the integral equation (1.8) as an operator equation in $L^2(\Gamma)$, which is a natural space for the practical solution of second-kind integral equations since it is self-dual. It is well-known that, when $\eta \neq 0$, $A'_{k, \eta}$ is a bounded and invertible operator on $L^2(\Gamma)$ (see [10 theorem 2.27]).
Instead of using Green’s integral representation to formulate the BVP as an integral equation, one can pose the ansatz

\[ u^S(x) = \int_{\Gamma} \frac{\partial \Phi_k(x, y)}{\partial n(y)} \phi(y) ds(y) - i\eta \int_{\Gamma} \Phi_k(x, y) \phi(y) ds(y). \]

with the sought density \( \phi \in L^2(\Gamma) \) and \( \eta \in \mathbb{R} \setminus \{0\} \); this is the so-called indirect method. Imposing the boundary condition \( u^S = -u^I \) on \( \Gamma \) leads to the integral equation

\[ A_{k, \eta} \phi = -u^I, \]

where

\[ A_{k, \eta} := \frac{1}{2} I + D_k - i\eta S_k \]

and \( D_k \) is the double-layer operator, which is defined for \( \psi \in L^2(\Gamma) \) by

\[ D_k \psi(x) = \int_{\Gamma} \frac{\partial \Phi_k(x, y)}{\partial n(y)} \psi(y) ds(y), \quad x \in \Gamma. \]

The operators \( A_{k, \eta} \) and \( A'_{k, \eta} \) are adjoint with respect to the real-valued \( L^2(\Gamma) \) inner product, and then it is straightforward to show that, firstly, \( \| A_{k, \eta} \| = \| A'_{k, \eta} \| \) and \( \| (A_{k, \eta})^{-1} \| = \| (A'_{k, \eta})^{-1} \| \), where \( \| \cdot \| \) denotes the operator norm from (complex-valued) \( L^2(\Gamma) \) to itself, and, secondly, if one of \( A_{k, \eta} \) or \( A'_{k, \eta} \) is coercive, then so is the other (with the same coercivity constant); see [10, eqs. 2.37–2.40 and remark 2.24] for more details.

The main difference between the direct and indirect integral equations, (1.8) and (1.10), respectively, is that the physical meaning of the unknown is clear in the direct equation (it is the normal derivative of the total field) but not in the indirect equation (it turns out that \( \phi \) is the difference of traces of certain exterior and interior Helmholtz BVPs; see [10, p. 132]).

Both the operators \( A'_{k, \eta} \) and \( A_{k, \eta} \) involve the arbitrary coupling parameter \( \eta \). By proving bounds on \( \| A'_{k, \eta} \| \) and \( \| (A_{k, \eta})^{-1} \| \), one can show that, when \( k \) is large, the choice \( |\eta| \sim k \) is optimal, in that it minimizes the condition number of \( A'_{k, \eta} \) (and hence also of \( A_{k, \eta} \)); see [10] remark 5.1.

There are several different ways of solving integral equations such as (1.8) and (1.10), but in this paper we focus on the Galerkin method. Concentrating on the direct equation (1.8) and denoting \( \partial u/\partial n \) by \( v \), we have that solving (1.8) is equivalent to the variational problem

find \( v \in L^2(\Gamma) \) such that \( (A'_{k, \eta} v, \phi)_{L^2(\Gamma)} = (f, \phi)_{L^2(\Gamma)} \) for all \( \phi \in L^2(\Gamma) \).
(where \( (\psi, \phi)_{L^2(\Gamma)} = \int_{\Gamma} \psi \bar{\phi} \, ds \)). Given a finite-dimensional approximation space \( V_N \subseteq L^2(\Gamma) \) (with \( N \) being the dimension), the Galerkin method is

\[
\text{(1.11) find } v_N \in V_N \text{ such that } (A_{k,\eta}' v_N, \phi_N)_{L^2(\Gamma)} = (f, \phi_N)_{L^2(\Gamma)} \text{ for all } \phi_N \in V_N.
\]

If one can prove that \( A_{k,\eta}' \) is coercive (i.e., (1.1) holds), then the Lax-Milgram theorem and Céa’s lemma give the following error estimate:

\[
\text{(1.12) } \|v - v_N\|_{L^2(\Gamma)} \leq \left( \frac{\|A_{k,\eta}'\|}{\alpha_{k,\eta}} \right) \inf_{\phi_N \in V_N} \|v - \phi_N\|_{L^2(\Gamma)},
\]

and the Galerkin method is then said to be quasi-optimal. (If the left-hand side of (1.12) were equal to the best approximation error, \( \inf_{\phi_N \in V_N} \|v - \phi_N\|_{L^2(\Gamma)} \), then the method would be optimal; instead we have optimality up to a constant.)

1.2 What Is Known about the Coercivity of \( A_{k,\eta}' \)?

The only domains so far for which coercivity is completely understood are balls (i.e., \( \Gamma \) is a circle or sphere); this is because the operator \( A_{k,\eta}' \) acts diagonally in the bases of trigonometric polynomials (in two dimensions) and spherical harmonics (in three dimensions). For the circle, Domínguez, Graham, and the third author showed in [19] that if \( \eta = k \) then there exists a \( k_0 \) such that, for all \( k \geq k_0 \), (1.1) holds with \( \alpha_{k,\eta} = \frac{1}{2} \); for the sphere they proved that if \( \eta = k \), then (1.1) holds for sufficiently large \( k \) with

\[
\alpha_{k,\eta} \geq \frac{1}{2} - O\left( \frac{1}{k^{2/3}} \right).
\]

These proofs relied on bounding below the eigenvalues of \( A_{k,\eta}' \), which are combinations of Bessel and Hankel functions. (Note that \( A_{k,\eta}' \) is not invertible, and hence is not coercive, when both \( \eta \) and \( k \) equal 0. Therefore, if we take \( \eta = k \), we cannot hope for \( A_{k,\eta}' \) to be coercive for all \( k \geq 0 \)—only for \( k \) sufficiently large.)

Although nothing has been proved until now about the coercivity of \( A_{k,\eta}' \) on domains other than the circle or sphere, weaker results about the norm of \( (A_{k,\eta}')^{-1} \) can be used to deduce information about possible values of the coercivity constant \( \alpha_{k,\eta} \) by using the fact that if \( A_{k,\eta}' \) is coercive then

\[
\alpha_{k,\eta} \leq \| (A_{k,\eta}')^{-1} \|^{-1}
\]

(this follows from (1.1) using the Cauchy-Schwarz inequality). Furthermore, if a part of \( \Gamma \) is \( C^1 \), then

\[
\| (A_{k,\eta}')^{-1} \| \geq 2
\]

in both two and three dimensions [9, lemma 4.1] (this follows from the fact that \( S_k \) and \( D_{k}' \) are compact operators when \( \Gamma \) is \( C^1 \)) and hence, if \( \alpha_{k,\eta} \) exists,

\[
\alpha_{k,\eta} \leq \frac{1}{2};
\]

\[
\text{(1.13)}
\]
therefore the bound obtained on $\alpha_{k,\eta}$ for the circle in [19] is sharp. Examples of two-dimensional trapping domains where $\|(A'_{k,\eta})^{-1}\|$ grows either polynomially or exponentially in $k$ through some increasing sequence of wavenumbers can be found in [9, theorem 5.1] and [3, theorem 2.8] (for a summary of these results and an outline of the general argument, see [10, §5.6.2]). Therefore, if $A'_{k,\eta}$ is coercive for these domains, then $\alpha_{k,\eta}$ must decay either polynomially or exponentially as $k$ increases.

Betcke and the first author undertook a numerical investigation of coercivity by computing the numerical range (also known as the field of values) of $A'_{k,\eta}$, denoted by $\mathcal{W}(A'_{k,\eta})$, for various two-dimensional domains in [5]. Recall that \[ \mathcal{W}(A'_{k,\eta}) := \{(A'_{k,\eta}\phi, \phi)_{L^2(\Gamma)} : \phi \in L^2(\Gamma) \text{ with } \|\phi\|_{L^2(\Gamma)} = 1\}, \] and thus if $A'_{k,\eta}$ is coercive, then $\alpha_{k,\eta} = \text{dist}(\mathcal{W}(A'_{k,\eta}), 0)$. These experiments (all conducted with $\eta = k$) indicated that if $\Omega_+$ is trapping, then $A'_{k,k}$ is not coercive at values of $k$ close to the “wavenumber” of the cavity that traps waves, and if $\Omega_+$ is nontrapping, then $A'_{k,k}$ is coercive with $\alpha_{k,k}$ independent of $k$, as long as $k$ is sufficiently large.

It is interesting to note that, although changing $\eta$ from $k$ to $-k$ does not affect the bounds on $\|(A'_{k,\eta})^{-1}\|$ (since they depend on $|\eta|$; see [10, §5.6.1] and [57, §1.3]), it completely changes the coercivity properties of $A'_{k,\eta}$. Indeed, whereas $A'_{k,k}$ appears to be coercive when $\Omega_+$ is nontrapping and $k$ is sufficiently large, $A'_{k,-k}$ is not coercive when $\Gamma$ is the unit circle and $k \geq 1$. (This can be seen by plotting the eigenvalues of $A'_{k,-k}$, which are given explicitly in terms of Bessel and Hankel functions by, e.g., [10, eq. 5.20c], and noting that they encircle the origin; thus the fact that $\mathcal{W}(A'_{k,-k})$ is convex [5, prop. 3.2] implies that $A'_{k,-k}$ is not coercive.)

Finally, to give some indication of why proving that $A'_{k,\eta}$ is coercive is difficult, we note that it appears that $A'_{k,\eta}$ is a normal operator if and only if $\Omega_-$ is a ball (i.e., $\Gamma$ is a circle or sphere). Indeed, it is straightforward to prove that if $\Gamma$ is the circle or sphere, then $A'_{k,\eta}$ is normal (via the diagonalization in trigonometric polynomials or spherical harmonics). The numerical experiments in [5] suggest that the converse is true, and the analogue of this result for the operator $S_k$ was proved in [4, theorem 3.1]. It is well known that, although the spectrum determines the behavior of normal operators, this is not the case for nonnormal operators; see, e.g., [62].

1.3 Main Result and Its Consequences

In this paper we prove that $A'_{k,\eta}$ is coercive for smooth, convex domains in two or three dimensions when $\eta \geq k$ and $k$ is sufficiently large. More precisely:

**Theorem 1.2.** Let $\Omega_-$ be a convex domain in either two dimensions or three dimensions whose boundary, $\Gamma$, has strictly positive curvature and is both $C^3$ and
piecewise analytic. Then there exists a constant $\eta_0 > 0$ such that, given $\delta > 0$, there exists $k_0 > 0$ (depending on $\delta$) such that, for $k \geq k_0$ and $\eta \geq \eta_0 k$,

(1.14)  \[ \Re(A_k' \phi, \phi)_{L^2(\Gamma)} \geq \left( \frac{1}{2} - \delta \right) \|\phi\|_{L^2(\Gamma)}^2 \]  
for all $\phi \in L^2(\Gamma)$. (By the remarks in §1.1 the bound also holds with $A_k'$ replaced by $A_k$.)

Note that the inequality (1.14) implies that $\alpha_{k, \eta} \geq \left( \frac{1}{2} - \delta \right)$, and then this bound on the coercivity constant is effectively sharp by (1.13) above. In fact, the proof of Theorem 1.2 shows that, as $k \to \infty$,

(1.15a)  \[ \alpha_{k, \eta} \geq \frac{1}{2} - O\left( \frac{1}{k^{1/2}} \right) \]  
when $d = 2$,

(1.15b)  \[ \alpha_{k, \eta} \geq \frac{1}{2} - O\left( \frac{\log k}{k^{1/13}} \right) \]  
when $d = 3$.

In the rest of this paper, we call a convex domain with strictly positive curvature a uniformly convex domain (motivated by the fact that if a convex function has $D^2 f \geq \theta$, in the sense of quadratic forms, for some $\theta > 0$, then it is sometimes described as being uniformly convex; see, e.g., [23, p. 621]). In three dimensions, by strictly positive curvature we mean that both of the principal curvatures are strictly positive.

We now outline the two main consequences of Theorem 1.2. Both of these need an upper bound on the norm of $A_k'$ as an operator on $L^2(\Gamma)$. The currently best available bound when $\Omega_-$ is a uniformly convex domain satisfying the conditions of Theorem 1.2 is

(1.16)  \[ \|A_k'\| \lesssim 1 + k^{(d-1)/2} \left( 1 + \frac{[\eta]}{k} \right) \]

for all $k > 0$ and $\eta \in \mathbb{R}$; see [9, theorem 3.6]. Note that we are using the notation $A \lesssim B$ if $A \leq cB$ with $c$ independent of $k$ and $\eta$. In fact, the bound (1.16) is valid when $\Omega_-$ is a general Lipschitz domain and appears not to be sharp when $\Omega_-$ is uniformly convex. Indeed, when $\Gamma$ is the circle or sphere, $\|A_k'\| \lesssim k^{1/3}$ when $\eta \sim k$; see [10, §5.4–5.5] for more details.

**$k$-Explicit Quasi-Optimality of the Galerkin Method for Any Finite-Dimensional Subspace.** The main application of Theorem 1.2 is that it implies that the Galerkin method (1.11) is quasi-optimal for any finite-dimensional subspace. Indeed, combining the result (1.14) with the estimates (1.16) and (1.12), we see that if $\Omega_-$ satisfies the conditions of Theorem 1.2 and the direct integral equation (1.8) is solved via (1.11) with $\eta$ chosen so that $\eta_0 k \leq \eta \leq k$, then, for all $k \geq k_0$,

(1.17)  \[ \|v - v_N\|_{L^2(\Gamma)} \lesssim k^{(d-1)/2} \inf_{\phi_N \in V_N} \|v - \phi_N\|_{L^2(\Gamma)}, \]
where $k_0$ and $\eta_0$ are as in Theorem 1.2 and $v := \partial u / \partial n$. An analogous result also holds for the indirect equation (1.10).

The key point is that the quasi-optimality (1.17) is established for any subspace $V_N \subset L^2(\Gamma)$ without any constraint on the dimension $N$. In contrast, the usual approach to the numerical analysis of Helmholtz problems is to prove coercivity up to a compact perturbation (i.e., a Gårding inequality). Even when these arguments can be made explicit in $k$, they yield quasi-optimality only when $N$ is larger than some $k$-dependent threshold. For example, if the integral equation (1.8) is solved using the Galerkin method with $V_N$ consisting of piecewise polynomials of degree $\leq p$ for some fixed $p \geq 0$ on shape regular meshes of diameter at most $h$ (so $N \sim h^{-(d-1)}$), then a $k$-explicit version of the classical compact perturbation argument shows that, if $\Omega_-$ is a $C^2$, star-shaped, two- or three-dimensional domain, then

$$\inf_{\phi_N \in V_N} \| v - \phi_N \|_{L^2(\Gamma)} \lesssim \inf_{\phi_N \in V_N} \| v - \phi_N \|_{L^2(\Gamma)}$$

provided that $hk^{(d+1)/2} \lesssim 1$; see [28, theorem 1.6]. The fact that the mesh threshold in (1.18) is more stringent than the $hk^{1/2}$ rule of thumb can be understood as a consequence of the pollution effect (see, e.g., [22, §1]).

To compare the error estimates (1.17) and (1.18) we need to understand how the best approximation error, $\inf_{\phi_N \in V_N} \| v - \phi_N \|_{L^2(\Gamma)}$, depends on $h$ and $k$. It is generally believed that this is $\lesssim \| v \|_{L^2(\Gamma)}$ if $hk \lesssim 1$, and this has been proved if $\Omega_-$ is a $C^\infty$, uniform convex, two-dimensional domain. Indeed, the results about the asymptotics of $v := \partial u / \partial n$ for this type of domain in, e.g., [42], adapted for a numerical analysis context in [19, theorem 5.4, cor. 5.5], imply that

$$\inf_{\phi_N \in V_N} \| v - \phi_N \|_{L^2(\Gamma)} \lesssim h k^{d/2} \| v \|_{L^2(\Gamma)}$$

see [28, theorem 1.2]. Using this bound in both (1.17) and (1.18), we see that both the estimate from coercivity and the estimate from the $k$-explicit compact perturbation argument show that, in the two-dimensional case, the relative error $\| v - v_N \|_{L^2(\Gamma)} / \| v \|_{L^2(\Gamma)}$ is bounded independently of $k$ when $hk^{3/2} \lesssim 1$.

In summary, since any quasi-optimality estimate for piecewise-polynomial subspaces will ultimately be considered under some $k$-dependent threshold for $N$ (coming from controlling the best approximation error), the advantage of the “no-threshold” quasi-optimality given by coercivity over the “threshold” quasi-optimality (usually called “asymptotic” quasi-optimality) of the compact perturbation arguments is not felt for these subspaces.

The advantage of coercivity is crucial, however, when seeking to establish quasi-optimality of hybrid numerical-asymptotic methods. As discussed above, there has been much recent research in designing $k$-dependent approximation spaces that incorporate the oscillation of the solution, with the result that the best approximation error for these spaces either is bounded or grows mildly as $k$ increases with $N$ fixed. If one applies the standard compact-perturbation arguments to try to establish quasi-optimality of Galerkin methods using these subspaces, it is not at all
clear how the threshold for quasi-optimality depends on \( k \) and whether \( N \) will ever be large enough to exceed this threshold (since the whole point of these methods is to keep \( N \) relatively small). Establishing coercivity, however, bypasses these difficulties.

For example, a \( k \)-dependent approximation space \( \mathcal{V}_{N,k} \) for sound-soft scattering by smooth, uniformly convex obstacles in two dimensions was designed in [19] by using knowledge of the \( k \to \infty \) asymptotics. The space \( \mathcal{V}_{N,k} \) divides \( \Gamma \) into the illuminated zone, the shadow zone, and two shadow boundary zones. The solution to the integral equation \( v := \partial u / \partial n \) is then approximated by an oscillatory factor multiplied by a polynomial of degree \( N \) in the illuminated zone and the two shadow boundary zones, and by zero in the shadow zone. Combining Theorem 1.2 with results about the best approximation error in \( \mathcal{V}_{N,k} \) proved in [19, theorem 6.7] (using results from [42] about the asymptotics of \( v \)), we obtain the following error estimate for the Galerkin method using \( \mathcal{V}_{N,k} \):

**Theorem 1.3.** Let \( \Omega_\omega \) be a uniformly convex, two-dimensional domain whose boundary is \( C^\infty \) and piecewise analytic. Suppose that the sound-soft scattering problem of Definition 1.1 is solved with the Galerkin method using the combined potential integral equation (1.8) and the hybrid approximation space introduced in [19] (and denoted by \( \mathcal{V}_{N,k} \) above).

Let \( N \) be the degree of the polynomials used in each of the three zones (so \( N \) is proportional to the total number of degrees of freedom of the method). Then there exist \( \eta_0, k_0, \delta, \) and \( c_0 \), all greater than 0, such that, if the coupling parameter \( \eta \) is chosen so that \( \eta \leq \eta_0 \leq \delta \), then

\[
\| v - v_N \|_{L^2(\Gamma)} \leq k^{19/18} \left\{ \left( \frac{k^{1/9}}{N} \right)^{N+1} + k^{4/9} \exp(-c_0 k^{\delta}) \right\}
\]

for all \( k \geq k_0 \). Therefore, provided that \( N \) grows like \( k^{1/9 + \varepsilon} \) for some \( \varepsilon > 0 \), the error is bounded as \( k \to \infty \).

Similar ideas were used in [26] to design a \( k \)-dependent approximation space for scattering by smooth, uniformly convex obstacles in three dimensions. Theorem 1.2 along with a bound on the best approximation error for this subspace, can then also give rigorous error estimates for this method.

**Bounding the Numerical Range of \( A_{k,\eta}^l \) and the Associated \( k \)-Explicit Bounds on GMRES Iterations.** Whereas the first consequence of coercivity (\( k \)-explicit quasi-optimality for any approximation space) is more relevant for the Galerkin method with hybrid, \( k \)-dependent subspaces, the second consequence is more applicable to the Galerkin method with conventional piecewise polynomial subspaces. In this case, the Galerkin matrices will be of size \( N \times N \) and, with \( N \) having to grow at least like \( k^{d-1} \) to maintain accuracy, the associated linear systems will usually be solved using iterative methods such as GMRES. (Note that the hybrid subspaces are specifically designed so that \( N \) grows mildly
with $k$, and thus, for geometries where these subspaces are available, the linear systems can be solved using direct, as opposed to iterative, methods.)

Although nothing has yet been proven about how GMRES behaves when applied to linear systems resulting from Galerkin discretizations of $A'_{k,n}$, it is usually believed that the number of iterations needed to achieve a prescribed accuracy must grow mildly with $k$, e.g., like $k^a$ for some $0 < a < 1$.\footnote{We could not find any relevant numerical results for Galerkin discretizations of $A'_{k,n}$ in the literature; however, results for Nyström discretizations of both the analogous operator for the Neumann problem and modifications of this operator that make it a compact perturbation of the identity on smooth domains can be found in \[21\] (see also \[20\] theorem 3.3)) and appears in this particular form in, e.g., \[2\] eq. 1.2].

**Theorem 1.4.** If the matrix equation $Av = f$ is solved using GMRES, then, for $m \in \mathbb{N}$, the $m^{th}$ GMRES residual, $r_m := Av_m - f$, satisfies

\[
(1.19) \quad \frac{\|r_m\|_2}{\|r_0\|_2} \leq \sin^m \beta \quad \text{where} \quad \cos \beta = \frac{\text{dist}(0, W(A))}{\|A\|_2},
\]

and where $W(A) := \{ (Av, v) : v \in \mathbb{C}^N, \|v\|_2 = 1 \}$ is the numerical range of $A$ and $\| \cdot \|_2$ denotes the $l_2$ (i.e., euclidean) vector norm.

Coercivity of the operator $A'_{k,n}$ implies that the numerical range of the associated Galerkin matrix $A$ is bounded away from zero, and thus allows us to obtain $k$-explicit bounds on the number of GMRES iterations needed to solve $Av = f$. Indeed, consider the $h$-version of the Galerkin method; i.e., $\mathcal{V}_N \subset L^2(\Gamma)$ is the space of piecewise polynomials of degree $\leq p$ for some fixed $p \geq 0$ on quasi-uniform meshes of diameter $h$, with $h$ decreasing to 0 (thus $N \sim h^{-(d-1)}$). Let $\mathcal{V}_N = \text{span}\{\phi_i : i = 1, \ldots, N\}$, let $v_N \in \mathcal{V}_N$ be equal to $\sum_{j=1}^N V_j \phi_j$, and define $v \in \mathbb{C}^N$ by $v := (V_j)_{j=1}^N$. Then, with $A_{ij} := (A'_{k,n} \phi_j, \phi_i)_{L^2(\Gamma)}$ and $f_i := (f, \phi_i)_{L^2(\Gamma)}$, the Galerkin method (1.11) is equivalent to solving the linear system $Av = f$.

If $A'_{k,n}$ is coercive with coercivity constant $\alpha_{k,n}$, then, combining this property with the boundedness of $A'_{k,n}$, we have that

\[
(1.20) \quad |(Au, v)_2| \lesssim \|A'_{k,n}\| h^{d-1} \|u\|_2 \|v\|_2 \quad \text{and} \quad |(Av, v)_2| \gtrsim \alpha_{k,n} h^{d-1} \|v\|_2^2
\]

for all $u, v \in \mathbb{C}^N$, where we have used the bound $\|v_N\|_{L^2(\Gamma)} \sim h^{d-1} \|v\|_2$ (see, e.g., \[55\] cor. 5.3.28)). The two bounds in (1.20) imply that the ratio $\cos \beta$ in (1.19)

\[
\cos \beta \gtrsim \frac{\alpha_{k,n}}{\|A'_{k,n}\|},
\]
and then Theorem 1.4 implies that, given \( \varepsilon > 0 \), there exists a \( C \), independent of \( k, \eta, \) and \( \varepsilon \), such that

\[
(1.21) \quad \text{if } m \geq C \left( \frac{\| A_{k,\eta}' \|}{\alpha_{k,\eta}} \right)^2 \log \left( \frac{1}{\varepsilon} \right) \text{ then } \| r_m \| / \| r_0 \| \leq \varepsilon.
\]

If \( \Omega_- \) satisfies the conditions of Theorem 1.2 and we take \( \eta \) as prescribed in that theorem, then \( \alpha_{k,\eta} \gtrsim 1 \) (and we know that this bound is sharp in its \( k \)-dependence from (1.13)). Whether the bound (1.21) tells us anything practical about \( m \) then rests on the \( k \)-explicit bounds for \( \| A_{k,\eta}' \| \) and their sharpness. (In the rest of this discussion we assume that \( \eta \) is taken so that \( \eta_0 k \leq \eta \leq k \).)

Using the upper bound on \( \| A_{k,\eta}' \| \) (1.16) in the bound (1.21), we find that choosing \( m \) so that \( m \gtrsim k^{d-1} \) is sufficient for \( \| r_m \| / \| r_0 \| \) to be bounded independently of \( k \) as \( k \) increases. However, \( N \) will be either of order \( k^{d-1} \) (if a fixed number of degrees of freedom per wavelength are chosen, i.e., \( h k \lesssim 1 \)) or of order \( k^{(d+1)(d-1)/2} \) (if we take \( h k^{(d+1)/2} \lesssim 1 \) to be sure of eliminating the pollution effect by (1.18)). Therefore, since GMRES always converges in at most \( N \) steps (in exact arithmetic), the bound \( m \gtrsim k^{d-1} \) either doesn’t tell us anything about the \( k \)-dependence of the number of iterations or is very pessimistic.

Nevertheless, since the bound (1.16) on \( \| A_{k,\eta}' \| \) appears not to be sharp when \( \Omega_- \) is smooth and uniformly convex, there is hope that more practical bounds on \( m \) can be obtained. Indeed, if \( \Gamma \) is a sphere, then \( \| A_{k,\eta}' \| \lesssim k^{1/3} \), and therefore (1.21) gives \( m \gtrsim k^{2/3} \). Since \( N \) will be at least proportional to \( k^2 \) in this case, this bound on \( m \) is now nontrivial, and comes much closer to proving the mild growth observed in practice.

### 1.4 The Classical Method of "Transferring" Coercivity Properties of the PDE to Boundary Integral Operators

The method used to prove the main result (Theorem 1.2) is closely linked to a well-established idea in the theory of boundary integral equations, namely that coercivity properties of the weak form of the PDE can be “transferred” to the associated boundary integral operators. This idea was introduced for first-kind integral equations independently by Nédélec and Planchard [51], Le Roux [34], and Hsiao and Wendland [31], and for second-kind equations by Steinbach and Wendland [60]. We briefly recap this idea here, and then explain in Section 1.5 how it can be modified to prove Theorem 1.2.

For the Helmholtz equation posed in a bounded domain \( D \) (with outward-pointing unit normal vector \( v \)), the weak form of the PDE is based on Green’s identity integrated over \( D \):

\[
(1.22) \quad - \int_D \bar{v} \mathcal{L} u \, dx = \int_D (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}) \, dx - \int_{\partial D} \frac{\partial u}{\partial \nu} \, d\sigma.
\]
(recall that $L = \Delta u + k^2 u$). The fact that the volume terms on the right-hand side of (1.22) are single-signed when $k = 0$ and $v = u$ means that the standard variational formulations of Laplace’s equation are coercive. The fact that the volume terms are not single-signed when $k > 0$ and $v = u$ means that the standard variational formulations of the Helmholtz equation are not coercive when $k$ is large, only coercive up to a compact perturbation (see, e.g., [44, §1.1]).

Let $\Omega_-$ be as in §1.1 (i.e., $\Omega_-$ is bounded and $\Omega_+ := \mathbb{R}^d \setminus \Omega_-$ is connected); the following argument is valid when $\Omega_-$ is Lipschitz, but we ignore the technicalities needed in this case. Given $2 \in L^2(\Gamma)$, let $u$ be the single-layer potential $S_k$ with density $2 \in L^2(\Gamma)$. Then $L u = 0$ in $\Omega_- \cup \Omega_+$, and the following jump relations hold on $\Gamma$:

$$
(1.23) \quad u_\pm(x) = S_k \phi(x) \quad \text{and} \quad \frac{\partial u_\pm}{\partial n}(x) = \left( \mp \frac{1}{2} I + D_k' \right) \phi(x) \quad \text{for } x \in \Gamma
$$

(where $S_k$ and $D_k'$ are defined by (1.6) and (1.7), respectively).

Let $B_R := \{ |x| < R \}$ and apply Green’s identity (1.22) with $v = u$, first with $D = \Omega_-$, then with $D = \Omega_+ \cap B_R$ (with $R > 0$ chosen large enough so that $\Omega_- \subset B_R$), and then add the resulting two equations. Using the jump relations (1.23), we find that

$$
(1.24) \quad (S_k \phi, \phi)_{L^2(\Gamma)} = \int_{\Omega_+ \cap B_R \cup \Omega_-} (|\nabla u|^2 - k^2 |u|^2) \, dx - \int_{\partial B_R} \frac{\partial u}{\partial r} \, ds.
$$

This last equation holds when $\phi \in H^{-1/2}(\Gamma)$ if the left-hand side is replaced by $\langle S_k \phi, \phi \rangle_\Gamma$, where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.

We now seek to relate the terms on the right-hand side of (1.24) to $\| \phi \|^2_{H^{-1/2}(\Gamma)}$, ideally proving that they are $\gtrsim \| \phi \|^2_{H^{-1/2}(\Gamma)}$, which would show that $S_k$ is coercive as a mapping from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

First consider the case when $k = 0$, i.e., the PDE is Laplace’s equation and $d = 3$ (the case $d = 2$ for Laplace’s equation is more complicated because the fundamental solution does not decay at infinity). In this case $u(x) = O(1/r)$ and $\nabla u(x) = O(1/r^2)$ as $r := |x| \to \infty$, and thus

$$
(1.25) \quad \int_{\partial B_R} \frac{\partial u}{\partial r} \, ds \to 0 \quad \text{as } R \to \infty.
$$
The definition of $\partial u / \partial n$ in $H^{-1/2}(\Gamma)$ (which is essentially Green’s identity; see, e.g., [40, lemma 4.3]) implies that

$$(1.26) \quad \int_{\Omega_\pm} |\nabla u|^2 \, dx \geq \left\| \frac{\partial u_\pm}{\partial n} \right\|_{H^{-1/2}(\Gamma)}^2 ;$$

see, e.g., [59, cor. 4.5]. The second jump relation in (1.23) implies that

$$(1.27) \quad \| \phi \|_{H^{-1/2}(\Gamma)}^2 \leq \left\| \frac{\partial u_+}{\partial n} \right\|_{H^{-1/2}(\Gamma)}^2 + \left\| \frac{\partial u_-}{\partial n} \right\|_{H^{-1/2}(\Gamma)}^2 ,$$

and so, using (1.25), (1.26), and (1.27) in (1.24), we obtain that

$$(S_0 \phi, \phi)_\Gamma \geq \| \phi \|_{H^{-1/2}(\Gamma)}^2 \quad \text{for all } \phi \in H^{-1/2}(\Gamma) .$$

In summary, we have just used Green’s identity to prove that the Laplace single-layer operator in three dimensions is coercive as a mapping from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ (i.e., we “transferred” the coercivity of the weak form of the PDE to the integral operator). A slightly more complicated argument yields the analogous result in two dimensions (see [59, theorem 6.23] and [40, theorem 8.16]), and repeating the same argument with $u$ equal to the double-layer potential yields an analogous result for the hypersingular operator as a mapping from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ (after its nonzero kernel is quotiented out; see [59, theorem 6.24], [55, theorem 3.5.3], and [40, theorem 8.21]). Furthermore, using these results Steinbach and Wendland showed that $\frac{1}{2} I - D_0'$ is coercive on $H^{-1/2}(\Gamma)$ in the sense that

$$\left( \left( \frac{1}{2} I - D_0' \right) \phi, S_0 \phi \right)_\Gamma \geq \| \phi \|_{H^{-1/2}(\Gamma)}^2 \quad \text{for all } \phi \in H^{-1/2}(\Gamma) ,$$

and that $\frac{1}{2} I - D_0$ is coercive on $H^{1/2}(\Gamma)$; analogous results also hold for $\frac{1}{2} I + D_0'$ and $\frac{1}{2} I + D_0$ after their nonzero kernels are quotiented out (see [60, theorems 3.1 and 3.2] and [32, theorem 5.6.11]). See [17] (in particular [17, theorems 1 and 2]) for an insightful overview of all these results.

When we try to repeat this argument for $k > 0$, we run into two difficulties:

(i) the integral over $\partial B_R$ does not tend to 0 as $R \to \infty$, and

(ii) the volume terms in Green’s identity (1.22) are not single-signed when $v = u$.

Indeed, if $u = S_k \phi$ then $u$ satisfies the radiation condition (1.2) and one can then show that, as $R \to \infty$,

$$(1.28) \quad \Re \int_{\partial B_R} \bar{u} \frac{\partial u}{\partial r} \, ds \to 0 \quad \text{and} \quad \Im \int_{\partial B_R} \bar{u} \frac{\partial u}{\partial r} \, ds \to k \int_{\partial S^d} |f_1(\SK)|^2 \, ds ,$$
where \( f_1(\mathbf{x}) \) is the far-field pattern of \( u \) and \( \mathbb{S}^{d-1} \) is the \( d \)-dimensional unit sphere. Letting \( R \to \infty \) in (1.24) and using these limits, we find that

\[
\Re (S_k \phi, \phi)_\Gamma = \int_{\Omega_+ \cup \Omega_-} (|\nabla u|^2 - k^2 |u|^2) \, dx \quad \text{and}
\]

\[
\Im (S_k \phi, \phi)_\Gamma = k \int_{\mathbb{S}^{d-1}} |f_1(\mathbf{x})|^2 \, ds
\]

for all \( \phi \in H^{-1/2}(\Gamma) \). Therefore, considering only \( \Re (S_k \phi, \phi)_\Gamma \) bypasses (for now) the difficulty (i) above. The jump relations (1.23) again imply the bound (1.27), so all we need to do is bound below by \( \| \partial u_\pm / \partial n \|_{H^{-1/2}(\Gamma)}^2 \) the volume terms in (1.29). However, the analogue for \( k > 0 \) of the bound (1.26) in \( \Omega_- \) now contains \( k^2 \int_{\Omega_-} |u|^2 \, dx \) on the left-hand side, so the sign-indefiniteness of the volume terms in (1.29) means they cannot be bounded below by \( \| \partial u_- / \partial n \|_{H^{-1/2}(\Gamma)}^2 \). The analogue for \( k > 0 \) of the bound (1.26) in \( \Omega_+ \) is more complicated; it shares the problem of the bound in \( \Omega_- \) just described and, additionally, the fact that the integral over \( \partial B_R \) in Green’s identity does not tend to zero as \( R \to \infty \).

Ultimately, all one can prove in the Helmholtz case is that there exists a compact operator \( T_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) such that

\[
\Re ((S_k + T_k) \phi, \phi)_\Gamma \geq \| \phi \|_{H^{-1/2}(\Gamma)}^2 \quad \text{for all } \phi \in H^{-1/2}(\Gamma),
\]

that is, \( S_k \) is coercive up to a compact perturbation (i.e., satisfies a Gårding inequality); see, e.g., [16, theorem 2], [32, theorem 5.6.8], or [40, theorem 7.6] for the details.

In summary, using the ideas sketched above, the coercivity properties of the weak form of the PDE, i.e., coercivity for Laplace’s equation and coercivity up to a compact perturbation for the Helmholtz equation, can be “transferred” to the first- and second-kind boundary integral operators for these PDEs.

### 1.5 Modifying the Classical Method Using Morawetz’s Identities

The previous subsection showed that there are two reasons why the classical “transfer of coercivity” method only proves coercivity up to a compact perturbation for the Helmholtz boundary integral operators, as opposed to proving coercivity for the Laplace ones:

1. the volume terms of Green’s identity (1.22) are not single-signed when \( v = u \), and
2. when Green’s identity is applied in \( \Omega_+ \cap B_R \) with \( v = u \) and \( u \) satisfying the radiation condition (1.2), the integral over \( \partial B_R \) does not tend to 0 as \( R \to \infty \).
This paper uses the idea, first introduced in [58], to replace Green’s identity in the argument of the previous section with another identity for solutions of the Helmholtz equation for which the problems outlined in (1) and (2) above do not apply.

Recall that Green’s identity arises from multiplying \( Lu \) by \( v \). The multiplier \( rM_u \), where 
\[
M_u := \frac{x}{r} \cdot \nabla u - ik u + \frac{d - 1}{2r} u
\]
and \( r := |x| \), was introduced by Morawetz and Ludwig in [47]. In that paper, the resulting identity,
\[
(1.31)
\]
\[
2\Re((rM_u Lu) = \nabla \cdot [2\Re(rM_u \nabla u) + (k^2 |u|^2 - |\nabla u|^2)x]
- (|\nabla u|^2 - |ur|^2) - |ur - ik u|^2.
\]
was used to bound the Dirichlet-to-Neumann map for the Helmholtz equation in the exterior of a star-shaped domain (and it can also be used to bound the energy norm of the solution of the exterior Dirichlet problem in this class of domains). This is possible because

1. the nondivergence terms on the right-hand side of (1.31) are single-signed, and
2. when the identity (1.31) is integrated over \( \Omega_+ \cap B_R \), the integral over \( \partial B_R \) tends to 0 as \( R \to \infty \) if \( u \) satisfies the radiation condition (1.2).

(To understand where the star-shapedness requirement comes from, note that when we integrate (1.31) over \( \Omega_+ \cap B_R \), we get a surface integral on \( \Gamma \) involving \( x \cdot n(x) \), where \( n(x) \) is the unit normal vector on \( \Gamma \) pointing into \( \Omega_+ \). It turns out that we need \( x \cdot n(x) > 0 \) for all \( x \in \Gamma \) for the bounds to hold, and this means that \( \Omega_+ \) must be star-shaped.)

Repeating the transfer-of-coercivity argument reviewed in Section 1.4 but with Green’s identity (1.22) replaced by the integrated form of the Morawetz-Ludwig identity (1.31), we obtain that
\[
(1.32)
\]
\[
\Re((x \cdot n) D_k' + x \cdot \nabla \Gamma S_k - i\eta S_k(\phi, \phi)_{L^2(\Gamma)} \geq 0
\]
for all \( k > 0 \) and \( \phi \in L^2(\Gamma) \) if \( \eta = kr + i(d - 1)/2 \) (where \( \nabla \Gamma \) in (1.32) is the surface gradient on \( \Gamma \)) [58]. This inequality shows that the integral operator
\[
(1.33)
\]
\[
\mathscr{A}_k := (x \cdot n) \left( \frac{1}{2} I + D_k' \right) + x \cdot \nabla \Gamma S_k - i\eta S_k
\]
is coercive as an operator on \( L^2(\Gamma) \) if \( \Omega_+ \) is a star-shaped Lipschitz domain and \( \eta \) is chosen as above. Using Green’s integral representation, one can show that
\[
(1.34)
\]
\[
\frac{\partial u}{\partial n} = x \cdot \nabla u I - i\eta u I,
\]
and so the operator \( \mathscr{A}_k \) can be used to solve the exterior Dirichlet problem. Note that if \( \Gamma \) is the unit circle or sphere then, on \( \Gamma \), \( x = n(x) \), and so \( \mathscr{A}_k = A_k' \).
COERCIVITY OF INTEGRAL EQUATIONS

Therefore, the coercivity of the so-called “star-combined” operator \( \mathcal{A}_k \) gives an alternative proof of the coercivity of \( A'_{k,\eta} \) on the circle and sphere (see [58, cor. 4.8]).

The main idea of this paper is to use the more general multiplier

\[
Zu = Z \cdot \nabla u - ik\beta u + \alpha u,
\]

essentially introduced by Morawetz in [46], where \( Z(x) \) is a vector field, and \( \beta(x) \) and \( \alpha(x) \) are scalar fields. Replacing the identity (1.31), coming from the multiplier \( r \mathcal{M}u \), by the more general identity coming from the multiplier \( Zu \), and repeating the argument that led to (1.32), we find in Section 3 that, if \( Z \) is continuous across \( \Gamma \),

\[
\mathfrak{R}(\partial_i Z_j(x) \xi_i \bar{\xi}_j) \geq 0 \quad \text{for all} \quad \xi \in \mathbb{C}^d \quad \text{and} \quad x \in \Omega_- \cup (\Omega_+ \cap B_R), \quad \text{and} \quad \eta \geq k, \quad \text{then}
\]

\[
(1.36) \quad \mathfrak{R}(\langle (Z \cdot n) D'_k + Z \cdot \nabla \Gamma S_k - i\eta S_k \rangle \phi, \varphi)_{L^2(\Gamma)} \geq - o(1) \| \phi \|^2_{L^2(\Gamma)} \quad \text{as} \quad k \to \infty.
\]

Since \( \nabla \Gamma S_k \) is a vector-valued operator that is tangent to \( \Gamma \), if \( Z \) is a constant multiple of \( n \) on \( \Gamma \) (and the condition on the derivative of \( Z \) in the domain is satisfied), then the inequality (1.36) proves that \( A'_{k,\eta} \) is coercive.

### 1.6 Vector-Field Conditions for Coercivity

The method outlined in Section 1.5 above shows that \( A'_{k,\eta} \) is coercive, for \( \eta \geq k \) and \( k \) sufficiently large, if there exists a vector field, \( Z \), defined in \( \Omega_- \) and \( \Omega_+ \cap B_R \) for some \( R > 0 \) such that

(1) \( Z \) and \( \nabla \cdot Z \) are continuous across \( \Gamma \),

(2) \( Z = C \Gamma n \) on \( \Gamma \) for some constant \( C \Gamma > 0 \),

(3) \( Z(x) = x \) in a neighborhood of \( \partial B_R \), and

(4) \( \mathfrak{R}(\partial_i Z_j(x) \xi_i \bar{\xi}_j) \geq 0 \quad \text{for all} \quad \xi \in \mathbb{C}^d \quad \text{and} \quad x \in \Omega_- \cup (\Omega_+ \cap B_R). \)

(For simplicity, we have ignored the smoothness requirements on \( Z \) at this stage; see Section 3.1 for the details.) These vector-field conditions are similar to those obtained by Morawetz to prove a bound on the energy norm of the solution to the Dirichlet problem for the Helmholtz equation in \( \Omega_+ \) (i.e., a local resolvent estimate; see [46, eq. 1.3] and [48, eq. 4.2]). However, Morawetz needed a vector field only in \( \Omega_+ \cap B_R \), satisfying conditions 3 and 4 above, and satisfying the weaker condition than 2 that \( Z \cdot n > 0 \) on \( \Gamma \). Morawetz, Ralston, and Strauss then showed in [48, §4] that such a vector field exists when \( \Omega_+ \) is a two-dimensional nontrapping domain.

From one perspective it is clear why we need a vector field in both \( \Omega_+ \cap B_R \) and \( \Omega_- \) to prove that \( A'_{k,\eta} \) is coercive: following the method outlined in Section 1.5 we applied the identity coming from the \( Zu \) multiplier (1.35) in both \( \Omega_+ \cap B_R \) and \( \Omega_- \). From another perspective, however, it is natural to ask the question: since the scattering problem that we are trying to solve is posed only in \( \Omega_+ \), why should \( \Omega_- \) be involved? The fact that we need a vector field in \( \Omega_- \) as well as
in \( \Omega_+ \cap B_R \) becomes clear when we recall that the integral operators \( A'_{k,\eta} \) and \( A_{k,\eta} \), in addition to being able to solve the exterior Dirichlet problem (i.e., the Helmholtz equation posed in \( \Omega_- \) with boundary condition \( \partial u / \partial n - \eta u = g \) on \( \Gamma \) for some \( g \in L^2(\Gamma) \) and \( \eta \in \mathbb{R} \setminus \{0\} \)). Indeed, the operator \( A_{k,\eta} \) arises from the direct formulation of the interior impedance problem (see \cite{10} theorem 2.30), and the operator \( A'_{k,\eta} \) arises from an indirect formulation of the interior impedance problem (assuming that \( u = S_k \phi \) for some \( \phi \in L^2(\Gamma) \)).

Returning to the conditions for coercivity, 1–4 above, we show in Section 5 below that if \( \Omega_- \) is nonconvex then there does not exist a \( Z \) satisfying these conditions; indeed for these geometries one can reach a contradiction between the nonnegativity condition on \( \partial_i Z_j \) in \( \Omega_+ \) and the condition that \( Z = C_\Gamma n \) on \( \Gamma \).

If \( Z = \nabla \phi \) for some \( \phi \), then the nonnegativity condition, 4, becomes the requirement that \( \phi \) is convex. In Section 4 we construct a \( Z \) satisfying conditions 1–4 above (by constructing a suitable \( \phi \)) when \( \Omega_- \) is a uniformly convex, two- or three-dimensional domain with \( \Gamma \) both \( C^3 \) and piecewise analytic, thus proving Theorem 1.2. The main idea of the construction is that \( \pm \text{dist}(x, \Gamma) \) is convex in \( \Omega_\pm \) if \( \Omega_- \) is convex (see, e.g., \cite{54} pp. 28, 34) and its gradient is the normal vector \( n \) on \( \Gamma \). There are then three issues:

(a) the derivative of \( \text{dist}(x, \Gamma) \) is not defined on the set of points in \( \Omega_- \) that do not have a unique closest point to \( \Gamma \) (this set is called the medial axis or ridge of \( \Omega_- \)),

(b) we need \( \phi \) to be equal to \( \frac{1}{2} r^2 \) in a neighborhood of \( \partial B_R \) (so that \( Z = x \)), and

(c) it turns out that if we have uniform convexity of \( \phi \), i.e., \( D^2 \phi \geq \theta \) for some \( \theta > 0 \), then we need less smoothness of \( \Gamma \) (\( C^3 \) instead of \( C^4 \)).

The idea is to then use
\[
\phi(x) = \pm C_\Gamma \text{dist}(x, \Gamma) + \frac{1}{2} \text{dist}(x, \Gamma)^2,
\]
smooth it in \( \Omega_- \) (to deal with (a) above), smoothly change it to \( \frac{1}{2} r^2 \) in \( \Omega_+ \cap B_R \) (to deal with (b)), and choose \( C_\Gamma \) and \( R \) large enough to maintain the uniform convexity in (c). The condition that \( \Gamma \) must be piecewise analytic is needed to control the geometry of the medial axis, since without analyticity this set can behave very strangely (see the paragraph below Theorem 4.2 for more details).

1.7 Outline of the Paper

In Section 2 we recall the identities introduced for solutions of the Helmholtz equation by Morawetz in \cite{46}. In Section 3 we translate the problem of proving that \( A'_{k,\eta} \) is coercive into that of constructing an appropriate vector field \( Z \) in the multiplier \( Zu \). In Section 4 we construct such a vector field for uniformly convex, two- and three-dimensional domains that are \( C^3 \) and piecewise analytic. (The main result, Theorem 1.2, is then proved by combining parts 1 and 2 of Theorem 3.2.
part 1 of Theorem 3.4, and Lemma 4.1.) In Section 5 we show that the vector-field conditions for coercivity obtained in Section 3 cannot be satisfied if $\Omega_-$ is nonconvex. In Section 6 we conclude by placing this paper’s use of Morawetz’s identities into a wider context.

2 Morawetz’s Identities for the Helmholtz Equation

In this section, we state and prove two identities for solutions of the Helmholtz equation that arise from the multiplier $Zu$. (In the rest of the paper we refer to these as “Morawetz 1” and “Morawetz 2,” respectively.)

Lemma 2.1 (First Morawetz identity for Helmholtz (“Morawetz 1”)). Let $u$ be a complex-valued $C^2$ function on some set $D \subset \mathbb{R}^d$. Let $Lv := \Delta v + k^2 v$ with $k \in \mathbb{R}$. Let $Z \in (C^1(D))^d$ and $\beta, \alpha \in C^1(D)$ (i.e., $Z$ is a vector and $\beta$ and $\alpha$ are scalars), and let all three be real-valued. Then, with the summation convention,

$$2\Re(\overline{Z}vLv) = \nabla \cdot [2\Re(\overline{Z}v\nabla v) + (k^2|v|^2 - |\nabla v|^2)Z]$$

$$= -2\Re(\partial_i Z_j \partial_r v \overline{\partial_j v}) + (2\alpha - \nabla \cdot Z)(k^2|v|^2 - |\nabla v|^2)$$

$$- 2\Re(\overline{\nabla} (ik \nabla \beta + \nabla \alpha) \cdot \nabla v),$$

where

$$Z := Z \cdot \nabla v - ik\beta v + \alpha v.$$  

(2.1)

Lemma 2.2 (Second Morawetz identity for Helmholtz (“Morawetz 2”)). If the assumptions of Lemma 2.1 hold and, additionally, $\alpha \in C^2(D)$, then

$$2\Re(\overline{Z}vLv) = \nabla \cdot [2\Re(\overline{Z}v\nabla v) + (k^2|v|^2 - |\nabla v|^2)Z - \nabla \alpha|v|^2]$$

$$= -2\Re(\partial_i Z_j \partial_r v \overline{\partial_j v}) + (2\alpha - \nabla \cdot Z)(k^2|v|^2 - |\nabla v|^2)$$

$$- 2\Re(\overline{\nabla} (ik \nabla \beta \cdot \nabla v) + \Delta \alpha|v|^2)$$

where $Zv$ is given by (2.2).

Note that Lemma 2.2 follows from Lemma 2.1 by using

$$2\Re(\overline{\nabla} \alpha \cdot \nabla v) = \nabla \cdot [|\nabla \alpha|v|^2] - \Delta \alpha|v|^2.$$  

Proof of Lemma 2.1 Splitting $Zv$ up into its component parts we see that the identity (2.1) is the sum of the following three identities:

$$2\Re(Z \cdot \overline{\nabla}Lv) = \nabla \cdot [2\Re(Z \cdot \overline{\nabla}v\nabla v) + (k^2|v|^2 - |\nabla v|^2)Z]$$

$$= -2\Re(\partial_i Z_j \partial_r v \overline{\partial_j v}) + (\nabla \cdot Z)(|\nabla v|^2 - k^2|v|^2),$$

(2.4)

$$2\Re(ik\beta \overline{\nabla}Lv) = \nabla \cdot [2\Re(ik\beta \overline{\nabla}v)] - 2\Re(ik \nabla \beta \cdot \nabla v),$$

(2.5)

and

$$2\Re(\alpha \overline{\nabla}Lv) = \nabla \cdot [2\Re(\alpha \overline{\nabla}v)] + 2\alpha(k^2|v|^2 - |\nabla v|^2) - 2\Re(\overline{\nabla} \alpha \cdot \nabla v).$$  

(2.6)
To prove (2.5) and (2.6), expand the divergences on the right-hand sides (remembering that $\alpha$ and $\beta$ are real). The basic ingredient of (2.4) is the identity
\begin{equation}
\mathbf{Z} \cdot \nabla v \Delta v = \nabla \cdot \left[ \mathbf{Z} \cdot \nabla \nabla v \right] - \partial_i Z_j \partial_i v \partial_j v - \nabla v \cdot (\mathbf{Z} \cdot \nabla) \nabla v;
\end{equation}
to prove this, expand the divergence on the right-hand side and use the fact that the second derivatives of $v$ commute. We would like each term on the right-hand side of (2.7) to either be single-signed or be the divergence of something. We cannot do anything at this stage about the $\partial_i Z_j \partial_i v \partial_j v$ term (and making this single-signed will be one of the key requirements later). To deal with the final term, we use the identity
\begin{equation}
2 \Re(\nabla v \cdot (\mathbf{Z} \cdot \nabla) \nabla v) = \nabla \cdot \left[ ||\nabla v||^2 \mathbf{Z} \right] - (\nabla \cdot \mathbf{Z}) ||\nabla v||^2
\end{equation}
(which can be proved by expanding the divergence on the right-hand side). Indeed, taking twice the real part of (2.7) and using (2.8) yields
\begin{equation}
2 \Re(\mathbf{Z} \cdot \nabla \Delta v) = \nabla \cdot \left[ 2 \Re(\mathbf{Z} \cdot \nabla v \nabla v) - ||\nabla v||^2 \mathbf{Z} \right]
- 2 \Re(\partial_i Z_j \partial_i v \partial_j v) + (\nabla \cdot \mathbf{Z}) ||\nabla v||^2.
\end{equation}
Now add $k^2$ times
\begin{equation}
2 \Re(v \mathbf{Z} \cdot \nabla v) = \nabla \cdot \left[ ||v||^2 \mathbf{Z} \right] - (\nabla \cdot \mathbf{Z}) ||v||^2
\end{equation}
(which is the analogue of (2.8) with the vector $\nabla v$ replaced by the scalar $v$) to (2.9) to obtain (2.4).

A particular special case of the identity (2.11) is obtained by taking $\mathbf{Z} = \mathbf{x}$, $\beta = r$, and $\alpha$ a constant. Then $\mathbf{Z} v = r \mathcal{M}_\alpha v$, where
\begin{equation}
\mathcal{M}_\alpha v := v_r - ik v + \frac{\alpha}{r} v,
\end{equation}
and (2.11) becomes the following identity:

**Lemma 2.3** (Morawetz-Ludwig identity, [47], eq. 1.2). Let $v$ and $\mathcal{L} v$ be as in Lemma 2.1. Define the operator $\mathcal{M}_\alpha$ by (2.10) where $\alpha \in \mathbb{R}$ and $v_r = \mathbf{x} \cdot \nabla v / r$. Then
\begin{equation}
2 \Re(r \mathcal{M}_\alpha \overline{v} \mathcal{L} v) = \nabla \cdot \left[ 2 \Re(r \mathcal{M}_\alpha \overline{v} \nabla v) + (k^2 ||v||^2 - ||\nabla v||^2) v \right]
+ (2 \alpha - (d - 1))(k^2 ||v||^2 - ||\nabla v||^2)
- (||\nabla v||^2 - ||v_r||^2) - \left| \mathcal{M}_\alpha v - \frac{\alpha}{r} v \right|^2.
\end{equation}

**Proof.** To see that the nondivergence terms of (2.11) and (2.11) are equivalent when $\mathbf{Z} = \mathbf{x}$, $\beta = r$, and $\alpha$ is a constant, note that in this case $\mathbf{Z} = \beta \nabla \beta$, and thus one can express $2 \Re(ik \nabla \beta \cdot \nabla v)$ in terms of $|\mathbf{Z} v - \alpha v|^2 / \beta^2$. 

As discussed in Section 1.5, the Morawetz-Ludwig identity (2.11) has two important features:
(1) If $\alpha = (d - 1)/2$, then all the nondivergence terms on the right-hand side are $\leq 0$.

(2) If $v$ is a solution of the Helmholtz equation outside a ball of radius $R_0$ satisfying the Sommerfeld radiation condition (1.2), then when (2.11) is integrated over $\{R_0 \leq |x| \leq R\}$, the surface integral on $|x| = R$ tends to 0 as $R \to \infty$ (independently of the value of $\alpha$ in $\mathcal{M}_\alpha$) ([47], proof of lemma 5), ([58], lemma 2.4).

When we apply the identities Morawetz 1 (2.1) and Morawetz 2 (2.3) in $\Omega_+ \cap B_R$ we also want the nondivergence terms to be $\leq 0$ and for there to be no contribution from the surface integral at infinity. One way to ensure the latter condition is to make $Z = x, \beta = r$, and $2\alpha = (d - 1)$ when $r \geq R_0$ for some $R_0 > 0$. In fact, the next lemma implies that there is no contribution from infinity when $Z = x$, $\beta = C_1 r$, and $2\alpha = C_2$ for $C_1, C_2 \geq 1$, which gives us a bit more flexibility.

**Lemma 2.4** (Inequality on $\partial B_R$ used to deal with the contribution from infinity). Let $u$ be a solution of the homogeneous Helmholtz equation in $\mathbb{R}^d \setminus B_{R_0}$, for some $R_0 > 0$, satisfying the Sommerfeld radiation condition. If $C_1$ and $C_2$ are both constants $\geq 1$, then, for $R > R_0$,

\[
(2.12) \quad \int_{\partial B_R} R \left( \left| \frac{\partial}{\partial r} u \right|^2 + k^2 |u|^2 - |\nabla_S u|^2 \right) \, ds
- 2C_1 k R \Im \int_{\partial B_R} \bar{u} \frac{\partial}{\partial r} \bar{u} \, ds + C_2 \Re \int_{\partial B_R} \bar{u} \frac{\partial u}{\partial r} \, ds \leq 0,
\]

where $\nabla_S$ is the surface gradient on $r = R$ (recall that this is such that $\nabla v = \nabla_S v + k\bar{v}_r$ on $r = R$).

**Sketch of Proof Including References.** The inequality (2.12) follows from combining the following three inequalities:

\[
(2.13) \quad \Re \int_{\partial B_R} \bar{u} \frac{\partial}{\partial r} u \, ds \leq 0, \quad \Im \int_{\partial B_R} \bar{u} \frac{\partial}{\partial r} u \, ds \geq 0,
\]

and

\[
(2.14) \quad \int_{\partial B_R} R \left( \left| \frac{\partial}{\partial r} u \right|^2 + k^2 |u|^2 - |\nabla_S u|^2 \right) \, ds
- 2k R \Im \int_{\partial B_R} \bar{u} \frac{\partial}{\partial r} \bar{u} \, ds + \Re \int_{\partial B_R} \bar{u} \frac{\partial u}{\partial r} \, ds \leq 0.
\]

The two inequalities (2.13) are well-known but (2.14) not so. All three can be proved using the explicit expression for the solution of the Helmholtz equation in the exterior of a ball (i.e., an expansion in either trigonometric polynomials for
$d = 2$ or spherical harmonics for $d = 3$ with coefficients given in terms of Bessel and Hankel functions) and then proving bounds on the particular combinations of Bessel and Hankel functions. For proofs of (2.13) via this method see [50, theorems 2.6.1 and 2.6.4] or [11, lemma 2.1], with the latter reference also proving (2.14). (Note that the second inequality in (2.13) can also be obtained from applying Green’s identity in $\mathbb{R}^d \setminus B_R$ and using the second equation in (1.28).)

The Morawetz-Ludwig identity (2.11) can be used to prove the inequality (2.14) for $d = 2$, and a slightly weaker inequality for $d = 3$. Indeed, integrating (2.11) with $\partial_i Z_j$ and $2\beta = d - 1$ over $B_R \setminus B_R$, using the divergence theorem, and then letting $R \to \infty$ (using the fact mentioned above that the surface integral on $|x| = R_1$ tends to 0 as $R_1 \to \infty$ [58, lemma 2.4]), yields

$$\int_{\partial B_R} R \left( \frac{|\partial u|^2}{r^2} + k^2 u^2 - |\nabla u|^2 \right) ds$$

(2.15)

$$- 2kR \int_{\partial B_R} \frac{\partial u}{\partial r} ds + (d - 1)R \int_{\partial B_R} \frac{\partial u}{\partial r} ds$$

$$= - \int_{\mathbb{R}^d \setminus B_R} \left( (|\nabla u|^2 - |\nabla u|^2) + |\nabla u|^2 - (d - 1)2R \right) dx \leq 0.$$

By looking at the coefficient of the final term on the left-hand side, we see that inequality (2.15) is weaker than (2.14) when $d = 3$. (See [10, §5.3.1] for more discussion on both inequality (2.14) and its proof in [11, lemma 2.1].)

In what follows we need the identities Morawetz 1 and Morawetz 2 integrated over domains. We make these into lemmas here in order to keep track of how smooth $Z$, $\beta$, $\alpha$, and $u$ need to be (later we make $\alpha$ a function of $Z$, so outlining these conditions now will be helpful).

**Lemma 2.5 (Integrated version of Morawetz 1).** Let $D$ be a bounded Lipschitz domain with outward-pointing unit normal vector $\nu$, and let $u \in C^2(D) \cap C^1(\bar{D})$ be a solution of the Helmholtz equation in $D$. If $Z \in (C^1(D))^d \cap (C(\bar{D}))^d$, $\partial_i Z_j \in L^1(D)$, for $i, j = 1, \ldots, d$, $\beta \in C^1(D) \cap C(\bar{D})$, $\nabla \beta \in (L^1(D))^d$, $\alpha \in C^1(D) \cap C(\bar{D})$, and $\nabla \alpha \in (L^1(D))^d$, then

$$\int_{\partial D} \left[ 2 \Im \left( \overline{Z u} \frac{\partial u}{\partial \nu} \right) + (k^2 |u|^2 - |
abla u|^2) (Z \cdot \nu) \right] ds$$

(2.16)

$$= \int_{D} \left( 2 \Im \left( \partial_i Z_j \partial_i u \overline{\partial_j u} \right) + 2 \Im \left( \overline{\nu} (ik \nabla \beta + \nabla \alpha) \cdot \nabla u \right) \right.$$

$$- (2\alpha - \nabla \cdot Z)(k^2 |u|^2 - |
abla u|^2) \biggr) dx.$$
PROOF. The divergence theorem

\[ \int_D \nabla \cdot \mathbf{F} \, dx = \int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds \]

is valid if \( D \) is Lipschitz and \( \mathbf{F} \in (C^1(\overline{D}))^d \) [40, theorem 3.34]. Limiting arguments involving approximating either \( \mathbf{F} \) or \( D \) show that (2.17) is in fact valid when

\[ \mathbf{F} \in (C^1(D))^d \cap (C(\overline{D}))^d \quad \text{and} \quad \nabla \cdot \mathbf{F} \in L^1(D). \]

When we apply the divergence theorem to the integrated Morawetz identity, we take

\[ \mathbf{F} = 2 \Re \left[ (\mathbf{Z} \cdot \nabla u + ik\beta \overline{u} + \alpha u) \nabla u \right] + (k^2 |u|^2 - |
abla u|^2) \mathbf{Z}. \]

Therefore, if the conditions on \( \mathbf{Z}, \beta, \) and \( \alpha \) in the assertion hold, then (2.19) satisfies (2.18), and (2.16) follows from integrating (2.1) over \( D \) and applying (2.17). □

**Lemma 2.6 (Integrated version of Morawetz 2).** The integrated version of Morawetz 2 (2.3) holds if the conditions of Lemma 2.5 are satisfied and, in addition, \( \alpha \in C^2(D) \cap C(\overline{D}) \) and \( \Delta \alpha \in L^1(D) \).

**Proof.** Almost identical to that of Lemma 2.5. □

**Remark 2.7 (Bibliographic remarks).** The multiplier \( \mathbf{Z} \cdot \nabla u \) is associated with the name Rellich due to Rellich’s introduction of the multiplier \( x \cdot \nabla \) for the Helmholtz equation in [52]. Rellich identities have been well used in the study of the Laplace, Helmholtz, and other elliptic equations; see, e.g., the references in [10, §5.3] and [44, §1.4].

The idea of using a multiplier that is a linear combination of derivatives of \( v \) and \( v \) itself, such as \( Zv \), is attributed by Morawetz in [45] to Friedrichs. The multiplier \( rM_{\alpha}v \) for the Helmholtz equation was introduced by Morawetz and Ludwig in [47], and the multiplier \( Zv \) (2.2) is implicit in Morawetz’s paper [46]. Indeed, using the multiplier \( Zv \) is discussed informally at the beginning of [46, §1.2], but the resulting identity (essentially equation (2.3)) is only written down with \( Z = \phi \nabla \chi, \beta = \phi \psi, \) and \( \alpha = \phi \Delta \chi / 2 \) for arbitrary \( \chi \) and particular \( \phi \) and \( \psi \) [46, lemma 3]. Finally, we note that the multiplier \( Z \cdot \nabla v + \alpha v \) was independently introduced by Maz’ya for Laplace’s equation in the context of linear water waves in [39] (see also [33, eq. 2.28]).

### 3 Formulation of Coercivity in Terms of Conditions on the Vector Field \( \mathbf{Z} \)

The main goal of this section is to prove Theorem 3.2 below, which gives sufficient conditions for \( A'_{k,\eta} \) to be coercive in terms of the existence of an appropriate vector field \( \mathbf{Z} \). We begin by defining exactly what we mean by “coercivity” in this section.
Condition 3.1 (Coercivity). There exists an $\eta_0 > 0$ such that, given $\delta$, there exists a $k_0(\delta) > 0$ such that, for any $k \geq k_0$ and $\eta \geq \eta_0k$,

\begin{equation}
\Re \left( A'_{k,\eta} \phi, \phi \right)_{L^2(\Gamma)} \geq \left( \frac{1}{2} - \delta \right) \left\| \phi \right\|_{L^2(\Gamma)}^2
\end{equation}

for all $\phi \in L^2(\Gamma)$.

3.1 Statement of the Two Different Formulations of Coercivity

In this subsection we give two sufficient conditions for coercivity: Condition A and Condition B below. These conditions concern the existence of certain vector fields $Z$ defined in both $\Omega$ and $\mathbb{C} \setminus B_R$ for some sufficiently large $R$ (recall that $B_R := \{ x | |x | < R \}$). The two conditions are similar except that Condition B demands higher smoothness of $Z$ (and thus ultimately of $\xi$) in exchange for a slightly less restrictive condition on $\partial_z Z_j$ in the domain. We show in Section 4 below that Condition A is satisfied when $\Gamma$ is a uniformly convex, two- or three-dimensional domain that is $C^3$ and piecewise analytic. The advantage of Condition B is that it is closer to the vector-field condition obtained by Morawetz in [46, eq. 1.3] (see also [48, eq. 4.2] and [61, eqs. 2–4]) for bounding the energy norm of the solution of the Helmholtz exterior Dirichlet problem (which can then be used to prove local energy decay of the wave equation).

Condition A (Concerning the vector field associated with Morawetz 1 (2.1)). \Gamma is $C^2$, there exists a constant $R$ with $\overline{\Omega} \subset B_R$, a vector field $Z : B_R \to \mathbb{R}^d$, and a constant $C_\Gamma > 0$ such that the following hold:

A1. $Z$ is piecewise $C^2$ up to the boundary, i.e.,

\begin{equation}
Z \in (C^2(\overline{\Omega}))^d \cap (C^2(\overline{\Omega} \cap B_R))^d.
\end{equation}

A2. $Z_+ = Z_- = C_\Gamma n$ and $(\nabla \cdot Z)_+ = (\nabla \cdot Z)_-$ on $\Gamma$.

A3. $Z = x$ in a neighborhood of $\partial B_R$.

A4. There exists a $\theta > 0$ such that $\Re(\partial_z Z_j(x)\xi_i,\overline{\xi_j}) \geq \theta |\xi|^2$ for all $\xi \in \mathbb{C}^d$ and $x \in \overline{\Omega} \cup (\Omega_+ \cap B_R)$.

(Note that both here and in the rest of the paper, we use $+$ and $-$ subscripts to denote the limit of a function, here $Z(x)$, as $x \to \Gamma$ from $\Omega_+$ and $\Omega_-$, respectively.)

By using the identity Morawetz 2, (2.3), instead of the identity Morawetz 1, (2.1), we can make Condition A4 less restrictive (i.e., $\partial_z Z_j$ only needs to be non-negative rather than uniformly positive) if Condition A1 is made more restrictive (i.e., increased smoothness of $Z$).

Condition B (Concerning the vector field associated with Morawetz 2 (2.3)). \Gamma is $C^2$, there exists a constant $R$ with $\overline{\Omega} \subset B_R$, a vector field $Z : B_R \to \mathbb{R}^d$, and a constant $C_\Gamma > 0$ such that the following hold:

B1. $Z$ is piecewise $C^3$ up to the boundary, i.e.,

\begin{equation}
Z \in (C^3(\overline{\Omega}))^d \cap (C^3(\overline{\Omega} \cap B_R))^d.
\end{equation}
B2. $Z_+ = Z_- = C \Gamma n$ and $(\nabla \cdot Z)_+ = (\nabla \cdot Z)_-$ on $\Gamma$.

B3. $Z = x$ in a neighborhood of $\partial B_R$.

B4. $\Re(\partial_i Z_j(x) \xi_i \xi_j) \geq 0$ for all $\xi_i \in \mathbb{C}^d$ and $x \in \Omega_- \cup (\Omega_+ \cap B_R)$.

The extra smoothness of $Z$ in Condition B comes from the fact that, in formulating these conditions, the function $\alpha$ in the multiplier $Z v$ is defined in terms of $Z$ (it turns out to involve $\nabla \cdot Z$). Therefore, if we use the identity Morawetz 2 (2.3) instead of Morawetz 1 (2.1), the additional smoothness of $\alpha$ needed for (2.3) to hold entails additional smoothness of $Z$.

The next theorem shows how Conditions A and B (along with some constraints on the norm of the single-layer potential when $d = 3$) are sufficient for coercivity.

**Theorem 3.2 (Sufficient conditions for coercivity).** Coercivity (i.e., Condition 3.1) holds if one of the following four criteria is met.

1. $d = 2$ and Condition A holds.
2. $d = 3$, Condition A holds, and $\|S_k\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} = o(1)$ as $k \to \infty$.
3. $d = 2$ and Condition B holds.
4. $d = 3$, Condition B holds, and $\|S_k\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} = o(1)$ as $k \to \infty$.

We prove this theorem in Section 3.2 below, but first we make some remarks.

**Remark 3.3 (Asymptotics of the coercivity constant).** Theorem 3.2 gives sufficient conditions for coercivity (in the sense of Condition 3.1) to hold; however, it is also interesting to then ask how the coercivity constant depends on $k$.

The proof of Theorem 3.2 below shows that if Condition A holds then there exist $\eta_0 > 0$ and $k_1 > 0$ such that, if $k \geq k_1$ and $\eta \geq \eta_0$, then $\alpha_{k, \eta}$ is coercive (i.e., (1.1) holds) with

$$\alpha_{k, \eta} \geq \frac{1}{2} - \mathcal{O}(\|S_k\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)})$$

(3.2)

Similarly, the proof of Theorem 3.2 shows that if Condition B holds then the asymptotics (3.2) hold with

$$-\mathcal{O}(\|\chi S_k\|_{L^2(\Gamma) \rightarrow L^2(\mathbb{R}^d)}) \quad \text{and} \quad -\mathcal{O}(\|S_k\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)})$$

(3.2) added to the right-hand side.

The previous remark shows us that, in order to prove coercivity via this method, we need to have $\|S_k\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)}$ tending to 0 as $k \to \infty$ (and if we use Condition B then we also need $\|\chi S_k\|_{L^2(\Gamma) \rightarrow L^2(\mathbb{R}^d)}$ tending to 0). The following theorem recaps bounds on these two quantities, which we then use in the proofs of Theorems 3.2 and 1.1.

**Theorem 3.4 (Bounds on $\|S_k\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)}$ and $\|\chi S_k\|_{L^2(\Gamma) \rightarrow L^2(\mathbb{R}^d)}$).**

1. If $\Omega_-$ is a bounded Lipschitz domain in two or three dimensions and $\chi \in C_\text{comp}^\infty(\mathbb{R}^d)$, then, given $k_0 > 0$,

$$\|S_k\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \lesssim k^{(d-3)/2} \quad \text{and} \quad \|\chi S_k\|_{L^2(\Gamma) \rightarrow L^2(\mathbb{R}^d)} \lesssim k^{-1/2}$$

(3.3)
for all \( k \geq k_0 \).

(2) If \( \Omega_- \) is a \( C^2 \), uniformly convex, three-dimensional domain, then, given \( k_0 > 0 \),

\[
||S_k||_{L^2(\Gamma) \to L^2(\Gamma)} \lesssim \frac{(\log k)^{1/2}}{k^{1/3}}
\]

for all \( k \geq k_0 \).

**Proof.** The first bound in (3.3) was proved in [9, theorem 3.3] and the second bound was proved in [57, lemma 4.3]. The bound (3.4) was proved in [56, theorem 1.5]. \( \square \)

**Remark 3.5 (Smoothness of \( \Gamma \) and \( Z \)).** To keep things simple, we have assumed in Conditions A and B that \( \Gamma \) is \( C^2 \). Conditions A2 and B2 then imply that \( \Gamma \) must additionally be \( C^3 \). Indeed, if \( Z = C_\Gamma n \) on \( \Gamma \) and \( Z \) is piecewise \( C^2 \) up to the boundary, then \( n \) must be \( C^2 \), which implies that \( \Gamma \) must be \( C^3 \).

An important feature of Rellich and Morawetz identities is that they can be applied when \( \Gamma \) is Lipschitz, but this requires extra technicalities such as the notion of nontangential limits and, when \( v = S_k \phi \) for \( \phi \in L^2(\Gamma) \), harmonic analysis results about the single-layer potential (see [58, remark 4.7] and [10, theorem 2.16] and the references therein). The paper [12] goes through the argument of Theorem 3.2 when \( \Gamma \) is Lipschitz and shows that requiring \( Z = C_\Gamma n \) on \( \Gamma \) means that \( \Gamma \) must be at least \( C^{2.1} \) (so we have not lost much here by avoiding these technicalities).

**Remark 3.6 (A third set of conditions for coercivity).** As discussed above, in the proof of Theorem 3.2 below, the scalar function \( \alpha \) in the multiplier \( Zv \) involves \( \nabla \cdot Z \). The difference in smoothness of \( Z \) in the two conditions A and B is then due to the fact that Condition A uses the identity Morawetz 1 (2.1), which needs \( \alpha \in C^1 \), whereas Condition B uses Morawetz 2 (2.3), which needs \( \alpha \in C^2 \).

There is an additional set of conditions for coercivity that arise from letting \( \alpha \) be a constant. In this case \( Z \) need only be \( C^1 \) (and thus, from Remark 3.5, \( \Gamma \) need only be \( C^2 \)), but these conditions are much more restrictive than Condition A with \( \nabla \cdot Z \) needing to be bounded in terms of \( d \) and the constant \( \theta \) in the positivity condition. The vector field \( x \) satisfies these conditions when \( \Gamma \) is a circle or sphere, but it is not at all clear whether they can be satisfied for more general domains.

**Remark 3.7 (A modified integral operator).** The proof of Theorem 3.2 below shows that if the condition \( Z_+ = Z_- = C_\Gamma n \) in B2 is replaced by \( Z_+ = Z_- \) and \( Z \cdot n > 0 \) on \( \Gamma \), this modified version of Condition B holds, and also \( ||S_k||_{L^2(\Gamma) \to L^2(\Gamma)} = o(1) \) as \( k \to \infty \), then the integral operator

\[
A'_{k,\eta,k} := (Z \cdot n)\left( \frac{1}{2} I + D_k \right) + Z \cdot \nabla \Gamma S_k - i\eta S_k
\]

is coercive for \( k \) sufficiently large. More precisely, we have that given \( \delta > 0 \) there exists a \( k_0 > 0 \) such that if \( \eta = kR + i|\nabla \cdot Z|\Gamma/2 \), then \( A'_{k,\eta,k} \) is coercive
for all $k \geq k_0$, with coercivity constant $\inf_{x \in \Gamma} (Z \cdot n) - \delta$. (Note that, firstly, the operator $A'_{k, \eta, Z}$ can be used to solve the exterior Dirichlet problem for the Helmholtz equation, see [10, theorem 2.36], and, secondly, if $Z = x$, then $A'_{k, \eta, Z}$ becomes the star-combined operator, (1.43), of [58].) The vector field constructed by Morawetz, Ralston, and Strauss in [48, §4] satisfies this modified version of Condition B in $\Omega_+$ if $\Omega_+$ is nontrapping, but it is not clear how to construct a continuation of this vector field into $\Omega_-$ satisfying the nonnegativity condition $\mathbb{B}$. 

3.2 Proof of Theorem 3.2

PROOF OF THEOREM 3.2. We first prove parts 1 and 2 (relating to Condition $\mathbb{A}$) using Morawetz 1 (2.1), and then discuss the changes needed to prove parts 3 and 4 (relating to Condition $\mathbb{B}$) using Morawetz 2 (2.3).

Our strategy is to mimic the classical method of “transferring” the coercivity properties of the PDE formulation to the associated boundary integral operators (as discussed in Section 1.4), but with Green’s identity (1.22) replaced by the identity Morawetz 1 (2.1). That is, we apply the integrated version of (2.1), namely (2.16), with $v$ replaced by $u = S_k \phi$ (with $\phi \in L^2(\Gamma)$), and $D$ first equal to $\Omega_-$ and then equal to $\Omega_+ \cap B_R$. The multiplier in the identity (2.16) is given by (2.2) with $Z$ the vector field in Condition $\mathbb{A}$, $\beta = R$, and $2 \alpha = (\nabla \cdot Z) - \theta$, where $\theta$ is the constant in Condition $\mathbb{B}$. As the proof develops, we see why we make these choices of $\beta$ and $\alpha$. We go through the majority of the proof without worrying about how smooth $Z$ needs to be, and then return to this question at the end.

With the identity (2.1) written as $\nabla \cdot Q = P$, integrating it over $\Omega_-$ and $\Omega_+ \cap B_R$ yields

$$\int_{\Gamma} Q_- \cdot n \, ds = \int_{\Omega_-} P \, dx$$

and

$$- \int_{\Gamma} Q_+ \cdot n \, ds + \int_{\partial B_R} Q_R \, ds = \int_{\Omega_+ \cap B_R} P \, dx,$$

where (remembering that $\mathcal{L} u = 0)$

$$P = 2\Re (\partial_i Z_j \partial_i u \overline{\partial_j u}) - (2\alpha - \nabla \cdot Z)(k^2 |u|^2 - |\nabla u|^2)$$

$$+ 2\Re (\overline{\alpha \nabla \beta + \overline{\nabla \alpha}} \cdot \nabla u),$$

$$Q_\pm \cdot n = (Z_\pm \cdot n) \left( \left| \frac{\partial u_\pm}{\partial n} \right|^2 + k^2 |u_\pm|^2 - |\nabla u_\pm|^2 \right)$$

$$+ 2\Re \left( (Z_\pm \cdot \nabla \Gamma u_\pm + i k \beta u_\pm + \alpha u_\pm) \frac{\partial u_\pm}{\partial n} \right).$$
for $x \in \Gamma$ (where we have used that $\nabla u = \nabla u + n \partial u / \partial n$ on $\Gamma$), and

$$Q_R = Q \cdot \hat{n} = R \left( \frac{\partial u}{\partial r}^2 + k^2 |u|^2 - |\nabla_S u|^2 \right) - 2k\beta \Im \left( \frac{\partial u}{\partial r} \right) + 2\alpha \Re \left( \frac{\partial u}{\partial r} \right)$$

for $x \in \partial B_R$ (where we have used that $Z = x$ on $\partial B_R$, i.e., $A^3$). Adding (3.5) and (3.6) yields

$$ \int_{\Gamma} (Q_- - Q_+) \cdot n \, ds + \int_{\partial B_R} Q_R \, ds = \int_{\Omega_-} P \, dx + \int_{\Omega_+ \cap \partial B_R} P \, dx. $$

DEALING WITH THE INTEGRAL ON $\partial B_R$. Using the inequality (2.12) from Lemma 2.4 we see that $\int_{\partial B_R} Q_R \, ds \leq 0$ if

$$ \beta \geq R \text{ and } 2\alpha \geq 1 \text{ on } \partial B_R.$$ 

Since $\beta = R$, the first inequality is satisfied. Recall that we chose $2\alpha = (\nabla \cdot Z) - \theta$. Since $Z = x$ in a neighborhood of $\partial B_R$ (Condition $A^3$), $\nabla \cdot Z = d$ in this neighborhood, and thus the second inequality in (3.8) is satisfied if $\theta \leq d - 1$.

This is not restrictive, since if we have constructed a $Z$ that satisfies the positivity condition $\nabla \cdot Z$ with a value of $\theta > d - 1$, we can just choose $\theta = d - 1$ for the remainder of this argument (we see later that all we need is $\theta > 0$). Therefore,

$$ \int_{\Gamma} (Q_- - Q_+) \cdot n \, ds \geq \int_{\Omega_-} P \, dx + \int_{\Omega_+ \cap \partial B_R} P \, dx. $$

DEALING WITH THE INTEGRAL ON $\Gamma$. We now show that

$$ \int_{\Gamma} (Q_- - Q_+) \cdot n \, ds = 2\Re \left( \left( C_\Gamma D_k' - ik\beta S_k + \alpha S_k \right) \phi, \phi \right)_{L^2(\Gamma)}. $$

Indeed, we first note that, by Condition $A^2$, $Z_{\pm} = C_\Gamma n$ and $(\nabla \cdot Z)_+ = (\nabla \cdot Z)_-$ on $\Gamma$. Therefore, $Z \cdot n = C_\Gamma$ and $Z \cdot \nabla u = 0$ on $\Gamma$, and $\alpha$ is continuous across $\Gamma$. We next simplify $(Q_- - Q_+) \cdot n$ using these facts along with the single-layer potential jump relations

$$ u_\pm(x) = S_k \phi(x), \quad \nabla u_+(x) = \nabla u_-(x), \quad \frac{\partial u_0}{\partial n}(x) = \left( \mp \frac{1}{2} I + D'_k \right) \phi(x). $$

for $x \in \Gamma$, which are given for $\Gamma \in C^2$ by, e.g., [15, theorems 2.12 and 2.17]. A key identity to help us do this is

$$ \left| \frac{\partial u_0}{\partial n}(x) \right|^2 = \left| \frac{\partial u_+}{\partial n}(x) \right|^2 + \frac{1}{2} \left[ \left( \frac{\partial u_-}{\partial n}(x) \right)^2 - \left( \frac{\partial u_0}{\partial n}(x) \right)^2 \right] = 2\Re \left( D'_k \phi(x) \phi(x) \right) $$

for $x \in \Gamma$, which can be established using $|a|^2 - |b|^2 = \Re[(a + b)(a - b)]$ and the jump relation for $\partial u_0/\partial n$. 


Putting together the inequality (3.9) and the equality (3.10) yields

\[
\Re \left( \left( D_k' - ik \frac{R}{C} S_k + \frac{\alpha}{C} S_k \right) \phi, \phi \right)_{L^2(\Gamma)} \geq \frac{1}{2C} \left( \int_{\Omega_-} P \, dx + \int_{\Omega_+ \cap B_R} P \, dx \right).
\]

The definition of \( A_{k, \eta} \), equation (1.9), implies that if we can show that

\[
\Re \left( \left( D_k' - i\eta_0 k S_k \phi, \phi \right)_{L^2(\Gamma)} + o(1) \|\phi\|^2_{L^2(\Gamma)} \geq 0 \quad \text{as } k \to \infty,
\]

then this establishes inequality (3.1) in Condition 3.1 for \( \eta = \eta_0 k \). Note that Condition 3.1 requires the inequality (3.1) to hold for \( \eta \geq \eta_0 k \) and not just for \( \eta = \eta_0 k \). However, the former case follows from the latter by first noting that

\[
\Re \left( \left( D_k' - i\eta S_k \phi, \phi \right)_{L^2(\Gamma)} + o(1) \|S_k\|^2_{L^2(\Gamma)} \geq \Re \left( \left( -i(\eta - \eta_0 k) S_k \phi, \phi \right)_{L^2(\Gamma)} \right),
\]

and then using the fact that \( \Re \left( -i S_k \phi, \phi \right)_{L^2(\Gamma)} \geq 0 \) from (1.30).

Choosing \( \eta_0 = R/C \), we see that (3.11) gives us (3.12) if \( \|S_k\|_{L^2(\Gamma)} \to L^2(\Gamma) = o(1) \) and

\[
\int_{\Omega_-} P \, dx + \int_{\Omega_+ \cap B_R} P \, dx \geq -o(1) \|\phi\|^2_{L^2(\Gamma)} \quad \text{as } k \to \infty.
\]

The decay \( \|S_k\|_{L^2(\Gamma)} \to L^2(\Gamma) = o(1) \) as \( k \to \infty \) is given by the first bound in (3.3) when \( d = 2 \) and is a hypothesis in the theorem when \( d = 3 \).

Therefore, all that remains to prove coercivity (Condition 3.1) is to establish that the inequality (3.13) holds.

**DEALING WITH THE VOLUME TERMS.** We need to establish the inequality (3.13) with \( P \) given by (3.7). Using (the positivity of \( \partial_j Z_j \)), the fact that \( 2\alpha = \nabla \cdot \mathbf{Z} - \theta \), and also the fact that \( \beta \) is a constant, we have that

\[
P \geq \theta (k^2 |u|^2 + |\nabla u|^2) + 2\Re(\overline{\nabla \alpha} \cdot \nabla u).
\]

The inequality

\[
2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon},
\]

for all \( a, b, \) and \( \varepsilon > 0 \), implies that

\[
2 \left| \int_{\Omega_+ \cap B_R} \overline{\nabla \alpha} \cdot \nabla u \, dx \right| \leq \frac{\|\nabla \alpha\|_{L^\infty(\Omega_+ \cap B_R)}}{k} \int_{\Omega_+ \cap B_R} (|\nabla u|^2 + k^2 |u|^2) \, dx,
\]

(and similarly for the integral over \( \Omega_- \)). The bound (3.14) implies that choosing \( k \) large enough ensures that the left-hand side of (3.13) is \( \geq 0 \); thus we have proved that \( A_{k, \eta} \) is coercive (Condition 3.1).
Smoothness of $Z$ We now go back through the above argument and see what smoothness we need from $Z$ (and this will give us Condition A1).

We first check the conditions on $u$, $Z$, $\beta$, and $\alpha$ required by Lemma 2.5 A proof that $u = S_k \phi$ is in $C^2(\Omega_\pm) \cap C^1(\overline{\Omega_\pm})$ when $\Gamma$ is $C^2$ and $\phi \in L^2(\Gamma)$ is given in [15, theorems 2.12 and 2.17] (this proof is for Hölder-continuous $\phi$, but since Hölder-continuous functions are dense in $L^2(\Gamma)$, this gives the result for $\phi \in L^2(\Gamma)$). Turning to the conditions on $Z$, $\beta$, and $\alpha$, those on $\beta$ are satisfied since $\beta$ is a constant. We need $Z \in (C^1(\Omega_-))^d \cap (C(\Omega_-))^d$, $\partial_i Z_j \in L^1(\Omega_-)$ (and similarly in $\Omega_+ \cap B_R$). Furthermore, the fact that $2\alpha = (\nabla \cdot Z) - \theta$ means that we also need $\nabla \cdot Z \in C^1(\Omega_-) \cap C(\Omega_-)$, $\nabla(\nabla \cdot Z) \in (L^1(\Omega_-))^d$ (and again in $\Omega_+ \cap B_R$). If $Z$ is piecewise $C^2$ up to the boundary (i.e., $Z$ satisfies Condition A1), then all these conditions are satisfied.

After using Lemma 2.5 the proof needed (i) $Z$ and $\alpha$ to be continuous across $\Gamma$, and (ii) $\nabla \alpha$ to be in both $L^\infty(\Omega_-)$ and $L^\infty(\Omega_+ \cap B_R)$. Regarding (i): this leads to A2. It turns out that we could drop the restriction that $\alpha$ is continuous if we added the extra condition that $\|S_k\|D_k^{1'} = o(1)$ as $k \to \infty$ (to deal with the term on $\Gamma$ resulting from the nonzero jump of $\alpha$). However, at least in our construction of $Z$ in Section 3 ensuring that $\nabla \cdot Z$ is continuous across $\Gamma$ is not the limiting factor, and so we retain the condition that $\alpha$ is continuous. Regarding (ii): this implies that we need $\nabla(\nabla \cdot Z) \in L^\infty$, which is ensured by $Z$ being piecewise $C^2$ up to the boundary.

Changes to the Above Argument That Are Necessary to Prove Parts 3 and 4. We now repeat the above argument using Morawetz 2 (2.3) instead of Morawetz 1 (2.1); the changes are as follows: We choose $2\alpha = \nabla \cdot Z$. To apply Lemma 2.6 (the analogue of Lemma 2.5 with Morawetz 1 replaced by Morawetz 2), we need $\alpha \in C^2(D) \cap C^1(\overline{D})$ and $\Delta \alpha \in (L^1(D))^d$; these conditions are satisfied if $Z$ is piecewise $C^3$ up to the boundary (i.e., $Z$ satisfies Condition B1). Similar to before, the fact that $Z$ and $\alpha$ must be continuous across $\Gamma$ leads to Condition B2.

$Q_R$ now contains the extra term $-\langle \nabla \alpha \cdot \xi \rangle u^2$. This is 0, however, since $2\alpha = \nabla \cdot Z = d$ (i.e., a constant) in a neighborhood of $\partial B_R$. The condition that $2\alpha \geq 1$ (necessary for controlling the integral on $\partial B_R$) is now satisfied automatically.

$Q_{\pm} \cdot n$ now contains the extra term $-\langle \partial \alpha_{\pm} / \partial n \rangle u_{\pm}^2$. If we assume that $\nabla \alpha$ is continuous across $\Gamma$, then this extra term does not contribute to (3.10) since there is no jump in $u$ across $\Gamma$. However, this would impose the extra condition that $\nabla(\nabla \cdot Z)$ is continuous across $\Gamma$. If we don’t assume that $\nabla \alpha$ is continuous, then, if $\|S_k\|L^2(\Gamma) \to L^2(\Gamma) = o(1)$ as $k \to \infty$, we obtain (3.10) with $o(1)\|\phi\|L^2(\Gamma)$ added to the right-hand side. Since we assume this decay in $\|S_k\|L^2(\Gamma) \to L^2(\Gamma)$ to go from (3.11) to (3.12), we choose this second option (i.e., $\nabla \alpha$ discontinuous and no extra restriction on $Z$).
Since we are using Morawetz 2 (2.3), $P$ is now given by
\[
P = 2\Re(\bar{\partial}_j Z_j \partial_i u \overline{\partial_j u}) - (2\alpha - \nabla \cdot \mathbf{Z})(k^2 |u|^2 - |\nabla u|^2)
\]
\[+ 2\Re(ik \overline{\nabla} \beta \cdot \nabla u) - \Delta \alpha |u|^2.
\]
Using $\mathcal{B}_\delta$, and the fact that $2\alpha = \nabla \cdot \mathbf{Z}$, we find that
\[
P + \Delta \alpha |u|^2 \geq 0.
\]
Taking the $L^\infty$ norm of $\Delta \alpha$ out of the integrals (noting that $\mathbf{Z}$ being piecewise $C^3$ up to the boundary means that this is allowed) we see that, since $u = \mathcal{S}_k \phi$, the inequality needed for coercivity (3.13) will hold if $\|\mathcal{S}_k\|_{L^2(\Gamma) \rightarrow L^2(\mathbb{R}^d)} = o(1)$ as $k \to \infty$, and this decay is ensured by the second bound in (3.3).

\[\Box\]

4 Construction of a Vector Field $\mathbf{Z}$ Satisfying Condition [A] for Uniformly Convex, Two- and Three-Dimensional Domains That Are $C^3$ and Piecewise Analytic

This section proves the following result:

**Lemma 4.1.** If $\Omega_-$ is a uniformly convex, two- or three-dimensional domain with $\Gamma$ both $C^3$ and piecewise analytic, then there exists a $\mathbf{Z}$ satisfying Condition [A].

The main result of this paper, Theorem 1.2, then follows by combining Lemma 4.1 with parts 1 and 2 of Theorem 3.2 and part 2 of Theorem 3.4. The asymptotics of $\alpha_{k,\eta}$ given in (1.15) then follow from using the first bound in (3.3) (for $d = 2$) and the bound (3.4) (for $d = 3$) in equation (3.2).

We first prove the result of Lemma 4.1 for the two-dimensional case (in Section 4.2), and then outline the small modifications needed to establish the result for the three-dimensional case (in Section 4.4).

4.1 Orthogonal Curvilinear Coordinates Defined by $\Gamma$ in Two Dimensions

We are going to use the orthogonal curvilinear coordinate system defined by $\Gamma$, and so it is convenient to recap some facts about this in an initial subsection. At this stage we only need that $\Omega_-$ is convex and $\Gamma$ is $C^2$ (the conditions that $\Omega_-$ is uniformly convex and $\Gamma$ is both $C^3$ and piecewise analytic will come later in connection with $\mathbf{Z}$).

**Coordinate System in the Exterior.** Let $\mathbf{r}_0(s)$ be the position vector of a point on $\Gamma$, parametrized by the arc length $s$. The fact that $\Gamma$ is $C^2$ means that $\mathbf{r}_0(s)$ is $C^2$ as a function of $s$. Recall that $(d\mathbf{r}_0/ds)(s)$ is the unit tangent vector to $\Gamma$ and denote the outward-pointing unit normal vector by $\mathbf{n}(s)$ (recall that this is proportional to $(d^2\mathbf{r}_0/ds^2)(s)$). Define the (signed) curvature $\kappa(s)$ by
\[
\frac{d^2\mathbf{r}_0}{ds^2}(s) = -\kappa(s)\mathbf{n}(s),
\]
and define $\kappa_*$ and $\kappa^*$ by
\[
\kappa_* := \min_s \kappa(s) \quad \text{and} \quad \kappa^* := \max_s \kappa(s).
\]
respectively. The fact that \( \Omega_- \) is convex then implies that \( \kappa_n \geq 0 \). The fact that \( n \) is perpendicular to both \( dn/ds \) and \( dr_0/ds \) can then be used to show that

\[
\frac{dn}{ds}(s) = \kappa(s) \frac{dr_0}{ds}(s).
\]

(4.3)

Given a point \( P \) in \( \Omega_+ \), let \( r_0(s) \) be the position vector of the closest point on \( \Gamma \) to \( P \) (this closest point is unique since \( \Omega_- \) is convex). The position vector of \( P \), namely \( r \), can then be written as

\[
r(s, n) = r_0(s) + n n(s) \quad \text{where} \quad n := \text{dist}(r, \Gamma).
\]

The basis vectors in the \((n, s)\)-coordinate system, \( e_n^+ \) and \( e_s \) (where we use the \(+\) superscript on \( e_n \) to emphasize that we are in \( \Omega_+ \)) are then defined by

\[
e_n^+(n, s) := \frac{\partial r}{\partial n}(n, s) = n(s)
\]

and

\[
e_s(n, s) := \frac{\partial r}{\partial s}(n, s) = \frac{dr_0}{ds}(s) + n \frac{dn}{ds}(s).
\]

\[
e_s(n, s) = \left(1 + n \kappa(s)\right) \frac{dr_0}{ds}(s) \quad \text{by (4.3)}.
\]

The scale factors \( h_n \) and \( h_s \) are then

\[
h_n(n, s) := |e_n^+(n, s)| = 1 \quad \text{and} \quad h_s(n, s) := |e_s(n, s)| = 1 + n \kappa(s),
\]

and thus

\[
\hat{e}_n^+(n, s) := \frac{1}{h_n(n, s)} e_n^+(n, s) = n(s),
\]

\[
\hat{e}_s(n, s) := \frac{1}{h_s(n, s)} e_s(n, s) = \frac{dr_0}{ds}(s).
\]

The \((n, s)\)-coordinate system with basis vectors \( e_n^+ \) and \( e_s \) is orthogonal and, given a vector \( v \), we write

\[
v = v^ne_n^+ + v^s e_s.
\]

If \( \psi : \Omega_+ \to \mathbb{R} \) is differentiable, then

\[
\nabla \psi = \frac{1}{h_n} \frac{\partial \psi}{\partial n} \hat{e}_n^+ + \frac{1}{h_s} \frac{\partial \psi}{\partial s} \hat{e}_s = \frac{\partial \psi}{\partial n} \hat{e}_n^+ + \frac{1}{1 + n \kappa(s)} \frac{\partial \psi}{\partial s} \hat{e}_s.
\]

(4.4)

If \( v \) is a differentiable vector field in general curvilinear coordinates \( u^i \) with basis \( e_i := \partial r/\partial u^i \), then

\[
\left( \frac{\partial v}{\partial u^j} \right)^i = \frac{\partial v^i}{\partial u^j} + \Gamma^i_{kj} v^k,
\]

(4.5)

where \( \Gamma^i_{kj} \) are the Christoffel symbols; see, e.g., [53, eq. 21.85]. It is straightforward to check that the derivative of the vector \( v \) as a linear map from \( \mathbb{R}^d \) with basis \( \{e_i\} \) to itself is given by \((Dv)_{ij} = (\partial v/\partial u^i)^j\). In what follows we consider vector
fields $\mathbf{v} : \Omega_+ \rightarrow \mathbb{R}^d$ with $v^s = 0$ and $v^n$ a function of $n$ only. For such vectors, after calculating the Christoffel symbols in (4.5) (using the fact that $h_n$ is constant), we find that

$$\frac{\partial \mathbf{v}}{\partial n} = \frac{\partial v^n}{\partial n} \mathbf{e}^+_n$$

and

$$\frac{\partial \mathbf{v}}{\partial s} = \frac{v^n}{h_s} \frac{\partial h_n}{\partial n} \mathbf{e}_s.$$

The derivative of the vector $\mathbf{v}$, as a linear map from $\mathbb{R}^2$ with basis $\{\mathbf{e}^+_n, \mathbf{e}_s\}$ to itself, is then

$$(4.6) \quad D \mathbf{v} = \begin{pmatrix} \frac{\partial v^n}{\partial n} & 0 \\ \frac{v^n}{h_s} \frac{\partial h_n}{\partial n} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial v^n}{\partial n} & 0 \\ 0 & \frac{v^n \kappa(s)}{1 + n \kappa(s)} \end{pmatrix}.$$  

The vector field $\mathbf{Z}$ that we construct below to satisfy Condition A will be of the above form (i.e., $Z^s = 0$ and $Z^n$ is only a function of $n$). To verify the positivity condition that $\Re (\partial_j \mathbf{Z}(\mathbf{x}) \xi_i \overline{\xi_j}) \geq \theta |\xi|^2$ for all $\xi \in \mathbb{C}^d$ and $\mathbf{x} \in \Omega_+ \cap B_R$, we claim that it is sufficient to prove that the matrix $D \mathbf{Z}$ (defined by the analogue of (4.6)) is $\geq \theta$ (in the sense of quadratic forms) for all $n$ and $s$. Indeed, $D \mathbf{Z}$ defined by the analogue of (4.6) is the derivative of $\mathbf{Z}$ as a linear map both from $\mathbb{C}^d$ to $\mathbb{C}^d$ with basis $\{\mathbf{e}^+_n, \mathbf{e}_s\}$ and from $\mathbb{C}^d$ to $\mathbb{C}^d$ with basis $\{\mathbf{e}^+_n, \mathbf{e}_s\}$ (this is a consequence of the matrix being diagonal and the facts that $\mathbf{e}^+_n = h_n \mathbf{e}^+_n$ and $\mathbf{e}_s = h_s \mathbf{e}_s$). Now, given an $\mathbf{x} \in \Omega_+ \cap B_R$, there exist $n_1, s_1$ such that $\mathbf{x} = (n_1, s_1)$ in the $(n, s)$-coordinate system defined by $\Gamma$. Since $\{\mathbf{e}^+_n(n_1, s_1), \mathbf{e}_s(n_1, s_1)\}$ form an orthonormal basis, there exists an orthogonal matrix $\mathbf{B}$ such that $(\mathbf{B}^T D \mathbf{Z}(n_1, s_1) \mathbf{B})_{ij} = \partial_j \mathbf{Z}_i(\mathbf{x})$. It then follows that if $D \mathbf{Z}(n_1, s_1) \geq \theta$ then $\Re (\partial_j \mathbf{Z}_i(\mathbf{x}) \xi_i \overline{\xi_j}) \geq \theta |\xi|^2$ for all $\xi \in \mathbb{C}^d$.

**COORDINATE SYSTEM IN THE INTERIOR.** Given a point $P$ in $\Omega_-$ that has a unique closest point on $\Gamma$, let $r_0(s)$ be the position vector of the closest point. (The set of points in $\Omega_-$ that do not have a unique closest point on $\Gamma$ is called the *medial axis*, and we discuss this set below.) The position vector of $P$, $\mathbf{r}$, can then be written as

$$\mathbf{r}(s, n) = r_0(s) - n \mathbf{n}(s)$$

where again $n = \text{dist}(\mathbf{r}, \Gamma)$. Proceeding in a similar manner to the exterior case, we have that

$$\mathbf{e}^-_n(n, s) = -\mathbf{n}(s)$$

and

$$\mathbf{e}_s(n, s) = (1 - n \kappa(s)) \frac{d r_0}{d s}(s).$$

Therefore,

$$(4.7) \quad h_n(n, s) = 1, \quad h_s(n, s) = 1 - n \kappa(s).$$

$$(4.6) \quad \mathbf{e}^-_n(n, s) = -\mathbf{n}(s) \quad \text{and} \quad \mathbf{e}_s(n, s) = (\frac{d r_0}{d s}(s)).$$

Equations analogous to (4.4) and (4.6) hold for the derivatives of scalar and vector fields.

For a given $s$, this coordinate system breaks down locally when $n = 1/\kappa(s)$, and thus the bounds on $\kappa$ (4.2) imply that the earliest breakdown is at $n = 1/\kappa^*$. This corresponds to reaching an interior point that does not have a unique nearest point on $\Gamma$. 
Following the notation in [13, §2.1], given $x \in \Omega_-$, let
$$B(x) := \{y \in \Gamma : \|x - y\| = \text{dist}(x, \Gamma)\},$$
and let the medial axis $M_{\Omega_-}$ be defined by
$$M_{\Omega_-} := \{x \in \Omega_- : \text{card}(B(x)) \geq 2\}$$
(note that, with this definition, the medial axis is not closed, and the closure of the medial axis is called the cut locus). Since $\text{dist}(x, \Gamma)$ is differentiable at $x \in \Omega_-$ if and only if $\text{card}(B(x)) = 1$ [24, theorem 3.3], $M_{\Omega_-}$ is the set of points at which $\text{dist}(x, \Gamma)$ is not differentiable.

There are several slightly different notions in the literature that go by the names of the medial axis or ridge. For example, the definition of the ridge in [24, def. 3.6] allows it to contain points with $\text{card}(B(x)) \geq 1$, and the definition of the ridge used by [35] is $M_{\Omega_-}$ in our notation.

The following theorem collects some geometric properties of $M_{\Omega_-}$ that we need later.

**THEOREM 4.2 (Properties of the medial axis in two dimensions).**

(i) If $\Omega_-$ is a bounded, two-dimensional domain such that $\Gamma$ is piecewise analytic (i.e., the finite union of analytic curves), then $M_{\Omega_-}$ is a connected graph with finitely many vertices and edges, and each edge is an analytic curve.

(ii) If $\Omega_-$ is as in (i) and is also simply connected, then $M_{\Omega_-}$ is a tree.

(iii) If $\Omega_-$ is as in (i) and is also $C^2$, then there exists a constant $0 < n_0 \leq 1/k^*$ such that $\text{dist}(M_{\Omega_-}, \Gamma) \geq n_0$.

**PROOF.**

(i) This is proved in [14, theorem 8.2], [37, theorem 5.6], and [13, theorem 2.1 and cor. 2.1].

(ii) This is a consequence of the main result in [36].

(iii) If $U$ is a bounded open set, then $\partial U \in C^k$ implies that $\text{dist}(-, \partial U)$ is $C^k$ in a neighborhood of $\partial U$ for $k \geq 2$ [27, lemma 14.16, p. 355], [25]. Therefore, $\text{dist}(x, \Gamma)$ is differentiable in a neighborhood of $\Gamma$, and then, since $M_{\Omega_-}$ has finitely many vertices and edges (by (i)), $\text{dist}(M_{\Omega_-}, \Gamma)$ is bounded below by a positive constant, which we denote by $n_0$. The inequality $n_0 \leq 1/k^*$ follows from the facts that the osculating circle to a point on the boundary has radius $1/k(s)$, and centers of osculating circles are in $M_{\Omega_-}$ [8, lemma 2.2].

Counterexamples to point (i) in the theorem above when $\Gamma$ is only $C^\infty$ and not analytic can be found in [14, §2], and an example of a $C^{1,1}$, convex domain such that $M_{\Omega_-}$ has positive Lebesgue 2-measure can be found in [37, §3]. These examples demonstrate how the “nice” behavior of $M_{\Omega_-}$ under piecewise analyticity can disappear for domains that are only $C^\infty$. 


4.2 Definition of a \( Z \) Satisfying Condition A

For a fixed \( R > 0 \), we construct a \( W \) satisfying Condition A and then let \( Z = \nabla \phi \). (Note that we always assume that the origin from which \( B_R \) is defined is inside \( \Omega_- \).

Under the assumption that \( Z = \nabla \phi \), the requirements of Condition A become as follows:

A1. \( \phi \) is piecewise \( C^3 \) up to the boundary, i.e., \( \phi \in C^3(\Omega_-) \cap C^3(\Omega_+ \cap B_R) \).
A2. \( (\nabla \phi)_+ = (\nabla \phi)_- = C_\Gamma n \) and \( (\Delta \phi)_+ = (\Delta \phi)_- \) on \( \Gamma \).
A3. \( \phi = \frac{1}{2} r^2 \) in a neighborhood of \( \partial B_R \).
A4. There exists a \( \theta > 0 \) such that \( D^2 \phi(x) \geq \theta \) (in the sense of quadratic forms) for all \( x \in \Omega_- \cup (\Omega_+ \cap B_R) \), where \( (D^2 \phi)_{ij} = \partial_i \partial_j \phi \). (Note that we have lost the \( \theta \) that was in front of the original condition in terms of \( Z \) since \( \phi \) is real and \( D^2 \phi \) is symmetric.)

Let \( \phi \) be defined piecewise by \( \phi^+ := \phi^+ \in \Omega_+ \) and \( \phi^- := \phi^- \in \Omega_- \). The overview of how \( \phi^\pm \) are defined is as follows:

\[ \phi^+ \text{ is a smooth transition between} \]
\[ \begin{cases} \phi_{ML}, \text{ which satisfies the requirement A3 on } \partial B_R, \\ \phi_{\Gamma}^+, \text{ which satisfies the requirement A2 on } \Gamma. \end{cases} \]

\[ \phi^- \text{ is a smooth transition between} \]
\[ \begin{cases} \phi_{\Gamma}^-, \text{ which satisfies the requirement A2 on } \Gamma, \\ \phi_x, \text{ which is } \phi_{\Gamma}^- \text{ smoothed near } M_{\Omega_-}. \end{cases} \]

The functions \( \phi_{ML}, \phi_{\Gamma}^+, \phi_{\Gamma}^-, \) and \( \phi_x \) are all uniformly convex, and from this we are able to ensure that the positivity condition A4 on \( D^2 \phi \) is satisfied. Indeed, \( \phi \) defined below depends on two parameters, \( R \) and \( \varepsilon \) (\( R \) is not quite \( R \), the radius of \( B_R \), but is closely related). We show in Section 4.3 below that A4 is satisfied if \( R \) is large enough and \( \varepsilon \) is small enough, and that taking \( R \) large enough is equivalent to taking \( R \) large enough.

**Definition of \( \phi^+ \).** Let \( \eta(x) = \text{dist}(x, \Gamma) \) and let \( \chi(n) \in C^\infty[0, \infty) \) be monotonically decreasing, equal to 1 in a neighborhood of \( n = 0 \), equal to 0 in a neighborhood of \( n = 1 \), and then identically 0 for \( n \geq 1 \). For a fixed \( R > 0 \), define \( \chi_R(n) = \chi(n/R) \).

Define \( \phi^+ \) in terms of two other functions, \( \phi_{\Gamma}^+ \) and \( \phi_{ML} \), by

\[ \phi^+(x) := \chi_R(n(x)) \phi_{\Gamma}^+(x) + \frac{1 - \chi_R(n(x))}{\phi_{ML}(x)}, \quad x \in \Omega_. \]

The function \( \phi_{\Gamma}^+ \) is defined by

\[ \phi_{\Gamma}^+(x) := C_{\Gamma} n(x) + \frac{1}{2} n(x)^2. \]
where $C_T = 1/\kappa_*$ (recall that $\Omega_-$ being uniformly convex implies that $\kappa_* > 0$).

The function $\phi_{ML}$ is defined by
\begin{equation}
\phi_{ML}(x) := \frac{1}{2} r^2
\end{equation}
where $r := |x|$. (The subscript ML stands for “Morawetz-Ludwig,” since the gradient of $\frac{1}{2} r^2$ is the vector field $x$ that appears in the Morawetz-Ludwig identity (2.11).)

**Definition of $\phi_-$.** Let $n_0$ be as in Theorem 4.2 (i.e., $n(x) = \text{dist}(x, \Gamma)$ is differentiable when $0 < n < n_0$). Let $\chi_-(n) \in C^\infty[0, \infty)$ be monotonically decreasing, equal to 1 for $n \in [0, n_0/3]$, equal to 0 for $n \in [2n_0/3, \infty)$, and such that all its derivatives are 0 at $n = n_0/3$ and $n = 2n_0/3$.

Define $\phi_-$ in terms of $\phi_{\Gamma}^-$ and $\phi_\varepsilon$ by
\begin{equation}
\phi_-(x) := \chi_-(n(x)) \phi_{\Gamma}^-(x) + (1 - \chi_-(n(x))) \phi_\varepsilon(x), \quad x \in \Omega_-
\end{equation}
(note that the definition of $\chi_-$ implies that $\phi_\varepsilon = \phi_{\Gamma}^-$ for $0 \leq n \leq n_0/3$, and $\phi_\varepsilon = \phi_\varepsilon$ for $n \geq 2n_0/3$).

The function $\phi_{\Gamma}^-$ is defined for all $x \in \Omega_-$ by
\begin{equation}
\phi_{\Gamma}^-(x) = -C_{\Gamma} n(x) + \frac{1}{2} n(x)^2.
\end{equation}
where (as above) $C_{\Gamma} = 1/\kappa_*$. To define $\phi_\varepsilon$, first define the set $D$ by
\begin{equation}
D := \{ x \in \Omega_-, \text{dist}(x, \Gamma) \geq n_0/3 \},
\end{equation}
and note that from the definitions of $\phi_\varepsilon$ and $\chi_-$ we only need to define $\phi_\varepsilon$ on $D$. For $x \in D$ and $\varepsilon < n_0/3$, $\phi_\varepsilon(x)$ is defined by
\begin{equation}
\phi_\varepsilon(x) := \int_{B_\varepsilon(0)} \phi_{\Gamma}^-(x - y) \eta_\varepsilon(y) dy = \int_{B_\varepsilon(x)} \phi_{\Gamma}^-(y) \eta_\varepsilon(x - y) dy,
\end{equation}
where (following, e.g., [23 §C.4])
\begin{equation}
\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}
\end{equation}
$C$ is selected so that $\int_{\mathbb{R}^d} \eta(x) dx = 1$, and $\eta_\varepsilon(x) := \eta(x/\varepsilon)/\varepsilon^d$.

**4.3 Proof of Lemma 4.1 in Two Dimensions (i.e., That $Z$ Defined in Section 4.2 Satisfies Condition [A])**

We first check Condition [A1] (the smoothness) for both $\phi^+$ and $\phi^-$, then Conditions [A2–A4] for $\phi^+$, and finally Conditions [A2–A4] for $\phi^-$. 

**Checking [A1] for Both $\phi^+$ and $\phi^-$.** We need $\phi$ to be piecewise $C^3$ up to the boundary. Recall that $\phi$ is a smooth transition between $\phi_{\Gamma}^+$ and $\phi_{ML}$ in $\Omega_+$ and $\phi_{\Gamma}^-$ and $\phi_\varepsilon$ in $\Omega_-$. Now $\phi_{ML} \in C^\infty(\mathbb{R}^d)$ and, by properties of mollifiers (see,
Therefore, if $R\in \mathbb{R}$ and thus $(4.17)$ n.

Since we are assuming that the origin is inside $B$ radius of $e.g., [23, §C.4, theorem 6]), there exists an $R$ convex if $C$ its value is $1$ is $\delta$ continuous across $(4.15)$ dient in $n.s/$. Therefore, if $(4.16)$ (4.10), implies that $R$ and so the choice $C$ of $C$. If $n$ is smallest when $n=0$ and the (2,2)-element is smallest when $n=1$. Indeed, looking at the (2,2)-element of $D^2\phi^T+$, given by (4.16), as a function of $n \in [0, \infty)$, and writing $(C_T + n)\kappa(s)$ as $C_T\kappa(s) - 1 + (1 + n\kappa(s))$, we see that if $C_T\kappa(s) \geq 1$ for all $s$, then the (2,2)-element is smallest when $n=\infty$ and its value is $1$. If $C_T\kappa(s) < 1$ for some $s$, then the (2,2)-element is smallest when $n=0$ and its value is $C_T\kappa(s)$. Therefore,

$$D^2\phi^T_+ \geq \min(1, C_T\kappa_*)$$

and so the choice $C_T = 1/\kappa_*$ gives $D^2\phi^T_+(x) \geq 1$ for all $x \in \Omega_+$. The definition of $\phi_{\text{ML}}$, (4.10), implies that $\nabla\phi_{\text{ML}}(x) = x$ and hence $D^2\phi_{\text{ML}} = I$.

Using the uniform convexity of $\phi^T_+$ and $\phi_{\text{ML}}$, we now show that $\phi^+$ is uniformly convex if $R$ is large enough.

**Lemma 4.3** ($\phi^+$ is uniformly convex if $R$ is large enough). Given $\delta > 0$ there exists an $R_0$ such that, for all $R \geq R_0$, $D^2\phi^+(x) \geq (1-\delta)$ for all $x \in \Omega_+ \cap B_R$. 

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PROOF. From (4.18) above, we only need to show that, given \( \delta > 0 \) there exists an \( R_0 \) such that, for all \( R \geq R_0 \), \( D^2 \phi^+ \geq (1 - \delta) \) for all \( x \in \Omega_+ \cap \{ n \leq R \} \), and then we set \( R_0 = R_0 + d_{\Omega_-} \).

Differentiating twice the definition of \( \phi^+ \), equation (4.8), yields that

\[
D^2 \phi^+ = \chi_R D^2 \phi^+_T + (1 - \chi_R) D^2 \phi_{ML} + D^2 \chi_R (\phi^+_T - \phi_{ML})
\]

(4.19)

where

\[
(a \otimes_s b)_{ij} := \frac{a_i b_j + a_j b_i}{2}.
\]

From the fact that \( D^2 \phi^+_T(x) \) and \( D^2 \phi_{ML}(x) \) are both \( \geq 1 \) for all \( x \in \Omega_+ \), we see that the first two terms of (4.19) are \( \geq 1 \). We now need to show that the third and fourth terms are \( o(1) \) as \( R \to \infty \), which gives the assertion. Equation (4.17) implies that

\[
r = n + \mathcal{O}(1) \quad \text{as } n \to \infty,
\]

and simple geometry gives us that

\[
\hat{e}_r = \hat{e}^+_n + \mathcal{O}

\]

(4.20)

Using these asymptotics in the definitions of \( \phi^+_T \) and \( \phi_{ML} \) and the expressions for \( \nabla \phi^+_T \) (4.15) and \( \nabla \phi_{ML} \), we find that

\[
\phi^+_T(x) - \phi_{ML}(x) = \mathcal{O}(n) \quad \text{as } n \to \infty,
\]

(4.20)

and

\[
\nabla \phi^+_T(x) - \nabla \phi_{ML}(x) = \mathcal{O}(1) \quad \text{as } n \to \infty.
\]

(4.21)

Using (4.20) and the fact that

\[
D^2 \chi_R(n(x)) = \frac{1}{R^2} D^2 \chi \left( \frac{n(x)}{R} \right) = \mathcal{O} \left( \frac{1}{R^2} \right) \quad \text{as } R \to \infty,
\]

uniformly for \( x \in \Omega_+ \cap \{ n \leq R \} \), we obtain the following bound on the third term in (4.19):

\[
|D^2 \chi_R(n(x))(\phi^+_T(x) - \phi_{ML}(x))| = \mathcal{O} \left( \frac{1}{R} \right) \quad \text{as } R \to \infty,
\]

(4.22)

uniformly for \( x \in \Omega_+ \cap \{ n \leq R \} \). Using (4.21) and the fact that

\[
|\nabla \chi_R(n(x))| = \mathcal{O} \left( \frac{1}{R} \right) \quad \text{as } R \to \infty,
\]

uniformly for \( x \in \Omega_+ \cap \{ n \leq R \} \), we obtain the following bound on the fourth term in (4.19),

\[
|\nabla \chi_R(n(x)) \otimes_s (\nabla \phi^+_T(x) - \nabla \phi_{ML}(x))| = \mathcal{O} \left( \frac{1}{R} \right) \quad \text{as } R \to \infty,
\]

(4.23)

uniformly for \( x \in \Omega_+ \cap \{ n \leq R \} \). Using (4.22) and (4.23) in (4.19) then proves that \( D^2 \phi^+(x) \geq (1 - o(1)) \) as \( R \to \infty \), uniformly for \( x \in \Omega_+ \cap \{ n \leq R \} \). □
Checking $\mathbb{A}2$, $\mathbb{A}3$ for $\phi^-$. By the discussion about the $(n, s)$-coordinate system in Section 4.1 and parts (ii) and (iii) of Theorem 4.2, given any $x \in \Omega_\gamma \setminus \mathcal{M}_{\Omega_\gamma}$ there exist $(n, s)$ such that $x = (n, s)$ in the orthogonal coordinate system defined by $\Gamma_\gamma$ and $n \in (0, 1/\kappa(s))$. The definition of $\phi_{\Gamma^-}$, (4.12), and the analogues of (4.4) and (4.6) for $\Omega_\gamma$ then imply that

$$
\nabla \phi_{\Gamma^-}(n, s) = (-C_{\Gamma} + n) \hat{e}_n(s)
$$

and

$$
D^2 \phi_{\Gamma^-}(n, s) = \begin{pmatrix}
1 & 0 \\
0 & \frac{(C_{\Gamma} - n)\kappa(s)}{1 - n\kappa(s)}
\end{pmatrix}.
$$

Therefore, on $\Gamma$ (i.e., $n = 0$) $\nabla \phi_{\Gamma^-} = -C_{\Gamma} \hat{e}_n = C_{\Gamma} \mathbf{n}$, which fulfills part of $\mathbb{A}2$. To check the final requirement of $\mathbb{A}2$, namely that $\nabla \cdot \mathbf{Z} = \Delta \phi$ is continuous across $\Gamma$, note that equations (4.16) and (4.25) imply that

$$
D^2 \phi_{\Gamma}^- (0, s) = \begin{pmatrix}
1 & 0 \\
0 & C_{\Gamma}
\end{pmatrix} = D^2 \phi_{\Gamma}^+(0, s),
$$

and so $\Delta \phi$ is continuous (being a particular linear combination of elements of the matrix).

For the uniform convexity condition, $\mathbb{A}4$, we need to show that there exists a $\theta > 0$ such that $D^2 \phi^-(x) \geq \theta$ for all $x \in \Omega_-$. Our first step is to show that $D^2 \phi_{\Gamma^-}(x) \geq 1$ for all $x \in \Omega_\gamma \setminus \mathcal{M}_{\Omega_\gamma}$. Writing $(C_{\Gamma} - n)\kappa(s)$ as $C_{\Gamma}\kappa(s) - 1 + (1 - n\kappa(s))$, we see that if $C_{\Gamma}\kappa(s) \geq 1$, then the smallest value of the $(2, 2)$-element of $D^2 \phi_{\Gamma^-}$ as a function of $n \in [0, 1/\kappa(s)]$ is 1, occurring when $n = 0$. If $C_{\Gamma}\kappa(s) < 1$, then the smallest value is 0, occurring when $n = C_{\Gamma}$; this corresponds to the quadratic term in $\phi_{\Gamma^-}$ “kicking in too soon” and making the derivative of $\phi_{\Gamma^-}$ in the $\hat{e}_n$-direction positive. The choice $C_{\Gamma} = 1/\kappa_\gamma$ therefore ensures that $D^2 \phi_{\Gamma^-}(n, s) \geq 1$ for all $s$ and for all $n \in (0, 1/\kappa(s))$.

The next step is to prove that $\phi_{\varepsilon}$ is uniformly convex.

**Lemma 4.4** ($\phi_{\varepsilon}$ is uniformly convex if $\varepsilon$ is small enough). With $\phi_{\varepsilon}$ defined by (4.14), there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \leq \varepsilon_0$, $D^2 \phi_{\varepsilon}(x) \geq 1$ for all $x \in D$ (where $D$ is the set defined by (4.13)).

We assume this result for the moment and use it to prove uniform convexity of $\phi^-$. 

**Lemma 4.5** ($\phi^-$ is uniformly convex if $\varepsilon$ is small enough). Given $\delta > 0$ there exists an $\varepsilon_1 > 0$ such that, for all $\varepsilon \leq \varepsilon_1$, $D^2 \phi^-(x) \geq (1 - \delta)$ for all $x \in \Omega_-$. 

**Proof of Lemma 4.5**. Differentiating twice equation (4.11), the definition of $\phi^-$, yields that

$$
D^2 \phi^- = \chi_- D^2 \phi_{\Gamma^-} + (1 - \chi_-) D^2 \phi_{\varepsilon} \\
+ D^2 \chi_- (\phi_{\Gamma^-} - \phi_{\varepsilon}) + 2 \nabla \chi_- \otimes \nabla (\phi_{\Gamma^-} - \nabla \phi_{\varepsilon}).
$$
Now $D^2\phi^-_g(x) \geq 1$ for all $x \in \Omega_\epsilon \setminus \overline{M_{\Omega_\epsilon}}$, and so certainly for all $x \in \supp \chi_- = \{n : 0 \leq n \leq 2n_0/3\}$. Furthermore, Lemma 4.4 implies that $D^2\phi_e(x) \geq 1$ for all $x \in D = \supp(1 - \chi_-)$. These two facts imply that the first two terms in (4.26) are $\geq 1$ for all $x \in \Omega_- \epsilon$.

We now prove that the third and fourth terms of (4.26) are $o(1)$ as $\epsilon \to 0$. Since $D^2\chi_- \text{ and } \nabla \chi_-$ have support only in $\{n : n_0/3 \leq n \leq 2n_0/3\}$, it is sufficient to prove that $\phi_\epsilon \to \phi^-_g \text{ and } \nabla \phi_\epsilon \to \nabla \phi^-_g$ as $\epsilon \to 0$ on this set. These limits follow from the facts that $\phi^-_g \in C(\Omega_-)$ and $\nabla \phi^-_g \in (C(\Omega_- \setminus \{n : n \geq n_0\}))^d$ using a standard property of mollifiers, namely that if $U$ is open and $f \in C(U)$, then $f_s \to f$ uniformly on compact subsets of $U$ (e.g., [24] Theorem 6).

All that remains is to prove Lemma 4.4 i.e., that $\phi_\epsilon$ is uniformly convex.

**Proof of Lemma 4.4** We split the proof up into cases: (i) $B_\epsilon(x) \cap \overline{M_{\Omega_\epsilon}} = \emptyset$ and (ii) $B_\epsilon(x) \cap \overline{M_{\Omega_\epsilon}} \neq \emptyset$.

In case (i), we differentiate under the integral sign in the expression for $\phi_\epsilon$ in (4.14) in which the $x$-dependence under the integral sign is in $\phi^-_g$. This is allowed since, from above, $\phi^-_g \in C^2(\Omega_- \setminus \overline{M_{\Omega_\epsilon}})$. Using the facts that (with $C_g = 1/\kappa_*$) $D^2\phi^-_g(x) \geq 1$ for all $x \in \Omega_- \setminus \overline{M_{\Omega_\epsilon}}$ and $\int_{B_\epsilon(x)} \eta_\epsilon(x-y)dy = 1$ in the resulting expression shows that $D^2\phi_\epsilon \geq 1$.

In case (ii), we begin by recalling from Theorem 4.2 that $M_{\Omega_\epsilon}$ is a tree with finitely many vertices and edges. Following [14], we introduce the terminology that a vertex with degree $\geq 3$ is a *bifurcation point*, and a vertex with degree equal to 1 is a *terminal point*.

If $B_\epsilon(x) \cap \overline{M_{\Omega_\epsilon}} \neq \emptyset$, then there are now three different cases:

1. there are no bifurcation points or terminal points of $M_{\Omega_\epsilon}$ in $B_\epsilon(x)$,
2. there are no bifurcation points of $M_{\Omega_\epsilon}$ in $B_\epsilon(x)$, but at least one terminal point,
3. there is at least one bifurcation point of $M_{\Omega_\epsilon}$ in $B_\epsilon(x)$ (and possibly also terminal points).

We first consider case (1) and then show afterwards how cases (2) and (3) can be reduced to the first case. We let $\Sigma := \overline{M_{\Omega_\epsilon}} \cap B_\epsilon(x)$ and differentiate under the integral sign in the expression for $\phi_\epsilon$ in (4.14) in which the $x$-dependence under the integral sign is in $\eta_\epsilon$. Since $\partial_x \eta_\epsilon(x-y) = -\partial_y \eta_\epsilon(x-y)$, we find that

$$\partial_i \partial_j \phi_\epsilon(x) = \int_{B_\epsilon(x)} \phi^-_g(y) \partial_i \partial_j \eta_\epsilon(x-y)dy,$$

where, to avoid an excess of notation, we have omitted the $x$- or $y$-dependence from the derivatives, but highlight that on the left-hand side they are in $x$, and under the integral on the right-hand side they are in $y$.

Our plan is to integrate the right-hand side of (4.27) by parts to move the differentiation from $\eta_\epsilon$ to $\phi^-_g$, and then use the fact that $D^2\phi^-_g(x) \geq 1$ for all $x \in \Omega_- \setminus \Sigma$. 


Let $\Sigma$ divide $B_\varepsilon(x)$ into $B^+$ and $B^-$, and let $v$ be the unit normal to $\Sigma$ pointing into $B^+$. In order to apply the divergence theorem in $B^+$ and $B^-$, we need some information about the smoothness of $\Sigma$. Theorem 4.2 and the fact that we are in case (1) above imply that $\Sigma$ is analytic; thus, for small enough $\varepsilon$, $\partial B^\pm$ are Lipschitz and applying the divergence theorem in $B^\pm$ is allowed by, e.g., [40, theorem 3.34]. Integrating by parts (and recalling that $v$ points into $B^+$), we have that

$$\int_{B^+} \phi^-_\Gamma(y) \partial_i \partial_j \eta_\varepsilon(x - y) dy = - \int_{B^+} \partial_i \phi^-_\Gamma(y) \partial_j \eta_\varepsilon(x - y) dy$$

$$- \int_{\Sigma} v_i(y) \phi^-_\Gamma(y) \partial_j \eta_\varepsilon(x - y) ds(y)$$

(the integral over $\partial B^+ \cap \partial B_\varepsilon(x)$ equals 0 as $\eta_\varepsilon$ is 0 here). A similar result holds for the integral over $B^-$ (with the sign of the integral over $\Sigma$ reversed), and thus, since $\phi^-_\Gamma$ is continuous across $\Sigma$,

(4.28) $$\int_{B_\varepsilon(x)} \phi^-_\Gamma(y) \partial_i \partial_j \eta_\varepsilon(x - y) dy = - \int_{B_\varepsilon(x)} \partial_i \phi^-_\Gamma(y) \partial_j \eta_\varepsilon(x - y) dy,$$

where $\partial_i \phi^-_\Gamma(y)$ in the integral on the right-hand side is understood piecewise.

Integrating by parts again we have that

(4.29) $$\int_{B_\varepsilon(x)} \partial_i \phi^-_\Gamma(y) \partial_j \eta_\varepsilon(x - y) dy = - \int_{B_\varepsilon(x)} \partial_j \partial_i \phi^-_\Gamma(y) \eta_\varepsilon(x - y) dy$$

$$- \int_{\Sigma} [\partial_i \phi^-_\Gamma(y)]^+ v_j(y) \eta_\varepsilon(x - y) ds(y),$$

and then putting (4.27), (4.28), and (4.29) together we obtain that

$$\partial_i \partial_j \phi_\varepsilon(x) = \int_{B_\varepsilon(x)} \partial_j \partial_i \phi^-_\Gamma(y) \eta_\varepsilon(x - y) dy$$

(4.30) $$+ \int_{\Sigma} [\partial_i \phi^-_\Gamma(y)]^+ v_j(y) \eta_\varepsilon(x - y) ds(y),$$

where $\partial_j \partial_i \phi^-_\Gamma(y)$ in the first integral on the right-hand side is understood piecewise.

If we can show that

(4.31) $$[\partial_i \phi^-_\Gamma(y)]^+ v_j(y) \xi_i \xi_j \geq 0$$

for all $\xi \in \mathbb{R}^d$ and $y \in \Sigma$, 
then, using this in (4.30) along with the facts that \( D^2_y \phi(y) \geq 1 \) for all \( x \in \Omega_- \setminus \Sigma \), \( \eta_e \geq 0 \), and \( \int_{B_e(x)} \eta_e(x-y)dy = 1 \), we find that

\[
D^2_x \phi_e(x) \geq \int_{B_e(x)} D^2_y \phi(y) \eta_e(x-y)dy \geq \int_{B_e(x)} \eta_e(x-y)dy = 1,
\]

which is the result.

We now prove that the inequality (4.31) holds. Since \( y \) is in \( \Sigma \) and is not a bifurcation point, then there exist \((n_1, s_1), (n_2, s_2)\) such that, in \((n, s)\)-coordinates, \( y = (n_j, s_j), \quad j = 1, 2, \) with \( n_1 = n_2 \) but \( s_1 \neq s_2 \). Let \((0, s_1)\) be the closest point on \( \Gamma \) to \( y \) on the + side of \( \Sigma \), and \((0, s_2)\) be the closest point on \( \Gamma \) to \( y \) on the – side of \( \Sigma \). The expression for \( \nabla \Phi^-_{\Gamma} \), (4.24), implies that

\[
[\partial_i \Phi^-_{\Gamma}(y)]^+ \nu_j(y) = -(C_{\Gamma} - n_1)(\vec{e}_n^- (s_1) - \vec{e}_n^- (s_2))_i \nu_j(y).
\]

Since \( n_1 \leq C_{\Gamma} \) (as \( n_1 \leq 1/k(s_1) \) and \( C_{\Gamma} = 1/k_\star \)) it is sufficient to prove that

(4.32) \( (\vec{e}_n^- (s_1) - \vec{e}_n^- (s_2)) \otimes \nu(y) \leq 0 \) for all \( y \in \Sigma \),

in the sense of quadratic forms. Recall that \( \nu(y) \) is the unit normal vector to \( \Sigma \) at \( y \) that points into \( B^+ \), and let \( \tau(y) \) be a unit tangent vector to \( \Sigma \) at \( y \) (there are two possible choices for \( \tau \), but which one we choose will not matter in what follows). Recall that if \( a \otimes b \leq 0 \) and \( B \) is an orthogonal matrix, then \( B a \otimes B b \leq 0 \).

Therefore, since \( \nu(y) \) and \( \tau(y) \) are orthonormal for every \( y \in \Sigma \), we can verify that (4.32) holds for a given \( y \in \Sigma \) by working in the \{\( \nu(y) \), \( \tau(y) \)\} basis. We find that the inequality (4.32) will hold if

(a) the component of \( (\vec{e}_n^- (s_1) - \vec{e}_n^- (s_2)) \) in the \( \nu(y) \)-direction is \( \leq 0 \), and

(b) the component of \( (\vec{e}_n^- (s_1) - \vec{e}_n^- (s_2)) \) in the \( \tau(y) \)-direction equals \( 0 \).

Since \( \nu \) points into \( B^+ \), (a) holds. Furthermore, since \((0, s_1)\) and \((0, s_2)\) lie on the circle with center \( y \), the definition of \( M_{\Omega_-} \) and elementary geometry imply that the tangent line to \( \Gamma \) at \((0, s_1)\) is the reflection of the tangent line to \( \Gamma \) at \((0, s_2)\) in the tangent line of \( \Sigma \) at \( y \); this implies that (b) holds.

We have now proved the result for the first of the three cases outlined above. Case (2) can be reduced to case (1) by extending \( \Sigma \) continuously so that the extended curve divides \( B_e(x) \) into two parts. Since \( \Phi^-_{\Gamma} \) and \( \nabla \Phi^-_{\Gamma} \) are continuous across the extension, the argument proceeds as before. For case (3), first extend \( \Sigma \) at all terminal points as in case (2). This extended curve now divides \( B_e(x) \) into a finite number of pieces (\( \geq 3 \)), and the argument in case (1) for two pieces generalizes in an obvious way.

\[ \square \]

### 4.4 Modifications Needed to the Above Arguments in Three Dimensions

The definition of \( \phi \) in three dimensions is exactly the same as the definition in two dimensions given in Section 4.2 (i.e., equations (4.8)–(4.11)). Indeed, \( \phi^\pm_{\Gamma} \) are defined only in terms of the distance function, \( \phi_{\text{ML}} \) only in terms of \( r \), and \( \phi_e \) only in terms of \( \phi^-_{\Gamma} \) and \( \eta_e \), and thus all these quantities are well-defined when \( d = 3 \). The only difference is that we now define \( \kappa^* \) and \( \kappa_\star \) to be the maximum and
minimum of the principal curvatures, respectively. (Recall that, given \( x \in \Gamma \), the two principal curvatures at \( x \) are such that the curvature of any geodesic curve on \( \Gamma \) passing through \( x \) lies between the principal curvatures.) As in the two-dimensional case, we choose \( C_{\Gamma} = 1/\kappa_+ \).

In the proof of Lemma 4.1 for \( d = 2 \) in Section 4.3 we used the \((n, s)\)-coordinate system defined by \( \Gamma \) to verify that

\[
\begin{align*}
\text{(i)} & \quad \nabla \phi^+_\Gamma = \nabla \phi^-_\Gamma = C_{\Gamma} \mathbf{n} \quad \text{and} \quad \Delta \phi^+_\Gamma = \Delta \phi^-_\Gamma \quad \text{on} \; \Gamma \quad \text{(this gave Condition A2)}, \\
\text{(ii)} & \quad D^2 \phi^+_\Gamma(x) \geq 1 \quad \text{for all} \; x \in \Omega_+ \quad \text{and} \quad D^2 \phi^-_\Gamma(x) \geq 1 \quad \text{for all} \; x \in \Omega_- \setminus \Sigma \quad \text{(this was needed for Condition A4)}.
\end{align*}
\]

The rest of the argument in Section 4.3 that \( \phi \) satisfies Condition A is valid both in two dimensions and in three dimensions. Indeed, the only other part of the argument that depended on the dimension was the proof of Lemma 4.4 (the uniform convexity of \( \Gamma \)). This proof relied on the results about the geometry of the medial axis in two dimensions given in Theorem 4.2. An appropriate analogue of Theorem 4.2 holds in the three-dimensional case. Indeed, the analogue of part (i) of Theorem 4.2 in three dimensions is that, roughly speaking, if \( \Gamma \) is piecewise analytic (i.e., is the finite union of analytic surfaces), then \( \mathcal{M}_{\Omega_-} \) is also piecewise analytic; see [13, theorem 2.1 and cor. 2.1] for a more precise statement of this result and its proof. The analogue of part (ii) is that \( \mathcal{M}_{\Omega_-} \) has the same homotopy type as \( \Omega_- \), and thus if \( \Omega_- \) is simply connected, then so is \( \mathcal{M}_{\Omega_-} \) (i.e., every closed curve on \( \mathcal{M}_{\Omega_-} \) can be continuously shrunk down to a point); see [36]. Finally, part (iii) of Theorem 4.2 holds in three dimensions as well as in two dimensions. Using this information about the geometry of \( \mathcal{M}_{\Omega_-} \), we can generalize the proof of Lemma 4.4 from two dimensions to three dimensions in a straightforward manner.

Therefore, to prove that \( \phi \) defined in Section 4.2 satisfies Condition A when \( d = 3 \), we only need to show that (i) and (ii) above hold. As in the two-dimensional case, we do this in coordinate systems defined by \( \Gamma \), but now these will only be local to each \( x \) instead of well-defined in all of \( \Omega_+ \) or \( \Omega_- \setminus \mathcal{M}_{\Omega_-} \). Indeed, whereas in two dimensions it is straightforward to construct orthogonal coordinate systems for all of \( \Omega_+ \) and \( \Omega_- \setminus \mathcal{M}_{\Omega_-} \), in three dimensions it is not. However, for (i), given an \( x \in \Gamma \) we can construct an orthogonal coordinate system defined by \( \Gamma \) in a neighborhood of that \( x \) and calculate \( \nabla \phi^+_\Gamma(x) \) and \( D^2 \phi^+_\Gamma(x) \) in this coordinate system; for (ii), given an \( x \in \Omega_+ \) or \( \Omega_- \setminus \mathcal{M}_{\Omega_-} \) we can construct an orthogonal coordinate system defined by \( \Gamma \) in a neighborhood of that \( x \) and calculate \( D^2 \phi^+_\Gamma(x) \) in this coordinate system.

We now give the details of the coordinate systems that we use in \( \Omega_+ \). Given a point \( P \) in \( \Omega_+ \), let \( r_0 \) be the position vector of the closest point on \( \Gamma \) to \( P \) (this is unique since \( \Omega_- \) is convex). Introduce a coordinate system on \( \Gamma \) in a neighborhood of \( r_0 \) with coordinates \((s, t)\) such that \( \partial r_0 / \partial s \) and \( \partial r_0 / \partial t \) are unit vectors in the principal directions at \( r_0 \) (and are hence orthonormal). (If \( r_0 \) is an umbilical point, i.e., \( \Gamma \) is locally spherical at \( r_0 \), then just choose \((s, t)\) such that \( \partial r_0 / \partial s \) and \( \partial r_0 / \partial t \)
are orthonormal tangent vectors.) Let \( \mathbf{n}(s, t) \) be the outward-pointing unit normal vector, and define \( \kappa_1(s) \) and \( \kappa_2(t) \) by

\[
\frac{\partial^2 \mathbf{r}_0}{\partial s^2}(s, t) = -\kappa_1(s) \mathbf{n}(s, t) \quad \text{and} \quad \frac{\partial^2 \mathbf{r}_0}{\partial t^2}(s, t) = -\kappa_2(t) \mathbf{n}(s, t),
\]

respectively. By the definition of the principal directions, \( \kappa_1(s) \) and \( \kappa_2(t) \) are the principal curvatures. Our definitions of \( \kappa_1 \) and \( \kappa_2 \) imply that \( \kappa_1 \leq \kappa_1(s), \kappa_2(t) \leq \kappa^* \), and the fact that \( \Omega_- \) is convex implies that \( \kappa_* \geq 0 \). We then have that

\[
(4.33) \quad \frac{\partial \mathbf{n}}{\partial s}(s, t) = \kappa_1(s) \frac{\partial \mathbf{r}_0}{\partial s}(s, t) \quad \text{and} \quad \frac{\partial \mathbf{n}}{\partial t}(s, t) = \kappa_2(t) \frac{\partial \mathbf{r}_0}{\partial t}(s, t)
\]

(compare to (4.3)).

The position vector \( \mathbf{r} \) of \( P \) can then be expressed as

\[
\mathbf{r}(n, s, t) = \mathbf{r}_0(s, t) + n \mathbf{n}(s, t),
\]

where, as before, \( n = \text{dist}(\mathbf{r}, \Gamma) \). The definition of the basis vectors and the relations in (4.33) imply that

\[
e^+_n(n, s, t) := \frac{\partial \mathbf{r}}{\partial n}(n, s, t) = \mathbf{n}(s, t),
\]

\[
e_s(n, s, t) := \frac{\partial \mathbf{r}}{\partial s}(n, s, t) = (1 + n\kappa_1(s)) \frac{\partial \mathbf{r}_0}{\partial s}(s, t),
\]

\[
e_t(n, s, t) := \frac{\partial \mathbf{r}}{\partial t}(n, s, t) = (1 + n\kappa_2(t)) \frac{\partial \mathbf{r}_0}{\partial t}(s, t),
\]

and thus

\[
h_n := \left| \frac{\partial \mathbf{r}}{\partial n} \right| = 1, \quad h_s := \left| \frac{\partial \mathbf{r}}{\partial s} \right| = 1 + n\kappa_1(s), \quad \text{and} \quad h_t := \left| \frac{\partial \mathbf{r}}{\partial t} \right| = 1 + n\kappa_2(t).
\]

Since the coordinate system is orthogonal, everything goes through as in the two-dimensional case, with

\[
\nabla \psi = \frac{1}{h_n} \frac{\partial \psi}{\partial n} e^+_n + \frac{1}{h_s} \frac{\partial \psi}{\partial s} e_s + \frac{1}{h_t} \frac{\partial \psi}{\partial t} e_t
\]

for scalar functions \( \psi : \Omega_+ \to \mathbb{R} \), and

\[
D\mathbf{v} = \begin{pmatrix}
\frac{\partial v^n}{\partial n} & 0 & 0 \\
0 & \frac{\partial v^s}{\partial s} & \frac{\partial v^t}{\partial s} \\
0 & 0 & \frac{\partial v^t}{\partial t}
\end{pmatrix}
\]

for vector fields \( \mathbf{v} : \Omega_+ \to \mathbb{R}^d \) such that \( v^s = v^t = 0 \) and \( v^n \) is a function of \( n \) only. The coordinate system in \( \Omega_- \) is analogous, except that now \( h_s = 1 - n\kappa_1(s) \) and \( h_t = 1 - n\kappa_2(t) \). Therefore, for a given \( (n, s, t) \), the coordinate system breaks down when \( n = 1/\max(\kappa_1(s), \kappa_2(t)) \), and so the earliest breakdown is at \( n = 1/\kappa^* \).

Performing the three-dimensional analogues of the two-dimensional calculations in Section 4.3, we see that (as in the two-dimensional case) (i) \( \nabla \phi^+ \) =
\[ \nabla \phi^- = C \nabla \mathbf{n} \text{ and } \Delta \phi^+ = \Delta \phi^- \text{ on } \Gamma, \text{ and (ii) the choice } C = 1/\kappa^* \text{ ensures that } D^2 \phi^+ \geq 1 \text{ in } \Omega_+ \text{ and } \Omega_- \setminus \mathcal{M}_{\Omega_-}. \]

5 Nonexistence of a Z Satisfying Either Condition [A] or Condition [B] for Nonconvex \( \Omega_- \)

In this section, we show that if \( \Omega_- \) is nonconvex, then the condition that \( Z = C \nabla \mathbf{n} \text{ on } \Gamma \) (Condition [A] or [B]) and the nonnegativity condition on \( \partial_i Z_j \) (Condition [A] or [B]) cannot be satisfied simultaneously. We restrict our attention to \( C^2 \) domains since both Conditions [A] and [B] assume this smoothness of \( \Gamma \).

**Lemma 5.1.** If \( \Omega_- \) is a bounded \( C^2 \) domain that is nonconvex, then there does not exist a real-valued \( Z \in (C^1(\overline{\Omega}_+ \cap B_R))^d \), for any \( R \) such that \( \overline{\Omega}_- \subset B_R \), satisfying both

- \( Z = C \nabla \mathbf{n} \text{ on } \Gamma \) for some constant \( C > 0 \) and
- \( \partial_i Z_j(x) \xi_i \xi_j \geq 0 \) for all \( \xi \in \mathbb{R}^d \) and \( x \in \overline{\Omega}_+ \cap B_R \).

**Proof.** Since \( \Omega_- \) is nonconvex and \( C^2 \), there exists a geodesic curve \( \Gamma^* \subset \Gamma \) that has negative curvature. That is, if \( \Gamma^* := \{ r_0(s) : a \leq s \leq b \} \) and \( \kappa(s) \) is defined by (4.1), then there exists a constant \( \kappa_0 < 0 \) such that \( \kappa(s) \leq \kappa_0 \) for all \( s \in (a, b) \).

The idea of the proof is to lift \( \Gamma^* \) off \( \Gamma \) in the normal direction, calculate the derivative of the length of the lifted curve with respect to the distance from \( \Gamma \) in two different ways (one using the curvature, the other using the fact that \( Z = C \nabla \mathbf{n} \text{ on } \Gamma \)), and reach a contradiction.

Let \( r(s; \varepsilon) := r_0(s) + \varepsilon \mathbf{n}(s) \), and thus \( \{ r(s; \varepsilon) : a \leq s \leq b \} \) is the curve \( \Gamma^* \) lifted outwards in the normal direction by \( \varepsilon \). This definition and the expression (4.3) for \( \mathbf{n}/ds \) imply that

\[
\frac{dr}{ds}(s; \varepsilon) = \frac{dr_0}{ds}(s) + \varepsilon \frac{\mathbf{n}}{ds} = (1 + \varepsilon \kappa(s)) \frac{dr_0}{ds}(s),
\]

and so

\[
\left( \frac{dr}{ds}(s; \varepsilon) \right)^2 = (1 + \varepsilon \kappa(s))^2.
\]

Let \( I(\varepsilon) \) denote the length of \( \{ r(s; \varepsilon) : a \leq s \leq b \} \), i.e.,

\[
I(\varepsilon) := \int_a^b \left| \frac{dr}{ds}(s; \varepsilon) \right| ds.
\]

Equation (5.1) implies that, for sufficiently small \( \varepsilon \),

\[
I(\varepsilon) = \int_a^b (1 + \varepsilon \kappa(s)) ds.
\]
and thus

\[
\frac{dI}{de}(0) = \int_a^b \kappa(s) \, ds \leq \kappa_0(b - a) < 0.
\]

On the other hand, since \( Z = C_r \mathbf{n} \) on \( \Gamma \),

\[
r(s; \varepsilon) = r_0(s) + \varepsilon C_r^{-1} Z(r_0(s)).
\]

To condense notation, let \( x_j \) denote the \( j \)th component of \( r_0 \). Differentiating (5.3) to obtain \( dr/ds \), we find that

\[
\left| \frac{dr}{ds}(s; \varepsilon) \right|^2 = 1 + 2 \varepsilon C_r^{-1} \partial_i Z_j(r_0(s)) \frac{dx_i}{ds}(s) \frac{dx_j}{ds}(s) + \mathcal{O}(\varepsilon^2)
\]

as \( \varepsilon \to 0 \).

Using this in the definition of \( I(\varepsilon) \) yields

\[
I(\varepsilon) = \int_a^b \left| \frac{dr}{ds}(s; \varepsilon) \right| ds
\]

\[
= \int_a^b ds + \varepsilon C_r^{-1} \int_a^b \partial_i Z_j(r_0(s)) \frac{dx_i}{ds}(s) \frac{dx_j}{ds}(s) ds + \mathcal{O}(\varepsilon^2),
\]

and thus

\[
\frac{dI}{de}(0) = C_r^{-1} \int_a^b \partial_i Z_j(r_0(s)) \frac{dx_i}{ds}(s) \frac{dx_j}{ds}(s) ds.
\]

The facts that (i) \( \partial_i Z_j(\mathbf{x}) \xi_i, \xi_j \geq 0 \) for all \( \xi, \xi_j \in \mathbb{R}^d \) and \( \mathbf{x} \in \Omega_+ \cap B_R \) and that (ii) \( C_r > 0 \), then imply that \((dI/de)(0) \geq 0\), contradicting (5.2).

\[\square\]

Remark 5.2 (Can one of Conditions \( A \) or \( B \) be satisfied when \( \Omega_- \) is only convex (as opposed to uniformly convex)?) In Lemma 4.1 we constructed a \( Z \) (equal to the gradient of a scalar function \( \phi \)) satisfying Condition \( A \) when \( \Omega_- \) is a uniformly convex domain, and we just showed in Lemma 5.1 above that there does not exist a \( Z \) satisfying either Condition \( A \) or Condition \( B \) when \( \Omega_- \) is nonconvex.

The task remains to construct a \( Z \) (either the gradient of some function \( \phi \) or otherwise) satisfying either Condition \( A \) or Condition \( B \) when \( \Omega_- \) is a smooth convex domain (i.e., with \( \Gamma \) allowed to contain straight line segments). In two dimensions such a \( Z \) was essentially constructed in \( \Omega_+ \) by [48, §4]; indeed (in the notation of that paper) the extension of the vector field \( l \) to \( B^e \) satisfies the parts of Condition \( B \) that concern \( \Omega_+ \). Given this fact, one might ask why we did not use the construction of [48, §4] in Section 4. The reason is that the construction of \( Z \) for uniformly convex domains in Section 4 is such that the two-dimensional version generalizes almost immediately to three dimensions, but this is not the case for the construction in [48, §4].
Green’s identity: \( \overline{v}(\Delta u + k^2 u) = \nabla \cdot [\overline{v} \nabla u] - \nabla u \cdot \overline{v} + k^2 u \overline{v} \)

When \( v = u \), nondivergence terms on the right-hand side are not single-signed.

A. Difficult to prove \( k \)-explicit bounds for Helmholtz problems.

B. Variational (i.e., weak) formulations of Helmholtz problems are not coercive for large \( k \).

C. Only prove boundary integral equations are coercive up to a compact perturbation (see Section 1.4).

**Figure 6.1.** The consequences of Green’s identity for the analysis and numerical analysis of the Helmholtz equation.

Morawetz’s identities: \( \overline{Z} v(\Delta u + k^2 u) = \nabla \cdot \{ \cdots \} + \cdots \), where \( \overline{Z} v = Z \cdot \nabla v - ik \beta v + \alpha v \)

(when \( v = u \) these are equations [23], [23]).

When \( v = u \), nondivergence terms on the right-hand side are single-signed.

(under nonnegativity condition on \( \partial_z \mathbf{u} \)).

A1. Can prove \( k \)-explicit bounds for exterior Helmholtz problems: [46, 47].

A2. Can prove \( k \)-explicit bounds for interior impedance problem: [41], [48], [29].

B. New coercive variational formulations of exterior Dirichlet and interior impedance problems: [44].

C1. New coercive boundary integral equation for exterior Dirichlet problem (equation (1.34)): [58].

C2. Coercivity of standard boundary integral equation for exterior Dirichlet problem: this paper.

**Figure 6.2.** The consequences of Morawetz’s identities for the analysis and numerical analysis of the Helmholtz equation.

6 Conclusion: Identities for the Helmholtz Equation

In this conclusion we attempt to place this paper’s use of Morawetz’s identities into a wider context. We do this with the two diagrams Figures [6.1] and [6.2] which contrast the properties and uses of Green’s identity with those of Morawetz’s identities.

We make the following two remarks regarding Figure [6.2].

(i) The \( k \)-explicit bounds in A2 are for the interior impedance problem, i.e., the problem of finding \( u \) such that \( \Delta u + k^2 u = -f \) in \( \Omega \) and \( \partial u / \partial n - i\eta u = g \)
on $\Gamma$ for given $f$, $g$, and $\eta$ (with $\eta \in \mathbb{R} \setminus \{0\}$). For this problem, one can use the multiplier $Z \cdot \nabla u + \alpha u$ (i.e., $\beta = 0$ in $Zu$) and, furthermore, in all the references in the figure ([41], [18], and [29]) $Z$ is chosen to be $x$. The resulting identity is then equivalent to adding the Rellich identity with multiplier $x \cdot \nabla u$ (introduced by Rellich in [52]) to Green’s identity multiplied by $\alpha$, and this is how this method of obtaining bounds was understood in [18, 29, 41]. Note that the analogue of these bounds for the time-harmonic Maxwell equations was obtained in [30, theorem 4.6] and [43, theorem 5.4.5] by using the Maxwell analogue of the $x \cdot \nabla u + \alpha u$ multiplier; see [43, §5.3]. (The review in [10, §5.3.2] contains more discussion of these results and these multipliers.)

(ii) To obtain the results in A1, A2, and B, one needs a vector field $Z$ in the domain where the PDE is posed (since the identity is applied in this domain). In contrast, to obtain the integral equation results, C1 and C2, one needs a vector field in both the interior and exterior domains (i.e., $\Omega_-$ and $\Omega_+$. This is because (as we discussed in Section 1.5–1.6 and saw in Section 3) the identity is applied in both $\Omega_-$ and $\Omega_+$.

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