Polymorphic Games and Program Equivalence

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Parametric Polymorphism, or genericity, is a key feature of high-level languages, often combined with state (e.g. dynamic dispatch).

By hiding information, it creates interesting problems for reasoning about program equivalence.

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Object of this work: To develop games models for programming languages with effects and higher-rank polymorphism, and use it to reason about program behaviour and equivalence. Most material from “Game Semantics for a Polymorphic Programming Language”, JACM, 2013
Types:

\[ S, T ::= X \mid 1 \mid S \rightarrow T \mid S \times T \mid \forall X.T \]

[Proofs and rules subsequent to the types definition are listed here, including rules for typing terms and freshness of variables.]

We write \( I \) for \( \forall X.X \rightarrow X \).
What are the inhabitants of $S \rightarrow T$?

- **Extensionally** — functions from the set of inhabitants of $S$ to the set of inhabitants of $T$.
- **Intensionally** — a blueprint for computing an inhabitant of $T$ by interaction with an inhabitant of $S$.

What are the inhabitants of $\forall X. T$:

- **Extensionally** — a product or intersection of the sets of inhabitants of $T[S]$ over all possible $S$. (Predicativity?)
- **Intensionally** — a blueprint for computing an inhabitant of $T[S/X]$ for any $S$.

“Type variables in generic programs act as placeholders for types which may be instantiated later”.
Genericity — capturing “uniformity” in a more intensional way than e.g. relational parametricity or dinaturality (although these play a role).

Full abstraction — capturing information hiding in polymorphism, e.g. in equivalences such as
\[ \lambda x.(x\{l\}) \lambda y.y \simeq \lambda x.x : \forall X.(X \to X) \to \forall X.(X \to X). \]

Effective presentability — holding the prospect of verification applications. (Note that inhabitation in System F is not decidable).
A type $T$ is generic (w.r.t. a given theory) if for all terms $s, t : \forall X.S$, $s\{T\} = t\{T\}$ implies $s = t$.

**Theorem** [Longo et. al.] Extending System F with genericity at all types* yields a consistent theory.

- The syntactic (Böhm tree) model is not generic.
- Any generic *extensional* model will contain “junk” — e.g. infinitely many elements at each type.
A type $T$ is *generic* (w.r.t. a given theory) if for all terms $s, t : \forall X.S$, $s\{T\} = t\{T\}$ implies $s = t$.

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*Note that including e.g. the unit type breaks genericity.
• Types with free variables are interpreted as “context arenas” with holes into which arenas may be plugged.

• Second-order type structure is captured by a generalization of question/answer labelling and the bracketing condition.

• To instantiate a generic strategy, the question/answer relation is used to infer a *copycat* relation between occurrences of the instantiated arena.
A context arena $A$ is a labelled bipartite forest $(M_A, \vdash_A, \lambda_A, \triangleright_A)$, where

- $M_A$ is a set of moves.
- $\vdash \subseteq M_A \times M_A$ is an enabling relation, making
- $\lambda_A : M_A \rightarrow \{Q, A\} \cup \mathbb{N}$ labels moves as questions, answers or “$i$-holes” for $i \in \mathbb{N}$.
- $\triangleright \subseteq Q_A \times M_A \times \text{Ans}_A$ is a scoped question/answer relation determining which (Player/Opponent) answers may be given in response to each (Opponent/Player) question.
We can interpret the syntax trees of second order types $X_1, \ldots, X_n \vdash A$ directly as context games:

- Moves are (paths to the) *leaves* of the tree (type variables).
- Each path *enables* the moves (paths) immediately to its left.
- A path which ends with a free variable $X_i$ is labelled with $i$ (an $i$-hole).

Leaves which are positively bound variables are *questions.*
Leaves which are negatively bound variables are *answers.*
Example: $\forall X. X \rightarrow X \rightarrow X$
\[ \forall X. (S \to X) \to (T \to X) \to X \] is the lifted sum arena \([S] \oplus [T] \].

\[ \forall X. (S \to X) \to (T \to X) \to X \]

\[ \text{O}_q \]

\[ P_A \]

\[ \text{O}_q \]

\[ \vdots \]
Given an alternating path through the forest, we match up answers and questions as closing and opening parentheses:

\[ \begin{align*}
&\ ( \leftarrow [ \leftarrow ( [ [ [ \ldots ] ] ] ] \right) \leftarrow [ \leftarrow [ ] ] \right) \end{align*} \]

and require that:

If answer move \( a \) closes question \( q \), there is a move \( m \) such that

\[ m \vdash^* q, \ m \vdash^* a \text{ and } q \triangleright_m a. \]
Play in $[[\forall X. \lambda f. f\lambda x.(f\lambda y.x)\{X\}]$:

$\forall X. (((X \rightarrow X) \rightarrow \forall Y.Y) \rightarrow \forall Y.Y)$
Scoped Bracketing

Play in $[[\lambda f. f \land X. \lambda x. (f \lambda y. y)\{X\}]].$

$[[((\forall X. (X \rightarrow X) \rightarrow \forall Y. Y) \rightarrow \forall Y. Y)]]$
Given a strategy (non-empty, even-branching, even-prefix-closed set of legal sequences) \( \sigma : \forall X_{n+1}.A \), define \( \sigma[B] : A[B/X_{n+1}] \) — the *instantiation* of a context arena \( B \) into \( \sigma \) at \( X_{n+1} \) by *replacing each question-answer pair* in \( \sigma \) with a *copycat link*. 
Example

Plugging the arena $\forall Y. Y \rightarrow Y \rightarrow Y$:

\[
\begin{array}{c}
\forall Y. Y \\
\downarrow \\
\forall Y. Y
\end{array}
\]

into the identity strategy on $\forall X. X \rightarrow X$:

\[
\begin{array}{c}
q \\
\downarrow \\
a
\end{array}
\]

gives the identity on $(\forall Y. Y \rightarrow Y \rightarrow Y) \rightarrow \forall Y. Y \rightarrow Y \rightarrow Y$:

\[
\begin{array}{c}
\forall Y. Y \\
\downarrow \\
\forall Y. Y
\end{array}
\]
We define an indexed category $\mathcal{G} : \mathcal{I}^{op} \to \text{CCC}$, where:

- $\mathcal{I}$ is the category in which objects are natural numbers and morphisms from $m$ to $n$ are $n$-tuples of second order types over $m$ free variables (composed by substitution).

- For each $n$, $\mathcal{G}(n)$ is the (cartesian closed) category in which object are second-order types over $n$ free variables and morphisms from $A$ to $B$ are strategies on $\forall X_1 \ldots \forall X_n.(A \Rightarrow B)$ (composed by parallel composition plus hiding).

- For each tuple $\langle B_1, \ldots, B_n \rangle : m \to n$,
  $\mathcal{G}\langle B_1, \ldots, B_n \rangle : \mathcal{G}(m) \to \mathcal{G}(n)$ is the corresponding substitution functor.

The inclusion $\{ J_{n+1} : \mathcal{G}(n) \to \mathcal{G}(n+1) \mid n \in \mathbb{N} \}$ has an indexed left adjoint $\forall X_{n+1} : \mathcal{G}(n+1) \to \mathcal{G}(n)$. 
**Proposition** For any type $A$, instantiation is a *retraction*

$$\forall X. A[X] \preceq A[I/X].$$
**Proposition** For any type \( A \), instantiation is a *retraction* \[ \forall X. A[X] \sqsubseteq A[I/X] \].

(Corollary \( I \) is a *generic* type for our model. (If \( \llbracket M : \forall X. T \rrbracket \neq \llbracket N : \forall X. T \rrbracket \) then \( \llbracket M[I] : T[I/X] \neq \llbracket N[I] : T[I/X] \rrbracket \).
Our model is not fully complete for System F — strategies do not satisfy the *visibility* condition (and so they are not *innocent* or *total*, either).

In particular: the instantiation strategy $\sigma : \forall X. A \rightarrow A[B]$ does not satisfy visibility.
Example of a violation of visibility

At type $\forall X.((\forall Y.((X \to Y \to X) \to Y \to X)) \to X)$:

$$\land X.\lambda f^{\forall Y.((X\to Y\to X)\to Y\to X)}.(f\{X \to X\} (\lambda x^X.\lambda g^{X\to X}.g\ x)) \lambda y^X.y$$

$$\big[\forall X.((\forall Y.((\ X \to \ Y \to \ X) \to \ Y \to \ X)) \to \ X)\big]$$
We add a constant \( \text{new} : \forall X.\forall Y.(Y \to (Y \to I) \to X) \to X \)
declaring a reference cell, and denoting a strategy with the same underlying play as the preceding example.
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Using \( \text{new} \), the section \( p : A[I/X] \to \forall X. A \) is definable as a term which stores pointers to preceding \( X \)-moves as references. We use this fact to prove finite definability/full abstraction for our model, reducing it to (essentially) the simply-typed model of Abramsky, Honda and McCusker.
Intensional semantics of polymorphic value types ("data driven") is fundamentally different to computational types ("demand driven").

Based on (re-engineering of) Honda and Yoshida’s semantics of call-by-value $\lambda$-calculus.
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Specifically, we need to break down moves into tuples of smaller “atoms”, including holes into which arenas may be plugged.
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Quantification over types is by linking these holes with *explicit* pointers.
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To instantiate a generic strategy, these pointers are used to infer a *copycat link* between occurrences of the instantiated arena.

Full abstraction is established by showing that nat is a generic type.
∀(X × X → X) has two initial Opponent • questions — which must be played together, and a single, contingent • answer which may point to either of them —

\[
X \times X \rightarrow X
\]

\[
\langle •Q, •Q \rangle
\]
∀(X \times X \rightarrow X) \text{ has two initial Opponent } \bullet \text{ questions — which must be played together, and a single, contingent } \bullet \text{ answer which may point to either of them —}

\begin{align*}
X & \times X \rightarrow X \\
\langle \bullet Q, \bullet Q \rangle & \rightarrow \bullet A
\end{align*}
Subtype polymorphism (System $F_{\leq}$) using dinaturality properties of instantiation to represent *bounded quantification*.

Type-operators — extending the syntactic representation of games with $\lambda$-abstraction and application of type variables in arenas.

Constraining the model — can we describe models with fewer/different effects, linear types, etc.

Verification — when the references can be tamed, the model appears to be suitable for an algorithmic approach.