Category Theoretic Semantics for Logic Programming: Laxness and Saturation

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1 Summary

Recent research on category theoretic semantics of logic programming has focused on two ideas: lax semantics [3] and saturated semantics [1]. Until now, the two ideas have been presented as alternatives, but that competition is illusory, the two ideas being two views of a single, elegant body of theory, reflecting different but complementary aspects of logic programming.

Given a set of atoms $At$, one can identify a variable-free logic program $P$ built over $At$ with a $P_f P_f$-coalgebra structure on $At$, where $P_f$ is the finite powerset functor on $Set$: each atom is the head of finitely many clauses in $P$, and the body of each clause contains finitely many atoms. If $C(P_f P_f)$ is the cofree comonad on $P_f P_f$, then, given a logic program $P$ qua $P_f P_f$-coalgebra, the corresponding $C(P_f P_f)$-coalgebra structure characterises the and-or derivation trees generated by $P$.

Extending this to arbitrary programs, given a signature $\Sigma$ of function symbols, let $L_{\Sigma}$ denote the Lawvere theory generated by $\Sigma$, and, given a logic program $P$ with function symbols in $\Sigma$, consider the functor category $[L_{\Sigma}^{\text{op}}, Set]$, extending the set $At$ of atoms in a variable-free logic program to the functor from $L_{\Sigma}^{\text{op}}$ to $Set$ sending a natural number $n$ to the set $At(n)$ of atomic formulae with at most $n$ variables generated by the function symbols in $\Sigma$ and the predicate symbols in $P$. We would like to model $P$ by a $[L_{\Sigma}^{\text{op}}, P_f P_f]$-coalgebra $p : At \longrightarrow P_f P_f At$ that, at $n$, takes an atomic formula $A(x_1, \ldots, x_n)$ with at most $n$ variables, considers all substitutions of clauses in $P$ into clauses with variables among $x_1, \ldots, x_n$ whose head agrees with $A(x_1, \ldots, x_n)$, and gives the set of sets of atomic formulae in antecedents. However, that does not work for two reasons. The first may be illustrated as follows.

Example 1 ListNat (for lists of natural numbers) denotes the logic program
1. $\text{nat}(0) \leftarrow$
2. $\text{nat}(s(x)) \leftarrow \text{nat}(x)$
3. $\text{list}(\text{nil}) \leftarrow$
4. $\text{list}(\text{cons}(x,y)) \leftarrow \text{nat}(x), \text{list}(y)$

ListNat has nullary function symbols $0$ and $\text{nil}$. So there is a map in $L_{\Sigma}$ of the form $0 \rightarrow 1$ that models the function symbol $0$. Naturality of $p : At \longrightarrow P_f P_f At$ in $[L_{\Sigma}^{\text{op}}, Set]$ would yield commutativity of the

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But consider \( \text{nat}(x) \in \text{At}(1) \): there is no clause of the form \( \text{nat}(x) \leftarrow \text{ListNat} \), so commutativity of the diagram would imply that there cannot be a clause in ListNat of the form \( \text{nat}(0) \leftarrow \) either, but in fact there is one. Thus \( p \) is not a map in the functor category \( [\mathcal{L}_\Sigma^{op}, \text{Set}] \).

Lax semantics addresses this by relaxing the naturality condition on \( p \) to a subset condition, so that, given, for instance, a map in \( \mathcal{L}_\Sigma \) of the form \( f : 0 \rightarrow 1 \), the diagram need not commute, but rather the composite via \( P_f \) need only yield a subset of that via \( P \).

In contrast, saturation semantics works as follows. Regarding \( \text{ob}(\mathcal{L}_\Sigma) \), equally \( \text{ob}(\mathcal{L}_\Sigma)^{op} \), as a discrete category with inclusion functor \( I : \text{ob}(\mathcal{L}_\Sigma) \rightarrow \mathcal{L}_\Sigma \), the functor

\[
[I, \text{Set}] : [\mathcal{L}_\Sigma^{op}, \text{Set}] \rightarrow [\text{ob}(\mathcal{L}_\Sigma)^{op}, \text{Set}]
\]

that sends \( H : \mathcal{L}_\Sigma^{op} \rightarrow \text{Set} \) to the composite \( HI : \text{ob}(\mathcal{L}_\Sigma)^{op} \rightarrow \text{Set} \) has a right adjoint \( R \), given by right Kan extension. So the data for \( p : \text{At} \rightarrow P_f P_f \text{At} \) may be seen as a map in \( [\text{ob}(\mathcal{L}_\Sigma)^{op}, \text{Set}] \), which, by the adjointness, corresponds to a map \( \bar{p} : \text{At} \rightarrow R(P_f P_f \text{At}) \) in \( [\mathcal{L}_\Sigma^{op}, \text{Set}] \), yielding saturation semantics. In this talk, we show that the two approaches can elegantly be unified, the relationship corresponding to the relationship between theorem proving and proof search in logic programming.

The second problem mentioned above is about existential variables, which we now illustrate.

**Example 2** GC (for graph connectivity) denotes the logic program

1. \( \text{connected}(x,x) \leftarrow \)
2. \( \text{connected}(x,y) \leftarrow \text{edge}(x,z), \text{connected}(z,y) \)

There is a variable \( z \) in the tail of the second clause of GC that does not appear in its head. Such a variable is called an existential variable, the presence of which challenges the algorithmic significance of lax semantics. In describing the putative coalgebra \( p : \text{At} \rightarrow P_f P_f \text{At} \) just before Example[1] we referred to all substitutions of clauses in \( P \) into clauses with variables among \( x_1, \ldots, x_n \) whose head agrees with \( A(x_1, \ldots, x_n) \). If there are no existential variables, that amounts to term-matching, which is algorithmically efficient; but if existential variables do appear, the mere presence of a unary function symbol generates an infinity of such substitutions, creating algorithmic difficulty, which, when first introducing lax semantics, we, also Bonchi and Zanasi, avoided modelling by replacing the outer instance of \( P_f \) by \( P_c \), thus allowing for countably many choices. Such infiniteness militates against algorithmic efficiency, and we resolve it by refining the functor \( P_f P_f \) while retaining finiteness.

This talk is based upon the paper [2].

**References**


[2]