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# A stability/instability trichotomy for non-negative Lur'e systems\*

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**Abstract**—We identify a stability/instability trichotomy for a class of non-negative continuous-time Lur'e systems. Asymptotic as well as input-to-state stability concepts (ISS) are considered. The presented trichotomy rests on Perron-Frobenius theory, absolute stability theory and recent ISS results for Lur'e systems.

## I. INTRODUCTION

Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  and  $b, c \in \mathbb{R}^n$  and consider the corresponding single-input single-output non-negative linear system

$$\dot{x} = Ax + bu, \quad x(0) = \xi \in \mathbb{R}_+^n; \quad y = c^T x. \quad (1)$$

We assume that

**(A1)**  $A$  is Metzler,  $b, c \in \mathbb{R}_+^n$  and  $b, c \neq 0$  holds.

We recall that  $A = (a_{ij})$  is Metzler if  $a_{ij} \geq 0$  for  $i \neq j$  (all off-diagonal elements are non-negative).

System (1) is said to be *non-negative* if (A1) holds and  $u \geq 0$ . Non-negative systems of the form (1) occur naturally in biological, ecological and economic contexts.

We impose the following assumptions.

**(A2)**  $A$  is Hurwitz.

**(A3)** There exist non-negative numbers  $\alpha$  and  $\kappa$  such that  $\alpha I + A + \kappa bc^T$  is primitive.

Recall that (A3) means that the matrix  $(\alpha I + A + \kappa bc^T)^k$  is a positive matrix for some  $k \in \mathbb{N}$ .

In the following, let  $G$  denote the transfer function of (1), that is,  $G(s) := c^T (sI - A)^{-1} b$ .

**Lemma 1.1:** Assume that (A1)-(A3) hold. Then  $G(0) > 0$  and  $\|G\|_{H^\infty} = G(0)$ .

A proof of Lemma 1.1 can be found in [1].

Applying nonlinear non-negative feedback  $u = f(y)$  to (1), where  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is locally Lipschitz, leads to the following non-negative Lur'e system

$$\dot{x} = Ax + bf(c^T x), \quad x(0) = \xi \in \mathbb{R}_+^n. \quad (2)$$

We assume that the following assumption holds.

**(A4)**  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is locally Lipschitz and  $f(0) = 0$ .

Whilst absolute stability of Lur'e systems is a classical topic in control theory (see, for example, [2], [3], [8]), it seems that non-negative Lur'e systems have not received

much attention (see however [7] which provides an analysis of the stability properties of a class of non-negative discrete-time Lur'e systems).

Assuming that (A1)-(A4) hold, we set

$$p := \frac{1}{G(0)},$$

and consider the following three cases.

**Case 1.**  $f(z)/z \leq p$  for all  $z > 0$ .

**Case 2.**  $\inf_{z>0} f(z)/z > p$ .

**Case 3.** There exists  $y^* > 0$  such that  $f(y^*) = py^*$  and

$$\left| \frac{f(z) - f(y^*)}{z - y^*} \right| \leq p \text{ for all } z > 0, z \neq y^*.$$

The condition in Case 3 means that the graph of  $f$  is “sandwiched” between the straight lines  $l_1$  and  $l_2$  given by  $l_1(z) = pz$  and  $l_2(z) = 2py^* - pz$ , see Figure 1.

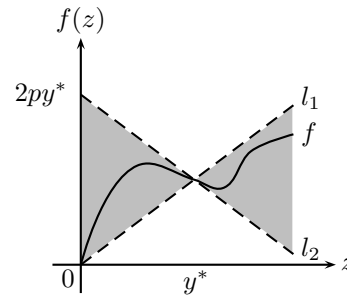


Fig. 1. Case 3: graph of  $f$  “sandwiched” between the lines  $l_1$  and  $l_2$ .

## II. LYAPUNOV STABILITY RESULTS

In this section, we present results which describe the stability/instability properties in each of three cases, where “stability” is interpreted in the sense of Lyapunov.

Let  $x(\cdot; \xi)$  denote the unique maximally defined forward solution of (2) with maximal interval of existence  $[0, \omega_\xi)$ , where  $0 < \omega_\xi \leq \infty$ .

The proposition below relates to Case 1. It follows from well known results in absolute stability theory, see, for example, [3].

**Proposition 2.1:** Assume that (A1)-(A4) hold.

(a) If  $f(z)/z \leq p$  for all  $z > 0$ , then the equilibrium 0 is stable in the large in the sense that there exists  $\Gamma \geq 1$  such that, for every  $\xi \in \mathbb{R}_+^n$ ,  $\omega_\xi = \infty$  and

$$\|x(t; \xi)\| \leq \Gamma \|\xi\| \quad \forall t \geq 0.$$

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(b) If  $f(z)/z < p$  for all  $z > 0$ , then the equilibrium 0 is globally asymptotically stable. In particular, for every  $\xi \in \mathbb{R}_+^n$ ,  $\omega_\xi = \infty$  and  $x(t; \xi) \rightarrow 0$  as  $t \rightarrow \infty$ .

(c) If  $\sup_{z>0} f(z)/z < p$ , then the equilibrium 0 is globally exponentially stable, that is, there exist  $N \geq 1$  and  $\nu > 0$  such that, for every  $\xi \in \mathbb{R}_+^n$ ,  $\omega_\xi = \infty$  and

$$\|x(t; \xi)\| \leq Ne^{-\nu t} \|\xi\| \quad \forall t \geq 0.$$

In Case 2, the solutions of (2) diverge to  $\infty$  for every non-zero initial condition. More precisely, we have the following result.

*Theorem 2.2:* Assume that (A1)-(A4) hold and  $\inf_{z>0} f(z)/z > p$ . Let  $\xi \in \mathbb{R}_+^n$ ,  $\xi \neq 0$ , be such that the solution  $x(t; \xi)$  exists for all  $t \geq 0$ . Then

$$\lim_{t \rightarrow \omega_\xi} x_i(t; \xi) = \infty \quad \forall i \in \{1, \dots, n\},$$

where  $x_i(\cdot; \xi)$  denotes the  $i$ -th component of  $x(\cdot; \xi)$ .

We proceed to consider Case 3.

*Theorem 2.3:* Assume that (A1)-(A4) hold.

(a) If there exists  $y^* > 0$  such that  $f(y^*) = py^*$  and

$$\left| \frac{f(z) - f(y^*)}{z - y^*} \right| \leq p \quad \forall z \geq 0, z \neq y^*$$

then  $x^* = -pA^{-1}by^* \in \mathbb{R}_+^n$  is an equilibrium of (2) and  $x^*$  is stable in the large in the sense that there exists  $\Gamma \geq 1$  such that, for every  $\xi \in \mathbb{R}_+^n$ ,  $\omega_\xi = \infty$  and

$$\|x(t; \xi) - x^*\| \leq \Gamma \|\xi - x^*\| \quad \forall t \geq 0.$$

(b) If there exists  $y^* > 0$  such that  $f(y^*) = py^*$  and

$$\left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p \quad \forall z > 0, z \neq y^*$$

then 0 and  $x^* = -pA^{-1}by^* \in \mathbb{R}_+^n$  are the only equilibria of (2) and  $x^*$  is globally asymptotically stable in the sense that  $x^*$  is stable in the large (see statement (a) of this theorem) and, for every  $\xi \in \mathbb{R}_+^n$  such that  $\xi \neq 0$ ,  $\omega_\xi = \infty$  and  $x(t; \xi) \rightarrow x^*$  as  $t \rightarrow \infty$ .

(c) If there exists  $y^* > 0$  such that  $f(y^*) = py^*$ ,

$$\left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p \quad \forall z > 0, z \neq y^*$$

and

$$\limsup_{y \rightarrow y^*} \left| \frac{f(z) - f(y^*)}{y - y^*} \right| < p,$$

and if

$$\liminf_{z \rightarrow 0} \frac{f(z)}{z} > p, \quad (3)$$

then 0 and  $x^* = -pA^{-1}by^* \in \mathbb{R}_+^n$  are the only equilibria of (2) and  $x^*$  is “semi-globally” exponentially stable in the sense that, for every compact set  $K \subset \mathbb{R}_+^n$  with  $0 \notin K$ , there exists  $N \geq 1$  and  $\nu > 0$  such that, for every  $\xi \in K$ ,  $\omega_\xi = \infty$  and

$$\|x(t; \xi) - x^*\| \leq Ne^{-\nu t} \|\xi - x^*\| \quad \forall t \geq 0.$$

(d) If (3) holds and there exists  $y^* > 0$  such that  $f(y^*) = py^*$  and, for every  $\varepsilon > 0$ ,

$$\sup_{z \geq \varepsilon, z \neq y^*} \left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p,$$

then 0 and  $x^* = -pA^{-1}by^* \in \mathbb{R}_+^n$  are the only equilibria of (2) and  $x^*$  is “quasi-globally” exponentially stable in the sense that, for every  $\delta > 0$  there exist  $N \geq 1$  and  $\nu > 0$  such that, for every  $\xi \in \mathbb{R}_+^n$  with  $\|\xi\| \geq \delta$ ,  $\omega_\xi = \infty$  and

$$\|x(t; \xi) - x^*\| \leq Ne^{-\nu t} \|\xi - x^*\| \quad \forall t \geq 0. \quad (4)$$

We remark that “global” exponential stability of  $x^*$  (in the sense that there exist  $N \geq 1$  and  $\nu > 0$  such that (4) is satisfied for all  $\xi \in \mathbb{R}_+^n$  with  $\xi \neq 0$ ) does not hold. This is an immediate consequence of the following result which follows from continuity properties of the flow generated by the Lur’e system (2).

*Proposition 2.4:* Assume that (A1)-(A4) hold and that there exists  $y^* > 0$  such that  $f(y^*) = py^*$ . Then, for every sequence  $(t_n)$  in  $\mathbb{R}_+$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ , there exists a sequence  $(\xi_n)$  in  $\mathbb{R}_+^n$  with  $\xi_n \neq 0$  and  $\xi_n \rightarrow 0$  as  $n \rightarrow \infty$  and such that

$$\lim_{n \rightarrow \infty} \frac{\|x(t_n; \xi_n) - x^*\|}{\|\xi_n - x^*\|} = 1,$$

where  $x^* = -pA^{-1}by^*$ .

Discrete-time results similar to statement (b) of Proposition 2.1, Theorem 2.2 and statement (b) of Theorem 2.3 can be found in [7].

Proofs of the results in Section II can be found in [1].

### III. INPUT-TO-STATE STABILITY RESULTS

Finally, we investigate the stability behaviour of (2) subject to non-negative disturbances, that is, we analyze input-to-state stability (ISS) properties of the forced Lur’e system

$$\dot{x} = Ax + b(f(c^T x) + d), \quad x(0) = \xi \in \mathbb{R}_+^n, \quad (5)$$

where  $d : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is locally essentially bounded. The unique maximally defined forward solution of (5) is denoted by  $x(\cdot; \xi, d)$ .

For an overview of ISS theory, the reader is referred to [6]. We recall some terminology and notation relating to comparison functions. Let  $\mathcal{K}$  denote the set of all continuous functions  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\varphi(0) = 0$  and  $\varphi$  is strictly increasing. Moreover, define  $\mathcal{K}_\infty := \{\varphi \in \mathcal{K} : \lim_{s \rightarrow \infty} \varphi(s) = \infty\}$ . We denote by  $\mathcal{KL}$  the set of functions in two variables  $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with the following properties:  $\psi(\cdot, t) \in \mathcal{K}$  for all  $t \geq 0$ , and  $\psi(s, \cdot)$  is nonincreasing with  $\lim_{t \rightarrow \infty} \psi(s, t) = 0$  for all  $s \geq 0$ .

The following proposition is a consequence of recent ISS results for Lur’e systems, see [4], [5].

*Proposition 3.1:* Assume that (A1)-(A4) hold. If there exists  $\rho \in \mathcal{K}_\infty$  such that

$$f(z) \leq pz - \rho(z) \quad \forall z \geq 0,$$

then the equilibrium 0 of the unforced Lur'e system (2) is ISS in the sense that there exist  $\psi \in \mathcal{KL}$  and  $\varphi \in \mathcal{K}$  such that for all  $\xi \in \mathbb{R}_+^n$  and all non-negative  $d \in L_{loc}^\infty(\mathbb{R}_+)$ ,  $x(\cdot; \xi, d)$  is defined on  $\mathbb{R}_+$  and

$$\|x(t; \xi, d)\| \leq \psi(\|\xi\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall t \geq 0.$$

The following theorem shows that, under suitable assumptions, the equilibrium  $x^*$  has stability properties which are similar to ISS.

*Theorem 3.2:* Assume that (A1)-(A4) hold and that there exists  $y^* > 0$  such that  $f(y^*) = py^*$  and

$$\left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p \quad \forall z > 0, z \neq y^*, \quad (6)$$

Furthermore, assume that (3) holds and

$$pz - f(z) \rightarrow \infty \quad \text{as } z \rightarrow \infty. \quad (7)$$

Then 0 and  $x^* = -pA^{-1}by^* \in \mathbb{R}_+^n$  are the only equilibria of the unforced Lur'e system (2) and  $x^*$  is "quasi ISS" in the sense that, for every  $\delta > 0$ , there there exist  $\psi \in \mathcal{KL}$  and  $\varphi \in \mathcal{K}$  such that for all  $\xi \in \mathbb{R}_+^n$  with  $\|\xi\| \geq \delta$  and all non-negative  $d \in L_{loc}^\infty(\mathbb{R}_+)$ ,  $x(\cdot; \xi, d)$  is defined on  $\mathbb{R}_+$  and

$$\|x(t; \xi, d) - x^*\| \leq \psi(\|\xi - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall t \geq 0. \quad (8)$$

To relate the conditions (6) and (7) to those in Proposition 3.1, we note that if (6) and (7) hold, then, for every  $\varepsilon > 0$ , there exists  $\rho \in \mathcal{K}_\infty$  such that

$$|f(z) - f(y^*)| \leq p|z - y^*| - \rho(|z - y^*|) \quad \forall z \geq \varepsilon, z \neq y^*.$$

The proof of Theorem 3.2 is based on Proposition 3.1 and the following lemma.

*Lemma 3.3:* Assume that (A1)-(A4) hold. If (3) is satisfied and there exists  $y^* > 0$  such that  $f(y^*) = py^*$  and (6) holds, then, for every  $\delta > 0$ , there exist constants  $\eta > 0$  and  $\tau \geq 0$  such that for all  $\xi \in \mathbb{R}_+^n$  with  $\|\xi\| \geq \delta$  and all non-negative  $d \in L_{loc}^\infty(\mathbb{R}_+)$ ,  $x(\cdot; \xi, d)$  is defined on  $\mathbb{R}_+$  and

$$c^T x(t; \xi, d) \geq \eta \quad \forall t \geq \tau.$$

This lemma also plays a key roll in the proof of statements (b)-(d) of Theorem 2.3 (with disturbance  $d = 0$ ). Detailed proofs of Proposition 3.1, Theorem 3.2 and Lemma 3.3 can be found in [1].

Finally, it follows from Proposition 2.4 that "global" ISS of  $x^*$  (in the sense that there exist  $\psi \in \mathcal{KL}$  and  $\varphi \in \mathcal{K}$  such that (8) is satisfied for all  $\xi \in \mathbb{R}_+^n$  with  $\xi \neq 0$  and all non-negative  $d \in L_{loc}^\infty(\mathbb{R}_+)$ ) does not hold.

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