On a Particular Construction of Skew-Selfadjoint Operator Matrices

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We consider a particular construction for skew-selfadjoint operator matrices, which are of central importance in initial boundary value problems of mathematical physics.

\section{Introduction}

Typical initial boundary value problems of mathematical physics can be represented in the general form

\[(\partial_0\mathcal{M} + A)U = F,\]

where \(A\) is skew-selfadjoint, indeed commonly of the specific block matrix form

\[A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},\]

with \(C : X_1 \subseteq X_0 \to Y\) a closed densely defined linear operator between Hilbert spaces \(X_0\) and \(Y\) with \(X_1 = D(C)\), see e.g. [4,5]. The operator \(\mathcal{M}\) is referred to as the material law operator, which in the situation of interest here is a suitable linear operator acting on a Hilbert space realizing the proper space-time framework for the problem at hand.

The main purpose of this paper is to focus on the operator \(C\) in this construction of the skew-selfadjoint operator \(A : D(C) \oplus D(C^*) \subseteq X_0 \oplus Y \to X_0 \oplus Y\) when \(Y\) is itself a direct sum of Hilbert spaces. In such a situation we shall loosely refer to a system of the form (1) as an abstract grad–div system. The guiding example, which at the same time motivates the name, is to take for \(C\) the differential operator \(\nabla\) with suitable domain \(X_1\). The idea in this paper is to replace the role of the partial derivatives in \(\nabla\) by general operators in general Hilbert spaces, hence the term abstract grad-div systems for the corresponding evolutionary systems associated with the skew-selfadjoint operator \(A\) constructed according to (2). To illustrate the utility of the concept we consider a case of interest in connection with the boundary constraint equations such as the Leontovich boundary condition of electrodynamics, see e.g. [2,3,7].

\section{Construction of Abstract grad–div Systems.}

In this section, we shall reconsider the concept of the adjoint operator of a densely defined, closed linear operator \(C\), specifically in order to deal with the fact that the image space \(Y\) of the operator \(C\) is given as an orthogonal sum of Hilbert spaces. Let us first provide a precise definition of what we would like to call an abstract grad–div system.

**Definition 2.1** Let \(C : X_1 \subseteq X_0 \to Y\), be a densely defined, closed linear operator with domain \(X_1\) between Hilbert spaces \(X_0, Y\). We shall refer to a system of the form (1) with \(A\) generated via (2), as an abstract grad–div system, if \(Y\) given as a direct sum, i.e. \(Y := \bigoplus_{k \in \{1,\ldots,n\}} Y_k\), for Hilbert spaces \(Y_k, k \in \{1,\ldots,n\}, n \in \mathbb{N}\).

As a matter of jargon we shall say that the abstract grad–div system is generated by \(C\). If \(\iota_k\) denotes the canonical isometric embedding of \(Y_k\) into \(Y\) then, with \(C_k := \iota_k^* C, k \in \{1,\ldots,n\}\), we have \(C x = C_0 x \oplus \cdots \oplus C_n x \equiv \begin{pmatrix} C_0 x \\ \vdots \\ C_n x \end{pmatrix}\).

Let \(X_1 \hookrightarrow X_0 \hookrightarrow X_1^*\) be a Gelfand triple. Consider a linear operator \(S : X_1 \subseteq X_0 \to Y\) such that \(L_S : X_1 \to Y, x \mapsto S x\), is a continuous linear operator (\(S\) need not be closable) and define the operator \(S^* : Y \to X_1^*\) by \(\langle S^* y | x \rangle_{X_0} = \langle y | S x \rangle_Y\) for all \(x \in X_1, y \in Y\), where here \(\langle \cdot | \cdot \rangle_{X_0}\) denotes here the continuous extension of the inner product of \(X_0\) to a duality pairing on \(X_1^* \times X_1\). With this concept of a dual we obtain the following result.

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Theorem 2.2 Let $C$ generate an abstract grad–div system with \( C = \left( \begin{array}{c} C_1 \\ \vdots \\ C_n \end{array} \right) \). Then

\[ C^* = \{(y_1, \ldots, y_n), x) \in Y \oplus X_0 \mid x = \sum_{k=1}^n C_k y_k \in X_0 \}. \]

3 An Application

We define the operator \textit{curl} as the closure of the classical curl as an operator on \( L^2(\Omega)^3 \) with domain \( \mathring{C}_\infty(\Omega)^3 \), the space of smooth vector fields with compact support in \( \mathbb{R}^3 \).

By integration by parts we see \( \text{curl} \subseteq \text{curl}^* =: \text{curl} \). We recall that \( w \in D(\text{curl}) \) encodes \( w|_{\partial \Omega} = 0 \) for domains \( \Omega \) with sufficiently smooth boundary, where \( n \) denotes the unit outward normal vector field on \( \partial \Omega \), see [1]. Let \( L^2_\tau(\Gamma) \) denote the space of tangential vector-fields on \( \Gamma \), i.e. \( L^2_\tau(\Gamma) := \{ f \in L^2(\Gamma)^3 \mid f \cdot n = 0 \} \), which is a closed subspace of \( L^2(\Gamma)^3 \). Let \( \pi \) be the continuous tangential component boundary trace operator \( \pi : H(\text{curl}, \Omega) \to V'_\tau \) and let \( \gamma \) be the continuous tangential boundary trace operator \( \gamma : H(\text{curl}, \Omega) \to V'_\tau \). Here \( V'_\tau \) and \( V_\tau \) are dual space of certain spaces \( V_\tau \) and \( V_\pi \), respectively, with \( L^2(\Gamma) \) as pivot space for the two corresponding Gelfand triples, see [1] for details. We take for this example \( X_0 := L^2(\Omega)^3 \).

Then \( A \) is constructed from \( A = \left( \frac{-\text{curl}}{\pi_\tau} \right) : X_1 \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3 \oplus L^2(\Gamma), \) where \( X_1 := \pi_\tau^{-1} [\text{id}_3 \left[ L^2_\tau(\Gamma) \right]] \)

equipped with the graph norm of \( A \) is a Hilbert space, \( \text{curl} = \text{curl}|_{X_1} : X_1 \to L^2(\Omega)^3 \) and \( \pi_\tau = \pi_\tau|_{X_1} : X_1 \to L^2(\Gamma) \).

For \( C \), which is indeed a closed operator, to generate an abstract grad–div system the only thing left to show is that \( X_1 \) is dense in \( L^2(\Omega)^3 \). This, however, is trivial as \( \mathring{C}_\infty(\Omega)^3 \subseteq X_1 \). In physical terms the operator \( A \) acts on the triple \((H, E, \eta)\), where \( H \) is the magnetic field, \( E \) the electric field and \( \eta \) represents, as we shall see, a quantity acting as the negative tangential boundary trace of \( E \). To characterize containment in the domain of \( \left( \frac{-\text{curl}}{\pi_\tau} \right)^* \) we need a prerequisite:

Lemma 3.1 Let \( E \in D(\text{curl}), \eta \in L^2(\Gamma) \). Then \( \text{curl} E = \text{curl}^* E \) if and only if \( \gamma \tau \varepsilon E + \eta = 0 \) on \( L^2(\Gamma) \).

Proof. We observe that for \( \Psi \in H^1(\Omega)^3 \subseteq X_1 \) the equation \( (\gamma \tau \varepsilon E + \eta) (\pi_\tau \Psi) = (\text{curl} E|\Psi)|_{L^2(\Omega)^3} - \left( \frac{-\text{curl}}{\pi_\tau} \right) (\pi_\tau \Psi) \) holds true. Thus, if \( \text{curl} E = \text{curl}^* E \), we get that \( (\gamma \tau \varepsilon E + \eta) (\pi_\tau \Psi) = 0 \) for each \( \Psi \in H^1(\Omega)^3 \). Hence, \( \gamma \tau \varepsilon E + \eta = 0 \) on \( L^2(\Gamma) \), due to the density of \( \pi_\tau \left[ H^1(\Omega)^3 \right] = V_\tau \) in \( L^2(\Gamma) \), see [1, p. 850]. On the other hand, if \( \gamma \tau \varepsilon E + \eta = 0 \), we immediately get \( \text{curl} E = \text{curl}^* E \) by the density of \( H^1(\Omega)^3 \) in \( L^2(\Omega)^3 \).

Theorem 3.2 We have \( \left( \frac{-\text{curl}}{\pi_\tau} \right)^* \subseteq \left( - \text{curl} \ 0 \right) \) and \( \text{D} \left( \left( \frac{-\text{curl}}{\pi_\tau} \right)^* \right) \) is given by the set

\[ \{(E, \eta) \in D(\text{curl}) \times L^2(\Gamma) \mid \gamma \tau \varepsilon E + \eta = 0 \} \]

Proof. Note that with \( \text{curl} = \text{curl}^* \) we have \( \left( \frac{-\text{curl}}{\pi_\tau} \right)^* \subseteq \left( \frac{-\text{curl}}{\pi_\tau} \right) =: C \). From this we get \( C^* \subseteq \left( - \text{curl} \ 0 \right) \).

Therefore, by Theorem 2.2, we obtain \( (E, \eta) \in D(C^*) \) if and only if \( E \in D(\text{curl}) \) and \( \text{curl} E = \left( \frac{-\text{curl}}{\pi_\tau} \right) \eta \in L^2(\Omega)^3 \), which, in turn, by Lemma 3.1 is equivalent to \( \gamma \tau \varepsilon E + \eta = 0 \) on \( L^2(\Gamma) \) and \( E \in D(\text{curl}) \).

The latter theorem tells us that the containment in the domain of \( A \) gives – in the presence of a suitable material law – a boundary equation involving \( \gamma \tau \varepsilon E \) and \( \pi_\tau \cdot H \). For a particular choice of material properties for example the Leontovich boundary condition can be recovered, for details see [6].

References