The backbone decomposition for spatially dependent supercritical superprocesses

A. E. Kyprianou, J-L, Pérez and Y.-X. Ren

Abstract Consider any supercritical Galton-Watson process which may become extinct with positive probability. It is a well-understood and intuitively obvious phenomenon that, on the survival set, the process may be pathwise decomposed into a stochastically 'thinner' Galton-Watson process, which almost surely survives and which is decorated with immigrants, at every time step, initiating independent copies of the original Galton-Watson process conditioned to become extinct. The thinner process is known as the *backbone* and characterizes the genealogical lines of descent of prolific individuals in the original process. Here, prolific means individuals who have at least one descendant in every subsequent generation to their own.

Starting with Evans and O'Connell [18], there exists a cluster of literature, [14, 32, 5, 2, 28], describing the analogue of this decomposition (the so-called *backbone decomposition*) for a variety of different classes of superprocesses and continuous-state branching processes. Note that the latter family of stochastic processes may be seen as the total mass process of superprocesses with non-spatially dependent branching mechanism.

In this article we consolidate the aforementioned collection of results concerning backbone decompositions and describe a result for a general class of supercritical superprocesses with spatially dependent branching mechanisms. Our approach exposes the commonality and robustness of many of the existing arguments in the literature.

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1 Superprocesses and Markov branching processes

This paper concerns a fundamental decomposition which can be found amongst a general family of superprocesses and has, to date, been identified for a number of specific sub-families thereof by a variety of different authors. We therefore start by briefly describing the general family of superprocesses that we shall concern ourselves with. The reader is referred to the many, and now classical, works of Dynkin for further details of what we present below; see for example [7, 8, 9, 10, 11]. The books of Le Gall [30], Etheridge [15] and Li [31] also serve as an excellent point of reference.

Let E be a domain of $\mathbb{R}^d$. Following the setting of Fitzsimmons [21], we are interested in strong Markov processes, $X=\{X_t : t\geq 0\}$ which are valued in $\mathcal{M}_F(E)$, the space of finite measures with support in E. The evolution of X depends on two quantities $\mathscr{P}$ and ψ. Here, $\mathscr{P}=\{\mathscr{P}_t : t\geq 0\}$ is the semi-group of an $\mathbb{R}^d$-valued diffusion killed on exiting E, and ψ is a so-called branching mechanism which, by assumption, takes the form

$$\psi(x,\lambda)=-\alpha(x)\lambda+\beta(x)\lambda^2+\int_{(0,\infty)}(e^{-\lambda z}-1+\lambda z)\pi(x,\mathrm{d}z), \tag{1}$$

where α and $\beta\geq 0$ are bounded measurable mappings from E to $\mathbb{R}$ and $[0,\infty)$ respectively and for each $x\in E$, $\pi(x,\mathrm{d}z)$ is a measure concentrated on $(0,\infty)$ such that $x\to\int_{(0,\infty)}(z\wedge z^2)\pi(x,\mathrm{d}z)$ is bounded and measurable. The latter ensure that the total mass of X is finite in expectation at each time. For technical reasons, we shall additionally assume that the diffusion associated to $\mathscr{P}$ satisfies certain conditions. These conditions are lifted from Section II.1.1 (Assumptions 1.1A and 1.1B) on pages 1218-1219 of [8][1]. They state that $\mathscr{P}$ has associated infinitesimal generator

$$L=\sum_{i,j}a_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_i b_i\frac{\partial}{\partial x_i},$$

where the coefficients $a_{i,j}$ and b_j are space dependent coefficients satisfying:

(Uniform Elliptically) There exists a constant $\gamma>0$ such that

$$\sum_{i,l}a_{i,j}u_iu_j\geq\gamma\sum_i u_i^2$$

for all $x\in E$ and $u_1,\cdots,u_d\in\mathbb{R}$.

(Hölder Continuity) The coefficients $a_{i,j}$ and b_i are uniformly bounded and Hölder continuous in such way that there exist a constants $C>0$ and $\alpha\in(0,1]$ with

$$|a_{i,j}(x)-a_{i,j}(y)|,\quad |b_i(x)-b_i(y)|\leq C|x-y|^\alpha$$

[1] The assumptions on $\mathscr{P}$ may in principle be relaxed. The main reason for this imposition here comes in the proof of Lemma 5 where a comparison principle is used for diffusions.

for all $x, y \in E$. Throughout, we shall refer to X as the $(\mathscr{P}, \psi)$-superprocess.

For each $\mu \in \mathscr{M}_F(E)$ we denote by $\mathbb{P}_\mu$ the law of X when issued from initial state $X_0 = \mu$. The semi-group of X, which in particular characterizes the laws $\{\mathbb{P}_\mu : \mu \in \mathscr{M}_F(E)\}$, can be described as follows. For each $\mu \in \mathscr{M}_F(E)$ and all $f \in \mathrm{bp}(E)$, the space of non-negative, bounded measurable functions on E,

$$\mathbb{E}_\mu(e^{-\langle f, X_t \rangle}) = \exp\left\{-\int_E u_f(x,t)\mu(\mathrm{d}x)\right\} \qquad t \geq 0, \tag{2}$$

where $u_f(x,t)$ is the unique non-negative solution to the equation

$$u_f(x,t) = \mathscr{P}_t[f](x) - \int_0^t \mathrm{d}s \cdot \mathscr{P}_s[\psi(\cdot, u_f(\cdot, t-s))](x) \qquad x \in E, t \geq 0. \tag{3}$$

See for example Theorem 1.1 on pages 1208-1209 of [8] or Proposition 2.3 of [21]. Here we have used the standard inner product notation,

$$\langle f, \mu \rangle = \int_E f(x)\mu(\mathrm{d}x),$$

for $\mu \in \mathscr{M}_F(E)$ and any f such that the integral makes sense.

Suppose that we define $\mathscr{E} = \{\langle 1, X_t \rangle = 0 \text{ for some } t > 0\}$, the event of *extinction*. For each $x \in E$ write

$$w(x) = -\log \mathbb{P}_{\delta_x}(\mathscr{E}). \tag{4}$$

It follows from (2) that

$$\mathbb{E}_\mu(e^{-\theta\langle 1, X_t \rangle}) = \exp\left\{-\int_E u_\theta(x,t)\mu(\mathrm{d}x)\right\} \qquad t \geq 0, \tag{5}$$

Note that $u_\theta(t,x)$ is increasing in θ and that $\mathbb{P}_\mu(\langle 1, X_t \rangle = 0)$ is monotone increasing. Using these facts and letting $\theta \to \infty$, then $t \to \infty$, we get that

$$\mathbb{P}_\mu(\mathscr{E}) = \lim_{t\to\infty} \mathbb{P}_\mu(\langle 1, X_t \rangle = 0) = \exp\left\{-\int_E \lim_{t\to\infty}\lim_{\theta\to\infty} u_\theta(x,t)\mu(\mathrm{d}x)\right\}. \tag{6}$$

By choosing $\mu = \delta_x$, with $x \in E$, we see that

$$\mathbb{P}_\mu(\mathscr{E}) = \exp\left\{-\int_E w(x)\mu(\mathrm{d}x)\right\}. \tag{7}$$

For the special case that ψ does not depend on x and $\mathscr{P}$ is conservative, $\langle 1, X_t \rangle$ is a continuous state branching process. If $\psi(\lambda)$ satisfy the following condition:

$$\int^\infty \frac{1}{\psi(\lambda)} \mathrm{d}\lambda < \infty,$$

then $\mathbb{P}_\mu$ almost surely we have $\mathscr{E} = \{\lim_{t\to\infty}\langle 1, X_t \rangle = 0\}$, that is to say the event of *extinction* is equivalent to the event of *extinguishing*, see [2] and [28] for examples.

By first conditioning the event $\mathscr{E}$ on $\mathscr{F}_t := \sigma\{X_s : s \le t\}$, we find that for all $t \ge 0$,

$$\mathbb{E}_\mu(e^{-\langle w, X_t\rangle}) = e^{-\langle w,\mu\rangle}.$$

The function w will play an important role in the forthcoming analysis and henceforth we shall assume that it respects the following property.

(A): w is locally bounded away from 0 and ∞.

Note that the notion of supercriticality is implicitly hidden in the assumption above, specifically in that w is locally bounded away from 0. This ensures that the extinction probability in (7) is not unity. We point out that we do not need local compact support property of X. The reason we consider diffusion as our spatial motion is that we will use the comparison principle of some integral equation, see (47) below. We expect that our results hold for more general superprocesses, for example, super-Lévy processes.

The pathwise evolution of superprocesses is somewhat difficult to visualise on account of their realisations at each fixed time being sampled from the space of finite measures. However a related class of stochastic processes which exhibit similar mathematical properties to superprocesses and whose paths are much easier to visualise is that of Markov branching processes. A Markov branching process $Z = \{Z_t : t \ge 0\}$ takes values in the space $\mathscr{M}_a(E)$ of finite atomic measures in E taking the form $\sum_{i=1}^n \delta_{x_i}$, where $n \in \mathbb{N}\cup\{0\}$ and $x_1, \cdots, x_n \in E$. To describe its evolution we need to specify two quantities, $(\mathscr{P}, F)$, where, as before, $\mathscr{P}$ is the semi-group of a diffusion on E and F is the so-called branching generator which takes the form

$$F(x,s) = q(x)\sum_{n\ge 0} p_n(x)(s^n - s), \qquad x \in E, s \in [0,1], \tag{8}$$

where q is a bounded measurable mapping from E to $[0,\infty)$ and, the measurable sequences $\{p_n(x) : n \ge 0\}$, $x \in E$, are probability distributions. For each $\nu \in \mathscr{M}_a(E)$, we denote by P_ν the law of Z when issued from initial state $Z_0 = \nu$. The probability P_ν can be constructed in a pathwise sense as follows. From each point in the support of ν we issue an independent copy of the diffusion with semi-group $\mathscr{P}$. Independently of one another, for $(x,t) \in E \times [0,\infty)$, each of these particles will be killed at rate $q(x)\mathrm{d}t$ to be replaced at their space-time point of death by $n \ge 0$ particles with probability $p_n(x)$. Relative to their point of creation, new particles behave independently to one another, as well as to existing particles, and undergo the same life cycle in law as their parents.

By conditioning on the first split time in the above description of a $(\mathscr{P}, F)$-Markov branching process, it is also possible to show that for any $f \in \mathrm{bp}(E)$,

$$\mathrm{E}_\nu(e^{-\langle f, Z_t\rangle}) = \exp\left\{-\int_E v_f(x,t)\nu(\mathrm{d}x)\right\} \qquad t \ge 0,$$

where $v_f(x,t)$ solves

$$e^{-v_f(x,t)} = \mathscr{P}_t[e^{-f}](x) + \int_0^t \mathrm{d}s\, \mathscr{P}_s[F(\cdot, e^{-v_f(\cdot,t-s)})](x) \qquad x \in E, t \geq 0. \quad (9)$$

Moreover, it is known, cf. Theorem 1.1 on pages 1208-1209 of [8], that the solution to this equation is unique. This shows a similar characterisation of the semi-groups of Markov branching processes to those of superprocesses.

The close similarities between the two processes become clearer when one takes account of the fact that the existence of superprocesses can be justified through a high density scaling procedure of Markov branching processes. Roughly speaking, for a fixed triplet, $(\mu, \mathscr{P}, \psi)$, one may construct a sequence of Markov branching processes, say $\{Z^{(n)} : n \geq 1\}$, such that the n-th element of the sequence is issued with an initial configuration of points which is taken to be an independent Poisson random measure with intensity $n\mu$ and branching generator F_n satisfying

$$F_n(x,s) = \frac{1}{n}[\psi(x, n(1-s)) + \alpha(x)n(1-s)], \qquad x \in E, s \in [0,1].$$

It is not immediately obvious that the right-hand side above conforms to the required structure of branching generators as stipulated in (8), however this can be shown to be the case; see for example the discussion on p.93 of [31]. It is now a straightforward exercise to show that for all $f \in \mathrm{bp}(E)$ and $t \geq 0$ the law of $\langle f, n^{-1}Z_t^{(n)}\rangle$ converges weakly to the law of $\langle f, X_t\rangle$, where the measure X_t satisfies (2). A little more work shows the convergence of the sequence of processes $\{n^{-1}Z^{(n)} : n \geq 1\}$ in an appropriate sense to a $(\mathscr{P}, \psi)$-superprocess issued from an initial state μ.

Rather than going into the details of this scaling limit, we focus instead in this paper on another connection between superprocesses and branching processes which explains their many similarities without the need to refer to a scaling limit. The basic idea is that, under suitable assumptions, for a given $(\mathscr{P}, \psi)$-superprocess, there exists a related Markov branching process, Z, with computable characteristics such that at each fixed $t \geq 0$, the law of Z_t may be coupled to the law of X_t in such a way that, given X_t, Z_t has the law of a Poisson random measure with intensity $w(x)X_t(\mathrm{d}x)$, where w is given by (4). The study of so-called *backbone decompositions* pertains to how the aforementioned Poisson embedding may be implemented in a pathwise sense at the level of processes.

The remainder of this paper is structured as follows. In the next section we briefly review the sense and settings in which backbone decompositions have been previously studied. Section 3 looks at some preliminary results needed to address the general backbone decomposition that we deal with in Sections 4, 5 and 6.

2 A brief history of backbones

The basic idea of a backbone decomposition can be traced back to the setting of Galton-Watson trees with ideas coming from Harris and Sevast'yanov; cf Harris [24]. Within any supercritical Galton-Watson process with a single initial ancestor for which the probability of survival is not equal to 0 or 1, one may identify prolific genealogical lines of descent on the event of survival. That is to say, infinite sequences of descendants which have the property that every individual has at least one descendant in every subsequent generation beyond its own. Together, these prolific genealogical lines of descent make a Galton-Watson tree which is thinner than the original tree. One may describe the original Galton-Watson process in terms of this thinner Galton-Watson process, which we now refer to as a *backbone*, as follows. Let $0 < p < 1$ be the probability of survival. Consider a branching process which, with probability $1-p$, is an independent copy of the original Galton-Watson process conditioned to become extinct and, with probability p, is a copy of the backbone process, having the additional feature that every individual in the backbone process immigrates an additional random number of offspring, each of which initiate independent copies of the original Galton-Watson process conditioned to become extinct. With an appropriate choice of immigration numbers, the resulting object has the same law as the original Galton-Watson process.

In Evans and O'Connell [18], and later in Engländer and Pinsky [14], a new decomposition of a supercritical superprocess with quadratic branching mechanism was introduced in which one may write the distribution of the superprocess at time $t \geq 0$ as the result of summing two independent processes together. The first is a copy of the original process conditioned on extinction. The second process is understood as the superposition of mass from independent copies of the original process conditioned on extinction which have immigrated 'continuously' along the path of an auxiliary dyadic branching particle diffusion which starts with a random number of initial ancestors whose cardinality and spatial position is governed by an independent Poisson point process. The embedded branching particle system is known as the *backbone* (as opposed to the *spine* or *immortal particle* which appears in another related decomposition, introduced in Roelly-Coppoletta and Rouault [33] and Evans [17]). In both [18] and [14] the decomposition is seen through the semi-group evolution equations which drive the process semi-group. However no pathwise construction is offered.

A pathwise backbone decomposition appears in Salisbury and Verzani [32], who consider the case of conditioning a super-Brownian motion as it exits a given domain such that the exit measure contains at least n pre-specified points in its support. There it was found that the conditioned process has the same law as the superposition of mass that immigrates in a Poissonian way along the spatial path of a branching particle motion which exits the domain with precisely n particles at the prespecified points. Another pathwise backbone decomposition for branching particle systems is given in Etheridge and Williams [16], which is used in combination with a limiting procedure to prove another version of Evan's immortal particle picture.

In Duquesne and Winkel [5] a version of the Evans-O'Connell backbone decomposition was established for more general branching mechanisms, albeit without taking account of spatial motion. In their case, quadratic branching is replaced by a general branching mechanism ψ which is the Laplace exponent of a spectrally positive Lévy process and which satisfies the conditions $0 < -\psi'(0+) < \infty$ and $\int^{\infty} 1/\psi(\xi)\mathrm{d}\xi < \infty$. Moreover, the decomposition is offered in the pathwise sense and is described through the growth of genealogical trees embedded within the underling continuous state branching process. The backbone is a continuous-time Galton Watson process and the general nature of the branching mechanism induces three different kinds of immigration. Firstly there is continuous immigration which is described by a Poisson point process of independent processes along the trajectory of the backbone where the rate of immigration is given by a so-called excursion measure which assigns zero initial mass, and finite life length of the immigrating processes. A second Poisson point process along the backbone describes the immigration of independent processes where the rate of immigration is given by the law of the original process conditioned on extinguishing and with a positive initial volume of mass randomised by an infinite measure. This accounts for so-called discontinuous immigration. Finally, at the times of branching of the backbone, independent copies of the original process conditioned on extinguishing are immigrated with randomly distributed initial mass which depends on the number of offspring at the branch point. The last two forms of immigration do not occur when the branching mechanism is purely quadratic.

Concurrently to the work of [5] and within the class of branching mechanisms corresponding to spectrally positive Lévy processes with paths of unbounded variation (also allowing for the case that $-\psi'(0+)=\infty$), Bertoin et al. [3] identify the aforementioned backbone as characterizing prolific genealogies within the underling continuous state branching process.

Berestycki et al. [2] extend the results of [18] and [5], showing that for superprocesses with relatively general motion and non-spatial branching mechanism corresponding to spectrally positive Lévy processes with finite mean, a pathwise backbone decomposition arises. The role of the backbone is played by a branching particle diffusion with the same motion operator as the superprocesses and, like Salisbury and Verzani [32], additional mass immigrates along the trajectory of the backbone in a Poissonian way. Finally Kyprianou and Ren [28] look at the case of a continuous-state branching process with immigration for which a similar backbone decomposition to [2] can be shown.

As alluded to in the abstract, our objective in this article is to provide a general backbone decomposition which overlaps with many of the cases listed above and, in particular, exposes the general effect on the backbone of spatially dependent branching. It is also our intention to demonstrate the robustness of some of the arguments that have been used in earlier work on backbone decompositions. Specifically, we are referring to the original manipulations associated with the semi-group equations given in Evans and O'Connell [18] and Engländer and Pinsky [14], as well as the use the Dynkin-Kuznetsov excursion measure, as found in Salisbury and Verzani [32], Berestyki et al. [2] and Kyprianou and Ren [28], to describe the rate of immigration along the backbone.

3 Preliminaries

Before stating and proving the backbone decomposition, it will first be necessary to describe a number of mathematical structures which will play an important role.

3.1 Localisation

Suppose that the stochastic process $\xi = \{\xi_t : t \geq 0\}$ on $E \cup \{\dagger\}$, where $\dagger$ is its cemetery state, is the diffusion in E corresponding to the semi-group $\mathscr{P}$. We shall denote its probabilities by $\{\Pi_x : x \in E\}$. Throughout this paper, we shall take $\mathrm{bp}(E \times [0,t])$ to be the space of non-negative, bounded measurable functions on $E \times [0,t]$, and it is implicitly understood that for all functions $f \in \mathrm{bp}(E \times [0,t])$, we extend their spatial domain to include $\{\dagger\}$ and set $f(\{\dagger, s\}) = 0$.

Definition 1. For any open, bounded set D compactly embedded in E (written $D \subset\subset E$), and $t \geq 0$, there exists a random measure $\widetilde{X}_t^D$ supported on the boundary of $D \times [0,t)$ such that, for all $f \in \mathrm{bp}(E \times [0,t])$ with the additional property that the value of $f(x,s)$ on $E \times [0,t]$ is independent of s and $\mu \in \mathscr{M}_F(D)$, the space of finite measures on D,

$$-\log \mathbb{E}_\mu \left(e^{-\langle f, \widetilde{X}_t^D \rangle} \right) = \int_E \widetilde{u}_f^D(x,t)\mu(\mathrm{d}x), \qquad t \geq 0, \tag{10}$$

where $\widetilde{u}_f^D(x,t)$ is the unique non-negative solution to the integral equation

$$\widetilde{u}_f^D(x,t) = \Pi_x[f(\xi_{t\wedge\tau^D}, t\wedge\tau^D)] - \Pi_x\left[\int_0^{t\wedge\tau^D} \psi(\xi_s, \widetilde{u}_f^D(\xi_s, t-s))\mathrm{d}s\right], \tag{11}$$

and $\tau^D = \inf\{t \geq 0, \xi_t \in D^c\}$. Note that, here, we use the obvious notation that $\langle f, \widetilde{X}_t^D \rangle = \int_{\partial(D\times[0,t))} f(x,s)\widetilde{X}_t^D(\mathrm{d}x,\mathrm{d}s)$. Moreover, with a slight abuse of notation, since their effective spatial domain is restricted to $D \cup \{\dagger\}$ in the above equation, we treat ψ and $\widetilde{u}_f^D$ as functions in $\mathrm{bp}(E \times [0,t])$ and accordingly it is clear how to handle a spatial argument equal to $\dagger$, as before. In the language of Dynkin [10], $\widetilde{X}_t^D$ is called an exit measure.

Now we define a random measure X_t^D on D such that $\langle f, X_t^D \rangle = \langle f, \widetilde{X}_t^D \rangle$ for any $f \in \mathrm{bp}(D)$, the space of non-negative, bounded measurable functions on D, where, henceforth, as is appropriate, we regard f as a function defined on $E \times [0,\infty)$ in the sense that

$$f(x,t) = \begin{cases} f(x), & x \in D \\ 0, & x \in E \setminus D. \end{cases} \tag{12}$$

Then for any $f \in \mathrm{bp}(D)$ and $\mu \in \mathscr{M}_F(D)$,

$$-\log\mathbb{E}_\mu\left(e^{-\langle f,X_t^D\rangle}\right)=\int_E u_f^D(x,t)\mu(\mathrm{d}x),\qquad t\geq 0,\tag{13}$$

where $u_f^D(x,t)$ is the unique non-negative solution to the integral equation

$$u_f^D(x,t)=\Pi_x[f(\xi_t);t<\tau^D]-\Pi_x\left[\int_0^{t\wedge\tau^D}\psi(\xi_s,u_f^D(\xi_s,t-s))\mathrm{d}s\right],\quad x\in D.\tag{14}$$

As a process in time, $\widetilde{X}^D=\{\widetilde{X}_t^D:t\geq 0\}$ is a superprocess with branching mechanism $\psi(x,\lambda)\mathbf{1}_D(x)$, but whose associated semi-group is replaced by that of the process ξ absorbed on ∂D. Similarly, as a process in time, $X^D=\{X_t^D:t\geq 0\}$ is a superprocess with branching mechanism $\psi(x,\lambda)\mathbf{1}_D(x)$, but whose associated semi-group is replaced by that of the process ξ killed upon leaving D. One may think of X_t^D as describing the mass at time t in X which *historically* avoids exiting the domain D. Note moreover that for any two open bounded domains, $D_1\subset\subset D_2\subset\subset E$, the processes $\widetilde{X}^{D_1}$ and $\widetilde{X}^{D_2}$ (and hence X^{D_1} and X^{D_2}) are consistent in the sense that

$$\widetilde{X}_t^{D_1}=(\widetilde{\widetilde{X}_t^{D_2}})^{D_1},\tag{15}$$

for all $t\geq 0$ (and similarly $X_t^{D_1}=(X_t^{D_2})^{D_1}$ for all $t\geq 0$).

3.2 Conditioning on extinction

In the spirit of the relationship between (10) and (11), we have that w is the unique solution to

$$w(x)=\Pi_x[w(\xi_{t\wedge\tau^D})]-\Pi_x\left[\int_0^{t\wedge\tau^D}\psi(\xi_s,w(\xi_s))\mathrm{d}s\right],\qquad x\in D,\tag{16}$$

for all open domains $D\subset\subset E$. Again, with a slight abuse of notation, we treat w with its spatial domain $E\cup\{\dagger\}$ as a function on $E\times[0,t]$ and $w(\dagger):=0$. From Lemma 1.5 in [8] we may transform (16) to the equation

$$w(x)=\Pi_x\left[w(\xi_{t\wedge\tau_D})\exp\left\{-\int_0^{t\wedge\tau_D}\frac{\psi(\xi_s,w(\xi_s))}{w(\xi_s)}\mathrm{d}s\right\}\right],\qquad x\in D,$$

which shows that for all open bounded domains D,

$$w(\xi_{t\wedge\tau^D})\exp\left\{-\int_0^{t\wedge\tau^D}\frac{\psi(\xi_s,w(\xi_s))}{w(\xi_s)}\mathrm{d}s\right\},\qquad t\geq 0,\tag{17}$$

is a martingale.

The function w can be used to locally describe the law of the superprocess when conditioned on *global extinction* (as opposed to extinction on the sub-domain D). The following lemma outlines standard theory.

Lemma 1. *Suppose that $\mu \in \mathscr{M}_F(E)$ satisfies $\langle w, \mu\rangle < \infty$ (so, for example, it suffices that μ is compactly supported). Define*

$$\mathbb{P}^*_\mu(\cdot) = \mathbb{P}_\mu(\cdot|\mathscr{E}).$$

Then for any $f \in \mathrm{bp}(E\times[0,t])$ with the additional property that the value of $f(x,s)$ on $E\times[0,t]$ is independent of s and $\mu \in \mathscr{M}_F(D)$,

$$-\log \mathbb{E}^*_\mu\left(e^{-\langle f, \widetilde{X}^D_t\rangle}\right) = \int_D \widetilde{u}^{D,*}_f(x,t)\mu(\mathrm{d}x),$$

where $\widetilde{u}^{D,*}_f(x,t) = \widetilde{u}^D_{f+w}(x,t) - w(x)$ and it is the unique solution of

$$\widetilde{u}^{D,*}_f(x,t) = \Pi_x[f(\xi_{t\wedge\tau^D})] - \Pi_x\left[\int_0^{t\wedge\tau_D} \psi^*(\xi_s, \widetilde{u}^{D,*}_f(\xi_s, t-s))\mathrm{d}s\right], \quad x\in D, \tag{18}$$

where $\psi^*(x,\lambda) = \psi(x,\lambda+w(x)) - \psi(x,w(x))$, restricted to D, is a branching mechanism of the kind described in the introduction and for each $\mu \in \mathscr{M}_F(E)$, $(\widetilde{X}, \mathbb{P}^*_\mu)$ is a superprocess. Specifically, on E,

$$\psi^*(x,\lambda) = -\alpha^*(x)\lambda + \beta(x)\lambda^2 + \int_{(0,\infty)}(e^{-\lambda z} - 1 + \lambda z)\pi^*(x,\mathrm{d}z), \tag{19}$$

where

$$\alpha^*(x) = \alpha(x) - 2\beta(x)w(x) - \int_{(0,\infty)}(1-e^{-w(x)z})z\pi(x,\mathrm{d}z)$$

and

$$\pi^*(x,\mathrm{d}z) = e^{-w(x)z}\pi(x,\mathrm{d}z) \ \textit{ on } E\times(0,\infty).$$

Proof. For all $f \in \mathrm{bp}(\partial(D\times[0,t)))$ with the additional property that the value of $f(x,s)$ on $E\times[0,t]$ is independent of s, we have

$$\begin{aligned}
\mathbb{E}^*_\mu(e^{-\langle f,\widetilde{X}^D_t\rangle}) &= \mathbb{E}_\mu(e^{-\langle f,\widetilde{X}^D_t\rangle}|\mathscr{E}) \\
&= e^{\langle w,\mu\rangle}\mathbb{E}_\mu(e^{-\langle f,\widetilde{X}^D_t\rangle}1_{\mathscr{E}}) \\
&= e^{\langle w,\mu\rangle}\mathbb{E}_\mu(e^{-\langle f,\widetilde{X}^D_t\rangle}\mathbb{E}_{\widetilde{X}^D_t}(1_{\mathscr{E}})) \\
&= e^{\langle w,\mu\rangle}\mathbb{E}_\mu(e^{-\langle f+w,\widetilde{X}^D_t\rangle}) \\
&= e^{-\langle \widetilde{u}^D_{f+w}(\cdot,t)-w,\mu\rangle}.
\end{aligned}$$

Using (11) and (16) then it is straightforward to check that $\widetilde{u}^{D,*}_f(x,t) = \widetilde{u}^D_{f+w}(x,t) - w(x)$ is a non-negative solution to (18), which is necessarily unique. The proof is complete as soon as we can show that $\psi^*(x,\lambda)$, restricted to D, is a branching mechanism which falls into the appropriate class. One easily verifies the formula (19) and that the new parameters α^* and π^*, restricted to D, respect the properties stipulated in the definition of a branching mechanism in the introduction. □

Corollary 1. *For any bounded open domain $D \subset\subset E$, any function $f \in \mathrm{bp}(D)$ and any $\mu \in \mathscr{M}_F(D)$ satisfying $\langle w, \mu\rangle < \infty$,*

$$-\log \mathbb{E}^*_\mu \left(e^{-\langle f, X^D_t\rangle}\right) = \int_D u^{D,*}_f(x,t)\mu(\mathrm{d}x),$$

where $u^{D,*}_f(x,t) = \widetilde{u}^D_{f+w}(x,t) - w(x)$ and it is the unique solution of

$$u^{D,*}_f(x,t) = \Pi_x[f(\xi_t); t < \tau^D] - \Pi_x\left[\int_0^{t\wedge\tau_D} \psi^*(\xi_s, u^{D,*}_f(\xi_s, t-s))\mathrm{d}s\right], \quad x \in D, \tag{20}$$

where ψ^* is defined by (19).

3.3 Excursion measure

Associated to the law of the processes X, are the measures $\{\mathbb{N}^*_x : x \in E\}$, defined on the same measurable space as the probabilities $\{\mathbb{P}^*_{\delta_x} : x \in E\}$ are defined on, and which satisfy

$$\mathbb{N}^*_x(1 - e^{-\langle f, X_t\rangle}) = -\log \mathbb{E}^*_{\delta_x}(e^{-\langle f, X_t\rangle}) = u^*_f(x,t), \tag{21}$$

for all $f \in \mathrm{bp}(E)$ and $t \geq 0$. Intuitively speaking, the branching property implies that $\mathbb{P}^*_{\delta_x}$ is an infinitely divisible measure on the path space of X, that is to say the space of measure-valued cadlag functions, $\mathbb{D}([0,\infty) \times \mathscr{M}(E))$, and (21) is a 'Lévy-Khinchine' formula in which $\mathbb{N}^*_x$ plays the role of its 'Lévy measure'. Such measures are formally defined and explored in detail in [13].

Note that, by the monotonicity property, for any two open bounded domains, $D_1 \subset\subset D_2 \subset\subset E$,

$$\langle f, X^{D_1}_t\rangle \leq \langle f, X^{D_2}_t\rangle \qquad \mathbb{N}^*_x\text{-a.e.},$$

for all $f \in \mathrm{bp}(D_1)$ understood in the sense of (12), $x \in D_1$ and $t \geq 0$. Moreover, for an open bounded domain D and f as before, it is also clear that $\mathbb{N}^*(1 - e^{-\langle f, X^D_t\rangle}) = u^{D,*}_f(x,t)$.

The measures $\{\mathbb{N}^*_x : x \in E\}$ will play a crucial role in the forthcoming analysis in order to describe the 'rate' of a Poisson point process of immigration.

3.4 A Markov branching process

In this section we introduce a particular Markov branching process which is built from the components of the $(\mathscr{P},\psi)$-superprocess and which plays a central role in the backbone decomposition.

Recall that we abuse our notation and extend the domain of w with the implicit understanding that $w(\dagger)=0$. Note, moreover, that thanks to (17), we have that, for $x\in E$, $w(x)^{-1}w(\xi_t)\exp\left\{-\int_0^t\psi(\xi_s,w(\xi_s))/w(\xi_s)\mathrm{d}s\right\}$ is in general a positive local martingale (and hence a supermartingale) under Π_x. For each $t\geq 0$, let $\mathscr{F}_t^{\xi}=\sigma(\xi_s: s\leq t)$. Let $\zeta=\inf\{t>0:\xi_t\in\{\dagger\}\}$ be the life time of ξ. The formula

$$\left.\frac{\mathrm{d}\Pi_x^w}{\mathrm{d}\Pi_x}\right|_{\mathscr{F}_t^{\xi}}=\frac{w(\xi_t)}{w(x)}\exp\left\{-\int_0^t\frac{\psi(\xi_s,w(\xi_s))}{w(\xi_s)}\mathrm{d}s\right\}\qquad \text{on } \{t<\zeta\},\quad t\geq 0, x\in E, \tag{22}$$

uniquely determines a family of (sub-)probability measures $\{\Pi_x^w: x\in E\}$. It is known that under these new probabilities, ξ is a right Markov process on E; see [6, Section 10.4], [25] or [34, Section 62]. We will denote by $\mathscr{P}^w$ the semi-group of the $E\cup\{\dagger\}$-valued process ξ whose probabilities are $\{\Pi_x^w: x\in E\}$.

Remark 1. The equation (16) may formally be associated with the equation $Lw(x)-\psi(x,w(x))=0$ on E, and the semi-group $\mathscr{P}^w$ corresponds to the diffusion with generator

$$L_0^w:=L^w-w^{-1}Lw=L^w-w^{-1}\psi(\cdot,w),$$

where $L^w u=w^{-1}L(wu)$ for any u in the domain of L. Intuitively speaking, this means that the dynamics associated to $\mathscr{P}^w$, encourages the motion of ξ to visit domains where the global survival rate is high and discourages it from visiting domains where the global survival rate is low. (Recall from (7) that larger values of $w(x)$ make extinction of the $(\mathscr{P},\psi)$-superprocess less likely under $\mathbb{P}_{\delta_x}$.)

Henceforth the process $Z=\{Z_t:t\geq 0\}$ will denote the Markov branching process whose particles move with associated semi-group $\mathscr{P}^w$. Moreover, the branching generator is given by

$$F(x,s)=q(x)\sum_{n\geq 0}p_n(x)(s^n-s), \tag{23}$$

where

$$q(x)=\psi'(x,w(x))-\frac{\psi(x,w(x))}{w(x)}, \tag{24}$$

$p_0(x)=p_1(x)=0$ and for $n\geq 2$,

$$p_n(x)=\frac{1}{w(x)q(x)}\left\{\beta(x)w^2(x)1_{\{n=2\}}+w^n(x)\int_{(0,\infty)}\frac{y^n}{n!}e^{-w(x)y}\pi(x,\mathrm{d}y)\right\}.$$

Here we use the notation

$$\psi'(x,w(x)) := \frac{\partial}{\partial\lambda}\psi(x,\lambda)\bigg|_{\lambda=w(x)}, \qquad x\in E.$$

Note that the choice of $q(x)$ ensures that $\{p_n(x): n\geq 0\}$ is a probability mass function. In order to see that $q(x)\geq 0$ for all $x\in E$ (but $q\neq 0$), write

$$q(x) = \beta(x)w(x) + \frac{1}{w(x)}\int_{(0,\infty)}(1-e^{-w(x)z}(1+w(x)z))\pi(x,\mathrm{d}z) \tag{25}$$

and note that $\beta\geq 0$, $w>0$ and $1-e^{-\lambda z}(1+\lambda z)$, $\lambda\geq 0$, are all non-negative.

Definition 2. In the sequel we shall refer to Z as the $(\mathscr{P}^w,F)$-backbone. Moreover, in the spirit of Definition 1, for all bounded domains D and $t\geq 0$, we shall also define $\widetilde{Z}^D_t$ to be the atomic measure, supported on $\partial(D\times[0,t))$, describing particles in Z which are first in their genealogical line of descent to exit the domain $D\times[0,t)$.

Just as with the case of exit measures for superprocesses, we define the random measure, $Z^D=\{Z^D_t: t\geq 0\}$, on D such that $\langle f, Z^D_t\rangle = \langle f,\widetilde{Z}^D_t\rangle$ for any $f\in \mathrm{bp}(D)$, where we remind the reader that we regard f as a function defined on $E\times[0,\infty)$ as in (12). As a process in time, Z^D is a Markov branching process, with branching generator which is the same as in (23) except that the branching rate $q(x)$ is replaced by $q^D(x) := q(x)\mathbf{1}_D(x)$, and associated motion semi-group given by that of the process ξ killed upon leaving D. Similarly to the case of superprocesses, for any two open bounded domains, $D_1\subset\subset D_2\subset\subset E$, the processes $\widetilde{Z}^{D_1}$ and $\widetilde{Z}^{D_2}$ (and hence Z^{D_1} and Z^{D_2}) are consistent in the sense that

$$\widetilde{Z}^{D_1}_t = (\widetilde{\widetilde{Z}^{D_2}_t})^{D_1}$$

for all $t\geq 0$ (and similarly $Z^{D_1}_t = (Z^{D_2}_t)^{D_1}$ for all $t\geq 0$).

4 Local backbone decomposition

We are interested in immigrating $(\mathscr{P},\psi^*)$-superprocesses onto the path of an $(\mathscr{P}^w,F)$-backbone within the confines of an open, bounded domain $D\subset\subset E$ and initial configuration $\nu\in\mathscr{M}_a(D)$, the space of finite atomic measures in D of the form $\sum_{i=1}^n\delta_{x_i}$, where $n\in\mathbb{N}\cup\{0\}$ and $x_1,\cdots,x_n\in D$. There will be three types of immigration: continuous, discontinuous and branch-point immigration which we now describe in detail. In doing so, we shall need to refer to individuals in the process Z for which we shall use classical Ulam-Harris notation, see for example p290 of Harris and Hardy [23]. Although the Ulam-Harris labelling of individuals is rich enough to encode genealogical order, the only feature we really need of the Ulam-Harris notation is that individuals are uniquely identifiable amongst $\mathscr{T}$, the set labels of individuals realised in Z. For each individual $u\in\mathscr{T}$ we shall write b_u and d_u for its birth and death times respectively, $\{z_u(r): r\in[b_u,d_u]\}$ for its spatial trajectory and N_u for the number of offspring it has at time d_u. We shall also write $\mathscr{T}^D$ for the set of labels of individuals realised in Z^D. For each $u \in \mathscr{T}^D$ we shall also define

$$\tau_u^D = \inf\{s \in [b_u, d_u], z_u(s) \in D^c\},$$

with the usual convention that $\inf\emptyset := \infty$. Note that if $u \in \mathscr{T}^D$, we denote by ω its historical path on $[0, d_u]$ (the spatial motion of its ancestors, including itself). Then we have $\inf\{t \geq 0 : \omega(t) \in D^c\} \geq b_u$.

Definition 3. For $\nu \in \mathscr{M}_a(D)$ and $\mu \in \mathscr{M}_F(D)$, let Z^D be a Markov branching process with initial configuration ν, branching generator which is the same as in (23), except that the branching rate $q(x)$ is replaced by $q^D(x) := q(x)\mathbf{1}_D(x)$, and associated motion semi-group given by that of $\mathscr{P}^w$ killed upon leaving D. Let $X^{D,*}$ be an independent copy of X^D under $\mathbb{P}^*_\mu$. Then we define the measure valued stochastic process $\Delta^D = \{\Delta^D_t : t \geq 0\}$ such that, for $t \geq 0$,

$$\Delta^D_t = X^{D,*}_t + I^{D,\mathbb{N}^*}_t + I^{D,\mathbb{P}^*}_t + I^{D,\eta}_t, \tag{26}$$

where $I^{D,\mathbb{N}^*} = \{I^{D,\mathbb{N}^*}_t : t \geq 0\}$, $I^{D,\mathbb{P}^*} = \{I^{D,\mathbb{P}^*}_t : t \geq 0\}$ and $I^{D,\eta} = \{I^{D,\eta}_t : t \geq 0\}$ are defined as follows.

i) (**Continuous immigration:**) The process $I^{D,\mathbb{N}^*}$ is measure-valued on D such that

$$I^{D,\mathbb{N}^*}_t = \sum_{u \in \mathscr{T}^D} \sum_{b_u < r \leq t \wedge d_u \wedge \tau^D_u} X^{(D,1,u,r)}_{t-r},$$

where, given Z^D, independently for each $u \in \mathscr{T}^D$ such that $b_u < t$,

$$\sum_{b_u < r \leq t \wedge d_u \wedge \tau^D_u} \delta_{(r, X^{(D,1,u,r)})}$$

is a Poisson point process on $[b_u, t \wedge d_u \wedge \tau^D_u] \times \mathbb{D}([0,\infty) \times \mathscr{M}(E))$ with intensity

$$\mathrm{d}r \times 2\beta(z_u(r))\mathrm{d}\mathbb{N}^*_{z_u(r)}.$$

ii) (**Discontinuous immigration:**) The process $I^{D,\mathbb{P}^*}$ is measure-valued on D such that

$$I^{D,\mathbb{P}^*}_t = \sum_{u \in \mathscr{T}^D} \sum_{b_u < r \leq t \wedge d_u \wedge \tau^D_u} X^{(D,2,u,r)}_{t-r},$$

where, given Z^D, independently for each $u \in \mathscr{T}^D$ such that $b_u < t$,

$$\sum_{b_u < r \leq t \wedge d_u \wedge \tau^D_u} \delta_{(r, X^{(D,2,u,r)})}$$

is a Poisson point process on $[b_u, t \wedge d_u \wedge \tau^D_u] \times \mathbb{D}([0,\infty) \times \mathscr{M}(E))$ with intensity

$$\mathrm{d}r \times \int_{y\in(0,\infty)} ye^{-w(z_u(r))y}\pi(z_u(r),\mathrm{d}y) \times \mathrm{d}\mathbb{P}^*_{y\delta_{z_u(r)}}.$$

iii)(**Branch point biased immigration:**) The process $I^{D,\eta}$ is measure-valued on D such that

$$I_t^{D,\eta} = \sum_{u\in\mathscr{T}^D} \mathbf{1}_{\{d_u\le t\wedge\tau_u^D\}} X_{t-d_u}^{(D,3,u)},$$

where, given Z^D, independently for each $u \in \mathscr{T}^D$ such that $d_u < t\wedge\tau_u^D$, the processes $X^{(D,3,u)}$ are independent copies of the canonical process X^D issued at time d_u with law $\mathbb{P}^*_{Y_u\delta_{z_u(d_u)}}$ such that, given u has $n\ge 2$ offspring, the independent random variable Y_u has distribution $\eta_n(z_u(d_u),\mathrm{d}y)$, where

$$\eta_n(x,\mathrm{d}y) = \frac{1}{q(x)w(x)p_n(x)}\left\{\beta(x)w^2(x)\delta_0(\mathrm{d}y)\mathbf{1}_{\{n=2\}} + w(x)^n\frac{y^n}{n!}e^{-w(x)y}\pi(x,\mathrm{d}y)\right\}. \tag{27}$$

It is not difficult to see that Δ^D is consistent in the domain D in the sense of (15). Accordingly, we denote by $\mathbf{P}_{(\mu,\nu)}$ the law induced by $\{\Delta_t^D, D\in\mathscr{O}(E), t\ge 0\}$, where $\mathscr{O}(E)$ is the collection of bounded open sets in E.

The so-called backbone decomposition of $(X^D,\mathbb{P}_\mu)$ for $\mu\in\mathscr{M}_F(D)$ entails looking at the process Δ^D in the special case that we randomise the law $\mathbf{P}_{(\mu,\nu)}$ by replacing the deterministic choice of ν with a Poisson random measure having intensity measure $w(x)\mu(\mathrm{d}x)$. We denote the resulting law by $\mathbf{P}_\mu$.

Theorem 1. *For any $\mu\in\mathscr{M}_F(D)$, the process $(\Delta^D,\mathbf{P}_\mu)$ is Markovian and has the same law as $(X^D,\mathbb{P}_\mu)$.*

5 Proof of Theorem 1

The proof involves several intermediary results in the spirit of the non-spatially dependent case of Berestycki et al. [2]. Localisation will be an important part of the process, allowing us to make use of Assumption A and uniqueness properties for certain integral equations. Accordingly, throughout we take D as an open, bounded domain such that $D\subset\subset E$. Any function f defined on D will be extended to E by defining $f=0$ on $E\setminus D$.

Lemma 2. *Suppose that $\mu\in\mathscr{M}_F(D)$, $\nu\in\mathscr{M}_a(D)$, $t\ge 0$ and $f\in \mathrm{bp}(D)$. We have*

$$\mathbf{E}_{(\mu,\nu)}\left(e^{-\langle f, I_t^{D,\mathbb{N}^*}+I_t^{D,\mathbb{P}^*}\rangle}|\{Z_s^D : s\le t\}\right) = \exp\left\{-\int_0^t \langle\phi(\cdot,u_f^{D,*}(\cdot,t-s)),Z_s^D\rangle\mathrm{d}s\right\},$$

where

$$\phi(x,\lambda) = 2\beta(x)\lambda + \int_{(0,\infty)}(1-e^{-\lambda y})ze^{-w(x)y}\pi(x,\mathrm{d}y), \qquad x\in D,\ \lambda\ge 0. \tag{28}$$

Proof. We write

$$\langle f, I_t^{D,\mathbb{N}^*} + I_t^{D,\mathbb{P}^*}\rangle = \sum_{u\in\mathcal{T}^D}\sum_{b_u<r\le t\wedge d_u\wedge\tau_u^D}\langle f, X_{t-r}^{(D,1,u,r)}\rangle + \sum_{u\in\mathcal{T}^D}\sum_{b_u<r\le t\wedge d_u\wedge\tau_u^D}\langle f, X_{t-r}^{(D,2,u,r)}\rangle.$$

Hence conditioning on Z^D, appealing to the independence of the immigration processes together with Campbell's formula and that $\mathbb{N}_x^*(1-e^{-\langle f, X_s^D\rangle}) = u_f^{D,*}(x,s)$, we have

$$\begin{aligned}
&\mathbf{E}_{(\mu,\nu)}(e^{-\langle f, I_t^{D,\mathbb{N}^*}\rangle}|\{Z_s^D : s\le t\})\\
&= \exp\left\{-\sum_{u\in\mathcal{T}^D} 2\int_{b_u}^{t\wedge d_u\wedge\tau_u^D}\beta(z_u(r))\cdot\mathbb{N}^*_{z_u(r)}(1-e^{-\langle f, X_{t-r}^D\rangle})\mathrm{d}r\right\}\\
&= \exp\left\{-\sum_{u\in\mathcal{T}^D} 2\int_{b_u}^{t\wedge d_u\wedge\tau_u^D}\beta(z_u(r))u_f^{D,*}(z_u(r),t-r)\mathrm{d}r\right\}.
\end{aligned} \tag{29}$$

On the other hand

$$\begin{aligned}
&\mathbf{E}_{(\mu,\nu)}(e^{-\langle f, I_t^{D,\mathbb{P}^*}\rangle}|\{Z_s^D : s\le t\})\\
&= \exp\left\{-\sum_{u\in\mathcal{T}^D}\int_{b_u}^{t\wedge d_u\wedge\tau_u^D}\int_0^\infty ye^{-w(z_u(r))y}\pi(z_u(r),\mathrm{d}y)\mathbb{E}^*_{y\delta_{z_u(r)}}(1-e^{-\langle f, X_{t-r}^D\rangle})\mathrm{d}r\right\}\\
&= \exp\left\{-\sum_{u\in\mathcal{T}^D}\int_{b_u}^{t\wedge d_u\wedge\tau_u^D}\int_0^\infty(1-e^{-u_f^{D,*}(z_u(r),t-r)y})ye^{-w(z_u(r))y}\pi(z_u(r),\mathrm{d}y)\mathrm{d}r\right\}.
\end{aligned} \tag{30}$$

Combining (29) and (30) the desired result follows. □

Lemma 3. *Suppose that the real-valued function $J(s,x,\lambda)$ defined on $[0,T)\times D\times\mathbb{R}$ satisfies that for any $c>0$ there is a constant $A(c)$ such that*

$$|J(s,x,\lambda_1) - J(s,x,\lambda_2)| \le A(c)|\lambda_1-\lambda_2|,$$

for all $s\in[0,T)$, $x\in D$ and $\lambda_1,\lambda_2\in[-c,c]$. Then for any bounded measurable function $g(s,x)$ on $[0,T)\times D$, the integral equation

$$v(t,x) = g(t,x) + \int_0^t \Pi_x\left[J(t-s,\xi_s,v(t-s,\xi_s)); s<\tau^D\right]\mathrm{d}x, \quad t\in[0,T),$$

has at most one bounded solution.

Proof. Suppose that v_1 and v_2 are two solutions, then there is a constant $c>0$ such that $-c\le v_1, v_2\le c$ and

$$\|v_1-v_2\|(t) \le A(c)\int_0^t\|v_1-v_2\|(s)\mathrm{d}s,$$

where $\|v_1 - v_2\|(t) = \sup_{x\in D}|v_1(t,x) - v_2(t,x)|$, $t \in (0,T)$. It follows from Gronwall's lemma (see, for example, Lemma 1.1 on page 1208 of [8]) that $\|v_1 - v_2\|(t) = 0, t \in [0,T)$.

Lemma 4. *Fix $t > 0$. Suppose that $f,h \in \mathrm{bp}(D)$ and $g_s(x)$ is jointly measurable in $(x,s) \in D \times [0,t]$ and bounded on finite time horizons of s such that $g_s(x) = 0$ for $x \in D^c$. Then for any $\mu \in \mathcal{M}_F(D)$, $x \in D$ and $t \geq 0$,*

$$e^{-W(x,t)} := \mathbf{E}_{(\mu,\delta_x)}\left[\exp\left(-\int_0^t \langle g_{t-s}, Z_s^D\rangle \mathrm{d}s - \langle f, I_t^{D,\eta}\rangle - \langle h, Z_t^D\rangle\right)\right]$$

is the unique $[0,1]$-valued solution to the integral equation

$$\begin{aligned} w(x)e^{-W(x,t)} = {} & \Pi_x\left[w(\xi_{t\wedge\tau_D})e^{-h(\xi_{t\wedge\tau_D})}\right] \\ & +\Pi_x\left[\int_0^{t\wedge\tau_D}[H_{t-s}(\xi_s, -w(\xi_s)e^{-W(\xi_s,t-s)}) - w(\xi_s)e^{-W(\xi_s,t-s)}g_{t-s}(\xi_s)\right. \\ & \left. -\psi(\xi_s, w(\xi_s))e^{-W(\xi_s,t-s)}]\mathrm{d}s\right], \end{aligned} \tag{31}$$

for $x \in D$, where

$$H_{t-s}(x,\lambda) = q(x)\lambda + \beta(x)\lambda^2 + \int_0^\infty (e^{-\lambda y} - 1 + \lambda y)e^{-(w(x)+u_f^{D,*}(x,t-s))y}\pi(x,\mathrm{d}y), \quad x \in D,$$

and $q(x)$ was defined in (24).

Proof. Following Evans and O'Connell [18] it suffices to prove the result in the case when g is time invariant. To this end, let us start by defining the semi-group $\mathscr{P}^{h,D}$ by

$$\mathscr{P}_t^{h,D}[k](x) = \Pi_x\left(e^{-\int_0^{t\wedge\tau^D} h(\xi_s)\mathrm{d}s}k(\xi_{t\wedge\tau^D})\right) \quad \text{for } h,k \in \mathrm{bp}(\overline{D}), \tag{32}$$

where, for convenience, we shall write

$$\mathscr{P}_t^D[k] = \mathscr{P}_t^{0,D}[k]. \tag{33}$$

Recall that for $h,k \in \mathrm{bp}(D)$, $h(x) = k(x) = 0$ for $x \notin D$. Then we have

$$\mathscr{P}_t^{h,D}[k](x) = \Pi_x\left(e^{-\int_0^t h(\xi_s)\mathrm{d}s}k(\xi_t); t < \tau^D\right) \quad \text{for } h,k \in \mathrm{bp}(D). \tag{34}$$

Define the function $\chi(x) = \psi(x,w(x))/w(x)$. Conditioning on the first splitting time in the process Z^D and recalling that the branching occurs at the spatial rate $q^D(x) = \mathbf{1}_D(x)(\psi'(x,w(x)) - \chi(x))$ we get that for any $x \in D$,

$$e^{-W(x,t)} = \frac{1}{w(x)}\mathscr{P}_t^{g+q+\chi,D}[we^{-h}](x)$$
$$+\Pi_x^w\left[\int_0^{t\wedge\tau^D}\exp\left(-\int_0^s(g+q)(\xi_r)\mathrm{d}r\right)\right.$$
$$\left.\left\{q(\xi_s)\sum_{n\geq 2}p_n(\xi_s)e^{-nW(\xi_s,t-s)}\int_{(0,\infty)}\eta_n(\xi_s,\mathrm{d}y)e^{-yu_f^{D,*}(\xi_s,t-s)}\right\}\mathrm{d}s\right] \quad (35)$$

From (27) we quickly find that for $x\in D$,

$$\sum_{n\geq 2}p_n(x)e^{-nW(x,t-s)}\int_{(0,\infty)}\eta_n(x,\mathrm{d}y)e^{-yu_f^{D,*}(x,t-s)}$$
$$=\frac{1}{q(x)w(x)}\left\{H_{t-s}(x,-w(x)e^{-W(x,t-s)})+w(x)q(x)e^{-W(x,t-s)}\right\}.$$

Using the above expression in (35) we have that

$$w(x)e^{-W(x,t)} = \mathscr{P}_t^{g+q+\chi,D}[we^{-h}](x)$$
$$+\Pi_x\left[\int_0^{t\wedge\tau^D}\exp\left(-\int_0^s(g+q+\chi)(\xi_r)\mathrm{d}r\right)\right.$$
$$\left.\left[(H_{t-s}(\xi_s,-w(\xi_s)e^{-W(\xi_s,t-s)})+w(\xi_s)q(\xi_s)e^{-W(\xi_s,t-s)})\right]\mathrm{d}s\right].$$

Now appealing to Lemma 1.2 in Dynkin [11] and recalling that $\chi(\cdot)=\psi(\cdot,w(\cdot))/w(\cdot)$ on D, we may deduce that for any $x\in D$,

$$w(x)e^{-W(x,t)} = \mathscr{P}_t^{D}[we^{-h}](x)$$
$$+\Pi_x\left[\int_0^{t\wedge\tau^D}[H_{t-s}(\xi_s,-w(\xi_s)e^{-W(\xi_s,t-s)})-w(\xi_s)g(\xi_s)e^{-W(\xi_s,t-s)}\right.$$
$$\left.-\psi(\xi_s,w(\xi_s))e^{-W(\xi_s,t-s)}]\mathrm{d}s\right] \quad (36)$$

as required. Note that in the above computations we have implicitly used that w is uniformly bounded away from ∞ on D.

To complete the proof we need to show uniqueness of solutions to (36). Lemma 3 offers sufficient conditions for uniqueness of solutions to a general family of integral equations which includes (36). In order to check these sufficient conditions, let us define $\overline{w}^D=\sup_{y\in D}w(y)$. Thanks to Assumption (A) we have that $0<\overline{w}^D<\infty$. For $s\geq 0$, $x\in D$ and $\lambda\in[0,\overline{w}^D]$, define the function $J(s,x,\lambda):=[H_s(x,-\lambda)-(g(x)+\chi(x))\lambda]$. We rewrite (36) as

$$w(x)e^{-W(x,t)}=\mathscr{P}_t^D[we^{-h}](x)+\int_0^t\Pi_x\left[J(t-s,\xi_s,w(\xi_s)e^{-W(\xi_s,t-s)});s<\tau^D\right]\mathrm{d}s.$$

Lemma 3 tells us that (36) has a unique solution as soon as we can show that J is continuous in s and that for each fixed $T>0$, there exists a $K>0$ (which may depend on D and T) such that

$$\sup_{s\le T}\sup_{y\in D}|J(s,y,\lambda_1)-J(s,y,\lambda_2)|\le K|\lambda_1-\lambda_2|,\quad \lambda_1,\lambda_2\in(0,\overline{w}^D].$$

Recall that $g(y)$ is assumed to be bounded, moreover, Assumption (A) together with the fact that

$$\sup_{y\in D}\left\{|\alpha(y)|+\beta(y)+\int_{(0,\infty)}(z\wedge z^2)\pi(y,\mathrm{d}z)\right\}<\infty \tag{37}$$

also implies that χ is bounded on D. Appealing to the triangle inequality, it now suffices to check that for each fixed $T>0$, there exists a $K>0$ such that

$$\sup_{s\le T}\sup_{y\in D}|H_s(y,-\lambda_1)-H_s(y,-\lambda_2)|\le K|\lambda_1-\lambda_2|,\quad \lambda_1,\lambda_2\in(0,\overline{w}^D]. \tag{38}$$

First note from Proposition 2.3 of Fitzsimmons [21] that

$$\sup_{s\le T}\sup_{x\in D}u_f^{D,*}(x,s)<\infty. \tag{39}$$

Straightforward differentiation of the function $H_s(x,-\lambda)$ in the variable λ yields

$$-\frac{\partial}{\partial\lambda}H_s(x,-\lambda)=q(x)-2\beta(x)\lambda+\int_{(0,\infty)}(1-\mathrm{e}^{\lambda z})\mathrm{e}^{-(w(x)+u_f^{D,*}(x,s))z}z\pi(x,\mathrm{d}z).$$

Appealing to (37) and (39) it is not difficult to show that the derivative above is uniformly bounded in absolute value for $s\le T$, $x\in D$ and $\lambda\in[0,\overline{w}^D]$, from which (38) follows by straightforward linearisation. The proof is now complete. □

Theorem 2. *For every $\mu\in\mathscr{M}_F(D)$, $\nu\in\mathscr{M}_a(D)$ and $f,h\in\mathrm{bp}(D)$*

$$\mathbf{E}_{(\mu,\nu)}\left(e^{-\langle f,\Delta_t^D\rangle-\langle h,Z_t^D\rangle}\right)=e^{-\langle u_f^{D,*}(\cdot,t),\mu\rangle-\langle v_{f,h}^D(\cdot,t),\nu\rangle}, \tag{40}$$

where $e^{-v_{f,h}^D(x,t)}$ is the unique $[0,1]$-solution to the integral equation

$$w(x)e^{-v_{f,h}^D(x,t)}=\Pi_x\left[w(\xi_{t\wedge\tau_D})e^{-h(\xi_{t\wedge\tau_D})}\right]$$
$$+\Pi_x\left[\int_0^{t\wedge\tau_D}[\psi^*(\xi_s,-w(\xi_s)e^{-v_{f,h}^D(\xi_s,t-s)}+u_f^{D,*}(\xi_s,t-s))-\psi^*(\xi_s,u_f^{D,*}(\xi_s,t-s))]\mathrm{d}s\right] \tag{41}$$

Proof. Thanks to Corollary 1 it suffices to prove that

$$\mathbf{E}_{(\mu,\nu)}(e^{-\langle f,I_t^D\rangle-\langle h,Z_t^D\rangle})=e^{-\langle v_{f,h}^D(\cdot,t),\nu\rangle},$$

where $I^D := I^{D,\mathbb{N}^*} + I^{D,\mathbb{P}^*} + I^{D,\eta}$, and $v^D_{f,h}$ solves (41). Putting Lemma 2 and Lemma 4 together we only need to show that, when $g_{t-s}(\cdot) = \phi(\cdot, u_f^{D,*}(\cdot, t-s))$ (where ϕ is given by (28)), we have that $\exp\{-W(x,t)\}$ is the unique $[0,1]$-valued solution to (41). Again following the lead of [2], in particular referring to Lemma 5 there, it is easy to see that on D

$$H_{t-s}(\cdot, -w(\cdot)e^{-W(\cdot,t-s)}) - \phi(\cdot, u_f^{D,*}(\cdot,t-s))w(\cdot)e^{-W(\cdot,t-s)} - \frac{\psi(\cdot, w(\cdot))}{w(\cdot)} w(\cdot)e^{-W(\cdot,t-s)}$$
$$= \psi^*(\cdot, w(\cdot)e^{-W(\cdot,t-s)} + u_f^{D,*}(\cdot,t-s)) - \psi^*(\cdot, u_f^{D,*}(\cdot,t-s)),$$

which implies that $\exp\{-W(x,t)\}$ is the unique $[0,1]$-valued solution to (41). □

Proof of Theorem 1:. The proof is guided by the calculation in the proof of Theorem 2 of [2]. We start by addressing the claim that $(\Delta^D, \mathbf{P}_\mu)$ is a Markov process. Given the Markov property of the pair (Δ^D, Z^D), it suffices to show that, given Δ^D_t, the atomic measure Z^D_t is equal in law to a Poisson random measure with intensity $w(x)\Delta^D_t$. Thanks to Campbell's formula for Poisson random measures, this is equivalent to showing that for all $h \in \mathrm{bp}(D)$,

$$\mathbf{E}_\mu(e^{-\langle h, Z_t^D\rangle} | \Delta^D_t) = e^{-\langle w\cdot(1-e^{-h}), \Delta^D_t\rangle},$$

which in turn is equivalent to showing that for all $f, h \in \mathrm{bp}(D)$,

$$\mathbf{E}_\mu(e^{-\langle f, \Delta_t^D\rangle - \langle h, Z_t^D\rangle}) = \mathbf{E}_\mu(e^{-\langle w\cdot(1-e^{-h}) + f, \Delta_t^D\rangle}). \tag{42}$$

Note from (40) however that when we randomize ν so that it has the law of a Poisson random measure with intensity $w(x)\mu(\mathrm{d}x)$, we find the identity

$$\mathbf{E}_\mu(e^{-\langle f, \Delta_t^D\rangle - \langle h, Z_t^D\rangle}) = \exp\left\langle -u_f^{D,*}(\cdot,t) - w\cdot(1 - e^{-v^D_{f,h}(\cdot,t)}), \mu\right\rangle.$$

Moreover, if we replace f by $w\cdot(1-e^{-h}) + f$ and h by 0 in (40) and again randomize ν so that it has the law of a Poisson random measure with intensity $w(x)\mu(\mathrm{d}x)$ then we get

$$\mathbf{E}_\mu\left(e^{-\langle w\cdot(1-e^{-h})+f, \Delta_t^D\rangle}\right) = \exp\left\langle -u^{D,*}_{w\cdot(1-e^{-h})+f}(\cdot,t) - w\cdot\left(1 - \exp\left\{-v^D_{w\cdot(1-e^{-h})+f,0}\right\}\right), \mu\right\rangle.$$

These last two observations indicate that (42) is equivalent to showing that, for all f, h as stipulated above and $t \geq 0$,

$$u_f^{D,*}(x,t) + w(x)(1 - e^{-v^D_{f,h}(x,t)}) = u^{D,*}_{w\cdot(1-e^{-h})+f}(x,t) + w(x)(1 - e^{-v^D_{w\cdot(1-e^{-h})+f,0}(x,t)}). \tag{43}$$

Note that both left and right-hand sides of the equality above are necessarily non-negative given that they are Laplace exponents of the left and right-hand sides of (42). Making use of (20), (16), and (41), it is computationally very straightforward to show that both left and right-hand sides of (43) solve (14) with initial condition

$f+w(1-e^{-h})$, which is bounded in $\overline{D}$. Since (14) has a unique solution with this initial condition, namely $u^D_{f+w\cdot(1-e^{-h})}(x,t)$, we conclude that (43) holds true. The proof of the claimed Markov property is thus complete.

Having now established the Markov property, the proof is complete as soon as we can show that $(\Delta^D, \mathbf{P}_\mu)$ has the same semi-group as $(X^D, \mathbb{P}_\mu)$. However, from the previous part of the proof we have already established that when $f,h \in \mathrm{bp}(D)$,

$$\mathbf{E}_\mu\left(e^{-\langle h, Z^D_t\rangle - \langle f, \Delta^D_t\rangle}\right) = e^{-\langle u^D_{w(1-e^{-h})+f}(\cdot,t),\mu\rangle} = \mathbb{E}_\mu\left(e^{-\langle f+w(1-e^{-h}), X^D_t\rangle}\right). \qquad (44)$$

In particular, choosing $h=0$ we find

$$\mathbf{E}_\mu\left(e^{-\langle f, \Delta^D_t\rangle}\right) = \mathbb{E}_\mu\left(e^{-\langle f, X^D_t\rangle}\right), \qquad t \geq 0,$$

which is equivalent to saying that the semi-groups of $(\Delta^D, \mathbf{P}_\mu)$ and $(X^D, \mathbb{P}_\mu)$ agree. □

6 Global backbone decomposition

So far we have localized our computations to an open bounded domain D. Our ultimate objective is to provide a backbone decomposition on the whole domain E. To this end, let D_n be a sequence of open bounded domains in E such that $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n \subseteq \cdots \subseteq E$ and $E = \cup_{n\geq 1} D_n$. Let X^{D_n}, Δ^{D_n} and Z^{D_n} be defined as in previous sections with D being replaced by D_n.

Lemma 5. *For any $h,f \in \mathrm{bp}(E)$ with compact support and any $\mu \in \mathcal{M}_F(E)$, we have that for any $t\geq 0$, each element of the pair $\{(\langle h, Z^{D_n}_s\rangle, \langle f, \Delta^{D_n}_s\rangle) : s \leq t\}$ pathwise increases $\mathbf{P}_\mu$-almost surely as $n\to\infty$. The limiting pair of processes, here denoted by $\{(\langle h, Z^{\min}_s\rangle, \langle f, \Delta^{\min}_s\rangle) : s\leq t\}$, are such that $\langle f, \Delta^{\min}_t\rangle$ is equal in law to $\langle f, X_t\rangle$ and, given $\Delta^{\min}_t$, the law of $Z^{\min}_t$ is a Poisson random field with intensity $w(x)\Delta^{\min}_t(\mathrm{d}x)$. Moreover, $Z^{\min}$ is a $(\mathscr{P}^w, F)$ branching process with branching generator as in (23) and associated motion semi-group given by (22).*

Proof. Appealing to the stochastic consistency of Z^D and Δ^D in the domain D, it is clear that both $\langle h, Z^{D_n}_t,\rangle$ and $\langle f, \Delta^{D_n}_t\rangle$ are almost surely increasing in n. It therefore follows that the limit as $n\to\infty$ exists for both $\langle h, Z^{D_n}_t\rangle$ and $\langle f, \Delta^{D_n}_t\rangle$, $\mathbf{P}_\mu$-almost surely. In light of the discussion at the end of the proof of Theorem 1, the distributional properties of the limiting pair are established as soon as we show that

$$-\log \mathbf{E}_\mu\left(e^{-\langle h, Z^{\min}_t\rangle - \langle f, \Delta^{\min}_t\rangle}\right) = \int_E u_{w(1-e^{-h})+f}(x,t)\mu(\mathrm{d}x), \quad t\geq 0. \qquad (45)$$

Assume temporarily that $\mathrm{supp}\mu$, the support of μ, is compactly embedded in E so that there exists an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have that $\mathrm{supp}\mu \subset D_n$ and $h=f=0$ on D^c_n. Thanks to (44) and monotone convergence (45) holds as soon as we can show that $u_g^{D_n} \uparrow u_g$ for all $g \in \mathrm{bp}(E)$ satisfying $g = 0$ on D_n^c for $n \geq n_0$. By (13) and (14), we know that $u_g^{D_n}(x,t)$ is the unique non-negative solution to the integral equation

$$u_g^{D_n}(x,t) = \Pi_x[g(\xi_{t\wedge\tau_{D_n}})] - \Pi_x\left[\int_0^{t\wedge\tau_{D_n}} \psi(\xi_s, u_g^{D_n}(\xi_s, t-s))\mathrm{d}s\right]. \tag{46}$$

Using Lemma 1.5 in [8] we can rewrite the above integral equation in the form

$$\begin{aligned} u_g^{D_n}(x,t) &= \Pi_x\left[g(\xi_{t\wedge\tau_{D_n}})\exp\left(\int_0^{t\wedge\tau_{D_n}} \alpha(\xi_s)\mathrm{d}s\right)\right] \\ -\Pi_x&\left[\int_0^{t\wedge\tau_{D_n}} \exp\left(\int_0^s \alpha(\xi_r)\mathrm{d}r\right)\left[\psi(\xi_s, u_g^{D_n}(\xi_s,t-s)) + \alpha(\xi_s)u_g^{D_n}(\xi_s,t-s))\right]\mathrm{d}s\right] \end{aligned} \tag{47}$$

Since $g = 0$ on D_n^c for $n \geq n_0$, we have

$$\Pi_x\left[g(\xi_{t\wedge\tau_{D_n}})\exp\left(\int_0^{t\wedge\tau_{D_n}} \alpha(\xi_s)\mathrm{d}s\right)\right] = \Pi_x\left[g(\xi_t)\exp\left(\int_0^t \alpha(\xi_s)\mathrm{d}s\right); t < \tau_{D_n}\right],$$

which is increasing in n. By the comparison principle, $u_g^{D_n}$ is increasing in n (see Theorem 3.2 in part II of [8]). Put $\tilde{u}_g = \lim_{n\to\infty} u_g^{D_n}$. Note that $\psi(x,\lambda) + \alpha(x)\lambda$ is increasing in λ. Letting $n \to \infty$ in (47), by the monotone convergence theorem,

$$\tilde{u}_g(x,t) = \mathscr{P}_t^\alpha g(x) - \Pi_x \int_0^t \mathscr{P}_s^\alpha\left[\psi(\cdot, \tilde{u}_g(\cdot,t-s)) + \alpha(\cdot)\tilde{u}_g(\cdot,t-s))\right]\mathrm{d}s,$$

where

$$\mathscr{P}_t^\alpha g = \Pi_x\left[g(\xi_t)\exp\left(\int_0^t \alpha(\xi_s)\mathrm{d}s\right)\right], \quad g \in \mathrm{bp}(E),$$

which in turn is equivalent to

$$\tilde{u}_g(x,t) = \mathscr{P}_s g(x) - \Pi_x \int_0^t \mathscr{P}_s \psi(\cdot, \tilde{u}_g(\cdot,t-s))\mathrm{d}s.$$

Therefore, $\tilde{u}_g$ and u_g are two solutions of (3) and hence by uniqueness they are the same, as required.

To remove the initial assumption that the support of μ is compactly embedded in E, suppose that μ_n is a sequence of compactly supported measures with mutually disjoint support such that $\mu = \sum_{k\geq 1}\mu_k$. By considering (45) for $\sum_{k=1}^n \mu_k$ and taking limits as $n \uparrow \infty$ we see that (45) holds for μ. Note in particular that the limit on the left hand side of (45) holds as a result of the additive property of the backbone decomposition in the initial state μ. □

Note that, in the style of the proof given above (appealing to monotonicity and the maximality principle) we can easily show that the processes $X^{D_n,*}$, $n \geq 1$, converge distributionally at fixed times, and hence in law, to the process $(X, \mathbb{P}_\mu^*)$; that is, a $(\mathscr{P}, \psi^*)$-superprocess. With this in mind, again appealing to the consistency and monotonicity of the local backbone decomposition, our main result follows as a simple corollary of Lemma 5.

Corollary 2. *Suppose that $\mu \in \mathscr{M}_F(E)$. Let Z be a $(\mathscr{P}^w, F)$-Markov branching process with initial configuration consisting of a Poisson random field of particles in E with intensity $w(x)\mu(\mathrm{d}x)$. Let X^* be an independent copy of $(X, \mathbb{P}^*_\mu)$. Then define the measure valued stochastic process $\Delta = \{\Delta_t : t \geq 0\}$ such that, for $t \geq 0$,*

$$\Delta_t = X^*_t + I^{\mathbb{N}^*}_t + I^{\mathbb{P}^*}_t + I^{\eta}_t, \tag{48}$$

where $I^{\mathbb{N}^*} = \{I^{\mathbb{N}^*}_t : t \geq 0\}$, $I^{\mathbb{P}^*} = \{I^{\mathbb{P}^*}_t : t \geq 0\}$ and $I^{\eta} = \{I^{\eta}_t : t \geq 0\}$ are defined as follows.

i) (**Continuum immigration:**) *The process $I^{\mathbb{N}^*}$ is measure-valued on E such that*

$$I^{\mathbb{N}^*}_t = \sum_{u \in \mathscr{T}} \sum_{b_u < r \leq t \wedge d_u} X^{(1,u,r)}_{t-r},$$

where, given Z, independently for each $u \in \mathscr{T}$ such that $b_u < t$,

$$\sum_{b_u < r \leq t \wedge d_u} \delta_{(r, X^{(1,u,r)})}$$

is a Poisson point process on $[b_u, t \wedge d_u] \times \mathbb{D}([0,\infty) \times \mathscr{M}(E))$ with intensity

$$\mathrm{d}r \times 2\beta(z_u(r))\mathrm{d}\mathbb{N}^*_{z_u(r)}.$$

ii) (**Discontinuous immigration:**) *The process $I^{\mathbb{P}^*}$ is measure-valued on E such that*

$$I^{\mathbb{P}^*}_t = \sum_{u \in \mathscr{T}} \sum_{b_u < r \leq t \wedge d_u} X^{(2,u,r)}_{t-r},$$

where, given Z, independently for each $u \in \mathscr{T}$ such that $b_u < t$,

$$\sum_{b_u < r \leq t \wedge d_u} \delta_{(r, X^{(2,u,r)})}$$

is a Poisson point process on $[b_u, t \wedge d_u] \times \mathbb{D}([0,\infty) \times \mathscr{M}(E))$ with intensity

$$\mathrm{d}r \times \int_{y \in (0,\infty)} y e^{-w(z_u(r))y} \pi(z_u(r), \mathrm{d}y) \times \mathrm{d}\mathbb{P}^*_{y\delta_{z_u(r)}}.$$

iii)(**Branch point biased immigration:**) *The process I^{η} is measure-valued on E such that*

$$I^{\eta}_t = \sum_{u \in \mathscr{T}^D} \mathbf{1}_{\{d_u \leq t\}} X^{(3,u)}_{t-d_u},$$

where, given Z, independently for each $u \in \mathscr{T}$ such that $d_u < t$, the processes $X^{(3,u)}$ are independent copies of the canonical process X issued at time d_u with *law* $\mathbb{P}^*_{Y_u\delta_{z_u(d_u)}}$ *such that, given* u *has* $n\geq 2$ *offspring, the independent random variable* Y_u *has distribution* $\eta_n(z_u(d_u),\mathrm{d}y)$*, where* $\eta_n(x,\mathrm{d}y)$ *is defined by (27).*

Then $(\Delta,\mathbf{P}_\mu)$ *is Markovian and has the same law as* $(X,\mathbb{P}_\mu)$*.*

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