On a Particular Construction of Skew-Selfadjoint Operator Matrices

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We consider a particular construction for skew-selfadjoint operator matrices, which are of central importance in initial boundary value problems of mathematical physics.

1 Introduction

Typical initial boundary value problems of mathematical physics can be represented in the general form

\[(\partial_0\mathcal{M} + A)U = F,\]  

where $A$ is skew-selfadjoint, indeed commonly of the specific block matrix form

\[A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \]  

with $C : X_1 \subseteq X_0 \to Y$ a closed densely defined linear operator between Hilbert spaces $X_0$ and $Y$ with $X_1 = D(C)$, see e.g. [4, 5]. The operator $\mathcal{M}$ is referred to as the material law operator, which in the situation of interest here is a suitable linear operator acting on a Hilbert space realizing the proper space-time framework for the problem at hand.

The main purpose of this paper is to focus on the operator $C$ in this construction of the skew-selfadjoint operator $A : D(C) \oplus D(C^*) \subseteq X_0 \oplus Y \to X_0 \oplus Y$ when $Y$ is itself a direct sum of Hilbert spaces. In such a situation we shall loosely refer to a system of the form (1) as an abstract grad–div system. The guiding example, which at the same time motivates the name, is to take for $C$ the differential operator $\nabla$ with suitable domain $X_1$. The idea in this paper is to replace the role of the partial derivatives in $\nabla$ by general operators in general Hilbert spaces, hence the term abstract grad-div systems for the corresponding evolutionary systems associated with the skew-selfadjoint operator $A$ constructed according to (2). To illustrate the utility of the concept we consider a case of interest in connection with the boundary constraint equations such as the Leontovich boundary condition of electrodynamics, see e.g. [2, 3, 7].


In this section, we shall reconsider the concept of the adjoint operator of a densely defined, closed linear operator $C$, specifically in order to deal with the fact that the image space $Y$ of the operator $C$ is given as an orthogonal sum of Hilbert spaces. Let us first provide a precise definition of what we would like to call an abstract grad–div system.

\textbf{Definition 2.1} Let $C : X_1 \subseteq X_0 \to Y$, be a densely defined, closed linear operator with domain $X_1$ between Hilbert spaces $X_0, Y$. We shall refer to a system of the form (1) with $A$ generated via (2), as an \textit{abstract grad–div system}, if $Y$ given as a direct sum, i.e. $Y := \bigoplus_{k \in \{1, \ldots, n\}} Y_k$, for Hilbert spaces $Y_k, k \in \{1, \ldots, n\}, n \in \mathbb{N}$.

As a matter of jargon we shall say that the abstract grad–div system is \textit{generated by $C$}. If $\iota_{Y_k}$ denotes the canonical isometric embedding of $Y_k$ into $Y$ then, with $C_k := \iota_{Y_k}^* C, k \in \{1, \ldots, n\}$, we have

\[
C x = C_0 x \oplus \cdots \oplus C_n x \equiv \begin{pmatrix} C_0 x \\ \vdots \\ C_n x \end{pmatrix}
\]

\[
x \in \begin{pmatrix} Y_0 \\ \vdots \\ Y_n \end{pmatrix} \quad \equiv Y \quad \text{for} \quad x \in X_1. \quad \text{To clarify notation, we need the following definition.}
\]

Let $X_1 \hookrightarrow X_0 \hookrightarrow X_1'$ be a Gelfand triple. Consider a linear operator $S : X_1 \subseteq X_0 \to Y$ such that $L_S : X_1 \to Y, x \mapsto S x$, is a continuous linear operator (S need not be closable) and define the operator $S^\circ : Y \to X_1'$ by $\langle S^\circ y | x \rangle_{X_0} = \langle y | S x \rangle_Y$ for all $x \in X_1, y \in Y$, where here $\langle \cdot | \cdot \rangle_{X_0}$ denotes here the continuous extension of the inner product of $X_0$ to a duality pairing on $X_1' \times X_1$. With this concept of a dual we obtain the following result.

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Theorem 2.2 Let $C$ generate an abstract grad–div system with $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$. Then

$$C^* = \{(y_1, \ldots, y_n), x \in Y \oplus X_0 \mid x = \sum_{k=1}^n C_k^* y_k \in X_0 \}.$$ 

3 An Application

We define the operator $\text{curl}$ as the closure of the classical curl as an operator on $L^2(\Omega)^3$ with domain $C_\infty(\Omega)^3$, the space of smooth vector fields with compact support in $\mathbb{R}^3$.

By integration by parts we see $\text{curl} \subseteq (\text{curl})^* =: \text{curl}$. We recall that $w \in D(\text{curl})$ encodes $w|_{\partial \Omega} \times n = 0$ for domains $\Omega$ with sufficiently smooth boundary, where $n$ denotes the unit outward normal vector field on $\partial \Omega$, see [1]. Let $L^2_2(\Gamma)$ denote the space of tangential vector-fields on $\Gamma$, i.e. $L^2_2(\Gamma) := \{ f \in L^2(\Gamma)^3 \mid f \cdot n = 0 \}$, which is a closed subspace of $L^2(\Gamma)^3$. Let $\pi_\tau$ be the continuous tangential component boundary trace operator $\pi_\tau : H(\text{curl}, \Omega) \to V'_\tau$ and let $\gamma_\tau$ be the continuous tangential boundary trace operator $\gamma_\tau : H(\text{curl}, \Omega) \to V''_\tau$. Here $V'_\tau$ and $V''_\tau$ are dual space of certain spaces $V'_\tau$ and $V''_\tau$, respectively, with $L^2_2(\Gamma)$ as pivot space for the two corresponding Gelfand triples, see [1] for details. We take for this example $X_0 := L^2(\Omega)^3$.

Then $A$ is constructed from $C = \begin{pmatrix} -\text{curl} \\ \pi_\tau \end{pmatrix}$:

$$X_1 \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3 \oplus L^2_2(\Gamma), \text{ where } X_1 := \pi_\tau^{-1} [\text{id} \big[ L^2_2(\Gamma) \big]]$$

equipped with the graph norm of $C$ is a Hilbert space, $\text{curl} := \text{curl}|_{X_1} : X_1 \to L^2(\Omega)^3$ and $\pi_\tau := \pi_\tau|_{X_1} : X_1 \to L^2_2(\Gamma)$.

For $C$, which is indeed a closed operator, to generate an abstract grad–div system the only thing left to show is that $X_1$ is dense in $L^2(\Omega)^3$. This, however, is trivial as $C_\infty(\Omega)^3 \subseteq X_1$.

In physical terms the operator $A$ acts on the triple $(H, E, \eta)$, where $H$ is the magnetic field, $E$ the electric field and $\eta$ represents, as we shall see, a quantity acting as the negative tangential boundary trace of $E$. To characterize containment in the domain of $\begin{pmatrix} -\text{curl} \\ \pi_\tau \end{pmatrix}$ we need a prerequisite:

Lemma 3.1 Let $E \in D(\text{curl})$, $\eta \in L^2_2(\Gamma)$. Then $C E = \text{curl} E - \pi_\tau^* \eta$ if and only if $\gamma_\tau E + \eta = 0$ on $L^2_2(\Gamma)$.

**Proof.** We observe that for $\Psi \in H^1(\Omega)^3 \subseteq X_1$ the equation $(\gamma_\tau E + \eta)(\pi_\Psi) = \langle \text{curl} E | \Psi \rangle_{L^2(\Omega)^3} - \langle \text{curl} E - \pi_\tau^* \eta | \Psi \rangle_{L^2_2(\Gamma)}$ holds true. Thus, if $\text{curl} E = \text{curl}^\circ E - \pi_\tau^* \eta$, we get that $(\gamma_\tau E + \eta)(\pi_\Psi) = 0$ for each $\Psi \in H^1(\Omega)^3$. Thus, $\gamma_\tau E + \eta = 0$ on $L^2_2(\Gamma)$, due to the density of $\pi_\tau[H^1(\Omega)^3] = V'_\tau$ in $L^2_2(\Gamma)$, see [1, p. 850]. On the other hand, if $\gamma_\tau E + \eta = 0$, we immediately get $\text{curl} E = \text{curl}^\circ E - \pi_\tau^* \eta$ by the density of $H^1(\Omega)^3$ in $L^2(\Omega)^3$.

Theorem 3.2 We have \begin{pmatrix} -\text{curl} \\ \pi_\tau \end{pmatrix}^* \subseteq ( - \text{curl} 0 ) and $D\left( \begin{pmatrix} -\text{curl} \\ \pi_\tau \end{pmatrix}^* \right)$ is given by the set \{(E, \eta) \in D(\text{curl}) \times L^2_2(\Gamma) \mid \gamma_\tau E + \eta = 0 on L^2_2(\Gamma) \}.

**Proof.** Note that with $\text{curl}^* = (\text{curl})^*$ we have $\begin{pmatrix} -\text{curl} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} -\text{curl} \\ \pi_\tau \end{pmatrix} =: C$. From this we get $C^* \subseteq ( - \text{curl} 0 )$.

Therefore, by Theorem 2.2, we obtain $(E, \eta) \in D(C^*)$ if and only if $E \in D(\text{curl})$ and $\text{curl} E = \text{curl}^\circ E + \pi_\tau^* \eta \in L^2(\Omega)^3$, which, in turn, by Lemma 3.1 is equivalent to $\gamma_\tau E + \eta = 0$ on $L^2_2(\Gamma)$ and $E \in D(\text{curl})$.

The latter theorem tells us that the containment in the domain of $A$ gives – in the presence of a suitable material law – a boundary equation involving $\gamma_\tau E$ and $\pi_\tau H$. For a particular choice of material properties for example the Leontovich boundary condition can be recovered, for details see [6].

References


