Low-gain Integral Control for Multi-Input, Multi-Output Linear Systems with Input Nonlinearities

Chris Guiver, Hartmut Logemann and Stuart Townley

Abstract—We consider the inclusion of a static anti-windup component in a continuous-time low-gain integral controller in feedback with a multi-input multi-output stable linear system subject to an input nonlinearity (from a class of functions that includes componentwise diagonal saturation). We demonstrate that the output of the closed-loop system asymptotically tracks every constant reference vector which is “feasible” in a natural sense, provided that the integrator gain is sufficiently small. Robustness properties of the proposed control scheme are investigated and three examples are discussed in detail.

Index Terms—Anti-windup methods, Constrained control, Robust control, Sufficiently small. Robustness properties of the proposed control scheme for a multi-input multi-output stable linear system subject to an input is “feasible” in a natural sense, provided that the integrator gain is sufficiently small. Robustness properties of the proposed control scheme is connected in series, as depicted in Figure 1, with an integrator $gK/s$, where the matrix $K$ and the positive scalar gain $g$ are design parameters.

Figure 1. Block diagram of a low-gain integral control scheme. Here $u$, $y$ and $r$ denote the input, output and reference signal, respectively.

It is known that input saturation may lead to an undesirable degradation of tracking performance of MIMO integral controlled systems, or even destabilise them; a phenomenon often called integrator windup [11]. Integrator windup is a consequence of basing controller design on the assumption of linearity, when in reality input saturation is an archetypal nonlinear effect. Anti-windup control refers to the study of mechanisms to alleviate or remove integrator windup and, owing to its importance in applications, is a well-studied topic. The chronological bibliography of the 1995 paper [12] contains already 250 references, for instance. We refer the reader to the tutorial [13], survey [14] or monograph [15] and the references therein for a thorough overview of anti-windup control. Briefly, as described there, many anti-windup mechanisms are designed under the assumption that the unsaturated system has the desired closed-loop stability and performance properties and an anti-windup compensator is subsequently included — a static or dynamical system driven by the error $z - \phi(z)$, where $z$ is the state or output of the controller and $\phi$ denotes the input nonlinearity.

We present a low-gain integral controller that includes a (direct linear, in the terminology of [13]) anti-windup component and prove that, for a large class of input nonlinearities, it achieves global exponential tracking for all feasible references provided that the integrator gain is sufficiently small. The class of nonlinearities is assumed to satisfy certain integral absolute stability theory. We demonstrate that the closed-loop system is connected in series, as depicted in Figure 1, with an integrator $gK/s$, where the matrix $K$ and the positive scalar gain $g$ are design parameters.

The resulting closed-loop system is known to be globally exponentially stable if: (i) $-KG(0)$ is a Hurwitz matrix, and; (ii) the gain $g > 0$ is sufficiently small, in which case the output of the closed loop system asymptotically tracks every constant reference $r$.

Low-gain integral control has been further developed by the present authors in, for example [7]–[10], to address discrete-time systems, sampled-data systems, classes of distributed parameter systems, to allow the gain parameter $g$ to be determined adaptively and to include input and output nonlinearities. One situation not addressed to date is low-gain integral control for multi-input, multi-output (MIMO) systems in the presence of input nonlinearities (such as saturation).

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endowed with the sup norm given by
\[
\|H\|_\infty := \sup_{x \in \mathbb{R}^n, x > 0} \|H(x)\| = \sup_{x \in \mathbb{R}^n} \|H(x)\|
\]
where \(|| \cdot \|\) is the operator norm induced by the 2-norm. As usual, a square complex matrix \(M\) is said to be Hurwitz if every eigenvalue of \(M\) has a negative real part. We let \(rk \gg M\) denote the rank of \(M\). The symbol \(I_q\) denotes the \(q \times q\) identity matrix, although the subscript shall be omitted when the dimension is clear from the context. Finally, for a function \(f: \mathbb{R}^p \to \mathbb{R}^m\) and a set \(S \subseteq \mathbb{R}^p\), \(f^{-1}(S)\) denotes the pre-image of \(S\) under \(f\), that is, \(f^{-1}(S) := \{z \in \mathbb{R}^p : f(z) \in S\}\).

If \(S = \{s\}\) is a singleton, then we write \(f^{-1}(s) = f^{-1}(\{s\}) = f^{-1}(S)\). Finally, for a vector \(v\), \(v_k\) denotes the \(k\)-th component of \(v\).

### II. LOW-GAIN INTEGRAL CONTROL WITH INPUT NONLINEARITIES

We focus on the linear control system with input nonlinearity
\[
\dot{x} = Ax + B\phi(u), \quad x(0) = x^0, \quad y = Cx, \tag{1}
\]
where \(A \in \mathbb{R}^{n \times n}\) is Hurwitz, \(B \in \mathbb{R}^{n \times m}\), \(C \in \mathbb{R}^{p \times n}\) and \(\phi: \mathbb{R}^m \to \mathbb{R}^m\) is locally Lipschitz continuous. As usual, \(x\) and \(y\) denote the state and output variables, respectively, whilst \(u\) is the control signal. There are \(n\) and \(m\) input, state and output variables, respectively. We let \(G\) denote the transfer function of the linear system specified by the triple \((A, B, C)\), that is, the matrix-valued function of the complex variable \(s\) given by \(G(s) := C(sI - A)^{-1}B\).

We seek to apply low-gain integral control to (1), with the aim that the output \(y(t)\) converges to a prescribed constant reference vector \(r \in \mathbb{R}^p\) as \(t \to \infty\). We say that \(r\) is feasible, if the set
\[
U^r := \{w \in \mathbb{R}^m : G(0)\phi(w) = r\},
\]
is non-empty. Obviously, if \(m = p\) and \(G(0)\) is invertible, then \(U^r = \phi^{-1}(G(0)^{-1}r)\). If the control signal \(u\) in (1) is such that \(\phi(u)\) has a limit \(\lambda\), that is, \(\phi(u(t)) \to \lambda\) as \(t \to \infty\), then, as \(A\) is Hurwitz, it is well-known that for any initial state \(x^0\), the output \(y(1)\) of (1) has limit
\[
\lim_{t \to \infty} y(t) = G(0)\lim_{t \to \infty} \phi(u(t)) = G(0)\lambda.
\]
Trivially, if \(r\) is feasible, then there exists a control signal \(u\) such that \(y(t) \to r\) (for example, \(u(t) \equiv w\) with \(w \in U^r\)). On the other hand, if \(r\) is not feasible, then there does not exist a bounded \(u\) such that \(\phi(u(t))\) converges and \(y(t) \to r\) as \(t \to \infty\). Moreover, under the additional assumption that \(m = p\) and \(G(0)\) is invertible, then, if \(r\) is not feasible, \(y(t)\) does not converge to \(r\) whenever the control signal \(u\) is bounded.

We say that a set \(R \subseteq \mathbb{R}^p\) of reference vectors is feasible if every \(r \in R\) is feasible. Given a feasible set \(R \subseteq \mathbb{R}^p\), we introduce the following assumption:

(F) there exists \(L > 0\) such that
\[
||\phi(w + \tilde{u}) - \phi(\tilde{u})|| \leq L||w|| \quad \forall w \in \mathbb{R}^m, \quad \forall \tilde{u} \in \bigcup_{r \in R} U^r.
\]

Assumption (F) is certainly satisfied if \(\phi\) is globally Lipschitz with Lipschitz constant \(L\).

A function \(\phi: \mathbb{R}^m \to \mathbb{R}^m\) with components \(\phi_i\) is called diagonal if, for all \(i = 1, 2, \ldots, m\) and all \(v \in \mathbb{R}^m\),
\[
\phi_i(v) = \phi_i(P_i v), \quad \text{where} \quad (P_i v)_j := \begin{cases} v_i & j = i \\ 0 & j \neq i. \end{cases}
\]

The next example illustrates the feasibility property for the familiar diagonal saturation function.

**Example 1.** For given \(v^1 < v^2\), set \(V := \{v^1, v^2\}\) and define the function \(\text{sat}_V\) by
\[
\text{sat}_V: \mathbb{R} \to \mathbb{R}, \quad \text{sat}_V(w) := \max\{v^1, \min\{w, v^2\}\} \tag{2},
\]
illustrated in Figure 2. The diagonal saturation function sat is defined as follows:
\[
\text{sat}: \mathbb{R}^m \to \mathbb{R}^m, \quad \text{sat}(w) := [\text{sat}_{v_1}(w_1), \ldots, \text{sat}_{v_m}(w_m)]^T, \tag{3}
\]
for \(v_k := \{v_k^1, v_k^2\}, k = 1, \ldots, m\). (Here, the set \(R := \{G(0)w : v_k^1 \leq w_k \leq v_k^2, k = 1, \ldots, m\}\), is feasible, and, since \(R = \{G(0)\text{sat}(w) : w \in \mathbb{R}^m\}\), \(R\) is the maximal feasible set. Furthermore, it is straightforward to see that sat satisfies (F) with \(L = 1\).)

\[\\]

![Figure 2](image)

**Figure 2.** Graph of the saturation function sat\(_V\) defined in (2), here with \(v^1 \leq 0 < v^2\).

Our second example contains a saturation function that satisfies (F) but is not diagonal.

**Example 2.** For fixed \(\theta > 0\), the function \(\rho: \mathbb{R}^m \to \mathbb{R}^m\) defined by
\[
\rho(v) := \begin{cases} v & \|v\| \leq \theta \\ \theta \frac{v}{\|v\|} & \|v\| > \theta, \end{cases}
\]
satisfies (F) with \(L \in [1, 2]\). For certain \(m\) and choice of norm, the upper bound 2 for \(L\) is achieved. The set \(R := \{G(0)w : w \in \mathbb{R}^m, \|w\| \leq \theta\}\), is the maximal feasible set.

Given the control system (1) and a feasible set \(R \subseteq \mathbb{R}^p\), let \(r \in R\) and consider the control law
\[
\dot{u} = gK(r - y) - g\tilde{\Gamma}(u - \phi(u) - u^\star + \phi(u^\star)), \tag{5}
\]
where \(g > 0\), \(K \in \mathbb{R}^{m \times p}\) and \(\Gamma \in \mathbb{R}^{m \times m}\) are design parameters and
\[
\|u^\star\| > \sup U^r.
\]

The assumption that \(R \subseteq \mathbb{R}^p\) is feasible implies that \(U^r\) is non-empty and hence (6) is meaningful. Note that if \(\Gamma = 0\) or \(\phi = \text{id}\), the identity function, then (5) reduces to “pure” integral control. Of course, in the case \(\phi = \text{id}\) the control system (1) is in fact linear. The term \(g\tilde{\Gamma}(w - \phi(w) - u^\star + \phi(u^\star))\) is the so-called anti-windup component of the controller.

In the following, we will focus on the analysis of the feedback interconnection of (1) and (5):
\[
\dot{x} = Ax + B\phi(u), \quad y = Cx, \quad \dot{u} = gK(r - y) - g\tilde{\Gamma}(u - \phi(u) - u^\star + \phi(u^\star)), \quad x(0) = x^0, \quad u(0) = u^0. \tag{7}
\]

**Remark 3.** Finding \(u^\star\) satisfying (6) requires knowledge of \(G(0)\), in general. However, in numerous applications this knowledge is not
required. For instance, when \( \phi = \text{sat} \), the saturation nonlinearity in (3), and \( R \) is given by (4), then, for every \( r \in R \), there exists \( u' \in U' \) such that \( \phi(u') = u' \) and so \( -u' + \phi(u') = 0 \) in (5). We note that in this case (7) may be placed in the anti-windup framework as presented in [13] with static anti-windup dynamics. \( \square \)

The following result is the main contribution of this note.

**Theorem 4.** Given the closed-loop integral control system (7) with feasible \( R \subseteq \mathbb{R}^p \), assume that

(a) \( A \) is Hurwitz,
(b) \( -\Gamma \) is Hurwitz,
(c) \( (F) \) holds.

Then, for all \( g > 0 \), \( r \in R \), \( u' \in U' \) and \( (x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m \), there exists a unique solution \((x, u)\) of (7) defined on \( \mathbb{R}_+ \). Furthermore, if

\[
\sup_{s \in \mathbb{R}_+} \left\| (sI + \Gamma)^{-1} \cdot (\| \Gamma - KG(0) \| < 1 / L \right\| < 1 / L, \quad (8)
\]

then there exists \( g^* > 0 \) such that, for all \( g \in (0, g^*) \), \( r \in R \), \( u' \in U' \) and \( (x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m \), the solution \((x, u)\) of (7) satisfies

(C1) \( u(t) \to u' \),
(C2) \( x(t) \to -A^{-1}B\phi(u') \),
(C3) \( y(t) = Cx(t) \to r \),
(C4) \( z(t) \to 0 \),

as \( t \to \infty \), and the rate of convergence are exponential.

Remark 5. At first glance, the hypotheses of Theorem 4 place very few constraints on \( m, p \) and \( KG(0) \). However, assuming that hypotheses (a)–(c) and inequality (8) are satisfied, it follows that, for nonlinearities \( \phi \) with \( L \geq 1 \) (which is the case for the diagonal and non-diagonal saturation functions in Examples 1 and 2, respectively),

\[
\| \Gamma - KG(0) \| < \frac{\rho}{L} \leq \rho, \quad (9)
\]

where \( \rho := 1 / \sup_{s \in \mathbb{R}_+} \| (sI + \Gamma)^{-1} \| \). Now \( \rho \) is the (unstructured) complex stability radius (see [32]) of the Hurwitz matrix \( -\Gamma \), and thus, since \( -KG(0) = -I + (\Gamma - KG(0)) \), it follows from (9) that \( -KG(0) \) is Hurwitz. In particular, if (a)–(c) are satisfied and \( L \geq 1 \), then \( rk(G(0)) = m \) (implying that \( p \geq m \)) is a necessary condition for (8) to hold. The assumption \( rk(G(0)) = p \), and thus necessarily \( m \geq p \), is typically made in output regulation problems so that in the unsaturated case (\( \phi = id \)) every reference vector in \( \mathbb{R}^p \) is feasible; see, for instance [35]. Our results do not apply when \( L \geq 1 \) and \( m > p \). \( \square \)

The proof of Theorem 4 is based on Lemma 6, stated and proven below.

**Lemma 6.** Given \( (A, B, C) \) as in (1) with \( A \) Hurwitz, and transfer function \( G \), let \( K \in \mathbb{R}^{n \times p}, \Gamma \in \mathbb{R}^{m \times m} \) and assume that \(-\Gamma \) is Hurwitz. For \( g > 0 \) define

\[
A := \begin{bmatrix} A & -gK \end{bmatrix}, \quad B := \begin{bmatrix} B \\ g\Gamma \end{bmatrix}, \quad C := \begin{bmatrix} 0 & I \end{bmatrix}, \quad (10)
\]

and let \( G \) denote the transfer function of the triple \((A, B, C)\). Then, \( G \in H^\infty \) for every \( g > 0 \). Moreover, for each \( \varepsilon > 0 \), there exists \( g^* > 0 \) such that for all \( g \in (0, g^*) \)

\[
\| G \|_\infty \leq \varepsilon + \sup_{s \in \mathbb{R}_+} \left\| (sI + \Gamma)^{-1} \cdot (\| \Gamma - KG(0) \| \right\|, \quad (11)
\]

**Proof.** The matrices \( A \) and \(-g\Gamma \) for \( g > 0 \) are Hurwitz by assumption, so that \( A \) in (10) is clearly Hurwitz. Hence, \( G \in H^\infty \) for all \( g > 0 \). An elementary calculation shows that

\[
G(s) = g(sI + g\Gamma)^{-1}(\Gamma - KG(s)),
\]

and so,

\[
G = G_1 + G_2, \quad (12)
\]

where \( G_1 \) and \( G_2 \) are given by

\[
G_1(s) := g(sI + g\Gamma)^{-1}K(G(0) - G(s)),
\]

\[
G_2(s) := -g(sI + g\Gamma)^{-1}(KG(0) - \Gamma).
\]

Since \(-g\Gamma \) is Hurwitz for \( g > 0 \) and \( G \in H^\infty \), it is clear that \( G_1, G_2 \in H^\infty \). To prove (11), let \( \varepsilon > 0 \) be given. We proceed to estimate \( \| G_1 \|_\infty \) and \( \| G_2 \|_\infty \). Since \(-\Gamma \) is Hurwitz it follows that there exists \( M > 0 \) such that

\[
\| (I + z\Gamma)^{-1} \| \leq M, \quad \forall z \in \mathbb{R}^r. \quad (13)
\]

Next, define \( J \in H^\infty \) by

\[
J(s) := \begin{cases} \frac{1}{s}K(G(s) - G(0)) & s \neq 0 \\ KG(0) & s = 0 \end{cases},
\]

where \( G'(0) \) denotes the derivative of \( G \) at \( s = 0 \). Let \( s \in \mathbb{R} \). If \( s \neq 0 \), we use (13) to estimate

\[
\| G_1(s) \| \leq g \left\| (I + (g/s)\Gamma)^{-1} \cdot (J(s)) \right\| \leq gM \| J \|_\infty.
\]

Since \( G_1(0) = 0 \), it follows that

\[
\| G_1 \|_\infty \leq gM \| J \|_\infty,
\]

and thus, setting \( g^* := \varepsilon/(M \| J \|_\infty) > 0 \), we conclude that

\[
\| G_1 \|_\infty \leq \varepsilon, \quad \forall g \in (0, g^*). \quad (14)
\]

To estimate \( \| G_2 \|_\infty \), we note that

\[
G_2(s) = -((s/g)I + \Gamma)^{-1}(KG(0) - \Gamma),
\]

and hence obtain, for every \( g > 0 \),

\[
\| G_2 \|_\infty \leq \sup_{s \in \mathbb{R}_+} \left\| (sI + \Gamma)^{-1} \cdot (\| KG(0) - \Gamma \| \right\|, \quad (15)
\]

Combining (12), (14) and (15) yields (11), completing the proof. \( \square \)

**Proof of Theorem 4.** Note that by hypothesis (F), the nonlinearity \( \phi \) is affinely linearly bounded, and thus, by [33, Proposition 4.12], it follows that, for all \( g > 0 \), \( r \in R, u' \in U' \) and all \( (x^0, u^0) \), the unique maximally defined solution \((x, u)\) of (7) exists on \( \mathbb{R}_+ \).

Let \( r \in R \) and \( u' \in U' \) and set

\[
z := x + A^{-1}B\phi(u') \quad \text{and} \quad v := u - u'. \quad (16)
\]

Then

\[
\dot{z} = \dot{\phi}(u') = -A + B[\phi(v + u') - \phi(u')]
\]

\[
= A + B\phi_{u'}(v), \quad (17)
\]

where \( \phi_{u'} : \mathbb{R}^m \to \mathbb{R} \) is defined by

\[
\phi_{u'}(w) := \phi(u' + w) - \phi(u'), \quad \forall w \in \mathbb{R}^m. \quad (18)
\]

Furthermore,

\[
\dot{v} = \dot{u} = g(K(r - Cx) - g\Gamma[u - \phi(u') - u' + \phi(u')])
\]

\[
= -gKCz - g\Gamma[v - \phi_{u'}(v)], \quad (19)
\]

will be used in the proof of the following result.
where we have used that \( r = G(0)\phi(u') = -CA^{-1}B\phi(u') \). We recast (17) and (19) as

\[
\begin{bmatrix}
\dot{z} \\
\dot{v}
\end{bmatrix} = \begin{bmatrix}
-A & 0 \\
-gKC & -g\Gamma 
\end{bmatrix} \begin{bmatrix}
\dot{z} \\
\dot{v}
\end{bmatrix} + \begin{bmatrix}
B \\
g\Gamma
\end{bmatrix} \phi_{\omega^r}(v)
= A \begin{bmatrix}
\dot{z} \\
\dot{v}
\end{bmatrix} + B\phi_{\omega^r}(C \begin{bmatrix}
\dot{z} \\
\dot{v}
\end{bmatrix}),
\]

(20)

where \( A, B \) and \( C \) are as in (10). By (F), the nonlinearity \( \phi_{\omega^r} \) satisfies

\[
\|\phi_{\omega^r}(w)\| \leq L\|w\| \quad \forall w \in \mathbb{R}^m.
\]

(21)

Invoking (8), it is clear that there exists \( \varepsilon > 0 \) such that

\[
\varepsilon + \sup_{s \in \mathbb{R}} \| (sI + \Gamma)^{-1} \| \cdot \| \Gamma - KG(0) \| < 1/L.
\]

(22)

Combining this with Lemma 6, we see that there exists \( g > 0 \) such that, for all \( g \in (0, g^*) \),

\[
\|G\|_\infty \leq \varepsilon + \sup_{s \in \mathbb{R}} \| (sI + \Gamma)^{-1} \| \cdot \| \Gamma - KG(0) \| < 1/L.
\]

(23)

Let \( g \in (0, g^*) \). The claims (C1)-(C3) follow once the zero equilibrium of the Lur’e system (20) is shown to be globally exponentially stable which in turn follows from (21) and (22) and an absolute stability result, such as [32, Corollary 5.6.50] or [34, Theorem 5 (iii)] (whilst the latter result is for SISO systems, it is not difficult to show that it extends to the MIMO case). To establish (C4), we note that \( \dot{x} = \dot{z} = Az + B\phi_{\omega^r}(u) \) and (C1), (C2), (17) and (21) to obtain that, for all \( t \geq 0 \),

\[
\|\dot{x}(t)\| \leq ce^{-\gamma t}(\|x(0)\| + \|v(0)\|)
\]

for suitable positive constants \( c > 0 \) and \( \gamma \), as required.

We continue with some remarks on Theorem 4, particularly the existence of a suitable matrices \( K \) and \( \Gamma \).

**Remark 7.**

(i) Recall from Remark 5 that \( p \geq m \) and \( \text{rk} G(0) = m \) are necessary conditions for (a)-(c) and (8) to hold (in the usual case that \( L \geq 1 \)). The rank condition on \( G(0) \) implies that \( G(0) \) has a left inverse: for example, \( G(0)^T G(0)^{-1} G(0)^T \). Choosing \( K \) as any left inverse of \( G(0) \) and \( \Gamma := KG(0) = I \), that (b) holds and (8) is trivially satisfied for every \( L > 0 \). In the special case that \( m = p \) and \( G(0) \) is invertible, then the above choices simplify to \( K := G(0)^{-1} \) and \( \Gamma := I \).

(ii) Assume that the system \( (A, B, C) \) is subject to parametric uncertainty, and that the “true” linear system is given by \( (A, B, C) \) with \( \hat{A} \) Hurwitz. If the “nominal” steady-state gain \( G(0) \) is such that there exists \( K \) with \( -G(0)K \) Hurwitz and

\[
\sup_{s \in \mathbb{R}} \| (sI + KG(0))^{-1} \| \cdot \| K (G(0) - G(0)) \| < 1/L,
\]

(24)

where \( G \) is the transfer function of \( (\hat{A}, \hat{B}, \hat{C}) \), then, with the choice \( \Gamma = KG(0) \), it is guaranteed that the conclusions of Theorem 4 hold in the context of the “true” system \( (A, B, C) \). Assuming that \( \text{rk} G(0) = m \), and in light of part (i), choosing \( K \) as any left inverse of \( G(0) \) and \( \Gamma := I \), the condition (23) simplifies to

\[
\|I - KG(0)\| < 1/L,
\]

(25)

which is certainly satisfied if

\[
\|G(0) - \tilde{G}(0)\| < \frac{1}{L\|K\|}.
\]

We comment that the estimate (25) may equivalently be formulated as a ball condition (with centre \( \tilde{G}(0) \) and known radius). To summarise the above discussion, Theorem 4 applies to all plants with (unknown) steady-state gain \( G(0) \) if the design of \( \Gamma \) and \( K \) is based on are the (nominal) steady-state gain \( G(0) \) and \( \tilde{G}(0) \) is sufficiently close in norm to \( G(0) \). In other words, the closed-loop system (7) is locally robust with respect to uncertainty in the steady-state gain, captured by the estimates (23)-(25).

(iii) We comment further that it is possible to augment (7) with an adaptation for the parameter \( g \), replacing it by a dynamic variable (in the spirit of [10]) and obviating the requirement that it is chosen “sufficiently small”. The conclusions of Theorem 4 may still be shown to hold, although a formal statement and proof of this claim is beyond the scope of the present note.

Theorem 4 can be applied to the problem of regulating the output of a stable linear control system to a prescribed constant reference vector in the presence of input constraints. To this end, consider the linear system

\[
\dot{x} = Ax + Bu, \quad x(0) = x^0, \quad y = Cx,
\]

(26)

where \( A \in \mathbb{R}^{n \times n} \) is Hurwitz, \( B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{m \times n} \), together with the control objective of asymptotic tracking of constant reference vectors subject to the input constraint \( v(t) \in V \subset \mathbb{R}^m \) for all \( t \geq 0 \), where \( V := \{v_1, v_2, \ldots, v_k\} \times \{v_{k+1}, v_{k+2}, \ldots, v_m\} \). This problem has been studied in [35], where a solution is proposed that determines \( v \) in (26) adaptively. We will show that Theorem 4 provides an alternative solution.

To that end, let \( r \in G(0)V \), where \( G \) denotes the transfer function matrix of (26). Defining sat as in Example 1, it follows from Theorem 4, that the control objective is achieved by the control law

\[
v = \text{sat}(u), \quad u = gK(r - y) - g\Gamma(u - \text{sat}(u)),
\]

(27)

provided that \(-\Gamma \) is Hurwitz, the estimate

\[
\sup_{s \in \mathbb{R}} \| (sI + \Gamma)^{-1} \| \cdot \| \Gamma - KG(0) \| < 1,
\]

(28)

holds and \( g > 0 \) is sufficiently small. Recall from Remark 7 that if \( K \) is equal to a left inverse of \( G(0) \) (requiring that \( \text{rk} G(0) = m \)) and \( \Gamma = I \), then (27) holds.

Compared to [35], the controller proposed here is much simpler and easier to implement, particularly in the (high-dimensional) MIMO case. However, more information — namely of the steady-state gain \( G(0) \) and a sufficiently small gain \( g \) — is required. We emphasise that the problems considered here and in [35] are related, but not identical, as we permit non-diagonal as well as unbounded input nonlinearities.

We next show that, under the assumptions of Theorem 4, the equilibrium \(-A^{-1}B\phi(u') \) of the system (7) is input-to-state stable (ISS) with respect to additive disturbances. We refer the reader to [27] and [28] for a detailed discussion of ISS.

**Proposition 8.** Consider the closed-loop system

\[
\begin{aligned}
\dot{x} &= Ax + B\phi(w) + d_1, \quad y = Cx + d_2, \\
\dot{w} &= gK(r - y) - g\Gamma(w - \phi(w) - u^r + \phi(u^r)) + d_3, \\
x(0) &= x^0, \quad u(0) = u^0,
\end{aligned}
\]

(29)

where \( r \in R \) for feasible \( R \subset \mathbb{R}^p \), \( u^r \in \mathbb{U} \), \( d_1 \in L^p_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^n) \), \( d_2 \in L^p_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^m) \) and \( d_3 \in L^p_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^n) \). Assume that \( A \) and \(-\Gamma \) are Hurwitz and (F) holds. Then, if (8) is satisfied, there exists \( g^* > 0 \) such that for all \( g \in (0, g^*) \), there exist constants \( c_1, c_2, c_3 > 0 \) such that for all \( (x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m \), all \( r \in R \) and all \( d_1, d_2, d_3 \in L^p_{\text{loc}}, \)
where \( \text{Proof.} \) Equation (30) is a Lur’e system with additive forcing. The assumption the solution \( g \) for all \( t \) and consequently, the claim follows from \([26, \text{Theorem 3.2 and Remark 9}]\).

\[
\frac{\dot{z}}{z} = Aw + Bu, \quad y(t) = Cw(t) \quad \forall t \geq 0.
\]

(29)

Proof. Let \( K \in \mathbb{R}^{m \times p} \), \( \Gamma \in \mathbb{R}^{m \times m} \) and assume that \( -\Gamma \) is Hurwitz and \((8)\) holds. Defining the variables \( z \) and \( v \) as in \((16)\) and setting

\[
\zeta := \begin{bmatrix} z \\ v \end{bmatrix}, \quad B_c := \begin{bmatrix} I & 0 \\ 0 & -gK \end{bmatrix} \quad \text{and} \quad d := \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},
\]

the system \((28)\) may be rewritten as

\[
\dot{\zeta} = A\zeta + B\phi \nu (C\zeta) + B_c d,
\]

where \((A, B, C)\) and \( \phi \nu \) are given by \((10)\) and \((18)\), respectively.

Equation \((30)\) is a Lur’e system with additive forcing. The assumptions of Theorem 4 are satisfied and so there exists \( g^* > 0 \) such that, for all \( g \in (0, g^*), \) \((22)\) holds. Now, by \((F), (21)\) is also satisfied, and consequently, the claim follows from \([26, \text{Theorem 3.2 and comment after the proof of Theorem 3.2}]\).

Remark 9. In the undisturbed case \((d_i = 0 \text{ for } i = 1, 2, 3), (-A^{-1}B\phi (u'), u')\) is an equilibrium of system \((28)\). Proposition 8 implies that this equilibrium is ISS (with respect to \( d_1, d_2 \) and \( d_3 \)). In particular, the tracking error is “small” for “small” disturbances. An important consequence of Proposition 8 is that “small” uncertainties in \( \phi \) and \( \nu \) (both quantities appear on the right-hand side of the control law \((5)\)) will cause only a small deterioration of the tracking performance.

III. Examples

In the absence of an input nonlinearity (that is, \( \phi = \text{id} \)), the control law \((5)\) reduces to

\[
\dot{u} = gK(r - y), \quad u(0) = u^0
\]

and it is well known that if \( A \) and \(-KG(0)\) are Hurwitz, then, for all sufficiently small \( g > 0 \), the integrator \((31)\) in feedback connection with \((1)\) achieves asymptotic tracking for all initial conditions \((x^0, u^0)\) and all \( r \) in the image of \( G(0) \). In Example 10 below we construct an example which demonstrates that this is in general not true when an input nonlinearity is present.

Example 10. Consider \((1)\) with

\[
A = -I_4, \quad B = C = I_4, \quad \phi = \text{sat},
\]

where sat is as in Example 1 with saturation bounds given by

\[
V_1 := \{0, 300\}, \quad V_2 := \{0, 300\}, \quad V_3 := \{0, 356\}, \quad V_4 := \{137, 300\}.
\]

Evidently \( A \) is Hurwitz and \( G(0) = I_4 \). We choose

\[
K := \begin{bmatrix} 23 & 20 & 18 & 17 \\ 23 & 26 & 29 & 26 \\ 56 & 56 & 62 & 68 \\ -90 & -90 & -96 & -97 \end{bmatrix},
\]

which has the property that \(-KG(0) = -K\) is Hurwitz. The reference vector

\[
r := 200 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T,
\]

is feasible since it trivially satisfies \( r = G(0)r = G(0)\text{sat}(r) \).

Defining

\[
u^0 := \begin{bmatrix} 291 & 8.5 & 357 & 136 \end{bmatrix}^T \quad \text{and} \quad x^0 := \text{sat}(u^0),
\]

we note that \( u^0_1 > 356 = v^0_2 \) and \( u^0_4 < 137 = v^0_3 \).

Now

\[
r - \text{sat}(u^0) = \begin{bmatrix} 200 \\ -291 \\ 200 \\ 137 \end{bmatrix} = \begin{bmatrix} -91 \\ -191.5 \\ -156 \\ 63 \end{bmatrix},
\]

and a calculation shows that, for every \( g > 0, \)

\[
gK(r - Cx^0) = gK(r - \text{sat}(u^0)) = g \begin{bmatrix} 0 \\ 0 \\ 240 \\ -180 \end{bmatrix}.
\]

(33)

Furthermore, we have that

\[
Ax^0 + B\text{sat}(u^0) = -x^0 + \text{sat}(u^0) = 0.
\]

(34)

Defining \((x, u)\) by

\[
x(t) := x = \text{sat}(u^0), \quad u(t) = u^0 + g t \begin{bmatrix} 0 \\ 0 \\ 240 \\ -180 \end{bmatrix} \quad \forall t \geq 0,
\]

we see that \((u(t)) = \text{sat}(u^0)\) for all \( t \geq 0 \) and therefore, it follows from \((33)\) and \((34)\) that \((x, u)\) solves the integral control system given by \((1)\) and \((31)\). Consequently, the feedback systems is unstable for any choice of \( g > 0 \). In particular, \( y(t) \not\to r \) as \( t \to \infty \). We conclude that, in the presence of input nonlinearities, the “pure” integral controller \((31)\) does not guarantee asymptotic tracking (actually, may fail to achieve global stability).

We now apply the controller \((7)\) to the model data \((32)\) with \( \Gamma = KG(0) = K \). Theorem 4 then guarantees convergence of \((y(t))\) to \( r \) for all sufficiently small integrator gains \( g \). Figure 3 contains the resulting simulation for \( g = 0.027 \). We see that, although the performance is somewhat sluggish (not unexpected because the \( K \) matrix has been chosen rather “badly”), the inputs and states (the latter are equal to the outputs) converge and the states track the reference.

Figure 3. Simulation: low-gain controller \((7)\) applied to model data \((32)\). (a) State variables. (b) Saturated inputs in solid lines, inputs in dashed-dotted lines and saturation bounds in dashed lines. In both panels, dotted lines indicate the limits.

Remark 11. When \( \phi = \text{id} \) and \(-KG(0)\) are Hurwitz, then the equilibrium \((-A^{-1}B\phi (u'), u')\) of the low-gain integral control feedback system \((1)\) and \((31)\) is known to be locally asymptotically stable for every feasible \( r \) and all sufficiently small \( g > 0 \). Under the additional assumptions that \( KG(0) \) is symmetric, the equilibrium is globally asymptotically stable (this can be proved using absolute stability arguments and \([36, \text{Proposition 3.9}]\)). However, the symmetry assumption is extremely non-robust to parametric uncertainty in...
$G(0)$ and hence is unrealistic (except in the SISO case, where it holds trivially). Finally, note that in Example 10, the matrix $KG(0)$ is not symmetric.

The next example relates to Remark 11. We provide a simulation which illustrates the fact that if $KG(0)$ is symmetric, then a “pure” integral controller does achieve asymptotic tracking for all sufficiently small $g > 0$. The example also shows that the rate of convergence may be worse than that obtained by using the integral/anti-windup controller (7).

Example 12. Consider (1) with

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = C = I_2,$$

(35a)

and where $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is the non-diagonal saturation function from Example 2 with $\theta = 2$ (and the usual Euclidean two-norm). Note that $\phi$ satisfies (F) with $L = 2$. It is readily verified that $A$ is Hurwitz with $A = A^T$ and $B = C = C^T$, so that

$$-G(0) = CA^{-1}B = A^{-1} = -\frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is Hurwitz and symmetric. Consider the data

$$r = \begin{bmatrix} 0.75 \\ 0 \end{bmatrix}, \quad u^0 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}, \quad x^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

(35b)

and note that $r$ is feasible since $r = G(0)u^0 = G(0)\phi(u^0)$, where $u^0 = [1.5 \ -0.75]^T$. Finally, we choose $K = I$, $g = 1$ and $\Gamma = G(0)$. Figure 4 shows simulations of the closed-loop dynamics generated by the integral controller (31) and controller (7) with anti-windup component. We observe that the rate of convergence of the saturated inputs and outputs is faster for the latter: in Figure 4 (b), the closed-loop dynamics “spend less time” at the saturation bounds than in the “pure” integral control scenario. \qed

Figure 4. Simulations of the low-gain integral control systems from Example 12. The dotted lines denote reference values and the limits of the saturated inputs. In each panel the solid and dashed-dotted lines correspond to the controllers (31) and (7), respectively. (a) Outputs and reference components. (b) Saturated inputs and saturation limits.

Example 13. As a final illustrative example of the theory presented, consider the electrical circuit depicted in Figure 5 with two inductors, two resistors, a single capacitor and two external voltage sources. For $k = 1, 2$, let $L_k$ and $i_{L_k}$ denote the inductance and current, respectively, of the $k$-th inductor, and let $C_a$ and $v_{C_a}$ denote the capacitance and voltage across the capacitor, respectively. Let $R_k$ denote the resistance of the $k$-th resistor for $k = 1, 2$. We assume that the voltage sources are subject to saturation.

Invoking Kirchhoff’s Laws with state variables

$$x_1 := L_1i_{L_1}, \quad x_2 := L_2i_{L_2}, \quad x_3 = C_a v_{C_a},$$

and input and output variables

$$u_1 := v_1, \quad u_2 := v_2, \quad y_1 := i_{L_2}, \quad y_2 := v_{C_a},$$

leads to the MIMO control system of the form (1) with

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & 0 \\ 0 & -\frac{R_2}{L_2} & -\frac{R_2}{C_a} \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(36a)

and $\phi = \text{sat}$, with saturation bounds $V_k$ to be specified. A routine calculation shows that the matrix $A$ is Hurwitz for all $C_a, L_1, L_2, R_1, R_2 > 0$.

For the following simulation, we assume that the actual parameter values are unknown, but within 10% of the nominal values:

$$C_a = 3 \times 10^{-3}, \quad L_1 = 0.01H, \quad L_2 = 0.05H,$$

$$R_1 = 1\Omega, \quad R_2 = 1.5\Omega.$$ (36b)

Let $G$ denote the transfer function of the triple in (36a) with the nominal values in (36b). As $-G(0)$ has two eigenvalues both with real part equal to $-0.4$, it is Hurwitz and so we choose $K = I$ in (5) and $\Gamma = KG(0)$. With saturation bounds

$$V_1 = \{0, 7.5\} \quad \text{and} \quad V_2 = \{-5, 7.5\},$$

(36c)

(units in volts) it follows from (4) that

$$R := \{G(0)v : v_1 \in [0, 7.5], v_2 \in [-5, 7.5]\},$$

is the maximal set of feasible references for the nominal system, and is depicted in Figure 6. The actual set of feasible references depends on the uncertain “true” system. For

$$r := \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad u^0 := \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad x^0 := 0 \quad \text{and} \quad g := 2,$$

(36d)

simulations of the dynamics generated by the integral/anti-windup controller (7) are plotted in Figure 7. We see that both the saturated inputs and outputs converge and that the outputs track the reference. The simulation shown in Figure 7 was performed by (pseudo)randomly drawing parameter values from within 10% of the nominal values. A calculation shows that the nominal and “true” steady-state gains are given by

$$G(0) = \frac{1}{R_1 + R_2} \begin{bmatrix} 1 & -1 \\ R_2 & R_1 \end{bmatrix}, \quad \tilde{G}(0) = \frac{1}{R_1 + R_2} \begin{bmatrix} 1 & -1 \\ R_2 & R_1 \end{bmatrix},$$

respectively. It is readily verified that the estimate (23) holds with $L = 1$, $K = I$ and $\Gamma = G(0)$, for all $R_1, R_2$ within 10% of the values in (36b), ensuring that Theorem 4 holds. Note that in fact no knowledge of $C_a, L_1$ or $L_2$ is required, and the imposed bounds on the parameter variation in $R_1$ and $R_2$ is sufficient to implement (7), meaning that exact knowledge of $G(0)$ is not required. \qed

Figure 5. Sample circuit diagram from Example 13.
Figure 6. Maximal feasible region (shaded area) of the nominal electrical circuit example (36) and reference (cross) $r$ in (36d).

Figure 7. Simulation: circuit example controlled by (7). (a) Outputs in solid lines with dotted lines denoting reference components. (b) Saturated inputs in solid lines, the dash-dotted lines denote the unsaturated inputs for comparison purposes, the dashed lines are selected saturation bounds and the dotted lines are the input limits.

REFERENCES


