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Vanishing Corrections for the Position in a Linear Model of FKPP Fronts

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Abstract: Take the linearised FKPP equation $\partial_t h = \partial_x^2 h + h$ with boundary condition $h(m(t), t) = 0$. Depending on the behaviour of the initial condition $h_0(x) = h(x, 0)$ we obtain the asymptotics—up to a $o(1)$ term $r(t)$—of the absorbing boundary $m(t)$ such that $\omega(x) := \lim_{t\to\infty} h(x + m(t), t)$ exists and is non-trivial. In particular, as in Bramson's results for the non-linear FKPP equation, we recover the celebrated $-3/2 \log t$ correction for initial conditions decaying faster than $x^\nu e^{-x}$ for some $\nu < -2$. Furthermore, when we are in this regime, the main result of the present work is the identification (to first order) of the $r(t)$ term, which ensures the fastest convergence to $\omega(x)$. When $h_0(x)$ decays faster than $x^\nu e^{-x}$ for some $\nu < -3$, we show that $r(t)$ must be chosen to be $-3\sqrt{\pi/t}$, which is precisely the term predicted heuristically by Ebert–van Saarloos (Phys. D Nonlin. Phenom. 146(1): 1–99, 2000) in the non-linear case (see also Mueller and Munier Phys Rev E 90(4):042143, 2014, Henderson, Commun Math Sci 14(4):973–985, 2016, Brunet and Derrida Stat Phys 1-20, 2015). When the initial condition decays as $x^\nu e^{-x}$ for some $\nu \in [-3, -2)$, we show that even though we are still in the regime where Bramson's correction is $-3/2 \log t$, the Ebert–van Saarloos correction has to be modified. Similar results were recently obtained by Henderson Commun Math Sci 14(4):973–985, 2016 using an analytical approach and only for compactly supported initial conditions.

1. Introduction

The celebrated Fisher–Kolmogorov–Petrovsky–Piscounof equation (FKPP) in one dimension for $h : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is:

$$\partial_t h = \partial_x^2 h + h - h^2, \quad h(x, 0) = h_0(x). \tag{1}$$

This equation is a natural description of a reaction-diffusion model [Fis37,KPP37, AW78]. It is also related to branching Brownian motion: for the Heaviside initial condition $h_0(x) = \mathbb{1}_{\{x<0\}}$, $h(x, t)$ is the probability that the rightmost particle at time t in a branching Brownian motion (BBM) is to the right of x.

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For suitable initial conditions where $h_0(x) \in [0, 1]$, $h_0(x)$ goes to 1 fast enough as $x \to -\infty$ and $h_0(x)$ goes to 0 fast enough as $x \to \infty$, it is known that $h(x, t)$ develops into a travelling wave: there exists a centring term $m(t)$ and an asymptotic shape $\omega_v(x)$ such that

$$\lim_{t\to\infty} h\big(m(t) + x, t\big) = \omega_v(x) \in (0, 1), \tag{2}$$

where $m(t)/t \to v$ and $\omega_v(x)$ is a travelling wave solution to (1) with velocity v: that is, the unique (up to translation) non-trivial solution to

$$\omega_v'' + v\,\omega_v' + \omega_v - \omega_v^2 = 0 \tag{3}$$

with $\omega_v(-\infty) = 1$ and $\omega_v(+\infty) = 0$.

In his seminal work [Bra83], Bramson showed how the initial condition h_0 (and in particular its large x asymptotic behaviour) determines $m(t)$ in (2). For the important example $h_0(x) = \mathbb{1}_{\{x<0\}}$ corresponding to the rightmost particle in BBM, he finds

$$m(t) = 2t - \frac{3}{2}\log t + a + o(1) \tag{4}$$

for some constant a, and a limiting travelling wave with (critical) speed $v = 2$. (Here and throughout, we use the notation $f(t) = o(1)$ to mean that $f(t) \to 0$ as $t \to \infty$.)

What makes Bramson's results extremely interesting is their universality; for instance Bramson proves [Bra83] that the previous result still holds if the reaction term $h - h^2$ in (1) is replaced by $f(h)$ with $f(0) = f(1) = 0$, $f'(0) = 1$ and $f(x) \le x$. The universality goes further than that, and for many other front equations, it is believed and sometimes known that the centring term $m(t)$ follows the same kind of behaviour as for (1): one needs to compute a function $v(\gamma)$ which has a minimum v_c at a point γ_c (in the FKPP case (1), $v(\gamma) = \gamma + 1/\gamma$, $\gamma_c = 1$, $v_c = 2$); then for an initial condition decreasing like $e^{-\gamma x}$, the front converges to a travelling wave with velocity $v(\gamma)$ if $\gamma \le \gamma_c$ and critical velocity v_c if $\gamma \ge \gamma_c$.

When the centring term $m(t)$ is defined as in (2), it is not uniquely determined: if $m(t)$ is any suitable centring term, then $m(t) + o(1)$ is also a suitable centring term. Instead one can try to give a more precise definition for $m(t)$. For example, one could reasonably ask for

$$\begin{aligned} &h\big(m(t), t\big) = \alpha \quad \text{for some } \alpha \in (0, 1) \quad \text{or} \quad \partial_x^2 h\big(m(t), t\big) = 0 \\ &\text{or} \quad m(t) = -\int \mathrm{d}x\, x\,\partial_x h(x, t) \end{aligned} \tag{5}$$

in addition to (2). In the case $h_0(x) = \mathbb{1}_{\{x<0\}}$, so that $h(x, t) = \mathbb{P}(R_t > x)$ where R_t is the position of the rightmost particle in a BBM at time t, the first definition in (5) would be the α-quantile of R_t, the second definition would be the mode of the distribution of R_t, and the third definition would be the expectation of R_t.

It has been heuristically argued [EvS00,MM14,Hen16,BD15] that any quantity $m(t)$ defined as in (5) behaves for large t as

$$m(t) = v_c t - \frac{3}{2\gamma_c}\log t + a - 3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}} \times \frac{1}{\sqrt{t}} + o\Big(\frac{1}{\sqrt{t}}\Big), \tag{6}$$

for any front equation of the FKPP type and for any initial condition that decays fast enough. In the FKPP case (1), one has $\gamma_c = 1$ and $v''(\gamma_c) = 2$ so that $m(t) = 2t - (3/2)\log t + a - 3\sqrt{\pi/t} + o(1/\sqrt{t})$.

Heuristically, the coefficient of the $1/\sqrt{t}$ term does not depend on the precise definition of $m(t)$ because the front $h(x,t)$ converges very quickly to its limiting shape in the region where h is neither very close to 0 nor very close to 1, so that the difference between any two reasonable definitions of $m(t)$ converges quickly (faster than $1/\sqrt{t}$) to some constant. Note that the constant term "a" is expected to be non-universal and to depend on the model, the initial condition and the precise definition of $m(t)$.

As argued in [EvS00], the reason why the "$\log t$" and the "$1/\sqrt{t}$" terms in (6) are so universal is that they are driven by the way the front develops very far on the right, in a region where it is exponentially small and where understanding the position $m(t)$ of the front is largely a matter of solving the linearised front equation. This idea has recently been exploited in the PDE literature: see [HNRR12,HNRR13].

However there is a catch: solving directly the linearised equation $\partial_t h = \partial_x^2 h + h$ with (for instance) a step initial condition $h_0(x) = \mathbb{1}_{\{x<0\}}$, one finds $h_{\text{linear}}(x,t) = \frac{1}{2}e^t \operatorname{erfc}(x/\sqrt{4t})$. Defining the position $m(t)$ by $h_{\text{linear}}\big(m(t),t\big) = 1$ gives $m(t) = 2t - \frac{1}{2}\log t + a + \mathcal{O}\big((\log^2 t)/t\big)$ rather than (4); the linearised equation has the same velocity 2 as for the FKPP equation, a logarithmic correction but with a different prefactor and no $1/\sqrt{t}$ correction. The problem is that with the linearised equation, the $h_{\text{linear}}(x,t)$ increases exponentially on the left of $m(t)$ and this "mass" pushes the front forward, leading to a $-\frac{1}{2}\log t$ rather than a $-\frac{3}{2}\log t$ correction. This means that in order to recover the behaviour of $m(t)$ for the FKPP equation, one must have a front equation with some saturation mechanism on the left. The behaviour of $m(t)$ is not expected to depend on which saturation mechanism is chosen, but one must be present. For these reasons, we consider in this paper a linearised FKPP with a boundary on the left, as in [Hen16].

We emphasize that, in the present work, the FKPP equation is only a motivation: we do not attempt to establish the equivalence between the FKPP equation and the linear model with a boundary. Our results are proved only for the linear model with boundary, and we can only conjecture that they do apply to the FKPP equation.

2. Statement of the Problem and Main Results

We study the following linear partial differential equation with initial condition $h_0(x)$ and a given boundary $m : [0,\infty) \to \mathbb{R}$:

$$\begin{cases} \partial_t h = \partial_x^2 h + h & \text{for} \quad x > m(t), \\ h\big(m(t),t\big) = 0, & h(x,0) = h_0(x). \end{cases} \tag{7}$$

Observe that without loss of generality we can (and will) insist that $m(0) = 0$ since otherwise we can simply shift the reference frame by $m(0)$ by the change of coordinate $x \mapsto x - m(0)$.

The same system was studied in [Hen16] by PDE methods for compactly supported initial conditions. In this paper, we use probabilistic methods, writing the solution of the heat equation as an expectation involving Brownian motion with a killing boundary. This model was introduced in a seminal paper by Kesten [Kes78] and has generated significant interest recently [HHK06,BBHM16,Mai13]. We give more general results, in particular lifting the compactly supported hypothesis.

If the boundary is linear, $m(t) = vt$, the problem is easily solved explicitly. However, as soon as $m(t)$ is no longer linear, gaining any explicit information about the solution is known to be hard (see for instance [HT15]) and there are few available results.

Motivated by the earlier FKPP discussion about convergence to a travelling wave as in (2), we are looking for functions $m : [0, \infty) \to \mathbb{R}$ and $\omega : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t\to\infty} h\big(m(t) + x, t\big) = \omega(x) \quad \text{for all } x \geq 0 \tag{8}$$

with ω non-trivial, $\omega(0) = 0$ and $\omega(x) > 0$ for all $x > 0$. Note that when $\frac{m(t)}{t} \to v$ such a function ω necessarily satisfies

$$\omega''(x) + v\omega'(x) + \omega(x) = 0, \quad \forall x \geq 0. \tag{9}$$

In this case, the boundary condition anchors the front. Requiring the convergence of $h(m(t) + x, t)$ to a limiting shape means that $m(t)$ must increase fast enough to prevent the mass near the front from growing exponentially, but not so fast that it tends to zero. This provides a saturation mechanism, and even though it might seem very unlike FKPP fronts to have $h\big(m(t), t\big) = 0$, as discussed earlier we do expect the two systems to behave similarly.

Throughout the article we use the following notation:

- $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to \infty$;
- $f(x) = \mathcal{O}\big(g(x)\big)$ means there exists $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all large x;
- $f(x) = o\big(g(x)\big)$ means $f(x)/g(x) \to 0$ as $x \to \infty$.
- A random variable G is said to have "Gaussian tails" if there exist two positive constants c_1, c_2 such that $\mathbb{P}(|G| > z) \leq c_1 \exp(-c_2 z^2)$ for all $z \geq 0$.

Our first theorem recovers the analogue of Bramson's results for the system (7), (8).

Theorem 1. *For each of the following bounded initial conditions h_0, a twice continuously differentiable function $m(t)$ such that $m(0) = 0$ and $m''(t) = \mathcal{O}(1/t^2)$ leads to a solution $h(x, t)$ to (7) with a non-trivial limit (8) if and only if $m(t)$ has the following large time asymptotics where a is an arbitrary constant:*

(a) *if $h_0(x) \sim Ax^\nu e^{-\gamma x}$ with $0 < \gamma < 1$ for large x,*

$$\begin{aligned} m(t) &= \Big(\gamma + \frac{1}{\gamma}\Big)t + \frac{\nu}{\gamma}\log t + a + o(1), \\ \text{and then} \quad \omega(x) &= \alpha\Big(e^{-\gamma x} - e^{-\frac{x}{\gamma}}\Big) \quad \text{with} \quad \alpha = Ae^{-\gamma a}\left(\frac{1}{\gamma} - \gamma\right)^\nu. \end{aligned} \tag{10a}$$

(b) *if $h_0(x) \sim Ax^\nu e^{-x}$ with $\nu > -2$ for large x,*

$$\begin{aligned} m(t) &= 2t - \frac{1-\nu}{2}\log t + a + o(1), \\ \text{and then} \quad \omega(x) &= \alpha x e^{-x} \quad \text{with} \quad \alpha = \frac{Ae^{-a}}{\sqrt{\pi}}2^\nu\Gamma\Big(1 + \frac{\nu}{2}\Big). \end{aligned} \tag{10b}$$

(c) *if $h_0(x) \sim Ax^{-2}e^{-x}$ for large x,*

$$\begin{aligned} m(t) &= 2t - \frac{3}{2}\log t + \log\log t + a + o(1), \\ \text{and then} \quad \omega(x) &= \alpha x e^{-x} \quad \text{with} \quad \alpha = \frac{Ae^{-a}}{4\sqrt{\pi}}. \end{aligned} \tag{10c}$$

(d) *if $h_0(x) = \mathcal{O}\big(x^\nu e^{-x}\big)$ with $\nu < -2$ for large x and such that the value of α below is non-zero,*

$$\begin{aligned} m(t) &= 2t - \frac{3}{2}\log t + a + o(1), \\ \text{and then} \quad \omega(x) &= \alpha x e^{-x} \quad \text{with} \quad \alpha = \frac{e^{-a-\Delta}}{2\sqrt{\pi}} \int_0^\infty \mathrm{d}y\, h_0(y) y e^y \psi_\infty(y), \end{aligned} \tag{10d}$$

where Δ and ψ_∞ are quantities depending on the whole function m (and not only the asymptotics) which are introduced (*in* (61) *and* (68)) *in the proofs.*

Remarks.

- From the probabilistic representation of $h(x,t)$ written later in the paper (21), it is clear that the solution $h(x,t)$ to (7) must be an increasing function of h_0 and a decreasing function of m (in the sense that if $m^{(1)}(t) \geq m^{(2)}(t)$ for all t, then $h^{(1)}(x,t) \leq h^{(2)}(x,t)$ for all x and t). This is also immediate from the maximum principle. This implies that the α given in Theorem 1 must be increasing functions of h_0 and decreasing functions of m. This was obvious from the explicit expression of α in cases (a), (b) and (c). In case (d), given the complicated expressions for Δ and ψ_∞, it is not obvious at all from its expression that α decreases with m.
- Consider now a twice differentiable function m without the assumption that $m''(t) = \mathcal{O}(1/t^2)$. The monotonicity of $h(x,t)$ with respect to m still holds, and by sandwiching such a m between two sequences of increasingly close functions that satisfy the $\mathcal{O}(1/t^2)$ condition, one can show easily in cases (a), (b) and (c) that if m has the correct asymptotics, then $h\big(m(t)+x,t\big)$ converges as in Theorem 1. Case (d) is more difficult as both Δ and ψ_∞ might be ill defined when one does not assume $m''(t) = \mathcal{O}(1/t^2)$.

We now turn to the analogue of the Ebert–van Saarloos correction (6) for our model (7). As explained in the introduction and shown in Theorem 1, with a characterization as in (8), $m(t)$ is only determined up to $o(1)$. If we wish to improve upon Theorem 1, then we need a more precise definition for $m(t)$, analogous to (5). Natural possible definitions could be

$$h\big(m(t)+1,t\big) = 1 \qquad \text{or} \qquad \partial_x h\big(m(t),t\big) = 1. \tag{11}$$

However, it is not obvious that such a function $m(t)$ even exists, would be unique or differentiable. We are furthermore interested only in the long time asymptotics of $m(t)$. Therefore, instead of requiring something like (11) we rather look, as in [Hen16], for the function $m(t)$ such that the convergence (8) is as fast as possible.

Our main result, Theorem 2, tells us how fast $h\big(m(t)+x,t\big)$ converges for suitable choices of m in case (d) of Theorem 1. This case is the most classical as it contains, for example, initial conditions with bounded support. It is the case studied by Ebert-Van Saarloos and Henderson, and is the case for which universal behaviour is expected. Theorem 2 is followed by two corollaries that highlight important consequences.

Theorem 2. *Suppose that h_0 is a bounded function such that $h_0(x) = \mathcal{O}\big(x^\nu e^{-x}\big)$ for large x for some $\nu < -2$, and such that α defined in (10) is non-zero. Suppose also that m is twice continuously differentiable with*

$$m(t) = 2t - \frac{3}{2}\log(t+1) + a + r(t) \tag{12}$$

where $r(0) = -a$, $r(t) \to 0$ as $t \to \infty$ and $r''(t) = \mathcal{O}(t^{-2-\eta})$ for large t for some $\eta > 0$. Then for any $x \geq 0$,

$$\begin{aligned} &h\big(m(t)+x,t\big) \\ &= \alpha x e^{-x}\left[1 - r(t) - \frac{3\sqrt{\pi}}{\sqrt{t}} + \mathcal{O}\left(t^{1+\frac{\nu}{2}}\right) + \mathcal{O}\left(\frac{1}{t^{\frac{1}{2}+\eta}}\right) + \mathcal{O}\left(\frac{\log t}{t}\right) + \mathcal{O}\big(r(t)^2\big)\right] \end{aligned} \tag{13}$$

with α as in (10).

If we further assume that $h_0(x) \sim Ax^\nu e^{-x}$ for large x for some $A > 0$ and $-4 < \nu < -2$, then

$$\begin{aligned} &h\big(m(t)+x,t\big) \\ &= xe^{-x}\left[\alpha\left(1 - r(t) - \frac{3\sqrt{\pi}}{\sqrt{t}}\right) - bt^{1+\frac{\nu}{2}} + o\left(t^{1+\frac{\nu}{2}}\right) + \mathcal{O}\left(\frac{1}{t^{\frac{1}{2}+\eta}}\right) + \mathcal{O}\big(r(t)^2\big)\right] \end{aligned} \tag{14}$$

with

$$b = -\frac{A}{\sqrt{4\pi}} e^{-a} 2^{\nu+1}\Gamma\left(\frac{\nu}{2}+1\right) > 0. \tag{15}$$

This result allows us to bound the rate of convergence $h\big(m(t)+x,t\big)$ to $\alpha x e^{-x}$: it is generically of order $\max\big(1/\sqrt{t}, |r(t)|, t^{1+\nu/2}\big)$.

This also suggests that for $m(t)$ defined as in either choice of (11), one should have $r(t) \sim -3\sqrt{\pi}/\sqrt{t}$ for $\nu < -3$ and $r(t)$ of order $t^{1+\nu/2}$ for $-3 \leq \nu < -2$. Note however that we are not sure that such a $m(t)$ exists and, if it exists, we do not know whether it satisfies the hypothesis on $m''(t)$ that we used in the Theorem.

In the following two corollaries we highlight the best rates of convergence of $h\big(m(t)+x,t\big) \to xe^{-x}$ that we can obtain from Theorem 2. For simplicity, we dropped the technical requirement that $m(0) = 0$ in the corollaries; the expression for α must therefore be adapted.

Corollary 3. *Suppose that h_0 is a bounded function such that $h_0(x) = \mathcal{O}\big(x^\nu e^{-x}\big)$ for large x with $\nu < -3$ and such that α is non-zero. If we choose*

$$m(t) = 2t - \frac{3}{2}\log(t+1) + a + \frac{c}{\sqrt{t+1}}, \tag{16}$$

then

$$\text{if } \nu \leq -4, \qquad c = -3\sqrt{\pi} \iff h\big(m(t)+x,t\big) = \alpha x e^{-x} + \mathcal{O}\left(\frac{\log t}{t}\right), \tag{17}$$

$$\text{if } -4 < \nu < -3, \quad c = -3\sqrt{\pi} \iff h\big(m(t)+x,t\big) = \alpha x e^{-x} + \mathcal{O}\left(t^{1+\frac{\nu}{2}}\right). \tag{18}$$

Note in particular that we have recovered the result of [Hen16], but with more general initial conditions ([Hen16] only considered compactly supported initial conditions).

Corollary 4. *Suppose that $h_0(x)$ is a bounded function such that $h_0(x) \sim Ax^\nu e^{-x}$ for large x with $-4 < \nu < -2$, with m, r and b as in Theorem 2. Then*

$$\begin{aligned}
&\textit{if } -3 < \nu < -2,\\
&\quad r(t) = -\frac{b}{\alpha} t^{1+\frac{\nu}{2}} + o\big(t^{1+\frac{\nu}{2}}\big) \iff h\big(m(t)+x, t\big) = \alpha x e^{-x} + o\big(t^{1+\frac{\nu}{2}}\big),\\
&\textit{if } -4 < \nu \le -3,\\
&\quad r(t) = -\frac{3\sqrt{\pi}}{\sqrt{t}} - \frac{b}{\alpha} t^{1+\frac{\nu}{2}} + o\big(t^{1+\frac{\nu}{2}}\big) \iff h\big(m(t)+x, t\big) = \alpha x e^{-x} + o\big(t^{1+\frac{\nu}{2}}\big).
\end{aligned}$$

Notice that for $h_0(x) \sim Ax^{-3}e^{-x}$ the position $m(t)$ still features a first order correction in $1/\sqrt{t}$ but with a coefficient $-\big(3\sqrt{\pi} + \frac{1}{4\alpha} A e^{-a}\big)$ which is different from the $\nu < -3$ case.

3. Writing the Solution as an Expectation of a Bessel

In this section, we write the solution to (7) as an expectation of a Bessel process.

We only consider functions $m(t)$ that are twice continuously differentiable. For each given $m(t)$, (7) is a linear problem. We first study the fundamental solutions $q(t, x, y)$ defined as

$$\begin{cases} \partial_t q = \partial_x^2 q + q & \text{if } x > m(t), \\ q(t, m(t), y) = 0, & q(0, x, y) = \delta(x - y); \end{cases} \tag{19}$$

where δ is the Dirac distribution. Then

$$h(x, t) = \int_0^\infty \mathrm{d}y\, q(t, x, y) h_0(y). \tag{20}$$

It is clear that $e^{-t} q(t, x, y)$ is the solution to the heat equation with boundary, and therefore

$$q(t, x, y)\mathrm{d}x = e^t \mathbb{P}\Big(B_t^{(y)} \in \mathrm{d}x,\ B_s^{(y)} > m(s)\, \forall s \in (0, t)\Big), \tag{21}$$

where $t \mapsto B_t^{(y)}$ is the Brownian motion started from $B_0^{(y)} = y$ with the normalization

$$\mathbb{E}\big[(B_{s+h}^{(y)} - B_s^{(y)})^2\big] = 2h. \tag{22}$$

Suppose $f : [0, \infty) \to \mathbb{R}$ is a continuous function, and $A_t(f)$ is a measurable functional that depends only on $f(s)$, $s \in [0, t]$. Then by Girsanov's theorem,

$$\mathbb{E}\big[A_t(B^{(y)})\big] = e^{-\frac{1}{4}\int_0^t \mathrm{d}s\, m'(s)^2} \mathbb{E}\Big[A_t(m + B^{(y)})\, e^{-\frac{1}{2}\int_0^t m'(s)\, \mathrm{d}B_s^{(y)}}\Big]. \tag{23}$$

Plugging into (21) at position $m(t) + x$ instead of x, we get

$$q(t, m(t)+x, y)\mathrm{d}x = e^{t - \frac{1}{4}\int_0^t \mathrm{d}s\, m'(s)^2} \mathbb{E}\Big[\mathbb{1}_{\{B_t^{(y)} \in \mathrm{d}x\}} \mathbb{1}_{\{B_s^{(y)} > 0\, \forall s \in (0,t)\}} e^{-\frac{1}{2}\int_0^t m'(s)\, \mathrm{d}B_s^{(y)}}\Big]. \tag{24}$$

We recall that, by the reflection principle, the probability that a Brownian path started from y stays positive and ends in dx is:

$$\mathbb{P}\big(B_t^{(y)} \in \mathrm{d}x,\ \ B_s^{(y)} > 0\,\forall s \in (0,t)\big) = \frac{1}{\sqrt{\pi t}}\sinh\Big(\frac{xy}{2t}\Big)e^{-\frac{x^2+y^2}{4t}}\,\mathrm{d}x. \tag{25}$$

Using (25), we write (24) as a conditional expectation:

$$q(t, m(t)+x, y) = \frac{\sinh\big(\frac{xy}{2t}\big)}{\sqrt{\pi t}}e^{-\frac{x^2+y^2}{4t}+t-\frac14\int_0^t \mathrm{d}s\, m'(s)^2}\,\mathbb{E}\Big[e^{-\frac12\int_0^t m'(s)\,\mathrm{d}\xi_s^{(t:y\to x)}}\Big], \tag{26}$$

where $\xi_s^{(t:y\to x)}$, $s\in[0,t]$ is a Brownian motion (normalized as in (22)) started from y and conditioned not to hit zero for any $s\in(0,t)$ and to be at x at time t. Such a process is called a Bessel-3 bridge, and we recall some properties of Bessel processes and bridges in Sect. 4.

It is convenient to think of the path $s\mapsto\xi_s^{(t:y\to x)}$ as the straight line $s\mapsto y+(x-y)s/t$ plus some fluctuations. This leads us to define

$$\begin{aligned}\psi_t(y,x) &:= \mathbb{E}\Big[e^{-\frac12\int_0^t m'(s)\left(\mathrm{d}\xi_s^{(t:y\to x)}-\frac{x-y}{t}\mathrm{d}s\right)}\Big] = \mathbb{E}\Big[e^{-\frac12\int_0^t m'(s)\,\mathrm{d}\xi_s^{(t:y\to x)}}\Big]e^{\frac{m(t)}{2t}(x-y)},\\ &= \mathbb{E}\Big[e^{\frac12\int_0^t m''(s)\left(\xi_s^{(t:y\to x)}-(y+\frac{x-y}{t}s)\right)\mathrm{d}s}\Big],\end{aligned} \tag{27}$$

where we have used integration by parts. With this quantity, (26) now reads

$$q(t,m(t)+x,y) = \frac{\sinh\big(\frac{xy}{2t}\big)}{\sqrt{\pi t}}e^{\frac{m(t)}{2t}(y-x)-\frac{x^2+y^2}{4t}+t-\frac14\int_0^t\mathrm{d}s\,m'(s)^2}\psi_t(y,x), \tag{28}$$

and the main part of the present work is to estimate $\psi_t(y,x)$.

4. The Bessel Toolbox

Before we begin our main task, we need some fairly standard estimates on Bessel-3 processes and Bessel-3 bridges. From here on, we refer to these simply as Bessel processes and Bessel bridges; the “3” will be implicit. We include proofs for completeness.

We build most of our processes on the same probability space. We fix a driving Brownian motion $(B_s, s\ge 0)$ started from 0 under a probability measure $\mathbb{P}$, with the normalization $\mathbb{E}[B_t^2]=2t$.

For each $y\ge 0$ we introduce a Bessel process $\xi^{(y)}$ started from y as the strong solution to the SDE

$$\xi_0^{(y)} = y, \qquad \mathrm{d}\xi_s^{(y)} = \mathrm{d}B_s + \frac{2}{\xi_s^{(y)}}\mathrm{d}s. \tag{29}$$

It is well-known that $\xi_s^{(y)}$ has the law of a Brownian motion conditioned to never hit zero.

We also introduce, for each $t\ge0$ and $y\ge0$

$$\xi_s^{(t:y\to0)} = \frac{t-s}{t}\xi^{(y)}_{\frac{st}{t-s}} \quad \text{for } s\in[0,t). \tag{30}$$

This process is a Bessel bridge from y to 0 in time t, which is a Brownian motion started from y and conditioned to hit 0 for the first time at time t. One can check by direct substitution that $\xi_s^{(t:y\to 0)}$ solves

$$\xi_0^{(t:y\to 0)} = y, \qquad \mathrm{d}\xi_s^{(t:y\to 0)} = \mathrm{d}\tilde{B}_{t,s} + \left(\frac{2}{\xi_s^{(t:y\to 0)}} - \frac{\xi_s^{(t:y\to 0)}}{t-s} \right) \mathrm{d}s, \tag{31}$$

where for each t, $(\tilde{B}_{t,s},\ s \in [0,t))$ is the strong solution to

$$\tilde{B}_{t,0} = 0, \qquad \mathrm{d}\tilde{B}_{t,s} = \frac{t-s}{t}\mathrm{d}\Big(B_{\frac{ts}{t-s}}\Big), \tag{32}$$

and is thus itself a Brownian motion.

One can compute directly the law of the Brownian motion conditioned to hit zero for the first time at time t using (25) and check that this law solves the forward Kolmogorov equation (or Fokker Planck equation) associated with the SDE (or Langevin equation) (31).

Similarly, we construct the Bessel bridge from y to x in time t, the Brownian motion conditioned not to hit zero for any $s \in (0,t)$ and to be at x at time t, through

$$\xi_0^{(t:y\to x)} = y, \qquad \mathrm{d}\xi_s^{(t:y\to x)} = \mathrm{d}\tilde{B}_{t,s} + \left(\frac{x}{t-s}\coth\frac{x\xi_s^{(t:y\to x)}}{2(t-s)} - \frac{\xi_s^{(t:y\to x)}}{t-s} \right)\mathrm{d}s. \tag{33}$$

The advantages of constructing all the processes from a single Brownian path $s \mapsto B_s$ is that they can be compared directly, realization by realization. In particular we use the following comparisons:

Lemma 5. *For any $y \ge z \ge 0$ and $s \ge 0$,*

$$\xi_s^{(z)} \le \xi_s^{(y)} \le \xi_s^{(z)} + y - z \quad \textit{and} \quad y + B_s \le \xi_s^{(y)}. \tag{34}$$

Furthermore, for any $y \ge 0$, $x \ge z \ge 0$, $t \ge 0$ and $s \in [0,t]$,

$$\begin{aligned} \xi_s^{(t:0\to 0)} &\le \xi_s^{(t:y\to 0)} \le \xi_s^{(t:0\to 0)} + y\frac{t-s}{t}, \\ \xi_s^{(t:y\to z)} &\le \xi_s^{(t:y\to x)} \le \xi_s^{(t:y\to z)} + \frac{(x-z)s}{t}. \end{aligned} \tag{35}$$

Proof. To prove (34) we make three observations.

- The processes $\xi_s^{(y)}$ and $y + B_s$ both start from y and

$$\mathrm{d}\big(\xi_s^{(y)} - (y + B_s)\big) = \frac{\mathrm{d}s}{\xi_s^{(y)}} > 0, \quad s > 0, \tag{36}$$

  so that $\xi_s^{(y)} > y + B_s$ for all $s > 0$ and $y \ge 0$.
- The processes $\xi_s^{(y)}$ and $\xi_s^{(z)}$ follow the same SDE (29) and $\xi_0^{(y)} \ge \xi_0^{(z)}$, so they must remain ordered at all times (see for instance [Kun97]).

- We have

$$d(\xi_s^{(y)} - \xi_s^{(z)}) = \left(\frac{1}{\xi_s^{(y)}} - \frac{1}{\xi_s^{(z)}}\right) ds, \tag{37}$$

  and since $\xi_s^{(y)} \geq \xi_s^{(z)}$ for all $s \geq 0$ we see that $\xi_s^{(y)} - \xi_s^{(z)}$ is decreasing, yielding $\xi_s^{(y)} - \xi_s^{(z)} \leq y - z$ for all $s \geq 0$.

The inequalities in the left part of (35) are a direct consequence of (34) through the change of time (30). We now focus on the inequalities in the right part of (35). First we assume that $z > 0$.

The fact that for $x \geq z$ we have $\xi_s^{(t:y\to x)} \geq \xi_s^{(t:y\to z)}$ follows from the fact that $x\coth(ax) \geq z\coth(az)$ for any $a > 0$ and $x \geq z$.

For the other inequality, the fact that $u(\coth u - 1)$ is decreasing yields that

$$\begin{aligned} d\xi_s^{(t:y\to x)} &= d\tilde{B}_{t,s} + \frac{2}{\xi_s^{(t:y\to x)}} \times \frac{x\xi_s^{(t:y\to x)}}{2(t-s)}\left(\coth\frac{x\xi_s^{(t:y\to x)}}{2(t-s)} - 1\right)ds + \frac{x - \xi_s^{(t:y\to x)}}{t-s}ds \\ &\leq d\tilde{B}_{t,s} + \frac{2}{\xi_s^{(t:y\to x)}} \times \frac{z\xi_s^{(t:y\to z)}}{2(t-s)}\left(\coth\frac{z\xi_s^{(t:y\to z)}}{2(t-s)} - 1\right)ds + \frac{x - \xi_s^{(t:y\to x)}}{t-s}ds \\ &\leq d\tilde{B}_{t,s} + \frac{z}{t-s}\left(\coth\frac{z\xi_s^{(t:y\to z)}}{2(t-s)} - 1\right)ds + \frac{x - \xi_s^{(t:y\to x)}}{t-s}ds, \end{aligned} \tag{38}$$

so that, writing $\zeta_s := \xi_s^{(t:y\to x)} - \xi_s^{(t:y\to z)} \geq 0$ for the difference process,

$$d\zeta_s \leq \frac{x - z - \zeta_s}{t-s}ds. \tag{39}$$

But the solution to $\frac{d\phi_s}{ds} = (x - z - \phi_s)/(t-s)$ and $\phi_0 = 0$ is $\phi_s = (x-z)s/t$, implying that $\zeta_s \leq (x-z)s/t$, which concludes the proof for $z > 0$. For the case $z = 0$ the proof is the same but uses the inequalities $1 \leq u\coth u \leq 1 + u$ for $u \geq 0$. □

We note that, intuitively, as the length of a Bessel bridge tends to infinity, on any compact time interval the bridge looks more and more like a Bessel process. Similarly, as the start point of a Bessel process tends to infinity, on any compact interval it looks more and more like a Brownian motion relative to its start position. We make this precise in the lemma below.

Lemma 6. *For all $s \geq 0$ and $y \geq 0$,*

$$\xi_s^{(t:y\to 0)} \to \xi_s^{(y)} \quad \text{as } t \to \infty. \tag{40}$$

For all $s \geq 0$

$$\xi_s^{(y)} - y \to B_s \quad \text{as } y \to \infty. \tag{41}$$

For all $s \geq 0$ and any $y_t \to \infty$ as $t \to \infty$,

$$\xi_s^{(t:y_t\to 0)} - y_t\frac{t-s}{t} \to B_s \quad \text{as } t \to \infty. \tag{42}$$

Proof. For (40), we simply recall (30) which defined

$$\xi_s^{(t:y\to 0)} = \frac{t-s}{t}\xi^{(y)}_{\frac{st}{t-s}} \quad \text{for } s \in [0,t), \tag{43}$$

so we are done by continuity of paths.

For (41), recall from Lemma 5 that $\xi_s^{(y)} - y \geq B_s$. This both gives us the required lower bound, and tells us that for any $s \geq 0$, $\inf_{u\in[0,s]} \xi_u^{(y)} \to \infty$ as $y \to \infty$. Thus

$$\xi_s^{(y)} - y = B_s + 2\int_0^s \frac{1}{\xi_u^{(y)}}\mathrm{d}u \leq B_s + \frac{2s}{\inf_{u\in[0,s]} \xi_u^{(y)}} \to B_s \quad \text{as } y \to \infty. \tag{44}$$

Finally, for (42), we write

$$\xi_s^{(t:y_t\to 0)} - y_t\left(\frac{t-s}{t}\right) = \left[\xi_s^{(y_t)} - y_t\right] - \left[\frac{s}{t}(\xi_s^{(y_t)} - y_t)\right] + \left[\left(\frac{t-s}{t}\right)(\xi^{(y_t)}_{s+\frac{s^2}{t-s}} - \xi_s^{(y_t)})\right]. \tag{45}$$

By (40), $\xi_s^{(y_t)} - y_t \to B_s$. By (34), $B_s \leq \xi_s^{(y_t)} - y_t \leq \xi_s^{(0)}$, so

$$\frac{s}{t}(\xi_s^{(y_t)} - y_t) \to 0 \quad \text{as} \quad t \to \infty. \tag{46}$$

Using our coupling between the Bessel processes and Brownian motion we have $\mathrm{d}B_u \leq \mathrm{d}\xi_u^{(y_t)} \leq \mathrm{d}\xi_u^{(0)}$ for all $u \geq 0$ and hence

$$B_{s+\frac{s^2}{t-s}} - B_s \leq \xi^{(y_t)}_{s+\frac{s^2}{t-s}} - \xi_s^{(y_t)} \leq \xi^{(0)}_{s+\frac{s^2}{t-s}} - \xi_s^{(0)} \tag{47}$$

so by continuity of paths,

$$\left(\frac{t-s}{t}\right)(\xi^{(y_t)}_{s+\frac{s^2}{t-s}} - \xi_s^{(y_t)}) \to 0 \quad \text{as } t \to \infty, \tag{48}$$

which concludes the proof of (42). □

We need the fact that the increments of a Bessel process over time s are roughly of order $s^{1/2}$. By paying a small price on the exponent, we obtain the following uniform bounds:

Lemma 7. *For any $\epsilon > 0$ small enough, there exists a positive random variable G with Gaussian tail such that uniformly in $s \geq 0$ and $y \geq 0$,*

$$\left|\xi_s^{(y)} - y\right| \leq G \max\left(s^{\frac{1}{2}-\epsilon}, s^{\frac{1}{2}+\epsilon}\right) \quad \text{and} \quad \left|B_s\right| \leq G \max\left(s^{\frac{1}{2}-\epsilon}, s^{\frac{1}{2}+\epsilon}\right). \tag{49}$$

Furthermore, uniformly in $x \geq 0$, $y \geq 0$, $t \geq 0$ and $0 \leq s \leq t$,

$$\left|\xi_s^{(t:y\to x)} - \left(y + \frac{x-y}{t}s\right)\right| \leq G \max\left(s^{\frac{1}{2}-\epsilon}, s^{\frac{1}{2}+\epsilon}\right). \tag{50}$$

Proof. From (34) we have $B_s \le \xi_s^{(y)} - y \le \xi_s^{(0)}$. Also by symmetry $\mathbb{P}(|B_s| > x) = 2\mathbb{P}(B_s > x)$. Thus to prove (49), it is sufficient to show that

$$\mathbb{P}\left(\sup_{s>0} \frac{\xi_s^{(0)}}{\max(s^{1/2-\epsilon}, s^{1/2+\epsilon})} > x\right) \le c_1 e^{-c_2 x^2} \tag{51}$$

for some positive c_1 and c_2. The proof is elementary and we defer it to an appendix.

To prove (50), notice that from (35) we have

$$\xi_s^{(t:y\to x)} - \left(y + \frac{x-y}{t}s\right) \le \xi_s^{(t:0\to 0)}. \tag{52}$$

But from the change of time (30) and (49),

$$\begin{aligned}\xi_s^{(t:0\to 0)} &= \frac{t-s}{t}\xi^{(0)}_{\frac{st}{t-s}} \\ &\le G\frac{t-s}{t}\max\left\{\left(\frac{st}{t-s}\right)^{\frac{1}{2}-\epsilon}, \left(\frac{st}{t-s}\right)^{\frac{1}{2}+\epsilon}\right\} \le G\max\left(s^{\frac{1}{2}-\epsilon}, s^{\frac{1}{2}+\epsilon}\right),\end{aligned} \tag{53}$$

where the last step is obtained by pushing the $(t-s)/t$ inside the max. This provides the upper bound of (50). For the lower bound, we introduce Brownian bridges $s \mapsto B_s^{(t:y\to x)}$ started from y and conditioned to be at x at time t. We couple the Brownian bridge to the Bessel bridges by building them over the family $\tilde{B}_{t,s}$ of Brownian motions defined in (32):

$$B_0^{(t:y\to x)} = y, \qquad \mathrm{d}B_s^{(t:y\to x)} = \mathrm{d}\tilde{B}_{t,s} + \frac{x - B_s^{(t:y\to x)}}{t-s}\mathrm{d}s. \tag{54}$$

One can check directly that

$$B_s^{(t:y\to x)} = y + \frac{x-y}{t}s + B_s^{(t:0\to 0)}. \tag{55}$$

Furthermore, by comparing (54) to (33), it is immediate from the fact that $\coth u \ge 1$ for all $u \ge 0$ that $\xi_s^{(t:y\to x)} \ge B_s^{(t:y\to x)}$. Therefore

$$\xi_s^{(t:y\to x)} - \left(y + \frac{x-y}{t}s\right) \ge B_s^{(t:0\to 0)}. \tag{56}$$

Also, as in (30), we can relate B_s and $B_s^{(t:0\to 0)}$ through a time change:

$$B_s^{(t:0\to 0)} = \frac{t-s}{t}B_{\frac{st}{t-s}} \quad \text{for } s \in [0, t), \tag{57}$$

and, as in (53),

$$\begin{aligned}\left|B_s^{(t:0\to 0)}\right| &= \frac{t-s}{t}\left|B_{\frac{st}{t-s}}\right| \le G\frac{t-s}{t}\max\left\{\left(\frac{st}{t-s}\right)^{\frac{1}{2}-\epsilon}, \left(\frac{st}{t-s}\right)^{\frac{1}{2}+\epsilon}\right\} \\ &\le G\max\left(s^{\frac{1}{2}-\epsilon}, s^{\frac{1}{2}+\epsilon}\right),\end{aligned} \tag{58}$$

which concludes the proof. □

5. Simple Properties of $\psi_t(y, x)$ and Proof of Theorem 1

As in the hypothesis of Theorem 1, we assume throughout this section that m is twice continuously differentiable with

$$m(0) = 0 \quad \text{and} \quad m''(s) = \mathcal{O}\Big(\frac{1}{s^2}\Big). \tag{59}$$

The large s behaviour of $m''(s)$ implies that there exists a v such that, for large s,

$$m'(s) = v + \mathcal{O}\Big(\frac{1}{s}\Big) \quad \text{and} \quad m(s) = vs + \mathcal{O}(\log s). \tag{60}$$

We define

$$\Delta = \frac{1}{4}\int_0^\infty \mathrm{d}s\, (m'(s) - v)^2, \tag{61}$$

which is finite because of (60).

5.1. Simple properties of $\psi_t(y, x)$. We recall from (27) that the main quantity we are interested in is

$$\psi_t(y, x) = \mathbb{E}\big[e^{I_t(y,x)}\big], \tag{62}$$

with

$$I_t(y, x) = \frac{1}{2}\int_0^t \mathrm{d}s\, m''(s)\Big(\xi_s^{(t:y\to x)} - \Big(y + \frac{x-y}{t}s\Big)\Big), \tag{63}$$

where we recall that $\xi_s^{(t:y\to x)}$, $s \in [0, t]$ is a Bessel bridge from y to x over time t. We mainly need to consider $x = 0$ so we use the shorthand

$$\psi_t(y) := \psi_t(y, 0). \tag{64}$$

We also define

$$I(y) = \frac{1}{2}\int_0^\infty \mathrm{d}s\, m''(s)\big(\xi_s^{(y)} - y\big) \tag{65}$$

where $\xi_s^{(y)}$, $s \geq 0$ is a Bessel process started from y.

Proposition 8. *The function $\psi_t(y, x)$ has the following properties:*

- *It is bounded away from zero and infinity: there exist two positive constants $0 < K_1 < K_2$ depending on the function $m''(s)$ such that for any x, y, t,*

$$K_1 \leq \psi_t(y, x) \leq K_2. \tag{66}$$

- *It hardly depends on x for large times: recalling that $\psi_t(y) := \psi_t(y, 0)$,*

$$\psi_t(y, x) = \psi_t(y)\Big(1 + x\,\mathcal{O}\Big(\frac{\log t}{t}\Big)\Big) \quad \textit{uniformly in } y \textit{ and } x. \tag{67}$$

- *For fixed y, it has a finite and positive limit as $t \to \infty$:*

$$\psi_\infty(y) := \lim_{t\to\infty} \psi_t(y) = \mathbb{E}\Big[e^{I(y)}\Big] > 0. \tag{68}$$

- *The large time limit $\psi_\infty(y)$ has a well-behaved large y limit: for any function $t \mapsto y_t$ that goes to infinity as $t \to \infty$,*

$$\lim_{y\to\infty} \psi_\infty(y) = \lim_{t\to\infty} \psi_t(y_t) = \mathbb{E}\Big[e^{\frac{1}{2}\int_0^\infty \mathrm{d}s\, m''(s) B_s}\Big] = e^{\Delta}. \tag{69}$$

Proof. For the first result, Lemma 7 tells us that

$$\left|\xi_s^{(t:y\to x)} - \Big(y + \frac{x-y}{t}s\Big)\right| \le G \max\Big(s^{\frac{1}{2}-\epsilon}, s^{\frac{1}{2}+\epsilon}\Big), \tag{70}$$

where $G > 0$ is a random variable with Gaussian tail independent of t, y and x. Then, since $m''(s) = \mathcal{O}(1/s^2)$,

$$\big|I_t(y,x)\big| \le \frac{1}{2}\int_0^\infty \mathrm{d}s\, \big|m''(s)\big| G \max\Big(s^{\frac{1}{2}-\epsilon}, s^{\frac{1}{2}+\epsilon}\Big) = G\mathcal{O}(1). \tag{71}$$

For the second result, we compare paths going to x with paths going to 0: we know from Lemma 5 that $0 \le \xi_s^{(t:y\to 0)} - \xi_s^{(t:y\to x)} + xs/t \le xs/t$, so

$$\begin{aligned}\big|I_t(y,0) - I_t(y,x)\big| &\le \frac{1}{2}\int_0^t \mathrm{d}s\, \big|m''(s)\big| \times \left|\xi_s^{(t:y\to 0)} - \xi_s^{(t:y\to x)} + \frac{x}{t}s\right| \\ &\le \frac{x}{2t}\int_0^t \mathrm{d}s\, \big|m''(s)\big| s = x\mathcal{O}\Big(\frac{\log t}{t}\Big). \end{aligned} \tag{72}$$

We now turn to the third result. For any fixed s and y, Lemma 6 tells us that $\xi_s^{(t:y\to 0)} \to \xi_s^{(y)}$ as $t \to \infty$. Thus, using (70) and (71), we can apply dominated convergence and obtain

$$I_t(y,0) = \frac{1}{2}\int_0^t \mathrm{d}s\, m''(s)\Big(\xi_s^{(t:y\to 0)} - y\frac{t-s}{t}\Big) \to \frac{1}{2}\int_0^\infty \mathrm{d}s\, m''(s)\big(\xi_s^{(y)} - y\big) = I(y). \tag{73}$$

Furthermore, as the bound (71) is a random variable with Gaussian tails, using dominated convergence again we get

$$\lim_{t\to\infty} \mathbb{E}\big[e^{I_t(y,0)}\big] = \mathbb{E}\big[e^{I(y)}\big]. \tag{74}$$

For the fourth statement, by Lemma 6 for any fixed s we have

$$\lim_{y\to\infty} \big(\xi_s^{(y)} - y\big) = B_s \quad \text{and} \quad \lim_{t\to\infty} \Big(\xi_s^{(t:y_t\to 0)} - y_t\frac{t-s}{t}\Big) = B_s. \tag{75}$$

Then, by dominated convergence using again a uniform Gaussian bound from Lemma 7,

$$\lim_{y\to\infty} \psi_\infty(y) = \lim_{t\to\infty} \psi_t(y_t) = \mathbb{E}\Big[e^{\frac{1}{2}\int_0^\infty \mathrm{d}s\, m''(s) B_s}\Big]. \tag{76}$$

It now remains to compute the right-hand-side. Let

$$X_t := \frac{1}{2}\int_0^t \mathrm{d}s\, m''(s) B_s. \tag{77}$$

By integration by parts,

$$X_t = \frac{1}{2} m'(t) B_t - \frac{1}{2}\int_0^t m'(s)\,\mathrm{d}B_s = \frac{1}{2}\int_0^t \big(m'(t) - m'(s)\big)\mathrm{d}B_s \tag{78}$$

so X_t is a time change of Brownian motion with

$$\mathbb{E}\Big[e^{X_t}\Big] = e^{\frac{1}{2}\operatorname{var}(X_t)} = e^{\frac{1}{8}\int_0^t (m'(t)-m'(s))^2\, 2\mathrm{d}s} \to e^{\frac{1}{4}\int_0^\infty (v-m'(s))^2\,\mathrm{d}s} = e^{\Delta}. \tag{79}$$

Therefore, by dominated convergence as in (76), $\mathbb{E}[e^{X_\infty}] = e^{\Delta}$. □

5.2. Proof of Theorem 1. Since $m(0) = 0$ and $m''(s) = \mathcal{O}(1/s^2)$, we can write $m(s) = vs + \delta(s)$ with $\delta(0) = 0$, $\delta(s) = \mathcal{O}(\log s)$, and $\delta'(s) = \mathcal{O}(1/s)$. Note that

$$\int_0^t \mathrm{d}s\, m'(s)^2 = \int_0^t \mathrm{d}s\Big(v^2 + 2v\delta'(s) + \delta'(s)^2\Big) = v^2 t + 2v\delta(t) + 4\Delta + \mathcal{O}\Big(\frac{1}{t}\Big), \tag{80}$$

where we recall that $\Delta = \frac{1}{4}\int_0^\infty \mathrm{d}s\,\delta'(s)^2$. We now fix $x > 0$, so that any terms written as $\mathcal{O}(f(t))$ might depend on x; since x is fixed this will not matter. For instance, instead of (67) we simply write that $\psi_t(y,x) = \psi_t(y) e^{\mathcal{O}(\frac{\log t}{t})}$.

We recall (28):

$$q(t, m(t) + x, y) = \frac{\sinh\big(\frac{xy}{2t}\big)}{\sqrt{\pi t}} e^{\frac{m(t)}{2t}(y-x) - \frac{x^2+y^2}{4t} + t - \frac{1}{4}\int_0^t \mathrm{d}s\, m'(s)^2} \psi_t(y,x). \tag{81}$$

Substituting in the estimate above, and using also (67), we get

$$q(t, m(t)+x, y) = \frac{1}{\sqrt{\pi t}} e^{t\left(1-\frac{v^2}{4}\right) - \frac{v}{2}\delta(t) - \Delta - \frac{v}{2}x + \mathcal{O}\left(\frac{\log t}{t}\right)} \sinh\Big(\frac{xy}{2t}\Big) e^{\frac{v}{2}y + \frac{\delta(t)}{2t}y}\psi_t(y) e^{-\frac{y^2}{4t}}. \tag{82}$$

Then since $h(x,t) = \int_0^\infty \mathrm{d}y\, q(t,x,y) h_0(y)$—see (20)—we have

$$h\big(m(t) + x, t\big) = \frac{1}{\sqrt{4\pi}\, t^{3/2}} e^{t\left(1-\frac{v^2}{4}\right) - \frac{v}{2}\delta(t) - \Delta - \frac{v}{2}x + \mathcal{O}\left(\frac{\log t}{t}\right)} H(x,t), \tag{83}$$

with

$$H(x,t) = \int_0^\infty \mathrm{d}y\, h_0(y) 2t \sinh\Big(\frac{xy}{2t}\Big) e^{\frac{v}{2}y + \frac{\delta(t)}{2t}y}\psi_t(y) e^{-\frac{y^2}{4t}}. \tag{84}$$

We now must choose v and $\delta(t)$, depending on the initial condition, such that (83) has a finite and non-zero limit as $t \to \infty$.

We use the following simple calculus lemma to evaluate $H(x,t)$. We defer the proof to the end of this section.

Lemma 9. *Let $\phi(y)$ a bounded function such that*

$$\phi(y) \sim Ay^{\alpha} \qquad \text{as } y \to \infty \tag{85}$$

for some $A > 0$ and some α. If $\epsilon_t = o\big(t^{-1/2}\big)$ then, as $t \to \infty$,

$$\int_0^\infty \mathrm{d}y\, \phi(y) e^{-\frac{y^2}{4t} + \epsilon_t y} \psi_t(y) \begin{cases} \sim A\, 2^{\alpha} e^{\Delta} \Gamma\Big(\dfrac{1+\alpha}{2}\Big) t^{\frac{1+\alpha}{2}} & \text{if} \quad \alpha > -1 \quad \text{(86a)} \\[2ex] \sim \dfrac{A}{2} e^{\Delta} \log t & \text{if} \quad \alpha = -1 \quad \text{(86b)} \\[2ex] \to \displaystyle\int_0^\infty \mathrm{d}y\, \phi(y) \psi_\infty(y) & \text{if} \quad \alpha < -1. \quad \text{(86c)} \end{cases}$$

If (85) is replaced by $\phi(y) = \mathcal{O}(y^{\alpha})$, then (86c) remains valid, and (86a) and (86b) are respectively replaced by $\mathcal{O}(t^{(1+\alpha)/2})$ and $\mathcal{O}(\log t)$.

We now continue with the proof of Theorem 1. We distinguish two cases.

Case 1: $h_0(y) = \mathcal{O}\Big(y^{\nu} e^{-\frac{v}{2}y}\Big)$ for some ν We introduce $H_1(t)$ such that $xH_1(t)$ is the same as $H(x,t)$ with the sinh expanded to first order:

$$H_1(t) = \int_0^\infty \mathrm{d}y\, \Big(h_0(y) e^{\frac{v}{2}y}\Big) y e^{\frac{\delta(t)}{2t} y} \psi_t(y) e^{-\frac{y^2}{4t}}. \tag{87}$$

For any $z \geq 0$, by Taylor's theorem (with the Lagrange remainder), there exists $w \in [0, z]$ such that $0 \leq \sinh(z) - z = \frac{z^3}{6} \cosh(w) \leq \frac{z^3}{6} e^{z}$. It follows that

$$\Big| H(x,t) - xH_1(t) \Big| \leq \frac{x^3}{24t^2} \int_0^\infty \mathrm{d}y\, \Big(\big|h_0(y)\big| e^{\frac{v}{2}y} \Big) y^3 e^{\frac{x+\delta(t)}{2t} y} \psi_t(y) e^{-\frac{y^2}{4t}}. \tag{88}$$

By applying Lemma 9 to $\phi(y) = \big|h_0(y)\big| e^{\frac{v}{2}y} y^3$ with $\alpha = \nu + 3$ we obtain

$$H(x,t) - xH_1(t) = \begin{cases} \mathcal{O}\left(t^{\nu/2}\right) & \text{if } \nu > -4, \\ \mathcal{O}\left(t^{-2}\log t\right) & \text{if } \nu = -4, \\ \mathcal{O}\left(t^{-2}\right) & \text{if } \nu < -4. \end{cases} \tag{89}$$

We now apply Lemma 9 to $H_1(t)$ with $\alpha = \nu + 1$ and obtain

$$xH_1(t) \sim \begin{cases} x\dfrac{A}{2} e^{\Delta} \log t & \text{if } h_0(y) \sim Ay^{-2} e^{-\frac{v}{2}y} \text{ with } A > 0, \\[2ex] xAe^{\Delta} 2^{\nu+1} \Gamma\Big(1 + \dfrac{\nu}{2}\Big) t^{1+\frac{\nu}{2}} & \text{if } h_0(y) \sim Ay^{\nu} e^{-\frac{v}{2}y} \text{ with } A > 0 \text{ and } \nu > -2, \\[2ex] x\displaystyle\int_0^\infty \mathrm{d}y\, h_0(y) y e^{\frac{v}{2}y} \psi_\infty(y) & \text{if } h_0(y) = \mathcal{O}\Big(y^{\nu} e^{-\frac{v}{2}y}\Big) \text{ for some } \nu < -2, \end{cases} \tag{90}$$

where we assumed that in the third case the right hand side is non-zero. As the difference (89) between $H(x,t)$ and $xH_1(t)$ is always asymptotically small compared to the values in the right hand side of (90), it follows that (90) also gives the asymptotic behaviour of $H(x,t)$.

We now plug this estimate of $H(x,t)$ into (83). To prevent $h\big(m(t)+x,t\big)$ from growing exponentially fast we need to take $v=2$. Then $\delta(t)$ must be adjusted (up to a constant a) to kill the remaining time dependence. We find

$$\delta(t)=\begin{cases}-\dfrac{1-\nu}{2}\log t+a+o(1) & \text{if } h_0(y)\sim Ay^{\nu}e^{-y} \text{ with } A>0 \text{ and } \nu>-2,\\[2ex] -\dfrac{3}{2}\log t+\log\log t+a+o(1) & \text{if } h_0(y)\sim Ay^{-2}e^{-y} \text{ with } A>0,\\[2ex] -\dfrac{3}{2}\log t+a+o(1) & \text{if } h_0(y)=\mathcal{O}\big(y^{\nu}e^{-y}\big) \text{ for some } \nu<-2.\end{cases} \tag{91}$$

In (83), when $t\to\infty$, all the t-dependence disappears and what remains is $\omega(x)$ from the theorem, with the claimed value of α. This proves cases (b), (c) and (d) of Theorem 1.

Case 2: $h_0(y)\sim Ay^{\nu}e^{-\gamma y}$ with $\gamma<v/2$ We write $h_0(y)=g_0(y)e^{-\gamma y}$ with $g_0(y)\sim Ay^{\nu}$ so that (84) becomes

$$H(x,t)=2t\int_0^{\infty}\mathrm{d}y\, g_0(y)\sinh\left(\frac{xy}{2t}\right)\psi_t(y)e^{\frac{\delta(t)}{2t}y}e^{\frac{v}{2}y-\gamma y-\frac{y^2}{4t}}. \tag{92}$$

The terms in the second exponential reach a maximum at $y=\lambda t$ with $\lambda=v-2\gamma$. We make the change of variable $y=\lambda t+u\sqrt{t}$; after rearranging we have

$$\begin{aligned}H(x,t)=2t^{\nu+\frac{3}{2}}e^{\frac{\lambda^2}{4}t+\lambda\frac{\delta(t)}{2}}&\int_{-\lambda\sqrt{t}}^{\infty}\mathrm{d}u\,\frac{g_0(\lambda t+u\sqrt{t})}{t^{\nu}}\\ &\times\sinh\left(\frac{\lambda x}{2}+\frac{ux}{2\sqrt{t}}\right)\psi_t(\lambda t+u\sqrt{t})e^{u\frac{\delta(t)}{2\sqrt{t}}-\frac{u^2}{4}}.\end{aligned} \tag{93}$$

We bound each term in the integral with the goal of applying dominated convergence.

- As g_0 is bounded for small y and $g_0\sim Ay^{\nu}$ for large y, we can take $\tilde{A}$ such that $\big|g_0(y)\big|\le\tilde{A}(y+1)^{\nu}$. Then

$$\begin{aligned}\frac{\big|g_0(\lambda t+u\sqrt{t})\big|}{t^{\nu}}&\le\tilde{A}\lambda^{\nu}\left(1+\frac{u\sqrt{t}+1}{\lambda t}\right)^{\nu}\\ &\le\tilde{A}\lambda^{\nu}e^{\frac{|\nu|(u\sqrt{t}+1)}{\lambda t}}\le 2\tilde{A}\lambda^{\nu}e^{u}\text{ for } t \text{ large enough.}\end{aligned} \tag{94}$$

- We have the simple bound

$$\sinh\left(\frac{\lambda x}{2}+\frac{ux}{2\sqrt{t}}\right)\le e^{\frac{\lambda x}{2}+\frac{ux}{2\sqrt{t}}}\le e^{\frac{\lambda x}{2}+u}\text{ for } t \text{ large enough.} \tag{95}$$

- $\psi_t(\cdot)$ is bounded by Proposition 8.
- Finally, $\exp\big(u\delta(t)/(2\sqrt{t})\big)\le e^{u}$ for t large enough.

We have bounded the integrand in (93) by a constant times $\exp(3u - u^2/4)$ for t large enough, so we can apply dominated convergence. As $t \to \infty$, the $g_0(\cdot)/t^\nu$ term converges to $A\lambda^\nu$, the $\sinh(\cdot)$ term to $\sinh(\lambda x/2)$, the $\psi_t(\cdot)$ term to e^Δ and the exponential to $e^{-u^2/4}$. We are left with some constants and the integral of $e^{-u^2/4}$, which is $\sqrt{4\pi}$, and finally:

$$H(x,t) \sim 2t^{\nu+\frac{3}{2}} e^{\frac{\lambda^2}{4}t+\lambda\frac{\delta(t)}{2}} A\lambda^\nu \sinh\Big(\frac{\lambda x}{2}\Big) e^\Delta \sqrt{4\pi}. \tag{96}$$

In (83), this gives

$$h\big(m(t)+x,t\big) = 2\sinh\Big(\frac{\lambda x}{2}\Big) e^{-\frac{v}{2}x} \times e^{t\big(1-\frac{v^2}{4}+\frac{\lambda^2}{4}\big)-\frac{v-\lambda}{2}\delta(t)+o(1)} t^\nu A\lambda^\nu. \tag{97}$$

Recall that $\lambda = v - 2\gamma$. To avoid exponential growth, we need $1 - v^2/4 + \lambda^2/4 = 0$, which implies $v = \gamma + 1/\gamma$ with $\gamma < 1$ because we started with the assumption $\gamma < v/2$. As $\frac{v-\lambda}{2} = \gamma$, to have convergence of $h\big(m(t)+x,t\big)$ we need $\delta(t)$ to be of the form

$$\delta(t) = \frac{\nu}{\gamma}\log t + a + o(1) \quad \text{for large } t. \tag{98}$$

Writing the $\sinh(\cdot)$ as the difference of two exponentials leads to $2\sinh(\lambda x/2)e^{-vx/2} = e^{-\gamma x} - e^{-(1/\gamma)x}$; we then recover case (a) of Theorem 1 with the claimed value of $\omega(x)$ and α.

This completes the proof of Theorem 1, subject to proving Lemma 9. □

Proof of Lemma 9. Recall from Proposition 8 that $\psi_t(y)$ is bounded in t and y, $\psi_\infty(y) := \lim_{t\to\infty}\psi_t(y)$ exists, $\lim_{y\to\infty}\psi_\infty(y)$ exists and equals e^Δ, and $\lim_{t\to\infty}\psi_t(t^\alpha) = e^\Delta$ for any $\alpha > 0$.

For $\alpha < -1$, the result is obtained with dominated convergence by noticing that $e^{-y^2/(4t)+\epsilon_t y}$ is bounded by $e^{t\epsilon_t^2}$ (value obtained at $y = 2t\epsilon_t$). With $\epsilon_t = o\big(t^{-1/2}\big)$, this is bounded by a constant.

For $\alpha > -1$, cut the integral at $y = 1$. The integral from 0 to 1 is bounded, and in the integral from 1 to ∞ we make the substitution $y = u\sqrt{t}$:

$$\int_0^\infty \mathrm{d}y\,\phi(y)e^{-\frac{y^2}{4t}+\epsilon_t y}\psi_t(y) = \mathcal{O}(1) + t^{\frac{1+\alpha}{2}}\int_{\frac{1}{\sqrt{t}}}^\infty \mathrm{d}u\,\frac{\phi(u\sqrt{t})}{t^{\alpha/2}}e^{-\frac{u^2}{4}+\sqrt{t}\epsilon_t u}\psi_t(u\sqrt{t}). \tag{99}$$

A simple application of dominated convergence then leads to

$$\int_0^\infty \mathrm{d}y\,\phi(y)e^{-\frac{y^2}{4t}+\epsilon_t y}\psi_t(y) = \mathcal{O}(1) + t^{\frac{1+\alpha}{2}}\bigg(\int_0^\infty \mathrm{d}u\,Au^\alpha e^{-\frac{u^2}{4}}e^\Delta + o(1)\bigg), \tag{100}$$

and the substitution $t = u^2/4$ gives (86a).

For $\alpha = -1$, we cut the integral at $y = \sqrt{t}$ and again make the change of variable $y = u\sqrt{t}$ in the second part:

$$\begin{aligned}&\int_0^\infty \mathrm{d}y\,\phi(y)e^{-\frac{y^2}{4t}+\epsilon_t y}\psi_t(y)\\ &\quad= \int_0^{\sqrt{t}} \mathrm{d}y\,\phi(y)e^{-\frac{y^2}{4t}+\epsilon_t y}\psi_t(y) + \int_1^\infty \mathrm{d}u\,\sqrt{t}\phi(u\sqrt{t})e^{-\frac{u^2}{4}+\sqrt{t}\epsilon_t u}\psi_t(u\sqrt{t}).\end{aligned} \tag{101}$$

Again by dominated convergence, the second integral has a limit; we simply write it as $\mathcal{O}(1)$. For the first, the integrand is bounded so the integral from 0 to 1 is certainly $\mathcal{O}(1)$, and we may concentrate on the integral from 1 to $\sqrt{t}$. Making the substitution $y = t^x$, we have

$$\int_1^{\sqrt{t}} \mathrm{d}y\, \phi(y) e^{-\frac{y^2}{4t} + \epsilon_t y} \psi_t(y) = (\log t) \int_0^{1/2} \mathrm{d}x\, t^x \phi(t^x) e^{-\frac{t^{2x-1}}{4} + \epsilon_t t^x} \psi_t(t^x). \quad (102)$$

The integrand on the right converges for each $x \in (0, 1/2)$ to Ae^{Δ} so by dominated convergence,

$$\int_1^t \mathrm{d}y\, \phi(y) e^{-\frac{y^2}{4t} + \epsilon_t y} \psi_t(y) \sim \frac{A}{2} e^{\Delta} \log t, \quad (103)$$

as required. □

6. Estimating ψ_t: Finer Bounds, and Proof of Theorem 2

We want to refine Proposition 8 and estimate the speed of convergence of $\psi_t(y, x)$ to its limit as $t \to \infty$. As we are only interested up to errors of order $\frac{\log t}{t}$, it suffices to consider the case $x = 0$ since by (67), $\psi_t(y, x) = \psi_t(y) e^{x\mathcal{O}(\frac{\log t}{t})}$.

Recall that

$$\psi_t(y) = \mathbb{E}\Big[e^{I_t(y)}\Big], \quad \psi_\infty(y) = \mathbb{E}\Big[e^{I(y)}\Big], \quad (104)$$

where, introducing $I_t(y) := I_t(y, 0)$,

$$\begin{aligned} I_t(y) &= \frac{1}{2}\int_0^t \mathrm{d}s\, m''(s)\Big(\xi_s^{(t:y\to 0)} - y\frac{t-s}{t}\Big) = \frac{1}{2}\int_0^t \mathrm{d}s\, m''(s)\frac{t-s}{t}\Big(\xi_{\frac{st}{t-s}}^{(y)} - y\Big), \\ I(y) &= \frac{1}{2}\int_0^\infty \mathrm{d}s\, m''(s)\big(\xi_s^{(y)} - y\big). \end{aligned} \quad (105)$$

We have used the change of time (30) to give the second expression of $I_t(y)$. As in the hypothesis (12) of Theorem 2, we suppose that m is twice continuously differentiable and

$$m''(t) = \frac{3}{2(t+1)^2} + r''(t) \quad \text{with} \quad r''(t) = \mathcal{O}\left(\frac{1}{t^{2+\eta}}\right), \quad \eta > 0. \quad (106)$$

Our estimate of $\psi_t(y)$ is based on the following two propositions. By writing $I_t(y) = I(y) - (I(y) - I_t(y))$ in the definition of $\psi_t(y)$, and expanding the exponential in the small correction term $I(y) - I_t(y)$, we show that:

Proposition 10. *Assuming* (106)*, the following holds uniformly in y:*

$$\psi_t(y) = \psi_\infty(y)\Big(1 - \mathbb{E}\big[I(y) - I_t(y)\big]\Big) + \mathcal{O}\Big(\frac{\log t}{t}\Big) + y\mathcal{O}\Big(\frac{1}{t}\Big). \quad (107)$$

Further, some straightforward computations give that:

Proposition 11. *Assuming* (106)*, the following holds uniformly in y:*

$$\mathbb{E}\big[I(y) - I_t(y)\big] = \frac{3\sqrt{\pi}}{\sqrt{t}} + y\mathcal{O}\left(\frac{\log t}{t}\right) + \begin{cases} \mathcal{O}\left(\frac{1}{t}\right) & \text{if } \eta > 1/2, \\ \mathcal{O}\left(\frac{\log t}{t}\right) & \text{if } \eta = 1/2, \\ \mathcal{O}\left(\frac{1}{t^{1/2+\eta}}\right) & \text{if } \eta < 1/2. \end{cases} \tag{108}$$

We prove Propositions 10 and 11 in Sects. 6.2 and 6.3, after some preparatory work in Sect. 6.1. We now show how to prove Theorem 2 from these two propositions.

Proof of Theorem 2. We assume that $m(t)$ satisfies the hypothesis (12) of Theorem 2:

$$m(t) = 2t - \frac{3}{2}\log(t+1) + a + r(t)$$
$$\text{with } r(t) = o(1) \text{ and } r''(t) = \mathcal{O}\Big(\frac{1}{t^{2+\nu}}\Big) \text{ for large } t. \tag{109}$$

As in the proof of Theorem 1, we recall that $h\big(m(t) + x, t\big)$ is related to $H(x, t)$ through (83) and that $H(x, t)$ is given by (84). With $v = 2$ and $\delta(t) = -(3/2)\log(t + 1) + a + r(t)$, these two equations read:

$$h\big(m(t)+x,t\big) = \frac{1}{\sqrt{4\pi}} e^{-a-r(t)-\Delta-x+\mathcal{O}\left(\frac{\log t}{t}\right)} H(x,t), \tag{110}$$

$$H(x,t) = \int_0^\infty \mathrm{d}y \left(h_0(y)e^y\right) 2t \sinh\left(\frac{xy}{2t}\right) e^{\frac{-(3/2)\log(t+1)+a+r(t)}{2t}y - \frac{y^2}{4t}} \psi_t(y), \tag{111}$$

We compute $H(x, t)$ for an initial condition $h_0(x) = \mathcal{O}\big(x^\nu e^{-x}\big)$ for some $\nu < -2$. In (87) in the proof of Theorem 1, we introduced $H_1(t)$ which is $H(x, t)/x$ with the sinh replaced by its first order expansion:

$$H_1(t) = \int_0^\infty \mathrm{d}y \left(h_0(y)e^y\right) y e^{\frac{-(3/2)\log(t+1)+a+r(t)}{2t}y - \frac{y^2}{4t}} \psi_t(y), \tag{112}$$

and we showed in (89) that the difference between $H(x, t)$ and $xH_1(t)$ is very small. We continue to simplify the integral by introducing successive simplifications

$$\begin{aligned} H_2(t) &= \int_0^\infty \mathrm{d}y \left(h_0(y)e^y\right) y e^{-\frac{y^2}{4t}} \psi_t(y), \\ H_3(t) &= \int_0^\infty \mathrm{d}y \left(h_0(y)e^y\right) y e^{-\frac{y^2}{4t}} \psi_\infty(y), \\ H_4 &= \int_0^\infty \mathrm{d}y \left(h_0(y)e^y\right) y \psi_\infty(y), \end{aligned} \tag{113}$$

and by writing

$$\begin{aligned} H(x,t) = {} & \big(H(x,t) - xH_1(t)\big) + x\big(H_1(t) - H_2(t)\big) + x\left(H_2(t) - \left[1 - \frac{3\sqrt{\pi}}{\sqrt{t}}\right] H_3(t)\right) \\ & + x\left[1 - \frac{3\sqrt{\pi}}{\sqrt{t}}\right]\big(H_3(t) - H_4\big) + x\left[1 - \frac{3\sqrt{\pi}}{\sqrt{t}}\right] H_4. \end{aligned} \tag{114}$$

We now bound the successive differences in the above expression, as we did in (89), for the first one.

For t large enough, $-\frac{3}{2}\log(t+1)+a+r(t) < 0$ and for $z > 0$ we have $0 \le 1-e^{-z} \le z$. Thus

$$\left|H_2(t) - H_1(t)\right| \le \frac{\frac{3}{2}\log(t+1) - a - r(t)}{2t} \int_0^\infty \mathrm{d}y \left(\left|h_0(y)\right|e^y\right) y^2 e^{-\frac{y^2}{4t}} \psi_t(y). \tag{115}$$

An application of Lemma 9 with $\phi(y) = h_0(y)e^y y^2$ and hence $\alpha = \nu + 2$ then gives

$$H_1(t) - H_2(t) = \begin{cases} \mathcal{O}\left(t^{\frac{1+\nu}{2}} \log t\right) & \text{if } \nu > -3, \\ \mathcal{O}\left(\frac{\log^2 t}{t}\right) & \text{if } \nu = -3, \\ \mathcal{O}\left(\frac{\log t}{t}\right) & \text{if } \nu < -3. \end{cases} \tag{116}$$

For the difference involving H_2 and H_3, we use Propositions 10 and 11 which give that uniformly in y,

$$\psi_t(y) = \psi_\infty(y)\left(1 - \frac{3\sqrt{\pi}}{\sqrt{t}}\right) + y\mathcal{O}\left(\frac{\log t}{t}\right) + \begin{cases} \mathcal{O}\left(\frac{1}{t}\right) & \text{if } \eta > 1/2, \\ \mathcal{O}\left(\frac{\log t}{t}\right) & \text{if } \eta = 1/2, \\ \mathcal{O}\left(\frac{1}{t^{1/2+\eta}}\right) & \text{if } \eta < 1/2. \end{cases} \tag{117}$$

We get

$$\begin{aligned} &H_2(t) - \left(1 - \frac{3\sqrt{\pi}}{\sqrt{t}}\right) H_3(t) \\ &= \int_0^\infty \mathrm{d}y \left(h_0(y)e^y\right) y e^{-\frac{y^2}{4t}} \left[\psi_t(y) - \psi_\infty(y)\left(1 - \frac{3\sqrt{\pi}}{\sqrt{t}}\right)\right], \\ &= \mathcal{O}\left(\frac{1}{t^{1/2+\eta}}\right) + \begin{cases} \mathcal{O}\left(t^{\frac{1+\nu}{2}} \log t\right) & \text{if } \nu > -3, \\ \mathcal{O}\left(\frac{\log^2 t}{t}\right) & \text{if } \nu = -3, \\ \mathcal{O}\left(\frac{\log t}{t}\right) & \text{if } \nu < -3. \end{cases} \end{aligned} \tag{118}$$

Indeed, the $y\mathcal{O}\left(\frac{\log t}{t}\right)$ gives the same correction as in (116) by another application of Lemma 9 with $\alpha = \nu + 2$. As $\int \mathrm{d}y \left|h_0(y)\right|e^y y < \infty$ because $\nu < -2$, the contribution of the $y\mathcal{O}\left(\frac{\log t}{t}\right)$ term subsumes the other $\mathcal{O}$ in (117) except in the case $\eta < \frac{1}{2}$.

Finally, notice that $\left|H_4\right| < \infty$ because we supposed $\nu < -2$. Recalling $\psi_\infty(y) \le K_2$, one has

$$\begin{aligned}\left|H_4 - H_3(t)\right| &\le \int_0^\infty \mathrm{d}y \left(\left|h_0(y)\right| e^y\right) y \left(1 - e^{-\frac{y^2}{4t}}\right) \psi_\infty(y), \\ &\le K_2 \int_0^{\sqrt{t}} \mathrm{d}y \left(\left|h_0(y)\right| e^y\right) y \frac{y^2}{4t} + K_2 \int_{\sqrt{t}}^\infty \mathrm{d}y \left(\left|h_0(y)\right| e^y\right) y, \\ &= \begin{cases} \mathcal{O}\left(t^{1+\frac{\nu}{2}}\right) & \text{if } -2 > \nu > -4, \\ \mathcal{O}\left(\frac{\log t}{t}\right) & \text{if } \nu = -4, \\ \mathcal{O}\left(\frac{1}{t}\right) & \text{if } \nu < -4, \end{cases}\end{aligned} \tag{119}$$

where we used $h_0(y)e^y = \mathcal{O}(y^\nu)$. The end result comes from the integral from 0 to $\sqrt{t}$; the other integral is always $\mathcal{O}(t^{1+\nu/2})$.

Finally, collecting the differences (89), (116), (118) and (119) leads with (114) to

$$H(x,t) = x H_4 \left[1 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \mathcal{O}\left(t^{1+\frac{\nu}{2}}\right) + \mathcal{O}\left(\frac{1}{t^{1/2+\eta}}\right) + \mathcal{O}\left(\frac{\log t}{t}\right)\right]. \tag{120}$$

Substituting into (110) and expanding $e^{-r(t)}$ leads to the main expression (13) of Theorem 2, with the value α given in Theorem 1.

We now turn to the second part of Theorem 2 and assume that $h_0(y) \sim A y^\nu e^{-y}$ with $-4 < \nu < -2$. We look for an estimate of $H_4 - H_3(t)$ which is more precise than (119).

Writing $H_4 - H_3(t)$ as a single integral and doing the change of variable $y = u\sqrt{t}$ one gets

$$H_4 - H_3(t) = t^{1+\frac{\nu}{2}} \int_0^\infty \mathrm{d}u \, \frac{h_0(u\sqrt{t})e^{u\sqrt{t}}}{t^{\nu/2}} u \left(1 - e^{-\frac{u^2}{4}}\right) \psi_\infty(u\sqrt{t}). \tag{121}$$

A simple application of dominated convergence then gives

$$H_4 - H_3(t) \sim t^{1+\frac{\nu}{2}} A e^\Delta \int_0^\infty \mathrm{d}u \, u^{\nu+1} \left(1 - e^{-\frac{u^2}{4}}\right) = -A e^\Delta 2^{\nu+1} \Gamma\left(\frac{\nu}{2} + 1\right) t^{1+\frac{\nu}{2}}, \tag{122}$$

and (120) becomes

$$H(x,t) = x H_4 \left[1 - \frac{3\sqrt{\pi}}{\sqrt{t}}\right] + x A e^\Delta 2^{\nu+1} \Gamma\left(\frac{\nu}{2} + 1\right) t^{1+\frac{\nu}{2}} + o\left(t^{1+\frac{\nu}{2}}\right) + \mathcal{O}\left(\frac{1}{t^{\frac{1}{2}+\eta}}\right). \tag{123}$$

Using (110), this yields (13). $\square$

6.1. Decorrelation between $I(y)$ and $\xi_s^{(y)}$. A large part of our argument relies on a statement that roughly says "$I(y)$ and $\xi_s^{(y)}$ are almost independent for large s". The following proposition makes this precise.

Proposition 12. *Suppose that m is twice continuously differentiable with $m''(t) = \mathcal{O}(1/t^2)$. Define*

$$w(y,s) = \mathbb{E}\Big[e^{I(y)}\big(\xi_s^{(y)} - y\big)\Big] - \mathbb{E}\Big[e^{I(y)}\Big]\mathbb{E}\big[\xi_s^{(y)} - y\big]. \tag{124}$$

There exists a constant $C > 0$ such that

$$\begin{aligned} |w(y,s)| &\le C\log(s+1) && \text{for all } s, y \ge 0,\\ |w(y,s)| &\le C\left(1 + y\frac{\log(s+1)}{\sqrt{s}}\right) && \text{for all } s, y \ge 0,\\ \Big|w(y,s+\delta) - w(y,s)\Big| &\le C\frac{\delta}{s+1} && \text{for all } y \ge 0, \text{ whenever } 0 \le \delta \le s^2. \end{aligned} \tag{125}$$

The proof of this result is quite involved. The first step is to prove two fairly accurate estimates on the difference between two bridges with different end points, the first of which is best when the starting point y is large and the second of which is more accurate when y is small.

It is well-known that a Bessel process started from y and conditioned to be at position x at time t is equal in law to a Bessel bridge from y to x in time t followed by an independent Bessel process started from x at time t. We defined $\xi_s^{(t:y\to x)}$ for $s \in [0,t]$ as a Bessel bridge from y to x in a time t. In this section, we extend the definition of $\xi_s^{(t:y\to x)}$ for $s > t$ by interpreting it as an independent Bessel started from x at time t, so that $\xi_s^{(t:y\to x)}$, $s \ge 0$ is a Bessel process conditioned to be at x at time t. We assume that the Bessel processes attached to $\xi_s^{(t:y\to x)}$ for $s \ge t$ are built for all x and t with the same noise, so that we can compare them to each other. In particular, we apply (34) and (49) to these Bessel processes.

Recall that $I(y) = \frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(u)\big(\xi_u^{(y)} - y\big)$ and define

$$\tilde{I}_t(y,z) = \frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(u)\big(\xi_u^{(t:y\to z)} - y\big). \tag{126}$$

Lemma 13. *If m is twice continuously differentiable with $m''(t) = \mathcal{O}(1/t^2)$, then there exists a constant c and random variables G_t with distribution independent of t and Gaussian tails such that:*

- *For any t, y, z and x,*

$$|\tilde{I}_t(y,z) - \tilde{I}_t(y,x)| \le c|z-x|\frac{\log(t+1)}{t}. \tag{127}$$

- *For any t, y and z,*

$$\Big|\tilde{I}_t(y,z) - \tilde{I}_t(y,0)\Big| \le \frac{z^2}{t^{3/2}}G_t + c\left(\frac{z}{t} + \frac{z^3}{t^2} + \frac{z^2 y}{t^2}\log(t+1)\right). \tag{128}$$

Proof. Recall from (34) and (35) that $\left|\xi_s^{(t:y\to z)} - \xi_s^{(t:y\to x)}\right| \le |z-x| \min(s/t, 1)$. Therefore

$$\left|\tilde{I}_t(y,z) - \tilde{I}_t(y,x)\right| \le \frac{1}{2}\int_0^\infty \mathrm{d}s\, |m''(s)| \left|\xi_s^{(t:y\to z)} - \xi_s^{(t:y\to x)}\right| \tag{129}$$

$$\le \frac{1}{2}|z-x|\left(\int_0^t \mathrm{d}s\, |m''(s)|\frac{s}{t} + \int_t^\infty \mathrm{d}s\, |m''(s)|\right). \tag{130}$$

The first integral is $\mathcal{O}\left(\frac{\log t}{t}\right)$ while the second is a $\mathcal{O}(1/t)$. Their sum can be bounded by $2c\log(t+1)/t$ for some c, which proves the simpler bound (127).

To prove (128) we consider $x = 0$ and split the integral at $t/2$ and t. For $s > t/2$, with the same simple bounds as above we have

$$\left|\int_{\frac{t}{2}}^\infty \mathrm{d}s\, m''(s)\, \left(\xi_s^{(t:y\to z)} - \xi_s^{(t:y\to 0)}\right)\right|$$
$$\le z\left(\int_{\frac{t}{2}}^t \mathrm{d}s\, |m''(s)|\frac{s}{t} + \int_t^\infty \mathrm{d}s\, |m''(s)|\right) = z\mathcal{O}\left(\frac{1}{t}\right). \tag{131}$$

From 0 to $t/2$, we claim that the following bound is true:

$$0 \le \int_0^{\frac{t}{2}} \mathrm{d}s \frac{\xi_s^{(t:y\to z)} - \xi_s^{(t:y\to 0)}}{(1+s)^2} \le \frac{z^3}{3t^2} + \frac{z^2}{t^{3/2}} G_t + \frac{2z^2 y}{3t^2}\log(t+1), \tag{132}$$

for some non-negative G_t with distribution independent of t and Gaussian tails. Then, as there exists some constant c' such that $|m''(s)| \le c'/(1+s)^2$, (131) and (132) give the result (128). Therefore it only remains to prove (132).

We use the bound $\coth(x) \le 1/x + x/3$, together with the SDEs (31) and (33). We already know from Lemma 5 that $\xi_s^{(t:y\to 0)} \le \xi_s^{(t:y\to z)} \le \xi_s^{(t:y\to 0)} + zs/t$ for any $s \in [0,t]$. Therefore for any $s \in [0,t)$,

$$\mathrm{d}\xi_s^{(t:y\to z)} - \mathrm{d}\xi_s^{(t:y\to 0)} \le \left(\frac{z}{t-s}\coth\frac{z\xi_s^{(t:y\to z)}}{2(t-s)} - \frac{2}{\xi_s^{(t:y\to 0)}}\right)\mathrm{d}s \tag{133}$$

$$\le \frac{z^2\xi_s^{(t:y\to z)}}{6(t-s)^2}\mathrm{d}s \tag{134}$$

$$\le \left(\frac{z^3 s}{6t(t-s)^2} + \frac{z^2\xi_s^{(t:y\to 0)}}{6(t-s)^2}\right)\mathrm{d}s. \tag{135}$$

By integration by parts,

$$\int_0^{\frac{t}{2}} \mathrm{d}s \frac{\xi_s^{(t:y\to z)}}{(s+1)^2} = \int_0^{\frac{t}{2}} \frac{1}{s+1}\mathrm{d}\xi_s^{(t:y\to z)} - \frac{\xi_{t/2}^{(t:y\to z)}}{t/2+1} + y. \tag{136}$$

Using (35), the estimate on $\mathrm{d}\xi_s^{(t:y\to z)} - \mathrm{d}\xi_s^{(t:y\to 0)}$ from above, and $t - s \geq t/2$ for $s \leq t/2$, we get

$$0 \leq \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{\xi_s^{(t:y\to z)} - \xi_s^{(t:y\to 0)}}{(s+1)^2}$$

$$\leq \int_0^{\frac{t}{2}} \frac{1}{s+1} \mathrm{d}\xi_s^{(t:y\to z)} - \int_0^{\frac{t}{2}} \frac{1}{s+1} \mathrm{d}\xi_s^{(t:y\to 0)} \tag{137}$$

$$\leq \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{z^3 s}{6t(t-s)^2(s+1)} + \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{z^2 \xi_s^{(t:y\to 0)}}{6(t-s)^2(s+1)} \tag{138}$$

$$\leq \frac{2z^3}{3t^3} \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{s}{s+1} + \frac{2z^2}{3t^2} \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{\xi_s^{(t:y\to 0)}}{s+1} \tag{139}$$

$$\leq \frac{z^3}{3t^2} + \frac{2z^2}{3t^2} \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{y + \xi_s^{(t:0\to 0)}}{s+1} \tag{140}$$

$$\leq \frac{z^3}{3t^2} + \frac{2z^2 y}{3t^2} \log(t+1) + \frac{2z^2 y}{3t^2} \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{\xi_s^{(t:0\to 0)}}{s}. \tag{141}$$

By the scaling property, we introduce another Bessel bridge $\tilde{\xi}^{(1:0\to 0)}$ by setting $\xi_{tu}^{(t:0\to 0)} = \sqrt{t}\tilde{\xi}_u^{(1:0\to 0)}$. By adapting Lemma 7 to the new Bessel bridge, there exists a random variable G_t with distribution independent of t and Gaussian tails such that $\tilde{\xi}_u^{(1:0\to 0)} \leq G_t u^{\frac{1}{4}}$. Hence

$$\int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{\xi_s^{(t:0\to 0)}}{s} = \int_0^{\frac{1}{2}} \mathrm{d}u \, \frac{\xi_{tu}^{(t:0\to 0)}}{u} \leq \sqrt{t}\, G_t \int_0^{\frac{1}{2}} \mathrm{d}u \, u^{-\frac{3}{4}} \leq 4G_t\sqrt{t}. \tag{142}$$

This bounds the last term in (137) and establishes (132), thereby completing the proof. □

Finally, given that we are using random variables with Gaussian tails, the following trivial result is useful.

Lemma 14. *Suppose that G is a random variable with Gaussian tails. Then for any real number a and any polynomial P,*

$$\left|\mathbb{E}[P(G)e^{aG}]\right| < \infty. \tag{143}$$

We can now prove Proposition 12.

Proof of Proposition 12. Recall the definition(126) of $\tilde{I}$. For any deterministic x, since $\mathbb{E}\big[\xi_s^{(y)} - \mathbb{E}\big[\xi_s^{(y)}\big]\big] = 0$ and $\mathbb{E}\big[e^{\tilde{I}_s(y,x)}\big]$ is deterministic, we have

$$w(y,s) = \mathbb{E}\Big[e^{I(y)}\Big(\xi_s^{(y)} - \mathbb{E}\big[\xi_s^{(y)}\big]\Big)\Big] \tag{144}$$

$$= \mathbb{E}\Big[\Big(e^{I(y)} - \mathbb{E}\big[e^{\tilde{I}_s(y,x)}\big]\Big)\Big(\xi_s^{(y)} - \mathbb{E}\big[\xi_s^{(y)}\big]\Big)\Big] \tag{145}$$

$$= \int_0^\infty \Big(\mathbb{E}\big[e^{I(y)} | \xi_s^{(y)} = z\big] - \mathbb{E}\big[e^{\tilde{I}_s(y,x)}\big]\Big)\Big(z - \mathbb{E}\big[\xi_s^{(y)}\big]\Big)\mathbb{P}\big(\xi_s^{(y)} \in \mathrm{d}z\big) \tag{146}$$

$$= \int_0^\infty \mathbb{E}\Big[e^{\tilde{I}_s(y,z)} - e^{\tilde{I}_s(y,x)}\Big]\Big(z - \mathbb{E}\big[\xi_s^{(y)}\big]\Big)\mathbb{P}\big(\xi_s^{(y)} \in \mathrm{d}z\big), \tag{147}$$

where we used that $\mathbb{E}\big[e^{I(y)}\big|\xi_s^{(y)}=z\big]=\mathbb{E}\big[e^{\tilde I_s(y,z)}\big]$. Then

$$\big|w(y,s)\big|\le\int_0^\infty\mathbb{E}\Big[\Big|e^{\tilde I_s(y,z)}-e^{\tilde I_s(y,x)}\Big|\Big]\times\Big|z-\mathbb{E}\big[\xi_s^{(y)}\big]\Big|\times\mathbb{P}\big(\xi_s^{(y)}\in\mathrm{d}z\big).\tag{148}$$

By the mean value theorem, $|e^a-e^b|\le|a-b|e^{\max(a,b)}\le|a-b|e^{b+|a-b|}$. Thus

$$\begin{aligned}&|w(y,s)|\\&\le\int_0^\infty\mathbb{E}\Big[e^{\tilde I_s(y,x)}\Big|\tilde I_s(y,z)-\tilde I_s(y,x)\Big|e^{|\tilde I_s(y,z)-\tilde I_s(y,x)|}\Big]\Big|z-\mathbb{E}[\xi_s^{(y)}]\Big|\times\mathbb{P}\big(\xi_s^{(y)}\in\mathrm{d}z\big)\end{aligned}\tag{149}$$

$$\le\int_0^\infty\mathbb{E}\Big[e^{\tilde I_s(y,x)}\Big|\tilde I_s(y,z)-\tilde I_s(y,x)\Big|\Big]e^{c|z-x|\frac{\log(s+1)}{s}}\Big|z-\mathbb{E}[\xi_s^{(y)}]\Big|\times\mathbb{P}\big(\xi_s^{(y)}\in\mathrm{d}z\big),\tag{150}$$

where we applied (127) of Lemma 13 in the exponential. Now, by Cauchy–Schwarz,

$$\begin{aligned}|w(y,s)|\le\mathbb{E}\Big[e^{2\tilde I_s(y,x)}\Big]^{\frac12}\int_0^\infty\mathbb{E}\Big[\Big|\tilde I_s(y,z)-\tilde I_s(y,x)\Big|^2\Big]^{\frac12}\\\times e^{c|z-x|\frac{\log(s+1)}{s}}\Big|z-\mathbb{E}[\xi_s^{(y)}]\Big|\mathbb{P}\big(\xi_s^{(y)}\in\mathrm{d}z\big).\end{aligned}\tag{151}$$

Decompose $\tilde I_s(y,x)$ in the following way:

$$\begin{aligned}2\tilde I_s(y,x)=\int_0^s\mathrm{d}u\,m''(u)\Big(\xi_u^{(s:y\to x)}-y-(x-y)\frac{u}{s}\Big)+\int_s^\infty\mathrm{d}u\,m''(u)\Big(\xi_u^{(s:y\to x)}-x\Big)\\+\int_0^s\mathrm{d}u\,m''(u)(x-y)\frac{u}{s}+\int_s^\infty\mathrm{d}u\,m''(u)(x-y).\end{aligned}\tag{152}$$

The first integral is $2I_s(y,x)$. Using (50) it can be bounded uniformly in y, x and s by a variable with Gaussian tails. The second integral, which does not depend on y, can also be bounded uniformly in x and s using (49) by an independent variable with Gaussian tails. The third integral is $(x-y)\mathcal{O}\big(\frac{\log s}{s}\big)$ and the fourth is $(x-y)\mathcal{O}\big(\frac1s\big)$; they can be bounded together by $2c|x-y|\frac{\log(s+1)}{s}$ for some constant c. Finally, there exists a C_1 and a c such that, uniformly in s, y and x:

$$\mathbb{E}\Big[e^{2\tilde I_s(y,x)}\Big]^{\frac12}\le C_1e^{c|x-y|\frac{\log(1+s)}{s}}.\tag{153}$$

Substituting back into (151), we get

$$\begin{aligned}|w(y,s)|\le C_1e^{c|x-y|\frac{\log(1+s)}{s}}\int_0^\infty\mathbb{E}\Big[\Big|\tilde I_s(y,z)-\tilde I_s(y,x)\Big|^2\Big]^{\frac12}\\\times e^{c|z-x|\frac{\log(1+s)}{s}}\Big|z-\mathbb{E}[\xi_s^{(y)}]\Big|\mathbb{P}\left(\xi_s^{(y)}\in\mathrm{d}z\right).\end{aligned}\tag{154}$$

First we concentrate on showing the first line of (125), i.e. that $|w(y,s)|\le C\log(s+1)$. Using (127) again,

$$\mathbb{E}[|\tilde I_s(y,z)-\tilde I_s(y,x)|^2]^{1/2}\le c|z-x|\frac{\log(s+1)}{s},\tag{155}$$

so we get, by choosing $x = \mathbb{E}\big[\xi_s^{(y)}\big]$,

$$\begin{aligned}|w(y,s)| &\leq C_1 c \frac{\log(s+1)}{s} e^{c\left|\mathbb{E}[\xi_s^{(y)}]-y\right| \frac{\log(1+s)}{s}} \\ &\quad \times \int_0^\infty e^{c\left|z-\mathbb{E}[\xi_s^{(y)}]\right| \frac{\log(1+s)}{s}} \left(z - \mathbb{E}[\xi_s^{(y)}]\right)^2 \mathbb{P}(\xi_s^{(y)} \in \mathrm{d}z), \\ &= C_1 c \frac{\log(s+1)}{s} e^{c\left|\mathbb{E}[\xi_s^{(y)}]-y\right| \frac{\log(1+s)}{s}} \mathbb{E}\Big[e^{c\left|\xi_s^{(y)}-\mathbb{E}[\xi_s^{(y)}]\right| \frac{\log(1+s)}{s}} \left(\xi_s^{(y)} - \mathbb{E}[\xi_s^{(y)}]\right)^2\Big].\end{aligned} \tag{156}$$

It remains to bound the expectations above. Note from (34) that for all $z \geq 0$ we have $B_1 \leq \xi_1^{(z)} - z \leq \xi_1^{(0)}$ and therefore

$$B_1 - \mathbb{E}[\xi_1^{(0)}] \leq \xi_1^{(z)} - \mathbb{E}[\xi_1^{(z)}] \leq \xi_1^{(z)} - z \leq \xi_1^{(0)}, \tag{157}$$

so, with Γ the positive random variable with Gaussian tail defined by

$$\Gamma := \max\left\{\left|B_1 - \mathbb{E}\big[\xi_1^{(0)}\big]\right|, \left|\xi_1^{(0)}\right|\right\}, \tag{158}$$

we have, uniformly in z,

$$\left|\xi_1^{(z)} - \mathbb{E}\big[\xi_1^{(z)}\big]\right| \leq \Gamma, \qquad \left|\xi_1^{(z)} - z\right| \leq \Gamma. \tag{159}$$

Therefore, by the scaling property,

$$|w(y,s)| \leq C_1 c \frac{\log(s+1)}{s} e^{c\sqrt{s}\,\mathbb{E}[\Gamma] \frac{\log(1+s)}{s}} \mathbb{E}\Big[e^{c\sqrt{s}\,\Gamma \frac{\log(1+s)}{s}} s\Gamma^2\Big] \leq C \log(s+1), \tag{160}$$

for some constant C, where we used Lemma 14 to bound the last expectation. This is the first line of (125).

We now turn to showing the second line of (125), that $|w(y,s)| \leq C\big(1 + y\frac{\log(s+1)}{\sqrt{s}}\big)$. Given that we have already proven that $|w(y,s)| \leq C \log(s+1)$, it suffices to consider $y \leq \sqrt{s}$.

Recall (128):

$$\left|\tilde{I}_s(y,z) - \tilde{I}_s(y,0)\right| \leq \frac{z^2}{s^{3/2}} G_s + c\left(\frac{z}{s} + \frac{z^3}{s^2} + \frac{z^2 y}{s^2} \log(s+1)\right). \tag{161}$$

By Cauchy-Schwarz, if $a, b \geq 0$ and X is a non-negative random variable with finite second moment, then

$$\mathbb{E}[(aX+b)^2]^{1/2} \leq a\mathbb{E}[X^2]^{1/2} + b. \tag{162}$$

This tells us that

$$\mathbb{E}[|\tilde{I}_s(y,z) - \tilde{I}_s(y,0)|^2]^{\frac{1}{2}} \leq C_2 a_{s,y,z} \quad \text{with} \quad a_{s,y,z} = \frac{z}{s} + \frac{z^2}{s^{3/2}} + \frac{z^3}{s^2} + \frac{z^2 y}{s^2} \log(s+1) \tag{163}$$

for some constant C since the distribution of G_s does not depend on s.

Now choosing $x = 0$ in (154) and substituting (163), we get

$$|w(y,s)| \le C_1 C_2 e^{cy\frac{\log(1+s)}{s}} \int_0^\infty a_{s,y,z} e^{cz\frac{\log(1+s)}{s}} \Big|z - \mathbb{E}[\xi_s^{(y)}]\Big| \mathbb{P}(\xi_s^{(y)} \in \mathrm{d}z). \tag{164}$$

$$\le C_3 \mathbb{E}\Big[a_{s,y,\xi_s^{(y)}}\, e^{c\xi_s^{(y)}\frac{\log(1+s)}{s}} \Big|\xi_s^{(y)} - \mathbb{E}[\xi_s^{(y)}]\Big|\Big], \tag{165}$$

where we used $y \le \sqrt{s}$ to bound the factor in front of the integral by a constant. Using the scaling property, writing $\tilde{\xi}_1 = \xi_s^{(y)}/\sqrt{s}$ we have as in (159)

$$\big|\tilde{\xi}_1 - \mathbb{E}[\tilde{\xi}_1]\big| \le \Gamma, \quad \big|\tilde{\xi}_1 - y/\sqrt{s}\big| \le \Gamma, \quad \tilde{\xi}_1 \le 1 + \Gamma \tag{166}$$

for some positive random variable Γ with Gaussian tails; we used $y \le \sqrt{s}$ in the last equation. Then

$$|w(y,s)| \le C_3 \mathbb{E}\Big[a_{s,y,\sqrt{s}(1+\Gamma)} e^{c\sqrt{s}(1+\Gamma)\frac{\log(1+s)}{s}} \sqrt{s}\,\Gamma\Big], \tag{167}$$

but

$$a_{s,y,\sqrt{s}X}\sqrt{s} = X + X^2 + X^3 + X^2 y \frac{\log(s+1)}{\sqrt{s}}, \tag{168}$$

so using Lemma 14 again we obtain $|w(y,s)| \le C\Big(1 + y\frac{\log(s+1)}{\sqrt{s}}\Big)$ for some constant C, which is the second line of (125).

Finally we turn to the last line of (125) and bound the increments of $w(y,s)$. Our approach is very similar to the above, conditioning on the value of $\xi_{s+\delta}^{(y)} - \xi_s^{(y)}$ instead of $\xi_s^{(y)}$.

Let $X = \xi_{s+\delta}^{(y)} - \xi_s^{(y)}$ and $\mu = \mathbb{E}[X]$, and also define

$$\mathcal{E}(x) = \mathbb{E}\Big[e^{I(y)}\Big|X = x\Big]. \tag{169}$$

Directly from the definition (124) of w, since $\mathcal{E}(\mu)$ is deterministic and $\mathbb{E}[X - \mu] = 0$, we have

$$w(y, s+\delta) - w(y,s) = \mathbb{E}\Big[e^{I(y)}(X - \mu)\Big], \tag{170}$$

$$= \mathbb{E}\Big[\Big(e^{I(y)} - \mathcal{E}(\mu)\Big)(X - \mu)\Big], \tag{171}$$

$$= \int_{-\infty}^{\infty} \big(\mathcal{E}(x) - \mathcal{E}(\mu)\big)(x - \mu)\mathbb{P}(X \in \mathrm{d}x). \tag{172}$$

Applying the Markov property at time s, we have

$$\mathcal{E}(x) - \mathcal{E}(\mu) = \int_0^\infty \mathbb{P}(\xi_s^{(y)} \in \mathrm{d}z)\mathbb{E}\Big[e^{\frac{1}{2}\int_0^s \mathrm{d}u\, m''(u)\left(\xi_u^{(s:y\to z)} - y\right)}\Big]$$
$$\times\, \mathbb{E}\Big[e^{\frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(s+u)\left(\xi_u^{(\delta:z\to z+x)} - y\right)} - e^{\frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(s+u)\left(\xi_u^{(\delta:z\to z+\mu)} - y\right)}\Big]. \tag{173}$$

We now use the simple bound

$$|\xi_u^{(\delta:z\to z+x)} - \xi_u^{(\delta:z\to z+x')}| \leq |x-x'| \quad \text{for all} \quad z, x, x', \delta, u \geq 0, \tag{174}$$

which follows from Lemma 5 and implies that

$$\begin{aligned}\Big| \int_0^\infty &\mathrm{d}u\, m''(s+u) \left(\xi_u^{(\delta:z\to z+x)} - \xi_u^{(\delta:z\to z+\mu)}\right)\Big| \\ &\leq \int_0^\infty \mathrm{d}u\, |m''(s+u)||x-\mu| \leq \frac{2c|x-\mu|}{s+1}\end{aligned} \tag{175}$$

for some constant c. This, together with the bound $|e^a - e^b| \leq |a-b|e^{b+|a-b|}$ for any $a, b \in \mathbb{R}$, tells us that

$$\begin{aligned}&\left|e^{\frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(s+u)\left(\xi_u^{(\delta:z\to z+x)}-y\right)} - e^{\frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(s+u)\left(\xi_u^{(\delta:z\to z+\mu)}-y\right)}\right| \\ &\quad\leq e^{\frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(s+u)\left(\xi_u^{(\delta:z\to z+\mu)}-y\right)} \frac{c|x-\mu|}{s+1} e^{\frac{c|x-\mu|}{s+1}}.\end{aligned} \tag{176}$$

Substituting this into (173), we have

$$\begin{aligned}|\mathcal{E}(x) - \mathcal{E}(\mu)| &\leq \int_0^\infty \mathbb{P}(\xi_s^{(y)} \in \mathrm{d}z)\mathbb{E}\Big[e^{\frac{1}{2}\int_0^s \mathrm{d}u\, m''(u)\left(\xi_u^{(s:y\to z)}-y\right)}\Big] \\ &\qquad\times \mathbb{E}\Big[e^{\frac{1}{2}\int_0^\infty \mathrm{d}u\, m''(s+u)\left(\xi_u^{(\delta:z\to z+\mu)}-y\right)}\Big]\frac{c|x-\mu|}{s+1} e^{\frac{c|x-\mu|}{s+1}} \qquad (177)\\ &= \mathcal{E}(\mu)\frac{c|x-\mu|}{s+1} e^{\frac{c|x-\mu|}{s+1}}. \qquad (178)\end{aligned}$$

Returning to (172), we obtain

$$\begin{aligned}|w(y, s+\delta) - w(y,s)| &\leq \int_{-\infty}^\infty \mathcal{E}(\mu)\frac{c|x-\mu|}{s+1} e^{\frac{c|x-\mu|}{s+1}} |x-\mu|\mathbb{P}(X \in \mathrm{d}x) \qquad (179)\\ &= \mathcal{E}(\mu)\mathbb{E}\Big[\frac{c(X-\mu)^2}{s+1} e^{\frac{c|X-\mu|}{s+1}}\Big]. \qquad (180)\end{aligned}$$

Finally, by scaling, conditionally on $\xi_s^{(y)} = z$ we have

$$|X-\mu| \stackrel{(d)}{=} \sqrt{\delta}\Big|\xi_1^{(z/\sqrt{\delta})} - \mathbb{E}\Big[\xi_1^{(z/\sqrt{\delta})}\Big]\Big| \leq \sqrt{\delta}\,\Gamma, \tag{181}$$

where Γ was defined in (158) and is a non-negative random variable with Gaussian tail. Therefore

$$\mathbb{E}\Big[\frac{c|X-\mu|^2}{s+1} e^{\frac{c|X-\mu|}{s+1}}\Big] \leq C\frac{\delta}{s+1} \tag{182}$$

for some constant C provided $\delta \leq s^2$, and one may check similarly to (153) that $\mathcal{E}(\mu)$ is also bounded uniformly in y, s and δ. This establishes the last line of (125) and completes the proof. $\square$

6.2. Proof of Proposition 10: decomposition of ψ_t. To prove Proposition 10 we proceed via three lemmas. We first write $I_t(y) = I(y) - \big(I(y) - I_t(y)\big)$, and show that the correction $I(y) - I_t(y)$ is small in the following sense:

Lemma 15. *Suppose that m is twice continuously differentiable and satisfies* (106). *Then there exist positive random variables G and G_t with Gaussian tails, where all the G_t have the same distribution, such that uniformly in y,*

$$I(y) = G\mathcal{O}(1) \quad \text{and} \quad I(y) - I_t(y) = G_t\mathcal{O}\big(t^{-\frac{1}{2}}\big). \tag{183}$$

Unsurprisingly, for random variables with Gaussian tails we can make series expansions rather easily:

Lemma 16. *Let G and G_t be positive random variables with Gaussian tails such that all the G_t have the same distribution. Suppose that A_t and B_t are random variables such that*

$$A_t = G\mathcal{O}(1), \quad B_t = G_t\mathcal{O}(\epsilon_t) \tag{184}$$

where $\epsilon_t \geq 0$ is a deterministic function with $\epsilon_t \to 0$ as $t \to \infty$. Then for any integer $n \geq 0$,

$$\mathbb{E}\big[e^{A_t+B_t}\big] = \sum_{p=0}^{n} \frac{1}{p!}\mathbb{E}\big[e^{A_t}B_t^p\big] + \mathcal{O}(\epsilon_t^{n+1}). \tag{185}$$

Taking $n = 1$, $\epsilon_t = t^{-1/2}$, $A_t = I(y)$ and $B_t = -\big(I(y) - I_t(y)\big)$, we find

$$\mathbb{E}[e^{I_t(y)}] = \mathbb{E}[e^{I(y)}] - \mathbb{E}\Big[e^{I(y)}\big(I(y) - I_t(y)\big)\Big] + \mathcal{O}\Big(\frac{1}{t}\Big). \tag{186}$$

The difficult part is then to show how the $I(y)$ decorrelates asymptotically from $I(y) - I_t(y)$:

Lemma 17. *Suppose that m is twice continuously differentiable with $m''(t) = \frac{3}{2(t+1)^2} + r''(t)$ where $r''(t) = \mathcal{O}(t^{-2-\eta})$ for some $\eta > 0$. Then*

$$\mathbb{E}\Big[e^{I(y)}\big(I(y) - I_t(y)\big)\Big] = \mathbb{E}\Big[e^{I(y)}\Big]\mathbb{E}\big[I(y) - I_t(y)\big] + \mathcal{O}\Big(\frac{\log t}{t}\Big) + y\mathcal{O}\Big(\frac{1}{t}\Big). \tag{187}$$

Of course $\psi_\infty(y) = \mathbb{E}[e^{I(y)}]$ and $\psi_t(y) = \mathbb{E}[e^{I_t(y)}]$, so these lemmas together give Proposition 10. It remains to prove the lemmas.

Proof of Lemma 15. The bound on $I(y)$ is easy by applying Lemma 7 since $|m''(s)| \leq \frac{c}{(1+s)^2}$ for all s and some constant c. We now turn to $I_t(y) - I(y)$.

Recall the expression (105) of $I_t(y)$, replace $m''(s)$ by its expression (106) and cut the integral into three pieces to obtain

$$I_t(y) = \frac{1}{2}\int_0^t \mathrm{d}s\ m''(s)\frac{t-s}{t}\Big(\xi^{(y)}_{\frac{st}{t-s}} - y\Big), \tag{188}$$

$$= \frac{3}{4}\,\frac{t+1}{t}\int_0^t \frac{\mathrm{d}s}{(s+1)^2}\Big(\xi^{(y)}_{\frac{st}{t-s}} - y\Big)$$

$$- \frac{3}{4t}\int_0^t \frac{\mathrm{d}s}{s+1}\Big(\xi^{(y)}_{\frac{st}{t-s}} - y\Big) + \frac{1}{2}\int_0^t \mathrm{d}s\, r''(s)\frac{t-s}{t}\Big(\xi^{(y)}_{\frac{st}{t-s}} - y\Big). \tag{189}$$

Recall that, by scaling,

$$\xi_{tu}^{(y)} = \sqrt{t}\,\tilde{\xi}_u^{(\tilde{y})} \quad \text{with} \quad \tilde{y} = y/\sqrt{t} \tag{190}$$

where $\tilde{\xi}_u^{(\tilde{y})}$ is another, t dependent (implicit in notation), Bessel process started from $\tilde{y}$. We can apply Lemma 7 to the Bessel process $\tilde{\xi}_u^{(\tilde{y})}$ but, as it depends on t, the random variable G must be replaced by some other random variable $\tilde{G}_t$ which has the same Gaussian tails as G. Then

$$\left|\tilde{\xi}_u^{(\tilde{y})} - \tilde{y}\right| \le \tilde{G}_t \max\left(u^{\frac{1}{2}-\epsilon}, u^{\frac{1}{2}+\epsilon}\right) \quad \text{so} \quad \left|\xi_{tu}^{(y)} - y\right| \le \tilde{G}_t\sqrt{t}\max\left(u^{\frac{1}{2}-\epsilon}, u^{\frac{1}{2}+\epsilon}\right). \tag{191}$$

In the second integral of (188), make the change of variable $u = s/t$ and use (191) to obtain

$$\begin{aligned}\left|\frac{1}{t}\int_0^t \frac{\mathrm{d}s}{s+1}\left[\xi_{\frac{st}{t-s}}^{(y)} - y\right]\right| &\le \frac{1}{t}\int_0^1 \frac{\mathrm{d}u}{u}\tilde{G}_t\sqrt{t}\max\left\{\left(\frac{u}{1-u}\right)^{\frac{1}{2}+\epsilon}, \left(\frac{u}{1-u}\right)^{\frac{1}{2}-\epsilon}\right\}\\ &= \tilde{G}_t\mathcal{O}\big(t^{-\frac{1}{2}}\big).\end{aligned} \tag{192}$$

In the first integral of (188), make the change of variable $u = st/(t-s)$ to obtain

$$\begin{aligned} I_t(y) = \frac{3}{4}\,\frac{t+1}{t}\int_0^\infty \frac{\mathrm{d}u}{(u+1+u/t)^2}\big(\xi_u^{(y)} - y\big)\\ +\frac{1}{2}\int_0^t \mathrm{d}s\, r''(s)\frac{t-s}{t}\left(\xi_{\frac{st}{t-s}}^{(y)} - y\right) + \tilde{G}_t\mathcal{O}\big(t^{-\frac{1}{2}}\big).\end{aligned} \tag{193}$$

We now turn to $I(y)$. In expression (105) of $I(y)$, use the expression (106) and cut the integral into the following pieces:

$$\begin{aligned} I(y) = \frac{3}{4}\int_0^\infty \frac{\mathrm{d}s}{(s+1)^2}\big(\xi_s^{(y)} - y\big) + \frac{1}{2}\int_0^t \mathrm{d}s\, r''(s)\frac{t-s}{t}\big(\xi_s^{(y)} - y\big)\\ +\frac{1}{2}\int_0^t \mathrm{d}s\, r''(s)\frac{s}{t}\big(\xi_s^{(y)} - y\big) + \frac{1}{2}\int_t^\infty \mathrm{d}s\, r''(s)\big(\xi_s^{(y)} - y\big).\end{aligned} \tag{194}$$

Applying Lemma 7 and the fact that $r''(s)$ is bounded (since it is continuous on $[0,\infty)$ and tends to 0) with $r''(s) = \mathcal{O}(s^{-2-\eta})$ for some $\eta > 0$, it is easy to check that the third and fourth integrals are bounded in modulus by $G\mathcal{O}(t^{-1/2})$ if $\epsilon < \eta$. Using Lemma 7 again, it is also easy to check that the first terms in (193) and (194) are equal up to an error of size $G\mathcal{O}(1/t)$ which we absorb in the $G\mathcal{O}(t^{-1/2})$ that we already have. Thus we get

$$I_t(y) - I(y) = \frac{1}{2}\int_0^t \mathrm{d}s\, r''(s)\frac{t-s}{t}\left(\xi_{\frac{st}{t-s}}^{(y)} - \xi_s^{(y)}\right) + \tilde{G}_t\mathcal{O}\big(t^{-\frac{1}{2}}\big) + G\mathcal{O}\big(t^{-\frac{1}{2}}\big). \tag{195}$$

We now focus on the remaining integral. The difference $\xi_{st/(t-s)}^{(y)} - \xi_s^{(y)}$ is the position at time $s^2/(t-s) = st/(t-s) - s$ of a new Bessel process started from $\xi_s^{(y)}$. It is also, by scaling, equal to $t^{-1/2}$ times the position at time $ts^2/(t-s)$ of another Bessel process started from $\sqrt{t}\xi_s^{(y)}$. Applying Lemma 7 again to this last Bessel process, we get

$$\begin{aligned}\left|\xi^{(y)}_{\frac{st}{t-s}} - \xi^{(y)}_s\right| &\le \frac{\hat{G}_t}{\sqrt{t}} \max\left\{\Big(\frac{ts^2}{t-s}\Big)^{\frac{1}{2}+\epsilon}, \Big(\frac{ts^2}{t-s}\Big)^{\frac{1}{2}-\epsilon}\right\} \\ &\le \frac{\hat{G}_t}{\sqrt{t}} \begin{cases} \dfrac{t}{t-s} s^{1+2\epsilon} & \text{if } 1 < s < t, \\ \dfrac{t}{t-1} & \text{if } 0 < s < 1. \end{cases}\end{aligned} \tag{196}$$

where $\hat{G}_t$ is another t-dependent positive random variable with the same Gaussian tail as G. Since $r''(s)$ is bounded and $r''(s) = \mathcal{O}(s^{-2-\eta})$, the integral $\int_1^\infty \mathrm{d}s\, r''(s) s^{1+2\epsilon}$ is finite provided $\epsilon < \eta/2$, and we obtain

$$I_t(y) - I(y) = G_t \mathcal{O}\big(t^{-\frac{1}{2}}\big), \tag{197}$$

with $G_t = \max(G, \tilde{G}_t, \hat{G}_t)$. This concludes the proof. □

Proof of Lemma 16. With the hypothesis of the lemma, write $|A_t| \le \alpha G$ and $|B_t| \le \beta\epsilon_t G_t$ for some $\alpha > 0$ and $\beta > 0$. Writing

$$e^{A_t+B_t} = \sum_{p=0}^{\infty} \frac{1}{p!} e^{A_t} B_t^p, \tag{198}$$

we can apply dominated convergence—since the partial sums are dominated by $\exp(A_t + |B_t|)$ which has finite expectation—and obtain

$$\mathbb{E}\big[e^{A_t+B_t}\big] = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbb{E}\big[e^{A_t} B_t^p\big]. \tag{199}$$

It only remains to show that the sum for $p \ge n+1$ is $\mathcal{O}(\epsilon_t^{n+1})$. To do this observe that

$$\left|\frac{1}{p!}\mathbb{E}\big[e^{A_t} B_t^p\big]\right| \le \frac{\epsilon_t^p}{p!}\mathbb{E}\big[e^{\alpha G}(\beta G_t)^p\big] \le \epsilon_t^p \mathbb{E}\big[e^{\alpha G + \beta G_t}\big], \tag{200}$$

where the last expectation is finite. Then, as soon as $\epsilon_t < 1$, we have

$$\left|\sum_{p=n+1}^{\infty} \frac{1}{p!}\mathbb{E}\big[e^{A_t} B_t^p\big]\right| \le \frac{\epsilon_t^{n+1}}{1-\epsilon_t}\mathbb{E}\big[e^{\alpha G+\beta G_t}\big], \tag{201}$$

which concludes the proof. □

Proof of Lemma 17. Define

$$J_t(y) = 2\mathbb{E}\Big[e^{I(y)}\big(I(y) - I_t(y)\big)\Big] - 2\mathbb{E}\Big[e^{I(y)}\Big]\mathbb{E}\big[I(y) - I_t(y)\big]. \tag{202}$$

We want to show that $J_t(y) = \mathcal{O}\big(\frac{\log t}{t}\big) + y\mathcal{O}\big(\frac{1}{t}\big)$. Clearly,

$$J_t(y) = 2\Big(\mathbb{E}[e^{I(y)}I(y)] - \mathbb{E}[e^{I(y)}]\mathbb{E}[I(y)]\Big) - 2\Big(\mathbb{E}[e^{I(y)}I_t(y)] - \mathbb{E}[e^{I(y)}]\mathbb{E}[I_t(y)]\Big) \tag{203}$$

$$= \int_0^\infty \mathrm{d}s\, m''(s)\Big(\mathbb{E}\Big[e^{I(y)}\big(\xi_s^{(y)} - y\big)\Big] - \mathbb{E}\Big[e^{I(y)}\Big]\mathbb{E}\Big[\xi_s^{(y)} - y\Big]\Big)$$
$$- \int_0^t \mathrm{d}s\, m''(s)\frac{t-s}{t}\Big(\mathbb{E}\Big[e^{I(y)}\Big(\xi^{(y)}_{\frac{ts}{t-s}} - y\Big)\Big] - \mathbb{E}\Big[e^{I(y)}\Big]\mathbb{E}\Big[\xi^{(y)}_{\frac{ts}{t-s}} - y\Big]\Big) \tag{204}$$

$$= \int_0^\infty \mathrm{d}s\, m''(s)w(y,s) - \int_0^t \mathrm{d}s\, m''(s)\frac{t-s}{t}w\Big(y, \frac{ts}{t-s}\Big), \tag{205}$$

where we recall the definition of w from (124). We now apply Proposition 12. Cut the integrals at $t/2$ and rearrange the terms:

$$\begin{aligned} J_t(y) = &\int_{\frac{t}{2}}^\infty \mathrm{d}s\, m''(s)w(y,s) + \int_0^{\frac{t}{2}} \mathrm{d}s\, m''(s)\frac{s}{t}w(y,s) \\ &- \int_{\frac{t}{2}}^t \mathrm{d}s\, m''(s)\frac{t-s}{t}w\Big(y, \frac{ts}{t-s}\Big) \\ &- \int_0^{\frac{t}{2}} \mathrm{d}s\, m''(s)\frac{t-s}{t}\Big(w\Big(y, \frac{ts}{t-s}\Big) - w(y,s)\Big). \end{aligned} \tag{206}$$

Using from Proposition 12 that $|w(y,s)| \le C\log(s+1)$ and of course $m''(s) = \mathcal{O}(1/s^2)$, the first and third integrals are both $\mathcal{O}(\frac{\log t}{t})$, uniformly in y. Now using from Proposition 12 that $|w(y,s)| \le C\big(1 + y\frac{\log(s+1)}{\sqrt{s}}\big)$, the second integral is $y\mathcal{O}\big(\frac{1}{t}\big) + \mathcal{O}\big(\frac{\log t}{t}\big)$.

We now turn to the fourth integral. Writing $\frac{st}{t-s} = s + \frac{s^2}{t-s}$ and noticing that for $s < t/2$ we have $\frac{s^2}{t-s} < s^2$ as soon as $t \ge 2$, the last part of Proposition 12 gives $\big|w\big(y, \frac{ts}{t-s}\big) - w(y,s)\big| \le C\frac{s}{t-s}$, and therefore the fourth integral is $\mathcal{O}\big(\frac{\log t}{t}\big)$, which concludes the proof. □

6.3. Proof of Proposition 11: asymptotic for $\mathbb{E}[I(y) - I_t(y)]$. For $y \ge 0$ we introduce the notation $\mu(y,t) := \mathbb{E}[\xi_t^{(y)}] - y$ and observe that

$$\mu(y,tu) = \sqrt{t}\mu\Big(\frac{y}{\sqrt{t}}, u\Big), \quad \mu(0,s) = \frac{4}{\sqrt{\pi}}\sqrt{s}, \quad \max\Big[0, \mu(0,s) - y\Big] \le \mu(y,s) \le \mu(0,s). \tag{207}$$

(The first equality is the scaling property, and the inequalities are from (34). The second equality can be calculated directly from the probability density function for a Bessel process; see for example [RY99, page 446].)

With this notation we can rewrite

$$\mathbb{E}[I(y)] = \frac{1}{2}\int_0^\infty \mathrm{d}s\, m''(s)\mu(y,s) \tag{208}$$

$$\mathbb{E}[I_t(y)] = \frac{1}{2}\int_0^t \mathrm{d}s\, m''(s)\frac{t-s}{t}\mu\Big(y, \frac{st}{t-s}\Big). \tag{209}$$

As usual we use the expression (106), decomposing $\mathbb{E}\big[I(y) - I_t(y)\big]$ into terms containing $3/2(s+1)^2$ and terms containing $r''(s)$. In the former we make our usual change of time $u = st/(t-s)$, but in the latter we do not.

$$\begin{aligned}\mathbb{E}\big[I(y)-I_t(y)\big] &= \frac{3}{4}\int_0^\infty \mathrm{d}s\,\frac{1}{(s+1)^2}\mu(y,s) - \frac{3}{4}\int_0^\infty \mathrm{d}u\,\frac{1}{(\frac{tu}{t+u}+1)^2}\Big(\frac{t}{t+u}\Big)^3\mu(y,u)\\ &\quad+\frac{1}{2}\int_0^\infty \mathrm{d}s\, r''(s)\mu(y,s) - \frac{1}{2}\int_0^t \mathrm{d}s\, r''(s)\,\frac{t-s}{t}\,\mu\Big(y, \frac{st}{t-s}\Big). \end{aligned}\tag{210}$$

Rearranging we get

$$\begin{aligned}\mathbb{E}\big[I(y) - I_t(y)\big] &= \frac{3}{4}\int_0^\infty \mathrm{d}s\,\Big(1 - \frac{t}{t+s}\Big)\frac{1}{(s+1)^2}\,\mu(y,s)\\ &\quad+\frac{3}{4}\int_0^\infty \mathrm{d}s\,\Big(\frac{1}{(s+1)^2} - \frac{1}{(s+1+s/t)^2}\Big)\frac{t}{t+s}\,\mu(y,s)\\ &\quad+\frac{1}{2}\int_0^t \mathrm{d}s\, r''(s)\Big(\mu(y,s) - \frac{t-s}{t}\,\mu\Big(y,\frac{st}{t-s}\Big)\Big)\\ &\quad+\frac{1}{2}\int_t^\infty \mathrm{d}s\, r''(s)\mu(y,s), \end{aligned}\tag{211}$$

and we treat each of the four integrals on the right-hand side in turn.

The first integral in the right hand side of (211). Making the change of variable $s = tu$ and using the first part of (207) we have

$$\int_0^\infty \mathrm{d}s\,\Big(1 - \frac{t}{t+s}\Big)\frac{1}{(s+1)^2}\,\mu(y,s) = \frac{1}{\sqrt{t}}\int_0^\infty \mathrm{d}u\,\frac{u}{(u+1)(u+1/t)^2}\mu\Big(\frac{y}{\sqrt{t}}, u\Big). \tag{212}$$

We now approximate $\mu(y/\sqrt{t}, u)$ by $\mu(0,u)$, bounding the error by using the last part of (207):

$$\begin{aligned}&\bigg|\frac{1}{\sqrt{t}}\int_0^\infty \mathrm{d}u\,\frac{u}{(u+1)(u+1/t)^2}\mu\Big(\frac{y}{\sqrt{t}},u\Big) - \frac{1}{\sqrt{t}}\int_0^\infty \mathrm{d}u\,\frac{u}{(u+1)(u+1/t)^2}\mu(0,u)\bigg|\\ &\quad\le \frac{1}{\sqrt{t}}\int_0^\infty \mathrm{d}u\,\frac{u}{(u+1)(u+1/t)^2}\,\frac{y}{\sqrt{t}}. \end{aligned}\tag{213}$$

The right hand side is $y\mathcal{O}\big(\frac{\log t}{t}\big)$, and using the second part of (207), we have

$$\frac{1}{\sqrt{t}}\int_0^\infty \mathrm{d}u\,\frac{u}{(u+1)(u+1/t)^2}\mu(0,u) = \frac{4\sqrt{\pi}}{\sqrt{t}} + \mathcal{O}(1/t). \tag{214}$$

We therefore conclude that

$$\int_0^\infty \mathrm{d}s\,\Big(1 - \frac{t}{t+s}\Big)\frac{1}{(s+1)^2}\,\mu(y,s) = \frac{4\sqrt{\pi}}{\sqrt{t}} + y\mathcal{O}\Big(\frac{\log t}{t}\Big) + \mathcal{O}\Big(\frac{1}{t}\Big). \tag{215}$$

The second integral in the right hand side of (211). We note that

$$\frac{1}{(s+1)^2} - \frac{1}{(s+1+s/t)^2} = \frac{1}{(s+1)^2}\mathcal{O}\Big(\frac{1}{t}\Big), \tag{216}$$

and $t/(t+s) \le 1$, so using the bound $\mu(y,s) \le \mu(y,0) = 4\sqrt{s}/\sqrt{\pi}$ from (207), we easily see that the second integral is $\mathcal{O}(1/t)$ uniformly in y.

The third integral in the right hand side of (211) We use the following result: for any $\delta > 0$,

$$0 \le \mu(y, s+\delta) - \mu(y,s) \le \mu(0, s+\delta) - \mu(0,s) = \frac{4}{\sqrt{\pi}}\Big(\sqrt{s+\delta} - \sqrt{s}\Big). \tag{217}$$

This follows from the Markov property plus (207). Then

$$\frac{t-s}{t}\mu\Big(y, \frac{st}{t-s}\Big) - \mu(y,s) = \frac{t-s}{t}\Big[\mu\Big(y, \frac{st}{t-s}\Big) - \mu(y,s)\Big] - \frac{s}{t}\mu(y,s), \tag{218}$$

so that

$$-\frac{4}{\sqrt{\pi}}\frac{s^{3/2}}{t} \le \frac{t-s}{t}\mu\Big(y, \frac{st}{t-s}\Big) - \mu(y,s) \le \frac{4}{\sqrt{\pi}}\frac{t-s}{t}\Big[\Big(\frac{st}{t-s}\Big)^{1/2} - s^{1/2}\Big] \tag{219}$$

But $(\frac{st}{t-s})^{1/2} = s^{1/2}(1+\frac{s}{t-s})^{1/2} \le s^{1/2}\big(1+\frac{s}{2(t-s)}\big)$ so the right hand side of the previous equation is at most $(4/\sqrt{\pi}) \times s^{3/2}/(2t)$. We conclude that

$$\begin{aligned} &\left|\int_0^t \mathrm{d}s\, r''(s)\left[\mu(y,s) - \frac{t-s}{t}\mu\left(y, \frac{st}{t-s}\right)\right]\right| \\ &\le \frac{4}{\sqrt{\pi}t}\int_0^t \mathrm{d}s\, s^{3/2}\big|r''(s)\big| = \begin{cases} \mathcal{O}\left(\dfrac{1}{t}\right) & \text{if } \eta > \frac{1}{2}, \\ \mathcal{O}\left(\dfrac{\log t}{t}\right) & \text{if } \eta = \frac{1}{2}, \\ \mathcal{O}\left(\dfrac{1}{t^{\frac{1}{2}+\eta}}\right) & \text{if } \eta < \frac{1}{2}, \end{cases} \end{aligned} \tag{220}$$

uniformly in y.

The fourth integral in the right hand side of (211). Since $r''(s) = O(s^{-2-\eta})$ for some $\eta > 0$, using (207) again we have

$$\left|\int_t^\infty \mathrm{d}s\, r''(s)\mu(y,s)\right| \le \frac{4}{\sqrt{\pi}}\int_t^\infty \mathrm{d}s\, |r''(s)|\sqrt{s} = \mathcal{O}\left(\frac{1}{t^{\frac{1}{2}+\eta}}\right) \tag{221}$$

uniformly in y.

Putting together the results from the four integrals give the proposition.

Appendix

Lemma 18. *For any* $\epsilon > 0$*, the non-negative random variable*

$$G := \sup_{s>0} \frac{\xi_s^{(0)}}{\max\left(s^{1/2-\epsilon}, s^{1/2+\epsilon}\right)} \tag{222}$$

has Gaussian tail under $\mathbb{P}$*.*

Proof. We do this in two parts, first considering the supremum over $s \in (0, 1]$. We have

$$\mathbb{P}\left(\sup_{s\in(0,1]} \frac{\xi_s^{(0)}}{s^{1/2-\epsilon}} > z\right) \le \sum_{n=2}^{\infty} \mathbb{P}\left(\sup_{s\in\left(\frac{1}{n},\frac{1}{n-1}\right]} \frac{\xi_s^{(0)}}{s^{1/2-\epsilon}} > z\right). \tag{223}$$

By scaling, this equals

$$\sum_{n=2}^{\infty} \mathbb{P}\left(\sup_{s\in\left(1,\frac{n}{n-1}\right]} \frac{\xi_s^{(0)}}{s^{1/2-\epsilon}} > zn^{\epsilon}\right) \le \sum_{n=2}^{\infty} \mathbb{P}\left(\sup_{s\in(1,2]} \xi_s^{(0)} > zn^{\epsilon}\right). \tag{224}$$

Now note that there exist $c_3 > 0$ and $c_4 > 0$ such that $\mathbb{P}\big(\sup_{s\in(1,2]} \xi_s^{(0)} > z\big) \le c_3 \exp\left[-c_4 z^2\right]$ for all $z > 0$, so

$$\mathbb{P}\left(\sup_{s\in(0,1]} \frac{\xi_s^{(0)}}{s^{1/2-\epsilon}} > z\right) \le c_3 \sum_{n=2}^{\infty} e^{-c_4 z^2 n^{2\epsilon}}, \tag{225}$$

and it is an easy exercise to show that there exist c_1 and c_2 (with c_1 depending on ϵ) such that $c_3 \sum_{n=2}^{\infty} e^{-c_4 z^2 n^{2\epsilon}} \le c_1 e^{-c_2 z^2}$.

Similarly for $s \in (1, \infty)$,

$$\mathbb{P}\left(\sup_{s\in(1,\infty)} \frac{\xi_s^{(0)}}{s^{1/2+\epsilon}} > z\right) \le \sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{s\in(n,n+1]} \frac{\xi_s^{(0)}}{s^{1/2+\epsilon}} > z\right). \tag{226}$$

By scaling, this equals

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{s\in(1,\frac{n+1}{n}]} \frac{\xi_s^{(0)}}{s^{1/2+\epsilon}} > zn^{\epsilon}\right) \le \sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{s\in(1,2]} \xi_s^{(0)} > zn^{\epsilon}\right). \tag{227}$$

and the end of the argument is the same as in the previous case. □

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