New families of subordinators with explicit transition probability semigroup

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Abstract

There exist only a few known examples of subordinators for which the transition probability density can be computed explicitly along side an expression for its Lévy measure and Laplace exponent. Such examples are useful in several areas of applied probability, for example, they are used in mathematical finance for modeling stochastic time change, they appear in combinatorial probability to construct sampling formulae, which in turn is related to a variety of issues in the theory of coalescence models, moreover, they have also been extensively used in the potential analysis of subordinated Brownian motion in dimension $d \geq 2$. In this paper, we show that Kendall's classic identity for spectrally negative Lévy processes can be used to construct new families of subordinators with explicit transition probability semigroups. We describe the properties of these new subordinators and emphasise some interesting connections with explicit and previously unknown Laplace transform identities and with complete monotonicity properties of certain special functions.

Keywords: subordinator, Kendall identity, explicit transition density, Laplace transform identity, Bessel functions, Lambert W-function, Gamma function, complete monotonicity, generalized gamma convolutions

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1 Introduction

Subordinators with explicit transition semigroups have proved to be objects of broad interest on account of their application in a variety of different fields. We highlight three of them here. The first case of interest occurs in mathematical finance, where subordinators are used to perform time-changes of other stochastic processes to model the effect of stochastic volatility in asset prices, see for example [5] and [7]. A second application occurs in the theory of potential analysis of subordinated Brownian motion in high dimensions, which has undergone significant improvements thanks to the study of a number of key examples, see for example [23] and [14]. A third area in which analytic detail of the transition semigroup of a subordinator can lead to new innovations is that of combinatorial stochastic processes. A variety of sampling identities are intimately related to the range of particular subordinators, see for example [10]. Moreover this can also play an important role in the analysis of certain coalescent processes, see [19].

In this paper we will use a simple idea based on Kendall's identity for spectrally negative Lévy processes to construct some new families of subordinators with explicit transition semigroup. Moreover, we describe their properties, with particular focus on the associated Lévy measure and Laplace exponent in each of our new examples. The inspiration for the main idea in this paper came about by digging deeper into [4], where a remarkable identity appears in the analysis of the relationship between the first passage time of a random walk and the total progeny of a discrete-time, continuous-state branching process.

The rest of the paper is organised as follows. In the next section we remind the reader of Kendall's identity and thereafter, proceed to our main results. These results give a simple method for generating examples of subordinators with explicit transition semigroups as well as simultaneously gaining access to analytic features of their Lévy measure and Laplace exponent. In Section 3 we put our main results to use in generating completely new examples. Finally, in Section 4 we present some applications of these results to explicit Laplace transform identities and complete monotonicity properties of certain special functions.

2 Kendall's identity and main results

Let ξ be a spectrally negative Lévy process with Laplace exponent defined by

$$\psi(z) = \ln \mathbb{E}[\exp(z\xi_1)], \qquad z \geq 0. \tag{1}$$

In general, the exponent ψ takes the form

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{(-\infty,0)} (e^{zx} - 1 - zx\mathbf{1}_{(x>-1)})\Pi_\xi(\mathrm{d}x)$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π_ξ is a measure concentrated on $(-\infty, 0)$ that satisfies $\int_{(-\infty,0)}(1 \wedge x^2)\Pi_\xi(\mathrm{d}x) < \infty$, and is called the Lévy measure. From this definition, it is easy to deduce that ψ is convex on $[0,\infty)$, and it satisfies $\psi(0) = 0$ and $\psi(+\infty) = +\infty$. Hence, for every $q > 0$, there exists a unique solution $z = \phi(q) \in (0,\infty)$ to the equation $\psi(z) = q$. We will define $\phi(0) = \phi(0^+)$. Note that $\phi(0) = 0$ if and only if $\psi'(0) \geq 0$, which, by a simple differentiation of (1), is equivalent to $\mathbb{E}[\xi_1] \geq 0$.

Let us define the first passage times

$$\tau_x^+ := \inf\{t > 0 \,:\, \xi_t > x\}, \qquad x \geq 0. \tag{2}$$

It is well-known (see Theorem 3.12 and Corollary 3.13 in [16]) that $\{\tau_x^+\}_{x\geq 0}$ is a subordinator, killed at rate $\phi(0)$, whose Laplace exponent, $\phi(q)$, satisfies

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_x^+}\mathbf{1}_{\{\tau_x^+<+\infty\}}\right] = \mathrm{e}^{-x\phi(q)}, \qquad q \geq 0.$$

In general, the Laplace exponent ϕ is a Bernstein function. In particular, it takes the form

$$\phi(z) = \kappa + \delta z + \int_{(0,\infty)} (1 - \mathrm{e}^{-zx})\Pi(\mathrm{d}x), \tag{3}$$

for some $\kappa, \delta \geq 0$ and measure, Π, concentrated on $(0, \infty)$, satisfying $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty$. The constant κ is called the killing rate and δ is called the drift coefficient.

Kendall's identity (see [3] and Exercise 6.10 in [16]) states that

$$\int_y^\infty \mathbb{P}(\tau_x^+ \leq t)\frac{\mathrm{d}x}{x} = \int_0^t \mathbb{P}(\xi_s > y)\frac{\mathrm{d}s}{s}. \tag{4}$$

If the distribution of ξ_t is absolutely continuous for all $t > 0$ then the measure $\mathbb{P}(\tau_x^+ \in \mathrm{d}t)$ is also absolutely continuous and has the density

$$\mathbb{P}(\tau_x^+ \in \mathrm{d}t) = \frac{x}{t}p_\xi(t, x)\mathrm{d}t, \qquad x, t > 0, \tag{5}$$

where $p_\xi(t, x)\mathrm{d}x = \mathbb{P}(\xi_t \in \mathrm{d}x)$. On the one hand, one may view Kendall's identity as an analytical consequence follows from the Wiener-Hopf factorisation for spectrally negative Lévy processes. On the other, its probabilistic roots are related to certain combinatorial arguments associated to random walks in the spirit of the classical ballot problem.

Kendall's identity gives a very simple way of constructing new subordinators with explicit transition semigroup. Indeed, if we start with a spectrally negative process ξ for which the transition probability density $p_\xi(t, x)$ is known, then τ_x^+ is the desired subordinator with the explicit transition density given by (5). One way to build a spectrally negative process with known transition density (as indeed we shall do below) is as follows: start with a subordinator X, which has an explicit transition probability density and then define the spectrally negative process $\xi_t = t - X_t$. This also gives us a spectrally negative process with explicit transition probability density.

Before stating our main theorem, let us introduce some notation and definitions. We write $\mathcal{N}$ for the class of all subordinators, started from zero, having zero drift and zero killing rate. The Laplace exponent of a subordinator $Y \in \mathcal{N}$ is defined by $\Phi_Y(z) := -\ln \mathbb{E}\left[\exp(-zY_1)\right]$, $z \geq 0$. From the Lévy-Khinchine formula we know that

$$\Phi_Y(z) = \int_{(0,\infty)} \left(1 - \mathrm{e}^{-zx}\right)\Pi_Y(\mathrm{d}x), \quad z \geq 0, \tag{6}$$

where Π_Y is the Levy measure of Y. When it exists, we will denote the transition probability density function of Y as $p_Y(t, x) := \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(Y_t \leq x)$, $x > 0$.

Theorem 1. *For $X \in \mathcal{N}$ and $q > 0$, define $\phi(q)$ as the unique solution to*

$$z - \Phi_X(z) = q. \tag{7}$$

Define $\phi(0) = \phi(0+)$. Then we have the following:

(i) The function $\Phi_Y(z) := \phi(z) - \phi(0) - z$ is the Laplace exponent of a subordinator $Y \in \mathcal{N}$.

(ii) If the transition semi-group of X is absolutely continuous with respect to Lebesgue measure, then the transition semi-group of Y is given by

$$p_Y(t,y) = \frac{t}{t+y} \mathrm{e}^{\phi(0)t} p_X(t+y, y), \quad y > 0. \tag{8}$$

and the Levy measure of Y is given by

$$\Pi_Y(\mathrm{d}y) = \frac{1}{y} p_X(y,y)\mathrm{d}y, \quad y > 0. \tag{9}$$

Proof. The function $\phi(q)$ defines the Laplace exponent of the subordinator corresponding to the first passage process (2). Moreover, appealing to the standard facts that the drift coefficient of ϕ is equal to $\lim_{q\to\infty} \phi(q)/q$ and that $\phi(\infty) = \infty$, as well as the fact that X has zero drift, one notes that

$$\lim_{q\to\infty} \frac{\phi(q)}{q} = \lim_{q\to\infty} \frac{\phi(q)}{\phi(q) - \Phi_X(\phi(q))} = \lim_{q\to\infty} \frac{1}{1 - \Phi(\phi(q))/\phi(q)} = 1.$$

Moreover, noting that $\phi(0)$ is another way of writing the killing rate of the subordinator corresponding to ϕ, it follows that the function $\phi_Y(z) = \phi(z) - \phi(0) - z$ belongs to the class $\mathcal{N}$. Formula (8) follows at once from Kendall's identity as it appears in (5). The formula (9) follows from the fact that

$$\Pi_X(\mathrm{d}x) = \lim_{t\to 0^+} \frac{1}{t} \mathbb{P}(X_t \in \mathrm{d}x), \qquad x > 0, \tag{10}$$

see for example the proof of Theorem 1.2 in [1]. □

In constructing new subordinators, the above theorem has deliberately eliminated certain scaling parameters. For example, one may consider working more generally with the spectrally negative process $\xi_t = \lambda t - X_t$, $t \geq 0$ for some $\lambda > 0$. However, this can be reduced to the case that $\lambda = 1$ by factoring out the constant λ from ξ and noting that $\lambda^{-1}X$ is still a subordinator.

Theorem 1 raises the following natural question concerning its iterated use. Suppose we have started from a spectrally negative process, say $\xi^{(1)}$ and have constructed a corresponding subordinator $Y^{(1)}$. Can we take this subordinator, define a new, spectrally negative Lévy process $\xi_t^{(2)} := t - Y_t^{(1)}$, $t \geq 0$, and feed it into back into Theorem 1 to obtain a new subordinator $Y^{(2)}$? The answer is essentially "no": one can check that the subordinator $Y^{(2)}$ could also be obtained by one application of this procedure starting from the scaled process $\theta\xi_{ct}$ for appropriate constants $\theta, c > 0$. In other words, applying the Kendall identity trick twice does not give us fundamentally new processes.

Recent potential analysis of subordinators has showed particular interest in the case of *complete subordinators*, following their introduction in [22]. The class of complete subordinators can be defined by the analytical structure of their Laplace exponents. In particular, Φ_X is the Laplace exponent of a complete subordinator X if and only if it takes the form (3) such that the Lévy measure is absolutely continuous with respect to Lebesgue measure with a completely monotone density. Our next main result below investigates sufficient conditions on ξ to ensure that the resulting subordinator, Y, generated by the procedure in Theorem 1, is a complete subordinator.

Theorem 2. *Let ξ be a spectrally negative process with a Lévy density $\pi_\xi(x)$, $x<0$. Let $\pi_Y(y)$ be the Lévy density of $Y_t = \tau_t^+$. If $\pi_\xi(-x)$ is a completely monotone function, then the same is true for $\pi_Y(x)$.*

The proof involves various technical notions of complete Bernstein functions, and requires an additional lemma. Let us denote $\mathbb{C}^+ := \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$ and similarly $\mathbb{C}^- := \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) < 0\}$. In addition to the representation in (3), a function f on $(0,\infty)$ is a *complete Bernstein function* (CBF in short) if

$$f(z) = c_0 + c_1 z + \frac{1}{\pi}\int_{(0,\infty)} \frac{z}{z+s}\,\frac{m(\mathrm{d}s)}{s}$$

for some $c_0, c_1 \geq 0$ and a σ-finite positive measure m on $(0,\infty)$ satisfying $\int_{(0,\infty)} \min(s^{-1}, s^{-2}) m(\mathrm{d}s) < \infty$. Equivalently, f is the Laplace exponent of a (possibly killed) subordinator X, whose Lévy measure has a completely monotone density

$$\pi_X(x) = \int_0^\infty e^{-xs} m(\mathrm{d}s).$$

It is known (see Theorem 6.2 in [21]) that CBFs extend to analytic functions that map $\mathbb{C} \setminus (-\infty, 0]$ into $\mathbb{C} \setminus (-\infty, 0]$ and belong to the class of *Pick functions*, that is functions g analytic in $\mathbb{C}^+$ such that if $g(z) \in \overline{\mathbb{C}^+}$ for all $z \in \mathbb{C}^+$. Conversely, any Pick function which takes real values on $(0,\infty)$ (and for which $\lim_{(0,\infty)\ni x \to 0} f(x)$ exists and is real) is a CBF. For more information on CBFs see [21].

Lemma 1. *If Φ is a CBF and ϕ is the inverse function of $z\Phi(z)$, then ϕ is also a CBF.*

Proof. For $a > 0$ we will define S_a to be the horizontal strip $\{z \in \mathbb{C} \ : \ |\mathrm{Im}(z)| < a\}$ and for $\varepsilon \in (0,\pi)$ we will denote

$$C_\varepsilon := \{z \in \mathbb{C} \ : \ |\mathrm{Re}(z)| < \tfrac{1}{\varepsilon}, \ |\mathrm{Im}(z)| < \pi - \varepsilon\}.$$

Since Φ is a CBF, it can be extended to an analytic function in $\mathbb{C}\setminus(-\infty,0]$. We define $\Psi(z) = \ln(e^z\Phi(e^z))$ for $z \in S_\pi$ (here $\ln(\cdot)$ denotes the principal branch of the logarithm).

We list the properties of Ψ. Clearly, Ψ maps $\mathbb{R}$ into $\mathbb{R}$, $\Psi'(z) = 1 + \mathrm{e}^z\Phi'(\mathrm{e}^z)/\Phi(\mathrm{e}^z) > 0$ for $z \in \mathbb{R}$, $\Psi(\bar{z}) = \overline{\Psi(z)}$ and $\Psi(-\infty) = -\infty$, $\Psi(\infty) = \infty$. Hence, the inverse function Ψ^{-1} is well-defined on $\mathbb{R}$ and it extends to an analytic function in a neighbourhood of $\mathbb{R}$. We will also need the following two properties of Ψ: for all $z \in S_\pi$

$$|\mathrm{Im}(\Psi(z))| \geq |\mathrm{Im}(z)|, \tag{11}$$

and for all $\epsilon \in (0,\pi)$ and $z \in S_{\pi-\varepsilon}$

$$|\mathrm{Re}(\Psi(z)) - \Psi(\mathrm{Re}\, z)| \leq -\ln\left(\sin\left(\tfrac{\varepsilon}{2}\right)\right). \tag{12}$$

To prove (11), we assume that $\mathrm{Im}\, z \in (0,\pi)$, then $\mathrm{e}^z \in \mathbb{C}^+$, therefore $\Phi(\mathrm{e}^z) \in \mathbb{C}^+$, which implies that $\arg\Phi(\mathrm{e}^z) \in (0,\pi)$ and

$$\mathrm{Im}(\Psi(z)) = \mathrm{Im}\left(\ln(\mathrm{e}^z\Phi(\mathrm{e}^z))\right) = \arg \mathrm{e}^z + \arg\Phi(\mathrm{e}^z) = \mathrm{Im}(z) + \arg\Phi(\mathrm{e}^z) \geq \mathrm{Im}(z).$$

Inequality (11) follows from the above result and the fact that $\Psi(\bar{z}) = \overline{\Psi(z)}$.

In order to prove (12) we will require the following property: If f is a CBF, $\varepsilon \in (0,\pi)$ and $|\arg(z)| \leq \pi - \varepsilon$, then

$$\sin(\tfrac{\varepsilon}{2}) f(|z|) \leq |f(z)| \leq \frac{f(|z|)}{\sin(\frac{\varepsilon}{2})}; \tag{13}$$

the upper bound is proved in [15, Prop. 2.21(c)], and the lower bound follows from the upper one applied to the CBF $z/f(z)$. If $z \in S_{\pi-\varepsilon}$ then $|\arg(\mathrm{e}^z)| \le \pi - \varepsilon$, and by (13),

$$\begin{aligned}\mathrm{Re}(\Psi(z)) &= \mathrm{Re}\left(\ln(\mathrm{e}^z\Phi(\mathrm{e}^z))\right) = \ln|\mathrm{e}^z\Phi(\mathrm{e}^z)| = \ln|\mathrm{e}^z| + \ln|\Phi(\mathrm{e}^z)| \le \ln|\mathrm{e}^z| + \ln\Phi(|\mathrm{e}^z|) - \ln\sin\tfrac{\varepsilon}{2} \\ &= \ln(\mathrm{e}^{\mathrm{Re}(z)}) + \log\Phi(\mathrm{e}^{\mathrm{Re}(z)}) - \ln\left(\sin\left(\tfrac{\varepsilon}{2}\right)\right) = \Psi(\mathrm{Re}(z)) - \ln\left(\sin\left(\tfrac{\varepsilon}{2}\right)\right),\end{aligned}$$

and in a similar manner we would obtain

$$\mathrm{Re}(\Psi(z)) \ge \Psi(\mathrm{Re}(z)) + \ln\left(\sin\left(\tfrac{\varepsilon}{2}\right)\right).$$

Due to properties (11) and (12) and the fact that $1/(2\varepsilon) > -\ln\left(\sin\left(\frac{\varepsilon}{2}\right)\right)$ for ε small enough we see that

$$C_{2\varepsilon} \subset \Psi(C_\varepsilon). \tag{14}$$

Letting $\varepsilon \to 0^+$ we conclude that the strip S_π is a subset of the image of this same strip under the analytic map Ψ. In other words, for every $q \in S_\pi$ there exists $z \in S_\pi$ such that $\Psi(z) = q$.

Let us now take ε to be a small positive number and consider the open set $\Psi^{-1}(C_\varepsilon)$. We define D_ε to be the connected component of $\Psi^{-1}(C_\varepsilon)$ which intersects $\mathbb{R}$. This component exists and is unique due to the fact that $\Psi(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$ (see (11)). By definition, Ψ maps D_ε onto C_ε. Formula (14) implies that

$$D_\varepsilon \subseteq \Psi^{-1}(C_\varepsilon) \subset C_{\varepsilon/2} \subset S_\pi.$$

Thus Ψ is an analytic function on a neighbourhood of $\overline{D}_\varepsilon$, and therefore Ψ maps ∂D_ε onto ∂C_ε.

For $q \in C_\varepsilon$, let $n(q)$ be the number of solutions of the equation $\Psi(z) = q$, $z \in D_\varepsilon$, counting multiplicity. According to the Cauchy argument principle, $n(q)$ is equal to the winding number of $\Psi(z) - q$ as z goes around the contour ∂D_ε. Since Ψ maps D_ε onto C_ε (and ∂D_ε onto ∂C_ε), this winding number does not depend on q. Since $n(q) = 1$ for any real $q \in C_\varepsilon$, we conclude that Ψ is a univalent map from D_ε onto C_ε. In particular, Ψ^{-1} extends to an analytic function on C_ε, with values in D_ε.

By taking $\varepsilon \to 0^+$, Ψ^{-1} extends to an analytic function $\Psi^{-1} : S_\pi \mapsto S_\pi$. Due to (11) it is true that if $0 < \mathrm{Im}(q) < \pi$ then $0 < \mathrm{Im}(\Psi^{-1}(q)) \le \mathrm{Im}(q) < \pi$

Observe that $\phi(q) := \exp(\Psi^{-1}(\ln(q)))$ (where $q \in (0,\infty)$) is the inverse function of $\exp(\Psi(\ln(z))) = z\Phi(z)$ (where $z \in (0,\infty)$). Furthermore, $\exp(\Psi^{-1}(\ln(q)))$ extends to an analytic function in $\mathbb{C} \setminus (-\infty, 0]$. Assuming that $\mathrm{Im}(q) > 0$ (which is equivalent to $0 < \mathrm{Im}(\ln(q)) < \pi$) we see that $0 \le \mathrm{Im}\,\Psi^{-1}(\log q) \le \pi$, therefore $\mathrm{Im}\exp(\Psi^{-1}(\ln q)) \ge 0$. Therefore, $\phi(z)$ is a CBF. □

Proof of Theorem 2. Let H denote the descending ladder height process for ξ (which is a subordinator, possibly a killed one), and let Φ_H be its Laplace exponent. Then $\psi(z) = (z - c)\Phi_H(z)$, where $c = \phi(0)$ (see formula (9.1) in [16]). Theorem 2 in [20] tells us that if $\pi_\xi(-x)$ is completely monotone, then $\pi_H(x)$ is completely monotone, and therefore Φ_H is a CBF. Let $\tilde{\Phi}_H(z) = \Phi_H(z + c)$ and $\tilde{\psi}(z) = \psi(z + c)$, so that $\tilde{\psi}(z) = z\tilde{\Phi}_H(z)$. Note that $z = \phi(q)$ if and only if $\psi(z) = q$, that is, $\tilde{\psi}(z - c) = q$. Therefore, $\phi(q) = \tilde{\psi}^{-1}(q) + c$. Since $\tilde{\Phi}_H(z)$ is a CBF, by Lemma 1, $\tilde{\psi}^{-1}$ is a CBF. It follows that also ϕ is a CBF, and therefore $\pi_Y(x)$ is completely monotone. □

One corollary of Theorem 2 is that the transformation described in Theorem 1, which maps a subordinator X into a subordinator $Y_t = \tau_t^+$, preserves the class of complete subordinators. As our next result shows, this transformation also preserves an important subclass of complete subordinators. We define the class of *Generalized Gamma Convolutions* (GGC) as the family of infinite divisible distributions on $(0, \infty)$ having Lévy density $\pi(x)$, such that the function $x\pi(x)$ is completely monotone. In other words,

$$x\pi(x) = \int_0^\infty e^{-xy} U(\mathrm{d}y),$$

for some σ-finite and positive measure U, which is called *Thorin measure*. The measure U must satisfy the following integrability condition

$$\int_0^\infty \left(|\ln(y)| \wedge \tfrac{1}{y} \right) U(\mathrm{d}y) < \infty$$

in order for $\Pi(\mathrm{d}x) = \pi(x)\mathrm{d}x$ to be a Lévy measure of a positive random variable. The class of GGC can also be defined as the smallest class of distributions on $(0, \infty)$, which contains all gamma distributions and which is closed under convolution and weak convergence. See [2] for additional information on the class of GGC and its distributional properties. We say that a subordinator X belongs to the Thorin class $\mathcal{T}$ if the distribution of X_1 is GGC. The family $\mathcal{T}_0$ is defined as the subclass of all subordinators in $\mathcal{T}$ which have zero linear drift.

Proposition 1. *Assume that $X \in \mathcal{T}_0$ and Y is a subordinator constructed in Theorem 1. Then $Y \in \mathcal{T}_0$, in particular the function $y\pi_Y(y) = p_X(y, y)$ is completely monotone.*

Proof. We will need the following result (see Theorem 3.1.2 in [2]): *Let η be a positive random variable and define $f(z) := \ln \mathbb{E}\left[e^{-z\eta}\right]$. Then η has a GGC distribution if and only if $f'(z)$ is a Pick function.*

Assume that $X \in \mathcal{T}_0$. According to the above result, $-\Phi_X'(z)$ is a Pick function. Let Y be a subordinator constructed from X in Theorem 1. We recall that $\phi(q)$ is defined as the solution to $z - \Phi_X(z) = q$ and $\Phi_Y(z) = \phi(z) - \phi(0) - z$. Since $X \in \mathcal{T}_0$, it has a completely monotone Lévy density, thus according to Theorem 2, the same is true for Y. Therefore, the three functions $\Phi_X(z)$, $\Phi_Y(z)$ and $\phi(z)$ are Pick functions. Taking derivative of the identity $\phi(q) - \Phi_X(\phi(q)) = q$ we find that

$$-\phi'(q) = -\frac{1}{1 - \Phi_X'(\phi(q))}.$$

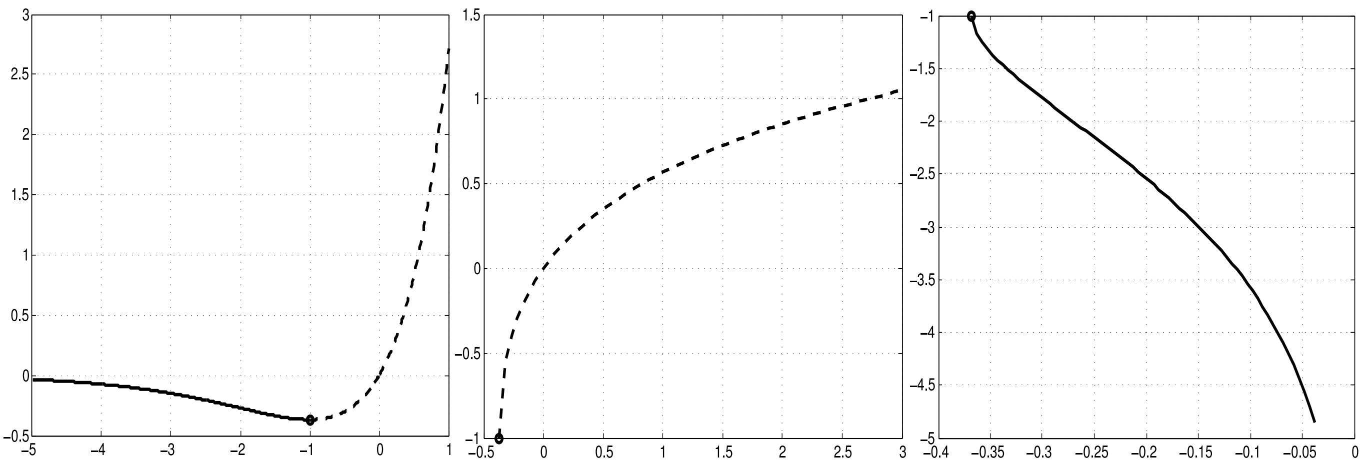
Since the composition of Pick functions is also a Pick function, and since the three functions

$$F : q \mapsto \phi(q), \quad G : z \mapsto -\Phi_X'(z), \quad H : w \mapsto -\frac{1}{1+w}$$

are Pick functions, we conclude that $-\phi'(q) = H(G(F(q)))$ is also a Pick function. Therefore, $-\Phi_Y'(q) = -\phi'(q) + 1$ is a Pick function, which implies $Y \in \mathcal{T}_0$. $\square$

3 Examples

In this section we present several new families of subordinators possessing explicit transition semigroups. Our first two examples are related to the Lambert W-function [8, 9], and we will start by reviewing some



(a) The function $z = we^w$ (b) $W_0(z)$: the solution to $we^w = z$ (c) $W_{-1}(z)$: the solution to $we^w = z$

Figure 1: The two real branches of the Lambert W-function: $W_0(z)$ is an increasing function which maps $[-1/e, \infty)$ onto $[-1, \infty)$, and $W_{-1}(z)$ is a decreasing function which maps $[-1/e, 0)$ onto $(-\infty, -1]$.

of its properties. Lambert W-function is defined as the inverse to the function $w \in \mathbb{C} \mapsto w\mathrm{e}^w$. When $z \neq 0$, the equation $w\mathrm{e}^w = z$ has infinitely many solutions, therefore we will have infinitely many branches of the Lambert W-function, which we will label by $W_k(z)$. See [8] for detailed discussion of branches of Lambert W-function. We will be only interested in two real branches of the Lambert W-function, $W_0(z)$ (the principal branch) and $W_{-1}(z)$. For $z > -1/e$, these are defined as the *real* solutions to $w\mathrm{e}^w = z$. It is easy to show that the function $w\mathrm{e}^w$ is increasing for $w > -1$ and decreasing for $w < -1$, see figure 1. Therefore, for $z \geq 0$ there is a unique real solution, corresponding to $W_0(z)$, while for $-1/e < z < 0$ there exist two real solutions $W_{-1}(z) < -1 < W_0(z) < 0$. The graphs of the two functions $W_0(z)$ and $W_{-1}(z)$ are presented on figures 1b and 1c. The function $W_0(z)$ is the principal branch of the Lambert W-function, and it has received considerably more attention compared to its other sibling, $W_{-1}(z)$. In many ways it is a simpler function, for example it one of the classical examples when Lagrange inversion formula gives a very simple and explicit Taylor series at $z = 0$ (see formula (3.1) in [8]).

$$W_0(z) = \sum_{n\geq 1} (-n)^{n-1} \frac{z^n}{n!}, \quad |z| < 1/e. \tag{15}$$

3.1 Poisson process

In this section we construct a subordinator starting from the spectrally negative process $\xi_t = t - N_{ct}$, where N is the standard Poisson process (i.e. with unit rate of arrival).

Proposition 2. *For $c > 0$ the function $\Phi_Y(z) = W_0\left(-c\mathrm{e}^{-c-z}\right) - W_0\left(-c\mathrm{e}^{-c}\right)$ is the Laplace exponent of a compound Poisson process. The distribution of Y_t is supported on $\{0, 1, 2, \cdots\}$ and is given by*

$$\mathbb{P}(Y_t = n) = ct \frac{(c(n+t))^{n-1}}{n!} \mathrm{e}^{-c(n+t)+at}, \quad n \geq 0, \tag{16}$$

where $a := 0$ if $c \leq 1$ and $a := c + W_0\left(-c\mathrm{e}^{-c}\right)$ if $c > 1$. The Lévy measure is given by

$$\Pi_Y(\{n\}) = \frac{n^{n-1}}{n!} c^n \mathrm{e}^{-cn}, \quad n \geq 1. \tag{17}$$

Proof. Consider the spectrally negative Lévy process $\xi_t = t - N_{ct}$, where N is the standard Poisson process. Our goal is to compute the Laplace exponent, transition semigroup and the Lévy measure of the subordinator $\{\tau_x^+\}_{x\geq 0}$. On account of the fact that the paths of ξ are piecewise linear, it is easy to see that $\{\tau_x^+\}_{x\geq 0}$ is necessarily a compound Poisson process. Moreover, as noted in the proof of Theorem 1, this subordinator must also have unit drift. Its jump size distribution must also supported on positive integers. This is intuitively clear on account of the fact that if exactly n jumps occur during an excursion of ξ from its maximum, then, since each jump is of unit size and ξ has a unit upward drift, then it requires precisely n units of time to return to the maximum. This is also clear from the analytical relation (9).

In order to find the Laplace exponent $\phi(q)$ we need to solve the following equation

$$z - c(1 - \mathrm{e}^{-z}) = q.$$

Changing variables $w = z - c - q$ we rewrite the above equation as

$$\mathrm{e}^w w = -c\mathrm{e}^{-c-q},$$

which gives us

$$z = \phi(q) = W\left(-c\mathrm{e}^{-c-q}\right) + c + q,$$

where W is one of the two real branches of the Lambert W-function. We need to choose the correct branch of the Lambert W-function. Since $\phi(q) - q - \phi(0)$ and hence $\phi(q) - q$ is the Laplace exponent of a subordinator, it must be increasing in q. Since $W_0(z)$ is increasing while $W_{-1}(z)$ is decreasing, this shows that the correct branch is $W = W_0$. Therefore we conclude

$$\phi(q) = W_0\left(-c\mathrm{e}^{-c-q}\right) + c + q, \qquad q \geq 0. \tag{18}$$

Note that $\{\tau_x^+\}_{x\geq 0}$ is killed at rate $\phi(0) = W_0\left(-c\mathrm{e}^{-c}\right) + c$ if $c > 1$, and, otherwise, at rate $\phi(0) = 0$ if $c \leq 1$.

Next, let us find the transition semi-group of $\{\tau_x^+\}_{x\geq 0}$. As we have discussed above, $\{\tau_x^+\}_{x\geq 0}$ has unit drift and its jump distribution is concentrated on the positive integers. This implies that the distribution of τ_x^+ is supported on $\{x, x+1, x+2, \cdots\}$. Let us define $p_n(x) = \mathbb{P}(\tau_x^+ = n + x)$. Then we find, for $t, y > 0$,

$$\int_y^\infty \mathbb{P}(\tau_x^+ \leq t)\frac{\mathrm{d}x}{x} = \int_y^\infty \sum_{n\geq 0} \mathbf{1}_{\{n+x\leq t\}} p_n(x)\frac{\mathrm{d}x}{x} = \sum_{0\leq n\leq t-y} \int_y^{t-n} p_n(x)\frac{\mathrm{d}x}{x}.$$

At the same time,

$$\begin{aligned}
\int_0^t \mathbb{P}(\xi_s > y)\frac{\mathrm{d}s}{s} &= \int_0^t \mathbb{P}(N_{cs} < s - y)\frac{\mathrm{d}s}{s} \\
&= \int_0^t \sum_{n\geq 0} \mathbf{1}_{\{n<s-y\}} \frac{(cs)^n}{n!}\mathrm{e}^{-cs}\frac{\mathrm{d}s}{s} \\
&= \sum_{0\leq n<t-y} \int_{n+y}^t \frac{(cs)^n}{n!}\mathrm{e}^{-cs}\frac{\mathrm{d}s}{s} \\
&= \sum_{0\leq n<t-y} \int_y^{t-n} cs\frac{(c(s+n))^{n-1}}{n!}\mathrm{e}^{-c(s+n)}\frac{\mathrm{d}s}{s}.
\end{aligned}$$

The above two equations combined with Kendall's identity (4) give us

$$\mathbb{P}(\tau_x^+ = n + x) = cx\frac{(c(n+x))^{n-1}}{n!}\mathrm{e}^{-c(n+x)}, \quad n \geq 0. \tag{19}$$

Now we define the subordinator Y, with zero drift coefficient and zero killing rate, via the Laplace exponent $\Phi_Y(z) = \phi(z) - z - \phi(0)$. The formula for the transition semigroup (16) follows from (19). $\square$

When $c \in (0,1)$, the distribution given in (16) was introduced in 1973 by Consul and Jain [6], who called it the generalized Poisson distribution. It is also comes under the name of Borel distribution. Note that this distribution changes behavior at $c = 1$. Using Stirling's approximation for $n!$ we find that

$$\Pi_Y(\{n\}) = \frac{1}{\sqrt{2\pi}}n^{-\frac{3}{2}}\mathrm{e}^{-(c-1-\ln(c))n}\left(1 + o(1)\right), \quad n \to +\infty,$$

therefore the jump distribution of Y has exponential tail when $c \neq 1$ and a power-law tail (with $\mathbb{E}[Y_1] = +\infty$) for $c = 1$.

3.2 Gamma process

In this section we construct a subordinator using Theorem 1 by starting from a gamma subordinator. We recall that a gamma subordinator X is defined by the Laplace exponent $\Phi_X(z) = c\ln(1+\theta z)$, $z \geq 0$, where the constants $c, \theta > 0$. It is well-known that X has zero drift and that the transition probability density and the density of the Lévy measure are given by

$$p_X(t,x) = \frac{x^{ct-1}\mathrm{e}^{-\frac{x}{\theta}}}{\theta^{ct}\Gamma(ct)}, \quad \pi_X(x) = \frac{c}{x}\mathrm{e}^{-\frac{x}{\theta}}, \qquad x, t > 0.$$

Proposition 3. *The function*

$$\Phi_Y(z) := -cW_{-1}\left(-\frac{1}{\theta c}\exp\left(-\frac{1+\theta z}{\theta c}\right)\right) + cW_{-1}\left(-\frac{1}{\theta c}\exp\left(-\frac{1}{\theta c}\right)\right) - z, \qquad z \geq 0, \tag{20}$$

is the Laplace exponent of a subordinator $Y \in \mathcal{T}_0$. The transition probability density of Y is

$$p_Y(t,y) = \frac{c\theta^{-1}t}{\Gamma(1+c(t+y))}\left(\frac{y}{\theta}\right)^{c(t+y)-1}\mathrm{e}^{-\frac{y}{\theta}+at}, \qquad y, t > 0,$$

where $a := 0$ if $\theta c \leq 1$ and $a := -1/\theta - cW_{-1}\left(-\frac{1}{\theta c}\mathrm{e}^{-\frac{1}{\theta c}}\right)$ if $\theta c > 1$. The density of the Lévy measure is given by

$$\pi_Y(y) = \frac{c\theta^{-1}}{\Gamma(1+cy)}\left(\frac{y}{\theta}\right)^{cy-1}\mathrm{e}^{-\frac{y}{\theta}}, \qquad y > 0.$$

Proof. This result is a straightforward application of Theorems 1 and 2, we only need to identify the function $\phi(q)$, which is the solution to $z - c\ln(1+\theta z) = q$. Making change of variables $u = -1/(\theta c) - z/c$ we can rewrite this equation as

$$u\mathrm{e}^u = -\frac{1}{\theta c}\mathrm{e}^{-\frac{1}{\theta c}-\frac{q}{c}},$$

therefore

$$u = W\left(-\frac{1}{\theta c}\mathrm{e}^{-\frac{1}{\theta c}-\frac{q}{c}}\right),$$

where W is one of the two real branches of the Lambert W-function. Again, we need to choose the correct branch, W_0 or W_{-1}. Let us consider

$$\phi(q) = -\frac{1}{\theta} - cu = -\frac{1}{\theta} - cW\left(-\frac{1}{\theta c}\mathrm{e}^{-\frac{1}{\theta c}-\frac{q}{c}}\right).$$

We know that $\phi(q)$ is the Laplace exponent of a subordinator with drift rate equal to one, therefore $\phi(q)$ is unbounded on $q \in (0, \infty)$. From the properties of W_0 and W_{-1} (see figure 1) this is only possible if we choose the branch $W = W_{-1}$. Thus we obtain

$$\phi(q) = -\frac{1}{\theta} - cu = -\frac{1}{\theta} - cW_{-1}\left(-\frac{1}{\theta c}\mathrm{e}^{-\frac{1}{\theta c}-\frac{q}{c}}\right).$$

Note that $\phi(0) = 0$ if and only if $\theta c \le 1$. The rest of the proof follows from Theorem 1 and from Proposition 1. □

Using Stirling's approximation for the Gamma function we find that

$$\pi_Y(y) = \sqrt{\frac{c}{2\pi}}y^{-\frac{3}{2}}\mathrm{e}^{-(\ln(\theta c)-1+\frac{1}{\theta c})cy}\left(1 + o(1)\right), \quad y \to +\infty,$$

therefore the Lévy density of Y has exponential tail when $\theta c \neq 1$ and a power-law tail (with $\mathbb{E}[Y_1] = +\infty$) for $\theta c = 1$.

3.3 Stable processes

In this section, we obtain new families of subordinators which are related to stable processes. We define

$$g(x;\alpha) := \frac{1}{\pi}\sum_{n\ge 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(\pi n\alpha)x^{-n\alpha-1}, \quad x > 0,\ 0 < \alpha < 1, \tag{21}$$

and

$$g(x;\alpha) := \frac{1}{\pi}\sum_{n\ge 1}(-1)^{n-1}\frac{\Gamma\left(1+n/\alpha\right)}{n!}\sin\left(\tfrac{\pi n}{\alpha}\right)x^{n-1}, \quad x \in \mathbb{R},\ \alpha > 1. \tag{22}$$

Note that, for $\alpha > 1$, the function $x \mapsto g(x;\alpha)$ is entire and satisfies the identity

$$xg(x;\alpha) = x^{-\alpha}g(x^{-\alpha};\alpha^{-1}), \quad x > 0, \alpha > 1. \tag{23}$$

The function $g(x;\alpha)$ has the following probalistic interpretation: for $\alpha \in (0,1)$ {resp. $\alpha \in (1,2)$} it is the probability density function of a strictly stable random variable U defined by $\mathbb{E}[\exp(-zU)] = \exp(-z^\alpha)$ {resp. $\mathbb{E}[\exp(zU)] = \exp(z^\alpha)$}, see Theorem 2.4.2 in [24]. Identity (23) is just a special case of the so-called Zolotarev duality, see Theorem 2.3.2 in [24]. It is known that U has a GGC distribution, see example 3.2.1 in [2].

When α is a rational number, the function $g(x;\alpha)$ can be given in terms of hypergeometric functions, for example:

$$g(x;\tfrac{1}{3}) = \frac{x^{-\frac{3}{2}}}{3\pi} K_{\frac{1}{3}}\left(\frac{2}{3\sqrt{3x}}\right), \quad g(x;\tfrac{2}{3}) = \frac{x^{-1}}{\sqrt{3}\pi} \mathrm{e}^{-\frac{2}{27x^2}} W_{\frac{1}{2},\frac{1}{6}}\left(\frac{4}{27x^2}\right), \qquad x>0,$$

where $K_\nu(x)$ denotes the modified Bessel function of the second type and $W_{a,b}(x)$ denotes the Whittaker function (see [12]). The above two formulas can be found in [24] (see formulas 2.8.31 and 2.8.33).

Proposition 4. *Assume that $\alpha \in (0,1)$ and $c>0$. For $q \geq 0$ define $\phi(q)$, $q \geq 0$, as the unique positive solution to the equation $z - cz^\alpha = q$. Then the function $\Phi_Y(z) = \phi(z) - c^{\frac{1}{1-\alpha}} - z$ is the Laplace exponent of a subordinator $Y \in \mathcal{T}_0$. The transition probability density of the subordinator Y is given by*

$$p_Y(t,y) = t\exp\left(c^{\frac{1}{1-\alpha}}t\right)\frac{(c(t+y))^{-\frac{1}{\alpha}}}{t+y} g\left(y(c(t+y))^{-\frac{1}{\alpha}};\alpha\right) \qquad x,t>0. \tag{24}$$

The density of the Lévy measure is given by

$$\pi_Y(y) = c^{-\frac{1}{\alpha}} y^{-\frac{1}{\alpha}-1} g\left(c^{-\frac{1}{\alpha}} y^{1-\frac{1}{\alpha}};\alpha\right), \qquad y>0. \tag{25}$$

Proof. Let X be an α-stable subordinator, having Laplace exponent $\Phi_X(z) = cz^\alpha$. Due to the scaling property $a^{-\frac{1}{\alpha}}X_{at} \stackrel{d}{=} X_t$ we find that the density of X_t is given by $p_X(t,x) = g(x(ct)^{-\frac{1}{\alpha}};\alpha)(ct)^{-\frac{1}{\alpha}}$. The rest of the proof is a straightforward application of Theorem 1, Proposition 1 and the fact that $\phi(0) = c^{\frac{1}{1-\alpha}}$. □

Remark 1. We can also compute the mean of the subordinator Y, but without having to consider the tail of the measure π_Y as in the previous examples. Recall that $\phi(q)$ satisfies $\psi_\xi(\phi(q)) = q$, for $q \geq 0$. Differentiating, it follows that, for $q>0$, $\phi'(q)\psi'_\xi(\phi(q)) = 1$ and hence,

$$\mathbb{E}[Y_1] = \lim_{q\to 0}\phi'(q) - 1 = \frac{1}{\psi'_\xi(\phi(0))} - 1.$$

It follows that the subordinator Y has infinite mean if and only if $\psi'(\phi(0)) = 0$. This happens if and only if $\phi(0) = 0$ and $\psi'(0+) = 0$. When that $\psi_\xi(z) = z - \Phi_X(z)$, Y has infinite mean if and only if $\phi(0) = 0$ and $\Phi'_X(0) = 1$. One easily shows in this example that

$$\mathbb{E}[Y_1] = \frac{1}{1 - c\alpha(c^{\frac{1}{1-\alpha}})^{\alpha-1}} - 1 = \frac{\alpha}{1-\alpha}.$$

In the next proposition, we use Theorem 1 in combination with a choice of ξ which is not the difference of a unit drift and a subordinator (and therefore a process of bounded variation). Instead we choose ξ directly to be a spectrally negative stable process with unbounded variation added to a unit positive drift.

Proposition 5. *Assume that $\alpha \in (1,2)$ and $c>0$. For $q \geq 0$ define $\Phi_Y(z)$ as the unique positive solution to the equation $z + cz^\alpha = q$. Then $\Phi_Y(z)$ is the Laplace exponent of an infinite mean subordinator $Y \in \mathcal{T}_0$. The transition probability density of the subordinator Y is given by*

$$p_Y(t,y) = c^{-\frac{1}{\alpha}} t y^{-\frac{1}{\alpha}-1} g\left((t-y)(cy)^{-\frac{1}{\alpha}};\alpha\right) \qquad y,t>0. \tag{26}$$

The density of the Lévy measure is given by

$$\pi_Y(y) = c^{-\frac{1}{\alpha}} y^{-\frac{1}{\alpha}-1} g\left(-c^{-\frac{1}{\alpha}} y^{1-\frac{1}{\alpha}}; \alpha\right), \qquad y > 0. \tag{27}$$

Proof. Let $\tilde{\xi}$ be a spectrally negative α-stable process, defined by the Laplace exponent $\mathbb{E}[\exp(z\tilde{\xi}_1)] = \exp(cz^\alpha)$, $z \geq 0$. Consider the spectrally negative process $\xi_t = \tilde{\xi}_t + t$. The density of ξ_t is

$$p_\xi(t,x) = (ct)^{-\frac{1}{\alpha}} g((x-t)(ct)^{-\frac{1}{\alpha}}; \alpha), \qquad x \in \mathbb{R}, t > 0.$$

We define the subordinator $Y_t = \tau_t^+$, $t \geq 0$. Formula (26) follows from Kendall's identity (5) and formula (27) follows from (9). Referring to the computations in Remark 1, it is straightforward to see that $\mathbb{E}[Y_1] = +\infty$. The proof that $Y \in \mathcal{T}_0$ is a result of essentially the same reasoning to that found in the proof of Proposition 1 and is left to the reader to check. □

Remark 2. The proof of Proposition 5 shows that the subordinator Y is the ascending ladder time subordinator of an unbounded variation spectrally negative stable process with unit *positive* drift. One could ask the following natural question: what if we consider the ascending ladder time subordinator of an unbounded variation spectrally negative stable process with unit *negative* drift, will we get a new family of subordinators? It turns out that in this case we would obtain (up to scaling) the same family of subordinators as in Proposition 4. The details are left to the reader. The case that we choose ξ to be just an unbounded variation spectrally negative stable process is uninteresting. In that case Theorem 1 simply delivers the classical result that Y is the ascending ladder time process which is a stable subordinator with index $1/\alpha$.

3.4 Bessel subordinator

A Bessel subordinator X is defined by the Laplace exponent

$$\Phi_X(z) = c \ln\left(1 + \theta z + \sqrt{(1+\theta z)^2 - 1}\right), \qquad z \geq 0, \tag{28}$$

where $c > 0$ and $\theta > 0$. This process was introduced in [17], and it was shown that its transition density and the density of the Lévy measure are respectively given by

$$p_X(t,x) = ctx^{-1} \mathrm{e}^{-\frac{x}{\theta}} I_{ct}\left(\tfrac{x}{\theta}\right), \quad \pi_X(x) = cx^{-1}\mathrm{e}^{-\frac{x}{\theta}} I_0\left(\tfrac{x}{\theta}\right), \qquad t, x > 0,$$

where $I_\nu(x)$ denotes the modified Bessel function of the first kind (see [12]). It is known that $X \in \mathcal{T}_0$, see example 1.6.b in [13]. Applying Theorem 1 and Proposition 1, as well as taking note of Remark 1, we obtain the following result.

Proposition 6. *For $q > 0$ define $\phi(q)$ as the unique solution to the equation*

$$z - c\ln\left(1 + \theta z + \sqrt{(1+\theta z)^2 - 1}\right) = q.$$

Then the function $\Phi_Y(z) = \phi(z) - \phi(0) - z$ is the Laplace exponent of a finite mean subordinator $Y \in \mathcal{T}_0$. The transition probability density of the subordinator Y is given by

$$p_Y(t,y) = cty^{-1}\mathrm{e}^{\phi(0)t - \frac{y}{\theta}} I_{c(t+y)}\left(\frac{y}{\theta}\right).$$

The density of the Lévy measure is given by

$$\pi_Y(y) = cy^{-1}\mathrm{e}^{-\frac{y}{\theta}} I_{cy}\left(\frac{y}{\theta}\right).$$

3.5 Geometric stable subordinator

Assume that $c > 0$, $\theta > 0$ and $\alpha \in (0, 1)$. Consider a geometric stable subordinator X, which is defined by the Laplace exponent $\Phi_X(z) = c \ln(1 + (\theta z)^\alpha)$ (see [23] and [18]). This process can be constructed by taking an α-stable subordinator and subordinating it with the Gamma process. The transition density and Lévy density of X are respectively given by

$$p_X(t,x) = \frac{\alpha c t}{x} \sum_{k\geq 0} \frac{(-1)^k (1+ct)_k}{\Gamma(1+\alpha(ct+k))k!} \left(\frac{x}{\theta}\right)^{\alpha(ct+k)}, \quad \pi_X(x) = c\alpha x^{-1} E_\alpha\left(-\left(\tfrac{x}{\theta}\right)^\alpha\right), \qquad t, x > 0,$$

where $(a)_k := a(a+1)\cdots(a+k-1)$ denotes the Pocchammer symbol and

$$E_\alpha(x) := \sum_{k\geq 0} \frac{x^k}{\Gamma(1+\alpha k)}$$

denotes the Mittag-Leffler function (see [23]). It is known that $x\pi_X(x)$ is a completely monotone function (see [11]), thus $X \in \mathcal{T}_0$. Applying Theorem 1 and Proposition 1, and again making use of Remark 1, we obtain the following family of subordinators.

Proposition 7. *Assume that $c > 0$, $\theta > 0$ and $\alpha \in (0, 1)$. For $q > 0$ define $\phi(q)$ as the unique solution to the equation*

$$z - c \ln(1 + (\theta z)^\alpha) = q.$$

Then the function $\Phi_Y(z) = \phi(z) - \phi(0) - z$ is the Laplace exponent of a finite mean subordinator $Y \in \mathcal{T}_0$. The transition probability density of the subordinator Y is given by

$$p_Y(t,y) = e^{\phi(0)t} \frac{\alpha c t}{y} \sum_{k\geq 0} \frac{(-1)^k (1+c(t+y))_k}{\Gamma(1+\alpha(c(t+y)+k))k!} \left(\frac{y}{\theta}\right)^{\alpha(c(t+y)+k)}, \qquad y, t > 0.$$

The density of the Lévy measure is given by

$$\pi_Y(y) = \frac{\alpha c}{y} \sum_{k\geq 0} \frac{(-1)^k (1+cy)_k}{\Gamma(1+\alpha(cy+k))k!} \left(\frac{y}{\theta}\right)^{\alpha(cy+k)}, \qquad y > 0.$$

3.6 Inverse Gaussian subordinator

If we consider an inverse Gaussian subordinator X, having Laplace exponent $\Phi_X(z) = c(\sqrt{1+\theta z} - 1)$, then it is easy to see that the subordinator $Y_t = \tau_t^+$, constructed from X via Theorem 1, is also in the class of inverse Gaussian subordinators. This is not surprising, since the inverse Gaussian subordinator itself appears as the first passage time of the Brownian motion with drift, and one can show that applying this construction repeatedly does not produce new families of subordinators (see the discussion on page 4).

4 Applications

The results that we have obtained in the previous sections have interesting and non-trivial implications for Analysis and Special Functions. Every family of subordinators that we have discussed above leads to an explicit Laplace transform identity of the form

$$\int_0^\infty e^{-zy}\mathbb{P}(Y_t \in dy) = e^{-t\Phi_Y(z)}, \qquad z \geq 0, \tag{29}$$

and it seems that in all of these cases (except for the first example involving Poisson process) we obtain new Laplace transform identities. We do not know of a simple direct analytical proof of these results (we have found one way to prove them, but this method is just a complex-analytical counterpart of the original probabilistic proof of Kendall's identity).

Below we present a number of analytical statements that follow from our results in Section 3.

Example 1: For $r < 0$ and $t \in (0, e^{-1})$

$$\left(\frac{W_{-1}(-t)}{-t}\right)^r = e^{-rW_{-1}(-t)} = -\int_{-r}^{\infty} r\frac{(w+r)^{w-1}}{\Gamma(1+w)}t^w dw. \tag{30}$$

This formula seems to be new, and it is a direct analogue of the known result

$$\left(\frac{W_0(-z)}{-z}\right)^r = e^{-rW_0(-z)} = \sum_{n\geq 0} r\frac{(n+r)^{n-1}}{n!}z^n, \quad r \in \mathbb{C},\ |z| < 1/e, \tag{31}$$

which can be found in [9]. Formula (31) can be obtained in two ways. The first is the classical analytical approach is via Lagrange inversion theorem (see [9]). The second approach is via proposition 2 and (29). This example seems to indicate that when the subordinator X in Theorem 1 has support on the lattice, then Kendall's identity is an analytical statement which is equivalent to Lagrange inversion formula. Formula (30) is obtained in a similar way from Proposition (3), and we hypothesize that in the general case Kendall's identity can be considered as an integral analogue of Lagrange inversion formula.

Example 2: Proposition 4 and (29) give us the following resut: For $q > 0$ we have

$$\int_0^\infty \sqrt{\frac{t+y}{y^3}} K_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{\frac{(t+y)^3}{3y}}\right) e^{-qy} dy = \frac{3\pi}{t} e^{t(q-\phi(q))}, \tag{32}$$

where $\phi(q)$ is the solution to $z - z^{\frac{1}{3}} = q$.

Example 3: Proposition 4 and (29) give us the following resut: For $q > 0$ we have

$$\int_0^\infty \frac{e^{-\frac{2}{27}\frac{(t+y)^3}{y^2}}}{y(t+y)} W_{\frac{1}{2},\frac{1}{6}}\left(\frac{4}{27}\frac{(t+y)^3}{y^2}\right) e^{-qy} dy = \frac{\sqrt{3\pi}}{t} e^{t(q-\phi(q))}, \tag{33}$$

where $\phi(q)$ is the solution to $z - z^{\frac{2}{3}} = q$.

Example 4: From formula (23) we find that

$$g(x; \tfrac{3}{2}) = x^{-\frac{5}{2}} g(x^{-\frac{3}{2}}; \tfrac{2}{3}) = \frac{1}{\sqrt{3\pi x}} e^{-\frac{2}{27}x^3} W_{\frac{1}{2},\frac{1}{6}}\left(\frac{4}{27}x^3\right).$$

Then Proposition 5 and (29) give us the following resut: For $q > 0$ we have

$$\int_0^\infty \frac{e^{-\frac{2}{27}\frac{(t-y)^3}{y^2}}}{y(t-y)} W_{\frac{1}{2},\frac{1}{6}}\left(\frac{4}{27}\frac{(t-y)^3}{y^2}\right) e^{-qy} dy = \frac{\sqrt{3\pi}}{t} e^{-t\phi(q)}, \tag{34}$$

where $\phi(q)$ is the solution to $z + z^{\frac{3}{2}} = q$.

Example 5: Proposition 6 and (29) give us the following resut: For $q > 0$, $c > 0$ we have

$$\int_0^\infty e^{-y(\frac{1}{\theta}+q)} I_{c(t+y)}\left(\frac{y}{\theta}\right) \frac{dy}{y} = \frac{1}{ct} e^{t(q-\phi(q))}, \tag{35}$$

where $\phi(q)$ is the solution to $z - c\ln\left(1 + \theta z + \sqrt{(1+\theta z)^2 - 1}\right) = q$.

Example 6: We recall that a subordinator X belongs to the Thorin class $\mathcal{T}_0$ if and only if $x\pi_X(x)$ is a completely monotone function (where $\pi_X(x)$ is the Lévy density of X). The fact that subordinators constructed in Propositions 3, 4, 5, 6 and 7 belong to the class $\mathcal{T}_0$ implies that the following functions

$$\begin{aligned}
f_1(y) &= \frac{y^{cy} e^{-y}}{\Gamma(1+cy)}, \quad c > 0,\ y > 0, \\
f_2(y) &= y^{-\frac{1}{\alpha}} g(y^{1-\frac{1}{\alpha}}; \alpha), \quad \alpha \in (0,1),\ y > 0, \\
f_3(y) &= y^{-\frac{1}{\alpha}} g(-y^{1-\frac{1}{\alpha}}; \alpha), \quad \alpha \in (1,2),\ y > 0, \\
f_4(y) &= e^{-y} I_{cy}(y), \quad c > 0,\ y > 0, \\
f_5(y) &= \sum_{k\geq 0} \frac{(-1)^k (1+cy)_k}{\Gamma(1+\alpha(cy+k))k!} y^{\alpha(cy+k)}, \quad c > 0,\ \alpha \in (0,1),\ y > 0,
\end{aligned}$$

are completely monotone. We are not aware of any simple analytical proof of this result.

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