ADJOINT ACTION OF AUTOMORPHISM GROUPS ON RADICAL ENDOMORPHISMS, GENERIC EQUIVALENCE AND DYNKIN QUIVERS

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ABSTRACT. Let Q be a connected quiver with no oriented cycles, k the field of complex numbers and P a projective representation of Q. We study the adjoint action of the automorphism group $\mathrm{Aut}_{kQ}P$ on the space of radical endomorphisms $\mathrm{radEnd}_{kQ}P$. Using generic equivalence, we show that the quiver Q has the property that there exists a dense open $\mathrm{Aut}_{kQ}P$-orbit in $\mathrm{radEnd}_{kQ}P$, for all projective representations P, if and only if Q is a Dynkin quiver. This gives a new characterisation of Dynkin quivers.

INTRODUCTION

Let $A = kQ$ be the path algebra of a connected quiver Q with no oriented cycles and k the field of complex numbers. Let P be a projective representation of Q. There is an adjoint action of $\mathrm{Aut}_A P$ on the Jacobson radical $\mathrm{radEnd}_A P$ of the endomorphism ring $\mathrm{End}_A P$, i.e. $g \cdot f = gfg^{-1}$ for any $g \in \mathrm{Aut}_A P$ and $f \in \mathrm{radEnd}_A P$. We are interested in generic $\mathrm{Aut}_A P$-orbits in $\mathrm{radEnd}_A P$, in particular, the existence of open orbits.

If Q is of type $\mathbb{A}$ with linear orientation, then $\mathrm{End}_A P$ is a parabolic Lie algebra and $\mathrm{radEnd}_A P$ has a dense open $\mathrm{Aut}_A P$-orbit by a theorem of Richardson [18]. The theorem of Richardson holds for parabolic Lie algebras of any reductive Lie-algebra, and in type $\mathbb{A}$, elements with dense orbits have been explicitly constructed by Brüstle, Hille, Ringel and Rörhle using representations of quivers [5], and in the classical types by Baur using a different approach [1]. These results and methods have been extended in various directions, see for example [3, 11, 12, 14, 19].

If Q is of type $\mathbb{A}$ with arbitrary orientation, then $\mathrm{End}_A P$ is a seaweed Lie algebra [9, 17] (also called a biparabolic Lie algebra [15]), and $\mathrm{radEnd}_A P$ has a dense open $\mathrm{Aut}_A P$-orbit by a theorem of Jensen, Su and Yu [14]. At present it is not known if seaweed Lie algebras of other classical types have dense orbits, however a counterexample exists for Lie algebras of type $\mathbb{E}_8$ [14].

If Q is of infinite representation type, i.e. Q is not Dynkin, then there is not necessarily a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$.

Lemma 1. *Let Q be a quiver that is not of Dynkin type. Then there exists a projective representation P of Q, such that no dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$ exists.*

The main result of this paper is as follows.

Theorem 1. *Let Q be a Dynkin-quiver and let P be an arbitrary projective representation of Q. Then there is a unique dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$.*

By Gabriel's Theorem [10], Dynkin quivers are exactly those quivers with finite representation type. Lemma 1 and Theorem 1 together give a new characterisation of Dynkin quivers, that is, a quiver is Dynkin if and only if there is a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$ for all projective representations P of Q.

We emphasize that although there is a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$, when Q is Dynkin, there are in general infinitely many $\mathrm{Aut}_A P$-orbits in $\mathrm{radEnd}_A P$. This is an interesting phenomenon that already exists in type $\mathbb{A}$ (see [2, 5]).

Note that the main result in [13] connects the adjoint action of $\mathrm{Aut}_A P$ on $\mathrm{radEnd}_A P$ with good representations of a double quiver of Q with relations. In particular, there is an open dense $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$ if and only if there is an open orbit in a variety of good representations. To prove our main result, we show that for some quivers Q, including Dynkin quivers other than type $\mathbb{A}$, the varieties of good representations are generically equivalent to representation varieties of a quiver Q' with the same underlying graph as Q. This means that, although the geometric properties of the varieties of good representations are in general very complicated, the behavior of generic good representations is simpler, and similar to that of generic representations of Q'. In particular, there is a dense open orbit in the variety of good representations if and only if there is an open orbit in the corresponding representation variety of Q'.

The paper is organized as follows. In Section 1 we recall basic facts on quivers and their representations, and introduce the notion of a generic section and generic equivalence. In Section 2 we recall results of Brüstle and Hille [4], and Hille and Vossieck [13] on the use of double quivers to parameterise $\mathrm{Aut}_A P$-orbits of radical endomorphisms of projective representations P. In Section 3 we recall Voigt's lemma for quiver representations and give a criterion for rigidity of good representations. In Section 4, using the criterion for rigidity, we give new simpler proofs to show that representations constructed for type $\mathbb{A}$ in [5, 14] have dense open orbits. Using this construction in type $\mathbb{A}$ and the criterion in Section 3, we prove the main results in Section 5 and Section 6. We also give various examples to illustrate the construction in the proofs of the technical lemmas in Section 6.

1. Representation varieties and generic sections

1.1. **Representation varieties.** Let Q be a quiver with $Q_0 = \{1, \cdots, n\}$ the set of vertices and Q_1 the set of arrows. Let $s, t : Q_1 \to Q_0$ be the functions mapping an arrow to its starting and terminating vertex, respectively. A vertex $i \in Q_0$ is called a sink if there are no arrows starting at i, and a source if there are no arrows terminating at i. It is called admissible if it is either a sink or a source, and interior if there are at least two arrows incident to i.

A representation M of Q consists of vector spaces $\{M_i\}_{i\in Q_0}$ and linear maps $\{M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}\}_{\alpha\in Q_1}$. A homomorphism of representations $h : M \to N$ is a collection of maps $h_i : M_i \to N_i$ satisfying $h_j M_\alpha = N_\alpha h_i$ for each arrow $\alpha : i \to j \in Q_1$. The direct sum of two representations is obtained by taking direct sums of vector spaces at each vertex and direct sum of linear maps at each arrow. A representation is indecomposable if it is not isomorphic to the direct sum of two nonzero representations.

For a representation M, let $\underline{\dim} M = (\dim_k M_i)_{i\in Q_0}$ denote the dimension vector of M. Let $c \in \mathbb{N}^n$ be a dimension vector and

$$\mathrm{Rep}(Q, c) = \prod_{\alpha\in Q_1} \mathrm{Hom}_k(k^{c_{s(\alpha)}}, k^{c_{t(\alpha)}})$$

be the space of representations. We fix a basis and view elements in $\mathrm{Rep}(Q, c)$ as tuples of matrices. Let Gl_{c_i} denote the general linear group of invertible $c_i \times c_i$-matrices. The group

$$\mathrm{Gl}(c) = \prod_{i\in Q_0} \mathrm{Gl}_{c_i}$$

acts on $\mathrm{Rep}(Q, c)$ by change of basis. There is a bijection between $\mathrm{Gl}(c)$-orbits in $\mathrm{Rep}(Q, c)$ and isomorphism classes of representations of Q with dimension vector c.

The path algebra $A = kQ$ is the algebra with basis the set of paths in Q. For two paths p and q, their product is defined to be the composition pq, if q terminates where p starts, and zero otherwise. For each vertex $i \in Q_0$, let e_i denote the trivial path of length zero at i. The trivial paths form a set of pairwise orthogonal idempotents for A. There is an equivalence of categories between left A-modules and representations of Q. Using this equivalence, any representation of Q can be viewed as an A-module, and vice versa.

A quiver Q is Dynkin if the underlying graph of Q is one of the Dynkin graphs $\mathbb{A}_i, \mathbb{D}_j, \mathbb{E}_l$ for $i \geq 1, j \geq 4, l = 6, 7, 8$. If Q is Dynkin, by Gabriel's Theorem [10] there are only finitely many orbits in $\mathrm{Rep}(Q, c)$ for any dimension vector c and thus there is always a dense orbit in $\mathrm{Rep}(Q, c)$. Moreover, the Dynkin quivers are characterised by this property. We summarise these properties as follows.

Theorem 2 ([10])**.** *If Q is a Dynkin quiver and c is a dimension vector, then there is a dense open $\mathrm{Gl}(c)$-orbit in $\mathrm{Rep}(Q, c)$. Moreover, if Q is not a Dynkin quiver, then there is a dimension vector c such that there is no dense open $\mathrm{Gl}(c)$-orbit in $\mathrm{Rep}(Q, c)$.*

Let $I \subseteq A$ be an ideal and let

$$\mathrm{Rep}(A/I, c) \subseteq \mathrm{Rep}(Q, c)$$

be the subset consisting of representations that are annihilated by I. This subset is a $\mathrm{Gl}(c)$-stable Zariski closed subvariety and is called the *variety of (Q, I)-representations* with dimension vector c.

1.2. **Generic sections.** We will relate $\mathrm{Aut}_A P$-orbits in $\mathrm{radEnd}_A P$ to representations of a quiver without relations. This relationship will be made precise using generic equivalence to be defined below.

Let G be a connected algebraic group. We call an affine space V a *G-space* if V admits a regular G-action.

Definition 1. *An H-space W is called a generic section of a G-space V if there is an injective morphism $\phi : W \longrightarrow V$ such that*

(1) *$\mathrm{Im}\phi \subseteq V$ is an affine subspace.*
(2) *$G \cdot \mathrm{Im}\phi$ contains a non-empty open subset of V.*
(3) *there is a non-empty open subset $W' \subseteq W$ such that $\phi(H \cdot w) = (G \cdot \phi(w)) \cap \mathrm{Im}\phi$ for all $w \in W'$.*

We give two examples of generic sections, where the first one will be used in the proof of Lemma 26 in Section 6.

Example 1. *Let $Q:\ 0 \longrightarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n$ and $c = (m, n, n-1, n-2, \cdots, 1)$ for $m, n > 0$. Let $Mat_{n\times m}$ be the space of $n \times m$ matrices, B_n the group of invertible upper triangular matrices and $B_n \times \mathrm{Gl}_m$ act on $Mat_{n\times m}$ by $(b, g)\cdot X = bXg^{-1}$. Let ϕ be the inclusion of vector spaces*

$$\phi : Mat_{n\times m} \to \mathrm{Rep}(Q, c),$$

sending a matrix $X \in Mat_{n\times m}$ to the representation that has X on the arrow $0 \to 1$ and is $\oplus_{i=1}^{n} Ae_i$ when restricted on the subquiver supported on $1, 2, 3, \cdots, n$. The stabiliser of $\oplus_{i=1}^{n} Ae_i$ is B_n. The subset $\mathrm{Gl}(c)\mathrm{Im}\phi$ contains all the representations in $\mathrm{Rep}(Q, c)$ with maps on each arrow $i \leftarrow i+1$ injective and thus is open in $\mathrm{Rep}(Q, c)$. Furthermore, X' is contained in the orbit $(B_n \times \mathrm{Gl}_m)X$ if and only if $\phi(X') \in \mathrm{Gl}(c)\phi(X)$. Therefore the $B_n \times \mathrm{Gl}_m$-space $Mat_{n\times m}$ is a generic section of the $\mathrm{Gl}(c)$-space $\mathrm{Rep}(Q, c)$.

Example 2. *Let $Q:\ 1 \rightrightarrows 2$ be the Kronecker quiver and consider $\mathrm{Rep}(Q, c)$ with $c = (n, n)$. The Σ_n-space k^n is a generic section of the $\mathrm{Gl}(c)$-space $\mathrm{Rep}(Q, c)$, where Σ_n is the symmetric group of order n which acts on k^n by the usual permutation of coordinates.*

The generic section in the example, as well as the generic sections we consider in this paper, preserve more information about the orbits than what we can deduce from Definition 1 alone. However, this definition will be sufficient to prove the main result of this paper.

Lemma 2. *Let V be a G-space and let the H-space W be a generic section in V. Then there is an open G-orbit in V if and only if there is an open H-orbit in W.*

Proof. We may assume that the map $\phi : W \to V$ is an inclusion $W \subseteq V$ of affine spaces. First, assume that $G \cdot x \subseteq V$ is open for some $x \in V$. By Definition 1 (2), $(G \cdot x) \cap W \subseteq W$ is nonempty and open. Then by Definition 1 (3), there exists $x' \in (G \cdot x) \cap W'$ with

$(G\cdot x')\cap W = H\cdot x'$, where W' is an open subset as in (3). This shows that W has an open H-orbit.

Conversely, assume that $H\cdot w\subseteq W$ is open. We may assume that $w\in W'$, which is an open subset as in Definition 1 (3) and so $(G\cdot w)\cap W = H\cdot w$. Since the map $G\times W\to V$ is dominant by Definition 1 (2), the restriction $G\times(H\cdot w)\to V$ is dominant, by comparison of fibre dimensions. This shows that V has an G-open orbit. □

In particular, the lemma proves that $\{x\}\subseteq V$ with the trivial action, is a generic section of the G-space V if and only if the orbit $G\cdot x\subseteq V$ is open.

Generic sections define a relation on spaces with group actions, and two spaces are said to be *generically equivalent* if they are related by a sequence of generic sections. Two spaces with a dense open orbit are generically equivalent.

The key technical result in this paper, proved in Section 6, is that for certain quivers Q and a projective representation P of Q, the $\mathrm{Aut}_A P$-space $\mathrm{radEnd}_A P$ is generically equivalent to a $\mathrm{Gl}(d')$-space $\mathrm{Rep}(Q',c')$, where Q' is a quiver with the same underlying graph as Q, and c' is a dimension vector constructed from Q and P. We do not know if this generic equivalence exists for all quivers Q, but they do exist for Dynkin quivers. As a consequence, the main theorem stated in the introduction will follow by the Theorem 2.

2. Double quivers, adjoint actions and two equivariantly isomorphic varieties

In this section we recall the construction of a finite dimensional quasi-hereditary algebra D, which can be used to study $\mathrm{Aut}_A P$-orbits in $\mathrm{radEnd}_A P$ [13]. We then recall and discuss some relevant properties of D and its representation varieties.

Let $\tilde{Q}$ be the double quiver of Q, i.e. $\tilde{Q}_0 = Q_0$ and $\tilde{Q}_1 = Q_1\cup Q_1^*$ with $Q_1^* = \{\alpha^* : i\to j|\alpha : j\to i\in Q_1\}$. Let $\mathcal{I}$ be the ideal of $k\tilde{Q}$ generated by

$$\alpha^*\alpha - \sum_{\beta\in Q_1, t(\beta)=s(\alpha)} \beta\beta^*$$

for any arrow $\alpha\in Q_1$, and

$$\alpha^*\beta$$

for pairs of arrows $\alpha\neq\beta$ in Q_1 terminating at the same vertex. The algebra D is defined as

$$D = k\tilde{Q}/\mathcal{I}.$$

We now define a grading on D to be used in the next section. For a path q in Q, we define $q^* = \alpha_n^*\dots\alpha_1^*$ if $q = \alpha_1\dots\alpha_n$ and $e_i = e_i^*$. The paths of the form ab^* for paths a, b in Q with $s(a) = s(b)$ form a basis for D. For any path p in $\tilde{Q}$, let $l^*(p)$ be the number of arrows from Q_1^* in p. As the defining relations of D are homogeneous with respect to l^*, we have the following lemma.

Lemma 3. *D is graded with respect to l^*.*

Let $c = \underline{\dim} P$ be the dimension vector of P, $\mathrm{Rep}(D,c)$ the variety of $(\tilde{Q},\mathcal{I})$-representations with dimension vector c and let $\mathrm{Gl}(c)$ act on $\mathrm{Rep}(D,c)$ by conjugation. As $A = kQ$ is a subalgebra of D, any D-module can be considered as an A-module. We call a D-module that is projective as an A-module an *A-projective D-module.* Let

$$\mathrm{Rep}(D,P)\subseteq \mathrm{Rep}(D,c)$$

be the subvariety consisting of the representations X with ${}_AX = P$. The group $\mathrm{Aut}_A P$ acts on $\mathrm{Rep}(D,P)$ and orbits correspond to isomorphism classes of A-projective D-modules X with ${}_AX\cong P$.

Lemma 4. (i) $\mathrm{Rep}(D,P)\subseteq\mathrm{Rep}(\tilde{Q},c)$ *is an affine space.*
(ii) $\mathrm{Gl}(c)\cdot\mathrm{Rep}(D,P)\subseteq\mathrm{Rep}(D,c)$ *is irreducible and open.*

Proof. Note that the maps on each arrow in Q are fixed for all point in $\mathrm{Rep}(D,P)$. So $\mathrm{Rep}(D,P)$ is the solution space of the system of linear equations given by the defining relations of D. Therefore $\mathrm{Rep}(D,P)$ is an affine space and by definition it is contained in $\mathrm{Rep}(\tilde{Q},c)$. This proves (i) and that $\mathrm{Gl}(c)\cdot\mathrm{Rep}(D,P)$ is irreducible.

Next observe that there is a natural morphism $\mathrm{Rep}(D,c)\to\mathrm{Rep}(Q,c)$ obtained by forgetting the Q_1^*-structure, i.e. the maps on each arrow in Q_1^*. Now $\mathrm{Gl}(c)\cdot\mathrm{Rep}(D,P)$ is the preimage of the open orbit of P, and so it is open. Thus (ii) follows. □

The following theorem can be deduced from its categorical version, Theorem 1.1 in [13]. This result enables us to explore the existence of open $\mathrm{Aut}_A P$-orbits in $\mathrm{radEnd}_A P$ using representations of quivers.

Theorem 3. *The varieties* $\mathrm{Rep}(D,P)$ *and* $\mathrm{radEnd}_A P$ *are* $\mathrm{Aut}_A P$*-equivariantly isomorphic.*

Consequently, there is a one-to-one correspondence between $\mathrm{Aut}_A P$-orbits in $\mathrm{radEnd}_A P$ and $(\tilde{Q},\mathcal{I})$-representations that are isomorphic to P as A-modules. Furthermore, an open $\mathrm{Aut}_A P$-orbit exists in $\mathrm{Rep}(D,P)$ if and only if an open $\mathrm{Aut}_A P$-orbit exists in $\mathrm{radEnd}_A P$.

Let $P_i = Ae_i$ be the indecomposable projective A-module associated to the vertex $i\in Q_0$. Given an A-projective D-module X, denote by d_i the multiplicity of P_i as a summand in ${}_AX$ and let the Δ-*dimension vector* of X be defined as

$$\underline{\dim}_\Delta X = (d_i)_i \in \mathbb{N}^n.$$

We let

$$\mathrm{Supp}_\Delta X = \{i\in Q_0 | d_i > 0\}$$

and call it the Δ-*support of* X. Note that the Δ-support is in general different from the usual support $\mathrm{Supp}X$ defined using the dimension vector $\underline{\dim}X$. Given $d=(d_i)_i\in\mathbb{N}^n$, let

$$P(d) = \bigoplus_{i=1}^n P_i^{d_i}.$$

Remark 2.1. *As mentioned earlier, the algebra D is a quasi-hereditary algebra with the indecomposable projective A-modules Ae_i as Verma modules* [13]*. The terminology Δ-dimension vector and Δ-support coincides with the one used in the setting of quasi-hereditary algebras, for example in* [5, 14]*.*

3. An exact sequence and a relative Voigt's lemma

In this section we assume that Q has no cycles. Let J be the ideal in D generated by the arrows in Q_1^*. Note that D is a split extension of A by J. We construct two useful exact sequences and give a criterion on the rigidity of A-projective D-modules, analogues to Voigt's Lemma [20].

Lemma 5. *As D-modules,* $J\cong\oplus_{\beta\in Q^*, t(\beta)=i} De_i$

Proof. Observe that J is the kernel of the natural surjective map

$$D\to A.$$

Now the lemma follows from the exact sequences (see [13, 8]),

$$0\longrightarrow \oplus_{\beta:i\to j\in Q^*} De_j \longrightarrow De_i \longrightarrow Ae_i \longrightarrow 0\ ,$$

for any vertex $i\in Q_0$. □

Recall the standard projective resolution of a representation M of Q (see [7]),

$$0\longrightarrow \oplus_{\alpha\in Q_1} Ae_{t(\alpha)}\otimes_k e_{s(\alpha)}M \longrightarrow \oplus_{i\in Q_0} Ae_i\otimes e_iM \longrightarrow M\longrightarrow 0.$$

This resolution can be interpreted as a short exact sequence

$$0\longrightarrow J'\otimes_S M\longrightarrow A\otimes_S M\longrightarrow M\longrightarrow 0\ ,$$

where $S = \oplus_{i\in Q_0} ke_i$ and J' is the kernel of the natural projection $A \to S$. In other words, J' is the Jacobson radical of A, which is the ideal of A generated by arrows in Q. We have an analogous resolution for A-projective D-modules as follows. The proof is similar to the proof of the standard resolution for quivers in [7].

Lemma 6. *Let X be an A-projective D-module. Then the following exact sequence is a projective resolution of X as a D-module,*

$$0 \longrightarrow J \otimes_A X \xrightarrow{f} D \otimes_A X \xrightarrow{g} X \longrightarrow 0 ,$$

where $g(d \otimes m) = dm$ and $f(d\beta \otimes x) = d\beta \otimes x - d \otimes \beta x$ for $d\beta \in J, \beta \in Q_1^*$, and $x \in X$.

Proof. Since X is a projective A-module and D and J are projective D-modules, both $J \otimes_A X$ and $D \otimes_A X$ are projective D-modules. So we only need to prove that the sequence is exact. By the definition of f and g, the map g is surjective and $gf = 0$. By applying $- \otimes_A X$ to the exact sequence

$$0 \longrightarrow J \longrightarrow D \longrightarrow A \longrightarrow 0 ,$$

we obtain the exact sequence

$$0 \longrightarrow J \otimes_A X \longrightarrow D \otimes_A X \longrightarrow {}_A X \longrightarrow 0 ,$$

and so $\dim_k J \otimes_A X = \dim_k \operatorname{Ker} g$. Therefore it remains to show that f is injective. Suppose that $\sum d\beta^* \otimes x$ is in the kernel of f. Since D is l^*-graded by Lemma 4.4, we may assume that $l^*(d\beta^*)$ is constant on the terms in the sum. Now $f(\sum d\beta^* \otimes x) = \sum(d\beta^* \otimes x - d \otimes \beta^* x) = 0$. Since $l^*(d) < l^*(d\beta^*)$, we have $\sum d\beta^* \otimes x = 0$. Hence f is injective, as required. □

Lemma 7. *Let X be an A-projective D-module. Then*

(1) $\operatorname{Hom}_D(D \otimes_A X, X) \cong \operatorname{End}_A X$.
(2) $\operatorname{Hom}_D(J \otimes_A X, X) \cong \operatorname{radEnd}_A X$.
(3) *we have an exact sequence*

$$0 \longrightarrow \operatorname{End}_D X \longrightarrow \operatorname{End}_A X \longrightarrow \operatorname{radEnd}_A X \longrightarrow \operatorname{Ext}^1_D(X, X) \longrightarrow 0.$$

Proof. First, $\operatorname{Hom}_D(D \otimes_A X, X) \cong \operatorname{Hom}_A(X, \operatorname{Hom}_D(D, X)) \cong \operatorname{Hom}_A(X, X) = \operatorname{End}_A X$. This proves (1).

Observe that for any indecomposable projective A-module Ae_i, $J \otimes_A Ae_i \cong Je_i$. We have $J = \oplus_{\beta\in Q_1} D\beta^*$, and so

$$\begin{aligned}\operatorname{Hom}_D(J \otimes_A Ae_i, Ae_j) &\cong \operatorname{Hom}_D(Je_i, Ae_j) \cong \oplus_{s\to i\in Q_1} \operatorname{Hom}_D(De_s, Ae_j)\\ &\cong \oplus_{s\to i\in Q_1} \operatorname{Hom}_A(Ae_s, Ae_j)\end{aligned}$$

As Q has no cycles,

$$\oplus_{s\to i\in Q_1} \operatorname{Hom}_A(Ae_s, Ae_j) = \begin{cases} k & \text{if } s = j \text{ or there is a path from } j \text{ to } s; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\operatorname{Hom}_D(J \otimes_A Ae_i, Ae_j) \cong \operatorname{radHom}_A(Ae_i, Ae_j)$$

and thus

$$\operatorname{Hom}_D(J \otimes_A X, X) \cong \operatorname{radEnd}_A X.$$

This proves (2).

Finally, applying $\operatorname{Hom}_D(-, X)$ to the sequence in Lemma 6 gives the exact sequence,

$$0 \longrightarrow \operatorname{End}_D X \longrightarrow \operatorname{Hom}_D(D \otimes X, X) \longrightarrow \operatorname{Hom}_D(J \otimes X, X) \longrightarrow \operatorname{Ext}^1_D(X, X) \longrightarrow 0.$$

Now by (1) and (2), we obtain the sequence in (3). □

Recall that a D-module is said to be *rigid* if $\mathrm{Ext}^1_D(X,X)=0$. We can now use Lemma 7 to prove the main result of this section on the rigidity of A-projective D-modules, similar to Voigt's Lemma [20]. Moreover, it includes an inequality for the dimension of stabilizers, which is very useful for determining whether an A-projective D-module is rigid or not.

Theorem 4. *Let X be an A-projective D-module with ${}_AX = P(d)$ for some $d \in \mathbb{N}^n$. Then*
(1) $\dim_k \mathrm{End}_D X \geq \sum_i d_i^2$.
(2) *the following are equivalent.*
  (i) $\mathrm{Aut}_A P \cdot X \subseteq \mathrm{Rep}(D,P)$ *is open.*
  (ii) X *is rigid.*
  (iii) $\dim_k \mathrm{End}_D X = \sum_i d_i^2$.

Proof. First observe that $\dim_k \mathrm{End}_A X - \dim_k \mathrm{radEnd}_A X = \sum {d_i}^2$. Now by Lemma 7

$$\dim_k \mathrm{End}_D X = \dim_k \mathrm{End}_A X - \dim_k \mathrm{radEnd}_A X + \dim_k \mathrm{Ext}^1_D(X,X) \geq \sum_i d_i^2.$$

This proves (1). Moreover, the equality holds if and only if $\mathrm{Ext}^1_D(X,X)=0$. Thus the equivalence of (ii) and (iii) follows.

By Theorem 3, $\dim_k \mathrm{Rep}(D,P) = \dim_k \mathrm{radEnd}_A X$. So

$$\dim_k \mathrm{Aut}_A P \cdot X = \dim_k \mathrm{End}_A X - \dim_k \mathrm{End}_D X = \dim_k \mathrm{Rep}(D,P) - \dim_k \mathrm{Ext}^1_D(X,X).$$

Therefore *i*) and *ii*) are equivalent. □

4. Type $\mathbb{A}$

As preparation for the next section, we recall results on rigid D-modules in type $\mathbb{A}$ from [5, 14]. At the same time we use Theorem 4 to give new, simpler and more uniform proofs of the two main results in [5] and [14], respectively. We also introduce a filtration of D, which induces a 'grading' on indecomposalbe rigid A-projective D-modules, describe homomorphisms in terms of this grading and define orders the rigid modules. This order will play a crucial role in the proofs of the main technical result in Section 6. Throughout this section Q is a quiver of type $\mathbb{A}$, with vertices $\{1,\cdots,n\}$, and arrows $\alpha_i : i \to i+1$ or $\alpha_i : i \leftarrow i+1$ for $i=1,\cdots,n-1$.

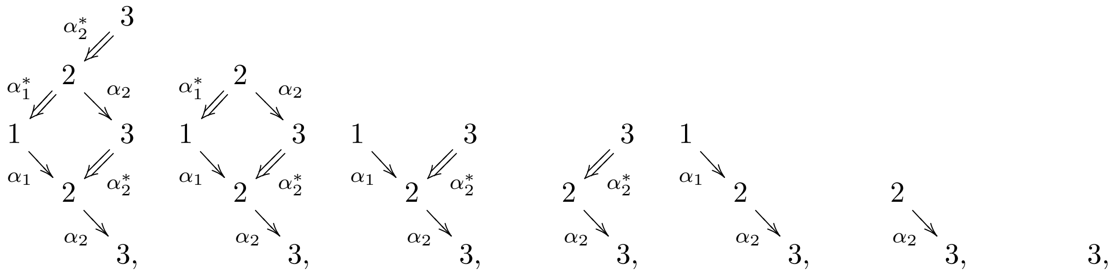
4.1. **Linear $\mathbb{A}_n$.** In this subsection Q is linearly oriented and the arrows are $\alpha_i : i \to i+1$ for $i=1,\cdots,n-1$. In this case $\mathrm{Aut}_A P$ is a parabolic and the existence of a dense $\mathrm{Aut}_A P$-orbit in $\mathrm{Rep}(D,P)$ follows from the classical result of Richardson [18]. We now recall the explicit construction of representations with dense open orbits by Brüstle, Hille, Ringel and Rörhle [5].

The projective D-module $Q_n = De_n$ at vertex n is injective and has a basis consisting of paths pq^*, where q is a path in Q ending at the vertex n. A submodule X of Q_n has a basis given by a subset of the paths pq^*, and it is uniquely determined by its A-structure ${}_AX \cong \oplus_{i=1}^n P_i^{d_i}$ with $d_i \in \{0,1\}$. Thus there is a natural bijection between subsets $I \subseteq Q_0$ and submodules of Q_n. More precisely, under this bijection a subset I corresponds to the unique submodule of Q_n with Δ-support I. Let $X(I)$ denote the submodule of Q_n corresponding to the subset I. For any vector $d \in \mathbb{N}^n$, define $X(d) = \sum_{i=1}^t X(I_i)$, with $\underline{\dim}_\Delta X(d) = d$ and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$. We give an example to illustrate the construction.

Example 3. *Let $n=3$ and $d=(2,1,2)$. The algebra D is given by the quiver*

$$1 \underset{\alpha_1^*}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\alpha_2^*}{\overset{\alpha_2}{\rightleftarrows}} 3\ ,$$

with the ideal $\mathcal{I}$ generated by $\alpha_1^*\alpha_1$ and $\alpha_1\alpha_1^* - \alpha_2^*\alpha_2$. The projective-injective D-module Q_3 has the following 7 nonzero submodules, where the first one is Q_3,



corresponding to the subsets $\{1,2,3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1\}$, $\{2\}$, $\{3\}$, *respectively. In the picture a number* i *indicate a one dimensional basis element at vertex* i *and the arrows indicate the nonzero action of the arrows in* $\tilde{Q}_1$.

For $d = (2,1,2)$, *we have* $X(d) = X(I_1) \oplus X(I_2)$, *where* $I_1 = \{1,2,3\}$ *and* $I_2 = \{1,3\}$.

Theorem 5 ([5])**.** *An* A*-projective* D*-module* X *is rigid if and only if* $X \cong X(d)$ *for some* $d \in \mathbb{N}^n$.

We will give a new proof of the above theorem. We first recall a lemma in [5] on the dimension of homomorphism spaces between submodules of Q_n, and give a new proof.

Lemma 8 ([5])**.** *Let* $X(I)$ *and* $X(J)$ *be submodules of* Q_n *with* Δ*-support* I *and* J*, respectively, and* $I \subseteq J$*. Then*

$$\dim_k \operatorname{Hom}_D(X(I), X(J)) = \dim_k \operatorname{Hom}_D(X(J), X(I)) = |I|.$$

Proof. Note that the module Q_n is generated by e_n and $\operatorname{End}_D(Q_n)$ is an n-dimensional vector space with basis $f_0, \ldots, f_{n-1}$, where $f_i(e_n) = (\alpha_{n-1}\alpha^*_{n-1})^i e_n$ for $0 \leq i \leq n-1$. In particular, f_0 is the identity map. For any two submodules X and Y of Q_n and any homomorphism $f\colon\ X \to Y$, since Q_n is injective, we have the following commutative diagram for some $g \in \operatorname{End}_D Q_n$

$$\begin{array}{ccc} X & \xrightarrow{\subseteq} & Q_n \\ {\scriptstyle f}\downarrow & & \vdots\,{\scriptstyle g} \\ Y & \xrightarrow{\subseteq} & Q_n. \end{array}$$

That is, $f = g|_{X(I)}$ and so f is a linear combination of the restrictions of $f_0, \ldots, f_{n-1}$ to $X(I)$.

By the definition, the nonzero restrictions of $f_0, \ldots, f_{n-1}$ in $\operatorname{Hom}_D(X(I), X(J))$ are linearly independent. Moreover, $f_i|_{X(I)} = 0$ if and only if $i \geq \sum d_i$, where $(d_i)_i = \underline{\dim}_\Delta X$. Therefore

$$\dim_k \operatorname{Hom}_D(X(I), X(J)) = \sum_i d_i = |I|.$$

Similarly, $\dim_k \operatorname{Hom}_D(X(J), X(I)) = |I|$. □

Proof of Theorem 5. By Theorem 4 and Lemma 8, each submodule $X(I)$ is rigid. So by the construction of $X(d)$, to show that $X(d)$ is rigid, we need only prove that

$$X = X(I) \oplus X(J)$$

with $I \subseteq J$ is rigid. We have ${}_AX \cong \oplus_i P(i)^{d_i}$, where $d_i = 2$ if $i \in I \cap J$, $d_i = 1$ if $i \in J \backslash I$ and $d_i = 0$ if $i \notin J$. By Lemma 8, $\dim_k \operatorname{End}_D(X) = 3|I| + |J| = 4|I| + |J\backslash I| = \sum d_i^2$, and so X is rigid by Theorem 4. This proves the existence of an open orbit in $\operatorname{Rep}(D,P)$ for any P. Now $\operatorname{Rep}(D,P)$ is irreducible by Lemma 4, and so by Theorem 4, any rigid D-module is isomorphic to $X(d)$ for some $d \in \mathbb{N}^n$. □

4.2. **Gluing modules at sinks and sources.** In this subsection Q is of type $\mathbb{A}$ with arbitrary orientation. We recall how to glue a pair of A-projective D-modules at an admissible interior vertex to obtain a new A-projective D-module [14].

Let

$$i_1 < i_2 < \cdots < i_{t-1} < i_t$$

be the complete list of interior admissible vertices in Q and let $i_0 = 1$ and $i_{t+1} = n$. Let $u = i_l$ be one of the interior admissible vertices, and let M' and M'' be two indecomposable A-projective D-modules with $\mathrm{Supp}_\Delta(M') \subseteq \{1, \dots, u\}$, $\mathrm{Supp}_\Delta(M'') \subseteq \{u, \dots, n\}$ and $(\underline{\dim}_\Delta M')_u = 1 = (\underline{\dim}_\Delta M'')_u$.

Assume first that u is a sink in Q. Let N' be the D-submodule of M' generated by M'_j for all $j < u$. Then N' is not Δ-supported at u, i.e. $(\underline{\dim}_\Delta N')_u = 0$. Since u is a sink and $(\underline{\dim}_\Delta M')_u = 1$, we have the short exact sequence

$$0 \longrightarrow N' \longrightarrow M' \xrightarrow{f'} P_u \longrightarrow 0,$$

where $P_u = S_u$ is the simple projective A-module associated to vertex u. Similarly,

$$0 \longrightarrow N'' \longrightarrow M'' \xrightarrow{f''} P_u \longrightarrow 0,$$

where $N'' \subseteq M''$ is the submodule generated by M''_j for all $j > u$. Let M be given by the pullback of f' and f'', that is, we have a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \oplus M'' \longrightarrow P_u \longrightarrow 0.$$

We say that M is obtained by *gluing* M' and M'' at the sink u.

We now define gluing of homomorphisms at the sink u. Assume that K is obtained by gluing K' and K'' at u and let $L' \subseteq K'$ and $L'' \subseteq K''$ be the submodules generated by the spaces K'_i for $i < u$ and K''_i for $i > u$, respectively. Let $g' : M' \to K'$ and $g'' : M'' \to K''$ be homomorphisms of D-modules. Then $g'(N') \subseteq L'$ and $g''(N'') \subseteq L''$, and so there are induced maps $\tilde{g'}, \tilde{g''} : P_u \to P_u$. Using the pullback sequence defining M, we see that there is an induced map $g : M \to K$ if $\tilde{g'} = \tilde{g''}$. In this case we say that g is obtained by gluing g' and g'' at the sink u.

Now assume that u is a source. Note that in this case, the projective A-module P_u is also a projective D-module. We have short exact sequences

$$0 \longrightarrow P_u \xrightarrow{f'} M' \longrightarrow N' \longrightarrow 0,$$

$$0 \longrightarrow P_u \xrightarrow{f''} M'' \longrightarrow N'' \longrightarrow 0,$$

where $P_u \subseteq M'$ and $P_u \subseteq M''$ are the submodules generated by M'_u and M''_u, respectively. Let M be given by the pushout of f' and f'',

$$0 \longrightarrow P_u \longrightarrow M' \oplus M'' \longrightarrow M \longrightarrow 0.$$

We then say that M is obtained by gluing M' and M'' at the source u. Assume K is obtained by gluing K' and K'' at the source u. Given a pair of homomorphisms $g' : M' \to K'$ and $g'' : M'' \to K''$ there is an induced map $g : M \to K$ if the induced maps $\tilde{g'} = g'|_{P_u}$ and $\tilde{g''} = g''|_{P_u}$ are equal. We then say that g is obtained by gluing g' and g'' at the source u.

Lemma 9. *Let u be an interior admissible vertex. Then*

$$\mathrm{Hom}_D(M, K) \cong \{(g', g'') \in \mathrm{Hom}_D(M', K') \oplus \mathrm{Hom}_D(M'', K'') | \tilde{g'} = \tilde{g''}\}$$

Proof. Assume u is a sink. Let

$$g : M \to K$$

be a homomorphism. Then there are the short exact sequences

$$0 \longrightarrow N' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow L' \longrightarrow K \longrightarrow K'' \longrightarrow 0,$$

where $N' \subseteq M$ and $L' \subseteq K$ are the submodule generated by M_i and K_i, respectively, for $i < u$. Then $g(N') \subseteq L'$ and so there is an induced map $g'' : M'' \to K''$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g|_{N'}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\ 0 & \longrightarrow & L' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0. \end{array}$$

Similarly, there is a map $g' : M' \to K'$. Moreover, $\tilde{g'} = \tilde{g''}$. So the map $g : M \to K$ is obtained by gluing maps g' and g'' at u. Thus there is an injection of vector spaces $\mathrm{Hom}_D(M, K) \to \mathrm{Hom}_D(M', K') \oplus \mathrm{Hom}_D(M'', K'')$ given by

$$g \mapsto (g', g'').$$

The image is equal to the pairs of maps (g', g'') with $\tilde{g'} = \tilde{g''}$, since each such pair can be glued at u. This proves the lemma in the case where u is a sink. The case where u is a source is similar and is left to the reader. □

4.3. **The construction of rigid modules.** For $d \in \mathbb{N}^n$ and any $0 \le s \le t$, let d^s be the vector defined by $(d^s)_j = d_j$ for $j = i_s, i_s + 1, \cdots, i_{s+1}$ and $(d^s)_j = 0$, otherwise. Using Theorem 5, we can construct a rigid representation with Δ-dimension vector d^s for the double quiver supported on $\{i_s, \cdots, i_{s+1}\}$ subject to the corresponding relations. This module, which we denote by $Y(d^s)$, may not be A-projective when considered as a D-module. However, if i_s is an interior source, we extend $Y(d^s)$ by $P_{i_s-1}^{d_{i_s}}$, and if i_{s+1} is an interior source, we extend $Y(d^s)$ by $P_{i_{s+1}+1}^{d_{i_{s+1}}}$, to obtain an A-projective D-module, which we denote by $X(d^s)$. Since the extension preserves the Δ-dimension vector and the dimension of the endomorphism ring, Theorem 4 implies that the D-module $X(d^s)$ is rigid. We will glue the indecomposable summands of $X(d^s)$ for all s with respect to an order defined below to obtain a rigid module with Δ-dimension vector d.

Following the construction of $X(d^s)$ each indecomposable summand of $X(d^s)$ is completely determined up to isomorphism by its Δ-support. Suppose that M is an indecomposable module obtained by gluing modules $X(I_s)$ at interior admissible vertices, where $I_s = \mathrm{Supp}_\Delta X \cap \{i_s, \ldots, i_{s+1}\}$ for all s, and that N is an indecomposable module obtained by gluing $X(J_s)$, where $J_s = \mathrm{Supp}_\Delta Y \cap \{i_s, \ldots, i_{s+1}\}$ for all s. For such indecomposable modules we define an order $\le_u$ for any vertex $u \in Q_0$.

Definition 2. *Let u be a vertex with $i_v < u \le i_{v+1}$. Suppose that both M and N are supported (but not necessarily Δ-supported) at u. We define $M \le_u N$ if, for any s with I_s and J_s nonempty, $I_s \subseteq J_s$ if $s - v$ is even and $I_s \supseteq J_s$ if $s - v$ is odd.*

The construction of a rigid A-projective module $X(d)$ with Δ-dimension vector d is done by induction on the number of interior admissible vertices in Q. Clearly, if there are no interior admissible vertices, then Q is linearly oriented, and we are done.

Now suppose that the number of interior admissible vertices is $t > 0$. For any $s \ge 1$, define a vector e^s given by $(e^s)_j = d_j$ if $j \le i_s$ and $(e^s)_j = 0$ otherwise. We have $e^1 = d^0$ and $e^{t+1} = d$. We suppose that

$$X(d^0) = X^{01} \oplus \cdots \oplus X^{0d_{i_1}} \oplus X'$$

and

$$X(d^1) = X^{11} \oplus \cdots \oplus X^{1d_{i_1}} \oplus X'',$$

where $i_1 \in \mathrm{Supp}_\Delta X^{01} \subseteq \cdots \subseteq \mathrm{Supp}_\Delta X^{0d_{i_1}}$ and $\mathrm{Supp}_\Delta X^{11} \supseteq \cdots \supseteq \mathrm{Supp}_\Delta X^{1d_{i_1}} \ni i_1$, each X^{ij} is indecomposable and X' and X'' are not Δ-supported at i_1. For $1 \le s \le d_{i_1}$, we glue X^{0i} and X^{1i} at i_1 and obtain a module, denoted by X^i. Let $X(e^2) = X^1 \oplus \cdots \oplus X^{d_{i_1}} \oplus$

$X' \oplus X''$. By construction, the summands of $X(e^2)$ that are Δ-supported at i_2 are totally ordered with respect to $\leq_{i_2}$.

By induction we may assume that we have constructed an A-projective D-module $X(e^s)$ with Δ-dimension vector e^s and that the indecomposable summands of $X(e^s)$ that have Δ-support at i_s are totally ordered using $\leq_{i_s}$. By abuse of notation we again use $X^1 \leq_{i_s} \cdots \leq_{i_s} X^{d_{i_s}}$ to denote the summands of $X(e^s)$ that are Δ-supported at i_s. We have $X(e^s) = X^1 \oplus \cdots \oplus X^{d_{i_s}} \oplus Z$ and $X(d^s) = X^{s1} \oplus \dots X^{sd_{i_s}} \oplus Z'$, where $\mathrm{Supp}_\Delta X^{s1} \supseteq \cdots \supseteq \mathrm{Supp}_\Delta X^{sd_{i_s}} \ni i_s$. For each i, glue X^i with X^{si} at vertex i_s and obtain a new module, which we again denote by X^i. Now let $X(e^{s+1}) = X^1 \oplus \cdots \oplus X^{d_{i_s}} \oplus Z \oplus Z'$.

Lemma 10. *Suppose that $X(e^s)$ above is rigid. Let M, N be two summands of $X(e^s)$, which are Δ-supported at i_s. Assume that $M \leq_{i_s} N$ and $i_s \in I \subseteq J \subseteq \{i_s, \cdots, i_{s+1}\}$. Let X be obtained by gluing M with $X(J)$ and let Y be obtained by gluing N with $X(I)$. Then $X \oplus Y$ is rigid.*

Proof. We claim that

$$\dim_k \mathrm{Hom}_D(X, Y) = \dim_k \mathrm{Hom}_D(M, N) + |I| - 1.$$

We first consider the case where i_s is a source. By Lemma 9,

$$\dim_k \mathrm{Hom}_D(X, Y) \leq \dim_k \mathrm{Hom}_D(M, N) + \dim_k \mathrm{Hom}_D(X(J), X(I)).$$

Again by Lemma 9 any pair

$$(g', g'') \in \mathrm{Hom}_D(M, N) \oplus \mathrm{Hom}(X(J), X(I))$$

with $g'|_{M_{i_s}} = 0$ and $g''|_{X(J)_{i_s}} = 0$ can be glued to a homomorphism $X \to Y$. Since $M \leq_{i_s} N$, there exists a map $g_1' : M \longrightarrow N$ with $g_1'|_{M_{i_s}} \neq 0$. Moreover, the subspace in $\mathrm{Hom}_D(M, N)$ consisting of the maps that are zero at i_s has codimension 1. In $\mathrm{Hom}_D(X(J), X(I))$, the subspace consisting of the maps that are zero at i_s has codimension at most 1, with equality if and only if $I = J$. So

$$\dim_k \mathrm{Hom}_D(X, Y) \geq (\dim_k \mathrm{Hom}_D(M, N) - 1) + (\dim_k \mathrm{Hom}_D(X(J), X(I)) - 1).$$

If $I = J$, then $g_1' : M \to N$ can be glued to an isomorphism $g_1'' : X(J) \to X(I)$. Therefore

$$\dim_k \mathrm{Hom}_D(X, Y) = \dim_k \mathrm{Hom}_D(M, N) + \dim_k \mathrm{Hom}_D(X(J), X(I) - 1.$$

If $I \subsetneq J$, then $g''|_{X(J)_{i_s}} = 0$ for all maps $g'' \in \mathrm{Hom}_D(X(J), X(I))$. Thus

$$\dim_k \mathrm{Hom}_D(X, Y) = (\dim_k \mathrm{Hom}_D(M, N) - 1) + \dim_k \mathrm{Hom}_D(X(J), X(I)).$$

By Lemma 8, $\dim_k \mathrm{Hom}_D(X(J), X(I)) = |I|$ and so the claim follows for i_s a source. The proof of the claim is similar for the case where i_s is a sink and we leave it to the reader.

Similarly, we have

$$\dim_k \mathrm{Hom}_D(X, X) = \dim_k \mathrm{Hom}_D(M, M) + |I| - 1,$$
$$\dim_k \mathrm{Hom}_D(Y, X) = \dim_k \mathrm{Hom}_D(N, M) + |I| - 1,$$
$$\dim_k \mathrm{Hom}_D(Y, Y) = \dim_k \mathrm{Hom}_D(N, N) + |J| - 1.$$

We show that $X \oplus Y$ is rigid. Suppose that $\underline{\dim}_\Delta(X \oplus Y) = (d_i)_i$. Then $d_i = 2$ for $i \in I$, $d_j = 1$ for $j \in J \backslash I$ and $d_i = (\underline{\dim}_\Delta M \oplus N)_i$ otherwise. By our assumption, $M \oplus N$ is rigid and so by Theorem 4,

$$\dim_k \mathrm{End}_D(M \oplus N) = \sum_{i=1}^{i_s} d_i^2.$$

Therefore

$$\dim_k \mathrm{End}_D(X \oplus Y) = \dim_k \mathrm{End}_D(M \oplus N) + 4|I| + |J \backslash I| - 4 = \sum_i d_i^2.$$

Again by Theorem 4, $X \oplus Y$ is rigid. □

Using the lemma we can finish the case of type $\mathbb{A}$.

Theorem 6 ([14]). *An A-projective D-module X is rigid if and only if $X \cong X(d)$ for some $d \in \mathbb{N}^n$.*

Proof. We use induction on s to prove that $X(e^{s+1})$ is rigid for any s, thus so is $X(d) = X(e^{t+1})$. Note that $X(e^1) = X(d^0)$ is rigid, and assume that $X(e^s)$ is rigid. By the previous lemma we need only to prove the rigidity of $X \oplus Y \oplus L \oplus L'$, where X and Y are as in the previous lemma, L and L' are, respectively, summands of $X(e^s)$ and $X(d^s)$ without Δ-support at i_s. Note that $\mathrm{Hom}_D(L, X \oplus Y) = \mathrm{Hom}_D(L, M \oplus N)$ and $\mathrm{Hom}_D(X \oplus Y, L) = \mathrm{Hom}_D(M \oplus N, L)$ and so $L \oplus X \oplus Y$ is rigid, by Theorem 4. Similarly, for $L \oplus L'$ and $L' \oplus X \oplus Y$. Therefore $X \oplus Y \oplus L \oplus L'$ is rigid.

Conversely, if $X \in \mathrm{Rep}(D, P(d))$ is rigid, then by Lemma 4 and Theorem 4, X is isomorphic to $X(d)$. □

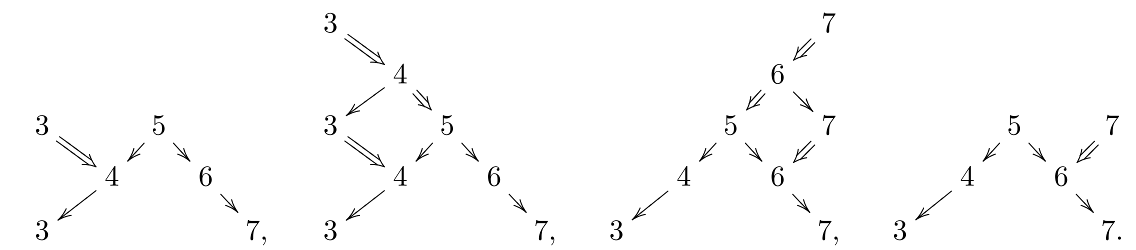
We give an example of the above construction.

Example 4. *Let Q be the quiver*

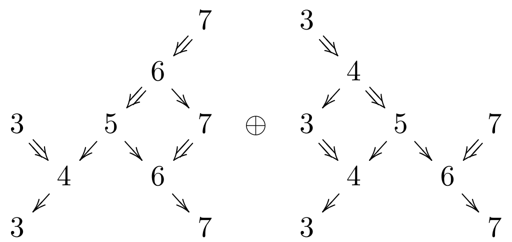
$$1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \longrightarrow 6 \longrightarrow 7.$$

(1) Let $d = (0,0,2,1,2,1,2)$. Then $d^0 = (0,0,2,0,0,00)$, $d^1 = (0,0,2,1,2,0,0)$ and $d^2 = (0,0,0,0,2,1,2)$, and $X(d^1) = M^1 \oplus M^2$ and $X(d^2) = N^1 \oplus N^2$ as follows.

M^1 : M^2 : N^1 : N^2 :

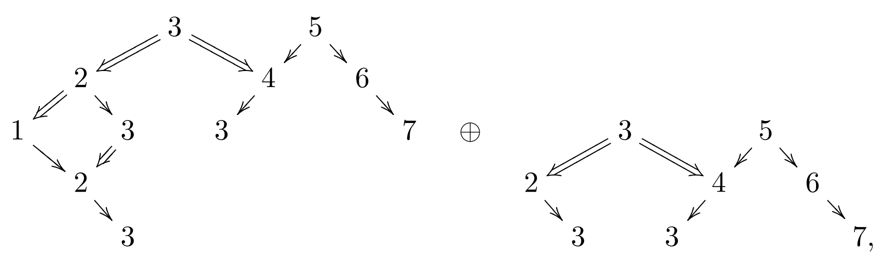


We have $M^1 \leq_5 M^2$, $N^1 = X(\{5,6,7\})$ and $N^2 = X(\{5,7\})$. So $X(d)$ is the direct sum of the gluings of M^1, M^2 with N^1 and N^2, respectively, as follows,

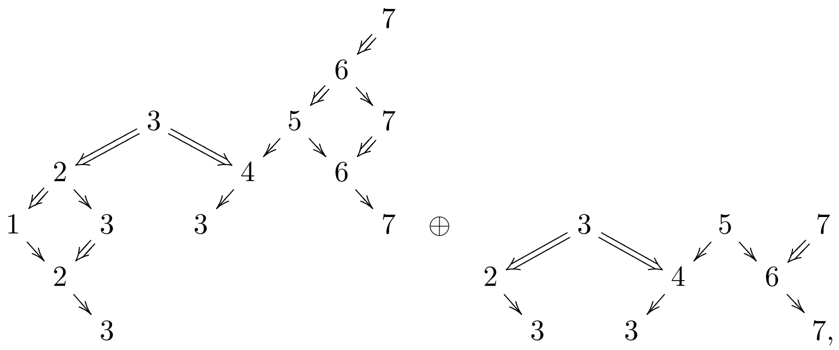


(2) Let $d = (1,2,2,0,2,1,2)$. In this case $e^2 = (1,2,2,0,2,0,0)$ and d^2 is the same as in (1). We have $X(e^2) = M \oplus N$ as follows.

M : N :



where $M \leq_5 N$. So $X(d)$ is the direct sum of the gluing of M and N with N^1 and N^2 from (1), respectively, as follows.



4.4. **A grading on rigid modules.** We now discuss a grading on indecomposable rigid modules $X(d)$ to be used in the next section. We remark that this does not make $X(d)$ into a graded module, but only decompose it into a direct sum of one dimensional subspaces. Let $D^i \subseteq D$ be the subspace spanned by paths of the form

$$a(\alpha\alpha^*)^i b$$

where $\alpha \in Q_1$ and a and b are paths in Q with no arrow in common. Then we have a decomposition

$$D = \bigoplus_{i \geq 0} D^i.$$

Let

$$\mathcal{J} = \oplus_{i \geq 1} D^i,$$

be the ideal generated by all paths of the form $\alpha\alpha^*$ for $\alpha \in Q_1$. We have a filtration

$$D = \mathcal{J}^0 \supseteq \mathcal{J} \supseteq \mathcal{J}^2 \supseteq \dots,$$

such that the inclusion $\mathcal{J}^i \subseteq D$ induces an isomorphism

$$\mathcal{J}^i/\mathcal{J}^{i+1} \cong D^i.$$

The decomposition of D induces a filtration on any indecomposable rigid A-projective D-module X,

$$X = \mathcal{J}^0 X \supseteq \mathcal{J} X \supseteq \mathcal{J}^2 X \supseteq \dots,$$

and a decomposition

$$X = \oplus_{i \geq 0} X^i,$$

in such a way that the component X^i is identified with $\mathcal{J}^i X/\mathcal{J}^{i+1}X$, in particular,

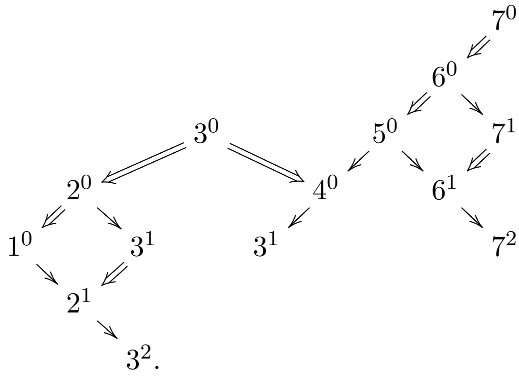
$$X^0 \cong X/\mathcal{J}X.$$

Each component X^i has a further decomposition

$$X^i = \oplus_{a \in Q_0} X^i_a,$$

where X^i_a is the subspace of X^i at vertex a.

Example 5. *The grading on the first summand of $X(d)$ in Example 4(2) is as follows.*



We collect some basic properties of the grading in the following lemmas.

Lemma 11. *Let X be a rigid indecomposable A-projective D-module supported at u. We have $X_u^0 \cong k$. Moreover,*

(1) *if u is a source, then $X_u^0 = X_u$,*
(2) *if u is an interior sink, then $X_u^i = (\alpha_{u-1}\alpha_{u-1}^*)^i X_u^0 \oplus (\alpha_u\alpha_u^*)^i X_u^0$ for $i \geq 1$, and*
(3) *if there is precisely one arrow $\alpha \in Q_1$ terminating at u, then $X_u^i = (\alpha\alpha^*)^i X_u^0$.*

Let u be an interior vertex of Q with $i_v < u \leq i_{v+1}$ and assume that i_v is a source and i_{v+1} is a sink. Let M and N be two indecomposable rigid D-modules supported (but not necessarily Δ-supported at u) such that $M <_u N$. Let

$$\mathrm{Hom}(M,N)_u^0 = \{f|_{M_u^0} \mid f \in \mathrm{Hom}_D(M,N), f(M_u^0) \subseteq N_u^0\}$$

denote the space of maps obtained by restricting a map f to M_u^0 where $f(M_u^0) \subseteq N_u^0$. We have $M_u^0 = k = N_u^0$, and so both $\mathrm{Hom}(N,M)_u^0$ and $\mathrm{Hom}(M,N)_u^0$ are at most one dimensional.

Lemma 12. *Let M and N be as above. Then*

(a) $\mathrm{Hom}(N,M)_u^0 = k$ *if and only if* $(\underline{\dim}_\Delta M)_w = (\underline{\dim}_\Delta N)_w$ *for all* $w > u$.
(b) $\mathrm{Hom}(M,N)_u^0 = k$ *if and only if* $(\underline{\dim}_\Delta M)_w = (\underline{\dim}_\Delta N)_w$ *for all* $w \leq u$.

Proof. (a) Suppose that $(\underline{\dim}_\Delta M)_w = (\underline{\dim}_\Delta N)_w$ for all $w > u$. Let I and J be the Δ-supports of M and N, respectively, and let $I_s = I \cap [i_s, i_{s+1}]$ and $J_s = J \cap [i_s, i_{s+1}]$, where $[i_s, i_{s+1}] = \{i_s, i_s{+}1, \ldots, i_{s+1}\}$. As $M <_u N$, by definition, $I_s \subseteq J_s$ if $s-v$ is even and $I_s \supseteq J_s$ if $s-v$ is odd, whenever both I_s and J_s are nonempty. Since $(\underline{\dim}_\Delta M)_w = (\underline{\dim}_\Delta N)_w$ for all $w > u$, we have $I_s = J_s$ for $s \geq i_{v+1}$. By construction, both M and N are obtained by gluing indecomposable rigid D-modules $X(I_s)$ and $X(J_s)$, respectively, and the identity map on the rigid indecomposable module obtained by gluing $X(J_s)$ with $s \geq v+1$ induces a homomorphism f: $M \to N$ with $f(N_u^0) = M_u^0$ and so $\mathrm{Hom}(N,M)_u^0 = k$

Conversely, assume there is a homomorphism $f : N \to M$ with $0 \neq f(N_u^0) \subseteq M_u^0$. Since i_{v+1} is a sink, there is the path $p : u \to i_{v+1}$ in Q. As N is an A-projective D-module and also by Lemma 11,

$$0 \neq pN_u^0 = (\alpha_{i_{v+1}-1}\alpha_{i_{v+1}-1}^*)^x N_{i_{v+1}}^0,$$

where x is the cardinality of $[u+1, i_{v+1}] \cap J_v$. Similarly,

$$0 \neq pM_u^0 = (\alpha_{i_{v+1}-1}\alpha_{i_{v+1}-1}^*)^y M_{i_{v+1}}^0,$$

where y is the cardinality of $[u+1, i_{v+1}] \cap I_v$. Since $I_v \subseteq J_v$ we have $x \geq y$. On the other hand, as $f(pN_u^0) = pM_u^0$, we have

$$(\alpha_{i_{v+1}-1}\alpha_{i_{v+1}-1}^*)^y M_{i_{v+1}}^0 \subseteq \mathcal{J}^x M,$$

and so $y \geq x$. This shows that $y = x$ and so $[u+1, i_{v+1}] \cap I_v = [u+1, i_{v+1}] \cap J_v$, that is $(\underline{\dim}_\Delta N)_w = (\underline{\dim}_\Delta M)_w$ for all $u < w \leq i_{v+1}$. If $i_{v+1} = n$ or if $i_{v+1} \notin J_v$, then we are done. Assume that $i_{v+1} < n$ and $i_{v+1} \in J_v$. Since $J_{v+1} \subseteq I_{v+1}$ and $f(N^0_{i_{v+1}}) \nsubseteq \mathcal{J}M$ we have $I_{v+1} = J_{v+1}$. Then it follows by induction that $I_s = J_s$ for all $s \geq v+1$, and so $(\underline{\dim}_\Delta N)_w = (\underline{\dim}_\Delta M)_w$ for all $w > u$.

(b) The proof is similar to (a). □

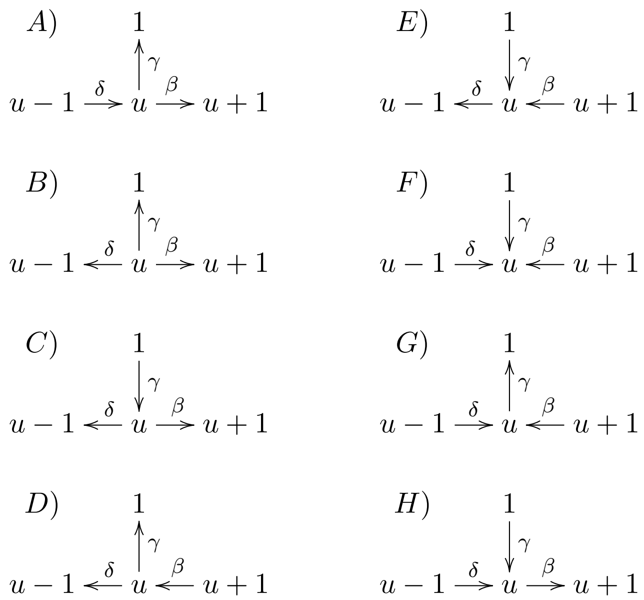
Let $u = i_v$ be a source and let $M \leq_u N$ be two indecomposable rigid D-modules supported, and therefore also Δ-supported at u. In this case, $\mathrm{Hom}(M,N)^0_u = \mathrm{Hom}(M,N)_u$, where $\mathrm{Hom}(M,N)_u$ are restrictions of homomorphisms to M_u. Again, assume that M and N are obtained by gluing $X(I_s)$ and $X(J_s)$, respectively. We have the following lemma, similar to Lemma 12. Note that $\leq_u$ is defined on the interval $i_{v-1} < u = i_v$, so the relative sizes of $X(I_s)$ and $X(J_s)$ are opposite to those in Lemma 12 and thus the inequalities are also opposite. Note also, as u is a source, $(\underline{\dim}_\Delta M)_u = (\underline{\dim} M)_u = 1$ and so is for N.

Lemma 13. *Let M and N be as above. Then*

(a) $\mathrm{Hom}(N,M)_u = k$ *if and only if* $(\underline{\dim}_\Delta M)_w = (\underline{\dim}_\Delta N)_w$ *for all* $w < u$.
(b) $\mathrm{Hom}(M,N)_u = k$ *if and only if* $(\underline{\dim}_\Delta M)_w = (\underline{\dim}_\Delta N)_w$ *for all* $w > u$.

5. Main results

In the remainder of this paper Q is a quiver obtained by attaching a vertex 1 with an arrow γ to an interior vertex u in a quiver of type $\mathbb{A}_{n-1}$ with vertices $\{2, \cdots, n\}$ and arrows between i and $i+1$, and Q' is the quiver with the same underlying graph as Q and a unique sink at u. Let P be a projective representation of Q with $P = P(d)$ for a vector $d \in \mathbb{N}^n$. We will study generic $\mathrm{Aut}_A P$-orbits in $\mathrm{Rep}(D,P)$, case by case, with respect to the following eight orientations of the three arrows incident to vertex u.



The following theorem is the main technical result of this paper. It shows that generic $\mathrm{Aut}_A P$-orbits in $\mathrm{Rep}(D,P)$ can be studied using generic representations of Q'. We emphasize that the theorem is true not only for Dynkin quivers, but also for some quivers of tame and even wild type.

Theorem 7. *Let P be a projective representation of Q. If the orientation at u is as in*

(1) *Case A, B or D, or*
(2) *Case C and $u = 3, 4, n-2$ or $n-1$,*

then there is a dimension vector c such that the $\mathrm{Aut}_A P$*-space* $\mathrm{Rep}(D,P)$ *is generically equivalent to the* $\mathrm{Gl}(c)$*-space* $\mathrm{Rep}(Q',c)$.

The proof of the theorem is long and we postpone the details to next section. In the remaining of this section, we prove the main result, using Theorem 7, which together with Lemma 2 gives the following corollary, a key step in the proof.

Corollary 14. *Let Q, Q', P and c be as in the theorem above. Then there exists a dense open* $\mathrm{Aut}_A P$*-orbit in* $\mathrm{Rep}(D, P)$ *if and only if there is a dense open* $\mathrm{Gl}(c)$*-orbit in* $\mathrm{Rep}(Q', c)$.

For a quiver Q, let Q^{op} be the opposite quiver of Q with vertices $Q_0^{op} = Q_0$ and arrows $Q_1^{op} = Q_1^*$, let $A^{op} = kQ^{op}$. Note that the orientations in Case E, F, G and H are opposite to those in Case A, B, C and D, respectively. For each vector $d \in \mathbb{N}^n$, there is a projective A^{op}-module denoted by $P^{op} = P^{op}(d)$, and an anti-isomorphism of algebras

$$\phi : \mathrm{End}_A P \to \mathrm{End}_{A^{op}} P^{op}$$

which maps $\mathrm{Aut}_A P$ and $\mathrm{radEnd}_A P$ onto $\mathrm{Aut}_{A^{op}} P^{op}$ and $\mathrm{radEnd}_{A^{op}} P^{op}$, respectively.

Lemma 15. *There is an open* $\mathrm{Aut}_A P$*-orbit in* $\mathrm{radEnd}_A P$ *if and only if there is an open* $\mathrm{Aut}_{A^{op}} P^{op}$*-orbit in* $\mathrm{radEnd}_{A^{op}} P^{op}$.

Proof. The anti-isomorphism ϕ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_A P \times \mathrm{radEnd}_A P & \xrightarrow{\phi \times \phi} & \mathrm{Aut}_{A^{op}} P^{op} \times \mathrm{radEnd}_{A^{op}} P^{op} \\ \downarrow & & \downarrow \\ \mathrm{radEnd}_A P & \xrightarrow{\phi} & \mathrm{radEnd}_{A^{op}} P^{op}, \end{array}$$

where the vertical maps are actions. The map $\phi|_{\mathrm{radEnd}_A P}$ sends an open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$ bijectively onto an open $\mathrm{Aut}_{A^{op}} P^{op}$-orbit in $\mathrm{radEnd}_{A^{op}} P^{op}$. This proves the lemma. □

The lemma allows us to reduce from the eight cases A-H to the four cases A-D to prove Theorem 1. We prove Lemma 1 and Theorem 1 together, which can be combined as follows.

Theorem 8. *Let Q be a quiver. Then there is a dense open* $\mathrm{Aut}_A P$*-orbit in* $\mathrm{radEnd}_A P$ *for all projective representations P of Q if and only if Q is a Dynkin quiver.*

Proof. First assume that Q is a Dynkin quiver and let $P = P(d)$ be a projective representation of Q. By Theorem 6 we may assume that Q is not of type $\mathbb{A}$ and Q is as in one of the eight cases A-H above.

In cases A-D, using Corollary 14 and Theorem 2, there is a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{Rep}(D, P)$, and therefore a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$, by Theorem 3. Thus, by Lemma 15, there is also a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{Rep}(D, P)$ in Case E, F, G and H.

Conversely, assume Q is not a Dynkin quiver. The space $(\mathrm{radEnd}_A P)^2$ is closed under the action on $\mathrm{Aut}_A P$, and so if there is a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$, then there is a dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P/(\mathrm{radEnd}_A P)^2$. Note that the kernel of the natural projection, π: $\mathrm{Aut}_A P \to \mathrm{Gl}(d) \cong \prod_i \mathrm{Aut} P_i^{d_i}$, acts trivially on $\mathrm{radEnd}_A P/(\mathrm{radEnd}_A P)^2$, and we have an isomorphism of vector spaces $\psi : \mathrm{radEnd}_A P/(\mathrm{radEnd}_A P)^2 \to \mathrm{Rep}(Q^{op}, d)$. Moreover π and ψ induce the following commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_A P \times (\mathrm{radEnd}_A P/(\mathrm{radEnd}_A P)^2) & \longrightarrow & \mathrm{Gl}(d) \times \mathrm{Rep}(Q^{op}, d) \\ \downarrow & & \downarrow \\ \mathrm{radEnd}_A P/(\mathrm{radEnd}_A P)^2 & \longrightarrow & \mathrm{Rep}(Q^{op}, d), \end{array}$$

where the vertical maps are actions. Since Q^{op} is not Dynkin, there is a dimension vector d such that there is no dense open $\mathrm{Gl}(d)$-orbit in $\mathrm{Rep}(Q^{op}, d)$. Therefore, for this d, there is no dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P/(\mathrm{radEnd}_A P)^2$. Consequently, there is no dense dense open $\mathrm{Aut}_A P$-orbit in $\mathrm{radEnd}_A P$. This completes the proof of the theorem. □

6. Proof of Theorem 7

Let Q, Q' and $P = P(d)$ for $d \in \mathbb{N}^n$ be as in the previous section. We prove Theorem 7 by considering Case A-D separately. The details of the proofs in different cases are quite different and thus the proof of the theorem is long, however in all the four cases the same strategy is used. We first fix genecially a module structure on the double $\tilde{\Gamma}$ of a subquiver $\Gamma \subseteq Q$ of type $\mathbb{A}$ supported on $\{1, \cdots, u\}$ in Case C, and on $\{2, \cdots, n\}$ in the other cases. We then show that the local module structure on arrows not in $\tilde{\Gamma}$ is generically equivalent to the $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D, P)$, and also generically equivalent to the $\mathrm{Gl}(c)$-space $\mathrm{Rep}(Q', c)$ for a dimension vector c constructed from P. To prove the generic equivalence we use the grading defined in Section 4.4 and the properties of homomorphisms with respect to this grading given in Lemma 12 and 13.

6.1. **Case A.** Let d' be the vector given by $(d')_1 = 0$ and $(d')_i = d_i$ for $i \neq 1$. Let Γ be the full subquiver of Q supported on $\{2, \cdots, n\}$. Since Γ is of type $\mathbb{A}$, there is a rigid module with Δ-dimension vector d', which we denote by $Y(d')$, for the double quiver $\tilde{\Gamma}$ with the corresponding relations. We expand $Y(d')$ to an A-projective D-module $X(d')$ with Δ-dimension vector d' as follows.

On $\tilde{\Gamma}$, $X(d')$ and $Y(d')$ are equal. Let $X(d')_1 = Y(d')_u$, $X(d')_\gamma = Id$ and $X(d')_{\gamma^*} = X(d')_\delta X(d')_{\delta^*}$, where $\delta : u-1 \to u \in Q_1$. Then $X(d')$ is an A-projective D-module, and it is rigid by Theorem 4, since the construction preserves the Δ-dimension vector and the dimension of the endomorphism ring.

Let $N^1, \cdots, N^p$ be indecomposable summands of $Y(d')$, one from each isomorphism class, supported (but not necessarily Δ-supported) at u, and ordered such that $N^i <_u N^{i+1}$, where $<_u$ is as in Definition 2. Let n_i denote the multiplicity of N^i as a summand in $Y(d')$, and let $M^1, \cdots, M^p$ be the corresponding summands of $X(d')$.

As discussed in Section 4.4, $Y(d')$ is graded using the ideal $\mathcal{J}$ with the graded components denoted by $Y(d')^i$ and with $\mathcal{J}Y(d')$ equal to the direct sum of the graded components with positive degrees. By construction $X(d')_1 = X(d')_u = Y(d')_u = Y(d')^0_u \oplus \mathcal{J}Y(d')_u$. We denote the two copies of the subspace $Y(d')^0_u$ in $X(d')_u$ and $X(d')_1$, by $X(d')^0_u$ and $X(d')^0_1$, respectively.

Following Section 4.4, we let $\mathrm{End}(X(d'))^0_u$ denote the space of maps which are restrictions
$$f_{|X(d')^0_u} : X(d')^0_u \to X(d')^0_u$$
of homomorphisms $f : X(d') \to X(d')$ with $f(X(d')^0_u) \subseteq X(d')^0_u$. The action of $\mathrm{Aut}_D(X(d'))$ on $X(d')$ induces by restriction an action of $\mathrm{Aut}_D(X(d'))^0_u$ on $X(d')^0_u$, where $\mathrm{Aut}_D(X(d'))^0_u \subseteq \mathrm{End}_D(X(d'))^0_u$ consists of the restrictions of invertible maps.

Lemma 16. *The* $\mathrm{Aut}_D(X(d'))^0_u \times \mathrm{Gl}_{d_1}$*-space* $\mathrm{Hom}_k(k^{d_1}, X(d')^0_u)$*, with the action given by* $(a, g)c = acg^{-1}$ *for* $(a, g) \in \mathrm{Aut}_D(X(d'))^0_u \times \mathrm{Gl}_{d_1}$ *and* $c \in \mathrm{Hom}_k(k^{d_1}, X(d')^0_u)$*, is a generic section of the* $\mathrm{Aut}_A P$*-space* $\mathrm{Rep}(D, P)$*.*

Proof. Any A-projective D-module X with ${}_A X = P$ has a unique submodule $X' \subseteq X$ which is generated by the subspaces X_i for $2 \leq i \leq n$. Then ${}_A X' = P(d')$ and $(X')_{\alpha^*} = X_{\alpha^*}$ for $\alpha \in \Gamma$. We consider the subset in $\mathrm{Rep}(D, P)$ consisting of representations X with the submodule $X' = X(d')$. Then as $(\tilde{Q}, \mathcal{I})$-representations such X are determined by $X(d') \subseteq X$ and the map X_{γ^*} as follows,

$$Y(d')^0_u \oplus (\mathcal{J}Y(d'))_u \; \underset{X_{\gamma^*} = \begin{pmatrix} c' & 0 & 0 \\ c'' & z_1 & z_2 \end{pmatrix}}{\overset{X_\gamma = \begin{pmatrix} 0 & Id & 0 \\ 0 & 0 & Id \end{pmatrix}^{tr}}{\rightleftarrows}} \; k^{d_1} \oplus Y(d')^0_u \oplus (\mathcal{J}Y(d'))_u,$$

where $\begin{pmatrix} 0 & 0 \\ z_1 & z_2 \end{pmatrix} = X(d')_{\gamma^*} = X(d')_\delta X(d')_{\delta^*}$ and $(z_1\ z_2)$ is surjective by the the structure of $Y(d')$. Furthermore, X is determined by the maps c' and c'' and so we denote it by $X(c', c'')$.

Let

$$\phi : \mathrm{Hom}_k(k^{d_1}, X(d')^0_u) \to \mathrm{Rep}(D, P)$$

be the morphism given by $\phi(c) = X(c, 0)$. We will show that ϕ satisfies the conditions in Definition 1 of a generic section. First, the subset $\mathrm{Im}\phi$ consists of representations of the form $X(c, 0)$ and is an affine subspace in $\mathrm{Rep}(D, P)$. Second, we will show that $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ is an open subset of $\mathrm{Rep}(D, P)$.

We first claim that any $X(c', c'')$ is isomorphic to $X(c', 0)$. Indeed, a map

$$f = (f_i)^n_{i=1} : X(c', c'') \to X(c, 0),$$

where $f_i : X(c', c'')_i \to X(c, 0)_i$, and c not necessarily equal to c', is a homomorphism if and only if the following 3 conditions are satisfied.

(1) $f|_{X(d')} = (a, f_2, \ldots, f_n) \in \mathrm{End}_D X(d')$, where $a = f_u = f_1|_{X(d')_1} = \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}$: $X(d')_1 \to X(d')_1$ with respect to the decomposition $X(d')_1 = Y(d')^0_u \oplus (\mathcal{J}Y(d'))_u$.

(2) $f_1 = \begin{pmatrix} g & 0 \\ b & a \end{pmatrix}$, where $g \in \mathrm{Gl}_{d_1}$ and $b : k^{d_1} \to X(d')_1$.

(3) (i) $a_1 c' = cg$ and (ii) $a_2 c' + a_3 c'' = (z_1\ z_2)b$.

Here (1) and (2) are due to $X(c', c'')$ and $X(c, 0)$ both having $X(d')$ as a submodule, and (3) is because of $f_u X(c', c'')_{\gamma^*} = X(c, 0)_{\gamma^*} f_1$.

When $c = c'$, since $(z_1\ z_2)$ is surjective, there exists a map $b : k^{d_1} \to X(d')_1$ such that $(z_1\ z_2)b = c''$. If we let $f : X(c', c'') \to X(c', 0)$ be given by $f|_{X(d')} = Id$ and $f_1 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, then f is an isomorphism. This shows that any $X(c', c'')$ is isomorphic to a representation in $\mathrm{Im}\phi$. Consequently, $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ is equal to the subset of representations X with the unique submodule $X' \cong X(d')$. Moreover, $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ is an open subset, because there is a morphism $\mathrm{Rep}(D, P) \to \mathrm{Rep}(D, P(d'))$ which maps X to the unique submodule X', with the preimage of the open orbit of $X(d')$ equal to $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$.

Third, if there is an isomorphism $f : X(c', 0) \to X(c, 0)$, then it follows from (1)-(3) above that $a_1 c' = cg$, i.e. $c, c' : k^{d_1} \to X(d')^0_u$ are conjugate under the action of $\mathrm{Aut}(X(d'))^0_u \times \mathrm{Gl}_{d_1}$ on $\mathrm{Hom}_k(k^{d_1}, X(d')^0_u)$. Conversely, suppose that c and c' are conjugate using $(a_1, g) \in \mathrm{Aut}(X(d'))^0_u \times \mathrm{Gl}_{d_1}$ and let $h : X(d') \to X(d')$ be an automorphism with a_1 equal to the restriction to $X(d')^0_u$. Then since $(z_1\ z_2)$ is surjective, there exists $b : k^{d_1} \to X(d')_1$ such that h, g and b together give an isomorphism $f : X(c', 0) \to X(c, 0)$. This shows that

$$\phi((\mathrm{Aut}_D(X(d'))^0_u \times \mathrm{Gl}_{d_1}) \cdot c) = \mathrm{Aut}_A P \cdot \phi(c) \cap \mathrm{Im}\phi,$$

for all $c \in \mathrm{Hom}_k(k^{d_1}, X(d')^0_u)$, and so the $\mathrm{Aut}_D(X(d'))^0_u \times \mathrm{Gl}_{d_1}$-space $\mathrm{Hom}_k(k^{d_1}, X(d')^0_u)$ is a generic section of the $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D, P)$. □

We compute the maps in $\mathrm{Aut}_D(X(d'))^0_u$.

Lemma 17. *Let $i, j \in \{1, \cdots, p\}$.*

(1) *If $\mathrm{Hom}_D(M^i, M^j)^0_u = k$ then*
   (a) *$i \geq j$ and $(\underline{\dim}_\Delta M^i)_w = (\underline{\dim}_\Delta M^j)_w$ for all $w > u$, or*
   (b) *$i < j$ and $(\underline{\dim}_\Delta M^i)_w = (\underline{\dim}_\Delta M^j)_w$ for all $w \leq u$.*

(2) *If both (a) and (b) fail then $\mathrm{Hom}_D(M^i, M^j)^0_u = 0$.*

Proof. By construction of M^i and M^j, we have $\mathrm{End}_D(X(d'))^0_u = \mathrm{End}_D(Y(d'))^0_u$, so the lemma follows from Lemma 12. □

For each interval $[i, j] = \{i, i + 1, \cdots, j\}$ for $2 \leq i \leq j \leq n$ there is an associated indecomposable representation $M[i, j]$ of Q' with support equal to $[i, j]$. We construct a representation

$$Z(d') = \bigoplus_{i=1}^{p} (Z^i)^{n_i}$$

of Q', with $\mathrm{Aut}_{kQ'}(Z(d'))_u \cong \mathrm{Aut}_D(X(d'))^0_u$, as follows. Let $Z^1 = M[2, u]$. Given $Z^i = M[j, j']$, let

$$Z^{i+1} = \begin{cases} M[j, j' + 1] & \text{if } \mathrm{Hom}_D(M^i, M^{i+1})^0_u = k, \\ M[j + 1, j'] & \text{if } \mathrm{Hom}_D(M^{i+1}, M^i)^0_u = k, \\ M[j + 1, j' + 1] & \text{otherwise.} \end{cases}$$

Lemma 18. *We have* $2 \leq j \leq u \leq j' \leq n$ *for any summand* $Z^i = M[j, j']$ *in* $Z(d')$*.*

Proof. The inequalities $2 \leq j$ and $u \leq j'$ are trivial. If $\mathrm{Hom}(M^{i+1}, M^i)^0_u = 0$, then by Lemma 17, M^i and M^{i+1} are Δ-supported differently on the interval $[u + 1, n]$. By the construction of $X(d')$ there are at most $n - u$ summands M^i with $\mathrm{Hom}(M^{i+1}, M^i)^0_u = 0$. Therefore $j' \leq n$. Similarly, $\mathrm{Hom}(M^i, M^{i+1})^0_u = 0$ for at most $u - 2$ summands M^i. Therefore $j \leq u$. □

Let $c(d') = \underline{\dim} Z(d')$ and $c(d) = (d_1, c(d')_2, \cdots, c(d')_n)$.

Lemma 19. *The* $\mathrm{Aut}_{kQ'}(Z(d')_u) \times \mathrm{Gl}_{d_1}$*-space* $\mathrm{Hom}_k(k^{d_1}, Z(d')_u)$ *is a generic section of the* $\mathrm{Gl}(c(d))$*-space* $\mathrm{Rep}(Q', c(d))$*.*

Proof. Let Γ' be the full subquiver of Q' supported on $\{2, \cdots, n\}$. The $\mathrm{Aut}_{kQ'}(Z(d')) \times \mathrm{Gl}_{d_1}$ orbits in $\mathrm{Hom}_k(k^{d_1}, Z(d')_u)$ parametrise the representations of Q' with restriction to Γ' equal to $Z(d')$. The lemma follows if we can prove that $Z(d')$ is a rigid representation.

The full subquiver Γ' has a unique sink u. Let $M = M[i, j] \oplus M[i', j']$, for $1 \leq i, i' \leq u \leq j, j' \leq n$. Then $\mathrm{Ext}^1_{kQ'}(M[i, j], M[i', j']) \neq 0$ if and only if $i < u < j$ and $[i', j'] \subseteq [i+1, j-1]$. The lemma follows. □

Finally we can prove Case A of Theorem 7.

Lemma 20. *The* $\mathrm{Gl}(c(d))$*-space* $\mathrm{Rep}(Q', c(d))$ *is generically equivalent to the* $\mathrm{Aut}_A P$*-space* $\mathrm{Rep}(D, P)$*.*

Proof. There are isomorphisms $(Z^i)_u \to (M^i)^0_u$, which extend to an isomorphism

$$\mathrm{Hom}_k(k^{d_1}, Z(d')_u) \to \mathrm{Hom}_k(k^{d_1}, X(d')^0_u)$$

of vector spaces. By the construction $\mathrm{Hom}(M^i, M^j)^0_u \cong \mathrm{Hom}_{kQ'}(Z^i, Z^j)_u$, and we have an isomorphism $\mathrm{Aut}_{kQ'}(Z(d'))_u \to \mathrm{Aut}(X(d'))^0_u$. Therefore there is a commutative diagram

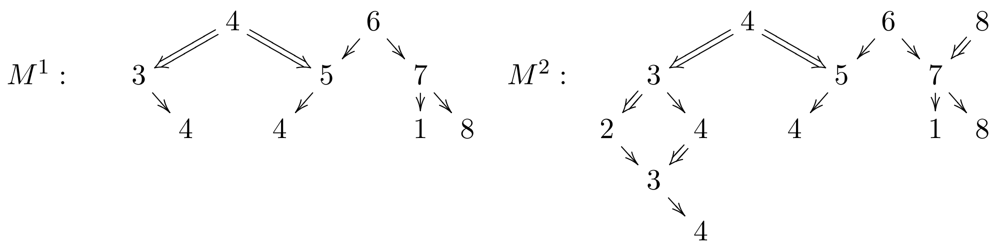
$$\begin{array}{ccc} Aut_{kQ'}(Z(d')) \times \mathrm{Gl}_{d_1} \times \mathrm{Hom}_k(k^{d_1}, Z(d')_u) & \longrightarrow & \mathrm{Aut}(X(d'))^0_u \times \mathrm{Gl}_{d_1} \times \mathrm{Hom}_k(k^{d_1}, X(d')^0_u) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_k(k^{d_1}, Z(d')_u) & \longrightarrow & \mathrm{Hom}_k(k^{d_1}, X(d')^0_u) \end{array}$$

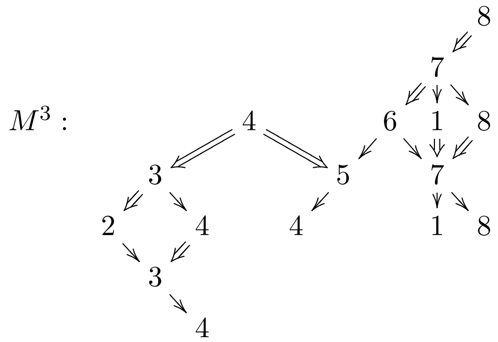
where the vertical maps are actions and the horizontal maps are isomorphisms. The lemma now follows from Lemmas 16 and 19. □

Example 6. *Let* Q *be the quiver*

$$\begin{array}{ccccccccccccc} & & & & & & & & & & 1 & & \\ & & & & & & & & & & \uparrow & & \\ 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longleftarrow & 5 & \longleftarrow & 6 & \longrightarrow & 7 & \longrightarrow & 8, \end{array}$$

$u = 7$ *and let* $d = (2, 2, 3, 3, 0, 3, 1, 2)$*. So* $d' = (0, 2, 3, 3, 0, 3, 1, 2)$*. Then* $X(d') = M_1 \oplus M_2 \oplus M_3$*, where* $M^1 <_7 M^2 <_7 M^3$ *are as follows.*





By computing directly or by Lemma 17,

$$\mathrm{Aut}(X(d'))^0_7 = \{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} | a_{ij} \in k \}.$$

(2) We have

$$Q' : \quad \begin{array}{ccccccccccccc} & & & & & & & & & & 1 & & \\ & & & & & & & & & & \downarrow & & \\ 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 & \longrightarrow & 7 & \longleftarrow & 8 \end{array}$$

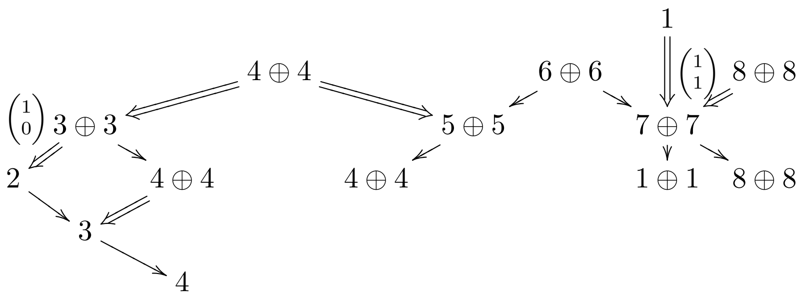
We construct the representation $Z(d') = Z^1 \oplus Z^2 \oplus Z^3$ of Q' with $Z^1 = M[2,7]$, $Z^2 = M[3,8]$ and $Z^3 = [4,8]$. Then $\mathrm{Aut}_{kQ'} Z \cong \mathrm{Aut}(X(d'))^0_7$. A representation of Q' with dimension vector $c(d) = (2,1,2,3,3,3,3,2)$ and with $Z(d')$ as a submodule has the following structure.

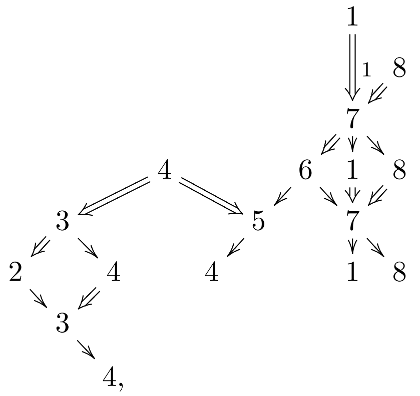
$$\begin{array}{ccccccccccccc} & & & & & & & & & & k^2 & & \\ & & & & & & & & & & \Big\downarrow {\scriptstyle c=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}} & & \\ k & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & k^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & k^3 & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & k^3 & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & k^3 & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & k^3 & \xleftarrow[\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}]{} & k^2. \end{array}$$

A rigid kQ'-module with dimension vector $(2,1,2,3,3,3,3,2)$ is isomorphic to the direct sum of the indecomposable representations with dimension vector $(1,1,2,2,2,2,2,1)$, $(1,0,0,1,1,1,1,1)$, respectively, where we may choose

$$c = \begin{pmatrix} 1 & | & 0 \\ 1 & | & 0 \\ - & + & - \\ 0 & | & 1 \end{pmatrix}$$

(3) The rigid kQ'-module in (2) induces a rigid A-projective D-module with Δ-dimension vector d as follows.





where $j \oplus j$ means that the vector space is 2-dimensional, the matrices on arrows connecting vector spaces of equal dimension are identity matrices. Also, the matrices for $1 \to 7$ and $1 \to 7 \oplus 7$ are the diagonal blocks of the matrix c in (2).

6.2. **Case B.** This case is very similar to Case A, but simpler. The main difference is that the grading in Section 4.4 is not essential in this section. For completeness and the convenience of the reader we include most of the details of the proof. Let d', Γ, $Y(d')$, N^i and n_i for $i = 1 \cdots p$ be as in Case A. We construct a rigid A-projective D-module $X(d')$ with Δ-projective dimension vector d' as follows. Let $X(d')$ be equal to $Y(d')$ on $\tilde{\Gamma}$. Let $X(d')_1 = X(d')_u$, $X(d')_\gamma = Id$ and $X(d')_{\gamma^*} = 0$ for $\gamma : u \to 1 \in Q_1$. As in Case A, $X(d')$ is rigid by Theorem 4. Let M^i be the summand in $X(d')$ corresponding to the summand N^i in $Y(d')$.

The following lemma is similar to Lemma 16 in Case A. Note that since u is a source, we have $X(d')_u = X(d')^0_u$ and $\mathrm{Aut}_D(X(d'))_u = \mathrm{Aut}(X(d'))^0_u$, by Lemma 11. This explains why the grading is not essential.

Lemma 21. *The* $\mathrm{Aut}_D(X(d'))_u \times \mathrm{Gl}_{d_1}$*-space* $\mathrm{Hom}_k(k^{d_1}, X(d')_u)$ *with the action* $(a, g)c = acg^{-1}$ *for* $g \in \mathrm{Gl}_{d_1}, a \in \mathrm{Aut}_D(X(d'))_u$ *and* $c \in \mathrm{Hom}_k(k^{d_1}, X(d')_u)$*, is a generic section of the* $\mathrm{Aut}_A P$*-space* $\mathrm{Rep}(D, P)$.

Proof. Any D-module X with ${}_AX = P(d)$ has a unique submodule $X' \subseteq X$ generated by X_i for all $i \neq 1$. We have ${}_AX' = P(d')$ and $(X')_{\alpha^*} = X_{\alpha^*}$ for any $\alpha \in \Gamma_1$. We consider the subset of $\mathrm{Rep}(D, P)$ consisting of those representations X with $X' = X(d')$. So as a $(\tilde{Q}, \mathcal{I})$-representation, X is determined by $X(d') \subseteq X$ and the map X_{γ^*} as follows,

$$Y(d')_u \underset{X_{\gamma^*}=(c\quad 0)}{\overset{X_\gamma = \binom{0}{Id}}{\rightleftarrows}} k^{d_1} \oplus Y(d')_u,$$

where $c : k^{d_1} \to Y(d')_u$. We let $X(c)$ denote such a representation.

Let
$$\phi : \mathrm{Hom}_k(k^{d_1}, Y(d')_u) \to \mathrm{Rep}(D,P)$$
be the morphism $\phi(c) = X(c)$. We show that the $\mathrm{Aut}_D(X(d'))_u \times \mathrm{Gl}_{d_1}$-space $\mathrm{Hom}_k(k^{d_1}, X(d')_u)$ is a generic section of $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D,P))$, using the map ϕ.

First, $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ is equal to the set of representations X with X' isomorphic to $X(d')$. There is a morphism $\mathrm{Rep}(D,P) \to \mathrm{Rep}(D,P(d'))$ such that $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ is equal to the preimage of the open orbit of $X(d')$. Therefore $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ is an open subset of $\mathrm{Rep}(D,P)$.

Second, $X(c) \cong X(c')$ if and only if there exists a pair
$$(a, \begin{pmatrix} g & 0 \\ b & a \end{pmatrix}),$$
for $a \in \mathrm{Aut}_D(X(d'))_u, g \in \mathrm{Gl}_{d_1}, b : k^{d_1} \to Y(d')_u$ such that $c'g = ac$, i.e. $c' = acg^{-1}$. In other words, $X(c) \cong X(c')$ if and only if c' and c are conjugate under the action of $\mathrm{Aut}_D(X(d'))_u \times \mathrm{Gl}_{d_1}$.

So the $\mathrm{Aut}_D(X(d'))_u \times \mathrm{Gl}_{d_1}$-space $\mathrm{Hom}_k(k^{d_1}, X(d')_u)$ is a generic section of the $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D,P)$. □

Similar to Case A, we compute the maps in $\mathrm{Aut}(X(d'))^0_u$. Note that the inequalities are opposite compared to Case A due to the definition of $\leq_u$, which is defined on the interval ending at u, and $\mathrm{Hom}_D(M^i, M^j)^0_u = \mathrm{Hom}_D(M^i, M^j)_u$.

Lemma 22. *Let $i, j \in \{1, \cdots, p\}$.*

(1) *If $\mathrm{Hom}_D(M^i, M^j)^0_u = k$ then*
  (a) *$i \geq j$ and $(\underline{\dim}_\Delta M^i)_w = (\underline{\dim}_\Delta M^j)_w$ for all $w \leq u$, or*
  (b) *$i < j$ and $(\underline{\dim}_\Delta M^i)_w = (\underline{\dim}_\Delta M^j)_w$ for all $w > u$.*

(2) *If (a) and (b) fail then $\mathrm{Hom}_D(M^i, M^j)^0_u = 0$.*

Proof. By construction, we have $\mathrm{End}_D(X(d'))_u = \mathrm{End}_D(Y(d'))_u$ and so we need only prove the corresponding lemma for N^i and N^j. Similar to Case A, the lemma now follows from Lemma 13. □

Using a similar procedure as in Case A, we construct a representation
$$Z(d') = \oplus (Z^i)^{n_i}$$
of Q', with $\mathrm{Aut}_{kQ'}(Z(d'))_u \cong \mathrm{Aut}_D(X(d'))_u$, as follows. Let $Z^1 = M[u,n]$. Given $Z^i = M[j,j']$, let
$$Z^{i+1} = \begin{cases} M[j-1, j'] & \text{if } \mathrm{Hom}_D(M^i, M^{i+1})_u = k, \\ M[j, j'-1] & \text{if } \mathrm{Hom}_D(M^{i+1}, M^i)_u = k, \\ M[j-1, j'-1] & \text{otherwise.} \end{cases}$$
Let $c(d') = \underline{\dim} Z(d')$ and $c(d) = (d_1, c(d')_2, \cdots, c(d')_n)$. Similar to Case A we have the following lemmas.

Lemma 23. (1) *For each $Z^i = M[j,j']$, we have $2 \leq j \leq u \leq j' \leq n$.*
(2) *$Z(d')$ is a rigid representation.*

Lemma 24. *The $\mathrm{Aut}_{kQ'}(Z(d')_u) \times \mathrm{Gl}_{d_1}$-space $\mathrm{Hom}_k(k^{d_1}, Z(d')_u)$ is a generic section of the $\mathrm{Gl}(c(d))$-space $\mathrm{Rep}(Q', c(d))$.*

As in Case A, using Lemma 21, Lemma 22 and Lemma 24, we have the following lemma, which proves Case B of Theorem 7.

Lemma 25. *The $\mathrm{Gl}(c(d))$-space $\mathrm{Rep}(Q', c(d))$ is generically equivalent to the $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D,P)$.*

6.3. **Case C.** Unlike case A and B, in this case we let Γ be the full subquiver of Q supported on the vertices $\{1, \cdots, u\}$. Moreover, we assume for now that $u = n-2$. The cases where $u = 3, 4$ or $n-1$ will be dealt with in Lemma 29 below. Let α and β be the arrows in Q as follows

$$\begin{array}{ccccccc} & & 1 & & & & \\ & & \downarrow{\scriptstyle\gamma} & & & & \\ \cdots & u-1 \longleftarrow & u & \xrightarrow{\beta} & u+1 & \overset{\alpha}{\longleftrightarrow} & u+2 \end{array}$$

where α could be of either orientation, that is, $u+2$ is either a sink or a source. Let d' be given by $d'_i = 0$ for $i = u+1, u+2$ and $d'_i = d_i$ otherwise. Let $Y(d')$ be a rigid module for the double quiver $\tilde{\Gamma}$ with the corresponding relations, which we extend to a rigid A-projective D-module $X(d')$ as follows.

On $\tilde{\Gamma}$ we let $X(d')$ be equal to $Y(d')$. Let $X(d')_{u+1} = Y(d')_u$, $X(d')_\beta = Id$, and $X(d')_{\beta^*} = Y(d')_\gamma Y(d')_{\gamma^*}$. If $u+2$ is a sink, then $X(d')_{u+2} = Y(d')_u$, $X(d')_\alpha = Id$ and $X(d')_{\alpha^*} = X(d')_{\beta^*}$, and if $u+2$ is a source, then $X(d')_{u+2} = 0$. The extension of $Y(d')$ to $X(d')$ preserves the Δ-dimension vector and the dimension of the endomorphism ring, and so $X(d')$ is rigid by Theorem 4.

Let $N^1, \cdots, N^p$ be indecomposable summands of $Y(d')$, one from each isomorphism class, supported at u and ordered such that $N^i <_u N^{i+1}$, and let n_i be the multiplicity of N^i as a summand in $Y(d')$. For each indecomposable summand N^i of $Y(d')$, let M^i be the corresponding indecomposable summand of $X(d')$.

Let d'' be the dimension vector supported on $\{u+1, u+2\}$, given by $d''_i = 0$ for $i \leq u$ and $d''_i = d_i$ otherwise. Let $X(d'')$ be the rigid D-module with Δ-dimension vector d''.

We will construct two groups H_V and H_W, which act on vector spaces V and W in such a way that the $H_V \times H_W$-space $\mathrm{Hom}_k(V, W)$ is a generic section of the $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D, P)$. The construction depends on whether $u+2$ is a sink or a source.

If $u+2$ is a sink, let

$$V = soc(X(d'')) \subseteq X(d'')_{u+2},\ W = X(d')^0_{u+1},$$

and

$$H_V = \{f|_V \mid f \in \mathrm{Aut}_D(X(d''))\},\ H_W = \mathrm{Aut}_D(X(d'))^0_{u+1}.$$

If $u+2$ is a source, let

$$V = (X(d'')/\mathrm{rad}X(d''))_{u+1} \cong X(d'')^0_{u+1} \cong k^{d_{u+1}},\ W = X(d')^0_u,$$

and

$$H_V = \{\overline{f} : (X(d'')/\mathrm{rad}X(d''))_{u+1} \to (X(d'')/\mathrm{rad}X(d''))_{u+1} \mid f \in \mathrm{Aut}_D(X(d''))\},$$
$$H_W = \mathrm{Aut}_D(X(d'))^0_u.$$

In either case, we have the following lemma.

Lemma 26. *The $H_V \times H_W$-space $\mathrm{Hom}_k(V, W)$ is generically equivalent to the $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D, P)$.*

Proof. Any A-projective D-module X has a unique submodule $X' \subseteq X$ generated by the spaces X_i for $i \leq u$, with corresponding quotient denoted by $X'' = X/X'$. If $d = \underline{\dim}_\Delta X$, then $\underline{\dim}_\Delta X' = d'$ and $\underline{\dim}_\Delta X'' = d''$. We consider the subset of $\mathrm{Rep}(D, P)$ consisting of D-modules X with $X' = X(d')$ and $X'' = X(d'')$, which is nonempty since it contains the direct sum $X(d') \oplus X(d'')$. Moreover, we fix a decomposition of vector spaces $X_i = X(d')_i \oplus X(d'')_i$ for all vertices i.

We decompose

$$X(d') = M \oplus N \oplus L,$$

where the indecomposable summands of M are Δ-supported at both 1 and u, those of N are Δ-supported at either 1 or u, but not both, and L is not supported at u. Also, we decompose

$$X(d'') = R \oplus T,$$

where the indecomposable summands of R are Δ-supported at both $u+1$ and $u+2$, and those of T are not Δ-supported at both $u+1$ and $u+2$.

The proof is divided into two parts, depending on whether $u+2$ is a source or a sink.

Part 1: We first consider the case where $u+2$ is a sink.

Let $W_1 = M^0_{u+1}$, $W_2 = M^1_{u+1}$ and $W_3 = N^0_{u+1}$. Note that $N^0_{u+1} = N_{u+1}$ and

$$W = W_1 \oplus W_3.$$

Let $V_1 = R^0_{u+2}$, $V_2 = R^1_{u+2}$ and $V_3 = T_{u+2}$. Then

$$V = V_2 \oplus V_3$$

By the relations of D, such a $(\tilde{Q}, \mathcal{I})$-representation X is determined by $X(d')$, $X(d'')$ and the maps between the vertices $u+1$ and $u+2$, which have the form

$$X(d')_{u+1} \oplus X(d'')_{u+1} \underset{X_{\alpha^*}=\begin{pmatrix} X(d')_{\alpha^*} & c \\ 0 & X(d'')_{\alpha^*}\end{pmatrix}}{\overset{X_\alpha=\begin{pmatrix} Id & 0 \\ 0 & X(d'')_\alpha\end{pmatrix}}{\rightleftarrows}} X(d')_{u+2} \oplus X(d'')_{u+2}$$

where $X(d')_{u+1} = X(d')_{u+2} = W_1 \oplus W_2 \oplus W_3$, $X(d'')_{u+2} = V_1 \oplus V_2 \oplus V_3$, $X(d'')_{u+1} = V_2 \oplus V_3$ if $d_{u+1} > d_{u+2}$, and $X(d'')_{u+1} = V_2$ if $d_{u+1} \le d_{u+2}$, and

$$c = (c_{ij})_{ij} : V_1 \oplus V_2 \oplus V_3 \to W_1 \oplus W_2 \oplus W_3.$$

We let $X(c)$ denote the representation in the diagram above.

Our first step is to show that with some restriction on the map c_{12}, $X(c)$ is isomorphic to $X(c_0)$, where

$$c_0 = \begin{pmatrix} 0 & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix}.$$

By fixing a basis we may assume we have matrices

$$X(d'')_\alpha = \begin{pmatrix} 0 & 0 \\ Id & 0 \\ 0 & Id \end{pmatrix}, X(d'')_{\alpha^*} = \begin{pmatrix} Id & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } X(d')_{\alpha^*} = \begin{pmatrix} 0 & 0 & 0 \\ Id & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that if $d_{u+1} \le d_{u+2}$ the second column of $X(d'')_\alpha$ and the second row of $X(d'')_{\alpha^*}$ are empty.

We choose a basis of W_1 (and of W_2) such that there is one basis element from each summand M^i, and order the basis elements according to the order $\le_u$.

A map $f = (f_i)_{i=1}^n : X(c) \to X(c')$, where $f_i : X(c)_i \to X(c')_i$, is a homomorphism if and only if the following 4 conditions are satisfied.

(1) $f|_{X(d')} = (f_1, f_2, \ldots, f_u, a, a) \in \mathrm{End}_D X(d')$.

(2) $(f_{u+1}, f_{u+2}) = \left(\begin{pmatrix} a & b' \\ 0 & g' \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & g \end{pmatrix} \right)$, where $b = (b_{ij})_{ij} : V_1 \oplus V_2 \oplus V_3 \to W_1 \oplus W_2 \oplus W_3$, and $b' = b|_{X(d'')_{u+1}}$.

(3) $\overline{f} = (0, 0, \ldots, 0, g', g) \in \mathrm{End}_D X(d'')$.

(4) $c'g = ac - X(d')_{\alpha^*} b + b' X(d'')_{\alpha^*}$, as $f_{u+1} X(c)_{\alpha^*} = X(c')_{\alpha^*} f_{u+2}$

Moreover, as matrices with respect to the fixed bases, we have

$$-X(d')_{\alpha^*} b + b' X(d'')_{\alpha^*} = \begin{pmatrix} b_{12} & 0 & 0 \\ -b_{11} + b_{22} & -b_{12} & -b_{13} \\ b_{32} & 0 & 0 \end{pmatrix} \text{ for } b = (b_{ij})_{ij},$$

$$g = \begin{pmatrix} g_1 & 0 & 0 \\ x_1 & g_1 & x_2 \\ x_3 & 0 & g_2 \end{pmatrix}$$

for any invertible matrices g_1 and g_2 and any matrices x_1, x_2 and x_3, and

$$a = \begin{pmatrix} a_1 & 0 & 0 \\ y_1 & a_1 & y_2 \\ y_3 & 0 & a_2 \end{pmatrix},$$

where a_1 and a_2 are invertible. In general, not all matrices occur as matrices a_i and y_i of an automorphism a.

The groups H_V and H_W consist of matrices of the form

$$\begin{pmatrix} g_1 & x_2 \\ 0 & g_2 \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & 0 \\ y_3 & a_2 \end{pmatrix},$$

respectively.

For a given map c we let $b(c)$ be

$$b(c) = \begin{pmatrix} c_{21} & c_{22} & c_{23} \\ 0 & 0 & 0 \\ 0 & -c_{31} & 0 \end{pmatrix}.$$

Then for any $X(c)$, using an isomorphism f with $a = Id$, $g = Id$ and $b = b(c)$, we see that $X(c) \cong X(c')$ for

$$c' = \begin{pmatrix} c'_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix},$$

where $c'_{11} = c_{11} + c_{22}$.

Using an isomorphism f with $b = 0$,

$$a = \begin{pmatrix} Id & 0 & 0 \\ y & Id & 0 \\ 0 & 0 & Id \end{pmatrix} : W_1 \oplus W_2 \oplus W_3 \to W_1 \oplus W_2 \oplus W_3,$$

and

$$g = \begin{pmatrix} Id & 0 & 0 \\ x & Id & 0 \\ 0 & 0 & Id \end{pmatrix} : V_1 \oplus V_2 \oplus V_3 \to V_1 \oplus V_2 \oplus V_3,$$

we have $X(c') \cong X(c'')$, where

$$c'' = ac'g^{-1} = \begin{pmatrix} c'_{11} - c_{12}x & c_{12} & c_{13} \\ y(c'_{11} - c_{12}x) & yc_{12} & yc_{13} \\ -c_{32}x & c_{32} & c_{33} \end{pmatrix}.$$

Now by an isomorphism with $a = Id$, $g = Id$ and an $b = b(c'')$ as above, $X(c'')$ is isomorphic to $X(c''')$ with

$$c''' = \begin{pmatrix} c'_{11} - c_{12}x + yc_{12} & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix}.$$

We identify W_1 with W_2 and V_1 with V_2. We claim that with some restriction on c_{12}, there exists maps $x : W_1 \to W_2$ and $y : V_1 \to V_2$ such that $c'_{11} - c_{12}x + yc_{12} = 0$. Indeed, by Example 1, the matrix space $\mathrm{Mat}_{\dim_k W_1 \times \dim_k V_1}$ has an open $\mathrm{Gl}_{\dim_k V_1} \times \mathrm{B}_{\dim_k W_1}$-orbit, where $\mathrm{B}_{\dim_k W_1}$ consists of all invertible lower triangular $\dim_k W_1 \times \dim_k W_1$-matrices, and the action is given by $(h, b)c_{12} = bc_{12}h^{-1}$. For a matrix c_{12} with an open $\mathrm{B}_{\dim_k W_1} \times \mathrm{Gl}_{\dim_k V_1}$-orbit, the associated map on tangent spaces

$$(*) \quad (\overline{b}, \overline{h}) \mapsto \overline{b}c_{12} - c_{12}\overline{h}$$

is surjective. Now by the structure of $X(d'')$, with respect to the fixed basis, the map $x : V_1 \to V_2$ can be any quadratic matrix. Also, by the structure of $X(d')$ and by the choice of basis of W_1 and W_2, any lower triangular quadratic matrix occur as a matrix of $y : W_1 \to W_2$. So there exists c_{12} such that the map $(*)$ is surjective. Thus there exist x and y such that $c'_{11} - c_{12}x + yc_{12} = 0$, which proves the claim.

Consequently, $X(c)$ is isomorphic to $X(c_0)$ where

$$c_0 = \begin{pmatrix} 0 & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix}.$$

Let $\iota : \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(V_1 \oplus V_2 \oplus V_3, W_1 \oplus W_2 \oplus W_3)$ be the map

$$\iota(\begin{pmatrix} z_{12} & z_{13} \\ z_{32} & z_{33} \end{pmatrix}) = \begin{pmatrix} 0 & z_{12} & z_{13} \\ 0 & 0 & 0 \\ 0 & z_{32} & z_{33} \end{pmatrix}.$$

Let

$$\phi : \mathrm{Hom}_k(V, W) \to \mathrm{Rep}(D, P)$$

be given by $\phi(z) = X(\iota(z))$. We first show that $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ contains a non-empty subset of $\mathrm{Rep}(D, P)$. Let F be the set of all representations of the form $X(c)$ for $c \in \mathrm{Hom}_k(V_1 \oplus V_2 \oplus V_3, W_1 \oplus W_2 \oplus W_3)$. The set $\mathrm{Aut}_A P \cdot F$ is equal to the subset of $\mathrm{Rep}(D, P)$ consisting of X with $X' \cong X(d')$ and $X'' \cong X(d'')$. Using the morphism of varieties $\mathrm{Rep}(D, P) \to \mathrm{Rep}(D, P(d')) \times \mathrm{Rep}(D, P(d''))$ given by $X \mapsto (X', X'')$, we have that $\mathrm{Aut}_A P \cdot F$ is equal to the preimage of $\mathrm{Aut}_A P(d') \cdot X(d') \times \mathrm{Aut}_A P(d'') \cdot X(d'')$ which is open. The condition that the map $(*)$ is surjective for c_{12} is an open condition on the set of representations in F. Therefore $(\mathrm{Aut}_A P \cdot \mathrm{Im}\phi) \cap F$ must contain an open subset of F, and so finally $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ contains an open subset of $\mathrm{Rep}(D, P)$.

Assume that $X(\iota(z)) \cong X(\iota(z'))$. Then there exists an isomorphism f with maps

$$a = \begin{pmatrix} a_1 & 0 & 0 \\ y_1 & a_1 & y_2 \\ y_3 & 0 & a_2 \end{pmatrix} \text{ and } g = \begin{pmatrix} g_1 & 0 & 0 \\ x_1 & g_1 & x_2 \\ x_3 & 0 & g_2 \end{pmatrix}$$

and some b such that conditions (1)-(4) are satisfied. Then by explicitly computing the matrices in (4), we see that z and z' are conjugate under the action of $H_V \times H_W$ using

$$\left(\begin{pmatrix} g_1 & x_2 \\ 0 & g_2 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ y_3 & a_2 \end{pmatrix}\right).$$

Conversely, if z and z' are conjugate under the action of $H_V \times H_W$ via

$$\left(\begin{pmatrix} g_1 & x_2 \\ 0 & g_2 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ y_3 & a_2 \end{pmatrix}\right),$$

then $X(\iota(z)) \cong X(\iota(z'))$ using an isomorphism f with $b = 0$,

$$a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ y_3 & 0 & a_2 \end{pmatrix} \text{ and } g = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_1 & x_2 \\ 0 & 0 & g_2 \end{pmatrix}.$$

Therefore, the $H_V \times H_W$-space $\mathrm{Hom}_k(V, W)$ is generically equivalent to the the $\mathrm{Aut}_A P$-space in $\mathrm{Rep}(D, P)$, in the case where $u + 2$ is a sink.

Part 2: We now consider the case where $u + 2$ is a source.

Let $W_1 = M_u^0$, $W_2 = M_u^1$, $W_3 = N_u^0$ and $W = W_1 \oplus W_3$. Let $V_1 = R_{u+1}^0$, $V_2 = R_{u+1}^1$ and $V_3 = T_{u+1}$. We have $V = V_1 \oplus V_3$ if $d_{u+1} > d_{u+2}$ and $V = V_1$ if $d_{u+1} \leq d_{u+2}$.

Recall that the representation X has a submodule $X(d')$ with quotient isomorphic to $X(d'')$, and that we have a decomposition of vector spaces $X_i = X(d')_i \oplus X(d'')_i$. By the relations of D, we see that as a $(\tilde{Q},\mathcal{I})$-representation X is determined by $X(d')$, $X(d'')$ and the maps between the vertices u and $u+1$ which have the form

$$X(d')_u \underset{X_{\beta^*}=(X(d')_{\beta^*}\ \ c)}{\overset{X_\beta=\begin{pmatrix}Id\\0\end{pmatrix}}{\rightleftarrows}} X(d')_{u+1}\oplus X(d'')_{u+1}$$

where $X(d')_u = X(d')_{u+1} = W_1 \oplus W_2 \oplus W_3$, $X(d'')_{u+1} = V_1 \oplus V_2 \oplus V_3$, and

$$c = (c_{ij})_{ij} : V_1 \oplus V_2 \oplus V_3 \to W_1 \oplus W_2 \oplus W_3.$$

By fixing a basis we may assume

$$X(d')_{\beta^*} = \begin{pmatrix} 0 & 0 & 0\\ Id & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Due to the relation $\beta^*\alpha = 0$ in D we have $c|_{V_2} = 0$ and so

$$c = \begin{pmatrix} c_{11} & 0 & c_{13}\\ c_{21} & 0 & c_{23}\\ c_{31} & 0 & c_{33}\end{pmatrix}.$$

Note that if $d_{u+1} \leq d_{u+2}$ then $V_3 = 0$ and so $c_{i3} = 0$ for all i. Denote by $X(c)$ the representation of the form above determined by $c = (c_{ij})_{ij}$, where $c_{i2} = 0$ for all i.

A map $f = (f_i)_{i=1}^n : X(c) \to X(c')$, where $f_i : X(c)_i \to X(c')_i$, is a homomorphism if and only if the following four conditions are satisfied.

(i) $f|_{X(d')} = (f_1, f_2, \dots, f_u, a, 0, 0) \in \mathrm{End}_D X(d')$.

(ii) $(f_u, f_{u+1}) = \left(a, \begin{pmatrix} a & b\\ 0 & g\end{pmatrix}\right)$, where $b = (b_{ij})_{ij} : V_1 \oplus V_2 \oplus V_3 \to W_1 \oplus W_2 \oplus W_3$.

(iii) $\overline{f} = (0, 0, \dots, 0, g, g') \in \mathrm{End}_D(X(d''))$.

(iv) $c'g = ac - X(d')_{\beta^*}b$, as $f_u X(c)_{\beta^*} = X(c')_{\beta^*} f_{u+1}$.

Moreover, as matrices we have

$$-X(d')_{\beta^*}b = \begin{pmatrix} 0 & 0 & 0\\ -b_{11} & -b_{12} & -b_{13}\\ 0 & 0 & 0\end{pmatrix} \text{ for } b = (b_{ij})_{ij},$$

$$g = \begin{pmatrix} g_1 & 0 & 0\\ x_1 & g_1 & x_2\\ x_3 & 0 & g_2\end{pmatrix},$$

where g_1 and g_2 can be any invertible matrices and x_1, x_2 and x_3 can be any matrices, and

$$a = \begin{pmatrix} a_1 & 0 & 0\\ y_1 & a_1 & y_2\\ y_3 & 0 & a_2\end{pmatrix}.$$

As in Part 1, the matrices a_i and y_i depend on the structure of $X(d')$, and so not all matrices occur. Now H_V is the group of invertible matrices

$$\begin{pmatrix} g_1 & 0\\ x_3 & g_2\end{pmatrix}$$

if $d_{u+1} > d_{u+2}$, and g_1, otherwise. The group H_W consists of matrices

$$\begin{pmatrix} a_1 & 0\\ y_3 & a_2\end{pmatrix},$$

induced by automorphisms of $X(d')$.

For a given map $c = (c_{ij})_{ij}$ with $c_{i2} = 0$ for any i, let

$$b(c) = \begin{pmatrix} c_{21} & 0 & c_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then for any $X(c)$, using an isomorphism f with $a = Id$, $g = Id$ and $b = b(c)$, we see that $X(c) \cong X(c')$ for

$$c' = \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & 0 & 0 \\ c_{31} & 0 & c_{33} \end{pmatrix}.$$

Let $\iota : \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(V_1 \oplus V_2 \oplus V_3, W_1 \oplus W_2 \oplus W_3)$ be the map

$$\iota(\begin{pmatrix} z_{11} & z_{13} \\ z_{31} & z_{33} \end{pmatrix}) = \begin{pmatrix} z_{11} & 0 & z_{13} \\ 0 & 0 & 0 \\ z_{31} & 0 & z_{33} \end{pmatrix}$$

if $d_{u+1} > d_{u+2}$, and

$$\iota(\begin{pmatrix} z_{11} \\ z_{31} \end{pmatrix}) = \begin{pmatrix} z_{11} & 0 & 0 \\ 0 & 0 & 0 \\ z_{31} & 0 & 0 \end{pmatrix}$$

otherwise. Let

$$\psi : \mathrm{Hom}_k(V, W) \to \mathrm{Rep}(D, P)$$

be given by $\psi(z) = X(\iota(z))$. First, $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ contains the nonempty subset of $\mathrm{Rep}(D, P)$ consisting of representations X such that $X' \cong X(d')$ and $X'' \cong X(d'')$, and as in the case where $u + 2$ is a sink, $\mathrm{Aut}_A P \cdot \mathrm{Im}\phi$ contains a nonempty open subset of $\mathrm{Rep}(D, P)$.

Assume that $X(\iota(z)) \cong X(\iota(z'))$. Then there exists an isomorphism f with maps b,

$$a = \begin{pmatrix} a_1 & 0 & 0 \\ y_1 & a_1 & y_2 \\ y_3 & 0 & a_2 \end{pmatrix} \text{ and } g = \begin{pmatrix} g_1 & 0 & 0 \\ x_1 & g_1 & x_2 \\ x_3 & 0 & g_2 \end{pmatrix}$$

such that conditions (i)-(iv) are satisfied. Then by explicitly computing the matrices in (iv), we see that z and z' are conjugate under the action of $H_V \times H_W$ via

$$\left(\begin{pmatrix} g_1 & 0 \\ x_3 & g_2 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ y_2 & a_2 \end{pmatrix} \right).$$

Conversely, if z and z' are conjugate under the action of $H_V \times H_W$ using

$$\left(\begin{pmatrix} g_1 & 0 \\ x_3 & g_2 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ y_2 & a_2 \end{pmatrix} \right),$$

then $X(\iota(z)) \cong X(\iota(z'))$ via the isomorphism f with $b = 0$,

$$a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ y_2 & 0 & a_2 \end{pmatrix} \text{ and } g = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_1 & 0 \\ x_3 & 0 & g_2 \end{pmatrix}.$$

Therefore the $H_V \times H_W$-space $\mathrm{Hom}_k(V, W)$ is generically equivalent to the $\mathrm{Aut}_A P$-space $\mathrm{Rep}(D, P)$. □

We compute $\mathrm{Aut}(X(d'))^0_u$.

Lemma 27. *Let $i, j \in \{1, \cdots, p\}$.*

1. *If $\mathrm{Hom}_D(M^i, M^j)^0_u = k$ then*
   - (a) *$i \geq j$ and $(\underline{\dim}_\Delta M^i)_w = (\underline{\dim}_\Delta M^j)_w$ for all $w < u$, or*
   - (b) *$i < j$ and $(\underline{\dim}_\Delta M^i)_w = (\underline{\dim}_\Delta M^j)_w$ for all $w = 1, u$.*
2. *If both (a) and (b) fail then $\mathrm{Hom}_D(M^i, M^j)^0_u = 0$.*

Proof. By relabeling the vertices $1 \mapsto 1$, $u \mapsto 2$ and $u-t \mapsto t+2$ we are in Case A, and the lemma follows from Lemma 17. □

Similar to Case A, we construct a rigid representation $Z(d')$ of Q' with summands supported on intervals in

$$2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow u-1 \longrightarrow u \longleftarrow 1 \ .$$

We have

$$Z(d') = \oplus (Z^i)^{n_i}$$

as follows, where n_i is the multiplicity of M^i in $X(d')$. Let $Z^1 = M[u,1]$, that is Z^1 is supported on 1 and u. Given $Z^i = M[j,j']$, let

$$Z^{i+1} = \begin{cases} M[j-1,j'] & \text{if } \mathrm{Hom}(M^i, M^{i+1})^0_u = k, \\ M[j,u] & \text{if } \mathrm{Hom}(M^{i+1}, M^i)^0_u = k, \\ M[j-1,u] & \text{otherwise.} \end{cases}$$

Only one of the two latter cases can occur, and it occurs at most once, and so every Z^i is supported at u. Let $c(d') = \underline{\dim} Z(d')$.

Recall $X(d'') = R \oplus T$, where the indecomposable summands of R has Δ-support at both $u+1$ and $u+2$, and T has Δ-support at $u+1$ or $u+2$, but not both. Let a and b be the multiplicities of the indecomposable summand in R and T, respectively. Similarly, we have a rigid representation $Z(d'')$ of Q' as follows.

If $u+2$ is a sink, let

$$Z(d'') = M[u+1,u+2]^a \oplus M[u+1,u+1]^b.$$

If $u+2$ is a source, let

$$Z(d'') = M[u+1,u+1]^a \oplus M[u+1,u+2]^b \text{ when } d_{u+1} > d_{u+2},$$

and

$$Z(d'') = M[u+1,u+1]^a \text{ when } d_{u+1} \leq d_{u+2}.$$

Let

$$c(d'') = \underline{\dim} Z(d'') \text{ and } c(d) = c(d') + c(d'').$$

Lemma 28. *The* $\mathrm{Aut}_{kQ'}(Z(d'')) \times \mathrm{Aut}_{kQ'}(Z(d'))$*-space* $\mathrm{Hom}_k(Z(d'')_{u+1}, Z(d')_u)$ *is a generic section of the* $\mathrm{Gl}(c(d))$*-space* $\mathrm{Rep}(Q', c(d))$.

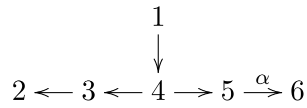
Proof. The $\mathrm{Aut}_{kQ'}(Z(d'')) \times \mathrm{Aut}_{kQ'}(Z(d'))$-orbits in $\mathrm{Hom}_k(Z(d'')_{u+1}, Z(d')_u)$ parameterise the representations in $\mathrm{Rep}(Q', c(d))$ with the restriction to the subquiver on $\{1, \cdots u\}$ equal to $Z(d')$ and the restriction to the subquiver on $\{u+1, u+2\}$ equal to $Z(d'')$. Clearly, $Z(d'')$ is rigid, and by the proof of Lemma 19, $Z(d')$ is rigid. So the $\mathrm{Aut}_{kQ'}(Z(d'')) \times \mathrm{Aut}_{kQ'}(Z(d'))$-space $\mathrm{Hom}_k(Z(d'')_{u+1}, Z(d')_u)$ is a generic section of the $\mathrm{Gl}(c(d))$-space $\mathrm{Rep}(Q', c(d))$. □

We can now prove Theorem 7 in case C).

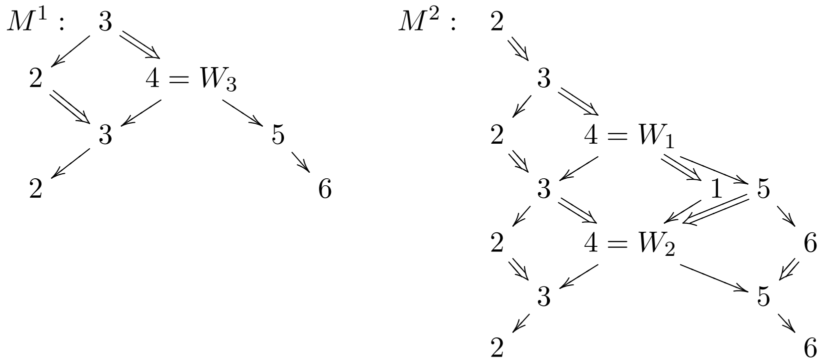
Lemma 29. *Let P be a projective representation of Q. If the orientation at u is as in Case C and $u = 3, 4, n-2$ or $n-1$, then there is a dimension vector c such that the* $\mathrm{Aut}_A P$*-space* $\mathrm{Rep}(D, P)$ *is generically equivalent to the* $\mathrm{Gl}(c)$*-space* $\mathrm{Rep}(Q', c)$.

Proof. In the case $u = n-2$ the proof follows from Lemma 26, 27, 28 and similar arguments as in Case A and B. If $u = 4$, we may relabel the vertices on the quiver, $1 \mapsto 1$ and $2+t \mapsto n-t$ for all $t \geq 0$. After relabeling, we are in the setting of $u = n-2$. If $u = n-1$ or $u = 3$, by relabeling with vertex u as vertex 1, we are in Case A, so the lemma follows from Lemma 20. □

Example 7. *In this example we illustrate the construction in Case C. Consider the following quiver of Case C and* $d = (1, 1, 2, 2, 1, 2)$.

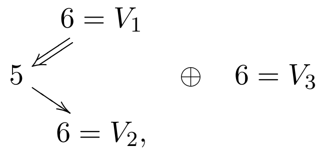


(1) We first construct a rigid A-projective D-module $M^1 \oplus M^2$ *of* Δ*-dimension vector* $d' = (1, 1, 2, 2, 0, 0)$ *as follows.*



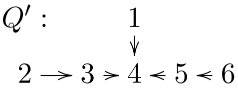
Here $M^1 <_4 M^2$ *and* $\mathrm{Hom}(M^1, M^2)^0_4 = 0 = \mathrm{Hom}(M^2, M^1)^0_4$. *The spaces* W_1, W_2, W_3 *are given in the picture,* $W = W_1 \oplus W_3$ *and* $H_W = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mid a_1, a_2 \in k^*\}$.

(2) A rigid A-projective D-module $R \oplus T$ *of* Δ*-dimension vector* $d'' = (0, 0, 0, 0, 1, 2)$ *is*



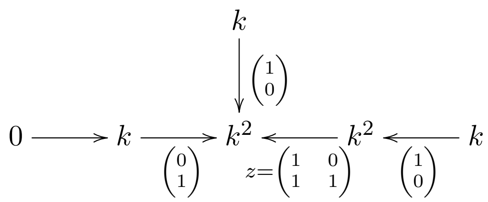
with V_1, V_2, V_3 *are indicated in the picture. We have* $V = V_2 \oplus V_3$ *and* $H_V = \{\begin{pmatrix} g_1 & x \\ 0 & g_2 \end{pmatrix} \mid g_1, g_2 \in k^*, x \in k\}$.

(3) Now the quiver Q' *is* Q' :

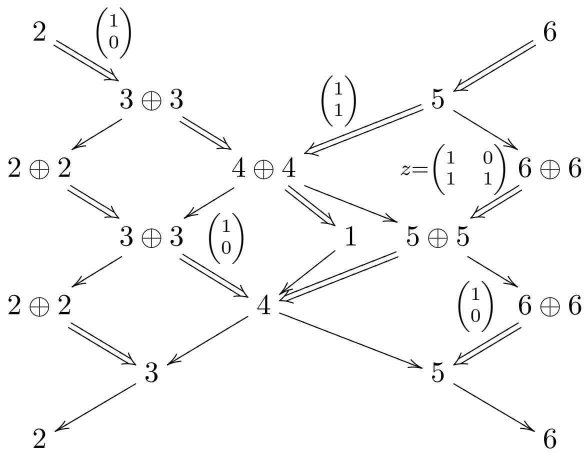


and the representations $Z(d')$ *and* $Z(d'')$ *of the quiver* Q' *are*

Clearly, $\mathrm{Aut}_{kQ'}Z(d') \cong H_W$ *and* $\mathrm{Aut}_{kQ'}Z(d'') \cong H_V$. *A rigid representation of* Q' *with dimension vector* $c(d) = \underline{\dim}Z(d') + \underline{\dim}Z(d'') = (1, 0, 1, 2, 2, 1)$ *is*



(4) The matrix z *in (3) induces a rigid A-projective D-module* M *of* Δ*-dimension vector* d *as follows.*



As before, $i \oplus i$ means that the vector space is 2-dimensional. If not specified, the matrices on the arrows connecting two vector spaces of the same dimensions are identity matrices, and the other maps are determined by the commutativity relations.

(5) As a test, one can compute directly the endomorphism ring of M,

$$\mathrm{End}_D M = \{f = (f_i) \mid (f_1,\, f_2,\, f_3,\, f_4,\, f_5|_{(M^1 \oplus M^2)_5},\, f_6|_{(M^1\oplus M^2)_6}) \in \mathrm{End}_D(M^1 \oplus M^2),$$
$$(0,\, 0,\, 0,\, 0,\, f_5|_{(R\oplus T)_5},\, f_6|_{(R\oplus T)_6}) \in \mathrm{End}_D(R \oplus T) \text{ and } f_5 M_{\beta^*} = M_{\beta^*} f_6\}$$

and obtain $\dim_k \mathrm{End}_D M = 15$, *which is* $\sum_{i=1}^{6} d_i^2$, *and so M is rigid, by Theorem 4.*

6.4. **Case D.** We have proven Theorem 7 for Case A, B and C, and it only remains to prove for Case D. By relabeling we are in Case A as follows.

Lemma 30. *Let Q be as in Case D and let P be a projective representation of Q. Then there is a dimension vector c such that the* $\mathrm{Aut}_A P$*-space* $\mathrm{Rep}(D, P)$ *is generically equivalent to the* $\mathrm{Gl}(c)$*-space* $\mathrm{Rep}(Q', c)$ *for some* $c \in \mathbb{N}^n$.

Proof. We relabel the vertices on the quiver, $1 \mapsto 1$ and $2 + t \mapsto n - t$ for all $t \geq 0$. We are then in Case A, and so the lemma follows from Lemma 20. □

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