Abstract. A Hecke endomorphism algebra is a natural generalisation of the \(q\)-Schur algebra associated with the symmetric group to a Coxeter group. For Weyl groups, B. Parshall, L. Scott and the first author \([9, 10]\) investigated the stratification structure of these algebras in order to seek applications to representations of finite groups of Lie type. In this paper we investigate the presentation problem for Hecke endomorphism algebras associated with arbitrary Coxeter groups. Our approach is to present such algebras by quivers with relations. If \(R\) is the localisation of \(\mathbb{Z}[q]\) at the polynomials with the constant term 1, the algebra can simply be defined by the so-called idempotent, sandwich and extended braid relations. As applications of this result, we first obtain a presentation of the 0-Hecke endomorphism algebra over \(\mathbb{Z}\) and then develop an algorithm for presenting the Hecke endomorphism algebras over \(\mathbb{Z}[q]\) by finding torsion relations. As examples, we determine the torsion relations required for all rank 2 groups and the symmetric group \(S_4\).

1. Introduction

Since I. Schur first introduced Schur algebras for linking representations of the general linear group \(GL_n(\mathbb{C})\), a continuous group, with those of the symmetric group \(\mathfrak{S}_r\), a finite group, this class of algebras has been generalised to various objects in several directions. For example, by the role they play in the Schur–Weyl duality, there are affine and super generalisations and their quantum analogues, which are known as (affine, super) \(q\)-Schur algebras (see, e.g., \([20]\), \([5]\), \([15]\), \([24]\)). As homomorphic images of a universal enveloping algebra, generalised Schur algebras and their quantum analogues are investigated not only for type \(A\) but also for other types (see, e.g., \([6]\), \([11]\)). By their definition as endomorphism algebras of certain permutation modules of symmetric groups, they have a natural generalisation to Hecke endomorphism algebras associated with finite Coxeter groups (see, e.g., \([8]\), \([9]\), \([12]\)).

Recently, with a completely different motivation, advances have been made in categorifying Hecke endomorphism algebras. For example, M. Mackaay, et al, obtained a diagrammatic categorification of the \(q\)-Schur algebra \([22]\) and some affine counterpart \([23]\), while G. Williamson \([28]\) investigated a more general class of Hecke categories.
associated with any Coxeter systems \((W,S)\) (called Schur algebroids loc.cit.), and categorified them in terms of singular Soergel bimodules. As seen from these works, a presentation of an algebra by generators and relations is related to the categorification of Hecke-endomorphism algebras. In fact, the question of presenting the Hecke endomorphism algebras by generators and relations was raised in Remark 2.2 in [28].

Unlike the \(q\)-Schur algebras, Hecke endomorphism algebras do not have a direct connection to Lie algebras and quantum groups and thus cannot be presented as a quotient of a quantum group. However, by viewing them as enlarged Hecke algebras, they can be presented with generators and relations, which are rooted in Hecke (algebra) relations. This paper is going to tackle the presentation problem in this direction.

Our approach is to present these algebras by quivers with relations, following closely the idea of using Hecke relations. First, by assuming the invertibility of all Poincaré polynomials in the ground ring \(R\), we consider in Sections 3 and 4 a quiver \(Q'\) with loops and impose the Hecke relations directly on the loops. We then replace the loops by cycles of length 2 to obtain a quiver \(Q\) with no loops and braid relations are imposed explicitly again. Now, by thinking of the generators displayed in [28, Cor. 2.12], we take a subset labelled by \(I, J \subseteq S\) satisfying \(I \subseteq J\) (see (2.5.1)) and introduce in Section 5 a Hasse quiver \(\tilde{Q}\) to replace \(Q\). It turns out that the algebra over \(R\) can simply be presented by \(\tilde{Q}\) together with the so-called extended braid relations and some obvious relations, called idempotent relations and sandwich relations (see Theorem 5.1).

The first application of this result is to obtain a presentation of the 0-Hecke endomorphism algebras by specialising \(q\) to 0 in Section 6. These degenerate algebras have attracted some attention (see, e.g., [25], [2], [13], [26],[18], [19], [17], [4]) and have also nice applications. For instance, J. Stembridge [27] used the 0-Hecke algebra to give a derivation of the Möbius function of the Bruhat order, while X. He [16] gave a more elementary construction of a monoid structure by A. Berenstein and D. Kazhdan [1].

We then investigate the integral case in Sections 7 and 8. We analyse the gap between the presentation over \(R\) and a possible presentation over the polynomial ring \(\mathbb{Z}[q]\). The idempotent relations can be replaced by the so-called quasi-idempotent relations, the sandwich relations are unchanged. The challenge is how to replace the extended braid relations by some torsion relations. We develop an algorithm and compute the examples of rank 2 and of type \(A_3\). The rank 2 case is relatively easy, the required torsion relations are simply the refined braid relations. Note that a recursive version of this case is done by B. Elias [14, Prop. 2.20]. However, the \(A_3\) case is more complicated. On top of the refined braid relations, there are two more sets of torsion relations, see Theorem 8.3. We believe that the algorithm can be used to compute the other lower rank cases.
2. Hecke algebras and Kazhdan–Lusztig generators

Let \((W, S)\) be a Coxeter system and let \(\ell : W \to \mathbb{N}\) be the length function with respect to \(S\) and \(\leq\) the Bruhat order.

For \(I \subseteq S\), let \(W_I = \langle s \mid s \in I \rangle\) be the parabolic subgroup generated by \(I\). We say \(I \subseteq S\) is finitary if \(W_I\) is finite, and in this case we denote by \(w_I\) the longest element in \(W_I\). Let

\[
\Lambda = \Lambda(W) = \{I \subseteq S \mid I \text{ is finitary}\} \quad \text{and} \quad \Lambda^\ast = \Lambda \setminus \emptyset,
\]

where \(\emptyset \in \Lambda\) is the empty set.

For \(I \in \Lambda\), we will also denote by \(D_I := D_{W_I}\) the set of all distinguished (or shortest) representatives of the right cosets of \(W_I\) in \(W\). Let \(D_{IJ} = D_I \cap D_J^{-1}\), where \(I, J \in \Lambda\). Then \(D_{IJ}\) is the set of shortest \(W_I-W_J\) double coset representatives. Symmetrically, let \(D_{IJ}^\ast\) be the set of longest representatives of \(W_I-W_J\) double cosets.

For \(d \in D_{IJ}\), the subgroup \(W_I^d \cap W_J = d^{-1}W_I d \cap W_J\) is a parabolic subgroup associated with an element in \(\Lambda\), which will be denoted by \(I^d \cap J \in \Lambda\). In other words, we define

\[
W_{I^d \cap J} = W_I^d \cap W_J.
\]

Moreover, the map

\[
W_I \times (D_{I^d \cap J} \cap W_J) \longrightarrow W_I d W_J, \quad (w, y) \longmapsto w d y
\]

is a bijection.

The Hecke algebra \(H_q = H_q(W)\) corresponding to \(W\) is a free \(\mathbb{Z}[q]\)-module, with basis \(\{T_w \mid w \in W\}\) and generators \(T_s, s \in S\), subject to the relations (see, e.g., [3, Ch. 7]):

\[
\begin{align*}
\text{(H1)} \quad & T_s^2 = (q - 1) T_s + q \quad \text{for any } s \in S; \\
\text{(H2)} \quad & T_s T_s t_1 T_s \ldots = T_s T_s t_1 T_s \ldots, \quad \text{where } s \neq t \text{ and } m_{s,t} \text{ is the order of } st \text{ in } W.
\end{align*}
\]

Here \(T_w = T_{s_1} T_{s_2} \ldots T_{s_l}\) if \(w = s_1 s_2 \ldots s_l\) is a reduced expression. The relations (H2) are called braid relations associated with \(W\) and we call the relations (H1) and (H2) Hecke relations. Note that \(H_q\) admits an anti-automorphism

\[
\iota : H_q \longrightarrow H_q, \quad T_w \longmapsto T_{w^{-1}}.
\]

For \(y, w \in W\), let \(P_{y,w} \in \mathbb{Z}[q]\) be the associated Kazhdan–Lusztig polynomial as defined in [21]. Then, \(P_{w,w} = 1\), \(P_{y,y} = 0\) unless \(y \leq w\), and \(P_{y,w}\) for \(y < w\) has degree \(\leq \frac{1}{2}(\ell(w) - \ell(y) - 1)\). Following [21] (cf. [9, p.189]), let

\[
C_w^+ = \sum_{y \leq w} P_{y,w} T_y.
\]

Then \(\{C_w^+\}_{w \in W}\) is a new basis for \(H_q\) and the following multiplication formula holds:

\[
C_s^+ C_w^+ = \begin{cases} (q + 1)C_w^+, & \text{if } sw < w, \\ C_s w^+ + q \sum_{y < w, y < y} \mu(y, w) q^{\frac{1}{2}(\ell(w) - \ell(y) - 1)} C_y^+, & \text{otherwise}, \end{cases}
\]

where \(\mu(y, w)\) is the Kazhdan–Lusztig polynomial.
where $\mu(y, w)q^{\frac{1}{2}(\ell(w) - \ell(y) - 1)}$ is the leading term of $P_{y, w}$ if $\mu(y, w) \neq 0$.

For each $w \in W$, fix a reduced expression $w = (s_1, s_2, \ldots, s_l)$ for $w$ (thus, $w = s_1 s_2 \cdots s_l$ and $\ell(w) = l$). Let

$$C_w^+ = C_{s_1}^+ C_{s_2}^+ \cdots C_{s_l}^+.$$  \hfill (2.0.4)

Then (2.0.3) implies that there exist $a_{y, w} \in \mathbb{Z}[q]$ such that

$$C_w^+ = C_{w'}^+ + q \sum_{y < w} a_{y, w} C_y^+,$$  \hfill (2.0.5)

where $y < w$ denotes a subsequence of $w$ obtained by removing some terms in the sequence.

For each $I \in \Lambda$, write

$$x_I := C_{w_I}^+ = \sum_{w \in W_I} T_w.$$  \hfill (2.0.6)

**Definition 2.1.** The $\mathbb{Z}[q]$-algebra $E_q = E_q(W) = \text{End}_{H_q}(\bigoplus_{I \in \Lambda} x_I H_q)$ is called the Hecke endomorphism algebra associated to the Coxeter group $W$. Specialising $q$ to 0, the resulting $\mathbb{Z}$-algebra $E_0 = E_q \otimes \mathbb{Z}$ is called the 0-Hecke endomorphism algebra associated with $W$.

Let

$$x_s = C_s^+ = T_s + 1.$$  

The elements $x_s, s \in S$, generate $H_q$. We now describe a presentation for $H_q$ with these Kazhdan–Lusztig generators.

For $\{s, t\} \in \Lambda$ and $m \leq m_{s,t}$, let

$$T_{[m]} = \cdots T_t T_s T_t$$  and  $$T_{[m]} = T_t T_s T_t \cdots,$$

and let

$$x_{(s,t)}^{(m)}(m) = \begin{cases} 1 + T_t, & \text{if } m = 1; \\ 1 + \sum_{i=1}^{m-1} (T_{[i]}s + T_{[i]}t) + T_{[m]}t, & \text{if } m \geq 2. \end{cases}$$

Since the generating relations for any Hecke-algebra are determined by the generating relations of its rank 2 subalgebras, we have the following.

**Lemma 2.2.** The Hecke algebra $H_q$ can be presented by generators $x_s, s \in S$ with

(a) Quasi-idempotent relations: $(x_s)^2 = (q + 1)x_s$ and

(b) Braid relations: $x_{(s,t)}^{(m)} = x_{(t,s)}^{(m)}$ for all $s, t \in S$.

We now write the elements $x_{(s,t)}^{(m)} (m \leq m_{s,t})$ as polynomials in $x_r$. Assume first that $\{s, t\} = \{1, 2\}$ generates an infinite dihedral subgroup in the next two lemmas. Thus, the condition $m \leq m_{s,t}$ can be dropped.
Lemma 2.3. We have the following recursive formulas for \( x^{(m)}_{(2,1)} \):

\[
x^{(m)}_{(2,1)} = \begin{cases} 
    x_2x_1, & \text{if } m = 2; \\
    x_1x_2x_1 - qx_1, & \text{if } m = 3; \\
    \left( x_2x_1 \right)^2 - 2qx_2x_1, & \text{if } m = 4; \\
    x^{(m-2)}_{(2,1)}(x_2x_1 - 2q) - q^2x^{(m-4)}_{(2,1)}, & \text{if } m \geq 5.
\end{cases}
\]

Proof. Let

\[
S_{[m]1} = \sum_{i=1}^{m} T[i]_1 \quad \text{and} \quad S_{[m]2} = \sum_{i=1}^{m} T[i]_2.
\]

Then

\[
x^{(m)}_{(2,1)} = 1 + S_{[m-1]2} + S_{[m]1}.
\]

Direct computation gives \( x^{(2)}_{(2,1)}, x^{(3)}_{(2,1)}, x^{(4)}_{(2,1)} \) and, for \( m \geq 3 \),

\[
x^{(m)}_{(2,1)}x_2x_1 = x^{(m)}_{(2,1)}(T_2T_1 + T_2 + T_1 + 1)
\]

\[
= x^{(m+2)}_{(2,1)} + 2q x^{(m)}_{(2,1)} + q^2x^{(m-2)}_{(2,1)}.
\]

Hence, the required formula follows. \( \square \)

By swapping the indices 1 and 2, we obtain an analogous version of Lemma 2.3 for \( x^{(m)}_{(1,2)} \). Consequently, \( x^{(m)}_{(2,1)} \) and \( x^{(m)}_{(1,2)} \) are polynomials in \( x_1 \) and \( x_2 \) without the constant term. Next, based on Lemma 2.3, we will deduce an explicit formula for \( x^{(m)}_{(2,1)} \) as a linear combination of monomials in \( x_1 \) and \( x_2 \),

\[
x^{(m)}_{(2,1)} = \sum_{j=1}^{m} b^m_j x_{[j]1}, \quad (2.3.1)
\]

where \( b^m_j \in \mathbb{Z}[q] \) and \( x_{[m]1} = \ldots x_2 x_1 \). Similarly, \( x_{[m]2} = \ldots x_1 x_2 \).

Lemma 2.4. The coefficient \( b^m_j \) is equal to \( \binom{j + s - 1}{j - 1}(-q)^s \) if \( s = \frac{m-j}{2} \) is an integer, and 0 otherwise. Consequently,

\[
x^{(m)}_{(2,1)} = \sum_{i=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m-i-1}{i}(-q)^i x_{[m-2i]1}.
\]

Proof. Use induction on \( m \). Observe from Lemma 2.3 that \( x^{(m)}_{(2,1)} \) has degree \( m \) and is a linear combination of monomials with degrees of the same parity as \( m \). So

\[
b^m_j = 0 \text{ if } \frac{m-j}{2} \notin \mathbb{Z}.
\]

Further, by Lemma 2.3 and (2.3.1), \( b^m_m = 1 \), \( b^m_{m-2} = -(m-2)q \) and \( b^m_1 = (-q)^{\frac{m-1}{2}} \) if \( m \) is odd (resp., \( b^m_2 = \frac{m}{2}(-q)^{\frac{m-2}{2}} \) if \( m \) is even). In general, for \( 3 \leq j \leq m-4 \), let
Lemma 2.6. For any \( s = \frac{m-j}{2} \), where \( j \) and \( m \) have the same parity. By Lemma 2.3 and induction,

\[
b_j^m = b_j^{m-2} - 2qb_j^{m-2} - q^2b_j^{m-4}
\]

\[
= (-q)^s \left( \frac{j+s-3}{j-3} \right) - 2q(-q)^{s-1} \left( \frac{j+s-2}{j-1} \right) - q^2(-q)^{s-2} \left( \frac{j+s-3}{j-1} \right)
\]

as required.

\[\square\]

Remark 2.5. The coefficients \( b_j^m \) also satisfies the recursive formula

\[
b_j^m = b_j^{m-1} - qb_j^{m-2}.
\]

This formula evaluated at \( q = 1 \) gives Elias' recursive formula in [14, Def. 2.10]

\[
d_j^m = d_j^{m-1} - d_{m-2},
\]

where \( d_j^m = b_j^m|_{q=1} \). So we can view \( b_j^m \) as a \( q \)-analogue of \( d_j^m \).

By definition, for \( m < m_{s,t} < \infty \), the elements \( x_{(s,t)}^{(m)} \) and \( x_{(t,s)}^{(m)} \) are distinct. However, if \( m = m_{s,t} \), then \( x_{(s,t)}^{(m_{s,t})} = x_{(t,s)}^{(m_{s,t})} = x_{(s,t)} \) are two distinct expressions in the Kazhdan–Lusztig generators.

In general, we define

\[
I \sqsubset J \iff I \subset J, |I| = |J| - 1.
\]  

(2.5.1)

Then for each \( I \in \Lambda \), every total ordering on \( I \):

\[
I_* : \emptyset = I_0 \sqsubset I_1 \sqsubset I_2 \sqsubset \cdots \sqsubset I_m = I,
\]  

(2.5.2)

gives an expression for \( x_I \) as follows. Fix a reduced expression \( w_{I_*} = s_is_{i-1}\cdots s_1 \) such that \( w_{I_i} = s_is_{i-1}\cdots s_1 \) for all \( i = 1, \ldots, m \) and let \( w_{I_*} = (s_i, s_{i-1}, \cdots, s_1) \). Let

\[
x_{w_{I_*}} = x_{s_i}x_{s_{i-1}}\cdots x_{s_1}.
\]

By (2.0.5), we have the following.

Lemma 2.6. For any \( I \in \Lambda \), any ordering \( I_* \) as in (2.5.2), and any reduced expression \( w_{I_*} \), there exist polynomials \( a_{y, w_{I_*}} \in \mathbb{Z}[q] \) such that

\[
x_I = x_{w_{I_*}} + q \sum_{y < w_{I_*}} a_{y, w_{I_*}} x_y.
\]

3. Two preliminary lemmas

In this section, \( R \) can be any commutative ring with 1. We prove two technical lemmas. Recall that a quiver \( Q \) is an oriented graph, consisting of a set \( Q_0 \) of vertices and a set \( Q_1 \) of arrows connecting the vertices. The path algebra \( RQ \) of \( Q \) is the free \( R \)-module with basis the set of (oriented) paths in \( Q \) and multiplication given by concatenation of paths, i.e. for any two paths \( \alpha \) and \( \beta \), \( \alpha \cdot \beta = \alpha \beta \) if the starting vertex of \( \alpha \) is the ending vertex of \( \beta \) and 0 otherwise. For any vertex \( i \), we denote by
Let relations. Let $p \in RQ$ be a quiver $Q$ with relations $\mathcal{I}$ and let $(Q', \mathcal{I}')$ be the quiver obtained from $Q$ by adding a loop $\alpha$ at a vertex to $Q_1$ and a relation $\alpha - p$ to $\mathcal{I}$, where $p \in RQ$ is a cycle through the vertex. Then the inclusion of $Q$ into $Q'$ induces an $R$-algebra isomorphism

$$RQ/\langle \mathcal{I} \rangle \to RQ'/\langle \mathcal{I}' \rangle.$$  

**Proof.** Since $RQ \cap \langle \mathcal{I}' \rangle = \langle \mathcal{I} \rangle$, the homomorphism $RQ \subset RQ' \to RQ'/\langle \mathcal{I}' \rangle$ induces a monomorphism $\phi : RQ/\langle \mathcal{I} \rangle \to RQ'/\langle \mathcal{I}' \rangle$. Now every coset in $RQ'/\langle \mathcal{I}' \rangle$ has a representative in $RQ$. Hence, this monomorphism is an isomorphism (and $\phi^{-1}$ is the map sending $x$ to $x$ and $\alpha$ to $p$, where $x$ is an arbitrary path in $RQ$). □

Let $A$ be an associative $R$-algebra defined by generators $a_1, \ldots, a_n$ and a set $\mathcal{I}$ of relations. Let $p_0 = 1, p_1, \ldots, p_s$ be idempotents in $A$. Note that each $p_i$ $(i \neq 0)$ or $p \in \mathcal{I}$ is a (noncommutative) polynomial $p_i(a_1, \ldots, a_n)$ or $p(a_1, \ldots, a_n)$ in $a_1, \ldots, a_n$. Let $\mathcal{J} \subseteq RQ'$ be the set of elements obtained from

1. $p' := p(a_1, \ldots, a_s)$ for all $p \in \mathcal{I}$;
2. $\delta_i v_i = e_i$ for all $1 \leq i \leq s$;
3. $v_i \delta_i = p'_i := p_i(a_1, \ldots, a_s)$ for all $1 \leq i \leq s$.

Let $R(\alpha_1, \ldots, \alpha_s)$ be the (centraliser) subalgebra generated by $\alpha_1, \ldots, \alpha_s$ with the identity $e_0$ and let

$$C = RQ'/\langle \mathcal{J}' \rangle.$$  

Since $R(\epsilon_0, \alpha_1, \ldots, \alpha_s) \cap \langle \mathcal{J}' \rangle$ is the ideal generated by $p'$ for $p \in \mathcal{I}$, it follows that the subalgebra $C_0$ of $C$ generated by all $\alpha_i + \langle \mathcal{J}' \rangle$ is isomorphic to $A$. Thus, as an image of $p_i$, each $p'_i$ is an idempotent in $C$.

**Lemma 3.2.** The algebra $B$ with the identity $1 = \sum_i 1_i$ is generated by $a_i, u_j$ and $d_j$ subject to the relations

1. $p(a_1, \ldots, a_n)$ for all $p \in \mathcal{I}$;
\[ d_i u_i = 1_i \quad \text{and} \quad u_i d_i = p_i \quad \text{for all} \quad 1 \leq i \leq s; \]

\[ u_1 u_i = u_i = 1_0 u_i \quad \text{and} \quad d_1 d_i = d_i = 1_d d_i \quad \text{for all} \quad 1 \leq i \leq s; \]

\[ 1_i 1_j = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad 1_i^2 = 1_i \quad \text{for all} \quad i. \]

**Proof.** Since \( \text{Hom}_A(p_i A, p_j A) \cong p_j A p_i \), the algebra \( B \) is generated by \( a_i, u_j \) and \( d_j \). We show that there is an isomorphism from \( C \) to \( B \), sending \( \alpha_i, \upsilon_j, \delta_j \) to \( a_i, u_j, d_j \), respectively, and thus conclude that the relations are the generating relations of \( B \).

First note that the identity \( 1_B \) in \( B \) is

\[ 1_B = 1_0 + 1_1 + \cdots + 1_s \]

and

\[ 1_0 B 1_0 \cong \text{Hom}_A(A, A) \cong A. \]

Define

\[ \phi : C \to B \]

with \( \alpha_i \mapsto a_i, e_i \mapsto 1_i, \upsilon_i \mapsto u_i \) and \( \delta_i \mapsto d_i \). By definition, \( a_j, u_i \) and \( d_i \) satisfy the relations \( J' \). So \( \phi \) is a well-defined surjective homomorphism. In particular, it induces a surjective map \( \phi_{ij} : e_i C e_j \to p_i A p_j (\cong 1_i B 1_j) \). We will show that all the \( \phi_{ij} \) are linear isomorphisms and thus conclude that \( \phi \) is an isomorphism. First, by comparing the generating relations and noting \( C_0 = e_0 C e_0 \), we have that the restriction of \( \phi \) to \( e_0 C e_0 \)

\[ \phi|_{e_0 C e_0} : e_0 C e_0 \to A \]

is an isomorphism. In particular, it induces an isomorphism from \( p'_i C p'_j \) to \( p_i A p_j \).

Using the relations between \( \upsilon_i, \delta_i, p_i \) and \( e_i \), we have

\[ e_i C e_j = \delta_i \upsilon_i C \delta_j \upsilon_j \subseteq \delta_i C \upsilon_j = e_i \delta_i C \upsilon_j e_j \subseteq e_i C e_j. \]

Thus

\[ e_i C e_j = \delta_i C \upsilon_j. \]

Similarly,

\[ p'_i C p'_j \cong \delta_i p'_i C p'_j \upsilon_j = \delta_i C \upsilon_j = e_i C e_j, \]

where the (linear) isomorphism, denoted by \( \psi \), is given by left multiplication with \( \delta_i \) and right multiplication with \( \upsilon_j \). We have the following commutative diagram

\[ \begin{array}{ccc}
\psi & \phi_{ij} & \epsilon \\
\downarrow & \downarrow & \\
e_i C e_j & p_i A p_j &
\end{array} \]

where \( \epsilon \) is an isomorphism induced by the isomorphism \( \phi|_{e_0 C e_0} \). Thus \( \phi_{ij} \) is a linear isomorphism, as required.

**Remark 3.3.** The corresponding relations of Lemma 3.2 (3) and (4) in \( C \) are implicit in the definition of the multiplication in the path algebra.
4. A quiver approach to Hecke endomorphism algebras

In the rest of the paper, let

\[ R = \left\{ \frac{f}{1 + qg} \bigg| f, g \in \mathbb{Z}[q] \right\} = \mathbb{Z}[q] P, \]  

be the ring obtained by localising \( \mathbb{Z}[q] \) at the multiplicative set \( P \) of all polynomials with the constant term 1.

We now apply the technical lemmas to give presentations of the Hecke endomorphism algebras over \( R \). Then applying the functors \(- \otimes_{\mathbb{Z}[q]} \mathbb{Q}(q)\) and \(- \otimes_{\mathbb{Z}[q]} \mathbb{Z}\) gives the presentations of the algebra over \( \mathbb{Q}(q) \) and \( \mathbb{Z} \), respectively.

By extending the ground ring to \( R \), we write \( RH_q = R \otimes_{\mathbb{Z}[q]} H_q \).

The elements in \( RH_q \) are \( R \)-linear combinations \( \sum_{w \in W} r_w T_w \), where \( r_w \in R \). Recall that the Poincaré polynomial associated to \( W_I \) is

\[ \pi(I) = \sum_{w \in W_I} q^{\ell(w)}. \]

Note that the elements \( x_I \) defined in (2.6) satisfies \( x_I^2 = \pi(I) x_I \). Since \( \frac{1}{\pi(I)} \in R \), the elements

\[ p_I = \frac{x_I}{\pi(I)} \]

are idempotents in \( RH_q \) and \( p_I RH_q = x_I RH_q \).

In the sequel, the notation \( p_I |_{\{t_s\}_{s \in S}} \) denotes the element in an \( R \)-algebra \( A \) obtained by replacing \( T_s \) in \( p_I \) with \( t_s \in A \) for all \( s \in S \).

We now apply Lemma 3.2 to the Hecke algebra \( A = RH_q \), the generators \( T_s, s \in S \), the set \( I \) of Hecke relations (cf. [18]), and the idempotents \( p_I, I \in \Lambda \), with \( p_0 = p_\emptyset = 1 \), and to give the first presentation of the Hecke endomorphism algebras \( B = RE_q \) over \( R \), where

\[ RE_q = R \otimes_{\mathbb{Z}[q]} E_q. \]

Let \( (Q', \mathcal{J}') \) be the quiver associated with \( RH_q(W) \) as constructed in the proof of Lemma 3.2. Thus, the vertices are \( e_I \) with \( I \in \Lambda \), arrows are \( \alpha_s, v_I, \delta_I \) with \( s \in S \) and \( I \in \Lambda^* \), and \( \mathcal{J}' \) consists of

\begin{itemize}
  \item[(J1')] Hecke relations on \( \alpha_s, s \in S \);
  \item[(J2')] \( \delta_I v_I = e_I \) for all \( I \in \Lambda^* \);
  \item[(J3')] \( v_I \delta_I = p_I := p_I |_{\{\alpha_s\}_{s \in S}} \) for all \( I \in \Lambda^* \).
\end{itemize}

In the case \( W = S_4 \), the symmetric group on 4 letters, and \( S = \{s_i \mid i = 1, 2, 3\} \) with \( s_i = (i, i + 1) \). The quiver \( Q' \), where \( s_i \) is identified with \( i \), has the form
We have the following relations in \( C = RQ'/⟨\mathcal{J}'⟩ \).

1. \( α_s = (1 + q)v_sδ_s − e_∅ \) for all \( s ∈ S \).
2. For any \( I \) and \( J \) in \( Λ^* \) with \( I ⊂ J \), \( p'_I = v_Iδ_I \) and \( p'_Ip'_J = p'_Jp'_I \).

Proof. (1) follows from (J3) by taking \( I = \{s\} \). (2) is true, by the definition of \( p'_I, p'_J \), the facts that \( x_Ix_J = π(I)x_J = x_Jx_I \) and \( α_s (s ∈ S) \) satisfy the Hecke relations.

We can now present the Hecke endomorphism algebra \( RE_q \) via the quiver \( Q' \).

**Proposition 4.2.** The Hecke endomorphism algebra \( RE_q(W) \) associated with a Coxeter system \((W, S)\) is isomorphic to the algebra \( RQ'/⟨\mathcal{J}'⟩ \). More precisely, \( RE_q(W) \) is generated by

\[ T_s, 1_I, u_J, d_I \quad (s ∈ S, I ∈ Λ, J ∈ Λ^*) \]

subject to the relations

1. Hecke relations on \( T_s, s ∈ S \);
2. \( d_Iu_I = 1_I \) and \( u_Id_I = p_I, I ∈ Λ^* \);
3. \( u_1d_1 = u_1δ_1 = 1_∅u_1 \) and \( d_1u_1 = d_1 = 1_∅d_1, I ∈ Λ^* \);
4. \( 1_I1_J = 0 \) for \( I ≠ J \) in \( Λ \) and \( 1^2_I = 1_I \) for all \( I ∈ Λ \).

Proof. It follows from Lemma 3.2 with the isomorphism

\[ φ : RQ'/⟨\mathcal{J}'⟩ → RE_q(W) \]

given by \( α_s ↦ T_s, e_I ↦ 1_I, v_I ↦ u_I \) and \( δ_I ↦ d_I \).

Let \( Q \) be the quiver obtained from \( Q' \) by removing the loops \( α_s, s ∈ S \), and let

\[ τ_s = (1 + q)v_sδ_s − e_∅ \] for all \( s ∈ S \). \hspace{1cm} (4.2.1)

Then \( τ_s ∈ RQ \) for all \( s ∈ S \). Define

\[ \mathcal{J} = \{ \text{Braid relations on } τ_s, s ∈ S \} ∪ \{ v_Iδ_I = p_I | (τ_s)_{s ∈ S}, δ_Iv_I = e_I \ | I ∈ Λ^* \}. \hspace{1cm} (4.2.2) \]

**Proposition 4.3.** The inclusion of \( Q \) into \( Q' \) induces an isomorphism of \( R \)-algebras

\[ RQ/⟨\mathcal{J}⟩ → RQ'/⟨\mathcal{J}'⟩. \]

In particular, the Hecke endomorphism algebra \( RE_q(W) \) is isomorphic to the path algebra \( RQ \) modulo the relations in \( \mathcal{J} \).
Proof. We show that \( \tau_s \) satisfies the quadratic Hecke relations.

\[
\tau_s^2 = (1 + q)^2(v_{s\{s\}}\delta_{s\{s\}})^2 - 2(1 + q)v_{s\{s\}}\delta_{s\{s\}} + e_\emptyset \\
= (1 + 2q + q^2)v_{s\{s\}}\delta_{s\{s\}} - 2(1 + q)v_{s\{s\}}\delta_{s\{s\}} + e_\emptyset \\
= (q^2 - 1)v_{s\{s\}}\delta_{s\{s\}} + e_\emptyset \\
= q(q + 1)v_{s\{s\}}\delta_{s\{s\}} - (q + 1)v_{s\{s\}}\delta_{s\{s\}} + e_\emptyset \\
= (q - 1)\tau_s + qe_\emptyset,
\]

Now the proposition follows from Lemma 3.1.

We now use the presentation above for \( RE_q \) to derive a presentation in terms of modified Williamson’s generators.

For any \( I \subset J \) in \( \Lambda \), define

\[
\begin{cases}
  v_{\emptyset, J} = v_J, & \text{if } I = \emptyset; \\
  v_{I, J} = \delta_I v_J, & \text{if } I \neq \emptyset.
\end{cases}
\]

Lemma 4.4. Suppose that \( I \subset J \subset K \) in \( \Lambda \). Then we have the following in \( RQ/\langle \mathcal{J} \rangle \).

1. \( v_{I, K} = v_{I, J}v_{J, K} \) and \( \delta_{I, K} = \delta_{I, J}\delta_{J, K} \). In particular,
   \[
   v_K = v_Jv_{J, K} \text{ and } \delta_K = \delta_{K, J}\delta_J.
   \]
2. \( \delta_{I, J}v_{I, J} = e_J \) and thus \( \delta_{I, J}v_{I, J} = e_J \) is an idempotent.
3. \( v_{I, J}\delta_{I, J} = \delta_{I, J}p_{IvI} = p_{Ie_I} \), where \( p_{J} = p_{J}\{\tau_s\}_{s \in S} \).

Proof. (1) By definition, \( v_{I, J}v_{J, K} = \delta_{I, J}\delta_{J, K}v_K \). Following the relations \( \delta_{I, J}v_I = e_I \) and \( v_I\delta_I = p_I \) for any \( I \in \Lambda^* \) in (4.2.2) and Lemma 4.1,

\[
v_{I, J}v_{J, K} = \delta_{I, J}(\delta_{I, J}\delta_{J, K}v_K)\delta_Kv_K = \delta_{I, J}p_{I}p_{J}p_{K}v_K = \delta_{I, J}p_{K}v_K = \delta_{I, J}v_K = \delta_{I, J}v_K,
\]

which is \( v_{I, K} \). So the first identity follows. Similarly, the second one holds.

(2) Also,

\[
\delta_{I, J}v_{I, J} = e_J\delta_{I, J}v_{I, J}e_J = \delta_{J}v_{Ie_I} = \delta_{J}v_{I}(\delta_{J}v_{Ie_I}) = \delta_{J}v_{I} = \delta_{J}v_{J},
\]

which is \( e_J^2 = e_J \), as required.

(3) It follows from (1) and Lemma 4.1. □

Lemma 4.4 gives the following sandwich relations.

Corollary 4.5. If \( I \subset J, J' \subset K \), then we have

\[
v_{I, J}v_{J, K} = v_{I, J'}v_{J', K}, \quad \delta_{K, J'}\delta_{J', J} = \delta_{K, J}\delta_{J, J'}.
\]

We now have the following quiver presentation for the Hecke endomorphism algebra \( RE_q(W) \cong RQ/\langle \mathcal{J} \rangle \). Recall the relation \( I \subset J \) defined in (2.5.1).

Corollary 4.6. The algebra \( RQ/\langle \mathcal{J} \rangle \) is generated by \( e_K, v_{I, J} \) and \( \delta_{I, J} \) for \( I, J, K \in \Lambda \) with \( I \subset J \), and the ideal \( \langle \mathcal{J} \rangle \) is generated by the following relations.

1. Braid relations on \( \{\tau_s | s \in S\} \);
\[(J2) \quad \delta_{I,I}v_{I,I} = e_J \text{ for all } I, J \in \Lambda \text{ with } I \sqsubseteq J;\]

\[(J3) \quad v_{I_0}v_{I_1}v_{I_2}v_{I_3} = e_J \text{ for all } I, J \in \Lambda^* \text{ and sequences } \]

\[
\emptyset = I_0 \sqsubseteq I_1 \sqsubseteq I_2 \sqsubseteq \cdots \sqsubseteq I_m = I.
\]

\[(J4) \quad v_{I,J}v_{I,K} = v_{I,J'}v_{I,K} \text{ and } \delta_{K,J} \delta_{J,I} = \delta_{K,J'} \delta_{J',I} \text{ for all } I, J, J', K \in \Lambda \text{ with } I \sqsubseteq J \sqsubseteq K \text{ and } I \sqsubseteq J' \sqsubseteq K.\]

**Proof.** It follows from Lemma 4.4 and the definition of $\mathcal{J}$ in (4.2.2). \qed

Note that (J3) and (J4) imply that $v_{I_0}v_{I_1}v_{I_2}v_{I_3} = p_I|_{\{\tau_s\}_{s \in S}}$ for all $I \in \Lambda^*$ and sequences

\[
\emptyset = I_0 \sqsubseteq I_1 \sqsubseteq I_2 \sqsubseteq \cdots \sqsubseteq I_m = I \text{ and } \emptyset = I'_0 \sqsubseteq I'_1 \sqsubseteq I'_2 \sqsubseteq \cdots \sqsubseteq I'_m = I.
\]

## 5. THE HASSE QUIVER PRESENTATION

Following Proposition 4.3 and Corollary 4.6, we now give a new presentation for $RE_d(W)$ in term of the Hasse quiver of the Coxeter system $(W,S)$.

Recall the set $\Lambda$ of finitary subsets of $S$. With the inclusion relation, $\Lambda$ is a poset. If we replace every edge in the Hasse diagram of the poset $\Lambda$ by a pair of arrows in opposite direction, the resulting quiver $\tilde{Q} = \tilde{Q}(W)$ is called the Hasse quiver associated to $W$.

Note that the new quiver $\tilde{Q}$ can be obtained from $Q$ by removing all arrows $v_I, \delta_I$ for $|I| > 1$, renaming the arrows $v_I, \delta_I$ for $|I| = 1$ to $\tilde{v}_{I,J}$, $\tilde{\delta}_{I,J}$, and introducing new arrows $\tilde{v}_{I,J}$ from $J$ to $I$ and $\tilde{\delta}_{J,I}$ from $I$ to $J$ for all $I, J \in \Lambda^*$ with $I \sqsubseteq J$. That is, the quiver has vertex set $\Lambda$ and arrow set

\[
\{\tilde{\delta}_{J,I}, \tilde{v}_{I,J} \mid I, J \in \Lambda, I \sqsubseteq J\}.
\]

For example, for $W = S_4$, the symmetric group on 4 letters, the quiver $\tilde{Q}$ is as follows, where the arrows between $\emptyset$ and $\{1\}, \{2\}, \{3\}$ are the same as those in $Q$.

$$\tilde{Q}:
\begin{array}{c}
\emptyset \\
\{1\} \\
\{1,2\} \\
\{1,2,3\} \\
\{2\} \\
\{1,3\} \\
\{2,3\} \\
\{1,2,3\}
\end{array}
$$

(5.0.1)

The path algebra $R\tilde{Q}$ of $\tilde{Q}$ over a ring $R$ admits an anti-involution

$$\tau : R\tilde{Q} \rightarrow R\tilde{Q}$$

(5.0.2)

that interchanges $\tilde{v}_{I,J}$ and $\tilde{\delta}_{I,J}$. 
Let $\tilde{e}_K$ denote the trivial path in $\tilde{Q}$ at the vertex $K \in \Lambda$. For each singleton \{s\} $\in \Lambda$, let

$$
\tilde{x}_s = (q + 1)\tilde{v}_{q,\{s\}}\tilde{e}_s, \in R\tilde{Q}.
$$

For each $I \in \Lambda$, a total ordering $I_\bullet$ on $I$ as in (2.5.2), and a reduced expression $w_{I_\bullet}$, let

$$
\tilde{x}_{I_\bullet} = \tilde{x}_{w_{I_\bullet}} + q \sum_{y < w_{I_\bullet}} a_{y,w_{I_\bullet}}\tilde{x}_y \quad \text{and} \quad \tilde{\rho}_{I_\bullet} = \frac{\tilde{x}_{I_\bullet}}{\pi(I)},
$$

where $\tilde{x}_w = \tilde{x}_{s_1}\tilde{x}_{s_{l-1}}\cdots \tilde{x}_{s_l}$ if $w = (s_j, s_{l-1}, \cdots, s_1)$ and $a_{y,w_{I_\bullet}} \in \mathbb{Z}[q]$ are given in Lemma 2.6.

**Theorem 5.1.** Maintain the notations $\Lambda$ and $\tilde{\rho}_{I_\bullet}$, etc., for $W$ as above. Let $\tilde{J}$ be the subset of $R\tilde{Q}$ defined by the following relations

(\text{J1}) Idempotent relations:

$$
\tilde{\delta}_{I,J}\tilde{v}_{I,J} = \tilde{e}_J
$$

for all $I, J \in \Lambda$ with $I \subseteq J$;

(\text{J2}) Sandwich relations:

$$
\tilde{v}_{I,J}\tilde{v}_{J,K} = \tilde{v}_{I,J'}\tilde{v}_{J',K} \quad \text{and} \quad \tilde{\delta}_{K,J}\tilde{v}_{I,J} = \tilde{\delta}_{K,J'}\tilde{v}_{I,J'}
$$

for all $I, J, J', K \in \Lambda$ with $I \subseteq J \subseteq K$ and $I \subseteq J' \subseteq K$.

(\text{J3}) Extended braid relations:

$$
\tilde{v}_{q,1} \cdots \tilde{v}_{q,m-1,1} \tilde{\delta}_{I,m-1} \tilde{\delta}_{I,m-2} \cdots \tilde{\delta}_{I,2,1} \tilde{\delta}_{I,1,0} = \tilde{\rho}_{I_\bullet}
$$

for all $I \in \Lambda^*$, $|I| > 1$, and given ordering $I_\bullet : \emptyset = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m = I$.

Then $RE_q(W) \cong RQ/(\langle J \rangle) \cong R\tilde{Q}/(\langle \tilde{J} \rangle)$.

**Proof.** By (J3), for any $s \in S$ we have $\tilde{x}_{\{s\}} = (q + 1)\tilde{v}_{q,\{s\}}\tilde{e}_s, \in \tilde{Q}.$ Thus, by (J1), we have $\tilde{x}_{\{s\}} = (q + 1)\tilde{x}_{\{s\}}$. For any $s, t \in S$, (J2) and (J3) imply $\tilde{\rho}_{(s,t)} = \tilde{\rho}_{(t,s)}$. Hence, $\tilde{x}_{\{s,t\}} = \tilde{x}_{\{t,s\}}$.

By Lemma 2.2, the subalgebra generated by $\tilde{x}_s$ is isomorphic to $R\tilde{H}_q(W)$. Note that this subalgebra is isomorphic to $\tilde{e}_q(R\tilde{Q}/(\langle \tilde{J} \rangle))\tilde{e}_q$ by (J3). Thus, $\tilde{\rho}_{I_\bullet} = \tilde{\rho}_{I}$ which is independent of the ordering on $I$. Further, if we put $\tilde{\tau}_s = \tilde{x}_s - 1$ for all $s \in S$, then $\tilde{\rho}_{I_\bullet} = \tilde{\rho}_{I}$ for all $s \in S$.

Consider the $R$-algebra homomorphism $\phi : R\tilde{Q} \rightarrow RQ$ sending $\tilde{v}_{I,J}, \tilde{\delta}_{I,J}, \tilde{e}_K$ to $v_{I,J}, \delta_{I,J}, e_K$, respectively. This map induces a surjective algebra homomorphism $\tilde{\phi} : R\tilde{Q} \rightarrow RQ/(\langle J \rangle)$. By Corollary 4.6, it is clear that $\phi^{-1}(\langle J \rangle) = \langle \tilde{J} \rangle$ and, hence, $\phi$ induces the required isomorphism.

The presentation in Theorem 5.1 is called the Hasse quiver presentation of $E_q(W)$.

We now give a direct description of the isomorphism from $R\tilde{Q}/(\langle \tilde{J} \rangle)$ to $RE_q(W)$ and turn the Hasse quiver presentations to a full presentation for $RE_q(W)$.

Note that, if $I \subseteq J$, then $x_J = h_{x_J} = x_I(h)$, where $h = h_{J,I} = \sum_{w \in D(i_{J,I})} T_w$ and $t$ is the anti-automorphism on $RH_q$ introduced in Section 2. So $x_JH_q$ is a submodule of $x_IH_q$ and

$$
p_J = \left( \frac{\pi(I)}{\pi(J)} h_{J,I} \right) p_I.
$$
Denote by

\[ u_{I,J} : p_I(RH_q) \to p_I(RH_q) \]

the natural inclusion and by

\[ d_{J,I} : p_I(RH_q) \to p_I(RH_q) \]

the splitting map of the inclusion given by left multiplication with \( \frac{\pi(I)}{\pi(J)} h_{J,I} \). Define the map

\[ h : R\tilde{Q}/\langle \tilde{J} \rangle \to RE_q(W), \]

sending \( \tilde{v}_{I,J}, \tilde{\delta}_{J,I}, e_K \) to \( u_{I,J}, d_{J,I} \) and \( 1_K \), respectively, where \( I, J, K \in \Lambda \) with \( I \subseteq J \).

**Corollary 5.2.** The map \( h \) induces an algebra isomorphism \( R\tilde{Q}/\langle \tilde{J} \rangle \cong RE_q(W) \). In other words, \( RE_q(W) \) is the \( R \)-algebra generated by \( u_{I,J}, d_{J,I} \) and \( 1_K \) subject to the relations \((\tilde{J}1)-(\tilde{J}3)\) obtained by replacing \( \tilde{v}_{I,J}, \tilde{\delta}_{J,I}, e_K \) with \( u_{I,J}, d_{J,I} \) and \( 1_K \), respectively, and the quiver relations

\[
(\tilde{J}4)\ u_{I,J} 1_J = u_{I,J} = 1_I u_{I,J}, \quad d_{J,I} 1_I = d_{J,I} = 1_J d_{J,I} \quad \text{and} \quad 1_I^2 = 1_I, \quad 1_I 1_J = 0 \quad \text{for} \ I \neq J.
\]

**Proof.** It follows from Propositions 4.2 and 4.3 and Theorem 5.1. \( \square \)

**Remark 5.3.** Let \( Z = Z[q^\frac{1}{2}, q^{-\frac{1}{2}}] \) and let \( ZE_q \) be the Hecke endomorphism algebra obtained by base change to \( Z \). The generators in Theorem 5.2 are scaled version of a subset of the generators for \( ZE_q \) given in [28] by Williamson. More precisely, \( u_{I,J}, \frac{\pi(J)}{\pi(I)} d_{J,I} \) with \( I \sqsubset J \) form a subset of Williamson’s generators. Let

\[ d'_{J,I} = \frac{\pi(J)}{\pi(I)} d_{J,I}. \]

As each \( \pi(I) \) is invertible in \( R \), if we replace the relations \((\tilde{J}1)-(\tilde{J}3)\) in the theorem by the following, we obtain a presentation of \( RE_q \) in terms of the subset of Williamson’s generators.

\[
(\tilde{J}1')\ d'_{J,I} u_{I,J} = \frac{\pi(J)}{\pi(I)} 1_{J}.
\]

\[
(\tilde{J}3')\ u_{\emptyset,I_1} \ldots u_{I_{m-1},I} d'_{I_m,I_{m-1}} \ldots d'_{I_2,I_1} d'_{I_1,\emptyset} = x_I|_{\{s\} \subset S} \quad \text{for any} \ I \in \Lambda \text{ and sequence} \\
\emptyset = I_0 \sqsubset I_1 \sqsubset I_2 \sqcup \ldots \sqcup I_m = I.
\]

Williamson’s other generators are monomials in \( d'_{J,I} \) and \( u_{I,J} \) with \( I \sqsubset J \). We will discuss the presentation of \( E_q \) over \( Z[q] \) in the last two sections.

**Remark 5.4.** Applying the functor \( - \otimes_R Q(q) \), Theorem 5.2 gives a presentation of the Hecke endomorphism algebra over \( Q(q) \), i.e.

\[ Q(q)E_q = Q(q) \otimes_R RE_q = Q(q) \otimes_{Z[q]} E_q. \]

Similarly, applying the functor \( Z \otimes_R - \), where \( Z \) is an \( R \)-module via the ring homomorphism \( R \to Z, q \mapsto 0 \), Theorem 5.2 gives a presentation of the 0-Hecke endomorphism algebra \( E_0 = Z \otimes_{Z[q]} E_q \). We will discuss 0-Hecke endomorphism algebras in the next section and show that the relations can be simplified.
6. 0-HECKE ENDMORPHISM ALGEBRAS

The first application of Theorem 5.2 is to give a presentation of the 0-Hecke endomorphism algebra $E_0$. Recall the Hecke algebra $H_q = H_q(W)$ over $\mathbb{Z}[q]$ and its canonical basis $\{C^+_w\}_{w \in W}$ defined in (2.0.2).

Consider the specialisation $\mathbb{Z}[q] \to \mathbb{Z}$ with $q \mapsto 0$ and let $H_0 = \mathbb{Z} \otimes_{\mathbb{Z}[q]} H_q$. We may regard $H_0$ as a degenerate Hecke algebra. Let $c_w = C_w^+ \otimes 1$, i.e. $c_w = C_w^+|_{q=0}$.

Recall that $H_0$ is a $\mathbb{Z}$-algebra generated by $T_1, \ldots, T_n$ subject to the relations $T_i^2 = -T_i$ and the braid relations. We call these relations 0-Hecke relations. Note that in the case of the longest element $w_I$, $c_w = x_I$. In particular, $c_s = T_s + 1 = x_{\{s\}}$ for all $s \in S$.

Let $w_{I,J}$ be the longest element in the double coset $W_I W_J$. Denote $c_{w_{I,J}}$ by $c_{I,J}$. Then $c_{I,I} = c_{w_I}$. Let $(W,*)$ be the Hecke monoid defined in [3, §4.5].

Lemma 6.1. (1) The elements $c_w, w \in W$, under the multiplication in $H_0$ form a monoid, which is isomorphic to the Hecke monoid $(W,*)$. In particular, they form a multiplicative basis for $H_0$.

(2) $c_I$ is an idempotent for every finitary $I \in \Lambda^*$.

(3) For any finitary subsets $I, J \subseteq S$, $c_{I,I} c_{J,J} = c_{I,J}$.

Proof. By (2.0.3), we have

$$c_s c_w = \begin{cases} c_w, & \text{if } sw < w; \\ c_{sw}, & \text{if } sw > w. \end{cases}$$

(6.1.1)

Thus, $\{c_s \mid s \in S\}$ generate a monoid whose associated monoid algebra is $H_0$. By [3, Prop. 4.34], the map $c : (W,*) \to H_0, w \mapsto c_w$ is the required monoid isomorphism, proving (1). Statements (2) and (3) are consequences of (6.1.1). $\square$

We call $\{c_w \mid w \in W\}$ the degenerate canonical basis of $H_0$. The following is clear from (6.1.1) and Lemma 6.1 (1).

Corollary 6.2. Let $I, J, K$ be finitary subsets of $S$ and let $x \in D^+_{I,J}$ and $y \in D^+_{J,K}$. Then $c_x c_y = c_{x*y}$. In particular, $x * y \in D^+_{I,K}$.

Lemma 6.3. The 0-Hecke algebra $H_0$ is generated by $c_s, s \in S$, subject to the relations.

(1) $c_s c_t \cdots = c_t c_s \cdots$, where $m_{s,t}$ is the order of $st$ in $W$.

(2) $c_s^2 = c_s$ for all $s \in S$. 

PRESENTING HECKE ENDMORPHISM ALGEBRAS
Proof. The equivalence between the relations in (1) and the braid relations follows from (6.1.1) and clearly $T_i^2 = -T_i$ implies that $(T_i + 1)^2 = (T_i + 1)$ and vice versa. So the claim holds.

We now give an interpretation of the 0-Hecke endomorphism algebra in terms of the degenerate canonical basis $c_w$, $w \in W$. Let

$$\mathcal{B} = \{(I, c_d, J) \mid I, J \in \Lambda, d \in D^+_{I,J}\}.$$ 

Form the free $\mathbb{Z}$-module $\mathbb{Z}\mathcal{B}$ with basis $\mathcal{B}$ and define multiplication with

$$(I, c_x, J) \ast (K, c_y, L) = \delta^K_J(I, c_xc_y, L),$$

where $\delta^K_J = 1$ if $J = K$ and 0 otherwise. Note that, by Corollary 6.2, the multiplication is well-defined. Note also that $f_I = (I, c_{I,I}, I)$ is an idempotent and $f_0\mathcal{B}f_0$ is isomorphic to the Hecke monoid $(\mathcal{W}, \ast)$.

**Theorem 6.4.** With this multiplication, $\mathbb{Z}\mathcal{B}$ becomes an associative algebra with the identity $1 = \sum_{I \in \Lambda} f_I$. Moreover, we have $E_0 \cong \mathbb{Z}\mathcal{B}.$

Proof. The isomorphism can be easily seen by using the degenerate canonical basis

$$\{\Theta^d_{I,J} \mid I, J \in \Lambda, d \in D^+_{I,J}\}$$

for $E_0$, where $\Theta^d_{I,J} \in E_0$ is defined by (cf. [7, Th. 2.3])

$$\Theta^d_{I,J}(c_{wK}h) = \delta^K_Jc_dh$$

for all $K \in \Lambda, h \in H_0$. By (6.1.1), it is easy to see that $\Theta^x_{I,J}\Theta^y_{J,K}(c_{wK}) = c_xc_y = c_{xy} = \Theta^{xy}_{I,K}(c_{wK}).$ Hence, $\Theta^x_{I,J}\Theta^y_{J,K} = \Theta^{xy}_{I,K}$ and the required isomorphism is the map $(I, c_x, J) \mapsto \Theta^x_{I,J}$. □

Following Corollary 5.2 and applying the functor $\mathbb{Z} \otimes_R -$ , we have the following presentation of the 0-Hecke endomorphism algebras $E_0$.

**Theorem 6.5.** The algebra $E_0$ with $1 = \sum_{I \in \Lambda} 1_I$ is generated by $u_{I,J}$, $d_{I,I}$ and $1_K$, where $I, J, K \in \Lambda$ with $I \subseteq J$ subject to the relations

1. $d_{I,I}u_{I,J} = 1_J$ for all $I, J \in \Lambda$ with $I \subseteq J$;
2. $u_{I,J}u_{J,K} = u_{I,J'}u_{J',K}$ and $d_{K,J}d_{J,I} = d_{K,J'}d_{J,I'}$ for all $I, J, J', K \in \Lambda$ with $I \subseteq J \subseteq K$ and $I \subseteq J' \subseteq K$;
3. $u_{I_1 \ldots I_m, I}d_{I_m, I_{m-1}} \ldots d_{I_2, I_1}d_{I_1, \emptyset} = p(s_1) \cdots p(s_2)p(s_1)$, for any $I \in \Lambda^*$, and any reduced expression of $w_I = s_1 \cdots s_2 s_1$ associated with the ordering $I_1 : \emptyset = I_0 \preceq I_1 \preceq I_2 \cdots \preceq I_m = I$;
4. $u_{I,J}1_I = u_{I,J} = 1_Iu_{I,J}, d_{I,I}1_I = d_{I,I} = 1_Jd_{J,I}, 1_J^2 = 1_I$ and $1_I1_J = 0 (I \neq J)$.

The generators $u_{I,J}$, $d_{I,I}$ and $1_K$ correspond to elements $(I, c_{I,J}, J), (J, c_{J,I}, I)$ and $f_K$ in $\mathcal{B}$. Theorem 6.5 gives a presentation for the algebra $\mathbb{Z}\mathcal{B}$ by the generators $(I, c_{I,J}, J), (J, c_{J,I}, I)$ for all $I, J \in \Lambda$ with $I \subseteq J$. 

We end this section with an algorithm for writing a basis element in \( B \) as product of these generator. For a \( W_I \)-double coset \( p \), let \( p^+, p^- \) be the longest, shortest elements in \( p \). For \( w \in W \), let

\[
R(w) = \{ s \in S \mid ws < w \}.
\]

The following algorithm gives a way to write a basis element \((I, c_{p^+}, J)\) as a product of generators. This algorithm is a degenerate version of the one given in the proof of [29, Prop. 1.3.4]. Recall the notation used in (2.0.1).

Given a double coset \((I, p, J)\) with \( I, J \in \Lambda \), let \( \bar{J} = R(p^+) \), \( J_1 = I^p \cap \bar{J} \) and \( p_1 = W_I p^- W_{J_1} \). Then

\[
c_{p^+} = c_{p_1^+} c_{w, J} = c_{p_1^+} c_{J_1, J}. \]

If \( J_1 = \bar{J} \), then \( I = \bar{J} \) and \( p^- = 1 \). So the algorithm stops. Otherwise, continuing the algorithm with \((I, p_1, J_1)\) yields \( \bar{J}_1 \in \Lambda \) and double coset \((I, p_2, J_2)\) such that \( J_2 \subset J_1 \supset J_1 \) and

\[
c_{p^+} = c_{p_2^+} c_{J_2, J_1} c_{J_1, J_1} c_{J_1, J}. \]

The algorithm stops if \( J_2 = \bar{J}_1 \). In this way, we eventually find sequences \( J_0 = J, J_1, \ldots, J_m = I \) and \( \bar{J}_0, \bar{J}_1, \ldots, \bar{J}_{m-1} = I \) in \( \Lambda \) such that \( J_i \subset \bar{J}_{i-1} \supset J_{i-1} \) \( (1 \leq i \leq m - 1) \) and

\[
c_{p^+} = c_{I, J_{m-1}} c_{J_{m-1}, J_{m-1}} \cdots c_{J_2, J_1} c_{J_1, J_1} c_{J_1, J}. \]

Thus, we write \((I, c_{p^+}, J)\) as a product of generators.

7. The integral case

It seems difficult to find the generating relations for the integral algebra \( E_q(W) \) over \( \mathbb{Z}[q] \). In this section, we first explain how to find torsion relations. We then compute the integral presentation in the rank 2 case (see Theorem 7.3). A more complicated example in the \( A_3 \) case is also computed in the last section.

For notational simplicity, we will denote the Hasse quiver \( \bar{Q} \) in Section 5 by \( Q \) and will drop the \( \bar{\cdot} \)’s on all its arrows and trivial paths.\(^1\) Thus, the vertex set of \( Q \) is \( \Lambda = \Lambda(W) \) and the arrow set is \( \{ \delta_{I,J}, v_{I,J} \mid I, J \in \Lambda, I \subset J \} \) with trivial paths \( e_K \), \( K \in \Lambda \).

Let \( Q_q = Q_q(W) := \mathbb{Z}[q] Q \) be the path algebra of \( Q \) over \( \mathbb{Z}[q] \). For \( s \in S \), let

\[
\chi_s = v_{\emptyset, (s)} \delta_{(s)}, \emptyset.
\]

Note that the factor \((q + 1)\) in \( \bar{\chi}_s \) is dropped here.

For any \( I \in \Lambda \) and given ordering \( I_s : \emptyset = I_0 \sqsubset I_1 \sqsubset \cdots \sqsubset I_m = I \), use \( \chi_s \) to define \( \chi_{I_s} \) similarly as defining \( \bar{\chi}_{I_s} \) in (5.0.3).

Let \( J \) be the subset of \( Q_q \) defined by the following relations

(\( J1 \) Quasi-idempotent relations: \( \delta_{I,J} v_{I,J} = \frac{\pi(J)}{\pi(I)} e_J \).\(^2\)

\(^1\)The reader should not be confused with the quiver \( Q \) discussed in Section 4.
(J2) Sandwich relations:
\[ v_{I,J,K} = v_{I,J'}v_{J',K} \text{ and } \delta_{K,J} = \delta_{K,J'}\delta_{J,I} \]
for all \( I,J,J',K \in \Lambda \) with \( I \subset J \subset K \) and \( I \subset J' \subset K \).

(J3) Extended braid relations:
\[ v_{\emptyset , \emptyset } \cdots v_{I_{m-1}, \emptyset }\delta_{I_{m-1},\emptyset } \cdots \delta_{I_{2},I_{1}}\delta_{I_{1},\emptyset } = \chi_{I}^* \]

for all \( I \in \Lambda^* \) and given ordering \( I^* \).

If we put \( \delta_{J,I} = \frac{\pi_{(J)}}{\pi_{(I)}}\delta_{J,I} \), then relations (J1) and (J3) give \( \tilde{J}1 \) and \( \tilde{J}3 \). Hence, in \( R\mathcal{Q}_{q} \), \( \tilde{J} \) generates the ideal \( \langle \tilde{J} \rangle \).

As in the proof of Corollary 5.2, we have a surjective algebra homomorphism
\[ \psi : \mathcal{Q}_{q}/\langle J \rangle \to E_{q}(W) \]
and thus a short exact sequence
\[ 0 \to \mathcal{K}/\langle J \rangle \to \mathcal{Q}_{q}/\langle J \rangle \to E_{q}(W) \to 0, \tag{7.0.1} \]
for some ideal \( \mathcal{K} \) of \( \mathcal{Q}_{q} \). Applying \( R \otimes_{\mathbb{Z}[q]} \mathcal{K}/\langle J \rangle \) gives the short exact sequence
\[ 0 \to R \otimes_{\mathbb{Z}[q]} \mathcal{K}/\langle J \rangle \to R \otimes_{\mathbb{Z}[q]} (\mathcal{Q}_{q}/\langle J \rangle) \to R \otimes_{\mathbb{Z}[q]} E_{q}(W) \to 0. \tag{7.0.2} \]
Since \( R \otimes_{\mathbb{Z}[q]} (\mathcal{Q}_{q}/\langle J \rangle) \cong R\tilde{Q}/\langle \tilde{J} \rangle \), by Corollary 5.2,
\[ R \otimes_{\mathbb{Z}[q]} \mathcal{K}/\langle J \rangle = 0. \]

Hence, to give a presentation of \( E_{q}(W) \), we need to find sufficient \( P \)-torsion elements, which are elements in \( \mathcal{K}/\langle J \rangle \) annihilated by a polynomial in \( P \). Note that \( P \) is the multiplicative set of polynomials in \( \mathbb{Z}[q] \) with the constant term 1. Below we explain how to find those elements.

For any \( I \in \Lambda \) and any given sequence \( \emptyset = I_0 \subset I_1 \subset \cdots \subset I_m = I \), let
\[ v_I = v_{\emptyset ,I} = v_{\emptyset ,I_1}v_{I_1,I_2} \cdots v_{I_{m-1},I} \]
\[ \delta_I = \delta_{I,\emptyset } = \delta_{I,I_{m-1}} \cdots \delta_{I_{2},I_{1}}\delta_{I_{1},\emptyset } \]
In general, for \( I \subset J \) in \( \Lambda \), let \( \delta_{J,I} \) (resp., \( v_{I,J} \)) denote the shortest paths from \( I \) (resp. \( J \)) to \( J \) (resp. \( I \)).

**Proposition 7.1.** Let \( p = \sum f_ip_i \) be a linear combination of paths from \( I \) to \( J \), where each \( f_i \in \mathbb{Z}[q] \). If \( v_{\emptyset ,I}p\delta_{J,\emptyset } = 0 \) in \( \mathcal{Q}_{q}/\langle J \rangle \), then \( p \) is \( P \)-torsion. Further, the ideal \( \mathcal{K} \) is generated by all such \( P \)-torsion elements and \( J \).

**Proof.** As \( \delta_{I}v_{I} = \pi(I)e_{I} \) for any \( I \in \Lambda \), then in \( \mathcal{Q}_{q}/\langle J \rangle \)
\[ \pi(J)\pi(I)p = \delta_{J}v_{J}p\delta_{I}v_{I} = 0. \]
So \( p \) is \( P \)-torsion.

Now suppose that \( p = \sum f_ip_i \) is \( P \)-torsion, we may assume that all the paths \( p_i \) start from \( I \) and end at \( J \). Then there is a polynomial \( f \in R \) such that \( fp = 0 \) and
\[ v_{J}fp\delta_{I} = 0 \in \bar{e}_{\emptyset }\left( \mathcal{Q}_{q}/\langle J \rangle \right) \bar{e}_{\emptyset }. \]
Note that (cf. the proof of Theorem 5.1)

\[ \tilde{e}_0 \left( Q_q / \langle J \rangle \right) \tilde{e}_0 \cong H_q, \]

which is a free \( \mathbb{Z}[q] \)-module. Therefore \( v_I p \delta_I = 0 \). This says that \( p \) is \( P \)-torsion as described and so the ideal \( \mathcal{K} \) is generated by all such \( P \)-torsion elements and \( \mathcal{J} \). \( \Box \)

The \( P \)-torsion elements in \( Q_q / \langle \mathcal{J} \rangle \) give us new relations. We call them torsion relations.

**Lemma 7.2.** If \((W, S)\) is finite, then any path from \( I \) to \( S \) (resp. from \( S \) to \( I \)) is a multiple of \( \delta_{I,S} \) (resp. \( \nu_{S,I} \)).

**Proof.** The proof follows from the sandwich relations and the quasi-idempotent relations. \( \Box \)

We now look for the torsion relations for dihedral Hecke algebras \( \mathbb{Z}[q] \)-algebra \( E_q(I_n) \). In this case, Elias gave an recursive presentation of the Hecke endomorphism algebras in [14, Prop. 2.20] over \( \mathbb{Z}[v, v^{-1}] \), where \( v^2 = q \). We work over \( \mathbb{Z}[q] \) and can write down explicit generating relations. Our method is different from Elias’.

Let \( S = \{1, 2\} \). Note that the Hasse quiver \( Q \) has the form:

\[ Q : \]

\[ \{1\} \]

\[ \delta_{(1)} \]

\[ \delta_{(1,2), (1)} \]

\[ \delta_{(1,2), (2)} \]

\[ \{2\} \]

\[ \begin{array}{c}
\delta_{(1,2), (1)} \\
\delta_{(1,2), (2)} \\
{1, 2} \end{array} \]

Let \( \chi_{1[j]} \) and \( \chi_{2[j]} \) be monomials in \( \chi_i \), defined in a similar way as \( x_{1[j]} \) and \( x_{2[j]} \).

Let \( \chi_{(1,2)}^{(m)} \) be the polynomial in the new \( \chi_1 \) and \( \chi_2 \), as given in Lemma 2.4, and \( \chi_{(2,1)}^{(m)} \) the polynomial obtained from \( \chi_{(1,2)}^{(m)} \) by swapping the indices 1 and 2. Thus, (2.3.1) and its counterpart for (2,1) can be combined into the following: for \((s, t) = (2, 1)\) or \((1, 2)\),

\[ \chi_{(s,t)}^{(m)} = \sum_{j=1}^{m} b_j^m \chi_{j[s]} = \sum_{i=0}^{\frac{m-1}{2}} \binom{m-i-1}{i} (-q)^i \chi_{[m-2i]t}. \]

**Theorem 7.3.** The \( \mathbb{Z}[q] \)-algebra \( E_q(I_n) \) is isomorphic to the quotient algebra of the path algebra \( Q_q \) modulo the following relations \( \mathcal{J}_n \).

1. **Quasi-idempotent relations:** for \( i = 1, 2 \),
   
   (i) \( \delta_{[i]} v_{[i]} = (1 + q) c_{[i]} \);
   
   (ii) \( \delta_{[1,2], [i]} v_{[i], [1,2]} = (1 + q + \cdots + q^{n-1}) c_{[1,2]} \).

2. **Sandwich relations:**
   
   (i) \( v_{[1]} v_{[1], [1,2]} = v_{[2]} v_{[2], [1,2]} \)
   
   (ii) \( \delta_{[1,2], [1]} \delta_{[1]} = \delta_{[1,2], [2]} \delta_{[2]} \);
Refined braid relations: for $(s, t) = (1, 2)$ or $(2, 1)$.

(i) $v_{(s,1,2)}(\delta_{(1,2)},s) = \delta_{(t)}\left(\sum_{j=2}^{n} b_j^n \chi_{[j-2]}(t) v_{s}\right)$, where $\chi_{[0]} = e_\emptyset$, if $n$ is even;

(ii) $v_{(s,1,2)}(\delta_{(1,2)},s) = \delta_{(s)}\left(\sum_{j=3}^{n} b_j^n \chi_{[j-2]}(s) v_{s}\right) + (-q)^{n-1} e_{s}$, if $n$ is odd.

Remark 7.4. Note that the braid relation $\chi^{(m)}_{(1,2)} = \chi^{(m)}_{(2,1)}$ in Lemma 2.2 can be easily derived by multiplying the refined braid relations with $v_{(t)}$ (or $v_{(s)}$) on the left and $\delta_{(s)}$ on the right and applying the sandwich relations.

Lemma 7.5. The following relations hold in $\mathcal{Q}_q/\langle \mathcal{J}_n \rangle$: for $(s, t) = (1, 2)$ or $(2, 1)$.

(a) $(q + 1) v_{(s,1,2)}(\delta_{(1,2)},s) = \delta_{(s)}(\sum_{j=1}^{n} b_j^n \chi_{[j-1]}(s) v_{s})$, if $n$ is even;

(b) $(q + 1) v_{(s,1,2)}(\delta_{(1,2)},s) = \delta_{(t)}(\sum_{j=1}^{n} b_j^n \chi_{[j-1]}(t) v_{s})$, if $n$ is odd,

where $\chi_{[0]} = e_\emptyset$.

Proof. We only prove (a). By the quasi-idempotent relation and the sandwich relation, the left hand side

\[ \text{LHS} = (\delta_{(s)} v_{(s,1,2)}(\delta_{(1,2)},s) = \delta_{(s)}(v_{(t,1,2)}(\delta_{(1,2)},s)). \]

Now, applying (3)(i) yields (a). \(\square\)

Proof of Theorem 7.3. Let $\phi : \mathcal{Q}_q/\langle \mathcal{J}_n \rangle \rightarrow E_q(\mathcal{I}_n)$ be the algebra homomorphism given by sending, for all $I, J, K \in \Lambda$ with $I \subseteq J$, $e_K$ to the identity map $1_K$ on $x_K H_q$, $\delta_{J,I}$ to the map $d_{J,I}' = \pi(I) d_{J,I} : x_I H_q \rightarrow x_J H_q$ (see Remark 5.3) and $v_{I,J}$ to the inclusion map $u_{I,J} : x_I H_q \rightarrow x_J H_q$. Then $x_{(i)} = u_{\emptyset,(i)} d_{(i),\emptyset} : H_q \rightarrow H_q$ is given by right multiplication with $x_{(i)}$. Direct computation shows that all three sets of relations are satisfied if $e_K$, $\delta_{J,I}$ and $u_{I,J}$ are replaced by $1_K$, $d_{J,I}'$ and $u_{I,J}$. For example, both sides of the refined braid relations become the maps sending $x_{(s)}$ to $x_{(1,2)}$.

So the map $\phi$ is indeed a well-defined algebra homomorphism. Note that $1_K$, $u_{I,J}$ and $d_{J,I}'$ for all $K, I, J \in \Lambda$ with $I \subseteq J$ generate the $\mathbb{Z}[q]$- algebra $E_q(\mathcal{I}_n)$ [28, Prop. 2.11]. Further, for any $I \subseteq J$, $u_{I,J}$ and $d_{J,I}'$ are monomials in $u_{K,I}$ and $d_{I,J}'$ with $K \subseteq L$. So $\phi$ is surjective.

We show now that $\phi$ is injective for even $n$, the odd case can be done similarly. Since $\bar{v}_\emptyset(\mathcal{Q}_q/\langle \mathcal{J}_n \rangle) \bar{v}_\emptyset \cong H_q$, it suffices by Lemma 7.2 and symmetry, to prove that the following sets $B_{J,I}$ span the space of paths from $I$ to $J$ in $\mathcal{Q}_q/\langle \mathcal{J}_n \rangle$,

\begin{align*}
B_{(1),\emptyset} &= \{\delta_{(1)} \chi_{[j]} \mid 0 \leq j \leq n - 1 \}; \\
B_{(1),(1)} &= \{e_{(1)}, v_{(1,1,2)}, \delta_{(1,2),(1)} \} \cup \{\delta_{(1)} \chi_{[j]} v_{(1)} \mid 1 \leq j \leq \frac{n}{2} - 1 \}; \\
B_{(2),(1)} &= \{\delta_{(2)} v_{(1)}, v_{(1,2,1)}, \delta_{(1,2),(1)} \} \cup \{\delta_{(2)} \chi_{[j]} v_{(1)} \mid 2 \leq j \leq \frac{n}{2} - 1 \}.
\end{align*}

Note that the cardinality of $B_{J,I}$ is the same as the cardinality of the double coset $W_J \backslash I_n / W_I$, i.e. the rank of the homomorphism space from $x_I H_q$ to $x_J H_q$, as a free $\mathbb{Z}[q]$-module. So we have the injectivity.

We now prove the claim for the set $B_{(1),(1)}$, the others can be done similarly. By Lemma 7.2, any path from $\{1\}$ to $\{1\}$ via $\{1, 2\}$ is a multiple of $v_{(1,1,2)} \delta_{(1,2),(1)}$. So
any path from \( \{1\} \) to \( \{1\} \) is a multiple of a path \( e_{\{1\}}, v_{\{1\}, \{1,2\}} \delta_{\{1,2\}, \{1\}} \) or \( \delta_{\{1\}} \chi_{j} v_{\{1\}} \), where \( j \geq 1 \) and \( s = 1 \) or 2. When \( s = 1 \), or \( j \) is even and \( s = 2 \), by the quasi-idempotent relations in (1), \( \delta_{\{1\}} \chi_{j} v_{\{1\}} \) is equal to a multiple of \( \delta_{\{1\}} \chi_{j} v_{\{1\}} \) with \( l < j \). So we need only to prove that \( \delta_{\{1\}} \chi_{j} v_{\{1\}} \), for odd \( l \geq n - 1 \), is a linear combination of the paths in \( B_{\{1\}, \{1\}} \). Indeed, by the relation in Lemma 7.5(a),

\[
\delta_{\{1\}} \chi_{n-1} v_{\{1\}} = (q + 1) v_{\{1\}, \{1,2\}} \delta_{\{1,2\}, \{1\}} - \delta_{\{1\}} \left( \sum_{j=1}^{n-2} b_{j}^{n} \chi_{[j-1]} v_{\{1\}} \right) \in \text{span}(B_{\{1\}, \{1\}})
\]

and, for odd \( l \geq n + 1 \),

\[
\delta_{\{1\}} \chi_{l} v_{\{1\}} = (\delta_{\{1\}} \chi_{n-1} v_{\{1\}}) \delta_{\{1\}} \chi_{l} v_{\{1\}}
\]

\[
= (q + 1) v_{\{1\}, \{1,2\}} \delta_{\{1,2\}, \{1\}} \chi_{l-1} v_{\{1\}} - \delta_{\{1\}} \left( \sum_{j=1}^{n-2} b_{j}^{n} \chi_{[j-1]} v_{\{1\}} \right) v_{\{1\}}
\]

Further, repeatedly applying \( \delta_{s} \chi_{s} = (q + 1) \delta_{s} \) and the downward sandwich relation on \( v_{\{1\}, \{1,2\}} \delta_{\{1,2\}, \{1\}} \delta_{\{2\}} \chi_{l} v_{\{1\}} \) leads to

\[
\delta_{\{1\}} \chi_{l} v_{\{1\}} = (q + 1)^{l-n+2} v_{\{1\}, \{1,2\}} \delta_{\{1,2\}, \{1\}} - \delta_{\{1\}} \left( \sum_{j=1}^{n-2} b_{j}^{n} \chi_{[j-1]} v_{\{1\}} \right) v_{\{1\}}.
\]

So, by induction, \( \delta_{\{1\}} \chi_{l} v_{\{1\}} \) is a linear combination of the paths in \( B_{\{1\}, \{1\}} \), as required. Therefore, we can conclude that the two algebras are isomorphic. \( \square \)

8. The integral case of type A3

In this section, we will compute the torsion relations required for presenting \( E_{q}(\mathcal{S}_{4}) \). Recall in this case \( S = \{1, 2, 3\} \), the Hasse quiver \( \mathcal{Q} \) given in (5.0.1), and the path algebra \( \mathcal{Q}_{q} = \mathbb{Z}[q] \mathcal{Q} \). Note that the Dynkin graph automorphism that interchanges 1 and 3 and fixes 2 induces a graph automorphism \( \sigma : \mathcal{Q}_{q} \rightarrow \mathcal{Q}_{q} \). We will also use subgraph automorphisms induced by the graph automorphism on parabolic subgroups.

Given a path \( p \) from \( I \) to \( J \), denote the path \( v_{\emptyset, I} p \delta_{J, \emptyset} \) by \( p^{\emptyset} \).

Lemma 8.1. Maintain the notation introduced in Section 7. The following are torsion relations in \( \mathcal{Q}_{q}/\langle \mathcal{J} \rangle \).\(^{2}\)

\begin{align*}
(T1) \quad & \delta_{\{1\}, \emptyset} \chi_{2} v_{\emptyset, \{1\}} = v_{\{1\}, \{1,2\}} \delta_{\{1,2\}, \{1\}} + q e_{\{1\}}; \\
(T2) \quad & \delta_{\{1\}, \emptyset} v_{\emptyset, \{3\}} = v_{\{1\}, \{1,3\}} \delta_{\{1,3\}, \{3\}}; \\
(T3) \quad & \delta_{\{1,2\}, \{1\}} v_{\{1\}, \{1,3\}} \delta_{\{1,3\}, \{1\}} v_{\{1\}, \{1,2\}} = v_{\{1\}, \{1\}, 2} s_{\delta_{S, \{1\}} = q (q + 1) e_{\{1,2\}}; \\
(T4) \quad & \delta_{\{1,2\}, \{2\}} v_{\{2\}, \{2,3\}} \delta_{\{2,3\}, \{3\}} v_{\{3\}, \{1,3\}} = v_{\{1\}, \{2\}, \{1\}} s_{\delta_{S, \{1\}} = q \delta_{\{1,2\}, \{1\}} v_{\{1\}, \{1,3\}}; \\
\end{align*}

\(^{2}\)Here each relation should be understood as the set of the relations obtained by applying (sub)graph automorphisms and the anti-involution (5.0.2) to the relation. Thus, (T1) really represents 4 relations.
Proof. (T1) and (T2) are rank 2 torsion relations from Theorem 7.3. We prove (T4), and (T3) can be done similarly. If we put

\[ p = v_{\{2\},\{2,3\}} \delta_{\{2,3\},\{3\}} v_{\{3\},\{1,3\}} \delta_{\{1,3\},\emptyset} \quad \text{and} \quad l = v_{\{3\},\{1,3\}} \delta_{\{1,3\},\emptyset}, \]

then

\[(q + 1)(LHS)^\wedge = v_{\emptyset,\{1,2\}} \delta_{\{1,2\},\{2\}} (q + 1)e_{\{2\}} \cdot p \]
\[= (\chi_2 \chi_1 \chi_2 - q \chi_2) v_{\emptyset,\{2\}} p, \quad \text{(by (J1), (J3))} \]
\[= (\chi_2 \chi_1 - qe_\emptyset) v_{\emptyset,\{2\}} (q + 1)p \]
\[= (\chi_2 \chi_1 - qe_\emptyset) v_{\emptyset,\{2\}} v_{\{2\},\{2,3\}} \delta_{\{2,3\},\{3\}} \cdot (q + 1) \cdot l \]
\[= (\chi_2 \chi_1 - qe_\emptyset)(\chi_3 \chi_2 \chi_3 - q \chi_3) v_{\emptyset,\{3\}} \cdot l, \quad \text{(by (J1), (J3))} \]
\[= (\chi_2 \chi_1 - qe_\emptyset)(\chi_3 \chi_2 - qe_\emptyset) v_{\emptyset,\{3\}} (q + 1) \cdot l, \quad \text{(by (J1))} \]
\[= (q + 1)(\chi_2 \chi_1 \chi_3 - q \chi_2 \chi_1 - q \chi_3 \chi_2 + q^2 e_\emptyset) v_{\emptyset,\{3\}} l \]
\[= (q + 1)(\chi_2 \chi_1 \chi_3 - q \chi_2 \chi_1 - q \chi_3 \chi_2 + q^2 e_\emptyset) \chi_3 \chi_1, \quad \text{(by (J3))} \]
\[= (q + 1)(\chi_3 + q \chi_2 \chi_3) \chi_1 \]
\[= (q + 1)[(\delta_{\{1,2\},\emptyset} v_{\emptyset,\{1,3\}})^\wedge + q(\delta_{\{1,2\},\emptyset} v_{\emptyset,\{1,3\}})^\vee]], \quad \text{(by (J3)), (J1)).} \]

So by definition and Proposition 7.1, (T4) is a torsion relation.

Let \( K \) be the ideal of \( Q_q \) generated by the quasi-idempotent relations and sandwich relations together the four sets relations (T1)–(T4). By footnote 2, both the anti-automorphism \( \tau \) and the graph automorphism \( \sigma \) stabilise \( K \). Let

\[ A = Q_q/K. \]

Then \( \tau \) and \( \sigma \) induces an anti-automorphism and an automorphism on \( A \). We will use the same letters to denote them. Observe that (cf. the proof of Theorem 5.1)

\[ e_\emptyset Ae_\emptyset \cong H_q. \quad \text{(8.1.1)} \]

Denote by \( P(J,I)_{\emptyset} \) the subspace in \( A \), consisting of paths from \( I \) to \( J \) via \( \emptyset \) and \( \chi_w = \chi_{s_{i_1}} \cdots \chi_{s_{i_t}} \) for a given reduced word \( w = s_{i_1} \cdots s_{i_t} \in W \). We call a path of the form \( v_{L,K} \delta_{K,I} \) a hook, where \( I, L \subset K \). Further, if \( |K| = i \), then the hook is called a level-\( i \) hook. The relations (T1) and (T2) show that a level-2 hook is a linear combination of paths without level-2 hooks.

Lemma 8.2. The space \( P(J,I)_{\emptyset} \) is spanned by

\[ \mathcal{P}^\emptyset_{J,I} = \{ \delta_{J,\emptyset} \chi_w v_{\emptyset,I} \mid w \in D_{J,I} \text{ and } w \text{ is a fixed reduced expression of } w \}. \]

Proof. If both \( I \) and \( J \) are the empty set \( \emptyset \), by the isomorphism in (8.1.1), it is done. Otherwise, by symmetry we may assume that \( I \neq \emptyset \). We claim that any path via the vertex \( \emptyset \) is a linear combination of paths \( \delta_{J,\emptyset} Y v_{\emptyset,I} \), where \( Y \) is a path from \( \emptyset \) to \( \emptyset \), and then the lemma again follows from the isomorphism in (8.1.1). Let \( p \) be a path in \( P(J,I)_{\emptyset} \). We may assume that \( p \) is not of the form \( \delta_{J,\emptyset} Y v_{\emptyset,I} \). Write \( p = lr \), where \( r \) is minimal ending at \( \emptyset \). If \( r \in \text{span}(\mathcal{P}^\emptyset_{\emptyset,I}) \), by symmetry, \( p \in \text{span}(\mathcal{P}^\emptyset_{J,I}) \). We
need only to prove the claim for \( r \). By the assumption, there is a hook in \( r \). Write \( r = v_{\emptyset,K} \delta_{K,L} r' \), where \( L \subset K \) and \( |L| \) is minimal. Further the cardinality \( |K| > 1 \), by assumption.

Now by the sandwich relations (if necessary) together with (T1) or (T2) when \( K = 2 \), or with (T4) when \( |K| = 3 \) (i.e. \( K = S \)), the hook \( v_{\emptyset,K} \delta_{K,L} \) is a linear combination \( \sum_Y a_Y Y v_{\emptyset,L} \), where \( a_Y \in \mathbb{Z}[q] \) and \( Y \) is a path from \( \emptyset \) to \( \emptyset \). So

\[
  r = \left( \sum_Y a_Y Y \right) v_{\emptyset,L} r'
\]

Note that for any \( r' \) in the sum, \( v_{\emptyset,L} r' \) is a path ending in \( \emptyset \) and the number of hooks in \( v_{\emptyset,L} r' \) is less than that of \( r \). By induction, the claim hold for each \( v_{\emptyset,L} r' \) and thus for \( r \). This finishes the proof. \( \square \)

**Theorem 8.3.** The ideal \( \mathcal{K} \) is generated by the sandwich relations, quasi-idempotent relations and the torsion relations of type (T1)–(T4). Thus we obtain a presentation of \( E_q(\mathfrak{S}_4) \) over \( \mathbb{Z}[q] \).

Before giving a proof of the theorem, we modify Williamson’s algorithm [29, 1.3.4, 2.2.7] that computes reduced translation sequences and a basis for \( \text{Hom}_{H_4}(x_I H_q, x_J H_q) \) for all \( I \) and \( J \). See also the discussion at the end of Section 6. We then use the modified algorithm to produce a set \( B_{IJ} \) of standard paths from \( I \) to \( J \) in the following list. We will show that those standard paths span the algebra \( A \) in the proof of Theorem 8.3.

\[
B_{1,1} = \{ \delta_{1,1} \}; \quad B_{\emptyset,\emptyset} = \{ \chi_w \mid w \in W \text{ and } w \text{ is a fixed reduced expression of } w \};
B_{1,2},1,2 = \{ e_{1,2}, v_{1,2}, s \delta_{1,2} \};
B_{1,3},1,3 = \{ e_{1,3}, \delta_{1,3}, \delta_{1,3} \alpha \chi_2 v_{1,3}, v_{1,3} s \delta_{1,3} \};
B_{1,2},1,3 = \{ \delta_{1,2}, v_{1,2}, v_{1,3} s \delta_{1,3} \};
B_{1,2},2,3 = \{ \delta_{1,2}, v_{1,2}, v_{2,3} s \delta_{1,3} \};
B_{1,3},1,3 = \{ v_{1,1}, v_{1,2}, v_{1,3} \delta_{1,3}, v_{1,1} s \delta_{1,3} \};
B_{1,2},1,2 = \{ v_{1,1}, v_{1,2}, v_{1,3} \delta_{1,3}, v_{1,1} s \delta_{1,2} \};
B_{1,2},2,3 = \{ v_{1,1}, v_{1,2}, v_{2,3} s \delta_{1,3} \};
B_{1,3},1,3 = \{ v_{1,1}, v_{1,2}, v_{1,3} \delta_{1,3}, v_{1,1} s \delta_{1,3} \};
B_{1,2},1,1 = \{ e_{1,1}, v_{1,1}, v_{1,1} \delta_{1,2}, v_{1,1} s \delta_{1,1} \};
B_{1,2},1,3 = \{ \delta_{1,1}, \delta_{1,3}, v_{1,1} \delta_{1,3}, v_{1,1} s \delta_{1,3} \};
B_{1,2},2,3 = \{ \delta_{1,1}, v_{1,1}, v_{2,3} s \delta_{1,3} \};
B_{1,3},1,3 = \{ \delta_{1,1}, v_{1,1}, v_{1,3} \delta_{1,3}, v_{1,1} s \delta_{1,3} \};
B_{1,2},1,1 = \{ \delta_{1,1}, v_{1,1}, v_{1,1} \delta_{1,2}, v_{1,1} s \delta_{1,1} \};
B_{1,2},1,3 = \{ \delta_{1,1}, v_{1,1}, v_{1,3} \delta_{1,3}, v_{1,1} s \delta_{1,3} \};
B_{1,2},2,3 = \{ \delta_{1,1}, v_{1,1}, v_{2,3} s \delta_{1,3} \};
B_{1,3},1,3 = \{ \delta_{1,1}, v_{1,1}, v_{1,3} \delta_{1,3}, v_{1,1} s \delta_{1,3} \};
B_{1,2},1,1 = \{ \delta_{1,1}, v_{1,1}, v_{1,1} \delta_{1,2}, v_{1,1} s \delta_{1,1} \};
B_{1,2},1,3 = \{ \delta_{1,1}, v_{1,1}, v_{1,3} \delta_{1,3}, v_{1,1} s \delta_{1,3} \};
B_{1,2},2,3 = \{ \delta_{1,1}, v_{1,1}, v_{2,3} s \delta_{1,3} \};
Further we only need to consider non-standard paths that do not contain 

The complete list for the sets of standard paths between any $I$ and $J$ can be obtained by applying (anti-)automorphisms $\sigma$ and $\tau$ to the sets above.

**Proof of Theorem 8.3.** We first claim that any path $\delta_{I,\emptyset} \chi \nu_0 \nu_0 J$ in $\mathcal{P}_{I,J}$ is a linear combination of the standard paths given above. Thus, by Lemma 8.2, any path via the empty set $\emptyset$ is a linear combination of the standard paths. This involves case-by-case checking. To illustrate, we show that

$$
\delta_{\{2\},\{3\}} \nu_0 \chi \chi_{2} \nu_0 \{3\} = \nu_{\{2\}}, s \delta_{S_{\{3\}}} + q \nu_{\{2\}}, \{2,3\} \delta_{\{2,3\},\{3\}} \nu_{\{3\},\{1,3\}} + q \nu_{\{2\},\{1,2\}} \delta_{\{1,2\},\{2\}} \nu_{\{1\},\{1,3\}} + q(q + 1) \delta_{\{2\}, \nu_0 \{1,3\}}.
$$

Indeed,

$$
\text{LHS} = \delta_{\{2\}, \nu_0 \{3\}} \cdot \delta_{\{3\}, \nu_0 \{3\}} \cdot \nu_{\{3\},\{1,3\}}
= \delta_{\{2\}, \nu_0 \{3\}} \cdot \delta_{\{3\}, \nu_0 \{3\}} (\nu_{\{3\}}, \{2,3\} \delta_{\{2,3\},\{3\}} + q e_{\{3\}}) \nu_{\{3\},\{1,3\}}
= \delta_{\{2\}, \nu_0 \{2\}} \nu_{\{2\},\{2,3\}} \delta_{\{2,3\},\{3\}} \nu_{\{3\},\{1,3\}} + q \delta_{\{2\}, \nu_0 \{3\}} \nu_{\{3\},\{1,3\}}
= \nu_{\{2\},\{1,2\}} \cdot \delta_{\{1,2\},\{2\}} \nu_{\{2\},\{2,3\}} \delta_{\{2,3\},\{3\}} \nu_{\{3\},\{1,3\}} + q \nu_{\{2\},\{1,2\}} \delta_{\{2,3\},\{3\}} \nu_{\{3\},\{1,3\}} + q(q + 1) \delta_{\{2\}, \nu_0 \{1,3\}}
+ q(q + 1) \delta_{\{2\}, \nu_0 \{1,3\}}
= \nu_{\{2\},\{1,2\}} \nu_{\{1,2\}} \nu_{\{2\},\{1,3\}} \nu_{\{1,3\}} + q \nu_{\{2\},\{1,2\}} \nu_{\{1,3\}} + q(q + 1) \delta_{\{2\}, \nu_0 \{1,3\}} = \text{RHS}.
$$

Next, by Lemma 7.2, any path from $I$ to $J$ via $S$ is a multiple of $\nu_{I,S} \delta_{S,I}$, which is a standard path. So it suffices to consider those paths that go via neither $S$ nor the empty set $\emptyset$, i.e., paths in the following graph

$$
\begin{align*}
\{1\} & \quad \{2\} & \quad \{3\} \\
\{1,2\} & \quad \{1,3\} & \quad \{2,3\}
\end{align*}
$$

Further we only need to consider non-standard paths that do not contain

1. $\delta_{I,J} \nu_{I,J}$, where $I \subset J$ (and $|I| = 1$). This equals $\pi_s(J) \hat{e}_J$ by the quasi-idempotent relations. So paths containing them can be shortened.
2. the hooks $\nu_{s,t} \delta_{s,t} \nu_{s,t}$ for $\{s, t\} = \{1, 2\}$ or $\{2, 3\}$, $\nu_{\{3\},\{1,3\}} \delta_{\{1,3\},\{1\}}$ and its reverse. As these hooks are standard and paths containing them are a linear
combinations of paths via the empty set $\emptyset$. By the claim above, this means that paths containing the hooks are linear combinations of standard paths.

In other words, we just consider non-standard paths in the above graph that do not contain cycles of length 2 and paths $\{s\} \rightarrow \{1,3\} \rightarrow \{t\}$ ($\{s,t\} = \{1,3\}$). Hence, any such non-standard path must contain

$$r = \delta_{\{1,2\},\{1\}}u_{\{1\}},\{1,3\}\delta_{\{1,3\},\{1\}}u_{\{1\}},\{1,2\} \quad \text{or} \quad t = \delta_{\{1,2\},\{2\}}u_{\{2\}},\{2,3\}\delta_{\{2,3\},\{3\}}u_{\{3\}},\{1,3\},$$

or one of the paths obtained from $r$ and $t$ by symmetry. Now, by relations (T3)–(T4) and the claim above, such a path is a linear combination of standard paths, as required. This finishes the proof. $\square$

### References


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