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Nominal rigidities equilibria in a non-Ricardian economy [1]

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Abstract

I consider a cash-in-advance economy with nominal price rigidities. Nominal interest rates are the cost of liquidity and fiscal policy sets nominal transfers that affect the distribution of wealth. Under a fiscal policy associated with an unequal distribution of wealth and for policies of low or even zero interest rates, coordination failures exist, that is, involuntary unemployment persist even if prices are set at full employment levels. Coordination failures exist if and only if nominal rates are below a threshold. Moreover, I demonstrate the following result on welfare: full employment allocations at a nominal rate equal to the threshold (high liquidity costs) are better, in terms of welfare, from unemployment allocations at any non-negative interest rates below the threshold. On the other hand, under a *sufficiently* progressive fiscal system that reduces the inequality in the wealth distribution, coordination failures do not exist.

1 Introduction

It is well known that nominal price rigidities can be associated with inefficiently low aggregate demand and involuntary unemployment. A usual argument is that monetary policies that set low nominal interest rates can stimulate aggregate demand and economic activity, as borrowing costs fall, and reduce unemployment. There is a limitation to this argument. I demonstrate that the effectiveness of monetary policy depends, crucially, on the type of the fiscal system. Under fiscal systems associated with large inequalities in the wealth distribution, low or zero interest rate policies may not be effective whereas under fiscal systems that reduce wealth inequalities, low nominal rates can be effective.

Here, in a cash-in-advance economy, nominal interest rates are the cost of liquidity and fiscal policy sets lump-sum transfers affecting the distribution of wealth. Under a fiscal system associated with an unequal distribution of wealth and at low or even zero nominal interest rates, coordination failures exist, that is, unemployment persist even if prices are set at full employment levels. Coordination failures exist if and only if nominal rates are below a threshold. Moreover, full employment allocations for high nominal rates (high liquidity costs) are better, in terms of welfare, from unemployment allocations for low or even zero nominal rates. Finally, a sufficiently progressive fiscal system suffices to "eliminate" coordination problems.

Nominal price rigidities are modelled by setting, exogenously, the price of goods. Arbitrary prices are associated with excess capacities in the market, i.e., excess supplies or excess demands. Prices associated with zero excess capacities are denoted as competitive prices and the respective allocations as competitive allocations. Nominal rigidities equilibria are classified into two types: Keynesian equilibria characterised by demand-determined output and involuntary unemployment and demand rationing equilibria where agents are constrained on demand.

A monetary-fiscal authority prints money balances, issues nominal bonds and distributes its revenue to households with transfers. Monetary policy sets nominal interest rates and fiscal policy specifies transfers. A fiscal policy is called "Ricardian" if it guarantees that the monetary-fiscal authority's terminal conditions are satisfied for all possible, equilibrium or non-equilibrium, values of the endogenous variables. This is equivalent to say that the monetary-fiscal authority *has* a budget constraint. This policy does not add enough restrictions for determinacy of equilibrium. If, however, fiscal policy is not of this form, it may add additional restrictions and determinacy may obtain. The monetary-fiscal authority's budget, in that case, is satisfied *only* for equilibrium values of the endogenous variables.

I specify a non-Ricardian fiscal policy to get determinacy. What type of non-Ricardian policy to specify is not obvious; various specifications can be found in the literature. Heterogeneity among households will depend on

whether households are rationed on the labor market or not, i.e., households can be either employed or fully unemployed. I consider a policy that sets nominal transfers to households depending on employment status. This policy, and what is important for the analysis, is equivalent to heterogeneity in non-labor *net* wealth of households. Higher transfers to unemployed households reflect a *sufficiently* progressive fiscal system that subsidises and taxes appropriately and manages to reduce the inequality in the wealth distribution. On the other hand, Auerbach et al. (2016) argued that in the U.S., where the fiscal system is highly progressive, financial wealth is distributed very unequally in the population, labor earnings are more equally distributed relative to financial wealth, high labor earning groups hold most of the financial wealth in the economy, and the inequality in the overall spending distribution[1] is large. The important contribution of financial wealth inequality to the overall spending inequality is captured here by assuming that non-labor net income of employed is higher relative to unemployed households that, in turn, reflects a progressive fiscal system that is associated with large inequalities in the overall distribution of wealth.

The non-Ricardian policy imposes restrictions on the equilibrium system that affect the qualitative properties of nominal rigidities equilibria. I characterise restrictions on policy parameters under which nominal rigidities allocations are far from competitive allocations, for prices close or equal to competitive prices.

Excess capacities under demand rationing equilibria vanish at competitive prices. Under the specific non-Ricardian policy, and irrespective of the characterisation of monetary policy, demand rationing equilibria can not be supported at competitive prices because the monetary-fiscal authority's budget is always violated.

The qualitative features of Keynesian equilibria are in sharp contrast to demand rationing. Under a policy of higher transfers to unemployed households, excess capacities vanish at competitive prices, or, equivalently, a Keynesian equilibrium can not be supported at competitive prices. On the other hand, under a fiscal system associated with a very unequal distribution of wealth, at low or even zero nominal rates, excess capacities do not vanish and one Keynesian equilibrium exists at competitive prices whereas at high nominal rates, excess capacities vanish. A threshold determines high and low nominal rates. It is derived from a relation (inequality) between nominal seignorage profits (revenue of the monetary authority), at competitive prices, and a cut-off point that depends on nominal transfers (liabilities). Seignorage revenue goes to zero with the interest rate. Non-vanishing excess capacities are equivalent to seignorage profits below the cut-off point. As interest rate increases, nominal price inflation drives the increase of seignorage revenue. Vanishing excess capacities is equivalent to

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[1]Defined as financial wealth plus labor earnings plus net taxes.

seignorage revenues equal or above the cut-off point.

Finally, setting prices at competitive values, I demonstrate that aggregate welfare under the competitive allocation for an interest rate equal to the threshold (coordination frictions do not exist), is greater than aggregate welfare under the Keynesian allocation for any non-negative interest rate below the threshold. An interesting trade-off emerges. Welfare under the competitive allocation is a decreasing function of the liquidity cost. On the other hand, under the Keynesian allocation, low liquidity costs are beneficial for employed households but are associated with high unemployment rates. The intuition behind the result is that at low or zero nominal rates, the unemployment rate is high enough such that the Keynesian allocation is worse, in terms of aggregate welfare, from the competitive allocation.

1.1 Literature

Money is modelled with cash-in-advance constraints. Notable contributions, among others, to the literature include: Lucas and Stokey (1987), Woodford (1994), Dubey and Geanakoplos (2003) and Nakajima and Polemarchakis (2005). The monetary economy is a one-period version of Nakajima and Polemarchakis (2005).

The non-Ricardian specification is closest to Dubey and Geanakoplos (2003) where agents are endowed with outside money. Other examples of non-Ricardian policies considered by Woodford (1994), Woodford (1996), Benhabib et al. (2002) and Benhabib et al. (2001) is to set real transfers and the composition of the monetary-fiscal authority's portfolio.

The modelling of nominal rigidities equilibria follows the fix-price literature. Notable contributions, among others, include: Barro and Grossman (1971), Benassy (1975), Dreze (1975), Malinvaud (1977), Laroque (1978), Silvestre (1982) and Herings (2014). Money is modelled as an argument of the utility function as in the static frameworks of Malinvaud (1977) and Laroque (1978) or as a store of value, and an argument of the indirect utility function, that links periods as in Benassy (1975). My motivation was to model money as *fiat* money, with cash-in-advance constraints. Following this approach, the specification of fiscal policy is *necessary* because revenues of the monetary-fiscal authority have to be distributed to agents and also, it is *crucial* for determinacy of equilibrium. Laroque (1978), in an environment without policy, argued that local unicity is not warranted, that is, there may exist price systems close to competitive prices, for which fix-price allocations are far from competitive allocations. He characterised conditions for local uniqueness or, equivalently, for fix-price allocations to be written as a function of the price system. Here, monetary and non-Ricardian policies impose restrictions under which Keynesian allocations are far from competitive allocations for prices close to competitive prices. Benassy (1975) did not consider policy. On the other hand, Malinvaud (1977) incorporated fiscal

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policy. A government finances real spending by taxing firm's profits and by printing money balances. The government *has* a budget constraint and as a result, fiscal policy does not add additional restrictions to the equilibrium system. He considered various comparative static experiments. Specific functional forms, rather than the specification of fiscal policy, guaranteed well-defined equilibria close to competitive prices, i.e., allocations can be written as a function of prices.

Silvestre (1980) and Citanna et al. (2001), in environments without policy, demonstrated existence of equilibria with Keynesian features as a result of coordination failures. Herings (2014) characterised sticky-price equilibria in an economy with a transaction technology for money that incorporates cash-in-advance as a special case, monetary policy sets the nominal interest rate and importantly, fiscal policy is Ricardian.

Unemployment is not a result of search and matching frictions as in Blanchard and Galí (2010) and the literature over there. On the contrary, unemployment is either a result of prices associated with excess supply in the labor market or a result of coordination failures. Moreover, following Malinvaud (1977), rationing on the labor market takes the following form: a fraction of households are excluded from the labor market and are fully unemployed.

The paper is organised as follows. Section 2 presents the economy. Section 3 discusses some features of nominal rigidities equilibria. Section 4 presents the main characterisation. Section 5 presents the welfare result. All derivation are in the "Appendix".

2   Economy

I consider a two-period economy with cash-in-advance constraints and nominal price rigidities; the second period is added *only* for accounting purposes. There are three commodities: consumption, labor and money. Money is numéraire. There is a continuum $[0, 1]$ of households, denoted by i, with identical preferences. Households are subject to cash-in-advance constraints. A firm employs labor, produces aggregate output but is not subject to cash-in-advance constraints. Finally, a monetary-fiscal authority prints money balances, issues nominal bonds, collects firm's profits and distributes its revenue to households. Monetary policy sets the nominal interest rate on bonds.

Nominal price rigidities are modelled by fixing exogenously the price of consumption and labor. At arbitrary prices, equality between buyers demands and sellers supplies may not be achieved. A rationing mechanism allocates goods between buyers and sellers. In particular, agents face quantity constraints on the amount of goods they trade and, consequently, they form their constrained demands and supplies. Equilibrium is characterised

by equality of constrained demands and supplies; quantities rather than prices adjust to balance markets.

Assumptions about rationing follow Malinvaud. Rationing of households on the labor market takes the following form: the household is either not rationed on the supply of labor (employed) or is fully unemployed. Rationing of households on the good's market is assumed to operate in a way that only employed households are constrained. Rationing of the firm on the good's market is modelled with a constraint on output sales and on the labor market with a constraint on labor demand.

I analyse three market regimes[2]: Keynesian regime where households and the firm are constrained on supply, classical regime where employed households are rationed on demand and unemployed on labor supply and excess demand regime where households are rationed on demand and the firm on labor demand.

The assumption below is made throughout the paper.

Assumption 1. (Non-Ricardian specification). Nominal transfers, δ^i, are specified as follows:

$$\delta^i = \begin{cases} \tau, & \text{if } i \text{ employed;} \\ \widehat{\tau}, & \text{if } i \text{ unemployed;} \end{cases}$$

$0 < \tau, \widehat{\tau} < \infty$ and $\tau \neq \widehat{\tau}$.

Fiscal policy sets nominal transfers that depend on the employment status of households. Determinacy of equilibria derives *only* from the fact that τ and $\widehat{\tau}$ are exogenously specified. Construction of the classical regime requires $\tau \neq \widehat{\tau}$[3]. I consider all other cases: $\tau > \widehat{\tau}$ or $\tau < \widehat{\tau}$.

Firm's profits accrue to the monetary-fiscal authority and are distributed to households with transfers. This assumption implies that non-labor net wealth of households is exogenous that, in turn, it allows me to characterise necessary and sufficient conditions for existence (non-existence) of Keynesian equilibria at competitive prices, that I can not do otherwise. Nevertheless, in a remark in section 4, I demonstrate that sufficient conditions for existence of Keynesian equilibria at competitive prices do not depend on this assumption.

The non-Ricardian specification is crucial. I provide an interpretation. According to the assumption on profits, τ and $\widehat{\tau}$ represent non-labor *net* wealth of employed and unemployed households, respectively. The distinction between employed and unemployed can be thought of as a proxy for the "rich" and the "poor", respectively, in the population, e.g., unemployed households might have been out of work for a prolonged period of time and rely exclusively on unemployment subsidies. A $\widehat{\tau} > \tau$ policy can be associated with a *sufficiently* progressive fiscal system that subsidies unemployed,

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[2]The names follow Malinvaud.
[3]This is an artefact of the set-up; I explain this point in the "Appendix".

taxes employed households and reduces the inequality in the overall distribution of wealth (although employed's wealth is higher because of labor income). Under $\widehat{\tau} > \tau$, coordination failures do not exist. On the other hand, a $\widehat{\tau} < \tau$ policy can be associated with a progressive fiscal system that does not manage to reduce the inequality in the overall distribution of wealth because financial wealth is very unequally distributed (the case of the U.S.). Under $\widehat{\tau} < \tau$, coordination failures may exist.

2.1 Households

Preferences are

$$U^i = \ln(c^i) + \ln(1 - l^i);$$

c stands for consumption and l for labor supply. Also, households are endowed with one unit of time.

The timing of transactions is as follows. The asset market opens first where households acquire money balances to buy goods. Subsequently, the good's and labor market open. At the beginning of the second period households redeem their debts. First I present the monetary transactions and then incorporate quantity constraints into the household's decision.

At the asset market, households receive transfers, trade nominal bonds with the monetary-fiscal authority and acquire money balances. The budget constraint of the household in the asset market is

$$m^i \le \delta^i + \frac{1}{1 + r} b^i; \tag{1}$$

m stands for money balances, b for nominal bonds and r for the nominal interest rate on bonds.

The market for the consumption good and labor open next. The purchase of the consumption good is subject to the cash-in-advance constraint

$$pc^i \le m^i; \tag{2}$$

p stands for the price of consumption. The household receives money by supplying labor to the firm. Money balances at the end of the period are

$$n^i = m^i - pc^i + wl^i; \tag{3}$$

w stands for the nominal wage and n for money at the end of the period. Lastly, households redeem their debts, i.e.,

$$b^i \le n^i. \tag{4}$$

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(4) is the transversality condition.

Substituting (3) into (2), the cash-in-advance constraint can be written as

$$n^i \geq wl^i. \tag{5}$$

Combining (1),(3) and (4), the budget constraint of the household is

$$pc^i + \frac{r}{1+r}n^i \leq \delta^i + wl^i. \tag{6}$$

The cash constraint (5) can be written as

$$\frac{r}{1+r}n^i = \frac{r}{1+r}wl^i,$$

because if $r > 0$, the cash constraint binds; if $r = 0$, the cash constraint does not bind and both sides of the equation are equal to zero. Substituting this into (6), the budget constraint modifies as

$$pc^i \leq \delta^i + \frac{w}{1+r}l^i.$$

The budget constraint should bind at an optimum; that is, the transversality condition is

$$b^i = n^i.$$

Households face quantity constraints on the labor and the good's market that are represented by nonnegative upper bounds $\bar{l}^i$ and $\bar{c}^i$, respectively.

The choice of each i is

$$\max_{c>0,\ 0 \leq l < 1}\ U^i,$$

s.t.

$$pc^i = \delta^i + \frac{w}{1+r}l^i,$$

$$l^i \leqslant \bar{l}^i,\ c^i \leqslant \bar{c}^i. \tag{7}$$

Given $p, w, r, \delta^i, \bar{l}^i, \bar{c}^i$, the household forms the constrained demands and supplies that solve (7).

2.2 Firm

The firm employs labor, L, produces output, y, operating a decreasing returns to scale technology $f(L) = \sqrt{L}$, and maximises profits π. Constraints

on the sales of output and on labor demand are represented by nonnegative upper bounds $\overline{y}$ and $\overline{L}$, respectively.

The choice of the firm is

$$\max_{y \geq 0, \ L \geq 0} \ \pi = py - wL,$$

s.t.

$$y = \sqrt{L},$$ \hfill (8)

$$y \leqslant \overline{y}, \ L \leqslant \overline{L}.$$

Given $p, w, \overline{y}, \overline{L}$, the firm forms the constrained demands and supplies that solve (8).

2.3 Monetary-fiscal authority

The budget constraint of the monetary-fiscal authority in the asset market is

$$M = T + \frac{1}{1+r}B;$$

M, B and T stand for aggregate money balances, aggregate nominal bonds and aggregate nominal transfers to households, respectively. Subsequently, the monetary-fiscal authority receives revenue from bond's issuance and from collecting firm's profits. The terminal condition at the beginning of period two is

$$M = B + \pi.$$

Substituting into the asset market budget constraint, I obtain the overall budget

$$\frac{r}{1+r}M + \frac{1}{1+r}\pi = T,$$

where revenue consists of seignorage and firm's profits and liabilities consist of transfers to households. The non-Ricardian policy guarantees that the terminal condition or, equivalently, the overall budget is satisfied *only* for equilibrium values of the endogenous variables. T is equal to the sum of τ and $\widehat{\tau}$. The sum is taken with respect to the measure of households rationed on the labor market; see the next section.

Monetary policy sets the nominal interest rate at nonnegative values[4]: $0 \leqslant r < \infty$; assumption 1 provides a characterisation of fiscal policy. Finally, I denote the nominal interest rate and nominal transfers as *policy parameters*.

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[4]Because of no-arbitrage considerations.

2.4 Equilibrium

To define an equilibrium, I need to specify precisely the rationing mechanism. As I demonstrate shortly, an agent can be rationed only on one market at a time. Thus, an agent is rationed or constrained on a market if the unconstrained schedule (optimal supply or demand under no restrictions) exceeds the respective upper bound.

Rationing of households on the labor market takes the following form: a household is either employed or is fully unemployed ($\bar{l}^i = 0$). I denote the measure of unemployed households by u; u is a proxy for the unemployment rate. Rationing on the good's market is assumed to operate in a way that only employed households are constrained and face an identical upper bound $\bar{c}$. The fact that only employed households are demand rationed simplifies the analysis. An intuition behind this assumption is as follows. The good's market opens for a fixed amount of time and households visit different shops to buy goods. An employed household splits into a shopper-worker whereas all members of an unemployed household visit the shops. I assume that the shopper of an employed household, alone, is not as efficient as all the members of an unemployed household to visit all the required shops and, hence, to buy all the "desired" consumption by the time the good's market closes. As a result, employed households are demand rationed whereas unemployed households are not.

The firm is constrained on sales by $\bar{y}$ and on labor demand by $\bar{L}$. A situation where the firm is constrained on both markets never occurs. Constrained demand and supply must belong to the production function. Dropping one constraint will not alter production possibilities and the firm is not constrained on that market.

An equilibrium is defined as follows:

Definition 1. A nominal price rigidities equilibrium at $(\tau, \hat{\tau}, r, w, p)$ consists of quantity constraints, allocations and portfolios of households, $\{\bar{c}^i, \bar{l}^i, c^i, l^i, m^i, b^i\}$, $i \in [0, 1]$, quantity constraints and allocations of the firm, $\{\bar{y}, \bar{L}, y, L\}$ and a portfolio of the monetary-fiscal authority, $\{M, B\}$, such that

1. given prices, policy parameters and quantity constraints, the household's problem (7) is solved by c^i, l^i, m^i, b^i;

2. given prices, technology and quantity constraints, the firm's problem (8) is solved by y, L;

3. the monetary authority accommodates the demand for money and bonds, $\int_0^1 m^i di = M$ and $\int_0^1 b^i di = B$;

4. good's and labor market clear, $\int_0^1 c^i di = y$ and $\int_0^1 l^i di = L$;

5. the market is in one of the following regimes:

(a) Keynesian: $l^i = \bar{l}^i = 0$, $c^i < \bar{c}^i$ for unemployed households, $l^i < \bar{l}^i$, $c^i < \bar{c}$ for employed households and $y = \bar{y}$, $L < \bar{L}$;

(b) Classical: $l^i = \bar{l}^i = 0$, $c^i < \bar{c}^i$ for unemployed households, $l^i < \bar{l}^i$, $c^i = \bar{c}$ for employed households and $y < \bar{y}$, $L < \bar{L}$;

(c) Excess demand: $l^i < \bar{l}^i$, $c^i = \bar{c}$ for $i \in [0,1]$ and $y < \bar{y}$, $L = \bar{L}$.

Conditions 1 and 2 require that agent's behaviour is optimal. Condition 3 requires that the monetary authority accommodates the demand for money and bonds. Condition 4 requires market clearing. Condition 5 describes the rationing mechanism; equalities imply agents are rationed on that market whereas strict inequalities imply absence of rationing. Rationing is one-sided: if the supply is constrained in a market, then the demand is not constrained in that market and vice versa. Finally, satisfaction of the monetary-fiscal authority budget follows from conditions 1-5.

I explain the details behind definition 1. To do that, it is necessary first to present the optimal decisions of agents under no rationing. According to (7), the optimal decision of each household is

$$c^i = \frac{1}{2p}\left(\tau + \frac{w}{1+r}\right), \quad l^i = 1 - \frac{1+r}{2w}\left(\tau + \frac{w}{1+r}\right). \tag{9}$$

(9) represents the unconstrained schedule of each household. According to (8), the optimal decision of the firm is

$$y = \frac{p}{2w}, \quad L = \frac{p^2}{4w^2}, \quad \pi = \frac{p^2}{4w}, \tag{10}$$

where (10) represents the unconstrained schedule of the firm.

Under a Keynesian equilibrium, the optimal decision of each employed household is given by (9) whereas unemployed are excluded from the labor market and their consumption is equal to $c^i = \hat{\tau}/p$. The firm produces up to $y = \bar{y}$, labor demand is determined by $\bar{y}$ and the technological possibilities, that is, it is equal to $\bar{y}^2$, and profits are $\pi = p\bar{y} - w\bar{y}^2$. Rationing on the good's market requires

$$\frac{p}{2w} > \bar{y}; \tag{11}$$

unconstrained supply exceeds $\bar{y}$. The measure of unemployed households must satisfy $0 < u < 1$; $u = 0$ is not consistent with rationing on the labor market and $u = 1$ is not consistent with good's market clearing because unemployed households demand a positive amount of the good but aggregate output is zero if aggregate labor supply is zero. At given policy parameters and prices, a Keynesian equilibrium consists of a measure u and a sales constraint $\bar{y}$ such that markets clear, (11) is satisfied and $0 < u < 1$. Finally, aggregate transfers of the monetary-fiscal authority are $T = (1-u)\tau + u\hat{\tau}$.

Under a classical equilibrium, employed households consume $c^i = \bar{c}$, labor supply is determined from the budget constraint and decisions of unemployed households are as before. Rationing on the good's market requires

$$\frac{1}{2p}(\tau + \frac{w}{1+r}) > \bar{c}; \tag{12}$$

unconstrained demand exceeds $\bar{c}$. The firm is not constrained and its decisions are given from (10). At given policy parameters and prices, a classical equilibrium consists of a pair $(\bar{c}, u)$ such that markets clear, (12) is satisfied and $0 < u < 1$. As before, aggregate transfers are $T = (1-u)\tau + u\hat{\tau}$.

Under an excess demand equilibrium, there are no unemployed households and optimal decisions of (employed) households are as in the classical equilibrium; condition 5(c) of definition 1 requires all households to consume $\bar{c}$, that, in turn, is consistent with the assumption on demand rationing. The firm demands labor up to $L = \overline{L}$, output is determined by $\overline{L}$ and the technological possibilities, that is, it is equal to $y = \sqrt{\overline{L}}$, and profits are $\pi = p\sqrt{\overline{L}} - w\overline{L}$. Rationing on the labor market requires

$$\frac{p^2}{4w^2} > \overline{L}; \tag{13}$$

unconstrained demand exceeds $\overline{L}$. At given policy parameters and prices, an equilibrium consists of a pair $(\bar{c}, \overline{L})$ such that markets clear and (12) and (13) are satisfied. Finally, aggregate nominal transfers are $T = \tau$ because of full employment.

The benchmark case is the competitive equilibrium where agents are not rationed (quantity constraints are not binding) and prices adjust to clear markets. I state the definition of competitive equilibrium without reference to quantity restrictions to point out that markets clear by price rather than quantity adjustments.

A competitive equilibrium is defined as follows:

Definition 2. A competitive equilibrium at $(\tau, \hat{\tau}, r)$ consists of prices, $\{w^*, p^*\}$, allocations and portfolios of households, $\left\{c^{*i}, l^{*i}, m^{*i}, b^{*i}\right\}$, $i \in [0,1]$, allocations of the firm, $\{y^*, L^*\}$ and a portfolio of the monetary-fiscal authority, $\{M^*, B^*\}$, such that

1. given prices and policy parameters, the household's problem (7) is solved by $c^{*i}, l^{*i}, m^{*i}, b^{*i}$;

2. given prices and the technology constraint, the firm's problem (8) is solved by y^*, L^*;

3. the monetary authority accommodates the demand for money and bonds, $\int_0^1 m^{*i} di = M^*$ and $\int_0^1 b^{*i} di = B^*$;

4. good's and labor market clear, $\int_0^1 c^{*i} di = y^*$ and $\int_0^1 l^{*i} di = L^*$.

Intuition behind conditions 1 to 4 is as before. The optimal decisions of households are given from (9) and (10). I denote competitive equilibrium values with a superscript $*$.

3 Preliminaries

For ease of exposition, I present some useful results before the main characterisation. I discuss existence of equilibria and sketch some of the qualitative features of each type of equilibrium; detailed derivations can be found in sections 6.1 and 6.2 of the "Appendix". Restrictions on policy parameters satisfy either $\tau < \widehat{\tau}$ and $0 < r < \infty$ or $\tau > \widehat{\tau}$ and $0 < r < \infty$. For $\tau < \widehat{\tau}$, it is guaranteed that the employed household's allocation is individually rational, that is, it provides higher utility compared to the utility that can be achieved with the unemployed's allocation[5]. In the opposite case, employed households may prefer to be unemployed and an equilibrium may not exist (the firm can not hire labor, output is zero, unemployed's demand is positive, and, as a result, market clearing is violated). On the other hand, for any $\tau > \widehat{\tau}$, any solution to the maximisation problem (7) satisfies individual rationality.

The following proposition characterises the competitive equilibrium:

Proposition 1. *(Competitive equilibrium). The competitive equilibrium is*

$$p^* = \frac{2\tau(1+r)\sqrt{3+2r}}{1+2r}, \quad w^* = \frac{\tau(1+r)(3+2r)}{1+2r},$$
$$y^* = c^* = \frac{1}{\sqrt{3+2r}}, \qquad L^* = l^* = \frac{1}{3+2r}.$$

Proof. See section 6.1 of the "Appendix". □

(w^*, p^*) are the competitive prices. Monetary and fiscal policy impose restrictions on competitive prices. The real allocation depends negatively on r; r is the cost of liquidity. For $\tau < \widehat{\tau}$, policy parameters are restricted such that the competitive allocation is individually rational, e.g, given r, $\widehat{\tau}$ close to τ suffices.

Policy parameters are set such that the competitive allocation is individually rational. Prices associated with a given type of equilibrium must satisfy the respective inequalities discussed in section 2.4. Moreover, since the focus of the paper is on whether allocations are close to or far from competitive allocations for prices close to (w^*, p^*), I restrict prices in a neighbourhood

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[5]Unemployment and, hence, transfers $\widehat{\tau}$, is the only outside option for employed households.

of (w^*, p^*). There exist a neighbourhood and prices in that neighbourhood associated with a given type of equilibrium.

Under an excess demand equilibrium, allocations are uniquely determined up to policy parameters and prices. As prices converge to competitive prices, allocations converge to competitive allocations. At (w^*, p^*), excess demands vanish, that is, they are equal to zero. A similar argument applies under a classical equilibrium. On the other hand, the qualitative features of Keynesian equilibria are in sharp contrast to the other equilibrium types. Given restrictions on policy parameters associated with individually rational competitive allocations, there always exist prices associated with a Keynesian equilibrium that is characterised by vanishing excess supplies. In particular, $\tau < \widehat{\tau}$, $0 < r < \infty$ and prices can *only* be associated with one equilibrium of the Keynesian type, characterised by vanishing excess supplies. On the other hand, $\tau > \widehat{\tau}$, $0 < r < \infty$ and prices may be associated (at most) with *two* equilibria of the Keynesian type, characterised by vanishing and non-vanishing excess supplies, respectively.

Finally, multiple equilibria exist. There exist prices associated with more than one type of equilibrium. Multiplicity obtains for any configuration of policy parameters discussed at the beginning of this section. Since the focus of the paper is on the qualitative properties of each type of equilibrium close to competitive prices, I elaborate more on multiplicity in section 6.2 of the "Appendix".

4   Characterisation

Section 4.1 demonstrates that under an excess demand or a classical type of equilibrium, excess capacities vanish at competitive prices and section 4.2 that a Keynesian type of equilibrium may exist at competitive prices.

4.1   Demand rationing

Consider the following proposition.

Proposition 2. *(Demand rationing). Under an excess demand or a classical type of equilibrium, excess capacities vanish at (w^*, p^*) or, equivalently, demand rationing equilibria can not be supported at (w^*, p^*).*

Proof. See section 6.2 of the "Appendix". □

The proof of proposition 2 is straightforward. I compute equilibria explicitly and demonstrate that as prices converge to (w^*, p^*), excess capacities converge to zero. At (w^*, p^*), allocations equal competitive allocations and demand rationing equilibria do not exist at competitive prices. This result obtains for any configuration of policy parameters discussed in the previous section.

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Non-existence of demand rationing equilibria at (w^*, p^*) derives from the specific fiscal policy rule I have postulated and does not depend on the characterisation of monetary policy. The intuition behind this result is that the fiscal policy rule does not take into account the possibility of demand rationing in the market. To see this, consider the (overall) budget constraint of the monetary-fiscal authority, i.e.,

$$\frac{r}{1+r}M + \frac{1}{1+r}\pi = T. \tag{14}$$

Liabilities under excess demand and competitive equilibrium are $T = \tau$. An excess demand equilibrium can not be supported at competitive prices because (14) is violated, that is, liabilities equal revenue at the competitive equilibrium that, in turn, is not equal to revenue under demand rationing. (14) is trivially violated. Under a classical equilibrium, liabilities are equal to $T = (1-u)\tau + u\hat{\tau}$. As before, (14) is violated at competitive prices. Combining good's and money market clearing and no rationing of the firm, it follows that $M = p^* y^*$ and $\pi = \pi^*$, respectively. (14) is satisfied only for $u = 0$ or, equivalently, is violated for any $0 < u < 1$.

The previous result is not modified if, in addition to nominal transfers, the monetary-fiscal authority finances exogenous real fiscal spending g, at prices p. As before, non-existence at competitive prices is equivalent to violation of (14). Moreover, relaxing the assumption on profits by assuming that profits are distributed to households will not modify the result. As before, under the specific fiscal policy rule, (14) is always violated at competitive prices. On the other hand, existence of demand rationing equilibria at competitive prices would require a fiscal policy that takes into account, also, demand rationing in the market. In that case, liabilities of the monetary-fiscal authority could adjust accordingly to allow satisfaction of the monetary-fiscal authority's budget and demand rationing might exist at competitive prices. Specification of such a rule that is simple, intuitive and guarantees existence of demand rationing equilibria is not obvious.

4.2 Keynesian equilibrium

Contrary to the previous result, a Keynesian equilibrium might exists at competitive prices. The following proposition provides a characterisation:

Proposition 3. *(Keynesian equilibrium).*

1. *Suppose $\hat{\tau} > \tau$ and $0 < r < \infty$. Excess supplies vanish at (w^*, p^*) or, equivalently, Keynesian equilibria can not be supported at (w^*, p^*).*

2. *Suppose $\hat{\tau} < \tau < 2\hat{\tau}$.*

    (a) *Suppose $0 < r < (\tau - \hat{\tau})/(2\hat{\tau} - \tau)$. There exists a Keynesian equilibrium characterised by non-vanishing excess supplies at (w^*, p^*).*

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(b) Suppose $(\tau - \widehat{\tau})/(2\widehat{\tau} - \tau) \leqslant r < \infty$. Excess supplies vanish at (w^*, p^*) or, equivalently, Keynesian equilibria can not be supported at (w^*, p^*).

3. Suppose $\tau \geq 2\widehat{\tau}$ and $0 < r < \infty$. There exists a Keynesian equilibrium characterised by non-vanishing excess supplies at (w^*, p^*).

Proof. See section 6.2 of the "Appendix". $\square$

Since existence has been discussed in the previous section (preliminaries), proposition 3 states only the qualitative properties of Keynesian equilibria for different configurations of policy parameters. To determine if Keynesian equilibria exist at competitive prices or, equivalently, if they are characterised by non-vanishing excess supplies, I proceed as follows. I compute the equilibrium explicitly as a function of prices and policy parameters and subsequently, I set prices at (w^*, p^*) and look for conditions on policy parameters under which a Keynesian equilibrium satisfies definition 1 (conditions 1-5(a)). Under case 1 or case 2(b) and $(\tau - \widehat{\tau})/(2\widehat{\tau} - \tau) < r < \infty$, prices in a neighbourhood of (w^*, p^*) are associated with only one equilibrium of the Keynesian type, characterised by vanishing excess supplies. Under case 2(a) or 3, prices in a neighbourhood of (w^*, p^*) can be associated with two equilibria of the Keynesian type, characterised by vanishing and non-vanishing excess supplies, respectively. Finally, case 2(b) and $r = (\tau - \widehat{\tau})/(2\widehat{\tau} - \tau)$ is a special case on policy parameters. At any prices associated with an equilibrium, the equilibrium is characterised by vanishing excess supplies; there exist prices associated with a Keynesian equilibrium.

To explain the intuition behind existence of equilibria at competitive prices, I proceed as follows. Prices are fixed at (w^*, p^*), for the rest of this section, and as before, I use the budget constraint of the monetary-fiscal authority. A simple relation (inequality) between nominal seignorage profits and a positive and exogenous cut-off point determines existence and non-existence at competitive prices. This cut-off point is equal to the difference of transfers between employed and unemployed households that, in turn, characterises the magnitude of inequality in the non-labor wealth distribution. Existence of a Keynesian equilibrium at low interest rates is equivalent to low profitability of the monetary authority, that is, seignorage revenues below the cut-off. On the other hand, as the interest rate increases, nominal price inflation drives the increase of seignorage revenue. Non-existence is equivalent to seignorage revenues equal or above the cut-off point. The argument is developed in two steps. First, I derive this relation and subsequently, I explain how it is related to the conditions of proposition 3. The advantage of this argument, in comparison to the proof of proposition 3, is that it "translates" restrictions on policy parameters to a simple relation between revenue (seignorage) and liabilities (transfers) of the monetary-fiscal authority.

To derive the relation between seignorage and transfers, it is convenient to consider the following representation of the monetary-fiscal authority budget:

$$\frac{r}{1+r}B + \pi = T. \tag{15}$$

(15) is equivalent to (14) if we take into account that $M = B + \pi$ (terminal condition). Under this representation, $(r/(1+r))B$ are the profits of the monetary authority from issuing bonds at a positive nominal rate. A Keynesian equilibrium at (w^*, p^*) requires (15) and

$$B = (1-u)w^*l^*, \ \pi = p^*\overline{y} - w^*\overline{y}^2, \ (1-u)l^* = \overline{y}^2, \ T = (1-u)\tau + u\widehat{\tau}, \tag{16}$$

where aggregate bonds equal aggregate labor income and labor market clears; money demand is accommodated and good's market clears residually. Substituting (16) into (15) and rearranging, I define Φ as

$$\Phi(u) \equiv$$

$$\underbrace{\frac{r}{1+r}(1-u)w^*l^* + p^*\sqrt{(1-u)l^*} - w^*(1-u)l^*}_{Revenue} - \underbrace{((1-u)\tau + u\widehat{\tau})}_{Liabilities}. \tag{17}$$

The competitive equilibrium requires $\Phi(0) = 0$, a Keynesian equilibrium requires $\Phi(u) = 0$ for $0 < u < 1$ and, also, Φ satisfies $\Phi(1) < 0$. Revenue is a decreasing function of u in the domain $[0, 1]$ because both seignorage and firm's profits are decreasing functions of u. Liabilities are a decreasing function of u if $\tau > \widehat{\tau}$ and increase with u if $\tau < \widehat{\tau}$. It is immediate that under case 1, a Keynesian equilibrium does not exist. Revenue decreases with u, liabilities increase with u, $\Phi(0) = 0$ and, as a result, $\Phi(u) < 0$ for $0 < u < 1$. On the other hand, under cases 2 and 3, both revenue and liabilities are decreasing functions of u and existence depends on the relative magnitude of the change of revenue and liabilities with respect to a change of u that, in turn, depends on conditions on policy parameters. Moreover, the derivative of revenue with respect to u, in absolute value, is increasing[6] in u and tends to infinity as $u \to 1$ whereas the derivative of liabilities with respect to u is independent of u. As a result, existence (non-existence) of a Keynesian equilibrium is equivalent to $\Phi'(0) > (\leq) 0$ that, in turn, is equivalent to

$$\frac{r}{1+r}w^*l^* < (\geq) \tau - \widehat{\tau}. \tag{18}$$

(18) is the aforementioned relation between nominal seignorage profits and transfers of households. The right and the left hand side, respectively, are

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[6]The derivative of firm's profits with respect to u, in absolute value, tends to infinity as u tends to one.

the absolute values of the derivatives of liabilities and revenue with respect to u, at $u = 0$ (the derivative of firm's profits at $u = 0$ is zero since profits are maximised at this point).

Seignorage profits depend on τ and r. For a given τ, seignorage is an increasing function of r and as $r \to \infty$, it converges to $\tau/2$. The competitive equilibrium nominal wage w^* is a decreasing function of r for $r < (\sqrt{2} - 1)(1/2)$ and increasing in r for $r > (\sqrt{2} - 1)(1/2)$. Labor supply l^* is decreasing in r (proposition 1). For low values of r, the increase of seignorage profits as r increases is determined entirely by the increase of the tax rate on money balances, i.e., $r/(1 + r)$. On the other hand, for high values of r, the increase of seignorage profits as r increases is determined entirely by the increase of nominal wages. For ease of exposition, I present these qualitative results in section 6.3 of the "Appendix".

Under case 2, $\tau - \widehat{\tau} < \tau/2$. 2(a) is equivalent to (18) with the inequality sign ($<$) and 2(b) with ($\geq$). Seignorage profits go to zero with r. As a result, for low values of r, existence of a Keynesian equilibrium is driven *entirely* by the low profitability of the monetary authority. On the other hand, as r increases, nominal wage inflation drives the increase of seignorage profits and (18) is satisfied with the inequality sign ($>$). Non-existence of Keynesian equilibria at high interests rates is driven by nominal wage inflation.

Under case 3, $\tau - \widehat{\tau} \geq \tau/2$. Seignorage profits are between zero and $\tau/2$ for any $0 < r < \infty$, (18) holds with the inequality sign ($<$) and a Keynesian equilibrium exists at competitive prices. The distribution of non-labor net wealth is very unequal and even a policy of sufficiently high interest rates, that creates nominal wage inflation that, in turn, implies that seignorage profits increase, can not "eliminate" Keynesian equilibria at competitive prices.

The following remarks are in order.

Remark 1. Incorporating exogenous real fiscal spending g does not modify the argument. Conditions on g exist such that the previous argument applies. I abstracted from fiscal spending since it does not contribute anything substantial to the argument and, moreover, it complicates computations.

Remark 2. The assumption on profits allows me to characterise clean conditions, as in proposition 3. Nevertheless, under $\tau > \widehat{\tau}$ and at low nominal rates, existence of a Keynesian equilibrium at competitive prices does not depend on this assumption. Suppose profits are taxed with $\zeta \in (0, 1)$ and *all* households receive, as end-of-period money balances, a fraction $1 - \zeta$ of profits. Aggregate bonds equal aggregate labor income plus profits that accrue to households (households borrow also against income from profits). I can construct Φ as a function of u such that a competitive equilibrium requires $\Phi(0) = 0$, a Keynesian equilibrium requires $\Phi(u) = 0$ for $0 < u < 1$ and $\Phi(1) < 0$. As before, $\Phi'(0) > 0$ implies existence of Keynesian equilibria at competitive prices and it is equivalent to expression (18) with a sign ($<$)

since the derivative of firm's profits with respect to u, at $u = 0$, is zero (profits are maximised at this point). Computing the competitive equilibrium explicitly, it follows that the left hand side of (18) goes to zero with r and, at low values of r, (18) can be satisfied with ($<$) and, hence, a Keynesian equilibrium exists at competitive prices. Characterising a threshold in that case is not clean since $\Phi'(0) \leqslant 0$ is no longer equivalent to non-existence while solving for a Keynesian equilibrium explicitly requires numerical analysis.

Remark 3. Laroque (1978) characterised necessary and sufficient conditions for allocations to be written as a function of the price system, for prices close to competitive prices. These conditions require that marginal propensities to consume and to supply labor, at competitive prices, are less than one. Here, the marginal propensity to consume (MPC) is defined as the derivative, in absolute value, of the value of aggregate consumption with respect to u divided by the derivative, in absolute value, of total income (the value of total output) with respect to u, at $u = 0$. MPC > 1 is equivalent to (18) with ($<$) whereas MPC $\leqslant 1$ is equivalent to (18) with ($\geqslant$). Conditions for non-vanishing excess supplies, equivalent to MPC > 1, violate the conditions of Laroque.

Remark 4. The argument extends to $r = 0$ where cash-in-advance constraints do not bind. Construction of a competitive equilibrium is as follows. Substitute $r = 0$ into equilibrium prices and allocations of proposition 1. At these prices and allocations, households solve (7) (ignoring the quantity constraints). Also, according to the the proof of proposition 1, markets clear. At the given prices and allocations, I can find money balances that satisfy (2) or, equivalently, (5).

Construction of rationing equilibria is similar to section 6.2 of the "Appendix". Given agents optimal decisions under each type of equilibrium, I can solve for equilibrium allocations as functions of prices and nominal transfers. Moreover, prices and transfers must satisfy the respective inequalities associated with each type of equilibrium. Subsequently, given allocations and prices, I can find money balances that satisfy (2) or, equivalently, (5).

Excess capacities under demand rationing vanish at competitive prices. As argued in section 4.1, the result is driven by the specification of fiscal policy rather than the characterisation of monetary policy.

Under $\tau < \widehat{\tau}$, Keynesian equilibria do not exist at competitive prices; the argument is similar to the previous discussion. Under any $\tau > \widehat{\tau}$, a Keynesian equilibrium exists at competitive prices. At $r = 0$, seignorage profits are zero and (18) is satisfied with ($<$) that, in turn, is equivalent to existence of a Keynesian equilibrium at competitive prices.

5 Coordination failures and welfare

A Keynesian equilibrium can be the result of coordination failures rather than the result of "wrong" prices. It is clear, under case 2 of proposition 3, that coordination frictions prevail if and only if nominal interest rates are below the threshold. At nominal rates below the threshold, the competitive allocation is Pareto superior to the Keynesian allocation: allocations of employed households are identical but unemployed households are worse-off. Also, welfare under the competitive allocation is decreasing in r and reaches the highest value at zero interest rates (first-best). I show that there exist restrictions on transfers under which aggregate welfare under the competitive allocation for a nominal rate equal to the threshold, is greater than aggregate welfare under the Keynesian equilibrium at any nonnegative nominal rate below the threshold. A policy of setting the interest rate at the threshold can be characterised as "not risky" in the sense that at nominal rates below the threshold, the worst equilibrium, in terms of aggregate welfare, might realise.

I denote nominal rates greater that the threshold by r^C and nominal rates below the threshold by r^K; superscript C denotes the competitive equilibrium and superscript K, the Keynesian equilibrium. r^C and r^K are defined as

$$r^C = (1 + \delta)\frac{\tau - \widehat{\tau}}{2\widehat{\tau} - \tau}, \quad r^K = (1 - \delta)\frac{\tau - \widehat{\tau}}{2\widehat{\tau} - \tau},$$

where $\delta \in [0, 1]$. At $\delta = 0$, $r^C = r^K$. Under case 2, $\widehat{\tau} < \tau < 2\widehat{\tau}$. I define $\tau = \theta\widehat{\tau}$, where $\theta \in (1, 2)$. Substituting into r^C and r^K, it follows that interest rates are functions of θ and δ. The advantage of these definitions is that competitive and Keynesian allocations depend only on δ and θ and I can present a graph of the welfare results in the (δ, θ)-space.

Aggregate welfare under the competitive allocation is

$$U^C = \ln\left(\frac{1}{\sqrt{3 + 2r^C}}\right) + \ln\left(1 - \frac{1}{3 + 2r^C}\right);$$

consumption and labor supply as in proposition 1. U^C depends on r^C that, in turn, depends on θ and δ.

Aggregate welfare under the Keynesian allocation is

$$U^K = (1 - u)\left(\ln\left(\frac{1}{\sqrt{3 + 2r^K}}\right) + \ln\left(1 - \frac{1}{3 + 2r^K}\right)\right) + u\ln\left(\frac{\widehat{\tau}}{p^*}\right);$$

the allocation of employed households is as in proposition 1 and consumption of unemployed households is $\widehat{\tau}/p^*$. The unemployment rate is equal to

$$u = \frac{4\theta\delta(\theta - 1)(2 - \theta)(1 - \delta(\theta - 1))}{(2\delta + 2\theta^2\delta - \theta(1 + 4\delta))^2};$$

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derivation of u can be found in section 6.4 of the "Appendix". The allocation of employed households depends on r^K that, in turn, depends on θ and δ. Substituting for p^* from proposition 1, it follows that the consumption of unemployed households depends on the ratio $\widehat{\tau}/\tau$ and r^K. Thus, U^K is a function of θ and δ.

At $\delta = 0$, $r^K = r^C = (\tau - \widehat{\tau})/(2\widehat{\tau} - \tau)$, $u = 0$ and $U^K = U^C$. At an interest rate equal to the threshold, competitive prices are associated only with the competitive equilibrium.

The following proposition states that the unemployment rate is a decreasing function of r^K or, equivalently, an increasing function of δ. This result is important to understand the welfare comparison that follows.

Proposition 4. *u is an increasing function of δ, that is,*

$$\frac{\partial u}{\partial \delta} > 0.$$

Proof. See section 6.4 of the "Appendix". $\qquad\square$

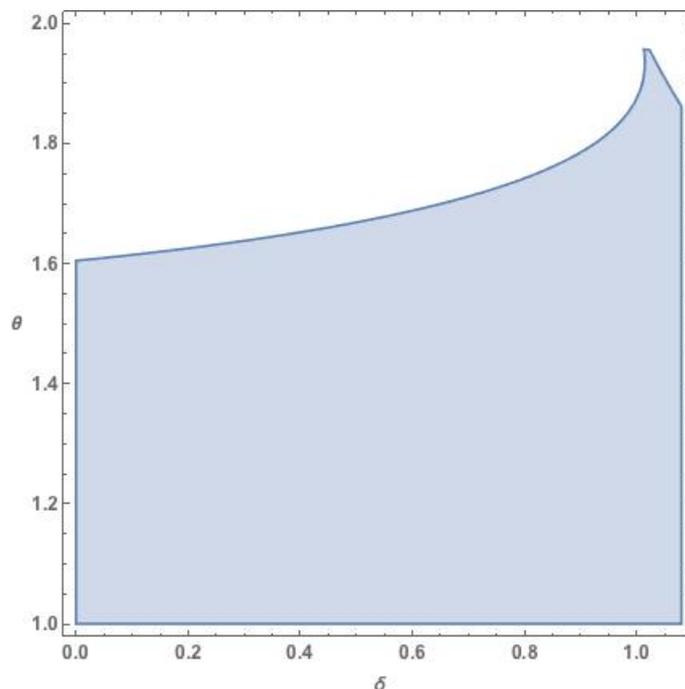Figure 1 presents the sign of $U^C - U^K$ in the (δ, θ)-space.

Figure 1: The sign of $U^C - U^K$

Inside the blue region of figure 1, $U^C - U^K > 0$, inside the white region, $U^C - U^K < 0$ and at the boundary, $U^C - U^K = 0$. Given any $\theta \in (1, 1.6)$,

U^C at the respective r^C is always greater than U^K at the respective r^K. Since U^C is a decreasing function of r, then U^C for $r = (\tau - \widehat{\tau})/(2\widehat{\tau} - \tau)$ is greater than U^K for any $0 \leq r < (\tau - \widehat{\tau})/(2\widehat{\tau} - \tau)$. Finally, I have extended the $\delta-$axis to values above 1 in order to demonstrate, graphically, that at $\delta = 1$ ($r^K = 0$), $U^C - U^K > 0$ for any $\theta \in (1, 1.6)$.

There is an interesting trade-off behind the previous result. Under U^K, a fall of the interest rate is beneficial for employed households but, at the same time, and according to proposition 4, lower nominal rates are associated with higher unemployment rates. Also, the utility of each unemployed household depends on the price level p^* which is a function of r. p^* is a decreasing function[7] of r for $r < 0.78$ and increases with r for $r > 0.78$. Suppose $\theta = 1.5$. It is straightforward to show that $(\tau - \widehat{\tau})/(2\widehat{\tau} - \tau) = 1$. Start from $\delta = 0$ such that $r^K = r^C = 1$ and $U^C = U^K$. As δ increases, r^K falls, employed households become better-off, the unemployment rate increases and the consumption of unemployed households increases until r^K reaches 0.78 and decreases after that point. Also, as δ increases, r^C increases and U^C falls. The increase of the unemployment rate dominates the increase of employed's utility and the increase of unemployed's consumption (until r^K reaches 0.78) such that U^K falls faster than U^C.

References

A. J. Auerbach, L. J. Kotlikoff, and D. R. Koehler. U.S. Inequality, Fiscal Progressivity, and Work Disincentives: An Intragenerational Accounting. NBER Working Papers 22032, National Bureau of Economic Research, Inc, Feb. 2016.

R. J. Barro and H. I. Grossman. A General Disequilibrium Model of Income and Employment. *American Economic Review*, 61(1):82–93, March 1971.

J.-P. Benassy. Neo-keynesian disequilibrium theory in a monetary economy. *The Review of Economic Studies*, 42(4):503–523, 1975.

J. Benhabib, S. Schmitt-Grohe, and M. Uribe. Monetary Policy and Multiple Equilibria. *American Economic Review*, 91(1):167–186, March 2001.

J. Benhabib, S. Schmitt-Grohe, and M. Uribe. Avoiding Liquidity Traps. *Journal of Political Economy*, 110(3):535–563, June 2002.

O. Blanchard and J. Galí. Labor Markets and Monetary Policy: A New Keynesian Model with Unemployment. *American Economic Journal: Macroeconomics, American Economic Association*, 2(2):1–30, April 2010.

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[7]See section 6.4 of the "Appendix".

A. Citanna, H. Cres, J. Dreze, P. J.-J. Herings, and A. Villanacci. Continua of underemployment equilibria reflecting coordination failures, also at Walrasian prices. *Journal of Mathematical Economics*, 36(3):169–200, December 2001.

J. H. Dreze. Existence of an Exchange Equilibrium under Price Rigidities. *International Economic Review*, 16(2):301–20, June 1975.

P. Dubey and J. Geanakoplos. Inside and outside fiat money, gains to trade, and IS-LM. *Economic Theory*, 21(2):347–397, 03 2003.

P. Herings. General equilibrium and the new neoclassical synthesis. *Economic Theory*, 57(3):437–477, November 2014.

G. Laroque. The Fixed Price Equilibria: Some Results in Local Comparative Statics. *Econometrica*, 46(5):1127–54, September 1978.

J. Lucas, Robert E and N. L. Stokey. Money and Interest in a Cash-in-Advance Economy. *Econometrica*, 55(3):491–513, May 1987.

E. Malinvaud. *The Theory of Unemployment Reconsidered*. Wiley, 1977.

T. Nakajima and H. Polemarchakis. Money and Prices Under Uncertainty. *Review of Economic Studies*, 72(1):223–246, 2005.

J. Silvestre. Continua of Hahn-unsatisfactory equilibria : Some examples. *Economics Letters*, 5(3):201–208, 1980.

J. Silvestre. Fixprice analysis in exchange economies. *Journal of Economic Theory*, 26(1):28–58, February 1982.

M. Woodford. Monetary Policy and Price Level Determinacy in a Cash-in-Advance Economy. *Economic Theory*, 4(3):345–80, 1994.

M. Woodford. Control of the Public Debt: A Requirement for Price Stability? NBER Working Papers 5684, National Bureau of Economic Research, Inc, July 1996.

6 Appendix

6.1 Proof of proposition 1

I solve for the competitive equilibrium. Agents optimal decisions are given by (9) and (10) of section 2.4. Superscript i is dropped since households make identical choices. The monetary authority accommodates the demand for money and bonds, good's and labor markets clear and the budget of the

monetary-fiscal authority is satisfied. It is convenient to use the monetary-fiscal authority's budget to solve for the equilibrium, i.e.,

$$\frac{r}{1+r}M + \frac{1}{1+r}\pi = \tau.$$

Combining cash-in-advance constraints, good's and money market clearing, it follows that $M = pc = py$. Substituting for M and for firm's optimal decisions into the monetary-fiscal authority's budget constraint and rearranging, I obtain

$$\frac{p^2}{4w}\frac{1+2r}{1+r} = \tau. \tag{19}$$

Substituting the optimal decision of agents into good's market clearing, I obtain

$$\frac{1}{2p}\left(\tau + \frac{w}{1+r}\right) = \frac{p}{2w}. \tag{20}$$

Combining (19) and (20) so as to eliminate τ, I obtain

$$\frac{w}{p} = \frac{\sqrt{3+2r}}{2}. \tag{21}$$

Substituting (21) into (19), I obtain

$$p^* = \frac{2\tau(1+r)\sqrt{3+2r}}{1+2r}. \tag{22}$$

Substituting (22) into (21), I obtain

$$w^* = \frac{\tau(1+r)(3+2r)}{1+2r}. \tag{23}$$

Finally, substituting (22) and (23) into household's optimal allocation, I obtain

$$c^* = \frac{1}{\sqrt{3+2r}}, \quad l^* = \frac{1}{3+2r}.$$

6.2   Equilibria under rationing

First, I compute each type of equilibrium explicitly and demonstrate that propositions 2 and 3 are true. Subsequently, I demonstrate that multiple equilibria exist. Prices are restricted in a neighbourhood of (w^*, p^*). A neighbourhood is defined as an open ball around (w^*, p^*) in the $(w, p)-$space. There exist a neighbourhood such that prices in the neighbourhood are associated with an equilibrium. To show that, I demonstrate that there exist prices arbitrarily close to competitive prices associated with an equilibrium.

6.2.1 Excess demand equilibrium

An excess demand equilibrium is characterised by rationing on demand. Household's consumption demand is $c = \bar{c}$ and from the household's budget constraint, labor supply is

$$l = \frac{1+r}{w}(p\bar{c} - \tau);$$

superscript i is dropped since households make identical choices. Firm's demand for labor is $L = \bar{L}$, supply of output is $y = \sqrt{\bar{L}}$ and profits are $\pi = p\sqrt{\bar{L}} - w\bar{L}$.

As before, I use the monetary-fiscal budget constraint to solve for an excess demand equilibrium. Combining cash-in-advance constraints, good's and money market clearing, it follows that $M = p\bar{c} = p\sqrt{\bar{L}}$. Substituting for M and π into the monetary-fiscal budget, I obtain

$$\frac{r}{1+r}p\sqrt{\bar{L}} + \frac{1}{1+r}(p\sqrt{\bar{L}} - w\bar{L}) = \tau.$$

Solving with respect to $\bar{L}$, I obtain

$$\bar{L} = \frac{\left(p(1+r) \pm \sqrt{p^2(1+r)^2 - 4w(1+r)\tau}\right)^2}{4w^2}. \tag{24}$$

I demonstrate shortly that only the small root of (24) is consistent with an excess demand equilibrium and that the term inside the square root is positive. Substituting the small root of (24) into good's market clearing, I solve for $\bar{c}$ as a function of parameters

$$\bar{c} = \frac{p(1+r) - \sqrt{p^2(1+r)^2 - 4w(1+r)\tau}}{2w}. \tag{25}$$

Rationing on the labor market requires

$$\frac{p^2}{4w^2} > \bar{L}. \tag{26}$$

Substituting the big root of $\bar{L}$ into (26) and rearranging, I obtain

$$\frac{p}{2w} > \frac{p(1+r)}{2w} + \frac{\sqrt{p^2(1+r)^2 - 4w(1+r)\tau}}{2w}. \tag{27}$$

(27) can not be satisfied. On the other hand, substituting the small root into (26) and rearranging, (26) is equivalent to

$$\frac{p^2}{4w}\frac{1+2r}{1+r} > \tau. \tag{28}$$

Also, it can be easily verified that if (28) is satisfied, then the square root term of (24) and (25) is positive.

Rationing on the good's market requires

$$\frac{1}{2p}(\tau + \frac{w}{1+r}) > \bar{c}. \tag{29}$$

Substituting (25) into (29) and rearranging, (29) is equivalent to

$$p^2 > \frac{w(\frac{w}{1+r} + \tau)^2}{2(1+r)(\frac{w}{1+r} - \tau)}. \tag{30}$$

(30) represents a restriction on parameters only if $w > (1+r)\tau$, that is, the right hand side is positive. Also, competitive equilibrium wages satisfy $w^*/(1+r) - \tau = 2\tau/(1+2r) > 0$. To sum up, policy parameters and prices associated with an excess demand equilibrium must satisfy (28) and (30).

At (w^*, p^*), excess demands vanish, i.e., (28) and (30) are equalities. The proof of proposition 1 and in particular, expression (19), implies that (28) is an equality at competitive prices that, in turn, implies zero excess demand of labor. If excess demand of labor is zero, then, from good's market clearing, excess demand on the good's market is zero, i.e., $\bar{c} = y^* = \sqrt{L^*} = c^*$. Substituting (w^*, p^*) into (30) and rearranging, I obtain that it is satisfied with equality or, equivalently, excess demand on the good's market is zero. Finally, according to (24) and (25), as prices converge to (w^*, p^*), $\bar{c}$ and $\bar{L}$ converge to competitive equilibrium demands.

For any given $\tau > \hat{\tau}$ and $0 < r < \infty$, prices arbitrarily close to (w^*, p^*), that satisfy (28) and (30), are associated with an excess demand equilibrium. Also, given $\tau < \hat{\tau}$ and $0 < r < \infty$ associated with an individually rational competitive allocation, prices arbitrarily close to (w^*, p^*), that satisfy (28) and (30), imply that excess demand allocations are very close to competitive allocations and, hence, excess demand equilibrium allocations are individually rational.

6.2.2 Classical equilibrium

A classical equilibrium is characterised by rationing of households on the good's and on the labor market. The decision of employed households is similar to the decision of households in the excess demand equilibrium. Unemployed households are rationed on the labor market and their consumption is equal to $\hat{\tau}/p$. The firm's optimal decisions are given by (10) of section 2.4.

Combining cash-in-advance constraints, good's and money market clearing, it follows that $M = (1-u)p\bar{c} + up(\hat{\tau}/p) = p(p/2w)$. Substituting for M and π into the monetary-fiscal budget, I obtain

$$\frac{r}{1+r}p\frac{p}{2w} + \frac{1}{1+r}\frac{p^2}{4w} = (1-u)\tau + u\hat{\tau}.$$

A classical equilibrium requires $\tau \neq \widehat{\tau}$ otherwise the above equilibrium relation depends only on exogenous parameters.

Suppose $\tau > \widehat{\tau}$. Solving for u, I obtain

$$u = \frac{\tau - \frac{p^2}{4w}\frac{1+2r}{1+r}}{\tau - \widehat{\tau}}.$$

Rationing on the labor market requires $0 < u < 1$, or equivalently,

$$\widehat{\tau} < \frac{p^2}{4w}\frac{1+2r}{1+r} < \tau. \qquad (31)$$

Substituting u into good's market clearing, it follows that

$$\overline{c} = \frac{\tau - \widehat{\tau}}{\frac{1+2r}{1+r}\frac{p^2}{4w} - \widehat{\tau}}\left(\frac{p}{2w} - \frac{\widehat{\tau}(\tau - \frac{1+2r}{1+r}\frac{p^2}{4w})}{p(\tau - \widehat{\tau})}\right).$$

Rationing on the good's market requires

$$\frac{1}{2p}\left(\tau + \frac{w}{1+r}\right) > \overline{c}. \qquad (32)$$

Policy parameters and prices associated with a classical equilibrium must satisfy (31) and (32).

Suppose $\tau < \widehat{\tau}$. u modifies as

$$u = \frac{\frac{p^2}{4w}\frac{1+2r}{1+r} - \tau}{\widehat{\tau} - \tau}$$

and rationing on the labor market requires

$$\tau < \frac{p^2}{4w}\frac{1+2r}{1+r} < \widehat{\tau}. \qquad (33)$$

Also, $\overline{c}$ modifies as

$$\overline{c} = \frac{\widehat{\tau} - \tau}{\widehat{\tau} - \frac{1+2r}{1+r}\frac{p^2}{4w}}\left(\frac{p}{2w} - \frac{\widehat{\tau}(\frac{1+2r}{1+r}\frac{p^2}{4w} - \tau)}{p(\widehat{\tau} - \tau)}\right).$$

and rationing on demand requires (32).

At (w^*, p^*), excess supply and demand vanish. The proof of proposition 1 and in particular, expression (19), implies $(p^{*2}/4w^*)((1+2r)/(1+r)) = \tau$ that, in turn, implies $u = 0$. Good's market clearing and $u = 0$ imply $\overline{c} = p^*/2w^* = y^* = c^*$ that, in turn, implies excess demand is zero. Substituting (w^*, p^*) into (32), it is straightforward to demonstrate that it is satisfied with equality and hence, excess demand is zero. Also, according to the explicit solutions obtained, as prices converge to (w^*, p^*), u converges to zero and $\overline{c}$ converges to c^*.

Given policy parameters, prices arbitrarily close to (w^*, p^*) that satisfy either (31) and (32) or (33) and (32), are associated with a classical equilibrium. These inequalities do not contradict one another. To see this, suppose $w > w^*$ and $(p^2/4w)((1+2r)/(1+r)) = \tau$ $(u = 0)$. Substituting the latter into (32) and rearranging, it follows that (32) is satisfied for any $w > w^*$. Thus, there exist prices very close to (w^*, p^*) such that the above inequalities are satisfied. Finally, classical equilibrium allocations satisfy individual rationality following a similar argument as in section 6.2.1.

6.2.3 Keynesian equilibrium

A Keynesian equilibrium is characterised by rationing on supply. Decisions of employed households are given by (9) of section 2.4 and the consumption of unemployed households is $\widehat{\tau}/p$. The firm produces $y = \overline{y}$, hires $\overline{y}^2$ units of labor and $\pi = p\overline{y} - w\overline{y}^2$.

Combining cash-in-advance constraints, good's and money market clearing, it follows that $M = (1-u)p((1/2p)/(\tau + w/(1+r))) + up(\widehat{\tau}/p) = p\overline{y}$. Substituting for M and π into the monetary-fiscal budget, I obtain

$$\frac{r}{1+r}p\overline{y} + \frac{1}{1+r}(p\overline{y} - w\overline{y}^2) = (1-u)\tau + u\widehat{\tau}.$$

Solving for $\overline{y}$ as a function of u, I obtain

$$\overline{y} = \frac{p(1+r) \pm \sqrt{p^2(1+r)^2 - 4w(1+r)((1-u)\tau + u\widehat{\tau})}}{2w}. \tag{34}$$

Following a similar argument as in section 6.2.1 (excess demand), *only* the small root of (34) is consistent with a Keynesian equilibrium. Contrary to the previous sections, it is more convenient to use labor market clearing, $(1-u)l = \overline{y}^2$, to solve for an equilibrium. Substituting (34) into labor market clearing and rearranging, I obtain

$$p\sqrt{p^2(1+r)^2 - 4w(1+r)z} = p^2(1+r) - 2wz - (1-u)\left(\frac{w^2}{1+r} - w\tau\right),$$

where $z = (1-u)\tau + u\widehat{\tau}$. Squaring both sides and rearranging, I obtain

$$\alpha u^2 + (-\beta)u - \gamma = 0$$

or, equivalently,

$$u = \frac{-(-\beta) - \sqrt{(-\beta)^2 + 4\alpha\gamma}}{2\alpha} \tag{35}$$

and

$$u = \frac{-(-\beta) + \sqrt{(-\beta)^2 + 4\alpha\gamma}}{2\alpha}, \tag{36}$$

where

$$\alpha = w(\frac{w}{1+r} - \tau)^2 + 4w(\tau - \widehat{\tau})^2 + 4w(\tau - \widehat{\tau})(\frac{w}{1+r} - \tau),$$

$$\beta = -2(1+r)p^2(\frac{w}{1+r} - \tau) + 4w(\frac{w}{1+r} - \tau)(\tau - \widehat{\tau})+$$

$$4w\tau(\frac{w}{1+r} - \tau) + 2w(\frac{w}{1+r} - \tau)^2 + 8w\tau(\tau - \widehat{\tau})$$

and

$$\gamma = 2(1+r)(\frac{w}{1+r} - \tau)\Big(p^2 - \frac{w(\frac{w}{1+r} + \tau)^2}{2(1+r)(\frac{w}{1+r} - \tau)}\Big).$$

Rationing on labor and good's market requires $0 < u < 1$ and

$$\frac{p}{2w} > \bar{y}, \tag{37}$$

respectively. Substituting (34) into (37) and rearranging, (37) is equivalent to

$$u > -\frac{\frac{p^2(1+2r)}{4w(1+r)} - \tau}{\tau - \widehat{\tau}}. \tag{38}$$

If (38) is satisfied, then the square root term of $\bar{y}$ is positive. Policy parameters and prices associated with a Keynesian equilibrium must satisfy $0 < u < 1$ and (38).

To proceed to the analysis of each case of proposition 3, I will use the following arguments.

I determine the signs of α, β and γ at (w^*, p^*) that, in turn, impose restrictions such that $0 < u < 1$ at prices close to (w^*, p^*). Substituting (w^*, p^*) into α and rearranging, $\alpha > 0$ is equivalent to

$$4 + \Big(\frac{\widehat{\tau}}{\tau}\Big)^2 - 4\frac{\widehat{\tau}}{\tau} + 4r\Big(\frac{\widehat{\tau}}{\tau} - 1\Big)^2(1+r) > 4r\Big(\frac{\widehat{\tau}}{\tau} - 1\Big)$$

and it is satisfied either for $\tau > \widehat{\tau}$ or $\tau < \widehat{\tau}$. Substituting (w^*, p^*) into β and rearranging, I obtain

$$\beta = \frac{16\tau^2(1+r)^2(3+2r)}{(1+2r)^3}\Big(\tau - \widehat{\tau} - r(2\widehat{\tau} - \tau)\Big). \tag{39}$$

$\beta < 0$ if $\tau < \widehat{\tau}$ and $\beta > 0$ only if $\tau > \widehat{\tau}$. Moreover, tedious algebra implies that $\alpha > \beta$ whenever $\beta > 0$. Lastly, according to the discussion after expression (30) in section 6.2.1, it follows that $\gamma = 0$.

Excess supplies vanish if $u = 0$ and if (37) or, equivalently, (38) is an equality. According to expression (19), proof of proposition 1, the right

hand side of (38) is zero at (w^*, p^*). As a result, if $u = 0$ at (w^*, p^*), then (38) is an equality and excess supply on the good's market is zero whereas if $0 < u < 1$ at (w^*, p^*), then (38) is satisfied and excess supply is positive.

Finally, consider the solutions of u. At (w^*, p^*), $\gamma = 0$ implies that at least one solution satisfies $u = 0$.

Consider case 1, i.e., $\hat{\tau} > \tau$ and $0 < r < \infty$. Prices close to (w^*, p^*) satisfy $\alpha > 0$ and $\beta < 0$ and *only* (36) can be a candidate for equilibrium.[8] At (w^*, p^*), $\gamma = 0$ implies $u = 0$. As a result, if there exist prices associated with a Keynesian equilibrium, then the equilibrium is characterised by vanishing excess supplies. Prices associated with a Keynesian equilibrium exist. Consider the following inequalities:

$$\frac{w(\frac{w}{1+r} + \tau)^2}{2(1+r)(\frac{w}{1+r} - \tau)} < p^2 \leq \frac{4w(1+r)\tau}{1+2r}. \tag{40}$$

For any $0 < \tau < \infty$ and $0 < r < \infty$, the upper and lower bounds of (40) cross at w^* and $\hat{w}$, where $\hat{w} > w^*$. (40) is a well-defined interval for any $w^* < w < \hat{w}$ and it is equivalent to $\gamma > 0$ and to the right hand side of (38) being non-positive. If prices arbitrarily close to (w^*, p^*) satisfy (40), then $\gamma > 0$ implies $0 < u < 1$ that, in turn, implies (38) is satisfied since its right the hand is non-positive. Finally, Keynesian allocations satisfy individual rationality following a similar argument as in section 6.2.1.

Consider case 2, i.e., $\hat{\tau} < \tau < 2\hat{\tau}$ and $0 < r < \infty$. Under 2(a), $\beta > 0$ at (w^*, p^*) because $0 < r < (\tau - \hat{\tau})/(2\hat{\tau} - \tau)$ and under 2(b), $\beta \leq 0$ at (w^*, p^*) because $(\tau - \hat{\tau})/(2\hat{\tau} - \tau) \leq r$.

Consider 2(a). At (w^*, p^*), $\gamma = 0$ implies $u = 0$ for (35) and $0 < u = \beta/\alpha < 1$ for (36). A Keynesian equilibrium exists at competitive prices. Moreover, there exist prices arbitrarily close to (w^*, p^*) associated with two Keynesian equilibria. Consider the following inequalities:

$$\frac{4w(1+r)\tau}{1+2r} \leq p^2 < \frac{w(\frac{w}{1+r} + \tau)^2}{2(1+r)(\frac{w}{1+r} - \tau)}. \tag{41}$$

(41) is a well-defined interval for $(1+r)\tau < w < w^*$, and it is equivalent to $\gamma < 0$ and to the right hand side of (38) being non-positive. If prices arbitrarily close to (w^*, p^*) satisfy (41), then $\gamma < 0$ implies $0 < u < 1$, for both solutions, that, in turn, implies (38) is satisfied. Also, there exist prices associated with only one equilibrium of the Keynesian type, that is, only (36) satisfies $0 < u < 1$. Consider the following inequalities:

$$p^2 > \frac{w(\frac{w}{1+r} + \tau)^2}{2(1+r)(\frac{w}{1+r} - \tau)}, \quad p^2 \geq \frac{4w(1+r)\tau}{1+2r}. \tag{42}$$

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[8] (35) implies $u < 0$.

(42) is equivalent to $\gamma > 0$ and to the right hand side of (38) being non-positive. Prices arbitrarily close to (w^*, p^*) that satisfy (42) are associated with one equilibrium of the Keynesian type.

Consider 2(b) and $(\tau - \widehat{\tau})/(2\widehat{\tau} - \tau) < r < \infty$. Prices close to (w^*, p^*) satisfy $\alpha > 0$ and $\beta < 0$ and *only* (36) can be a candidate for equilibrium. At (w^*, p^*), $\gamma = 0$ implies $u = 0$. As a result, if prices are associated with a Keynesian equilibrium, then the equilibrium is characterised by vanishing excess supplies. Prices associated with a Keynesian equilibrium exists, e.g., prices arbitrarily close to (w^*, p^*) that satisfy (42) are associated with an equilibrium of a Keynesian type.

Consider 2(b) and $r = (\tau - \widehat{\tau})/(2\widehat{\tau} - \tau)$, where $\beta = 0$ at (w^*, p^*). At (w^*, p^*), $\gamma = 0$ implies $u = 0$ for both solutions. This is a tedious case to analyse since β can take either sign for prices close to (w^*, p^*). Nevertheless, if prices arbitrarily close to competitive prices satisfy (42), then (36) satisfies $0 < u < 1$, irrespective of the sign of β, and a Keynesian equilibrium exists.

The argument under case 3 is similar to 2(a) and will not be repeated.

6.2.4   Multiple equilibria

Policy parameters and prices are associated with equilibria of different types.

Suppose $\tau > \widehat{\tau}$ and $0 < r < \infty$. Prices that satisfy (28) and (30) and hence, are associated with an excess demand equilibrium, satisfy also (42) and are associated with a Keynesian equilibrium.

Suppose $\tau < \widehat{\tau}$ and $0 < r < \infty$. There exist prices that satisfy $w > w^*$ and $(p^2/4w)((1+2r)/(1+r)) > \tau$ and are associated with a classical equilibrium. These prices are associated, also, with an excess demand equilibrium since $(p^2/4w)((1+2r)/(1+r)) > \tau$ is equivalent to (28), $w > w^*$ and (28) imply (30) (see the discussion after (40)) and an excess demand equilibrium exist.

6.3   Seignorage profits and nominal wages

I demonstrate the comparative static results of seignorage and nominal wages with respect to r.

Substituting for w^* and l^*, it follows that seignorage profits are equal to

$$\frac{r}{1+r} w^* l^* = \tau \frac{r}{1+2r}.$$

Profits increase with r and converge to $\tau/2$ as $r \to \infty$.

The derivative of w^* with respect to r is

$$\frac{\partial w^*}{\partial r} = \tau \frac{5 + 4r}{1 + 2r} - \tau \frac{2(1+r)(3+2r)}{(1+2r)^2}$$

and for $r = (\sqrt{2} - 1)/2$ it is zero, for $r < (\sqrt{2} - 1)/2$ negative and for $r > (\sqrt{2} - 1)/2$ positive.

6.4 Section 5 (Welfare)

I compute the unemployment rate, prove proposition 4 and compute the derivative of p^* with respect to r.

According to section 6.2.3 (Keynesian equilibrium), the unemployment rate, at competitive prices, under 2(a), is equal to $u = \beta/\alpha$. Substituting for α and β and rearranging, u is equal to

$$u = \frac{4(1+r)(\tau - \widehat{\tau} - r(2\widehat{\tau} - \tau))}{\tau + \tau(1+2r)^2(1 - \frac{\widehat{\tau}}{\tau}) + 2(\tau - \widehat{\tau})(1+2r)}.$$

Substituting for $r = r^K$ and $\tau = \theta\widehat{\tau}$ and rearranging, it follows that

$$u = \frac{4\theta\delta(\theta-1)(2-\theta)(1-\delta(\theta-1))}{(2\delta + 2\theta^2\delta - \theta(1+4\delta))^2}.$$

The derivative with respect to δ is

$$\frac{\partial u}{\partial \delta} = -\frac{4\theta(2 - \theta(3-\theta))(2\delta(\theta-1) - \theta)}{(2\delta + 2\delta\theta^2 - \theta(1+4\delta))^3}$$

and it is positive for any $\theta \in (1,2)$ and $\delta \in [0,1]$.

The derivative of p^* with respect to r is equal to

$$\frac{\partial p^*}{\partial r} = -\tau\frac{4(1+r)\sqrt{2r+3}}{(1+2r)^2} + \tau\frac{2(1+r)}{(1+2r)\sqrt{2r+3}} + \tau\frac{2\sqrt{2r+3}}{1+2r}$$

and for $r = 0.780776$ it is zero, for $r < 0.780776$ negative and for $r > 0.780776$ positive.