A stability/instability trichotomy for non-negative Lur’e systems*

Adam Bill 1, Chris Guiver 2, Hartmut Logemann 1 and Stuart Townley 2

Abstract—We identify a stability/instability trichotomy for a class of non-negative continuous-time Lur’e systems. Asymptotic as well as input-to-state stability concepts (ISS) are considered. The presented trichotomy rests on Perron-Frobenius theory, absolute stability theory and recent ISS results for Lur’e systems.

I. INTRODUCTION

Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) and \( b, c \in \mathbb{R}^n \) and consider the corresponding single-input single-output non-negative linear system

\[
\dot{x} = Ax + bu, \quad x(0) = \xi \in \mathbb{R}^n_+, \quad y = c^T x.
\]  

We assume that

(A1) \( A \) is Metzler, \( b, c \in \mathbb{R}^n_+ \) and \( b, c \neq 0 \) holds.

We recall that \( A = (a_{ij}) \) is Metzler if \( a_{ij} \geq 0 \) for \( i \neq j \) (all off-diagonal elements are non-negative).

System (1) is said to be non-negative if (A1) holds and \( u \geq 0 \). Non-negative systems of the form (1) occur naturally in biological, ecological and economic contexts.

We impose the following assumptions.

(A2) \( A \) is Hurwitz.

(A3) There exist non-negative numbers \( \alpha \) and \( \kappa \) such that \( \alpha I + A + \kappa bc^T \) is primitive.

Recall that (A3) means that the matrix \( (\alpha I + A + \kappa bc^T)^k \) is a positive matrix for some \( k \in \mathbb{N} \).

In the following, let \( G \) denote the transfer function of (1), that is, \( G(s) := c^T (sI - A)^{-1} b \).

Lemma 1.1: Assume that (A1)-(A3) hold. Then \( G(0) > 0 \) and \( \|G\|_{\infty} = G(0) \).

A proof of Lemma 1.1 can be found in [1].

Applying nonlinear non-negative feedback \( u = f(y) \) to (1), where \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) is locally Lipschitz, leads to the following non-negative Lur’e system

\[
\dot{x} = Ax + bf(c^T x), \quad x(0) = \xi \in \mathbb{R}^n_+.
\]

We assume that the following assumption holds.

(A4) \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) is locally Lipschitz and \( f(0) = 0 \).

II. LYAPUNOV STABILITY RESULTS

In this section, we present results which describe the stability/instability properties in each of three cases, where “stability” is interpreted in the sense of Lyapunov.

Let \( x(\cdot; \xi) \) denote the unique maximally defined forward solution of (2) with maximal interval of existence \([0, \omega\xi]\), where \( 0 < \omega\xi \leq \infty \).

The proposition below relates to Case 1. It follows from well known results in absolute stability theory, see, for example, [3].

Proposition 2.1: Assume that (A1)-(A4) hold.

(a) If \( f(z)/z \leq p \) for all \( z > 0 \), then the equilibrium \( 0 \) is stable in the large in the sense that there exists \( \Gamma \geq 1 \) such that, for every \( \xi \in \mathbb{R}^n_+ \), \( \omega\xi = \infty \) and

\[
\|x(t; \xi)\| \leq \Gamma \|\xi\| \quad \forall t \geq 0.
\]
(b) If \( f(z)/z < p \) for all \( z > 0 \), then the equilibrium 0 is globally asymptotically stable. In particular, for every \( \xi \in \mathbb{R}_+^n \), \( \omega_\xi = \infty \) and \( x(t; \xi) \to 0 \) as \( t \to \infty \).

(c) If \( \sup_{z > 0} f(z)/z < p \), then the equilibrium 0 is globally exponentially stable, that is, there exist \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in \mathbb{R}_+^n \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi)\| \leq N e^{-\nu t} \|\xi\| \quad \forall t \geq 0.
\]

In Case 2, the solutions of (2) diverge to \( \infty \) for every non-zero initial condition. More precisely, we have the following result.

**Theorem 2.2:** Assume that \( (A1)-(A4) \) hold and let \( \inf_{z > 0} f(z)/z > p \). Let \( \xi \in \mathbb{R}_+^n \), \( \xi \neq 0 \), be such that the solution \( x(t; \xi) \) exists for all \( t \geq 0 \). Then

\[
\lim_{t \to \infty} x_i(t; \xi) = \infty \quad \forall i \in \{1, \ldots, n\},
\]

where \( x_i(t; \xi) \) denotes the \( i \)-th component of \( x(t; \xi) \).

We proceed to consider Case 3.

**Theorem 2.3:** Assume that \( (A1)-(A4) \) hold.

(a) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and

\[
\frac{f(z) - f(y^*)}{z - y^*} \leq p \quad \forall z \geq 0, \quad z \neq y^*
\]

then \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) is an equilibrium of (2) and \( x^* \) is stable in the large in the sense that there exists \( \Gamma \geq 1 \) such that, for every \( \xi \in \mathbb{R}_+^n \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi) - x^*\| \leq \Gamma \|\xi - x^*\| \quad \forall t \geq 0.
\]

(b) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and

\[
\frac{f(z) - f(y^*)}{z - y^*} < p \quad \forall z > 0, \quad z \neq y^*
\]

then \( 0 \) and \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) are the only equilibria of (2) and \( x^* \) is globally asymptotically stable in the sense that \( x^* \) is stable in the large (see statement (a) of this theorem) and, for every \( \xi \in \mathbb{R}_+^n \) such that \( \xi \neq 0 \), \( \omega_\xi = \infty \) and \( x(t; \xi) \to x^* \) as \( t \to \infty \).

(c) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \),

\[
\frac{f(z) - f(y^*)}{z - y^*} < p \quad \forall z > 0, \quad z \neq y^*
\]

and

\[
\limsup_{y \to y^*} \frac{f(z) - f(y^*)}{z - y^*} < p,
\]

and if

\[
\liminf_{z \to 0^+} \frac{f(z)}{z} > p,
\]

then \( 0 \) and \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) are the only equilibria of (2) and \( x^* \) is “semi-globally” exponentially stable in the sense that, for every compact set \( K \subset \mathbb{R}_+^n \) with \( 0 \notin K \), there exists \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in K \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi) - x^*\| \leq N e^{-\nu t} \|\xi - x^*\| \quad \forall t \geq 0.
\]

(d) If (3) holds and there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and, for every \( \varepsilon > 0 \),

\[
\sup_{z > \varepsilon, z \neq y^*} \frac{f(z) - f(y^*)}{z - y^*} < p,
\]

then \( 0 \) and \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) are the only equilibria of (2) and \( x^* \) is “quasi-globally” exponentially stable in the sense that, for every \( \delta > 0 \) there exist \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in \mathbb{R}_+^n \) with \( \|\xi\| \geq \delta \), \( \omega_\xi = \infty \) and

\[
\|x(t; \xi) - x^*\| \leq Ne^{-\nu t} \|\xi - x^*\| \quad \forall t \geq 0.
\]

We remark that “global” exponential stability of \( x^* \) (in the sense that there exist \( N \geq 1 \) and \( \nu > 0 \) such that (4) is satisfied for all \( \xi \in \mathbb{R}_+^n \) with \( \xi \neq 0 \)) does not hold. This is an immediate consequence of the following result which follows from continuity properties of the flow generated by the Lur’e system (2).

**Proposition 2.4:** Assume that \( (A1)-(A4) \) hold and that there exists \( y^* > 0 \) such that \( f(y^*) = py^* \). Then, for every sequence \( (t_n) \in \mathbb{R}_+ \) with \( t_n \to \infty \) as \( n \to \infty \), there exists a sequence \( (\xi_n) \) in \( \mathbb{R}_+^n \) with \( \xi_n \neq 0 \) and \( \xi_n \to 0 \) as \( n \to \infty \) and such that

\[
\lim_{n \to \infty} \frac{\|x(t_n; \xi_n) - x^*\|}{\|\xi_n - x^*\|} = 1,
\]

where \( x^* = -pA^{-1}by^* \).

Discrete-time results similar to statement (b) of Proposition 2.1, Theorem 2.2 and statement (b) of Theorem 2.3 can be found in [7].

Proofs of the results in Section II can be found in [1].

### III. INPUT-TO-STATE STABILITY RESULTS

Finally, we investigate the stability behaviour of (2) subject to non-negative disturbances, that is, we analyze input-to-state stability (ISS) properties of the forced Lur’e system

\[
\dot{x} = Ax + b(f(x^T x) + d), \quad x(0) = \xi \in \mathbb{R}_+^n,
\]

where \( d : \mathbb{R}_+ \to \mathbb{R}_+ \) is locally essentially bounded. The unique maximally defined forward solution of (5) is denoted by \( x(\cdot; \xi, d) \).

For an overview of ISS theory, the reader is referred to [6]. We recall some terminology and notation relating to comparison functions. Let \( K \) denote the set of all continuous functions \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \varphi(0) = 0 \) and \( \varphi \) is strictly increasing. Moreover, define \( K_\infty := \{ \varphi \in K : \lim_{s \to \infty} \varphi(s) = \infty \} \). We denote by \( K \) the set of functions in two variables \( \psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) with the following properties: \( \psi(\cdot, t) \in K \) for all \( t \geq 0 \), and \( \psi(s, \cdot) \) is nonincreasing with \( \lim_{s \to \infty} \psi(s, t) = 0 \) for all \( s \geq 0 \).

The following proposition is a consequence of recent ISS results for Lur’e systems, see [4], [5].

**Proposition 3.1:** Assume that \( (A1)-(A4) \) hold. If there exists \( \rho \in K_\infty \) such that

\[
f(z) \leq pz - \rho(z) \quad \forall z \geq 0,
\]
then the equilibrium 0 of the unforced Lur’e system (2) is ISS in the sense that there exist \( \psi \in KL \) and \( \phi \in K \) such that for all \( \xi \in \mathbb{R}_+^n \) and all non-negative \( d \in L_{loc}^\infty(\mathbb{R}_+) \), 
\[
    x(\cdot; \xi, d) \text{ is defined on } \mathbb{R}_+ \text{ and }
    \|x(t; \xi, d)\| \leq \psi(\|\xi\|, t) + \phi(\|d\|_{L^\infty(0, t)}) \quad \forall \ t \geq 0.
\]
The following theorem shows that, under suitable assumptions, the equilibrium \( x^* \) has stability properties which are similar to ISS.

**Theorem 3.2:** Assume that (A1)-(A4) hold and that there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and
\[
    \left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p \quad \forall \ z > 0, \ z \neq y^*, \quad (6)
\]
Furthermore, assume that (3) holds and
\[
    pz - f(z) \to \infty \quad \text{as} \quad z \to \infty. \quad (7)
\]

Then \( 0 \) and \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) are the only equilibria of the unforced Lur’e system (2) and \( x^* \) is “quasi ISS” in the sense that, for every \( \delta > 0 \), there exist \( \psi \in KL \) and \( \phi \in K \) such that for all \( \xi \in \mathbb{R}_+^n \) with \( \|\xi\| \geq \delta \) and all non-negative \( d \in L_{loc}^\infty(\mathbb{R}_+) \), \( x(\cdot; \xi, d) \) is defined on \( \mathbb{R}_+ \) and
\[
    \|x(t; \xi, d) - x^*\| \leq \psi(\|\xi - x^*\|, t) + \phi(\|d\|_{L^\infty(0, t)}) \quad \forall \ t \geq 0.
\]

To relate the conditions (6) and (7) to those in Proposition 3.1, we note that if (6) and (7) hold, then, for every \( \varepsilon > 0 \), there exists \( \rho \in \mathcal{K}_\infty \) such that
\[
    |f(z) - f(y^*)| \leq p|z - y^*| - \rho(|z - y^*|) \quad \forall \ z \geq \varepsilon, \ z \neq y^*.
\]
The proof of Theorem 3.2 is based on Proposition 3.1 and the following lemma.

**Lemma 3.3:** Assume that (A1)-(A4) hold. If (3) is satisfied and there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and (6) holds, then, for every \( \delta > 0 \), there exist constants \( \eta > 0 \) and \( \tau \geq 0 \) such that for all \( \xi \in \mathbb{R}_+^n \) with \( \|\xi\| \geq \delta \) and all non-negative \( d \in L_{loc}^\infty(\mathbb{R}_+) \), \( x(\cdot; \xi, d) \) is defined on \( \mathbb{R}_+ \) and
\[
    c^T x(t; \xi, d) \geq \eta \quad \forall \ t \geq \tau.
\]
This lemma also plays a key role in the proof of statements (b)-(d) of Theorem 2.3 (with disturbance \( d = 0 \)). Detailed proofs of Proposition 3.1, Theorem 3.2 and Lemma 3.3 can be found in [1].

Finally, it follows from Proposition 2.4 that “global” ISS of \( x^* \) (in the sense that there exist \( \psi \in KL \) and \( \phi \in K \) such that (8) is satisfied for all \( \xi \in \mathbb{R}_+^n \) with \( \xi \neq 0 \) and all non-negative \( d \in L_{loc}^\infty(\mathbb{R}_+) \)) does not hold.

**REFERENCES**