21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

A stability/instability trichotomy for non-negative Lur'e systems*

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Abstract— **We identify a stability/instability trichotomy for a class of non-negative continuous-time Lur'e systems. Asymptotic as well as input-to-state stability concepts (ISS) are considered. The presented trichotomy rests on Perron-Frobenius theory, absolute stability theory and recent ISS results for Lur'e systems.**

I. INTRODUCTION

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $b, c \in \mathbb{R}^n$ and consider the corresponding single-input single-output non-negative linear system

$$\dot{x} = Ax + bu,\ x(0) = \xi \in \mathbb{R}^n_+; \quad y = c^T x. \tag{1}$$

We assume that

(A1) A is Metzler, $b, c \in \mathbb{R}^n_+$ and $b, c \neq 0$ holds.

We recall that $A = (a_{ij})$ is *Metzler* if $a_{ij} \geq 0$ for $i \neq j$ (all off-diagonal elements are non-negative).

System (1) is said to be *non-negative* if (A1) holds and $u \geq 0$. Non-negative systems of the form (1) occur naturally in biological, ecological and economic contexts.

We impose the following assumptions.

(A2) A is Hurwitz.

(A3) There exist non-negative numbers α and κ such that $\alpha I + A + \kappa bc^T$ is primitive.

Recall that (A3) means that the matrix $(\alpha I + A + \kappa bc^T)^k$ is a positive matrix for some $k \in \mathbb{N}$.

In the following, let G denote the transfer function of (1), that is, $G(s) := c^T(sI - A)^{-1}b$.

Lemma 1.1: Assume that (A1)-(A3) hold. Then $G(0) > 0$ and $\|G\|_{H^\infty} = G(0)$.
A proof of Lemma 1.1 can be found in [1].

Applying nonlinear non-negative feedback $u = f(y)$ to (1), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz, leads to the following non-negative Lur'e system

$$\dot{x} = Ax + bf(c^T x), \quad x(0) = \xi \in \mathbb{R}^n_+. \tag{2}$$

We assume that the following assumption holds.

(A4) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz and $f(0) = 0$.

Whilst absolute stability of Lur'e systems is a classical topic in control theory (see, for example, [2], [3], [8]), it seems that non-negative Lur'e systems have not received much attention (see however [7] which provides an analysis of the stability properties of a class of non-negative discrete-time Lur'e systems).

*This work was supported by EPSRC (Grant EP/I019456/1)
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Assuming that (A1)-(A4) hold, we set

$$p := \frac{1}{G(0)},$$

and consider the following three cases.

Case 1. $f(z)/z \leq p \quad$ for all $z > 0$.

Case 2. $\inf_{z>0} f(z)/z > p$.

Case 3. There exists $y^* > 0$ such that $f(y^*) = py^*$ and

$$\left| \frac{f(z) - f(y^*)}{z - y^*} \right| \leq p \ \text{ for all } z > 0,\ z \neq y^*.$$

The condition in Case 3 means that the graph of f is "sandwiched" between the straight lines l_1 and l_2 given by $l_1(z) = pz$ and $l_2(z) = 2py^* - pz$, see Figure 1.

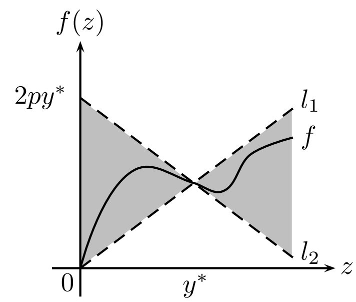

Fig. 1. Case 3: graph of f "sandwiched" between the lines l_1 and l_2.

II. LYAPUNOV STABILITY RESULTS

In this section, we present results which describe the stability/instability properties in each of three cases, where "stability" is interpreted in the sense of Lyapunov.

Let $x(\cdot\,; \xi)$ denote the unique maximally defined forward solution of (2) with maximal interval of existence $[0, \omega_\xi)$, where $0 < \omega_\xi \leq \infty$.

The proposition below relates to Case 1. It follows from well known results in absolute stability theory, see, for example, [3].

Proposition 2.1: Assume that (A1)-(A4) hold.
(a) If $f(z)/z \leq p$ for all $z > 0$, then the equilibrium 0 is stable in the large in the sense that there exists $\Gamma \geq 1$ such that, for every $\xi \in \mathbb{R}^n_+$, $\omega_\xi = \infty$ and

$$\|x(t; \xi)\| \leq \Gamma\|\xi\| \quad \forall\, t \geq 0.$$

ISBN: 978-90-367-6321-9

(b) If $f(z)/z < p$ for all $z > 0$, then the equilibrium 0 is globally asymptotically stable. In particular, for every $\xi \in \mathbb{R}^n_+$, $\omega_\xi = \infty$ and $x(t;\xi) \to 0$ as $t \to \infty$.
(c) If $\sup_{z>0} f(z)/z < p$, then the equilibrium 0 is globally exponentially stable, that is, there exist $N \geq 1$ and $\nu > 0$ such that, for every $\xi \in \mathbb{R}^n_+$, $\omega_\xi = \infty$ and

$$\|x(t;\xi)\| \leq Ne^{-\nu t}\|\xi\| \quad \forall\, t \geq 0.$$

In Case 2, the solutions of (2) diverge to ∞ for every non-zero initial condition. More precisely, we have the following result.

Theorem 2.2: Assume that (A1)-(A4) hold and $\inf_{z>0} f(z)/z > p$. Let $\xi \in \mathbb{R}^n_+$, $\xi \neq 0$, be such that the solution $x(t;\xi)$ exists for all $t \geq 0$. Then

$$\lim_{t \to \omega_\xi} x_i(t;\xi) = \infty \quad \forall\, i \in \{1, \ldots, n\},$$

where $x_i(\cdot\,;\xi)$ denotes the i-th component of $x(\cdot\,;\xi)$.

We proceed to consider Case 3.

Theorem 2.3: Assume that (A1)-(A4) hold.
(a) If there exists $y^* > 0$ such that $f(y^*) = py^*$ and

$$\left|\frac{f(z) - f(y^*)}{z - y^*}\right| \leq p \quad \forall\, z \geq 0,\ z \neq y^*$$

then $x^* = -pA^{-1}by^* \in \mathbb{R}^n_+$ is an equilibrium of (2) and x^* is stable in the large in the sense that there there exists $\Gamma \geq 1$ such that, for every $\xi \in \mathbb{R}^n_+$, $\omega_\xi = \infty$ and

$$\|x(t;\xi) - x^*\| \leq \Gamma\|\xi - x^*\| \quad \forall\, t \geq 0.$$

(b) If there exists $y^* > 0$ such that $f(y^*) = py^*$ and

$$\left|\frac{f(z) - f(y^*)}{z - y^*}\right| < p \quad \forall\, z > 0,\ z \neq y^*$$

then 0 and $x^* = -pA^{-1}by^* \in \mathbb{R}^n_+$ are the only equilibria of (2) and x^* is globally asymptotically stable in the sense that x^* is stable in the large (see statement (a) of this theorem) and, for every $\xi \in \mathbb{R}^n_+$ such that $\xi \neq 0$, $\omega_\xi = \infty$ and $x(t;\xi) \to x^*$ as $t \to \infty$.
(c) If there exists $y^* > 0$ such that $f(y^*) = py^*$,

$$\left|\frac{f(z) - f(y^*)}{z - y^*}\right| < p \quad \forall\, z > 0,\ z \neq y^*$$

and

$$\limsup_{y \to y^*} \left|\frac{f(z) - f(y^*)}{y - y^*}\right| < p,$$

and if

$$\liminf_{z \to 0} \frac{f(z)}{z} > p, \tag{3}$$

then 0 and $x^* = -pA^{-1}by^* \in \mathbb{R}^n_+$ are the only equilibria of (2) and x^* is "semi-globally" exponentially stable in the sense that, for every compact set $K \subset \mathbb{R}^n_+$ with $0 \notin K$, there exists $N \geq 1$ and $\nu > 0$ such that, for every $\xi \in K$, $\omega_\xi = \infty$ and

$$\|x(t;\xi) - x^*\| \leq Ne^{-\nu t}\|\xi - x^*\| \quad \forall\, t \geq 0.$$

(d) If (3) holds and there exists $y^* > 0$ such that $f(y^*) = py^*$ and, for every $\varepsilon > 0$,

$$\sup_{z \geq \varepsilon,\, z \neq y^*} \left|\frac{f(z) - f(y^*)}{z - y^*}\right| < p,$$

then 0 and $x^* = -pA^{-1}by^* \in \mathbb{R}^n_+$ are the only equilibria of (2) and x^* is "quasi-globally" exponentially stable in the sense that, for every $\delta > 0$ there exist $N \geq 1$ and $\nu > 0$ such that, for every $\xi \in \mathbb{R}^n_+$ with $\|\xi\| \geq \delta$, $\omega_\xi = \infty$ and

$$\|x(t;\xi) - x^*\| \leq Ne^{-\nu t}\|\xi - x^*\| \quad \forall\, t \geq 0. \tag{4}$$

We remark that "global" exponential stability of x^* (in the sense that there exist $N \geq 1$ and $\nu > 0$ such that (4) is satisfied for all $\xi \in \mathbb{R}^n_+$ with $\xi \neq 0$) does not hold. This is an immediate consequence of the following result which follows from continuity properties of the flow generated by the Lur'e system (2).

Proposition 2.4: Assume that (A1)-(A4) hold and that there exists $y^* > 0$ such that $f(y^*) = py^*$. Then, for every sequence (t_n) in $\mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$, there exists a sequence (ξ_n) in $\mathbb{R}^n_+$ with $\xi_n \neq 0$ and $\xi_n \to 0$ as $n \to \infty$ and such that

$$\lim_{n \to \infty} \frac{\|x(t_n;\xi_n) - x^*\|}{\|\xi_n - x^*\|} = 1,$$

where $x^* = -pA^{-1}by^*$.
Discrete-time results similar to statement (b) of Proposition 2.1, Theorem 2.2 and statement (b) of Theorem 2.3 can be found in [7].

Proofs of the results in Section II can be found in [1].

III. INPUT-TO-STATE STABILITY RESULTS

Finally, we investigate the stability behaviour of (2) subject to non-negative disturbances, that is, we analyze input-to-state stability (ISS) properties of the forced Lur'e system

$$\dot{x} = Ax + b(f(c^T x) + d), \quad x(0) = \xi \in \mathbb{R}^n_+, \tag{5}$$

where $d : \mathbb{R}_+ \to \mathbb{R}_+$ is locally essentially bounded. The unique maximally defined forward solution of (5) is denoted by $x(\cdot\,;\xi,d)$.

For an overview of ISS theory, the reader is referred to [6]. We recall some terminology and notation relating to comparison functions. Let $\mathcal{K}$ denote the set of all continuos functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and φ is strictly increasing. Moreover, define $\mathcal{K}_\infty := \{\varphi \in \mathcal{K} : \lim_{s\to\infty} \varphi(s) = \infty\}$. We denote by $\mathcal{KL}$ the set of functions in two variables $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties: $\psi(\cdot, t) \in \mathcal{K}$ for all $t \geq 0$, and $\psi(s, \cdot)$ is nonincreasing with $\lim_{t\to\infty} \psi(s,t) = 0$ for all $s \geq 0$.

The following proposition is a consequence of recent ISS results for Lur'e systems, see [4], [5].

Proposition 3.1: Assume that (A1)-(A4) hold. If there exists $\rho \in \mathcal{K}_\infty$ such that

$$f(z) \leq pz - \rho(z) \quad \forall\, z \geq 0,$$

then the equilibrium 0 of the unforced Lur'e system (2) is ISS in the sense that there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}_+^n$ and all non-negative $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$, $x(\cdot\,;\xi,d)$ is defined on $\mathbb{R}_+$ and

$$\|x(t;\xi,d)\| \leq \psi(\|\xi\|,t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall\, t \geq 0.$$

The following theorem shows that, under suitable assumptions, the equilibrium x^* has stability properties which are similar to ISS.

Theorem 3.2: Assume that (A1)-(A4) hold and that there exists $y^* > 0$ such that $f(y^*) = py^*$ and

$$\left|\frac{f(z)-f(y^*)}{z-y^*}\right| < p \quad \forall\, z > 0,\ z \neq y^*, \tag{6}$$

Furthermore, assume that (3) holds and

$$pz - f(z) \to \infty \quad \text{as } z \to \infty. \tag{7}$$

Then 0 and $x^* = -pA^{-1}by^* \in \mathbb{R}_+^n$ are the only equilibria of the unforced Lur'e system (2) and x^* is "quasi ISS" in the sense that, for every $\delta > 0$, there there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}_+^n$ with $\|\xi\| \geq \delta$ and all non-negative $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$, $x(\cdot\,;\xi,d)$ is defined on $\mathbb{R}_+$ and

$$\|x(t;\xi,d)-x^*\| \leq \psi(\|\xi-x^*\|,t)+\varphi(\|d\|_{L^\infty(0,t)}) \quad \forall\, t \geq 0. \tag{8}$$

To relate the conditions (6) and (7) to those in Proposition 3.1, we note that if (6) and (7) hold, then, for every $\varepsilon > 0$, there exists $\rho \in \mathcal{K}_\infty$ such that

$$|f(z)-f(y^*)| \leq p|z-y^*|-\rho(|z-y^*|) \quad \forall\, z \geq \varepsilon,\ z \neq y^*.$$

The proof of Theorem 3.2 is based on Proposition 3.1 and the following lemma.

Lemma 3.3: Assume that (A1)-(A4) hold. If (3) is satisfied and there exists $y^* > 0$ such that $f(y^*) = py^*$ and (6) holds, then, for every $\delta > 0$, there exist constants $\eta > 0$ and $\tau \geq 0$ such that for all $\xi \in \mathbb{R}_+^n$ with $\|\xi\| \geq \delta$ and all non-negative $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$, $x(\cdot\,;\xi,d)$ is defined on $\mathbb{R}_+$ and

$$c^T x(t;\xi,d) \geq \eta \quad \forall\, t \geq \tau.$$

This lemma also plays a key roll in the proof of statements (b)-(d) of Theorem 2.3 (with disturbance $d = 0$). Detailed proofs of Proposition 3.1, Theorem 3.2 and Lemma 3.3 can be found in [1].

Finally, it follows from Proposition 2.4 that "global" ISS of x^* (in the sense that there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that (8) is satisfied for all $\xi \in \mathbb{R}_+^n$ with $\xi \neq 0$ and all non-negative $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$) does not hold.

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