Nilpotent symplectic alternating algebras II

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In this paper and its sequel we continue our study of nilpotent symplectic alternating algebras. In particular we give a full classification of such algebras of dimension 10 over any field. It is known that symplectic alternating algebras over GF(3) correspond to a special rich class $\mathcal{C}$ of 2-Engel 3-groups of exponent 27 and under this correspondance we will see that the nilpotent algebras correspond to a subclass of $\mathcal{C}$ that are those groups in $\mathcal{C}$ that have an extra group theoretical property that we refer to as being powerfully nilpotent and can be described also in the context of p-groups where p is an arbitrary prime.

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1 Introduction

A symplectic alternating algebra (SAA) is a symplectic vector space L, whose associated alternating form is non-degenerate, that is furthermore equipped with a binary alternating product $\cdot : L \times L \to L$ with the extra requirement that

$$(x \cdot y, z) = (y \cdot z, x)$$

for all $x, y, z \in L$. This condition can be expressed equivalently by saying that $(u \cdot x, v) = (u, v \cdot x)$ for all $u, v, x \in L$ or in other words that mulitplication from the right is self-adjoint with respect to the alternating form.

Symplectic alternating algebras originate from a study of powerful 2-Engel groups [1], [4] and there is in a 1-1 correspondence between a certain rich class of powerful 2-Engel 3-groups of exponent 27 and SAAs over the field GF(3). We will consider this in more detail later in the introduction.

Let $2n$ be a given even integer and $\mathbb{F}$ a fixed field. Let V be the symplectic vector space over the field $\mathbb{F}$ with a non-degenerate alternating form. Fix some basis $u_1, u_2, \ldots, u_{2n}$ for V. An alternating product $\cdot$ that turns V into a symplectic alternating algebra is uniquely determined by the values

$$\mathcal{P}: \quad (u_i \cdot u_j, u_k), \quad 1 \leq i < j < k \leq 2n.$$

Let L be the resulting symplectic alternating algebra. We refer to the data above as a presentation for L with respect to the basis $u_1, \ldots, u_{2n}$.

If $m(n)$ is the number of symplectic alternating algebras over a finite field $\mathbb{F}$ then $m(n) = |\mathbb{F}|^{\frac{4n^3}{3}+O(n^2)}$ [2]. Because of the sheer growth, a general classification does not seem to be within reach although this has been done for small values of n. Thus it is not difficult to see that $m(0) = m(1) = 1$ and $m(2) = 2$. For higher dimensions the classification is already difficult. It is though known that when $\mathbb{F} = \mathrm{GF}(3)$ we have $m(3) = 31$ [5]. Some general structure theory is developed in [2], [3] and [5]. In particular there is dichotomy result that is an analog to a corresponding theorem for Lie algebras, namely that L either contains an abelian ideal or is a direct sum of simple symplectic alternating algebras [5]. We also have that any symplectic algebra that is abelian-by-nilpotent must be nilpotent while this is not the case in general for solvable algebras [3].

Here we focus on the subclass of nilpotent symplectic alternating algebras (NSAAs) and this paper is a sequel to [2] where a rich general structure theory was developed for NSAAs with a number of beautiful properties. Before discussing these we need to introduce some notation. Firstly we can always pick a basis $x_1, y_1, \ldots, x_n, y_n$ with the property that $(x_i, x_j) = (y_i, y_j) = 0$ and $(x_i, y_j) = \delta_{ij}$ for $1 \leq i \leq j \leq n$. We refer to a basis of this type as a standard basis. It turns out that for any nilpotent symplectic alternating algebra one can always choose a suitable standard basis such that the chain of subspaces

$$0 = I_0 < I_1 < \ldots < I_n < I_{n-1}^{\perp} < \cdots < I_0^{\perp} = L,$$

with $I_k = \mathbb{F}x_n + \cdots + \mathbb{F}x_{n-k+1}$ for $k > 0$, is a central chain of ideals. One can furthermore see from this that $x_i y_j = 0$ if $j \leq i$ and that $I_{n-1}^{\perp}$ is abelian. It follows that a number of the triple values (uv, w) are trivial. Listing only the values that are possibly non-zero it suffices to consider

$$\mathcal{P}: \quad (x_i y_j, y_k) = \alpha_{ijk}, \quad (y_i y_j, y_k) = \beta_{ijk}$$

for some $\alpha_{ijk}, \beta_{ijk} \in \mathbb{F}$ where $1 \leq i < j < k \leq n$. Such a presentation is called a *nilpotent presentation*. Conversely any such presentation describes a nilpotent SAA. The algebras that are of maximal class turn out to have a rigid ideal structure. In particular when $2n \geq 10$ we can choose our chain of ideals above such that they are all characteristic and it turns out that $I_0, I_2, I_3, \ldots, I_{n-1}, I_{n-1}^{\perp}, I_{n-2}^{\perp}, \ldots, I_0^{\perp}$ are unique and equal to both the terms of the lower and upper central series (see [2] Theorems 3.1 and 3.2). The algebras of maximal class can be identified easily from their nilpotent presentations. In fact, if $\mathcal{P}$ is any nilpotent presentation of L with respect to a standard basis $\{x_1, y_1, \ldots, x_n, y_n\}$, and $2n \geq 8$, we have that L is of maximal class if and only if $x_i y_{i+1} \neq 0$ for all $i = 2, \ldots, n-2$, and $x_1 y_2, y_1 y_2$ are linearly nilpotent (see [2] Theorem 3.4).

From the general theory of nilpotent SAAs one can also determine their growth. Thus if $k(n)$ is the number of nilpotent SAAs of dimension $2n$ over a finite field $\mathbb{F}$ then $k(n) = |\mathbb{F}|^{n^3/3+O(n^2)}$ [2]. Again the growth is too large for a general classification to be feasable. The algebras of dimension $2n$ for $n \leq 4$ are classified in [2] over any field. In this paper and its sequel we deal with the challenging classification of algebras of dimension 10 over any field. As we will see the classification depends very much on the underlying field. It turns out that the classification of nilpotent symplectic alternating algebras of dimension 10 with a centre that is not isotropic can be easily reduced to the known classifcation of algebras of dimension 8. The main bulk of the work is thus about algebras with isotropic centre that must lie between 2 and 5. In this paper we consider the situation when the isotropic centre is of dimension 3 or 5 leaving the remaining cases to the sequel. As we said above, the classification depends on the underlying field. We can read in particular from the classification that over a field that is algebraically closed there are

22 NSAAs of dimension 10.

2 The correspondence between SAAs and groups

Before starting the work on the classification we consider here in more detail the correspondence between SAAs and 2-Engel groups, mentioned in the introduction. The study in [4] reveals that there is one-one correspondence between symplectic alternating algebras over the field GF(3) and a certain class $\mathcal{C}$ of powerful 2-Engel 3-groups of exponent 27. These groups form a class that consists of all powerful 2-Engel 3-groups G with the following extra properties:

(a) $G = \langle x, H\rangle$ where $H = \{g \in G : g^9 = 1\}$ and $Z(G) = \langle x\rangle$ with $O(x) = 27$.
(b) G is of rank $2r + 1$ and has order 3^{3+4r}.

The associated symplectic alternating algebra $L(G)$ is constructed as follows. First we consider $L(G) = H/G^3$ as a vector space over GF(3). To this we associate a bilinear alternating form (,) and an alternating binary multiplication as follows: for any $\bar{a} = aG^3, \bar{b} = bG^3$ and $\bar{c} = cG^3$ in $L(G)$,

$$[a, b]^3 = x^{9(\bar{a},\bar{b})}$$
$$\bar{a} \cdot \bar{b} = \bar{c} \text{ where } [a, b]Z(G) = c^3 Z(G).$$

One can show that these are well defined and turn $L(G)$ into a SAA. Furthermore $L(G) \cong L(K)$ if and only if $G \cong K$ [4].

In order to indentify the groups in $\mathcal{C}$ that correspond to the SAAs that are nilpotent, we introduce some new terms.

Definition. A finite p-group G is *powerfully nilpotent* if there exists an ascending chain

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that $[H_i, G] \leq H_{i-1}^p$ for $i = 1, \ldots, n$. We refer to such a chain as a powerfully central chain and n is the length of the chain. If G is powerfully nilpotent then the smallest possible length of a powerfully central chain for

G is called its *powerful nilpotence class.*

Let us now return to our special class $\mathcal{C}$ of powerful 2-Engel 3-groups. Let $G \in \mathcal{C}$. For any K such that $G^3 \leq K \leq G$ we let $\overline{K} = K/G^3$. Notice that

$$\overline{A} \cdot L(G) \leq \overline{B} \text{ if and only if } [\langle A, x\rangle, G] \leq \langle B, x\rangle^3.$$

Thus if $G^3 \leq H_i$ for $i = 1, \ldots, n$, then

$$\{0\} = \overline{H_0} \leq \overline{H_1} \leq \cdots \leq \overline{H_n} = L(G)$$

is a central chain of ideals in $L(G)$ if and only if

$$\{1\} \leq \langle x\rangle \leq \langle H_0, x\rangle \leq \cdots \leq \langle H_n, x\rangle = G$$

is a powerfully central chain. The classification of the NSAAs of dimension 10 over GF(3) gives us thus the classification for the powerfully nilpotent groups in $\mathcal{C}$ that are of rank 11. The classification reveals that there are 25 such groups.

3 Algebras with a non-isotropic centre and algebras with an isotropic centre of dimension 5

We consider first the algebras with a non-isotropic centre. Let L be such an algebra. In this case we can assume that we have a standard basis where $x_5, y_5 \in Z(L)$. We then have that L is a direct sum of the abelian algebra $\mathbb{F}x_5 + \mathbb{F}y_5$ and a NSAA of dimension 8. The algebras of dimension 8 were however classified in [2] and according to this classification there are, apart from the abelian algebra, two algebras and one family of algebras. From this we can read that there are the following non-abelian NSAAs of dimension 10 with a non-isotropic centre.

$$\begin{array}{ll}
\mathcal{Q}_{10}^{(7,1)}: & (y_1y_2, y_3) = 1. \\
\mathcal{Q}_{10}^{(5,1)}: & (y_1y_2, y_3) = 1,\ (x_1y_3, y_4) = 1. \\
\mathcal{Q}_{10}^{(4,1)}(r): & (x_2y_3, y_4) = r,\ (x_1y_2, y_4) = 1,\ (y_1y_2, y_3) = 1,
\end{array}$$

where $r \in \mathbb{F} \setminus \{0\}$ and $\mathcal{Q}_{10}^{(4,1)}(s) \cong \mathcal{Q}_{10}^{(4,1)}(r)$ if and only if $r/s \in (\mathbb{F}^*)^3$. Here the notation $\mathcal{Q}_{10}^{(m,1)}$ indicates that the algebra has dimension 10 with centre of dimension m. From now on we can thus assume that all our algebras have an isotropic centre and we start considering the case when the centre has dimension 5. Let L be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 5. We can then choose a standard basis $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5$ such that

$$Z(L) = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1.$$

Here $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5$ will be determined later such that some further conditions hold. The elements $y_1, \ldots, y_5$ are not in $Z(L)$ and without loss of generality we can assume that $(y_1y_2, y_3) = 1$. Now suppose that $(y_iy_j, y_4) = \alpha_{ij}$ and $(y_iy_j, y_5) = \beta_{ij}$ for $1 \leq i, j \leq 3$. Replacing x_1, x_2, x_3, y_4, y_5 by

$$\begin{array}{ll} \tilde{x}_1 = x_1 + \alpha_{23}x_4 + \beta_{23}x_5, & \tilde{y}_4 = y_4 - \alpha_{12}y_3 - \alpha_{23}y_1 - \alpha_{31}y_2, \\ \tilde{x}_2 = x_2 + \alpha_{31}x_4 + \beta_{31}x_5, & \tilde{y}_5 = y_5 - \beta_{12}y_3 - \beta_{23}y_1 - \beta_{31}y_2, \\ \tilde{x}_3 = x_3 + \alpha_{12}x_4 + \beta_{12}x_5, & \end{array}$$

we can assume that our standard basis has the further property that $(y_iy_j, y_4) = (y_iy_j, y_5) = 0$ for $1 \leq i < j \leq 3$. As $y_4 \notin Z(L)$, we know that one of $(y_1y_4, y_5), (y_2y_4, y_5), (y_3y_4, y_5)$ is nonzero. Without loss of generality we can assume that $(y_1y_4, y_5) = 1$. The only triples whose values are not known are then $\alpha = (y_2y_4, y_5)$ and $\beta = (y_3y_4, y_5)$. Replacing x_1, y_2, y_3 by $\tilde{x}_1 = x_1 + \alpha x_2 + \beta x_3$, $\tilde{y}_2 = y_2 - \alpha y_1, \tilde{y}_3 = y_3 - \beta y_1$, we get a new standard basis where the only nonzero triple values are $(y_1y_2, y_3) = 1$ and $(y_1y_4, y_5) = 1$. We have thus proved the following result.

Proposition 3.1 *There is a unique nilpotent SAA of dimension* 10 *that has isotropic centre of dimension* 5*. This algebra can be described by the nilpotent presentation*

$$\mathcal{P}_{10}^{(5,1)} : \ (y_1y_2, y_3) = 1, \quad (y_1y_4, y_5) = 1.$$

4 Algebras with an isotropic centre of dimension 3

In this section we will be assuming that $Z(L)$ is isotropic of dimension 3. First we derive some properties that hold for these algebras. Here throughout

$$\begin{aligned} Z(L) &= \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3. \\ L^2 &= Z(L) + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_2. \end{aligned}$$

Lemma 4.1 $Z(L) \leq L^3$.

Proof Otherwise $Z_2(L) = (L^3)^\perp \not\leq Z(L)^\perp = L^2$. Without loss of generality we can suppose that $y_3 \in Z_2(L) \setminus L^2$. As $Z_2(L) \cdot L^2 = \{0\}$, we then have $y_3 \cdot L^2 = \{0\}$. Now also $x_2 \cdot L^2 = \{0\}$. Let $\alpha = (x_2y_4, y_5)$ and $\beta = (y_3y_4, y_5)$. Notice that $\alpha, \beta \neq 0$ as $x_2, y_3 \notin Z(L)$. But then

$$((\beta x_2 - \alpha y_3)y_4, y_5) = 0$$

that implies that $\beta x_2 - \alpha y_3 \in Z(L)$. This is absurd. $\Box$

Lemma 4.2 *dim* $L^3 \geq 5$.

Proof Otherwise $\dim L^3 \leq 4$ and as $Z(L) \leq L^3 \leq L^2 = Z(L)^\perp$ we can choose our standard basis such that $Z(L) = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3$ and

$$L^3 \leq \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2.$$

This implies that $\mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 \leq (L^3)^\perp = Z_2(L)$ and (notice that $Z_2(L) \leq L^2$ as $Z(L) \leq L^3$) $L^2 = Z_2(L) + Fy_2$ that implies that L^2 is abelian. Then for any $x \in L^2$ and $a, b, c \in L$, we have

$$(x, abc) = -(x(ab), c) = -(0, c) = 0$$

and $L^3 \leq (L^2)^\perp = Z(L)$. Hence $L^3 = Z(L)$ and $Z_2(L) = L^2$. Suppose $L = Z_2(L) + \mathbb{F}u_1 + \mathbb{F}u_2 + \mathbb{F}u_3$. Then $L^2 = Z(L) + \mathbb{F}u_1u_2 + \mathbb{F}u_1u_3 + \mathbb{F}u_2u_3$ and we get the contradiction that $4 = \dim L^2 - \dim Z(L) \leq 3$. $\Box$

Lemma 4.3 *If dim* $L^3 = 5$*, then* L^3 *is isotropic.*

Proof Otherwise we can choose our basis such that $L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_1 + \mathbb{F}y_1$ and then $Z_2(L) = (L^3)^\perp = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}y_2$ and as $L^2 \cdot Z_2(L) = \{0\}$, it follows that $x_1y_2 = y_1y_2 = 0$. Then L^2 is abelian and thus we get the contradiction that $L^3 \leq Z(L)$. $\Box$

Lemma 4.4 $Z(L) \leq L^4$.

Proof We have seen that $\dim L^3 \geq 5$. So we can choose our standard nilpotent basis such that either

$$L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1$$

or

$$L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1.$$

We consider the two cases in turn beginning with the first case. If $Z(L) \not\leq L^4$, then $\dim Z(L) \cap L^4 \leq 2$ and thus $\dim L^2 + Z_3(L) = \dim (Z(L) \cap L^4)^\perp \geq 8$. Suppose $L = L^2 + Z_3(L) + \mathbb{F}u + \mathbb{F}v$. Then $L^2 = L^3 + Z_2(L) + \mathbb{F}uv = L^3 + Fuv$ and we get the contradiction that $\dim L^2 \leq 5 + 1 = 6$. We now turn to the second case where $L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1$. We argue by contradiction and suppose that $Z(L) \cap L^4 < Z(L)$. Then we can choose our basis such that

$$L^4 \leq \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_2$$

and $Z(L) \cap L^4 \leq \mathbb{F}x_5 + \mathbb{F}x_4$. Now $y_3 \in (L^4)^\perp = Z_3(L)$ and as $Z_3(L) \cdot L^3 = \{0\}$, it follows that

$$x_1 y_3 = x_2 y_3 = y_1 y_3 = 0.$$

It follows from this that $x_1 y_2, y_1 y_2, y_3 y_2 \in \mathbb{F}x_5 + \mathbb{F}x_4$. Thus in particular these three elements are linearly dependent and we have $(\alpha x_1 + \beta y_1 + \gamma y_3) y_2 = 0$ where not all of α, β, γ are zero. Then $x_2, \alpha x_1 + \beta y_1 + \gamma y_3$ commute with all the basis elements except possibly y_4 and y_5. Suppose

$$\begin{aligned} (x_2 y_4, y_5) &= r \\ ((\alpha x_1 + \beta y_1 + \gamma y_3) y_4, y_5) &= s. \end{aligned}$$

If $r = 0$ then we get the contradiction that $x_2 \in Z(L)$ and if $r \neq 0$, we get the contradiction that $-s x_2 + r\alpha x_1 + r\beta y_1 + r\gamma y_3 \in Z(L)$. $\Box$

After these more general results we classify all the algebras where $Z(L)$ is isotropic of dimension 3. We consider the two subcases $\dim L^3 = 5$ and $\dim L^3 = 6$ separately.

4.1 The algebras where $\dim L^3 = 5$

We have seen that L^3 must be isotropic and thus in particular we have that $L^3 = (L^3)^\perp = Z_2(L)$ that implies that $L^4 \leq Z(L)$. By Lemma 4.4 we thus have $L^4 = Z(L)$. We have thus determined the terms of the lower and the upper central series

$L^2 \cdot L^2$	x_5	y_5		
	x_4	y_4	$(L^2 \cdot L^2)^\perp$	$Z(L) = L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3$
$Z(L) = L^4$	x_3	y_3		$Z_2(L) = L^3 = Z(L) + \mathbb{F}x_2 + \mathbb{F}x_1$
	x_2	y_2	$Z_3(L) = L^2$	$Z_3(L) = L^2 = Z_2(L) + \mathbb{F}y_1 + \mathbb{F}y_2$
$Z_2(L) = L^3$	x_1	y_1		

Remark. As $L^2 \cdot Z_2(L) = \{0\}$ we see that $x_1 y_2 = 0$. Now L^2 is not abelian as this would imply that $L^3 \leq Z(L)$. It follows that $y_1 y_2 \neq 0$ and we get a one dimensional characteristic subspace

$$L^2 \cdot L^2 = \mathbb{F}y_1 y_2.$$

Notice that $y_1 y_2 \in Z(L)$. We choose our basis such that $y_1 y_2 = x_5$. We will also work with the 9 dimensional characteristic subspace

$$V = (L^2 \cdot L^2)^\perp = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3 + \mathbb{F}y_4.$$

As $x_1 y_2 = 0$ we have that $x_1 y_3, x_1 y_4 \perp y_2$. As $y_1 y_2 = x_5$ we also have that $y_1 y_3, y_1 y_4 \perp y_1, y_2$ and $y_2 y_3, y_2 y_4 \perp y_1, y_2$. It follows that

$$V^2 + L^4 = (L^2 + \mathbb{F}y_3 + \mathbb{F}y_4)(L^2 + \mathbb{F}y_3 + \mathbb{F}y_4) + L^4 = \mathbb{F}y_3 y_4 + L^4.$$

We consider few subcases.

4.1.1 Algebras where $V^2 \leq L^4$

Notice then that $y_3 y_4 \in \mathbb{F}x_5$ and thus $x_1 y_3, x_2 y_3, x_1 y_4, x_2 y_4 \in \mathbb{F}x_5$. As $L^3 = L^4 + \mathbb{F}x_2 + \mathbb{F}x_1$, we have

$$L^4 = (\mathbb{F}x_2 + \mathbb{F}x_1)(\mathbb{F}y_3 + \mathbb{F}y_4 + \mathbb{F}y_5) = \mathbb{F}x_5 + \mathbb{F}x_2 y_5 + \mathbb{F}x_1 y_5.$$

Pick x_5, y_5, x_2, x_1, y_1 satisfying the conditions above and let

$$x_4 = -x_2 y_5,\ x_3 = -x_1 y_5.$$

We can then extend $x_5, x_4, x_3, y_1, y_2, y_5$ to a standard basis $x_5, x_4, x_3, x_2, x_1, y_1, y_2, y_3, y_4, y_5$ satisfying the conditions above. All triples involving both x_1 and

y_2 are 0. The remaining ones are

$$\begin{array}{lll}
(x_1y_3, y_5) = 1 & (x_1y_3, y_4) = 0 & (x_1y_4, y_5) = 0 \\
(x_2y_3, y_4) = 0 & (x_2y_3, y_5) = 0 & (x_2y_4, y_5) = 1 \\
(y_1y_2, y_3) = 0 & (y_1y_2, y_4) = 0 & (y_1y_2, y_5) = 1 \\
(y_2y_3, y_4) = 0 & (y_2y_3, y_5) = \alpha & (y_2y_4, y_5) = \beta \\
(y_1y_3, y_4) = 0 & (y_1y_3, y_5) = \gamma & (y_1y_4, y_5) = \delta \\
(y_3y_4, y_5) = r & &
\end{array}$$

Now let

$$\begin{array}{ll}
\tilde{y}_3 = y_3 + \alpha y_1 - \gamma y_2 - s x_2 - s\gamma x_3 - s\delta x_4 & \tilde{y}_2 = y_2 - s x_3 \\
\tilde{x}_1 = x_1 - \alpha x_3 - \beta x_4 & \tilde{y}_4 = y_4 + \beta y_1 - \delta y_2 \\
\tilde{x}_2 = x_2 + \gamma x_3 + \delta x_4 &
\end{array}$$

where $s = r + \alpha\delta - \beta\gamma$. One checks readily that we get a new standard basis with a presentation like the one above where $\tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} = \tilde{\delta} = \tilde{r} = 0$.

So we arrive at a unique algebra with presentation

$\mathcal{P}_{10}^{(3,1)}$: $(x_1y_3, y_5) = 1,\ (x_2y_4, y_5) = 1,\ (y_1y_2, y_5) = 1.$

One can check that the centre has dimension 3 and that L^3 has dimension 5. Also $((L^2 \cdot L^2)^{\perp})^2 = \mathbb{F}x_5 \leq L^4$.

4.1.2 Algebras where $V^2 \not\leq L^4$ but $V^2 \leq L^3$

Here we can pick our basis such that

$$V^2 + L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2.$$

Notice that $V^3 = \mathbb{F}x_2y_3 + \mathbb{F}x_2y_4$ and as $(y_3y_4, x_2) = 0$ we have that $V^3 \leq \mathbb{F}x_5$. As $x_2 \notin Z(L)$, we furthermore must have that $\dim V^3 = 1$. This means that there is a characteristic ideal W of codimension 1 in V such that $x_2W = V^2W = \{0\}$. We choose our basis such that

$$W = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3.$$

It follows that we have a chain of characteristic ideals:

$L^2 \cdot L^2$	x_5	y_5	
$W^\perp$	x_4	y_4	V
$Z(L) = L^4$	x_3	y_3	W
$V^2 + L^4$	x_2	y_2	$Z_3(L) = L^2$
$Z_2(L) = L^3$	x_1	y_1	$(V^2)^\perp \cap L^2$

$$\begin{aligned}
&L^2 \cdot L^2 = \mathbb{F}x_5 \\
&W^\perp = \mathbb{F}x_5 + \mathbb{F}x_4 \\
&Z(L) = L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 \\
&V^2 + L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 \\
&Z_2(L) = L^3 = Z(L) + \mathbb{F}x_2 + \mathbb{F}x_1 \\
&(V^2)^\perp \cap L^2 = L^3 + \mathbb{F}y_1 \\
&Z_3(L) = L^2 = Z_2(L) + \mathbb{F}y_1 + \mathbb{F}y_2 \\
&W = L^3 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3 \\
&V = L^3 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3 + \mathbb{F}y_4
\end{aligned}$$

We want to show that there is again a unique algebra satisfying these conditions. We modify the basis and reach a unique presentation. Notice that $V^2W = \{0\}$ and $L^2 \cdot Z_2(L) = \{0\}$ imply that

$$x_1y_2 = x_2y_3 = 0.$$

We have also chosen our basis such that

$$y_1y_2 = x_5. \tag{1}$$

Notice next that $x_2y_3 = 0$ implies that x_2y_4 is orthogonal to y_3 and y_4 and thus $x_2y_4 = ry_5$ where r must be nonzero as $x_2 \notin Z(L)$. By replacing y_4 and x_4 by ry_4 and $\frac{1}{r}x_4$, we can assume that

$$x_2y_4 = x_5. \tag{2}$$

As $y_3y_4 \in V^2 \leq L^3$ and as $x_1y_2 = 0$ we have that x_1y_4 is orthogonal to y_2, y_3, y_4. Thus $x_1y_4 = \alpha x_5$ for some $\alpha \in \mathbb{F}$. Replacing x_1, y_2 by $x_1 - \alpha x_2$ and $y_2 + \alpha y_1$ we get a new standard basis where

$$x_1y_4 = 0. \tag{3}$$

Notice that the change in y_2 does not affect (1). We next turn our attention to x_1y_3. As $x_1y_2 = 0$ and $x_1y_4 = 0$, we have that x_1y_3 is orthogonal to y_1, y_2, y_3, y_4 and thus $x_1y_3 = rx_5$ where r is nonzero since $x_1 \notin Z(L)$. By replacing y_3 and x_3 by ry_3 and $\frac{1}{r}x_3$ we get

$$x_1y_3 = x_5. \tag{4}$$

Now we see, as $y_1y_2 = x_5$ and $y_3y_4 \in L^4 + V^2$, that y_1y_3 is orthogonal to y_1, y_2, y_3 and y_4. Thus $y_1y_3 = ax_5$ for some $a \in \mathbb{F}$. Replacing y_1 by $y_1 - ax_1$ we can assume that

$$y_1y_3 = 0. \tag{5}$$

As $x_1y_2 = 0$ the change in y_1 does not affect (1). From the discussion above we know that y_1y_4 is orthogonal to y_1, y_2, y_3 and y_4 and thus $y_1y_4 = ax_5$ for some $a \in \mathbb{F}$. Replacing y_4, x_2 by $y_4 - ay_2$ and $x_2 + ax_4$, we get a new standard basis where

$$y_1y_4 = 0. \tag{6}$$

These changes do not affect (2) and (3). As $y_3y_4 \in V^2 + L^4$ but not in L^4 we know that $(y_2y_3, y_4) = r$ for some nonzero $r \in \mathbb{F}$. Suppose also that $(y_3y_4, y_5) = \alpha$. Then $y_3y_4 = rx_2 + \alpha x_5$. Replace x_2 and y_5 by $x_2 + \frac{\alpha}{r}x_5$ and $y_5 - \frac{\alpha}{r}y_2$. Then

$$y_3y_4 = rx_2. \tag{7}$$

The changes do not affect (2). Then consider the triples

$$(y_2y_3, y_5) = a, \quad (y_2y_4, y_5) = b.$$

Replacing y_5, x_4, x_3 by $y_5 - \frac{a}{r}y_4 + \frac{b}{r}y_3$, $x_4 + \frac{a}{r}x_5$ and $x_3 - \frac{b}{r}x_5$ we can assume that

$$(y_2y_3, y_5) = (y_2y_4, y_5) = 0. \tag{8}$$

We have then arrived at a presentation where the only nonzero triples are

$$(x_2y_4, y_5) = 1,\ (x_1y_3, y_5) = 1,\ (y_1y_2, y_5) = 1,\ (y_2y_3, y_4) = r.$$

Replacing $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4$, by $\frac{1}{r}x_1, ry_1, rx_2, \frac{1}{r}y_2, \frac{1}{r}x_3, ry_3, rx_4, \frac{1}{r}y_4$, we get a unique algebra with presentation:

$$\mathcal{P}_{10}^{(3,2)} :\ (x_2y_4, y_5) = 1,\ (x_1y_3, y_5) = 1,\ (y_1y_2, y_5) = 1,\ (y_2y_3, y_4) = 1.$$

One can easily check that conversely this algebra belongs to the category that we have been studying.

4.1.3 Algebras where $V^2 \leq L^2$ but $V^2 \not\leq L^3$

Pick our basis such that

$$V^2 + L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}y_1.$$

Notice then that

$$\mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 = (V^2 + L^4)^{\perp} \cap L^3 = (V^2)^{\perp} \cap L^3$$

is a characteristic ideal of L. As $y_3y_4 \in V^2 + L^4$ we have that $x_2y_3 \perp y_4$ and $x_2y_4 \perp y_3$. Thus $x_2V \leq \mathbb{F}x_5$. As $x_2 \not\in Z(L)$, we must furthermore have that $x_2V = ((V^2)^{\perp} \cap L^3)V = \mathbb{F}x_5$. This implies that the centraliser of $(V^2)^{\perp} \cap L^3$ in V is a characteristic ideal W of codimension 1. We can choose our basis such that

$$W = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3.$$

We now get a chain of characteristic ideals as before

$L^2 \cdot L^2$	x_5	y_5	
$W^{\perp}$	x_4	y_4	V
$Z(L) = L^4$	x_3	y_3	W
$(V^2)^{\perp} \cap L^3$	x_2	y_2	$Z_3(L) = L^2$
$Z_2(L) = L^3$	x_1	y_1	$V^2 + L^3$

$$\begin{aligned}
&L^2 \cdot L^2 = \mathbb{F}x_5 \\
&W^{\perp} = \mathbb{F}x_5 + \mathbb{F}x_4 \\
&Z(L) = L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 \\
&(V^2)^{\perp} \cap L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 \\
&Z_2(L) = L^3 = Z(L) + \mathbb{F}x_2 + \mathbb{F}x_1 \\
&V^2 + L^3 = L^3 + \mathbb{F}y_1 \\
&Z_3(L) = L^2 = Z_2(L) + \mathbb{F}y_1 + \mathbb{F}y_2 \\
&W = L^3 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3 \\
&V = L^3 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3 + \mathbb{F}y_4
\end{aligned}$$

As $((V^2)^{\perp} \cap L^3)W = \{0\}$ and $L^2 \cdot Z_2(L) = \{0\}$, we see that

$$x_1y_2 = x_2y_3 = 0.$$

We have also chosen our basis such that

$$y_1y_2 = x_5. \tag{9}$$

Notice next that $x_2y_3 = 0$ implies that x_2y_4 is orthogonal to y_3 and y_4 and thus $x_2y_4 = ry_5$ where r must be nonzero as $x_2 \not\in Z(L)$. By replacing y_4 and x_4 by ry_4 and $\frac{1}{r}x_4$, we can assume that

$$x_2y_4 = x_5. \tag{10}$$

As $y_3y_4 \in V^2 + L^4$ and $y_1y_2 = x_5$, we have that y_1y_4 is orthogonal to y_2, y_3, y_4. Thus $y_1y_4 = ay_5$ for some $a \in \mathbb{F}$. Replacing y_4, x_2 by $y_4 - ay_2$ and $x_2 + ax_4$ we get

$$y_1y_4 = 0. \tag{11}$$

Notice that the change does not affect (10). Next notice similarly that y_1y_3 is orthogonal to y_2, y_3, y_4 and thus $y_1y_3 = ay_5$ for some $a \in \mathbb{F}$. Replacing y_3 and x_2 by $y_3 - ay_2$ and $x_2 + ax_3$ we get

$$y_1y_3 = 0. \tag{12}$$

Notice that (11) is not affected by this change. We know that $x_1y_2 = 0$. The possible nonzero triples involving x_1 are then

$$(x_1y_3, y_4) = r,\ (x_1y_3, y_5) = a,\ (x_1y_4, y_5) = b.$$

Notice that as $y_3y_4 \in (Z(L) + Fy_1) \setminus Z(L)$ we must have that $r \neq 0$. Replace y_5, x_4, x_3 by $y_5 - \frac{a}{r}y_4 + \frac{b}{r}y_3$, $x_4 + \frac{a}{r}x_5$ and $x_3 - \frac{b}{r}x_5$ and we get a new standard basis where

$$x_1y_3 = rx_4, \quad x_1y_4 = -rx_3.$$

Replacing y_3, x_3 by $ry_3, \frac{1}{r}x_3$ gives

$$x_1y_3 = x_4, \quad x_1y_4 = -x_3. \tag{13}$$

It follows that $(y_2y_3, y_4) = (y_3y_4, y_5) = 0$. Suppose $(y_2y_3, y_5) = a,\ (y_2y_4, y_5) = b$. Replace y_3, y_4, x_1 by $y_3 + ay_1,\ y_4 + by_1$ and $x_1 - ax_3 - bx_4$. Notice that these changes do not affect the equations above and we now arrive at a unique algebra with presentation:

$\mathcal{P}_{10}^{(3,3)}:\ (x_2y_4, y_5) = 1,\ (x_1y_3, y_4) = 1,\ (y_1y_2, y_5) = 1.$

Calculations show that conversely this algebra belongs to the relevant category. There are thus exactly three algebras where $Z(L)$ is isotropic of dimension 3 and where the dimension of L^3 is 5.

Proposition 4.5 *There are exactly three NSAAs of dimension* 10 *that have an isotropic centre of dimension* 3 *and where dim* $L^3 = 5$. *These are given by the presentations:*

$\mathcal{P}_{10}^{(3,1)}:\ (x_1y_3, y_5) = 1,\ (x_2y_4, y_5) = 1,\ (y_1y_2, y_5) = 1.$

$\mathcal{P}_{10}^{(3,2)}:\ (x_2y_4, y_5) = 1,\ (x_1y_3, y_5) = 1,\ (y_1y_2, y_5) = 1,\ (y_2y_3, y_4) = 1.$

$\mathcal{P}_{10}^{(3,3)}:\ (x_2y_4, y_5) = 1,\ (x_1y_3, y_4) = 1,\ (y_1y_2, y_5) = 1.$

4.2 The algebras where $\dim L^3 = 6$

Here we are thus assuming that

$$L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1.$$

Lemma 4.6 *We have dim* $L^4 = 4$.

Proof By Lemma 4.4, we know that $Z(L) \leq L^4$ and we also have $L^4 \leq \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2$. Thus if the dimension of L^4 is not 4, then $L^4 = Z(L) = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3$ and $Z_3(L) = (L^4)^{\perp} = L^3 + \mathbb{F}y_2$. As $Z_3(L) \cdot L^3 = \{0\}$, it follows that $x_1 y_2 = y_1 y_2 = 0$ and L^2 is abelian. Hence we get the contradiction that $L^3 \leq Z(L)$. $\Box$

It follows that we have $L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2$.

Lemma 4.7 *We have dim* $L^5 = 2$.

Proof We have an alternating form

$$\phi : L/L^2 \times L/L^2 \to \mathbb{F}$$

given by $\phi(\bar{y}, \bar{z}) = (x_2 y, z)$. As L/L^2 has odd dimension we know that the isotropic part must be non-trivial. Thus we can then choose our standard basis such that $(x_2 y_3, y_4) = (x_2 y_3, y_5) = 0$ and thus $x_2 y_3 = 0$. It follows that $L^5 = \mathbb{F}x_2(y_3 + y_4 + y_5) = \mathbb{F}x_2 y_4 + \mathbb{F}x_2 y_5$ and thus it is of dimension at most 2. As $L^4 \not\leq Z(L)$ we have $\dim L^5 > 0$ and as we know [3, Proposition 3.10] that $\dim L^5 \neq 1$ we must have that $\dim L^5 = 2$. $\Box$.

We thus have determined the lower and upper central series of L. We have

$$\begin{array}{r|c|c|l}
\cline{2-3}
 & x_5 & y_5 & \\
L^5 & x_4 & y_4 & \\
\cline{2-3}
Z(L) & x_3 & y_3 & Z_4(L) \\
\cline{2-3}
L^4 = Z_2(L) & x_2 & y_2 & L^2 \\
\cline{2-3}
 & x_1 & y_1 & L^3 = Z_3(L) \\
\cline{2-3}
\end{array}$$

$$\begin{aligned}
&L^5 = \mathbb{F}x_5 + \mathbb{F}x_4 \\
&Z(L) = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 \\
&Z_2(L) = L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 \\
&Z_3(L) = L^3 = Z(L) + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 \\
&L^2 = L^3 + \mathbb{F}y_2 \\
&Z_4(L) = L^3 + \mathbb{F}y_2 + \mathbb{F}y_3
\end{aligned}$$

Notice that $x_2 \in L^4$ and $y_3 \in Z_4(L)$ and thus $x_2 y_3 = 0$. Also

$$L^3 L^2 = \mathbb{F}x_1 y_2 + \mathbb{F}y_1 y_2 \leq Z(L).$$

Furthermore x_1y_2 and y_1y_2 are linearly independent. To see this we argue by contradiction and suppose that $0 = ax_1y_2 + by_1y_2$ for some $a, b \in \mathbb{F}$ where not both a, b are zero. Then $(ax_1 + by_1)L \leq Z(L)$ that would give us the contradiction that $ax_1 + by_1 \in Z_2(L)$.

We thus have that L^3L^2 is a 2-dimensional subspace of $Z(L)$ and we consider two possible cases namely $L^3L^2 = L^5$ and $L^3L^2 \neq L^5$. We consider the latter first.

4.2.1 Algebras where $L^3L^2 \neq L^5$

Here $L^3L^2 \cap L^5$ is one dimensional and we can choose our standard basis such that $L^3L^2 = L^3y_2 = \mathbb{F}x_5$. In order to clarify the structure further we introduce the following isotropic characteristic ideal of dimension 5:

$$U = \{x \in L^3 : xL^2 \leq L^3L^2 \cap L^5\}.$$

Now L^3L^2 is of dimension 2 and $L^4L^2 = 0$ and thus U is of codimension 1 in L^3 and contains L^4. We can thus choose our standard basis such that $U = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}x_1$. We thus have the following picture

$L^5 \cap L^3L^2$	x_5	y_5	
L^5	x_4	y_4	$(L^3L^2 \cap L^5)^\perp$
$Z(L)$	x_3	y_3	$Z_4(L)$
$L^4 = Z_2(L)$	x_2	y_2	L^2
U	x_1	y_1	$L^3 = Z_3(L)$

Notice that $UZ_4(L) = \mathbb{F}x_1y_2 + \mathbb{F}x_1y_3 = \mathbb{F}x_5 + \mathbb{F}x_1y_3$, where $x_1y_3 \in L^5$. Again we consider two possible cases.

I. Algebras where $UZ_4(L)$ is 1-dimensional

Here $UZ_4(L) = x_1Z_4(L) = \mathbb{F}x_5$ and there is a characteristic subspace V of codimension 1 in $Z_4(L)$ that contains L^3 given by the formula

$$V = \{x \in Z_4(L) : Ux = 0\}.$$

We can then choose our standard basis such that

$$V = L^3 + \mathbb{F}y_3 = U + \mathbb{F}y_1 + \mathbb{F}y_3.$$

Notice that in particular $x_1y_3 = 0$. From this we also get a 1-dimensional characteristic subspace $V^2 = \mathbb{F}y_1y_3$. Notice that $(y_1y_3, y_2) \neq 0$ as otherwise $y_1y_2 \in L^5 \cap L^3L^2$ that contradicts our assumption that $L^3L^2 \neq L^5$. Thus $y_1y_3 \in L^4 \setminus Z(L)$ and we can choose our standard basis such that $\mathbb{F}y_1y_3 = \mathbb{F}x_2$. In fact it is not difficult to see that with the data we have acquired so far we can choose our standard basis such that

$$x_1y_2 = x_5,\ y_1y_2 = x_3,\ x_1y_3 = 0,\ y_1y_3 = -x_2. \tag{14}$$

This deals with all triple values apart from

$$\begin{array}{llll} (x_1y_4, y_5) = a, & (y_2y_3, y_4) = c, & (y_2y_4, y_5) = e, & (x_2y_4, y_5) = r, \\ (y_1y_4, y_5) = b, & (y_2y_3, y_5) = d, & (y_3y_4, y_5) = f. & \end{array}$$

Notice that $r \neq 0$ as $x_2y_3 = 0$ but $x_2 \not\in Z(L)$. We will show that we can choose a new standard basis so that the values of $a = b = c = d = e = f = 0$ and $r = 1$. Replace $x_1, x_4, y_1, y_2, y_3, y_4, y_5$ by $x_1 - (a/r)x_2 + cx_4 + dx_5, rx_4, y_1 - (b/r)x_2, y_2 - (b/r)x_1 - (e/r)x_2 - (f/r)x_3 - (bc/r)x_4 + (a/r)y_1, y_3 - (f/r)x_2, (1/r)y_4 - (c/r)y_1, y_5 - dy_1 + (bd/r)x_2$. Thus we have that we get a unique algebra.

Proposition 4.8 *There is a unique nilpotent SAA L with an isotropic centre of dimension 3 and where $\dim L^3 = 6$ that has the further properties that $L^3L^2 \neq L^5$ and $\dim UZ_4(L) = 1$. This algebra is given by the presentation*

$$\mathcal{P}_{10}^{(3,4)} :\ (x_1y_2, y_5) = 1,\ (y_1y_2, y_3) = 1,\ (x_2y_4, y_5) = 1.$$

Remark. As before, inspection shows that the algebra with the presentation above satisfies all the properties listed.

II. Algebras where $UZ_4(L)$ is 2-dimensional

Here we can pick our standard basis such that $UZ_4(L) = \mathbb{F}x_5 + \mathbb{F}x_4$. As $L^3L^2 \neq L^5$ we know that $(y_1y_2, y_3) \neq 0$ and from this one sees that $L^3Z_4(L) = L^4$. Furthermore it is not difficult to see that we can choose our standard basis such that

$$x_1y_2 = x_5,\ x_1y_3 = x_4,\ y_1y_2 = x_3,\ y_1y_3 = -x_2. \tag{15}$$

In order to clarify the structure further we are only left with the triple values

$$\begin{array}{llll} (x_1y_4, y_5) = a, & (y_2y_3, y_4) = c, & (y_2y_4, y_5) = e, & (x_2y_4, y_5) = r, \\ (y_1y_4, y_5) = b, & (y_2y_3, y_5) = d, & (y_3y_4, y_5) = f. & \end{array}$$

Let $\alpha = b - cr$ and replace $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$ by $x_1 - (a/r)x_2 + (\alpha/r)x_4 + dx_5, (1/r)x_2, rx_3, (1/r)x_4, rx_5, y_1 - (b/r)x_2, ry_2 - bx_1 - ex_2 - fx_3 + ay_1, (1/r)y_3 - (f/r^2)x_2, ry_4 + \alpha(b/r)x_2 - \alpha y_1, (1/r)y_5 + d(b/r^2)x_2 - (d/r)y_1$. One checks readily that we get a new standard basis such that $a = b = c = d = e = f = 0$ and $r = 1$. We thus see that we get a unique algebra.

Proposition 4.9 *There is a unique nilpotent SAA L with an isotropic centre of dimension 3 and where $\dim L^3 = 6$ that has the further properties that $L^3L^2 \neq L^5$ and $\dim UZ_4(L) = 2$. This algebra is given by the presentation*

$$\mathcal{P}_{10}^{(3,5)} : \ (x_1y_2, y_5) = 1, \ (y_1y_2, y_3) = 1, \ (x_1y_3, y_4) = 1, \ (x_2y_4, y_5) = 1.$$

4.2.2 The algebras where $L^3L^2 = L^5$

	x_5	y_5	
L^5	x_4	y_4	
$Z(L)$	x_3	y_3	$Z_4(L)$
$L^4 = Z_2(L)$	x_2	y_2	L^2
	x_1	y_1	$L^3 = Z_3(L)$

$$\begin{aligned}
&L^5 = \mathbb{F}x_5 + \mathbb{F}x_4 \\
&Z(L) = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 \\
&Z_2(L) = L^4 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 \\
&Z_3(L) = L^3 = Z(L) + \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 \\
&L^2 = L^3 + \mathbb{F}y_2 \\
&Z_4(L) = L^3 + \mathbb{F}y_2 + \mathbb{F}y_3
\end{aligned}$$

Here we are assuming that $L^5 = L^3L^2 = \mathbb{F}x_1y_2 + \mathbb{F}y_1y_2$ and thus in particular we know that x_1y_2, y_1y_2 is a basis for L^5. We will now introduce some linear maps that will help us in understanding the structure. Consider first the linear maps

$$\begin{aligned}
\phi &: \ L^3/L^4 \longrightarrow L^5, \ \bar{u} = u + L^4 \mapsto u \cdot y_2 \\
\psi &: \ L^3/L^4 \longrightarrow L^5, \ \bar{u} = u + L^4 \mapsto u \cdot y_3.
\end{aligned}$$

As $L^4Z_4(L) = \{0\}$, these maps are well defined. As $L^3L^2 = L^5$ we also know that ϕ is bijective. We thus have the linear map

$$\tau = \psi\phi^{-1} : \ L^5 \longrightarrow L^5.$$

It is the map τ that will be our key towards understanding the structure of the algebra.

Lemma 4.10 *The minimal polynomial of $\tau = \psi\phi^{-1}$ must be of degree 2.*

Proof We argue by contradiction and suppose that $\tau = \lambda\mathrm{id}$. Replacing y_3, x_2 by $y_3 - \lambda y_2, x_2 + \lambda x_3$ gives us a new standard basis where $\tau = 0$. Pick our standard basis such that $\bar{x}_1 = x_1 + L^4 = \phi^{-1}(x_4)$ and $\bar{y}_1 = y_1 + L^4 = \phi^{-1}(x_5)$. We then have

$$x_1 y_2 = x_4,\ y_1 y_2 = x_5,\ x_1 y_3 = 0,\ y_1 y_3 = 0.$$

Now $y_2 y_3 \perp x_1, y_1, y_2, y_3$ and thus

$$y_2 y_3 = a x_4 + b x_5$$

for some $a, b \in \mathbb{F}$. Replacing y_3 by $y_3 + a x_1 + b y_1$, x_1 by $x_1 - b x_3$ and y_1 by $y_1 + a x_3$, we can assume that $y_2 y_3 = 0$.

Now suppose that $(y_3 y_4, y_5) = a$ and $(x_2 y_4, y_5) = b$. Notice that $b \neq 0$ as $x_2 \notin Z(L)$ and $x_2 y_3 = 0$. Replace y_3, y_2 by $y_3 - (a/b) x_2,\ y_2 - (a/b) x_3$ and we get a new standard basis where all the previous identities hold but also $(y_3 y_4, y_5) = 0$. We thus get the contradiction that $y_3 \in Z(L)$. $\Box$

Notice next that if we have an alternative standard basis $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{y}_5$, then $\tilde{y}_2 = c y_2 + u$ and $\tilde{y}_3 = a y_3 + b y_2 + v$ where $a, c \neq 0$ and where $u, v \in L^3$. If the minimal polynomial of τ with respect to the old basis is $f(t)$ then the minimal polynomial with respect to the new basis is a multiple of $f((c/a)(t - (b/c)))$. In particular we have the following possible distinct scenarios that do not depend on what standard basis we choose.

A. The minimal polynomial of τ has two distinct roots in $\mathbb{F}$.
B. The minimal polynomial of τ has a double root in $\mathbb{F}$.
C. The minimal polynomial of τ is irreducible in $\mathbb{F}[t]$.

I. Algebras of type A

Suppose the two distinct roots of the minimal polynomial of $\tau = \psi\phi^{-1}$ are λ and μ. Pick some eigenvectors x_4 and x_5 with respect to the eigenvalues λ and μ respectively. Thus

$$\begin{aligned} \psi\phi^{-1}(x_4) &= \lambda x_4, \\ \psi\phi^{-1}(x_5) &= \mu x_5. \end{aligned}$$

Replacing y_3, x_2 by $y_3 - \lambda y_2, x_2 + \lambda x_3$ we see that $\psi\phi^{-1}(x_4) = 0$ and we can thus assume that $\lambda = 0$. Then replace y_3, x_3 by $(1/\mu) y_3, \mu x_3$ and we get that

$\psi\phi^{-1}(x_5) = x_5$ and we can now assume that $\mu = 1$.

We would like to pick our standard basis such that $\bar{x}_1 = x_1 + L^4 = \phi^{-1}(x_4)$ and $\bar{y}_1 = y_1 + L^4 = \phi^{-1}(x_5)$. The only problem here is that we need $(x_1, y_1) = 1$ but this can be easily arranged. If $(x_1, y_1) = \sigma$ then we just need to replace y_1, x_5, y_5 by $(1/\sigma)y_1, (1/\sigma)x_5, \sigma y_5$. We have thus seen that we can choose our standard basis such that

$$x_1y_2 = x_4,\ y_1y_2 = x_5,\ x_1y_3 = 0,\ y_1y_3 = x_5. \tag{16}$$

Recall also that $x_2y_3 = 0$ since $L^4Z_4(L) = \{0\}$. In order to fully determine the structure of the algebra we are only left with the following triple values

$$\begin{array}{llll} (x_1y_4, y_5) = a, & (y_2y_3, y_4) = c, & (y_2y_4, y_5) = e, & (x_2y_4, y_5) = r, \\ (y_1y_4, y_5) = b, & (y_2y_3, y_5) = d, & (y_3y_4, y_5) = f. & \end{array}$$

Notice that $r \neq 0$ as $x_2y_3 = 0$ but $x_2 \notin Z(L)$. Let $\alpha = (1/r)(e - f - bd + ab/r - ac)$ and replace y_4, y_3, y_2, y_1, x_1 by $y_4 + ey_1 - e(b/r)x_2 + cex_3, y_3 + cx_1 - \alpha x_2 + (d + a/r)y_1, y_2 + (a/r)y_1 - (b/r)x_1 + ((e-f)/r)x_3, y_1 - (b/r)x_2 + cx_3, x_1 - (a/r)x_2 - (d + a/r)x_3 - ex_4$. One checks readily that we get a new standard basis such that $a = b = c = d = e = f = 0$. We have thus seen that L has a presentation of the form $\mathcal{P}_{10}^{(3,6)}(r)$ as described in the next proposition.

Proposition 4.11 *Let L be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 3 that has the further properties that dim $L^3 = 6$, $L^3L^2 = L^5$ and L is of type A. Then L has a presentation of the form*

$$\mathcal{P}_{10}^{(3,6)}(r):\ (x_2y_4, y_5) = r,\ (x_1y_2, y_4) = 1,\ (y_1y_2, y_5) = 1,\ (y_1y_3, y_5) = 1$$

where $r \neq 0$. Furthermore the presentations $\mathcal{P}_{10}^{(3,6)}(r)$ and $\mathcal{P}_{10}^{(3,6)}(s)$ describe the same algebra if and only if $s/r \in (\mathbb{F}^*)^3$.

Proof We have already seen that all such algebras have a presentation of the form $\mathcal{P}_{10}^{(3,6)}(r)$ for some $0 \neq r \in \mathbb{F}$. Straightforward calculations show that conversely any algebra with such a presentation has the properties stated in the Proposition. It remains to prove the isomorphism property. To see that the property is sufficient, suppose we have an algebra L with presentation $\mathcal{P}_{10}^{(3,6)}(r)$ with respect to some given standard basis. Let s be any element in $\mathbb{F}^*$ such that $s/r = b^3 \in (\mathbb{F}^*)^3$. Replace the basis with a new standard basis $\tilde{x}_1, \ldots, \tilde{y}_5$ where $\tilde{x}_1 = x_1$, $\tilde{y}_1 = y_1$, $\tilde{x}_2 = bx_2$, $\tilde{y}_2 = (1/b)y_2$, $\tilde{x}_3 = bx_3$,

$\tilde{y}_3 = (1/b)y_3$, $\tilde{x}_4 = (1/b)x_4$, $\tilde{y}_4 = by_4$, $\tilde{x}_5 = (1/b)x_5$, $\tilde{y}_5 = by_5$. Direct calculations show that L has presentation $\mathcal{P}_{10}^{(3,6)}(s)$ with respect to the new basis.

It remains to see that the property is necessary. Consider again an algebra L with presentation $\mathcal{P}_{10}^{(3,6)}(r)$ and suppose that L has also a presentation $\mathcal{P}_{10}^{(3,6)}(s)$ with respect to some other standard basis $\tilde{x}_1, \ldots, \tilde{y}_5$. We want to show that $s/r \in (\mathbb{F}^*)^3$. We know that $L = \mathbb{F}\tilde{y}_5 + \mathbb{F}\tilde{y}_4 + Z_4(L) = \mathbb{F}y_5 + \mathbb{F}y_4 + Z_4(L)$. Thus

$$\begin{aligned}\tilde{y}_4 &= ay_4 + by_5 + u\\ \tilde{y}_5 &= cy_4 + dy_5 + v\end{aligned}$$

for some $u, v \in Z_4(L)$ and $a, b, c, d \in \mathbb{F}$ where $ad - bc \neq 0$. As $L^3L^2 = L^5 \perp Z_4(L)$ and as $Z_4(L)L^4 = \{0\}$ we have $(Z_4(L)L^2, L^3) = (Z_4(L)L, L^4) = 0$ and thus $Z_4(L)L^2 \leq (L^3)^\perp = L^4$ and $Z_4(L)L \leq (L^4)^\perp = L^3$. It follows that

$$\begin{aligned}\tilde{y}_4\tilde{y}_5\tilde{y}_5 &= (ay_4 + by_5)(cy_4 + dy_5)(cy_4 + dy_5) + w\\ \tilde{y}_5\tilde{y}_4\tilde{y}_4 &= (cy_4 + dy_5)(ay_4 + by_5)(ay_4 + by_5) + z\end{aligned}$$

for some $w, z \in L^4$. Using the fact that $(L^4, L^3) = 0$, as $L^6 = \{0\}$, we then see that

$$s^2 = (\tilde{y}_4\tilde{y}_5\tilde{y}_5, \tilde{y}_5\tilde{y}_4\tilde{y}_4) = r^2(ad - bc)^3.$$

Hence $s/r \in (\mathbb{F}^*)^3$. $\Box$

Remark. Notice that it follows that we have only one algebra if $(\mathbb{F}^*)^3 = \mathbb{F}^*$. This includes all fields that are algebraically closed as well as $\mathbb{R}$. For a finite field of order p^n there are 3 algebras if $3|p^n - 1$ but otherwise one. For $\mathbb{Q}$ there are infinitely many algebras.

II. Algebras of type B

Suppose that the double root of the minimal polynomial of $\tau = \psi\phi^{-1}$ is λ. We can then have a basis x_4, x_5 for L^5 such that

$$\begin{aligned}\psi\phi^{-1}(x_4) &= \lambda x_4\\ \psi\phi^{-1}(x_5) &= \lambda x_5 + x_4.\end{aligned}$$

If we replace y_3, x_2 by $y_3 - \lambda y_2, x_2 + \lambda x_3$ then we can furthermore assume that $\lambda = 0$. We want to pick our standard basis such that $\bar{x}_1 = x_1 +$

$L^4 = \phi^{-1}(x_4)$ and $\bar{y}_1 = y_1 + L^4$. Again the only problem is to arrange for $(x_1, y_1) = 1$. But if $(x_1, y_1) = \sigma$ then we replace x_5, x_3, y_1, y_3, y_5 by $(1/\sigma)x_5, (1/\sigma)x_3, (1/\sigma)y_1, \sigma y_3, \sigma y_5$ and that gives $(x_1, y_1) = 1$. We have thus seen that we can choose our standard basis such that

$$x_1y_2 = x_4,\ y_1y_2 = x_5,\ x_1y_3 = 0,\ y_1y_3 = x_4. \tag{17}$$

As before we have furthermore $x_2y_3 = 0$ and we are only left with the following triple values

$$\begin{array}{llll} (x_1y_4, y_5) = a, & (y_2y_3, y_4) = c, & (y_2y_4, y_5) = e, & (x_2y_4, y_5) = r, \\ (y_1y_4, y_5) = b, & (y_2y_3, y_5) = d, & (y_3y_4, y_5) = f. & \end{array}$$

Notice that $r \neq 0$ as $x_2y_3 = 0$ but $x_2 \notin Z(L)$. Let $\alpha = (a^2 + acr + bdr)/r$, $\beta = cr + a$ and replace $x_1, y_3, y_1, y_4, y_2, y_5$ by $x_1 - (a/r)x_2 - dx_3 - ex_4 + fx_5, y_3 - (\alpha/r)x_2 + (\beta/r)x_1 + dy_1, y_1 - (b/r)x_2 + (\beta/r)x_3, y_4 - e(b/r)x_2 + e(\beta/r)x_3 + ey_1, y_2 + (a/r)y_1 - (b/r)x_1, y_5 + f(b/r)x_2 - f(\beta/r)x_3 - fy_1$. One checks readily that these changes do not affect (17) and we get a new standard basis where $a = b = c = d = e = f = 0$. We thus arrive at at a presentation of the form $\mathcal{P}_{10}^{(3,7)}(r)$ as given in the next proposition.

Proposition 4.12 *Let L be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 3 that has the further properties that $\dim L^3 = 6$, $L^3L^2 = L^5$ and L is of type B. Then L has a presentation of the form*

$$\mathcal{P}_{10}^{(3,7)}(r):\ (x_2y_4, y_5) = r,\ (x_1y_2, y_4) = 1,\ (y_1y_2, y_5) = 1,\ (y_1y_3, y_4) = 1$$

where $r \neq 0$. Furthermore the presentations $\mathcal{P}_{10}^{(3,6)}(r)$ and $\mathcal{P}_{10}^{(3,6)}(s)$ describe the same algebra if and only if $s/r \in (\mathbb{F}^*)^3$.

Proof Similar to the proof of Proposition 4.11.

III. Algebras of type C

It turns out to be useful to consider the cases $char\,\mathbb{F} \neq 2$ and $char\,\mathbb{F} = 2$ separately.

a. The algebras where $char\,\mathbb{F} \neq 2$

Suppose the minimal polynomial of $\tau = \psi\phi^{-1}$ is $t^2 + at + b$ with respect to some y_2, y_3. Replacing y_3 by $y_3 + (a/2)y_2$, one gets a minimal polynomial of the form $t^2 - s$ with $s \notin \mathbb{F}^2$.

Remark. Let $\tilde{y}_3 = \alpha y_3 + u$ where $\alpha \neq 0$ and $u \in L^2$. For the minimal polynomial of τ to have trivial linear term we must have $u \in L^3$. Thus $\mathbb{F}y_3 + L^3$ is a characteristic subspace of L.

Pick any $0 \neq x_5 \in L^5$ and let $x_4 = \psi\phi^{-1}(x_5)$. Then $\psi\phi^{-1}(x_4) = sx_5$. We want to pick our standard basis such that $\phi^{-1}(x_4) = x_1 + L^4, \phi^{-1}(x_5) = y_1 + L^4$. For this to work out we need $(x_1, y_1) = 1$. Again this can be easily arranged. If $(x_1, y_1) = \sigma$, then we replace x_5, y_1, y_3 by $(1/\sigma)x_5$, $(1/\sigma)y_1$, σy_3 and we get $(x_1, y_1) = 1$ and $\psi\phi^{-1}(x_4) = (\sigma^2 s)x_5$. We have thus seen that we choose our standard basis such that

$$x_1y_2 = x_4,\ y_1y_2 = x_5,\ x_1y_3 = sx_5,\ y_1y_3 = x_4 \tag{18}$$

for some $s \notin \mathbb{F}^2$. In order to clarify the structure further we are only left with the following triple values

$$\begin{array}{llll} (x_1y_4, y_5) = a, & (y_2y_3, y_4) = c, & (y_2y_4, y_5) = e, & (x_2y_4, y_5) = r, \\ (y_1y_4, y_5) = b, & (y_2y_3, y_5) = d, & (y_3y_4, y_5) = f. & \end{array}$$

Notice that $r \neq 0$ as $x_2y_3 = 0$ but $x_2 \notin Z(L)$. Let $\alpha = (a^2 + acr - b^2s + bdr)/r$, $\beta = c + a/r$, $\gamma = (s(b/r) - d)$ and replace $x_1, y_4, y_1, y_5, y_2, y_3$ by $x_1 - (a/r)x_2 - ex_4 + fx_5 + \gamma x_3, y_4 - e(b/r)x_2 + e\beta x_3 + ey_1, y_1 - (b/r)x_2 + \beta x_3, y_5 + f(b/r)x_2 - f\beta x_3 - fy_1, y_2 - (b/r)x_1 + (a/r)y_1, y_3 - (\alpha/r)x_2 + \beta x_1 - \gamma y_1$. We see then that the remaining triple values are zero. Thus L has a presentation of the form $\mathcal{P}_{10}^{(3,8)}(r, s)$ as described in the next proposition.

Proposition 4.13 *Let L be a nilpotent SAA of dimension 10 over a field of characteristic that is not 2 that has an isotropic centre of dimension 3. Suppose also that L has the further properties that $\dim L^3 = 6$, $L^3L^2 = L^5$ and L is of type C. Then L has a presentation of the form*

$$\mathcal{P}_{10}^{(3,8)}(r, s) :\ (x_2y_4, y_5) = r,\ (x_1y_3, y_5) = s,\ (x_1y_2, y_4) = 1,\ (y_1y_2, y_5) = 1,\ (y_1y_3, y_4) = 1$$

where $r \neq 0$ and $s \notin \mathbb{F}^2$. Furthermore the presentations $\mathcal{P}_{10}^{(3,8)}(\tilde{r}, \tilde{s})$ and $\mathcal{P}_{10}^{(3,8)}(r, s)$ describe the same algebra if and only if $\frac{\tilde{r}}{r} \in (\mathbb{F}^*)^3$ and $\frac{s}{\tilde{s}} \in G(s)$ where $G(s) = \{(x^2 - y^2s)^2 :\ (x, y) \in \mathbb{F} \times \mathbb{F} \setminus \{(0, 0)\}\}$.

Proof We have already seen that any such algebra has a presentation of the given form. Direct calculations show that an algebra with a presentation $\mathcal{P}_{10}^{(3,8)}(r,s)$ has the properties stated. We turn to the isomorphism property. To see that the condition is sufficient, suppose we have an algebra L that has presentation $\mathcal{P}_{10}^{(3,8)}(r,s)$ with respect to some standard basis $x_1, y_1, \ldots, x_5, y_5$. Suppose that $\frac{\tilde{r}}{r} = \frac{1}{\beta^3}$ and $\frac{s}{\tilde{s}} = [(b/\beta)^2 - s(a/\beta)^2]^2$ for some $\beta \in \mathbb{F} \setminus \{0\}$ and $(a,b) \in \mathbb{F} \times \mathbb{F} \setminus \{(0,0)\}$. Let $\alpha = \frac{\beta^3}{b^2 - a^2 s}$ and consider a new standard basis

$$
\begin{array}{ll}
\tilde{x}_1 = (\alpha/\beta^2)(bx_1 + asy_1), & \tilde{y}_1 = (1/\beta)(by_1 + ax_1), \\
\tilde{x}_2 = (1/\beta)x_2, & \tilde{y}_2 = \beta y_2, \\
\tilde{x}_3 = (1/\alpha)x_3, & \tilde{y}_3 = \alpha y_3, \\
\tilde{x}_4 = (\alpha/\beta)(bx_4 + asx_5), & \tilde{y}_4 = (1/\beta^2)(by_4 - ay_5), \\
\tilde{x}_5 = ax_4 + bx_5, & \tilde{y}_5 = (\alpha/\beta^3)(by_5 - asy_4).
\end{array}
$$

Calculations show that L has then presentation $\mathcal{P}_{10}^{(3,8)}(\tilde{r},\tilde{s})$ with respect to the new standard basis.

It remains to see that the conditions are also necessary. Consider an algebra L with presentation $\mathcal{P}_{10}^{(3,8)}(r,s)$ with respect to some standard basis $x_1, y_1, \ldots, x_5, y_5$. Take some arbitrary new standard basis $\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_5, \tilde{y}_5$ such that L satisfies the presentation $\mathcal{P}_{10}^{(3,8)}(\tilde{r},\tilde{s})$ with respect to the new basis for some $0 \neq \tilde{r} \in \mathbb{F}$ and $\tilde{s} \notin \mathbb{F}^2$. Notice that

$$
\begin{aligned}
\tilde{x}_5 &= ax_4 + bx_5 \\
\tilde{y}_2 &= \beta y_2 + u \\
\tilde{y}_3 &= \gamma y_3 + v,
\end{aligned}
$$

such that $u, v \in L^3$ and $0 \neq \beta, \gamma \in \mathbb{F}$. The reader can convince himself that $\tilde{r}/r \in (\mathbb{F}^*)^3$ and $s/\tilde{s} \in G(s)$. $\Box$

Examples. (1) If $\mathbb{F} = \mathbb{C}$ then as any quadratic polynomial is reducible, there are not algebras of type C. This holds more generally for any field $\mathbb{F}$ of characteristic that is not 2 and where all the elements in $\mathbb{F}$ have a square root in $\mathbb{F}$.

(2) Suppose $\mathbb{F} = \mathbb{R}$. Let $s \notin \mathbb{R}^2$ and $0 \neq r \in \mathbb{R}$. Then $1/r \in (\mathbb{R}^*)^3$ and $s < 0$. Also $s/(-1) = a^4 = (a^2 - 0^2 s)^2$ for some $a \in \mathbb{R} \setminus \{0\}$. We thus have that $\mathcal{P}_{10}^{(3,8)}(r,s)$ describes the same algebra as $\mathcal{P}_{10}^{(3,8)}(1,-1)$. There is thus a

unique algebra in this case.

(3) Let $\mathbb{F}$ be a finite field of some odd characteristic p. Suppose that $|\mathbb{F}| = p^n$. Let s be any element that is not in $(\mathbb{F}^*)^2$. Notice then that $\mathbb{F}^* = (\mathbb{F}^*)^2 \cup s(\mathbb{F}^*)^2$ and thus for any $\tilde{s}$ that is not in $\mathbb{F}^2$, we have $s/\tilde{s} \in (\mathbb{F}^*)^2 = G(s)$. We can thus keep s fixed and each algebra has a presentation of the form $\mathcal{Q}(r) = \mathcal{P}_{10}^{(3,8)}(r,s)$ where $\mathcal{Q}(\tilde{r})$ and $\mathcal{Q}(r)$ describe the same algebra if and only if $\tilde{r}/r \in (\mathbb{F}^*)^3$. There are thus either three or one algebra according to whether 3 divides $p^n - 1$ or not. $\Box$

b. The algebras where $char\, \mathbb{F} = 2$

If the irreducible minimal polynomial of $\psi\phi^{-1}$ is $t^2 + rt + s$ with respect to y_2, y_3 then the minimal polynomial with respect to $ay_2, by_3 + cy_2$, where $a, b \neq 0$, is

$$t^2 + r(b/a)t + [(c/a)^2 + r(c/a)(b/a) + (b/a)^2 s].$$

Thus we have two distinct subcases (that do not depend on the choice of basis). Let $m = m(y_2, y_3)$ be the minimal polynomial of $\psi\phi^{-1}$ with respect to a given standard basis for L.

(1) The minimal polynomial m is of the form $t^2 - s$ for some $s \notin \mathbb{F}^2$.

(2) The minimal polynomial m is of the form $t^2 + rt + s$ where $r \neq 0$ and the polynomial is irreducible.

For case (1) we get the same situation as in Proposition 4.13.

Proposition 4.14 *Let L be a nilpotent SAA of dimension 10 over a field of characteristic 2 that has an isotropic centre of dimension 3. Suppose also that L has the further properties that $dim\, L^3 = 6$, $L^3L^2 = L^5$ and L is of type C where the minimal polynomial $m(y_2, y_3)$ is of the form $t^2 - s$ for some $s \notin \mathbb{F}^2$. Then L has a presentation of the form*

$$\mathcal{P}_{10}^{(3,8)}(r,s):\ (x_2y_4, y_5) = r,\ (x_1y_3, y_5) = s,\ (x_1y_2, y_4) = 1,\ (y_1y_2, y_5) = 1,\ (y_1y_3, y_4) = 1$$

where $r \neq 0$ and $s \notin \mathbb{F}^2$. Furthermore the presentations $\mathcal{P}_{10}^{(3,8)}(\tilde{r}, \tilde{s})$ and $\mathcal{P}_{10}^{(3,8)}(r,s)$ describe the same algebra if and only if $\frac{\tilde{r}}{r} \in (\mathbb{F}^*)^3$ and $\frac{s}{\tilde{s}} \in G(s)$ where $G(s) = \{(x^2 - y^2 s)^2 :\ (x,y) \in \mathbb{F} \times \mathbb{F} \setminus \{(0,0)\}\}$.

Example. Consider the field $\mathbb{Z}_2(x)$ of rational functions in one variable over $\mathbb{Z}_2$. Notice that

$$\mathbb{Z}_2(x)^* = \{f(x)^2 + xg(x)^2 : f(x), g(x) \in \mathbb{Z}_2(x)^2 \setminus \{(0,0)\}\}.$$

Thus $G(x) = (\mathbb{Z}_2(x)^*)^2$ and the last proposition tells us that $\mathcal{P}_{10}^{(3,8)}(\tilde{r}(x), \tilde{s}(x))$ and $\mathcal{P}_{10}^{(3,8)}(r(x), s(x))$ describe the same algebra if and only if $\tilde{r}(x)/r(x) \in (\mathbb{Z}_2(x)^*)^3$ and $s(x)/\tilde{s}(x) \in (\mathbb{Z}_2(x)^*)^2$. We thus have infinitely many algebras here.

We then move on to the latter collection of algebras. For the rest of this subsection we will be assuming that the minimal polynomial of $\psi\phi^{-1}$ is an irreducible polynomial of the form $t^2 + rt + s$ where $r \neq 0$.

Pick $0 \neq x_5 \in L^5$ and let $x_4 = \psi\phi^{-1}(x_5)$. Then $\psi\phi^{-1}(x_4) = rx_4 + sx_5$. We want to pick our standard basis such that $x_1 + L^4 = \phi^{-1}(x_4)$ and $y_1 + L^4 = \phi^{-1}(x_5)$. The only constraint to worry about is, as before, that $(x_1, y_1) = 1$. If $(x_1, y_1) = \sigma$, we just need to then replace y_3 by $(1/\sigma)y_3$. Notice that this changes the minimal polynomial of $\psi\phi^{-1}$ to $t^2+(r/\sigma)t+(s/\sigma^2)$. In any case this shows that we can choose our standard basis such that

$$x_1y_2 = x_4,\ y_1y_2 = x_5,\ x_1y_3 = rx_4 + sx_5,\ y_1y_3 = x_4 \tag{19}$$

for some $r, s \in \mathbb{F}$ where $r \neq 0$ and $t^2 + rt + s$ is irreducible. As before we also know that $x_2y_3 = 0$. In order to clarify the structure further we are only left with the following triple values:

$$\begin{array}{llll} (x_1y_4, y_5) = a, & (y_2y_3, y_4) = c, & (y_2y_4, y_5) = e, & (x_2y_4, y_5) = \gamma, \\ (y_1y_4, y_5) = b, & (y_2y_3, y_5) = d, & (y_3y_4, y_5) = f. & \end{array}$$

Notice that $\gamma \neq 0$ as $x_2y_3 = 0$ but $x_2 \notin Z(L)$. Let $z_1 = (a + c\gamma - br)/\gamma$, $z_2 = (d - s(b/\gamma))$, $z_3 = z_1(a/\gamma) + z_2(b/\gamma)$ and replace $x_1, y_3, y_1, y_4, y_2, y_5$ by $x_1 - (a/\gamma)x_2 - z_2x_3 - ex_4 + fx_5, y_3 + z_2y_1 + z_1x_1 - z_3x_2, y_1 - (b/\gamma)x_2 + z_1x_3, y_4 - e(b/\gamma)x_2 + ey_1 + ez_1x_3, y_2 + (a/\gamma)y_1 - (b/\gamma)x_1, y_5 + f(b/\gamma)x_2 - fy_1 - fz_1x_3$. This shows that we can choose a new standard basis so that the remaining values are zero. We have thus arrived at a presentation of the form $\mathcal{P}_{10}^{(3,9)}$ as described in the next proposition. Before stating that proposition we introduce two groups that are going to play a role.

Definition. For each minimal polynomial $t^2 + rt + s$, we let

$$\begin{aligned} H(r) &= \{x^2 + rx : x \in \mathbb{F}\} \\ G(r,s) &= \{x^2 + rxy + sy^2 : (x,y) \in \mathbb{F} \times \mathbb{F} \setminus \{(0,0)\}\}. \end{aligned}$$

Remarks. (1) $H(r)$ is a subgroup of the additive group of $\mathbb{F}$.

(2) Consider the splitting field $\mathbb{F}[\alpha]$ of the polynomial $t^2 + rt + s$ in $\mathbb{F}[t]$. Then $a^2 + abr + b^2 s$ is the norm $N(a + b\alpha) = (a + b\alpha)(a + b(\alpha + r))$ of $a + b\alpha$. As this is a multiplicative function we have that $G(r,s)$ is a multiplicative subgroup of $\mathbb{F}^*$.

Proposition 4.15 *Let L be a nilpotent SAA of dimension 10 over a field of characteristic 2 that has an isotropic centre of dimension 3. Suppose also that L has the further properties that $\dim L^3 = 6$, $L^3 L^2 = L^5$ and L is of type C where the minimal polynomial $m(y_2, y_3)$ is irreducible with a non-zero linear term. Then L has a presentation of the form*

$$\begin{aligned} \mathcal{P}_{10}^{(3,9)}(\gamma, r, s) : \quad & (x_2 y_4, y_5) = \gamma,\ (x_1 y_3, y_4) = r,\ (x_1 y_3, y_5) = s, \\ & (x_1 y_2, y_4) = 1,\ (y_1 y_2, y_5) = 1,\ (y_1 y_3, y_4) = 1 \end{aligned}$$

where $\gamma, r \neq 0$ and $t^2 + rt + s$ is irreducible. Furthermore the presentations $\mathcal{P}_{10}^{(3,9)}(\tilde{\gamma}, \tilde{r}, \tilde{s})$ and $\mathcal{P}_{10}^{(3,9)}(\gamma, r, s)$ describe the same algebra if and only if $\frac{\tilde{\gamma}}{\gamma} \in (\mathbb{F}^*)^3$, $\frac{\tilde{r}}{r} \in G(r,s)$ and $\tilde{s} - (\frac{\tilde{r}}{r})^2 s \in H(\tilde{r})$.

Proof We have already seen that any such algebra has a presentation of the given form. Direct calculations show that conversely any algebra with a presentation of this type satisfies all the properties listed. It remains to deal with the isomorphism property. To see that the condition is sufficient, suppose we have an algebra L that has a presentation $\mathcal{P}_{10}^{(3,9)}(\gamma, r, s)$ with respect to some standard basis $x_1, y_1, \ldots, x_5, y_5$. Suppose that $\frac{\tilde{\gamma}}{\gamma} = \frac{1}{\beta^3}$, $\frac{\tilde{r}}{r} = (\frac{b}{\beta})^2 + (\frac{b}{\beta})(\frac{a}{\beta})r + (\frac{a}{\beta})^2 s$ and $\tilde{s} - (\frac{\tilde{r}}{r})^2 s = (\frac{\delta}{\beta})^2 + (\frac{\delta}{\beta})\tilde{r}$ for some $a, b, \delta, \beta \in \mathbb{F}$ where $(a,b) \neq (0,0)$ and $\beta \neq 0$. We let $\alpha = \beta/((\frac{b}{\beta})^2 + (\frac{b}{\beta})(\frac{a}{\beta})r + (\frac{a}{\beta})^2 s)$. Consider the new standard basis

$$\begin{aligned} \tilde{x}_1 &= \tfrac{1}{\beta^2}[(\alpha a r + \alpha b + \delta a)x_1 + (\delta b + \alpha a s)y_1], & \tilde{y}_1 &= \tfrac{1}{\beta}(a x_1 + b y_1), \\ \tilde{x}_2 &= \tfrac{1}{\alpha\beta}(\alpha x_2 + \delta x_2), & \tilde{y}_2 &= \beta y_2, \\ \tilde{x}_3 &= \tfrac{1}{\alpha} x_3, & \tilde{y}_3 &= \alpha y_3 + \delta y_2, \\ \tilde{x}_4 &= \tfrac{1}{\beta}[(\alpha a r + \alpha b + \delta a)x_4 + (\delta b + \alpha a s)x_5], & \tilde{y}_4 &= \tfrac{1}{\beta^2}(b y_4 + \alpha y_5), \\ \tilde{x}_5 &= a x_4 + b x_5, & \tilde{y}_5 &= \tfrac{1}{\beta^3}[(\alpha a r + \alpha b + \delta a)y_5 + (\delta b + \alpha a s)y_4]. \end{aligned}$$

Calculations show that L has then presentation $\mathcal{P}_{10}^{(3,9)}(\tilde{\gamma},\tilde{r},\tilde{s})$ with respect to the new standard basis.

It remains to see that the conditions are also necessary. Consider an algebra L with presentation $\mathcal{P}_{10}^{(3,9)}(\gamma,r,s)$ with respect to some standard basis $x_1,y_1,\ldots,x_5,y_5$. Take some arbitrary new standard basis $\tilde{x}_1,\tilde{y}_1,\ldots,\tilde{x}_5,\tilde{y}_5$ such that L has presentation $\mathcal{P}_{10}^{(3,9)}(\tilde{\gamma},\tilde{r},\tilde{s})$ with respect to the new basis where $\tilde{\gamma},\tilde{r}\neq 0$ and where $t^2+\tilde{r}t+\tilde{s}$ is irreducible. Then $\tilde{x}_5 = ax_4+bx_5$, $\tilde{y}_2 = \beta y_2 + u$ and $\tilde{y}_3 = \alpha y_3 + \delta y_2 + v$ for some $u,v \in L^3$, $\alpha,\beta,\delta \in \mathbb{F}$ and $(a,b)\in\mathbb{F}\times\mathbb{F}\setminus\{(0,0)\}$ where $\alpha,\beta\neq 0$. The reader can convince himself that the new basis that we get satisfies the conditions stated. $\Box$

Before we give an example of an algebra of this form, we list some useful properties of the groups $G(r,s)$ and $H(r)$.

Lemma 4.16 *For any irreducible polynomials t^2+rt+s and $t^2+\tilde{r}t+\tilde{s}$, we have that*

(1) $H(\tilde{r}) = (\tilde{r}/r)^2 H(r)$.
(2) $G(\tilde{r},\tilde{s}) = G(r,s)$ if $\tilde{s}-(\tilde{r}/r)^2 s \in H(\tilde{r})$.

Proof Straightforward calculations.

Example. Let $\mathbb{F}$ be the finite field of order 2^n. Let $r,s,\tilde{r},\tilde{s}$ be as in the last lemma. Then $G(\tilde{r},\tilde{s}) = G(r,s) = \mathbb{F}^*$ and thus $\tilde{r}/r \in G(r,s)$. Also $[\mathbb{F}:H(\tilde{r})] = 2$ and thus $\tilde{s}-(\tilde{r}/r)^2 s \in H(\tilde{r})$. It follows from the last proposition that the presentations $\mathcal{P}_{10}^{(3,9)}(\gamma,r,s)$ and $\mathcal{P}_{10}^{(3,9)}(\tilde{\gamma},\tilde{r},\tilde{s})$ describe the same algebra if and only if $\tilde{\gamma}/\gamma \in (\mathbb{F}^*)^3$. There are thus either three algebras or one algebra according to whether 3 divides 2^n-1 or not.

We end this section by giving a direct explanation why the relation

$$(\tilde{r},\tilde{s}) \sim (r,s) \text{ if } \frac{\tilde{r}}{r} \in G(r,s), \tilde{s}-(\frac{\tilde{r}}{r})^2 s \in H(\tilde{r})$$

is an equivalence relation.

First it is easy to see that $(r,s)\sim(r,s)$ as $1\in G(r,s)$ and $0\in H(r)$. Next if $(\tilde{r},\tilde{s})\sim(r,s)$ then, as $G(r,s)=G(\tilde{r},\tilde{s})$ is a group, we have that $r/\tilde{r}\in G(\tilde{r},\tilde{s})$

and $s - (r/\tilde{r})^2\tilde{s} = (r/\tilde{r})^2(-\tilde{s} + (\tilde{r}/r)^2 s) \in (r/\tilde{r})^2 H(\tilde{r}) = H(r)$. This shows that $\sim$ is symmetric. Finally suppose $(r^*, s^*) \sim (\tilde{r}, \tilde{s})$ and $(\tilde{r}, \tilde{s}) \sim (r, s)$. Then we have that $r^*/r = r^*/\tilde{r} \cdot \tilde{r}/r \in G(r, s)$ and $s^* - (r^*/r)^2 s = [s^* - (r^*/\tilde{r})^2\tilde{s}] + [(r^*/\tilde{r})^2\tilde{s} - (r^*/r)^2 s] = [s^* - (r^*/\tilde{r})^2\tilde{s}] + (r^*/\tilde{r})^2[\tilde{s} - (\tilde{r}/r)^2 s]$ is in $H(r^*) + (r^*/\tilde{r})^2 H(\tilde{r}) = H(r^*)$. Hence $\sim$ is also transitive and we have an equivalence relation.

References

[1] P. Moravec and G. Traustason, Powerful 2-Engel groups, *Comm. Algebra*, **36** (2008), no. 11, 4096–4119.

[2] L. Sorkatti and G. Traustason, Nilpotent symplectic alternating algebras I, *J. Algebra* **423** (2015), 615–635.

[3] A. Tortora, M. Tota and G. Traustason, Symplectic alternating nil-algebras, *J. Algebra* **357** (2012), 183-202.

[4] G. Traustason, Powerful 2-Engel groups II, *J. Algebra* **319** (2008), no. 8, 3301–3323.

[5] G. Traustason, Symplectic Alternating Algebras, *Internat. J. Algebra Comput.* **18** (2008), no. 4, 719–757.