Fixed Points In Quantitative Semantics

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Abstract

We describe an interpretation of recursive computation in a symmetric monoidal category with infinite biproducts and *cofree commutative comonoids* (for instance, the category of free modules over a complete semiring). Such categories play a significant role in "quantitative" models of computation: they bear a canonical complete monoid enrichment, but may not be cpo-enriched, making standard techniques for reasoning about fixed points unavailable. By constructing a *bifree algebra* for the cofree exponential, we obtain fixed points for morphisms in its co-Kleisli category without requiring any order-theoretic structure. These fixed points corresponding to infinite sums of finitary approximants indexed over the nested finite multisets, each representing a unique call-pattern for computation of the fixed point. We illustrate this construction by using it to give a denotational semantics for PCF with non-deterministic choice and scalar weights from a complete semiring, proving that this is computationally adequate with respect to an operational semantics which evaluates a term by taking a weighted sum of the residues of its terminating reduction paths.

Categories and Subject Descriptors F.3.2 [*Semantics of Programming Languages*]: Denotational semantics

1. Introduction

We introduce a fixed point construction for certain models of linear type theory with infinite sums — specifically, symmetric monoidal categories with infinite biproducts, in which the exponentials are *cofree commutative comonoids*. These models have a quantitative flavour in two (related) senses — the cofree exponential allows a precise representation of the multiplicity of resources used by a program, and countable biproducts imply a natural enrichment over the category of $\mathcal{R}$-modules, for a complete (commutative) semiring $\mathcal{R}$, which can be used to capture properties of computation such as probability of failure, minimal or maximal cost, security level etc.

However, these models may not be continuously ordered. This poses two fundamental problems for interpreting recursive higher-order programs: how to construct *fixed points* for endomorphisms, and how to establish that they correspond to an operational interpretation of fixed points. We address both of these questions: the first by defining a fixed point operator for the Cartesian category defined by our model of linear type theory (the co-Kleisli category of the cofree exponential), which is unique in satisfying a *uniformity* property. To investigate the second, we define a nondeterministic functional language with scalar weights from a complete semiring, and give a denotational semantics in our categorical model which is *computationally adequate*: each program denotes the weighted sum (over all reduction paths) of the values to which it reduces. In establishing these results, we develop a new way of representing fixed points in quantitative semantics

1.1 Related Work

Quantitative interpretations of programs as linear functionals, with the direct sum and product as additive and multiplicative connectives, and cofree comonoids as exponentials, played key roles (explicitly and implicitly) in the development of linear logic (Girard 1988). Self-duality of the direct sum (i.e. biproducts) implies the existence of finite sums in these models, but construction of the cofree exponential requires certain infinite sums, forcing a choice between designing a model in which only those which are needed exist (cf. for example Ehrhard's finiteness spaces (Ehrhard 2010)), and a more general approach in which they all do (i.e. in this case, the requirement to have countable biproducts). Although this excludes some interesting cases, we might hope to arrive at a more complete understanding of the relationship between these alternatives.

To construct fixed points in the absence of any order theoretic structure, we turn to the abstract characterization of such operators provided by *axiomatic domain theory*. Specifically, we use the observation (Freyd 1990; Simpson and Plotkin 2000) that uniform fixed point operators exist (and are unique) for any comonad which is an *algebraically compact* functor (Barr 1993) — i.e. it has a *bifree algebra* (an initial algebra for which the inverse is a terminal coalgebra). In a nice example of the success of this theory, we define such an algebra for the cofree exponential by iterating Lafont's construction of a cofree commutative comonoid as a sum of finite multisets (Lafont 1988; Melliès et al. 2009), to obtain a bifree algebra which is a biproduct over the *nested finite multisets*. We observe, however, that many instances of our categorical model contain distinct (non-uniform) fixed points computed using a *trace operator* (Hasegawa 2002), so further investigation is needed to identify the operator which correctly captures program behaviour.

Our study of the operational behaviour of our fixed points generalizes and extends the results presented in (Laird et al. 2013b), which describes a denotational model for nondeterministic PCF with scalar weights from a *continuous* semiring, $\mathcal{R}$ (based on a semantics of linear logic introduced by Lamarche (Lamarche 1999)) in the category of sets and matrices over $\mathcal{R}$ (also known as the category of free $\mathcal{R}$-modules and their homomorphisms). Omitting the assumption of continuity allows some interesting examples, such as finite semirings characterizing *strong convergence* of programs, and a simpler and more general notion of categorical model. Moreover, it forces the development of a new way to characterize fixed points, based on a precise representation of the

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computational resources consumed in evaluating a function in the style of the resource λ-calculus (Boudol 1993) — in this case extended with *nested multiset* resource bounds capturing the call patterns of recursively defined procedures. This suggests further relationships with the rich theory underlying quantitative models, including the differential λ-calculus (Ehrhard and Regnier 2003): on the syntactic side, via their correspondence with the resource calculus (Tranquilli 2009), and on the semantic side, via the notion of differential category (Blute et al. 2006), which shares many properties and examples with our notion of categorical model.

2. Weighted PCF in a Complete Semiring

To illustrate our quantitative semantics of fixed points, we recall from (Laird et al. 2013b) the syntax and operational semantics of an extension of PCF with bounded non-deterministic choice and "scalar" weights from a given commutative monoid $\mathcal{M}$. Typing is Church-style — i.e. every variable has a fixed type (generated with the constructor $\rightarrow$ from a single ground type `nat`). The well-typed terms are defined with respect to contexts (finite sequences) of variables according to the rules in Table 1. The operational semantics for $\mathrm{PCF}^{\mathcal{M}}$ determines a weight in $\mathcal{M}$ for each *reduction path* from a program (closed term of type `nat`) to a terminal value (numeral), by assigning a weight to each reduction step and multiplying together the weights from each step in the path. Since the only computation step at which more than one rule may be applied is reduction of erratic choice (where reduction may take the right or the left branch), each reduction path for a given program corresponds to a unique element of the free monoid $\mathcal{B} = \{l, r\}^*$. Thus we define a reduction relation with labels from $\mathcal{B} \times \mathcal{M}$.

DEFINITION 2.1. *The operational semantics of* $\mathrm{PCF}^{\mathcal{M}}$ *is the* labelled transition system *(LTS) in which the* states *are the programs (closed terms of type* `nat`*) of* $\mathrm{PCF}^{\mathcal{M}}$*, the* labels *are the elements of* $\mathcal{B} \times \mathcal{M}$ *and the* actions *are as defined in Table 2.*

DEFINITION 2.2. *Let* $\overset{u,a}{\Longrightarrow}$ *be the reflexive, transitive closure of* $\overset{u,a}{\longrightarrow}$*. Equivalently (and conveniently for inductive proofs),* $P \overset{u,a}{\Longrightarrow} Q$ *if there exists* $i \in \mathbb{N}$ *such that* $P \overset{u,a}{\Longrightarrow}_i Q$*, where:*

$$\frac{}{P \overset{\varepsilon,1}{\Longrightarrow}_i P} \qquad \frac{P \overset{u,a}{\longrightarrow} P' \quad P' \overset{v,b}{\Longrightarrow}_i P''}{P \overset{u\cdot v, a\cdot b}{\Longrightarrow}_j P''} i < j$$

LEMMA 2.3. $P \overset{u,a}{\Longrightarrow}_i \mathtt{m}$ *and* $P \overset{u,b}{\Longrightarrow} \mathtt{n}$ *implies* $a = b$ *and* $m = n$*.*

PROOF: By induction on i, using the fact that $P \overset{u,a}{\longrightarrow} P'$ and $P \overset{u,b}{\longrightarrow} P''$ implies $a = b$ and $P' = P''$. $\square$

Thus, if $(P \Rightarrow n) = \{u \in \mathcal{B} \mid \exists a. P \overset{u,a}{\Longrightarrow} \mathtt{n}\}$ is the set of paths from P to $\mathtt{n}$, we may define the *path weight* function $w : (P \Rightarrow n) \rightarrow \mathcal{M}$ by: $w(u) = a$ if $P \overset{u,a}{\Longrightarrow} \mathtt{n}$.

Given a suitable notion of (countably) infinite indexed *sum* for values in $\mathcal{M}$ we may evaluate P at n by taking the sum of the weights for for all paths from P to n: as discussed in (Laird et al. 2013b) this value can represent aspects of the evaluation behaviour of P, such as the number of distinct paths or length of the shortest or longest path to a given value or the probability of reaching it, depending on the choice of sum. The properties we shall require of the sum are that it is a *complete monoid*, and with the monoid $\mathcal{M}$ as multiplication forms a *complete semiring*.

DEFINITION 2.4. *A (countably) complete monoid is a pair* (S, Σ) *of a set* S *with a sum operation* Σ *on (countably) indexed families of elements of* S*, satisfying the axioms:*

Partition Associativity *For any partitioning of the indexing set* I *into* $\{I_j \mid j \in J\}$*,* $\Sigma_{i\in I} a_i = \Sigma_{j\in J}\Sigma_{i\in I_j} a_i$*.*

Unary Sum $\Sigma_{i\in\{j\}} a_i = a_j$.

We write 0 for the sum of the empty family, which is well-defined and a neutral element for the sum by the above axioms. Any complete monoid is a commutative monoid in the usual sense (with binary sum $a_1 + a_2 = \Sigma_{i\in\{1,2\}} a_i$).

DEFINITION 2.5. *A complete semiring* $\mathcal{R}$ *is a tuple* $(|\mathcal{R}|, \Sigma, \cdot, 1)$ *such that* $(|\mathcal{R}|, \Sigma)$ *is a complete monoid and* $(|\mathcal{R}|, \cdot, 1)$ *is a monoid which distributes over* Σ *— i.e.* $\Sigma_{i\in I}(a \cdot b_i) = a \cdot \Sigma_{i\in I} b_i$ *and* $\Sigma_{i\in I}(b_i \cdot a) = (\Sigma_{i\in I} b_i) \cdot a$*.* $\mathcal{R}$ *is* commutative *if* $(|\mathcal{R}|, \cdot, 1)$ *is a commutative monoid.*

The operational and denotational semantics of $\mathrm{PCF}^{\mathcal{R}}$, where $\mathcal{R}$ is a *continuous semiring*, were studied in (Laird et al. 2013b).

DEFINITION 2.6. *A (commutative) continuous semiring is a commutative semiring* $(|\mathcal{R}|, +, 0, \cdot, 1)$ *with a directed complete partial order on* $|\mathcal{R}|$ *with least element* 0 *such that* $+$ *and* $\cdot$ *are continuous.*

Any continuous semiring extends to a complete semiring by setting $\Sigma_{i\in I} a_i = \bigvee_{J\subseteq_{fin} I} \Sigma_{j\in J} a_j$: say that $(|\mathcal{R}|, \Sigma, \cdot, 1)$ is continuous if it arises in this way — i.e. its underlying set can be ordered so that 0 is a least element and the sum and product are continuous. Examples of continuous semirings are described in (Laird et al. 2013b), including any complete lattice, the natural or positive real numbers completed with a greatest element ∞, and the so-called *exotic* semirings. Examples of complete commutative semirings which cannot be continuously ordered include finite semirings with the *formal sum*:

DEFINITION 2.7. *A semiring is* positive *if it has no additive inverses — i.e.* $a + b = 0$ *implies* $a = b = 0$*.*[1]
Given a positive semiring $\mathcal{R} = (|\mathcal{R}|, +, 0, \cdot, 1)$*, define the complete semiring* $\mathcal{R}^\infty = (|\mathcal{R}| \cup \{\infty\}, \Sigma, \cdot, 1)$*, where* $\infty \notin |\mathcal{R}|$ *and:*

- $\Sigma_{i\in I} a_i = \Sigma_{j\in J} a_j$*, where* $J = \{i \in I \mid a_i \neq 0\}$*, if* J *is finite.*
- $\Sigma_{i\in I} a_i = \infty$*, otherwise.*

and $a.\infty = \infty.a = 0$ *if* $a = 0$*;* $a.\infty = \infty.a = \infty$*, otherwise.*

PROPOSITION 2.8. *If* $|\mathcal{R}|$ *is finite then* $\mathcal{R}^\infty$ *is not continuous.*

PROOF: Suppose $\mathcal{R}$ is partially ordered with $0 \leq a$ for all a. Let $\underline{n} = \Sigma_{1\leq i\leq n} 1$. Then $\underline{0} \leq \underline{1} \leq \ldots$ and so $\bigvee_{J\subseteq_{fin}\mathbb{N}} \Sigma_{j\in J} 1 = \bigvee_{m\in\mathbb{N}} \underline{m} = \underline{n}$ for some n. But $\Sigma_{i\in\mathbb{N}} 1 = \infty$. $\square$

Observe that we may extend any (commutative) semiring $\mathcal{R}$ (even one without an additive zero) to a positive semiring $\mathcal{R}_0$ by adding a (new) zero element — i.e. $0' \notin |\mathcal{R}|$ such that $0' + a = a + 0' = a$ and $0'.a = a.0' = 0'$ for all $a \in |\mathcal{R}|$.

Relaxing the continuity requirement on the sum yields further examples and refinements of the quantitative semantics programs in $\mathrm{PCF}^{\mathcal{R}}$. For example, observe that if P is *strongly convergent* (i.e. there is no infinite reduction sequence starting from P) then by König's lemma, $(P \Rightarrow n)$ is finite for all n. Thus for any commutative semiring $\mathcal{R}$ (not necessarily complete) we may define the following notion of testing:

$P \Downarrow (n, a)$ *if* P *is strongly convergent and* $\Sigma_{u\in(P\Rightarrow n)} w(u) = a$*.*

Minimally, if $\mathcal{R}$ is the Boolean semiring this characterizes *must-convergence*: P must converge to n if and only if $P \Downarrow (n, \mathtt{1})$ and $P \Downarrow (m, \mathtt{0})$ for all $m \neq n$.

We may represent this notion of testing by evaluating programs in the complete semiring $\mathcal{R}_0^\infty$, using an adaptation of the characterization of strong convergence obtained in (Laird et al. 2013b) via

[1] Every complete monoid satisfies the positivity property — if $a + b = 0$ let $a_i = a, b_i = b$ for all $i \in \mathbb{N}$, then $0 = \Sigma_{i\in\mathbb{N}}(a_i + b_i) = a_0 + \Sigma_{i\in\mathbb{N}}(b_i + a_{i+1}) = a + 0 = a$.

$$\frac{}{\Gamma,x{:}T,\Gamma'\vdash x{:}T} \qquad \frac{\Gamma\vdash M{:}S\to T \quad \Gamma\vdash N{:}S}{\Gamma\vdash M\ N{:}T} \qquad \frac{\Gamma\vdash M{:}\mathtt{nat}}{\Gamma\vdash \underline{a}.M{:}\mathtt{nat}}\, a\in\mathcal{R} \qquad \frac{}{\Gamma\vdash \mathtt{0}{:}\mathtt{nat}} \qquad \frac{\Gamma\vdash M{:}\mathtt{nat}}{\Gamma\vdash \mathtt{succ}(M){:}\mathtt{nat}}$$

$$\frac{\Gamma,x{:}T\vdash M{:}T}{\Gamma\vdash \mu x.M{:}T} \qquad \frac{\Gamma,x{:}S\vdash M{:}T}{\Gamma\vdash \lambda x.M{:}S\to T} \qquad \frac{\Gamma\vdash M{:}\mathtt{nat} \quad \Gamma\vdash N{:}\mathtt{nat}}{\Gamma\vdash M\ \mathtt{or}\ N{:}\mathtt{nat}} \qquad \frac{\Gamma\vdash M{:}\mathtt{nat}}{\Gamma\vdash \mathtt{Ifz}(M){:}\mathtt{nat}\to\mathtt{nat}\to\mathtt{nat}} \qquad \frac{\Gamma\vdash M{:}\mathtt{nat}}{\Gamma\vdash \mathtt{pred}(M){:}\mathtt{nat}}$$

Table 1. Typing Judgments for $\mathrm{PCF}^{\mathcal{M}}$

$$\begin{array}{lcl@{\qquad\qquad}lcl}
E[(\lambda x.M)\ N] & \xrightarrow{\varepsilon,1} & E[M[N/x]] & E[\mu x.M] & \xrightarrow{\varepsilon,1} & E[M[\mu x.M/x]] \\
E[\mathtt{Ifz}(\mathtt{0})] & \xrightarrow{\varepsilon,1} & E[\lambda x.\lambda y.x] & E[M\ \mathtt{or}\ N] & \xrightarrow{l,1} & E[M] \\
E[\mathtt{Ifz}(\mathtt{n}+\mathtt{1})] & \xrightarrow{\varepsilon,1} & E[\lambda x.\lambda y.y] & E[M\ \mathtt{or}\ N] & \xrightarrow{r,1} & E[N] \\
E[\mathtt{pred}(\mathtt{n}+\mathtt{1})] & \xrightarrow{\varepsilon,1} & E[\mathtt{n}] & E[\underline{a}.M] & \xrightarrow{\varepsilon,a} & E[M]
\end{array}$$

$$E[\bullet] ::= [\bullet] \mid E[\bullet]\ N \mid \mathtt{Ifz}(E[\bullet]) \mid \mathtt{succ}(E[\bullet]) \mid \mathtt{pred}(E[\bullet])$$

Table 2. Labelled Transitions for $\mathrm{PCF}^{\mathcal{M}}$

a translation $(\_)^\circ$ from terms of $\mathrm{PCF}^{\mathcal{R}}$ to terms of $\mathrm{PCF}^{\mathcal{R}_0^\infty}$. This replaces λ and μ abstractions as follows:

$$(\lambda x.M)^\circ = \lambda x.\lambda y_1 \dots y_n.(M^\circ\ y_1 \dots y_n\ \mathtt{or}\ \underline{0}.\mathtt{1})$$
$$(\mu x.M)^\circ = \mu X.\lambda y_1 \dots y_n.(M^a\ y_1 \dots y_n\ \mathtt{or}\ \underline{0}.\mathtt{1})$$

(where $M : T_1 \to \dots \to T_n \Rightarrow \mathtt{nat}$) so that fixed point unwinding or β-reduction of function application creates a new path with weight 0 (recall that 0 is a zero for finite sums, but $\Sigma_{i\in I} 0 = \infty$ if I is infinite). Any infinite reduction sequence for a program P will contain infinitely many fixed point unwindings and thus create infinitely many such paths from P°. So, in particular, if $P_n =_{df} \mathtt{Ifz}(P^\circ)\ \mathtt{then}\ \underline{0}.\mathtt{n}\ \mathtt{else}\ \underline{0}.\mathtt{n}$:

LEMMA 2.9. *If P is strongly convergent then $\Sigma_{u\in(P_n\Rightarrow n)} w(u) = 0$. Otherwise, $\Sigma_{u\in(P_n\Rightarrow n)} w(u) = \infty$.*

Thus we obtain the following characterization of $\Downarrow$:

PROPOSITION 2.10. *$P \Downarrow (n,a)$ iff $\Sigma_{u\in((P\ \mathtt{or}\ P_n)\Rightarrow n)} w(u) = a$.*

In this paper we study quantitative denotational semantics of programs which capture their "sum-of-path-weights" in a complete, but not necessarily continuous semiring. We generalize to an abstract setting the model presented in (Laird et al. 2013b), in which each program denotes a matrix of values in a continuous semiring, by showing that any *Lafont category* with *countable biproducts* (defined below) contains a computationally adequate model of $\mathrm{PCF}^{\mathcal{R}}$. To do so we will require an interpretation of fixed points without a cpo-enrichment.

3. Lafont Categories with Countable Biproducts

We now recall some aspects of the relationship between (countably) complete monoids and (countable) *biproducts*[2].

DEFINITION 3.1. *A category $\mathcal{C}$ has (countable)* biproducts *if it has all (countable) set-indexed products and coproducts, and these coincide on objects and morphisms — i.e. for for any (countably) indexed family of objects $\{A_i \mid i \in I\}$ there is an object $\bigoplus_{i\in I} A_i$ which is a product and coproduct of the A_i, and for any family of morphisms $\{f_i : A_i \to B_i \mid i \in I\}$, $\langle \pi_i; f_i \mid i \in I\rangle = [f_i; \iota_i \mid i \in I]$ (denote this by $\bigoplus_{i\in I} f_i : \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i$).*

It is well known that any category with finite biproducts carries a canonical enrichment in the category of commutative monoids.

[2] The restriction to countable biproducts (and countably complete monoids and semi-rings) can be largely ignored, save for Lemma 4.22.

Similarly, categories with infinite biproducts bear an enrichment over the category of complete monoids and their homomorphisms (in which we may define symmetric monoidal structure such that the morphisms from the tensor product are *bilinear* functions (Bahamonde 1985)).

PROPOSITION 3.2. *Any category with countable biproducts bears an enrichment over the category of countably complete monoids.*[3]

PROOF: Given a family of morphisms $\{f_i : A \to B \mid i \in I\}$, let $\Sigma_{i\in I} f_i = \Delta_A^I; \bigoplus_{i\in I} f_i; \nabla_A^I$, where $\Delta_I^A : A \to \bigoplus_{i\in I} A = \langle \mathsf{id}_A \mid i \in I\rangle$ and $\nabla_A^I : \bigoplus_{i\in I} A \to A = [\mathsf{id}_B \mid i \in I]$. This satisfies the unary sum axiom by definition, and partition associativity as follows: for any partitioning of I into $\{I_j \mid j \in J\}$:
$\Sigma_{j\in J}\Sigma_{i\in I_j} f_i = \Delta_A^J; \bigoplus_{j\in J}(\Delta_A^{I_j}; (\bigoplus_{i\in I_j} f_i); \nabla_A^{I_j}); \nabla_A^J$
$= \Delta_A^J; (\bigoplus_{j\in J} \Delta_A^{I_j}); (\bigoplus_{j\in J} \bigoplus_{i\in I_j} f_i); (\bigoplus_{j\in J} \nabla_A^{I_j}); \nabla_A^J$
$= \Delta_A^I; (\bigoplus_{i\in I} f_i); \nabla_A^I = \Sigma_{i\in I} f_i$
Similarly, we may show that composition is *bilinear* — i.e. $(\Sigma_{i\in I} f_i); (\Sigma_{j\in J} g_j) = \Sigma_{\langle i,j\rangle\in I\times J} f_i; g_j$.[4] □

Conversely, any (countably) complete monoid enriched category may be *completed* with (countable) biproducts.

DEFINITION 3.3. *If $\mathcal{C}$ is complete monoid enriched, let $\mathcal{C}^\Pi$ be the category in which objects are (countable) set-indexed families of objects of $\mathcal{C}$, and morphisms from $\{A_i \mid i \in I\}$ to $\{B_j \mid j \in J\}$ are $I \times J$-indexed sets of morphisms $\{f_{i,j} : A_i \to B_j \mid \langle i,j\rangle \in I \times J\}$, composed by setting $(f;g)_{ik} = \Sigma_{j\in J}(f_{ij}; g_{jk})$.*

Observe that the endomorphisms on any object of a complete-monoid-enriched category form a complete semiring (with composition as multiplication). Conversely, any complete semiring $\mathcal{R}$ may be viewed as a one-object complete-monoid-enriched category in which composition is multiplication in $\mathcal{R}$. Thus we obtain a key example of a category with (countable) biproducts: $\mathcal{R}^\Pi$ is the category in which objects are (countable) sets, and morphisms from A to B are $A \times B$ *matrices* with entries in $\mathcal{R}$, composed by matrix multiplication.

[3] In particular, any category with biproducts has a *zero object*, which is both terminal and initial, and thus for objects A, B, a *zero map* — $0_{A,B} : A \to B$ obtained by composing the terminal map from A with the initial map into B.

[4] By the Eckmann-Hilton argument, any choice of biproduct gives the same complete monoid enrichment.

We extend these definitions to symmetric monoidal categories[5] by requiring that the tensor *distributes* over biproducts — i.e. $(\bigoplus_{i\in I} A_i)\otimes B = \bigoplus_{i\in I}(A_i\otimes B)$[6]. The complete monoid enrichment thus extends to the monoidal structure — i.e. $(\Sigma_{i\in I} f_i)\otimes g = \Sigma_{i\in I}(f_i\otimes g)$. If $\mathcal{C}$ is a complete monoid enriched SMC in this sense, then we may define the (distributive) tensor product on $\mathcal{C}^\Pi$: $\{A_i\,|i\in I\}\otimes\{B_j\mid j\in J\} = \{A_i\otimes B_j\,|\langle i,j\rangle\in I\times J\}$, with $(f\otimes g)_{ikjl} = f_{ij}\otimes g_{kl}$.
Thus for any complete commutative semiring $\mathcal{R}$ (a one-object SMCC in which the tensor product is multiplication) we have a SMCC with distributive biproducts $\mathcal{R}^\Pi$. Conversely:

PROPOSITION 3.4. *In a complete-monoid-enriched SMC $(\mathcal{C},\otimes,I)$, the endomorphisms on I form a complete,* commutative *semiring.*

PROOF: It remains to observe that if $\mathcal{C}$ is symmetric monoidal then for any $f,g : I\to I$, $f;g = (f\otimes I);(I\otimes g) = (f\otimes g) = g\otimes f = g;f$. (The symmetry isomorphism $\theta_{I,I} : I\otimes I\to I\otimes I$ is the identity, since the left and right unitors $l_I, r_I : I\otimes I\to I$ are equal and so by the unit coherence of a symmetric monoidal category, $\theta_{I,I} = r_I; l_I^{-1} = \mathsf{id}_{I\otimes I}$.) $\square$

We call this category $\mathcal{R}_\mathcal{C}$.[7] Since it is by definition a full subcatgory of $\mathcal{C}$, and the latter is equivalent to $\mathcal{C}^\Pi$ if $\mathcal{C}$ has countable biproducts:

LEMMA 3.5. *If $\mathcal{C}$ is a SMC with distributive biproducts there is a fully faithful (strong monoidal) functor $\widetilde{(\_)} : \mathcal{R}_\mathcal{C}^\Pi\to\mathcal{C}$.*

Specifically, $\widetilde{A} = \bigoplus_{x\in A} I$ (i.e. the A-indexed biproduct of copies of I). Note that if $\mathcal{C}$ is a complete monoid enriched SMC which is also *cpo-enriched* then so is its biproduct completion $\mathcal{C}^\Pi$ — so in particular if $\mathcal{R}$ is a continuous semiring then $\mathcal{R}^\Pi$ bears a cpo-enrichment which may be used to define fixed points and prove their key properties (Laird et al. 2013b). Conversely:

PROPOSITION 3.6. *In any cpo-enriched category with biproducts the induced sum is continuous.*

PROOF: For $J\subseteq I$, let $\delta_A^J : A\to\oplus_{i\in I}A = \langle g_i\mid i\in I\rangle$, where $g_i = \mathsf{id}_A$ if $i\in J$, and $g_j = 0_{A,A}$, otherwise. By uniqueness, zero morphisms are least elements, and so $\delta_A^I = \bigvee_{J\subseteq_{fin} I}\delta_A^J$. So for any $\{f_i : A\to B\mid i\in I\}$, $\Sigma_{i\in I} f_i = \delta_A^I;[f_i\mid i\in I] = (\bigvee_{J\subseteq_{fin} I}\delta_A^J);[f_i\mid i\in I] = \bigvee_{J\subseteq_{fin} I}\delta_A^J;[f_i\mid i\in I] = \bigvee_{J\subseteq_{fin} I}\Sigma_{j\in J} f_j$. $\square$

Hence if $\mathcal{C}$ is a category with biproducts such that $\mathcal{R}_\mathcal{C}$ is *not* continuous then $\mathcal{C}$ cannot be cpo-enriched and so standard domain-theoretic techniques for defining and reasoning about fixed points in $\mathcal{C}$ are not available — one of the objects of our investigation is to develop alternatives.

3.1 Lafont Categories

We will describe a construction of fixed points for models of intuitionistic linear type theory with biproducts, in which the exponential $!B$ is the *cofree commutative comonoid* on B.

DEFINITION 3.7. *A cofree commutative comonoid on an object B in a symmetric monoidal category $\mathcal{C}$ is an object $(!B,\delta,\epsilon)$ in* $\mathsf{comon}(\mathcal{C})$ *(the category of commutative commonoids and comonoid morphisms of $\mathcal{C}$) with a natural equivalence between $\mathcal{C}(A,B)$ and* $\mathsf{comon}(\mathcal{C})(A,!B)$ *for each commutative comonoid (A,δ_A,ϵ_A). This is given by a morphism* $\mathsf{der}_B\in\mathcal{C}(!B,B)$ *and a map* $(\_)^\dagger : \mathcal{C}(A,B)\to\mathsf{comon}(A,!B)$ *such that* $f^\dagger;\mathsf{der}_B = f$ *for all* $f\in\mathcal{C}(A,B)$ *and* $(g;\mathsf{der}_B)^\dagger = g$ *for all* $g\in\mathsf{comon}(\mathcal{C})(A,!B)$*, and for any comonoid morphism $h : C\to A$, $h;f^\dagger = (h;f)^\dagger$.*

Thus $\mathcal{C}$ has (all) cofree commutative comonoids if and only if the forgetful functor from $\mathsf{comon}(\mathcal{C})$ into $\mathcal{C}$ has a right adjoint. This (monoidal) adjunction resolves a monoidal comonad $! : \mathcal{C}\to\mathcal{C}$, with action on morphisms sending $f : A\to B$ to $!f : !A\to\, !B = (\mathsf{der}_A;f)^\dagger$. We will refer to such a comonad as the *cofree exponential*. A *Lafont category* (Lafont 1988) is a symmetric monoidal *closed* category $\mathcal{C}$ with finite products and a cofree exponential: its co-Kleisli category $\mathcal{C}_!$ is therefore Cartesian closed.

Distributive biproducts are a key element of a general construction of cofree commutative comonoids (the so-called *Lafont exponential*) which may be presented as follows (Melliès et al. 2009).

DEFINITION 3.8. *A family of objects $\{B^i\mid i\in\mathbb{N}\}$ in a symmetric monoidal category are* symmetric tensor powers *of B if:*

- *For each n there is a morphism $\mathsf{eq}_n : B^n\to B^{\otimes n}$ such that (B^n,eq_n) is an* equalizer *for the set of automorphisms on $B^{\otimes n}$ derived from the permutations $\{1,\dots,n\}$.*
- *These equalizers are preserved by the tensor product — i.e. $(B^m\otimes B^n,\mathsf{eq}_m\otimes\mathsf{eq}_n)$ is an equalizer for the tensor products of pairs of permutation automorphisms.*

An object $!B$ in a SMC with biproducts is a Lafont exponential *of B if it is the biproduct of symmetric tensor powers of B — i.e. $!B = \bigoplus_{n\in\mathbb{N}} B^n$.*

We may equip the Lafont exponential with commutative comonoid structure by defining $\epsilon_{!B} : !B\to I = \pi_0$ and $\delta_{!B} : !B\to !B\otimes !B = \langle\pi_{m+n};\delta_{m,n}\mid m,n\in\mathbb{N}\rangle$, where $\delta_{m,n} : B_{m+n}\to B^m\otimes B^n$ is the unique morphism such that $\mathsf{eq}_{m+n} = \delta_{m,n};(\mathsf{eq}_m\otimes\mathsf{eq}_n)$.

PROPOSITION 3.9. *If $!B$ is the Lafont exponential of B then $(!B,\delta_{!B},\epsilon_{!B})$ is the cofree commutative comonoid on B.*

PROOF: For details, see (Melliès et al. 2009): $\mathsf{der}_B : !B\to B = \pi_1$ and for any $f : A\to B$, $f^\dagger : A\to !B = \langle f^i\mid i\in\mathbb{N}\rangle$, where $f^n : A\to B^n$ is the unique morphism such that $f^n;\mathsf{eq}_n = \delta(n);f^{\otimes n}$ (where $\delta(n) : A\to A^{\otimes n}$ is the unique k-ary co-multiplication for A). $\square$

As observed in (Lamarche 1999; Laird et al. 2013b), for any set S the set $\mathcal{M}_*(S)$ of *finite multisets* over S is the Lafont exponential of S in $\mathcal{R}^\Pi$ (which is therefore a Lafont category). Moreover, we may recast this construction in *any* symmetric monoidal category $\mathcal{C}$ with (distributive) biproducts, using our previous observation that in a symmetric monoidal category, every permutation automorphism on $I^{\otimes k}$ is the identity — i.e. I is a k-ary tensor power of itself. For each set S, let $\mathcal{M}_k(S)$ denote the set of finite multisets over S of cardinality k.

LEMMA 3.10. *For any set S, the objects $\{\widetilde{\mathcal{M}_k(S)}\mid k\in\mathbb{N}\}$ are symmetric tensor powers of $\widetilde{S}$.*

PROOF: $\mathcal{M}_k(S)$ corresponds to the set of permutation equivalence classes of elements of $\Pi_{i\leq k}S$ and so for each $X\in\mathcal{M}_k(S)$ there exists $\widehat{X}\in\Pi_{i\leq k}S$ such that $[\widehat{X}] = X$.

$\widetilde{S}^{\otimes k} = \widetilde{\Pi_{i\leq k}S}$ by distributivity of $\otimes$ over biproducts. Thus for each permutation θ on $\{1,\dots,k\}$, the corresponding automorphism $\theta_{\widetilde{S}} : \widetilde{S}^{\otimes k}\to\widetilde{S}^{\otimes k}$ is the map $\langle\pi_{\theta^{-1}(s)}\mid s\in\Pi_{i\leq k}S\rangle$. Let $\mathsf{eq}_k : \widetilde{\mathcal{M}_k(S)}\to\widetilde{S}^{\otimes k} = \langle\pi_{[x]}\mid x\in\Pi_{i\leq k}S\rangle$. Then $(\widetilde{\mathcal{M}_k(S)},\mathsf{eq}_k)$

[5] In the following, we will generally elide associativity and unit isomorphisms as if in a strict monoidal category.

[6] This holds in any symmetric monoidal *closed* category as $\otimes$ preserves colimits.

[7] Any complete-monoid-enriched SMC is enriched over the category of $\mathcal{R}_\mathcal{C}$-modules — i.e. each hom-set is a complete monoid with a distributive "scalar multiplication" by elements of $R_\mathcal{C}$ ($k.f = k\otimes f$): composition and the tensor product are evidently bilinear with respect to this structure.

is an equalizer for the permutation automorphisms on $\widetilde{S}^{\otimes k}$ (and these equalizers are preseved by the tensor product):

- $\mathsf{eq}_n; \theta_{\widetilde{S}} = \langle \pi_{[x]} \mid x \in \Pi_{i \leq k} S\rangle; \langle \pi_{\theta^{-1}(s)} \mid s \in S^k\rangle = \langle \pi_{[\theta^{-1}(s)]} \mid s \in S^k\rangle = \mathsf{eq}_n$.
- For any $f : B \to \widetilde{\Pi_{i \leq k} S}$ such that $f; \theta_{\widetilde{S}} = f$ for all θ, let $u : B \to \widetilde{\mathcal{M}_k(S)} = \langle f; \pi_{\widehat{X}} \mid X \in \mathcal{M}_k(S)\rangle$, so that $g; \mathsf{eq}_n = f$ if and only if $g = u$.

□

For any set S, $\widetilde{\mathcal{M}_*(S)} = \bigoplus_{k \in \mathbb{N}} \widetilde{\mathcal{M}_k(S)}$ and hence:

PROPOSITION 3.11. *$\widetilde{\mathcal{M}_*(S)}$ is a Lafont exponential for $\widetilde{S}$.*

By definition, the functor $\widetilde{(\_)} : \mathcal{R}^{\Pi}_{\mathcal{C}} \to \mathcal{C}$ preserves Lafont exponentials (if $\mathcal{C}$ is a Lafont category then it is a map of adjunctions in the sense of (Mac Lane 1971)).

The following identities relating the biproduct structure of the Lafont exponential to its comonoid structure follow directly from its definition ($X \subseteq Y$ denotes multiset inclusion of X in Y, $X + Y$ denotes their multiset union and $X - Y$ their multiset difference).

LEMMA 3.12. *For any set S, the Lafont exponential $!\widetilde{S}$ satisfies:*

- $\iota_{[\_]}; \epsilon_{!\widetilde{S}} = \mathsf{id}_I$ *and* $\pi_{[\_]} = \epsilon_{!\widetilde{S}}$.
  If $X \notin \mathcal{M}_1(S)$ *then* $\iota_X; \epsilon_{!\widetilde{S}} = 0_{I,I}$.
- *For all* $X \in \mathcal{M}_{*(S)}$, $\iota_X; \delta_{!\widetilde{S}} = \Sigma_{Y \subseteq X}(\iota_Y \otimes \iota_{X-Y})$ *and for all* $Y \subseteq X$, $\pi_X = \delta_{!\widetilde{S}}; (\pi_Y \otimes \pi_{X-Y})$
- *For all* $x \in S$, $\iota_{[x]}; \mathsf{der}_{\widetilde{S}} = \iota_x$ *and* $\pi_{[x]} = \mathsf{der}_{\widetilde{S}}; \pi_x$.
  If $X \notin \mathcal{M}_1(S)$ *then* $\iota_X; \mathsf{der}_{\widetilde{S}} = 0_{I,\widetilde{S}}$.

Explicitly, in $\mathcal{R}^{\Pi}$ $!S$ is the set of finitely multisets of S, and $\delta : !S \to !S \otimes !S$, $\epsilon : !S \to I$ and $\mathsf{der} : !S \to S$ are the matrices with:

- $\delta_{XYZ} = 1$ if $X = Y + Z$ and 0 otherwise.
- $\epsilon_{X*} = 1$ if $X = [\_]$ and 0 otherwise.
- $\mathsf{der}_{XY} = 1$ if $X = [Y]$ and 0 otherwise.

The promotion of a morphism $f : !S \to S$ is the matrix with $f^{\dagger}_{X,[y_1,\ldots,y_n]} = \Sigma\{f_{X_1,y_1} \cdot \ldots \cdot f_{X_n,y_n} \mid X = X_1 + \ldots + X_n\}$. Further examples of Lafont categories with countable biproducts may be given using two-player games and strategies (see (Laird et al. 2013a) for a deconstruction of such a category into the co-Kleisli category of the Lafont exponential on the biproduct completion of a category of schedules).

4. Uniform Fixed Points

We will show that if a Lafont category $\mathcal{C}$ has countable biproducts then $\mathcal{C}_!$ has a *uniform fixed point operator*.

DEFINITION 4.1. *A fixed point operator for a category $\mathcal{C}$ with a terminal object is a map taking each endomorphism $f \in \mathcal{C}(A, A)$ to a morphism* $\mathsf{fix}(f) \in \mathcal{C}(1, A)$ *satisfying* $\mathsf{fix}(f) = \mathsf{fix}(f); f$.

Let $L : \mathcal{C} \to \mathcal{C}$ be a comonad with co-Kleisli triple $(L, (\_)^{\dagger}, \mathsf{der})$. A fixed point operator for the co-Kleisli category $\mathcal{C}_L$ is uniform *if for any morphisms (in $\mathcal{C}$) $f : LA \to A$, $g : LB \to B$ and $h : A \to B$ which satisfy $f; h = L(h); g$, we have* $\mathsf{fix}(g) = \mathsf{fix}(f); h$.

PROPOSITION 4.2. *If $\mathcal{C}$ is cpo-enriched and L is continuous on morphisms then $\mathcal{C}_L$ has a uniform fixed point operator.*

PROOF: Taking $\mathsf{fix}(f)$ to be the least fixed point of f — the supremum of the ω-chain $f_0 \leq f_1 \leq \ldots$ defined $f_0 = \bot_{L1,B}$ and $f_{i+1} = f_i^{\dagger}; f$, we may show by induction that if $f; h = Lh; g$ then $f_i; h = g_i$ for all i and hence $\mathsf{fix}(f); h = (\bigvee_{i \in \omega} f_i); h = \bigvee_{i \in \omega}(f_i; h) = \bigvee_{i \in \omega} g_i = \mathsf{fix}(g)$. □

By Proposition 3.6, categories with biproducts in which the induced sum is *not* continuous cannot be cpo-enriched, and therefore the above fixed point construction does not apply. Instead, we shall use the observation of (Freyd 1990), further developed in (Simpson and Plotkin 2000), that any comonad which has a *bifree algebra* has a unique uniform fixed point operator on its co-Kleisli category.

DEFINITION 4.3. *A bifree algebra for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is an* initial algebra *$\alpha : FA \to A$ for F such that $\alpha^{-1} : A \to FA$ is a* final coalgebra *for F*[8]*. In other words, F is* algebraically compact *(Barr 1993).*

PROPOSITION 4.4. *If a comonad $L : \mathcal{C} \to \mathcal{C}$ has a bifree algebra $\psi : L\Psi \to \Psi$ then $\mathcal{C}_L$ has a unique uniform fixed point operator.*

PROOF: The comultiplication $\mathsf{id}^{\dagger}_{L1} : L1 \to L^2 1$ is a L-coalgebra and therefore has a unique *anamorphism* $\infty : L1 \to \Psi$ such that $\infty; \psi^{-1} = \mathsf{id}^{\dagger}_{L1}; L(\infty)$ — i.e. ∞ is the unique morphism such that $\infty = \infty^{\dagger}; \psi$.

Any endomorphism $f \in \mathcal{C}_L(A, A)$ is an L-algebra in $\mathcal{C}$ and therefore has a unique *catamorphism* $(\![f]\!) : \Psi \to A$ such that $\psi; (\![f]\!) = L(\![f]\!); f$. Define $\mathsf{fix}(f) = \infty; (\![f]\!)$, which satisfies:

- Fixed Point Property: $\mathsf{fix}(f) = \infty; (\![f]\!) = \infty^{\dagger}; \psi; (\![f]\!) = \infty^{\dagger}; L(\![f]\!); f = (\infty; (\![f]\!))^{\dagger}; f = \mathsf{fix}(f)^{\dagger}; f$.
- Uniformity: If $f; h = Lh; g$ then $\psi; ((\![f]\!); h) = L(\![f]\!); f; h = L(\![f]\!); Lh; g = L((\![f]\!); h); g$ — and so by uniqueness of catamorphisms, $(\![g]\!) = (\![f]\!); h$ and $\mathsf{fix}(g) = \infty; (\![g]\!) = \mathsf{fix}(f); h$.
- Uniqueness: for any uniform fixed point operator, $\mathsf{fix}(\psi)^{\dagger}; \psi = \mathsf{fix}(\psi)$ and so $\mathsf{fix}(\psi) : L1 \to \Psi = \infty$ by uniqueness of anamorphisms. Hence for any $f : LA \to A$, $\mathsf{fix}(f) = \mathsf{fix}(\psi); (\![f]\!) = \infty; (\![f]\!)$ by uniformity, since $\psi; (\![f]\!) = L(\![f]\!); f$.

□

4.1 The Bifree Algebra of Nested Finite Multisets

We now show that in any category with distributive biproducts, the cofree exponential has a bifree algebra, and thus a uniform fixed point operator. The finite multiset operation $\mathcal{M}_*(\_)$ is $\subseteq$-continuous on sets, and thus has a $\subseteq$-least fixed point — the (countable) set $\mathbb{M}$ of *nested finite multisets*.

DEFINITION 4.5. *Let $\mathbb{M} = \bigcup_{i \in \mathbb{N}} \mathbb{M}_i$, where $\mathbb{M}_0 = \varnothing$ and $\mathbb{M}_{i+1} = \mathcal{M}_*(\mathbb{M}_i)$. Then $\mathbb{M}_i \subseteq \mathbb{M}_{i+1}$ for all i and $\mathcal{M}_*(\mathbb{M}) = \mathbb{M}$. The* depth *$d(X)$ of an element $X \in \mathbb{M}$ is the least $n \in \mathbb{N}$ such that $X \in \mathbb{M}_n$. The* flattening *of $X \in \mathbb{M}$ to $|X| \in \mathbb{N}$ is defined recursively: $|[X_1, \ldots, X_k]| = |X_1| + \ldots + |X_k| + k + 1$.*

Suppose $\mathcal{C}$ is a symmetric monoidal category with distributive biproducts and cofree commutative comonoids. By Proposition 3.11, $\widetilde{\mathbb{M}} = \bigoplus_{i \in \mathbb{M}} I = \bigoplus_{i \in \mathcal{M}_*(\mathbb{M})} I$ is the Lafont exponential of itself — and thus $!\widetilde{\mathbb{M}} \cong \widetilde{\mathbb{M}}$. We will show that this isomorphism is a bifree algebra for $! : \mathcal{C} \to \mathcal{C}$, assuming without loss of generality that it is the identity.

For each $X \in \mathbb{M}$ we have a projection $\pi_X : \widetilde{\mathbb{M}} \to I$ and injection $\iota_X : I \to \widetilde{\mathbb{M}}$: define $\phi_X : \widetilde{\mathbb{M}} \to \widetilde{\mathbb{M}} = \pi_X; \iota_X$. Then by definition of biproducts and the associated sum, $\Sigma_{X \in \mathbb{M}} \phi_X = \mathsf{id}_{\widetilde{\mathbb{M}}}$, and $\phi_X; \phi_Y = \phi_X$ if $X = Y$ and $\phi_X; \phi_Y = 0$, otherwise. The following identities derive from the corresponding properties of the injections and projections (Lemma 3.12):

P1 $\phi_{[\_]}; \epsilon_{\widetilde{\mathbb{M}}} = \pi_{[\_]}; \iota_{[\_]}; \epsilon_{\widetilde{\mathbb{M}}} = \pi_{[\_]} = \epsilon_{\widetilde{\mathbb{M}}}$,
$\phi_X; \epsilon_{\widetilde{\mathbb{M}}} = \pi_X; 0_{I,I} = 0_{\widetilde{\mathbb{M}},I}$, if $X \neq [\_]$.

[8] α is an isomorphism by Lambek's lemma.

P2 $\phi_X; \delta_{\widetilde{\mathbb{M}}} = \pi_X; \Sigma_{Y\subseteq X}(\iota_Y \otimes \iota_{X-Y})$
$= \Sigma_{Y\subseteq X}(\pi_X; (\iota_Y \otimes \iota_{X-Y}))$
$= \delta_{\widetilde{\mathbb{M}}}; \Sigma_{Y\subseteq X}((\pi_Y; \iota_Y) \otimes (\pi_{X-Y}; \iota_{X-Y}))$
$= \Sigma_{Y\subseteq X}\delta_{\widetilde{\mathbb{M}}}; (\phi_Y \otimes \phi_{X-Y})$ for all $X \in \mathbb{M}$.

P3 $\phi_{[Y]}; \mathsf{der}_{\widetilde{\mathbb{M}}} = \pi_{[Y]}; \iota_Y = \mathsf{der}_{\widetilde{\mathbb{M}}}; \phi_Y$ for all $Y \in \mathbb{M}$.
$\phi_X; \mathsf{der}_{\widetilde{\mathbb{M}}} = \pi_X; 0_{I,\widetilde{\mathbb{M}}} = 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}$ if $X \notin \widetilde{\mathbb{M}}_1(\mathbb{M})$.

LEMMA 4.6. *For all $n \in \mathbb{N}$, $\Sigma_{X\in\mathbb{M}_n}\phi_X = !^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}$.*

PROOF: For all n, $\Sigma_{X\in\mathbb{M}_{n+1}}\phi_X$ is a comonoid morphism: $(\Sigma_{X\in\mathbb{M}_{n+1}}\phi_X); \epsilon_{\widetilde{\mathbb{M}}} = \phi_{[\_]}; \epsilon_{\widetilde{\mathbb{M}}} + \Sigma_{X\in\mathbb{M}_{n+1}\setminus\{[\_]\}}\phi_X; \epsilon_{\widetilde{\mathbb{M}}} = \epsilon_{\widetilde{\mathbb{M}}} + 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}} = \epsilon_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}$ by **P1**), and
$(\Sigma_{X\in\mathbb{M}_{n+1}}\phi_X); \delta_{\widetilde{\mathbb{M}}} = \Sigma_{X\in\mathbb{M}_{n+1}}(\phi_X; \mathsf{der})$
$= \Sigma_{X\in\mathbb{M}_{n+1}}\Sigma_{Y\subseteq X}(\delta_{\widetilde{\mathbb{M}}}; (\phi_Y \otimes \phi_{X-Y}))$ by **P2**
$= \delta_{\widetilde{\mathbb{M}}}; \Sigma_{X\in\mathbb{M}_{n+1}}\Sigma_{Y\subseteq X}(\phi_Y \otimes \phi_{X-Y})$
$= \delta_{\widetilde{\mathbb{M}}}; ((\Sigma_{Y\in\mathbb{M}_{n+1}}\phi_Y) \otimes (\Sigma_{Z\in\mathbb{M}_{n+1}}\phi_Z))$.

Hence $\Sigma_{X\in\mathbb{M}_{n+1}}\phi_X = !(\Sigma_{X\in\mathbb{M}_n}\phi_X)$, since
$(\Sigma_{X\in\mathbb{M}_{n+1}}\phi_X); \mathsf{der}_{\widetilde{\mathbb{M}}} = \Sigma_{X\in\mathbb{M}_{n+1}}\phi_X; \mathsf{der}_{\widetilde{\mathbb{M}}}$
$= \Sigma_{X\in\mathbb{M}_n}(\phi_{[X]}; \mathsf{der}_{\widetilde{\mathbb{M}}}) + \Sigma_{X\in\mathbb{M}\setminus\widetilde{\mathbb{M}}_1(\mathbb{M})}(\phi_X; \mathsf{der}_{\widetilde{\mathbb{M}}})$
$= \Sigma_{X\in\mathbb{M}_n}(\mathsf{der}_{\widetilde{\mathbb{M}}}; \phi_X) + 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}$ by **P3**.
$= \mathsf{der}_{\widetilde{\mathbb{M}}}; \Sigma_{X\in\mathbb{M}_n}\phi_X$.

So $\Sigma_{X\in\mathbb{M}_n}\phi_X = !^n\Sigma_{X\in\mathbb{M}_0}\phi_X = !^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}$ □

LEMMA 4.7. *For $X \in \mathbb{M}_n$, $!^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; \phi_X = \phi_X; !^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}} = \phi_X$.*

PROOF: $\phi_X; !^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}} = \phi_X; \Sigma_{Y\in\mathbb{M}_n}\phi_Y = \Sigma_{Y\in\mathbb{M}_n}(\phi_X; \phi_Y) = \phi_X$ and similarly $!^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; \phi_X = \Sigma_{Y\in\mathbb{M}_n}(\phi_Y; \phi_X)$. □

We now show that $\mathsf{id}_{\widetilde{\mathbb{M}}} : \widetilde{\mathbb{M}} \to \widetilde{\mathbb{M}}$ is a bifree algebra for the cofree exponential. Given $f : !A \to A$ we will define a unique morphism $(\![f]\!)_! : \widetilde{\mathbb{M}} \to !A$ such that $(\![f]\!)_! = !((\![f]\!); f)$, and given $g : A \to !A$, a unique morphism $[\![g]\!]_! : !A \to \widetilde{\mathbb{M}}$ such that $[\![g]\!] = g; ![\![g]\!]$. Define the approximants $(\![f]\!)_i : \widetilde{\mathbb{M}} \to A$ and $[\![g]\!]_i : A \to \widetilde{\mathbb{M}}$ for each i by $(\![f]\!)_0 = 0_{\widetilde{\mathbb{M}},!A}$, $[\![g]\!]_0 = 0_{!A,\widetilde{\mathbb{M}}}$ and $(\![f]\!)_{i+1} = !((\![f]\!)_i; f)$, $[\![g]\!]_{i+1} = !(g; [\![g]\!]_i)$.

LEMMA 4.8. *If $i \leq j$ then $!^i 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; (\![f]\!)_j = (\![f]\!)_i$ and $[\![g]\!]_j; !^i 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}} = [\![g]\!]_i$.*

PROOF: By induction on i: e.g. $!^0 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; (\![f]\!)_j = 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}} = (\![f]\!)_0$ and $!^{i+1}0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; (\![f]\!)_{j+1} = !!^i 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; !((\![f]\!)_j; f) = !(!^i 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}(\![f]\!)_j; f)$
$= !((\![f]\!)_i; f) = (\![f]\!)_{i+1}$. □

LEMMA 4.9. *If $X \in \mathbb{M}_j$ then $\phi_X; (\![f]\!)_j = \phi_X; (\![f]\!)_{d(X)}$ and $[\![g]\!]_j; \phi_X = [\![g]\!]_{d(X)}; \phi_X$.*

PROOF: $\phi_X; (\![f]\!)_{j+1} = \phi_X; !^{d(X)}0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; (\![f]\!)_j$ by Lemma 4.7
$= \phi_X; (\![f]\!)_{d(X)}$ by Lemma 4.8.
Similarly $![\![g]\!]_j; \phi_X = ![\![g]\!]_j; !^{d(X)}0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; \phi_X = [\![g]\!]_{d(X)}; \phi_X$. □

Define $(\![f]\!)_! : \widetilde{\mathbb{M}} \to !A = \Sigma_{X\in\mathbb{M}}(\phi_X; (\![f]\!)_{d(X)})$ and $[\![g]\!]_! : !A \to \widetilde{\mathbb{M}} = \Sigma_{X\in\mathbb{M}}((\![g]\!)_{d(X)}; \phi_X)$.

LEMMA 4.10. *$(\![f]\!)_!$ and $[\![g]\!]_!$ are comonoid morphisms.*

PROOF: For example:

- $(\![f]\!)_!; \epsilon_{!A} = (\Sigma_{X\in\mathbb{M}}\phi_X; (\![f]\!)_{d(X)}); \epsilon_{!A}$
$= \Sigma_{X\in\mathbb{M}}(\phi_X; (\![f]\!)_{d(X)}; \epsilon_{!A}) = \Sigma_{X\in\mathbb{M}}(\phi_X; \epsilon_{\widetilde{\mathbb{M}}})$
$= \phi_{[\_]}; \epsilon_{\widetilde{\mathbb{M}}} + \Sigma_{X\in\mathbb{M}\setminus\{[\_]\}}\phi_{[\_]}; \epsilon_{\widetilde{\mathbb{M}}} = \epsilon_{\widetilde{\mathbb{M}}} + 0 = \epsilon_{\widetilde{\mathbb{M}}}$ by **P1**.
- $(\![f]\!)_!; \delta_{!A} = (\Sigma_{X\in\mathbb{M}}\phi_X; (\![f]\!)_{d(X)}); \delta_{!A} =$
$\Sigma_{X\in\mathbb{M}}(\phi_X; \delta_{\widetilde{\mathbb{M}}}; ((\![f]\!)_{d(X)} \otimes (\![f]\!)_{d(X)}) =$
$\Sigma_{X\in\mathbb{M},Y\subseteq X}\delta_{\widetilde{\mathbb{M}}}; ((\phi_Y; (\![f]\!)_{d(X)}) \otimes (\phi_{X-Y}; (\![f]\!)_{d(X)}))$ by **P2**,
$= \Sigma_{X\in\mathbb{M},Y\subseteq X}\delta_{\widetilde{\mathbb{M}}}; ((\phi_Y; (\![f]\!)_{d(Y)}) \otimes (\phi_{X-Y}; (\![f]\!)_{d(X-Y)}))$, by Lemma 4.9,
$= \Sigma_{Y,Z\in\mathbb{M}}\delta_{\widetilde{\mathbb{M}}}; ((\phi_Y; (\![f]\!)_{d(Y)}) \otimes (\phi_Z; (\![f]\!)_{d(Z)}))$,
$= \delta_{\widetilde{\mathbb{M}}}; ((\![f]\!)_! \otimes (\![f]\!)_!)$.

□

LEMMA 4.11. *$(\![f]\!)_! = !((\![f]\!)_!; f)$ and $[\![g]\!]_! = !(g; [\![g]\!]_!)$.*

PROOF: $(\![f]\!)_!; \mathsf{der}_A = (\Sigma_{X\in\mathbb{M}}\phi_X; (\![f]\!)_{d(X)}); \mathsf{der}_A$
$= (\Sigma_{X\in\mathbb{M}}\phi_X; (\![f]\!)_{d(X)+1}); \mathsf{der}_A$ by Lemma 4.9
$= \Sigma_{X\in\mathbb{M}}(\phi_X; !((\![f]\!)_{d(X)}; f); \mathsf{der}_A)$
$= \Sigma_{X\in\mathbb{M}}(\phi_X; \mathsf{der}_{\widetilde{\mathbb{M}}}; (\![f]\!)_{d(X)}; f)$
$= \Sigma_{Y\in\mathbb{M}}(\phi_{[Y]}; \mathsf{der}_{\widetilde{\mathbb{M}}}; (\![f]\!)_{d(Y)+1}; f)$
$+ \Sigma_{X\in\mathbb{M}\setminus\mathbb{M}_1(\mathbb{M})}(\phi_X; \mathsf{der}_{\widetilde{\mathbb{M}}}; (\![f]\!)_{d(X)}; f)$
$= \Sigma_{Y\in\mathbb{M}}(\mathsf{der}_{\widetilde{\mathbb{M}}}; \phi_Y; (\![f]\!)_{d(Y)+1}; f) + 0_{\widetilde{\mathbb{M}},A}$ by **P3**
$= \mathsf{der}_{\widetilde{\mathbb{M}}}; (\Sigma_{Y\in\mathbb{M}}(\phi_Y; (\![f]\!)_{d(Y)})); f = \mathsf{der}_{\widetilde{\mathbb{M}}}; (\![f]\!)_!; f$.

So $(\![f]\!)_! = ((\![f]\!)_!; \mathsf{der}_A)^\dagger = (\mathsf{der}_{\widetilde{\mathbb{M}}}; (\![f]\!)_!; f)^\dagger = !((\![f]\!); f)$, since $(\![f]\!)_!$ is a comonoid morphism (Lemma 4.10).
Similarly, $[\![g]\!]_!; \mathsf{der}_{\widetilde{\mathbb{M}}} = \mathsf{der}_A; [\![g]\!]$ and hence $[\![g]\!]_! = !(g; [\![g]\!]_!)$. □

LEMMA 4.12. *If $h : \widetilde{\mathbb{M}} \to !A = !(h; f)$ then $h = (\![f]\!)_!$ and if $k : !A \to \widetilde{\mathbb{M}} = !(g; k)$ then $k = [\![g]\!]_!$.*

PROOF: First, we show by induction that $!^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; h = (\![f]\!)_n$: $0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; h = 0_{\widetilde{\mathbb{M}},!A} = f_0$ and $!^{i+1}0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; h = !^{i+1}0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; !(h; f) = !(!^i 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; h; f) = !((\![f]\!)_i; f) = (\![f]\!)_{i+1}$.
Thus $(\![f]\!)_! = \Sigma_{X\in\mathbb{M}}(\phi_X; (\![f]\!)_{d(X)}) = \Sigma_{X\in\mathbb{M}}(\phi_X; !^{d(X)}0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}}; h)$
$= \Sigma_{X\in\mathbb{M}}(\phi_X; h)$ (by Lemma 4.7)
$= (\Sigma_{X\in\mathbb{M}}\phi_X); h = h$ (since $\Sigma_{X\in\mathbb{M}}\phi_X = \mathsf{id}_{\widetilde{\mathbb{M}}}$).
Similarly, $k; !^n 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}} = (\![g]\!)_n$, and hence $k = [\![g]\!]_!$. □

PROPOSITION 4.13. $\mathsf{id}_{\widetilde{\mathbb{M}}} : \widetilde{\mathbb{M}} \to \widetilde{\mathbb{M}}$ *is a bifree algebra for the cofree exponential.*

PROOF: Given $f : !A \to A$ define the *catamorphism* $(\![f]\!) : \widetilde{\mathbb{M}} \to A = (\![f]\!)_!; f$, so that $!(\![f]\!); f = !((\![f]\!)_!; f); f = (\![f]\!)_!; f = (\![f]\!)$ by Lemma 4.11. This is unique, since if $!h; f = h$ then $!(!h; f) = !h$ and so $!h = (\![f]\!)_!$ by Lemma 4.12 and $h = (\![f]\!)_!; f = (\![f]\!)$. Similarly, the unique *anamorphism* of $g : A \to !A$ is $[\![g]\!] = g; [\![g]\!]_!$. □

So by Proposition 4.4, $\mathcal{C}_!$ has a unique uniform fixed point operator. If $\mathcal{C}$ is cpo-enriched (e.g. $\mathcal{R}^\Pi$ for a continuous semiring $\mathcal{R}$), by Proposition 4.2, this sends each endomorphism to its least fixed point (note that the cofree exponential is always continuous on morphisms).

4.2 Nested multiset approximants

We illustrate the fixed point construction via a more direct characterization as a sum of nested multiset approximants, relating it more closely to the resource λ-calculus (Boudol 1993) and differential λ-calculus (Ehrhard and Regnier 2003). Given $f : !A \to A$ and $X \in \mathbb{M}$ let $f^X : I \to !A = \iota_X; (\![f]\!)_!$.

LEMMA 4.14. $\mathsf{fix}(f) = \Sigma_{X\in\mathbb{M}}f^X; f$

PROOF: We show that $\Sigma_{X\in\mathbb{M}}\iota_X$ is a comonoid morphism and $\Sigma_{X\in\mathbb{M}}\iota_X); \mathsf{der}_M = \Sigma_{X\in\mathbb{M}}\iota_X$ by the properties of the ι_X in Lemma 3.12 (as in Lemma 4.11). Hence $\Sigma_{X\in\mathbb{M}}\iota_X : I \to M = (\Sigma_{X\in\mathbb{M}}\iota_X)^\dagger = \Sigma_{X\in\mathbb{M}}\iota_X = \infty$ by uniqueness of anamorphisms, and $\mathsf{fix}(f) = \infty; (\![f]\!) = \infty; (\![f]\!)_!; f = (\Sigma_{X\in\mathbb{M}}\iota_X); (\![f]\!)_!; f = \Sigma_{X\in\mathbb{M}}f^X; f$. □

We can think of each nested finite multiset as representing a unique forest of nested calls to f, which compute the fixed point approximant $f^X; f$ — i.e. $f^{[X_1,\ldots,X_k]}$ corresponds to k calls to

f at top level, each of which makes nested calls to f with call-patterns $X_1, \ldots, X_k$. Analogously, given $f : I \to B$ we may express $f^\dagger : I \to !B$ as a sum of approximants (over $\mathbb{N}$). Recalling that $!I = \bigoplus_{i \in \mathbb{N}} I$, let $f^i : I \to !B = \iota_i; !f$ for $i \in \mathbb{N}$. Then $f^\dagger = \mathsf{id}_I^\dagger; !f = (\Sigma_{i \in \mathbb{N}} \iota_i); !f = \Sigma_{i \in \mathbb{N}} f^i$.

The following identities derive from the corresponding properties of the Lafont exponential and its bifree algebra:

LEMMA 4.15. *For any $f : I \to B$ and $g :!B \to B$:*

1. $f^0; \epsilon_{!B} = \mathsf{id}_I$ *and* $f^{i+1}; \epsilon_{!B} = 0_{I,I}$.
   $g^{[\_]}; \epsilon_{!B} = \mathsf{id}_I$ *and* $g^X; \epsilon_{!B} = 0_{I,I}$ *if* $X \neq [\_]$.
2. $f^n; \delta_{!B} = \Sigma_{i \leq n} f^i \otimes f^{n-i}$,
   $g^X; \delta_{!B} = \Sigma_{Y \subseteq X} (g^Y \otimes g^{X-Y})$.
3. $f^1; \mathsf{der}_B = f$ *and* $f^i; \mathsf{der}_B = 0_{I,B}$ *if* $i \neq 1$
   $g^{[Y]}; \mathsf{der}_B = g^Y; g$ *and* $g^X; \mathsf{der}_B = 0_{I,B}$ *if* $X \notin \mathcal{M}_*(\mathbb{M})$.

These approximants, and their properties are used directly in the semantics of the abstract machine with integer and nested multiset resource bounds defined in Section 6, which may be considered as a form of the resource λ-calculus (Boudol 1993), in which functions are supplied with multisets of arguments.

REMARK 4.16. *Where $!B$ is a Lafont exponential, we may compute $f^X : I \to !B$ using the following explicit formula:*
Suppose the support of X is $\{x_1, \ldots, x_k\}$, and the multiplicity *of each x_i in X is $n_i \in \mathbb{N}$. Then*

$$f^X = ((f_1^{x_1}; f)^{n_1} \otimes \ldots \otimes (f^{x_k}; f)^{n_k})); \mu_B^k$$

where $\mu^k : (!B)^{\otimes k} \to !B$ is the k-ary multiplication of the commutative monoid $(!B, \mu, \eta)$, where $\mu :!B \otimes !B \to !B = ((\mathsf{der} \otimes \epsilon) + (\epsilon \otimes \mathsf{der}))^\dagger$ and $\eta : I \to !B = 0_{I,B}^\dagger$.

EXAMPLE 4.17. *Consider $f :!\mathbb{N} \to \mathbb{N}$ in $\mathcal{R}^\Pi$, defined:*
$f_{Yn} = 1$ if $Y = [\_]$ and $n = 0$ or $Y = [n-1]$
$f_{Yn} = 0$ otherwise.
($\mathsf{fix}(f)$ is the denotation of the $\mathrm{PCF}^\mathcal{R}$ term $\mu x.(\mathtt{0}\ \mathtt{or}\ \mathtt{succ}(x))$.)
Define $[\_]^n \in \mathbb{M}$ by $[\_]^0 = [\_]$ and $[\_]^{n+1} = [[\_]^n]$. For each $X \in \mathbb{M}$, $f_{*n}^X; f = 1$ if $X = [\_]^n$, and $f_{*n}^X = 0$ otherwise. So $\mathsf{fix}(f)_{*n} = \Sigma_{X \in \mathbb{M}} (f^X; f)_{*n} = 1$ for all $n \in \mathbb{N}$.

EXAMPLE 4.18. *Consider $g :!\mathbb{N} \to \mathbb{N}$ in $\mathcal{R}^\Pi$ defined:*
$g_{Yn} = 1$ if $(Y, n) = ([\_], 1)$ or $(Y, n) = ([i, j], i + j)$ for $i, j > 0$
$g_{Yn} = 0$ otherwise.
($\mathsf{fix}(g)$ is the denotation of the $\mathrm{PCF}^\mathcal{R}$ term $\mu x.(\mathtt{1}\ \mathtt{or}\ (x + x))$.)

Note that $\mathbb{M}$ is (isomorphic to) the quotient of the set of finite trees *under the equivalence induced by permuting subtrees at each node. For each $X \in \mathbb{M}$ let $\#(X)$ be the the number of* binary trees *in the equivalence class corresponding to X — i.e. $\#(X) = 1$ if $X = [\_]$, $\#(X) = \#(Y)^2$ if $X = [Y, Y]$), $\#(X) = 2\#(Y).\#(Z)$ if $X = [Y, Z]$ where $Y \neq Z$, and $\#(X) = 0$ otherwise.*
Then $g_{*n}^X; g = \#(X)$ if $n = |X| + 1$ (and $g_{*n}^X; g = 0$, otherwise). So for $n > 0$, $\mathsf{fix}(g)_{*n} = \Sigma_X (g_{*n+1}^X; g)$ is the number of binary trees with n leaves (i.e. the nth Catalan number *$\frac{(2n)!}{n!(n+1)!}$) as an element of $\mathcal{R}$.*

4.3 Parameterized Fixed Points

We now show that the fixed point operator derived from a bifree algebra for the cofree exponential may be *parameterised*[9].

[9] In fact, any Cartesian closed category with a fixed point operator has a parameterised fixed point operator: given $f : A \times B \to B$, define $\mathsf{fix}_A(f)$ to be the uncurrying of the fixed point of $\Lambda(\langle \pi_r, \mathsf{eval}_{A,B} \rangle; f) : (A \Rightarrow B) \to A \Rightarrow B$.

DEFINITION 4.19. *A parameterised fixed point operator for a category $\mathcal{C}$ with Cartesian products is a family of operators $\mathsf{fix}_A : \mathcal{C}(A \times B, B) \to \mathcal{C}(A, B)$ indexed over the objects of $\mathcal{C}$, such that:*

- *Fixed Point Property: For each A, $\langle A, \mathsf{fix}_A(f) \rangle; f = \mathsf{fix}_A(f)$.*
- *Naturality: If $g : C \to A$ then $g; \mathsf{fix}_A(f) = \mathsf{fix}_C((g \times B); f)$.*

For each object A, we have a comonad $!A \otimes \_$ on $\mathcal{C}$ with a *distributive law* (Beck 1969) $l :!A \otimes !\_ \to !(!A \otimes \_)$, yielding a comonad $!_A$ on the category $\mathcal{C}_{!A\otimes\_}$: it is sufficient to show that $\widetilde{\mathbb{M}}$ is a bifree algebra for each of these.

THEOREM 4.20. *If $\mathcal{C}$ is a symmetric monoidal category with distributive biproducts and cofree exponential then $\mathcal{C}_!$ has a uniform parameterised fixed point operator.*

PROOF: For any object A, the monoidal comonad $!A \otimes \_$ preserves biproducts and thus its co-Kleisli category $\mathcal{C}_{!A\otimes\_}$ is a symmetric monoidal category with distributive biproducts and cofree commutative comonoids. So by Proposition 4.13 the comonad $!_A : \mathcal{C}_{!A\otimes\_} \to \mathcal{C}_{!A\otimes\_}$ has the identity on $\widetilde{\mathbb{M}}$ in $\mathcal{C}_{!A\otimes\_}$ (which is $\epsilon_{!A} \otimes \mathsf{id}_{\widetilde{\mathbb{M}}} :!A \otimes \widetilde{\mathbb{M}} \to \widetilde{\mathbb{M}}$) as its bifree algebra. Hence by Proposition 4.4, for each A there is a uniform fixed point operator fix_A on the co-Kleisli category of $!_A$.

This family is natural in A: given $f \in \mathcal{C}_{!A\otimes\_}(!B, B)$ with a catamorphism $(\![f]\!)_A \in \mathcal{C}_{!A\otimes\_}(\widetilde{\mathbb{M}}, B)$, for any $g :!C \to A$, $g^\dagger \otimes \widetilde{\mathbb{M}}); (\![f]\!)_A$ is a morphism of $!_C$ algebras from $\epsilon_{!C} \otimes \mathsf{id}_{\widetilde{\mathbb{M}}}$ to $(g^\dagger \otimes B); f$ and is therefore equal to $(\![(g^\dagger \otimes B); f]\!)_C$. Hence $\mathsf{fix}_C(g^\dagger \otimes B); f) = (!C \otimes \infty); (\![(g^\dagger \otimes \widetilde{\mathbb{M}}); f]\!)_C = (!C \otimes \infty); (g^\dagger \otimes \widetilde{\mathbb{M}}); (\![f]\!)_A = g^\dagger; (!A \otimes \infty); (\![f]\!)_A = g^\dagger; \mathsf{fix}_A(f)$.

By the natural isomorphism $!A \otimes !\_ \cong !(A \times \_)$ the co-Kleisli category of $!_A : \mathcal{C}_{!A\otimes\_} \to \mathcal{C}_{!A\otimes\_}$ is isomorphic to the co-Kleisli category of the comonad $A \times \_ : \mathcal{C}_! \to \mathcal{C}_!$, Thus we have a family of uniform fixed point operators for the latter which is natural in A and is therefore a parameterised fixed point operator for $\mathcal{C}_!$. Uniformity in $!_A$ implies *parametric uniformity* in $!$ — i.e. for any $f :!(A \times B) \to B$, $g :!(A \times C) \to C$, and $h : B \to C$ such that $f; h = !(A \times h); g$, $\mathsf{fix}_A(g) = \mathsf{fix}_A(f); h$. $\square$

4.4 Non-Uniform Fixed Points

We now give an example of a fixed point not computed by our construction, which exists in any Lafont category with countable biproducts which cannot be cpo-enriched. For any complete commutative semiring $\mathcal{R}$, we may define a *trace operator* (Joyal et al. 1996) on the symmetric monoidal category $\mathcal{R}^\Pi$ — viz. $\mathrm{tr}_{A,B}^C : \mathcal{R}^\Pi(A \otimes C, B \otimes C) \to \mathcal{R}^\Pi(A, B)$ where $\mathrm{tr}(f)_{ab} = \Sigma_{c \in C} f_{acbc}$. This yields a fixed point operator on the co-Kleisli category $\mathcal{R}_!^\Pi$ (cf. (Hasegawa 2002)): for any $f :!A \to A$, $\mathrm{tr}_{I,!A}^{!A}(f^\dagger; \delta_{!A}) = \mathrm{tr}_{I,!A}^{!A}(\delta_{!A}; (f^\dagger \otimes f^\dagger)) = \mathrm{tr}_{I,!A}^{!A}(\delta_{!A}; (f^\dagger \otimes !A); (!A \otimes f^\dagger)) = \mathrm{tr}_{I,!A}^{!A}(f^\dagger; \delta_{!A}; (f^\dagger \otimes !A)) = \mathrm{tr}_{I,!A}^{!A}(f^\dagger; \delta_{!A}); f^\dagger$ by naturality/dinaturality of the trace operator. Moreover, we may show that this is a comonoid morphism, and thus $\mathsf{fix}(f) = \mathrm{tr}_{I,!A}^{!A}(f^\dagger; \delta_{!A}); \mathsf{der}_A$ satisfies $\mathsf{fix}(f)^\dagger; f = \mathsf{fix}(f)$. However, this operator is non-uniform whenever $\mathcal{R}$ is not *idempotent*.

DEFINITION 4.21. *A (countably) complete commutative semiring is idempotent if $\Sigma_{i \in I} a_i = b$ whenever $a_i = b$ for all $i \in I$ (non-empty).*

Note that any idempotent semiring is continuous with respect to the (countably complete) order $a \leq b$ if $a + b = b$.

LEMMA 4.22. *$\mathcal{R}$ is idempotent if and only if $\Sigma_{i \in \mathbb{N}} 1 = 1$.*

PROOF: If $\Sigma_{i \in \mathbb{N}} 1 = 1$ then $\Sigma_{i \in I} 1 = 1$ for all $I \neq \varnothing$, since we may partition $\mathbb{N}$ into I copies of itself. Hence $\mathcal{R}$ is idempotent: for all b, $\Sigma_{i \in I} b = b.\Sigma_{i \in I} 1 = b$. $\square$

PROPOSITION 4.23. *If the traced fixed point operator on $\mathcal{R}^\Pi$ is uniform then $\mathcal{R}$ is idempotent.*

PROOF: We establish the contrapositive by counterexample. Consider the morphisms $f, g : !\mathbb{B} \to \mathbb{B}$ and $h : \mathbb{B} \to \mathbb{B}$ such that:

- $f_{X,v} = 1$ if $X = [\_]$ and $v = \mathtt{1}$ or $X = [\mathtt{0}]$ and $v = \mathtt{0}$; $f_{X,v} = 0$ otherwise
- $g_{X,v} = 1$ if $X = [\_]$ and $v = \mathtt{1}$; $g_{X,v} = 0$ otherwise.
- $h_{u,v} = 1$ if $u = v = \mathtt{1}$; $h_{u,v} = 0$, otherwise.

Then $!h; g = g = f; h$ but $\mathsf{fix}(g) \neq \mathsf{fix}(f); h$ — $(g^\dagger; \delta_B)_{X,[\mathtt{1}],X} = 1$ if $X = [\_]$ (and 0 otherwise), and so $\mathsf{fix}(g)_{[\_],v} = 1$ if $v = \mathtt{1}$ and 0 otherwise[10], whereas $(f^\dagger; \delta_B)_{X,[\mathtt{1}],X} = 1$ if $X \in \mathcal{M}_*(\{\mathtt{0}\})$ (and 0 otherwise).
Hence $\mathsf{fix}(f)_{[\_],\mathtt{1}} = \Sigma_{X \in \mathcal{M}_*(\{\mathtt{0}\})} 1 \neq 1$ by Lemma 4.22, and so $(\mathsf{fix}(f); h)_{[\_],\mathtt{1}} \neq 1 = \mathsf{fix}(g)_{[\_],\mathtt{1}}$.

□

Hence by Propositions 3.5 and 3.6, if $\mathcal{C}$ is a Lafont category with biproducts which is not cpo-enriched, then $\mathcal{C}_!$ has non-uniform fixed points. In the following, we show that it is the uniform fixed point operator which is the "right" one from an operational perspective, since it allows us define a computationally adequate model of $\mathrm{PCF}^{\mathcal{R}}$.

5. Computational Adequacy for $\mathrm{PCF}^{\mathcal{R}}$

Let $\mathcal{C}$ be a Lafont category with biproducts, and $\mathcal{R}$ a complete semiring with an inclusion $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{C}}$. Following (Laird et al. 2013b) we may interpret $\mathrm{PCF}^{\mathcal{R}}$ in $\mathcal{C}$ by fixing an "object of numerals" N in $\mathcal{C}$ with morphisms $z : I \to \mathsf{N}$, $s, p : \mathsf{N} \to \mathsf{N}$ and $c : \mathsf{N} \to !\mathsf{N} \multimap !\mathsf{N} \multimap \mathsf{N}$ satisfying $s; p = \mathsf{id}_{\mathsf{N}}$, $z; c = \lambda x.\lambda y.x$ and $z; s^{n+1}; c = \lambda x.\lambda y.y$.

For instance, any *monoid* $(A, \mu : A \otimes A \to A, \eta : I \to A)$ in $\mathcal{C}$ yields an object of numerals $\mathsf{N} = \bigoplus_{i \in \mathbb{N}} A$, with each numeral $\mathtt{n}$ denoting $\eta; \iota_n$. Minimally (as in *loc. cit.*), we may take the monoid $I \cong I \otimes I$ — i.e. $\mathsf{N} = \widetilde{\mathbb{N}}$ (a game semantics example of an alternative is to take A to be the game with a single question and answer, with the sequential composition monoid). We interpret each type as an object of $\mathcal{C}$ by setting $[\![\mathtt{nat}]\!] = \mathsf{N}$ and $[\![S \to T]\!] = ![\![S]\!] \multimap [\![T]\!]$, each context $\Gamma = x_1 : S_1, \ldots, x_n : S_n$ as $![\![S_1]\!] \otimes \ldots \otimes ![\![S_n]\!] \cong !([\![S_1]\!] \times \ldots [\![S_n]\!])$, and each term-in-context $\Gamma \vdash P : T$ as a morphism $[\![P]\!]_\Gamma : [\![\Gamma]\!] \to [\![T]\!]$ in $\mathcal{C}$, so that the Cartesian closed structure of $\mathcal{C}_!$ yields interpretations of the operations of the λ-calculus and:

- μ-abstraction denotes a parameterised fixpoint — $[\![\mu x.M]\!]_\Gamma = \mathsf{fix}_{[\![\Gamma]\!]}([\![M]\!]_\Gamma)$.
- Choice and scalar weighting denote the corresponding $\mathcal{R}_{\mathcal{C}}$-module operations — $[\![M \;\mathtt{or}\; N]\!]_\Gamma = [\![M]\!]_\Gamma + [\![N]\!]_\Gamma$ and $[\![a.M]\!]_\Gamma = a \otimes [\![M]\!]_\Gamma$.
- Other operations denote composition (in $\mathcal{C}$) with the corresponding morphisms — $[\![\mathtt{0}]\!]_\Gamma = \epsilon_\Gamma; z$, $[\![\mathtt{Ifz}(M)]\!]_\Gamma = [\![M]\!]_\Gamma; c$, $[\![\mathtt{succ}(M)]\!]_\Gamma = [\![M]\!]_\Gamma; s$ and $[\![\mathtt{pred}(M)]\!]_\Gamma = [\![M]\!]_\Gamma; p$

If $\mathcal{R}$ is a continuous semiring then by Lemma 4.2, $\mu x.P$ denotes a *least* fixed point in R^Π, and therefore this interpretation is equivalent in this case to the semantics in (Laird et al. 2013b).

The key result relating operational and denotational semantics is a form of *computational adequacy* — the denotation of a program is given by the weighted sum of the denotations of the values to which it reduces.

DEFINITION 5.1. *By Lemma 2.3 we may define the* path interpretation *of a program P with respect to $u \in \mathcal{B}$ to be the morphism* $\langle\!|P|\!\rangle_u : I \to N$ *such that:*
$\langle\!|P|\!\rangle_u = a \otimes (z; s^n)$ *if* $P \overset{u,a}{\Longrightarrow} \mathtt{n}$; $\langle\!|P|\!\rangle_u = 0$ *otherwise.*
Computational adequacy *is the property that* $\Sigma_{u \in \mathcal{B}} \langle\!|P|\!\rangle_u = [\![P]\!]$ *for all programs P.*

So, in particular, if $\mathsf{N} = \widetilde{\mathbb{N}}$ then computational adequacy is equivalent to requiring that $[\![P]\!]; \pi_n = \Sigma_{u \in (P \Rightarrow n)} w(u)$ for all programs P and values n.

The proof of adequacy in (Laird et al. 2013b) depends on cpo-enrichment: its absence necessitates a different approach. We define an operational semantics more directly related to our interpretation of fixed points — an *abstract machine* in which the environment is instrumented with bounds characterizing a particular call-pattern for each variable, and show that (a) this gives an equivalent notion of weighted reduction path to the unbounded semantics, and (b) its denotational semantics is computationally adequate in the above sense.

A bounded *environment* $\mathcal{E}$ is a finite sequence of triples, $(x_1, M_1, r_1), \ldots, (x_n, M_n, r_n)$, where each x_i is a variable, M_i is a term, and r_i is a resource bound — either a natural number or a nested finite multiset. (Note that these are not upper (or lower bounds) but upper *and* lower bounds — precise specifications of how many times a procedure may be called or a fixed point unfolded.) We write $|\mathcal{E}|$ for the sequence of variables $x_1 : T_1, \ldots, x_n : T_n$ and define typing judgements $\Gamma \vdash \mathcal{E}$ for well formed environments as follows:

$$\frac{}{\Gamma \vdash \varepsilon} \qquad \frac{\Gamma \vdash \mathcal{E} \quad \Gamma, |\mathcal{E}| \vdash M : T}{\Gamma \vdash \mathcal{E}, (x^T, M, k)} k \in \mathbb{N} \qquad \frac{\Gamma \vdash \mathcal{E} \quad \Gamma, |\mathcal{E}|, x : T \vdash M : T}{\Gamma \vdash \mathcal{E}, (x^T, M, X)} X \in \mathbb{M}$$

A *configuration* is a pair $\mathcal{E}; P$ of an environment $\_ \vdash \mathcal{E}$ and a term P such that $|\mathcal{E}| \vdash P : \mathtt{nat}$.

DEFINITION 5.2. *The* bounded abstract machine *for* $\mathrm{PCF}^{\mathcal{R}}$ *is the LTS in which the* states *are configurations (up to α-equivalence),* labels *are elements of $\mathcal{B} \times \mathcal{R}$ and* actions *are defined in Table 3.*

Every reduction path of this LTS is, in fact, terminating (by Lemma 5.12); we say that a reduction path terminates *successfully* if the environment in its final configuration is empty — i.e. all bounds are zero or the empty multiset: let Env_0 be the set of such environments. We define the "many-step" evaluation relation $\mathcal{E}; P \overset{u,a}{\Longrightarrow} n$ ($\mathcal{E}; P$ successfully reduces to $\mathtt{n}$ along the path u with weight a) if $\mathcal{E}; P \overset{u,a}{\Longrightarrow}_i n$ for some $i \in \mathbb{N}$, where:

$$\frac{}{\mathcal{E}_0; \mathtt{n} \overset{\varepsilon,1}{\Longrightarrow}_i n} \mathcal{E}_0 \in Env_0 \qquad \frac{\mathcal{E}; P \overset{u,a}{\longrightarrow} \mathcal{E}'; P' \quad \mathcal{E}'; P' \overset{v,b}{\Longrightarrow}_i n}{\mathcal{E}; P \overset{u \cdot v, a \cdot b}{\Longrightarrow}_j n} \; i < j$$

Although the abstract machine semantics is nondeterministic in the sense that a state may have one-step reductions with the same label to countably many different states, only one of those states (at most) is on a successfully terminating reduction path. Define $\mathcal{E} \sim \mathcal{E}'$ if the environments $\mathcal{E}$ and $\mathcal{E}'$ may differ only in the bounds assigned to each variable.

PROPOSITION 5.3. *If $\mathcal{E} \sim \mathcal{E}'$, $\mathcal{E}; P \overset{u,a}{\Longrightarrow}_i m$ and $\mathcal{E}'; P \overset{u,b}{\Longrightarrow} n$ then $\mathcal{E} = \mathcal{E}'$, $m = n$ and $a = b$.*

PROOF: By induction on i, based on the observation that if $\mathcal{E}; P \overset{u,a}{\longrightarrow} \mathcal{E}''; Q$ and $\mathcal{E}'; P \overset{u,a'}{\longrightarrow} \mathcal{E}''; Q'$, where $\mathcal{E} \sim \mathcal{E}'$, then $\mathcal{E} = \mathcal{E}'$, $a = a'$ and $Q = Q'$. □

Thus we may define the path interpretation of configurations: if $\mathcal{E}; P \Longrightarrow^{u,a} n$ then $\langle\!|\mathcal{E}; P|\!\rangle_u = a \otimes (z; s^n)$; otherwise $\langle\!|\mathcal{E}; P|\!\rangle_u = 0$. We now show that this agrees with the path interpretation of programs, proving the following lemmas by induction on i:

LEMMA 5.4. *For any $(x, N, X), \mathcal{E}; P$, where $X \in \mathbb{M}$:*

[10] This is also the fixed point of f given by the uniform operator.

$$\begin{array}{lll}
\mathcal{E}; E[(\lambda x.M)\ N] & \xrightarrow{\varepsilon,1} & \mathcal{E},(x,N,k); E[M] \quad (x \notin |\mathcal{E}|, k \in \mathbb{N}) \\
\mathcal{E}; E[\mu x.M] & \xrightarrow{\varepsilon,1} & \mathcal{E},(x,M,X); E[M] \quad (x \notin |\mathcal{E}|, X \in \mathbb{M}) \\
\mathcal{E},(x,M,k+1),\mathcal{E}'; E[x] & \xrightarrow{\varepsilon,1} & \mathcal{E},(x,M,k),\mathcal{E}'; E[M] \\
\mathcal{E},(x,M,X+[Y]),\mathcal{E}'; E[x] & \xrightarrow{\varepsilon,1} & \mathcal{E},(x,M,X),\mathcal{E}',(y,M[y/x],Y); E[M[y/x]] \quad y \notin |\mathcal{E}|,|\mathcal{E}'|
\end{array}$$

$$\begin{array}{lll|lll}
\mathcal{E}; E[\mathtt{Ifz}(0)] & \xrightarrow{\varepsilon,1} & \mathcal{E}; E[\lambda x.\lambda y.x] & \mathcal{E}; E[M \ \mathtt{or}\ N] & \xrightarrow{l,1} & \mathcal{E}; E[M] \\
\mathcal{E}; E[\mathtt{Ifz}(\mathtt{n}+1)] & \xrightarrow{\varepsilon,1} & \mathcal{E}; E[\lambda x.\lambda y.y] & \mathcal{E}; E[M \ \mathtt{or}\ N] & \xrightarrow{r,1} & \mathcal{E}; E[N] \\
\mathcal{E}; E[\mathtt{pred}(\mathtt{n}+1)] & \xrightarrow{\varepsilon,1} & \mathcal{E}; E[\mathtt{n}] & \mathcal{E}; E[k.M] & \xrightarrow{\varepsilon,k} & \mathcal{E}; E[M]
\end{array}$$

Table 3. Labelled Transitions for Configurations

- *If* $(x,M,X),\mathcal{E};P \overset{u,a}{\Longrightarrow}_i n$ *then* $(\mathcal{E};M)[\mu x.M/x] \overset{u,a}{\Longrightarrow}_i n$.
- *If* $(\mathcal{E};P)[\mu x.N/x] \overset{u,a}{\Longrightarrow}_i n$ *then there exists* $X' \in \mathbb{M}$ *such that* $(x,N,X),\mathcal{E};P \overset{u,a}{\Longrightarrow} n$

For any configuration $(x,N,k),\mathcal{E};P$, *where* $k \in \mathbb{N}$:

- *If* $(x,N,k),\mathcal{E};P \overset{u,a}{\Longrightarrow}_i n$ *then* $(\mathcal{E};P)[N/x] \overset{u,a}{\Longrightarrow}_i n$.
- *If* $(\mathcal{E};P)[N/x] \overset{u,a}{\Longrightarrow}_i n$ *then there exists* $k \in \mathbb{N}$ *such that* $(x,N,k),\mathcal{E};P \overset{u,a}{\Longrightarrow} n$

LEMMA 5.5. $P \overset{u,a}{\Longrightarrow} \mathtt{n}$ *if and only if* $\_;P \overset{u,a}{\Longrightarrow} n$.

Hence by the definitions of $\langle\!\langle \_ \rangle\!\rangle$ for programs and configurations:

PROPOSITION 5.6. *For any* $u \in \mathcal{B}$, $\langle\!\langle P \rangle\!\rangle_u = \langle\!\langle \_;P \rangle\!\rangle_u$.

5.1 Denotational Semantics for Configurations

We now extend the denotational semantics of programs to configurations, and show that $\Sigma_{u\in\mathcal{B}}\langle\!\langle \mathcal{E};P\rangle\!\rangle_u = [\![\mathcal{E};P]\!]$. Environments are interpreted using parameterized versions of the integer and nested finite multiset indexed approximants derived from the construction of the uniform fixed point operator in Section 4. Given $g : !A \otimes !B \to B$ — i.e. $g \in \mathcal{C}_{!A\otimes\_}(!B,B)$ — and $X \in \mathbb{M}$, let $g^X : !A \to !B = (!A \otimes \iota_X); (\![g]\!)_{!A}$, so that $\mathsf{fix}_A(g) = \delta_{!A}; (\mathsf{id}_{!A} \otimes (\Sigma_{X\in\mathbb{M}} g^X)); g$. Similarly, and more directly, given $f : !A \to B$ and $i \in \mathbb{N}$, let $f^i : !A \to !B = (\iota_i \otimes !A); ((\mathsf{der}_I \otimes !A); f)^\dagger$.

An environment $\Gamma \vdash \mathcal{E}$ is interpreted as a morphism $[\![\mathcal{E}]\!]_\Gamma : [\![\Gamma]\!] \to [\![|\mathcal{E}|]\!]$ using these approximants. Let $[\![\varepsilon]\!] : I \to I = \mathsf{id}_I$ and:

- $[\![\mathcal{E},(x,M,k)]\!]_\Gamma = [\![\mathcal{E}]\!]_\Gamma; \delta_{[\![|\mathcal{E}|]\!]}; ([\![|\mathcal{E}|]\!] \otimes [\![M]\!]^k_{|\mathcal{E}|}), k \in \mathbb{N}$.
- $[\![\mathcal{E},(x,M,X)]\!]_\Gamma = [\![\mathcal{E}]\!]_\Gamma; \delta_{[\![|\mathcal{E}|]\!]}; ([\![|\mathcal{E}|]\!] \otimes [\![M]\!]^X_{|\mathcal{E}|,x:T}), X \in \mathbb{M}$.

Thus (up to coherence isomorphisms), $[\![\mathcal{E},\mathcal{E}']\!] = [\![\mathcal{E}]\!]; \delta_{|\mathcal{E}|}; ([\![|\mathcal{E}|]\!] \otimes [\![\mathcal{E}']\!])$. The configuration $\mathcal{E};P$ denotes $[\![\mathcal{E}]\!]; [\![P]\!]_{|\mathcal{E}|} : I \to N$. To prove soundness for the reduction rules we require an interpretation of evaluation contexts as morphisms in $\mathcal{C}$ (rather than the co-Kleisli category), which is derived from the following result.

LEMMA 5.7. *For any* $\Gamma \vdash E[M] : T$ *and* $\Gamma \vdash M : S$, *there exists a morphism* $[\![E[\bullet]]\!]_\Gamma : [\![\Gamma]\!] \otimes [\![S]\!] \to [\![T]\!]$ *such that* $[\![E[M]]\!]_\Gamma = \delta_\Gamma; ([\![\Gamma]\!] \otimes [\![M]\!]_\Gamma); [\![E[\bullet]]\!]_\Gamma$.

PROOF: By structural induction, as follows:

- If $E[\bullet] = [\bullet]$ then $T = S$ and $[\![[\bullet]]\!]_\Gamma = (\epsilon_{[\![\Gamma]\!]} \otimes [\![T]\!])$
- If $E[\bullet] = E'[\bullet]\ N$ then for some type $T', \Gamma \vdash E'[M] : T' \to T$ and $\Gamma \vdash N : T'$ — let $[\![E[\bullet]]\!]_\Gamma = ((\delta_{[\![\Gamma]\!]} \otimes [\![S]\!]); ([\![N : T']\!]_\Gamma \otimes [\![E'[\bullet]_\Gamma]\!]); \mathsf{eval}_{[\![T]\!],[\![T']\!]}$.
- $[\![\mathtt{Ifz}(E[\bullet])]\!]_\Gamma = [\![E[\bullet]]\!]_\Gamma; c$, $[\![\mathtt{succ}(E[\bullet])]\!]_\Gamma = [\![E[\bullet]]\!]_\Gamma; s$, and $[\![\mathtt{pred}(E[\bullet])]\!]_\Gamma = [\![E[\bullet]]\!]_\Gamma; p$.

□

We apply this result, together with Lemma 4.15 and the properties of our categorical model to establish the following properties:

LEMMA 5.8. *The denotational semantics satisfies:*

1. $[\![\mathcal{E}; E[\mu x.M]]\!] = \Sigma_{X\in\mathbb{M}}[\![\mathcal{E},(x,M,X); E[M]]\!]$.
2. $[\![\mathcal{E}; E[(\lambda x.M)\ N]]\!] = \Sigma_{k\in\mathbb{N}}[\![\mathcal{E},(x,N,k); E[M]]\!]$.
3. $[\![\mathcal{E},(x,M,X),\mathcal{E}'; E[x]]\!] = \Sigma_{Y\in s(X)}$ $[\![\mathcal{E},(x,M,X-[Y]),(y,M[y/x],Y),\mathcal{E}'); E[M[y/x]]]\!]$.
4. $[\![\mathcal{E},(x,M,k+1),\mathcal{E}'; E[x]]\!] = [\![\mathcal{E},(x,M,k),\mathcal{E}'; E[M]]\!]$ *and* $[\![\mathcal{E},(x,M,0),\mathcal{E}'; E[x]]\!] = 0$.
5. $[\![\mathcal{E}; E[M \ \mathtt{or}\ N]]\!] = [\![\mathcal{E}; E[M]]\!] + [\![\mathcal{E}; E[N]]\!]$ *and* $[\![\mathcal{E}; E[a.M]]\!] = a \otimes [\![\mathcal{E}; E[M]]\!]$.
6. $[\![\mathcal{E}; \mathtt{n}]\!] = z; s^n$ *if* $\mathcal{E} \in Env_0$; *otherwise* $[\![\mathcal{E}; \mathtt{n}]\!] = 0$.

To give an inductive proof of the adequacy of the resource-bounded semantics, we show that reduction is strictly decreasing with respect to a measure on terms based on the *nested multiset order* (Dershowitz and Manna 1979).

DEFINITION 5.9. *For each* $i \in \mathbb{N}$, $(\mathbb{M}_{i+1}, \ll_{i+1})$ *is the* multiset order *generated by* $(\mathbb{M}_i, \ll_i)$ — *i.e.* $X \ll_{i+1} Y$ *if for all* $x \in \mathsf{s}(X - Y)$ *there exists* $y \in \mathsf{s}(Y)$ *with* $x \ll_i y$.
This is a well-founded partial order (Dershowitz and Manna 1979), and $\ll_i \subseteq \ll_{i+1}$. *Hence we may define a well-founded order* $\ll^* = \bigcup_{i\in\mathbb{N}} \ll_i$ *on* $\mathbb{M}$. *We write* $<^*$ *for the corresponding strict inequality.*

Note that the nested multiset order satisfies $X \ll^* X'$ and $Y \ll^* Y'$ implies $X + Y \ll^* X' + Y'$, and the following key property (where $k.X$ denotes the k-fold multiset union of Y with itself):

LEMMA 5.10. $k.X <^* [X]$ *for all* $k \in \mathbb{N}$ *and* $X \in \mathbb{M}$.

DEFINITION 5.11. *Define a map* ℓ *from* $\mathrm{PCF}^{\mathcal{R}}$ *terms into* $\mathbb{M}$ *by:*

$$\begin{array}{ll}
\ell(x) = \ell(0) = 1 & \ell(M \ \mathtt{or}\ N) = \ell(N) + \ell(N) \\
\ell(\mathtt{op}(M)) = [\ell(M)] & \ell(M\ N) = \ell(M) + [\ell(N)]
\end{array}$$

(where $\mathtt{op}$ *is any unary operation). Extend* ℓ *to environments by setting* $\ell(\varepsilon) = 0$, $\ell(\mathcal{E},(x,M,k)) = \ell(\mathcal{E}) + k.\ell(M)$ *for* $k \in \mathbb{N}$ *and* $\ell(\mathcal{E},(x,M,X)) = \ell(\mathcal{E}) + |X|.\ell(M)$ *for* $X \in \mathbb{M}$. *Hence we define* $\ell(\mathcal{E};P) = \ell(\mathcal{E}) + \ell(P)$.

LEMMA 5.12. $\mathcal{E};P \xrightarrow{u,a} \mathcal{E}';P'$ *implies* $\ell(\mathcal{E}';P') <^* \ell(\mathcal{E};P)$.

PROOF: Extend ℓ to evaluation contexts by setting $\ell(\bullet) = \varnothing$, so $\ell(E[M]) = \ell(E[\bullet]) + \ell(M)$. Then e.g.

- Suppose $P = E[\mu x.N]$, so $\mathcal{E}';P' = (\mathcal{E},(x,N,X); E[N]$ for some $X \in \mathbb{M}$. Then $\ell(\mathcal{E}';P')$
$= \ell(\mathcal{E}) + |X|.\ell(N) + \ell(E[\bullet]) + \ell(N)$
$= \ell(\mathcal{E}) + \ell(E[\bullet]) + (|X|+1).\ell(N)$
$<^* \ell(\mathcal{E}) + \ell(E[\bullet]) + [\ell(N)]$ by Lemma 5.10
$= \ell(\mathcal{E};P)$.
- Suppose $\mathcal{E} = \mathcal{E}'',(x,N,X+[Y]),\mathcal{E}'''$ and $P = E[x]$, so $\mathcal{E}' = \mathcal{E}'',(x,N,X),\mathcal{E}''',(y,N[y/x],Y),\mathcal{E}'''$ and $P = E[N[y/x]]$. Then $\ell(\mathcal{E}';P') =$
$\ell(\mathcal{E}'') + |X|.\ell(N) + |Y|.\ell(N) + \ell(\mathcal{E}''') + \ell(E[\bullet]) + \ell(N[y/x])$

$$= \ell(\mathcal{E}'') + (|X| + |Y| + 1).\ell(N) + \ell(\mathcal{E}''') + \ell(E[\bullet])$$
$$<^* \ell(\mathcal{E}'') + |X + [Y]|.\ell(N) + \ell(\mathcal{E}''') + \ell(E[\bullet]) + \ell(x)$$
$$= \ell(\mathcal{E}; P).$$

$\square$

We now establish adequacy for the bounded abstract machine.

PROPOSITION 5.13. *For any* $\mathcal{E}; P$, $\Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = [\![\mathcal{E}; P]\!]$.

PROOF: By nested multiset induction on $\ell(\mathcal{E}; P)$:

- Suppose $P = \mathtt{n}$. If $\mathcal{E} \in Env_0$ then $\Sigma_{s \in \mathcal{B}} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \langle\!\langle \mathcal{E}; P \rangle\!\rangle_\varepsilon = z; s^n = [\![\mathcal{E}; P]\!]$ by Lemma 5.8.
  Otherwise $\Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \langle\!\langle \mathcal{E}; P \rangle\!\rangle_\varepsilon = 0 = [\![\mathcal{E}; P]\!]$ (Lemma 5.8).
- Suppose $P = E[N_l \texttt{ or } N_r]$, so $\mathcal{E}; P \xrightarrow{(\varepsilon,a)} \mathcal{E}; E[N_a])$ for $a \in \{l, r\}$. $\mathcal{B} = \{\varepsilon\} \cup \{lu, ru \mid u \in \mathcal{B}\}$, and therefore
  $\Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; P \rangle\!\rangle = \langle\!\langle \mathcal{E}; P \rangle\!\rangle_\varepsilon + \Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_{lu} + \Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_{ru}$
  $= 0 + \Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; E[N_l] \rangle\!\rangle_u + \Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; E[N_r] \rangle\!\rangle_u$
  $= [\![\mathcal{E}; E[N_l]]\!] + [\![\mathcal{E}; E[N_r]]\!]$ (by the induction hypothesis)
  $= [\![\mathcal{E}; P]\!]$ by Lemma 5.8.
- Suppose $P = E[\mu x.N]$, so $\mathcal{E}; P \xrightarrow{(\varepsilon,1)} \mathcal{E}, (x, N, X); E[N])$ for each $X \in \mathbb{M}$.
  By Proposition 5.3, for each $u \in \mathcal{B}$ there is at most one $X \in \mathbb{M}$ such that $\langle\!\langle \mathcal{E}, (x, N, X); E[N] \rangle\!\rangle_u \neq 0$, and hence $\langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \Sigma_{X \in \mathbb{M}} \langle\!\langle \mathcal{E}, (x, N, X); E[N] \rangle\!\rangle_u$.
  So $\Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \Sigma_{u \in \mathcal{B}} \Sigma_{X \in \mathbb{M}} \langle\!\langle \mathcal{E}, (x, N, X); E[N] \rangle\!\rangle_u =$
  $\Sigma_{X \in \mathbb{M}} \Sigma_{u \in \mathcal{B}} \langle\!\langle \mathcal{E}, (x, N, X); E[N] \rangle\!\rangle_u$ by partition associativity
  $= \Sigma_{X \in \mathbb{M}} [\![\mathcal{E}, (x, N, X); E[N]]\!]$ by the induction hypothesis
  $= [\![\mathcal{E}; P]\!]$ by Lemma 5.8.

The remaining cases are similar, or follow directly from Lemmas 5.8 and 5.12. $\square$

By Proposition 5.6, we have proved that $[\![P]\!] = \Sigma_{u \in \mathcal{B}} \langle\!\langle P \rangle\!\rangle_u$ for all programs P. In other words:

THEOREM 5.14. *If $\mathcal{C}$ is a Lafont category with biproducts with $\mathcal{R} \subseteq \mathcal{R}_\mathcal{C}$, the semantics of $\mathrm{PCF}^\mathcal{R}$ in $\mathcal{C}$ is computationally adequate.*

6. Conclusions

We have established two results for quantitative semantics: that Lafont categories with countable biproducts have uniform fixed points, and that these provide a computationally adequate interpretation of erratic PCF with weights from a complete commutative semiring. These results have been proved in rather different ways — using the principles of axiomatic domain theory in the first instance, and some basic operational techniques in the latter. It is not yet clear how closely these approaches may be combined: whether computational adequacy may be established by purely axiomatic means.

By placing further conditions on our model (such as requiring that the cofree exponential is a Lafont exponential) we may establish further properties of the uniform fixed point operator — for example, if $\mathcal{C}$ has "sufficiently many" bifree algebras (in the formal sense of (Simpson and Plotkin 2000)) then it is a *Conway Operator*, giving further equational reasoning principles for fixed points. Other avenues include extension of our results to recursive types using principles from (Simpson 2004) or models of linear logic which are not Lafont categories — for example, the notion of "new Lafont category" in (Melliès 2009).

We have not addressed questions of *full abstraction*: while computational adequacy implies soundness with respect to observational equivalence by a standard argument, as established in (Laird et al. 2013b) by counterexample, the semantics of $\mathrm{PCF}^\mathcal{R}$ in $\mathcal{R}^\Pi$ is not fully abstract for any non-trivial $\mathcal{R}$. It may be possible to identify the $\mathrm{PCF}^\mathcal{R}$-definable elements in a games model, however. Alternatively, we may consider fixed points in other settings such as stateful or concurrent languages. The representation of fixed point approximants using nested finite multisets suggests that we could extend the resource λ-calculus, and related formalisms such as the differential λ-calculus (Ehrhard and Regnier 2003) and differential nets (Ehrhard and Regnier 2006) to reason about fixed points.

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