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Consensus time and conformity in the adaptive voter model

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The adaptive voter model is a paradigmatic model in the study of opinion formation. Here we propose an extension for this model, in which conflicts are resolved by obtaining another opinion, and analytically study the time required for consensus to emerge. Our results shed light on the rich phenomenology of both the original and extended adaptive voter models, including a dynamical phase transition in the scaling behavior of the mean time to consensus.

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In nature, collective intelligence is observed in a wide variety of species. Quite generally groups of animals are able to aggregate information and make decisions jointly [1]. The most impressive example is perhaps human culture, which is created through the aggregation and transmission of individual insights and opinions. However, while collective decision making seems to be universally beneficial in animals, it can have an adverse effect in humans, where the exchange of opinions can lead to the propagation of counterfactual rumors and can even give rise to the formation of radicalized groups. A deeper understanding of the process of collective opinion formation is needed if we are to determine the conditions under which it leads reliably to a beneficial outcome. Significant progress is starting to be made on this problem, with several recent studies linking statistical physics models of opinion dynamics to experimental data [1–4]. For this effort to continue, the theoretical understanding of these systems must be expanded and systematic tools developed to reach analytical results.

A paradigmatic model in this field is the adaptive voter model [5,6], describing a collection of individual agents whose opinions and social contacts may change over time. Agents hold one of two opinions, say, A and B, and are linked together by a sparse network of social interactions. The system evolves in time as follows: Pairs of connected agents with opposing opinions are randomly chosen and either (i) the conflict is resolved by one agent adopting the opinion of the other, or (ii) one agent breaks the contact and forms a new link to a different agent. After a sufficiently long time the system reaches one of two types of absorbing state: a consensus state in which all agents hold the same opinion, or a fragmented state in which both opinions survive in disconnected groups [6–9].

While the adaptive voter model has been explored in several recent studies, a larger body of previous work focuses on opinion dynamics on static networks (where the rewiring process does not occur and hence fragmentation is impossible). The main question addressed in these studies is the time taken for consensus to emerge [10–12], which in general grows as N^{μ}, where the exponent $\mu \leqslant 1$ depends on the degree distribution of the underlying network [12]. In contrast to this work, all the major analytical results in the adaptive networks literature have been concerned with the occurrence of fragmentation [6–9]. The question of consensus time has so far been largely neglected, although some interesting results have been obtained via simulations [13] and heuristic scaling arguments [7].

Here we describe a systematic and generally applicable analytical method to compute the time taken for consensus to emerge in the adaptive voter models. Furthermore, we show that when we allow pairs of agents to resolve conflicts by seeking another agent's opinion, this can either speed up or slow down the formation of consensus. This extension of the model exhibits a dynamical phase transition between exponential and logarithmic scaling laws, depending on the probability of accepting the other agent's opinion. The original adaptive voter model is the critical case, exhibiting linear growth of consensus time.

Consider a network of N nodes (agents) joined by a total of K edges, which represent social interactions. Initially each agent is randomly assigned either opinion A or B, and the edges are placed randomly. In each time step an edge (i,j) is selected at random. If the focal edge connects agents that hold different opinions, then it said to be active and the corresponding conflict is resolved in either of two ways. With probability ϕ the edge is rewired, with node i cutting the edge and creating a new one to another node, selected at random from the set of all nodes holding the same opinion as i. If the edge is not rewired then a third opinion is sought: Another node is selected at random from the rest of the graph and with probability ρ both i and j adopt the opinion of the new node, otherwise both i and j adopt the opposite opinion. This three-body interaction is a notable departure from the traditional voter model, although related systems have been studied previously [14].

The parameter ρ can be interpreted as a measure of social conformity [15] and may range from $\approx$1 for a strongly conformist opinion formation process, to $\approx$0 for nonconformist processes. For $\rho = 1/2$ the decision is not biased by the other agent, and the standard adaptive voter model is recovered. In most relevant contexts the stochastic response to another agent should not be interpreted as an actual consultation, but rather as an influence from the cultural "mean field" propagated by mainstream mass media.

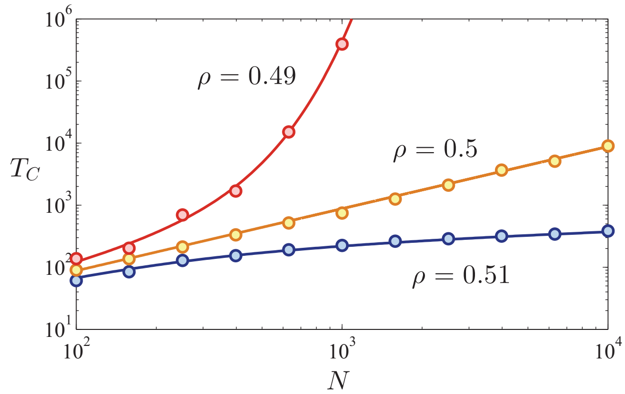

FIG. 1. (Color online) Dynamical phase transition in the scaling behavior of the mean time to reach consensus T_C as a function of network size N. At the critical value $\rho = 1/2$ the growth of T_C is linear, while being exponential for ρ above the critical point, and logarithmic below. In each case circles give the average over 100 simulation runs with $k = 10$ and $\phi = 0.1$, while the solid lines show the result of the theory developed in the main text—see Eq. (7).

As we will see, the model exhibits a dynamical phase transition as ρ crosses the threshold $\rho_c = 1/2$. Simulation results (Fig. 1) show that when agents accept the third opinion with probability $\rho > \rho_c$, the time to consensus only grows logarithmically with N, whereas in the case that $\rho < \rho_c$ it exponentially diverges. At precisely $\rho = \rho_c$ the original adaptive voter model is recovered, and we find a linear growth of the consensus time.

For understanding the emergence of consensus in the model, we capture the dynamics of the system by a set of system-level variables that indicate the abundance of individual nodes and linked pairs of nodes with given opinion states. We denote the numbers of agents with each opinion by $[A]$ and $[B]$, and the numbers of edges between different agents by $[AA]$, $[AB]$, and $[BB]$. Because of the conservation laws for nodes, $[A]+[B]=N$, and edges [16], $[AA]+[AB]+[BB]=K$, the state of the system can be summarized by just three independent quantities: $X=[A]-[B]$, $Y=[AA]-[BB]$, and $Z=[AB]$. In analogy with spin glasses, the first two of these describe the magnetization of nodes and edges, while the third specifies the number of active edges and therefore controls the overall reaction rate of the system.

Our analysis proceeds by deriving a closed set of rules for the stochastic dynamics of the variables X, Y, and Z, which approximate the evolution of the full network model. Introducing the system state vector $\mathbf{\Omega} = (X,Y,Z)^T$ (T denotes transpose), we consider the effect of the four possible events which may occur in a given time step: rewiring or updating of an A or B agent. For each, we write down the probability r_i of occurrence in a given time step, and the average net change $\boldsymbol{s}_i$ to $\mathbf{\Omega}$ caused by the event. For example, an AB link is chosen to be rewired to create an AA link with probability $r_1 = \phi Z/2K$ and the change to the system is $\boldsymbol{s}_1 = (0,1,-1)^T$

Following Ref. [17], we approximate the dynamics of $\mathbf{\Omega}$ by a Markov jump process known as the pair-based proxy (PBP). It is defined as follows: In each time step a jump vector $\boldsymbol{s}_i$ is chosen randomly with probability r_i, and the summary vector is updated by $\mathbf{\Omega} \mapsto \mathbf{\Omega} + \boldsymbol{s}_i$. The PBP represents a considerable reduction in complexity from the original adaptive network, and yet retains the essential stochastic nature of the system.

In the limit of large network size, the PBP can be further reduced to a low-dimensional system of stochastic differential equations (SDEs). For simplicity, we package the update vectors into a stoichiometric matrix $\mathbf{S} = (\boldsymbol{s}_1 \cdots \boldsymbol{s}_4)$ and collect the event probabilities in a vector $\boldsymbol{r} = (r_1 \cdots r_4)^T$. Defining the rescaled variables $x = X/N$, $y = Y/K$, and $z = Z/K$, we apply Kurtz' theorem [18] to obtain the following SDE:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{S}\boldsymbol{r} + \frac{1}{\sqrt{N}}\boldsymbol{\eta}(t), \tag{1}$$

where $\langle \eta_i(t)\eta_j(t')\rangle = \delta(t-t')B_{ij}$, and the noise correlation matrix $\mathbf{B}$ is given by

$$B_{ij} = \sum_k S_{ik} r_k S_{kj}. \tag{2}$$

These equations can be written explicitly in terms of x, y, and z if necessary. Let us consider the expression for the magnetization x in detail,

$$\frac{dx}{dt} = 4(1-\phi)(\rho - 1/2)xz + \frac{1}{\sqrt{N}}\eta_1(t). \tag{3}$$

The factor of $(\rho - 1/2)x$ constitutes either positive or negative feedback depending on the value of ρ, which already suggests a transition in behavior around point $\rho_c = 1/2$. The nonlinear interaction xz shows that the dynamics requires the presence of active edges, as well as an overall imbalance of opinions.

We make analytical progress by exploring the behavior of (1) in the neighborhood of the transition, introducing $\varepsilon = \rho - 1/2$. Let us first consider the case $\varepsilon = 0$ in the deterministic limit $N \to \infty$. In this limit, the system (1) possesses two manifolds of fixed points. The first is the plane $z = 0$, which represents the state in which there are no active edges and thus the model is frozen. These states are also absorbing states of the finite-size network model, corresponding to fragmentation. The second manifold of fixed points defines a parabola

$$y = x, \quad z = \frac{1}{2}(1-x^2)\frac{\phi_* - \phi}{1-\phi}, \tag{4}$$

where $\phi_* = (k-2)/(k-1)$ is the approximate critical rewiring rate for the fragmentation transition identified in Ref. [7]. We note that Eq. (4) is a pair-level approximation which becomes poor close to the fragmentation point; see Ref. [19] and Fig. 2(b). However, the question of consensus time concerns only values of ϕ below the transition, where we find the approximation to be sufficient for a large parameter range.

Local stability is governed by a linearization that is provided by the Jacobian matrix $\mathbf{J}$ of (1). Computing the eigenvalues of the Jacobian on the parabola of active states we find $\lambda_1 = 0$, $\lambda_2 = 2(\phi - \phi_*)/(2-\phi_*)$, and $\lambda_3 = \phi - \phi_*$. The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ x\mu_1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ x\mu_2 \end{pmatrix}, \tag{5}$$

where the constants are given by $\mu_1 = -1 + 1/(k-1)(1-\phi)$ and $\mu_2 = -1 - 2/(k-1)(k-2)(1-\phi)$. The second two eigenvalues are negative, and large in comparison to $\lambda_1 = 0$, meaning that trajectories close to the parabola collapse quickly in the directions of $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ [Fig. 2(b)]. This behavior, which will play a central role below, was previously noted in Ref. [7] and is reminiscent of similar observations in the voter model on a static network [12,20].

Although the full stochastic system (1) cannot be easily solved, we can derive an "effective" solvable one-dimensional system by restricting our attention to behavior in the neighborhood of the slow manifold, in analogy with Refs. [21,22]. We reason as follows: In short time windows small Gaussian perturbations described by the noise correlation matrix defined in (2) may move the system away from the slow manifold; for sufficiently small perturbations the net drift is then governed by the fast eigenvectors $\boldsymbol{v}_{2,3}$. We formalize this idea by fixing y and z to the values in (4) and replacing the noise matrix $\boldsymbol{B}$ with $\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^T$, where $\boldsymbol{P}$ is the linear projection whose range is spanned by $\boldsymbol{v}_1$ and kernel by $\boldsymbol{v}_{2,3}$.

We thus obtain a reduced equation for motion on the manifold, in which x is the only remaining variable,

$$\frac{dx}{dt} = 2\varepsilon(\phi_* - \phi)x(1-x^2) + \sqrt{\frac{2(\phi_* - \phi)(1-x^2)}{N}}\,\xi(t), \quad (6)$$

where ξ is a standard Gaussian white noise variable. The picture we have now is as follows: From the initial condition the system state moves rapidly to the parabolic slow manifold (4), where it then drifts stochastically according to (6) until eventually reaching one of the absorbing consensus states at $x = \pm 1$.

We are interested in the mean waiting time before consensus is reached. Since our theory is one dimensional, we follow Ref. [23] to derive

$$T_C = \frac{N}{(\phi_* - \phi)} \int_0^1 \int_0^y \frac{e^{\varepsilon N(x^2 - y^2)}}{1 - x^2}\,dx\,dy. \quad (7)$$

In the special case $\varepsilon = 0$ the integral above can be computed easily to obtain

$$T_C = N\frac{\log(2)}{\phi_* - \phi}. \quad (8)$$

In Fig. 2 we show a comparison between this prediction and the results of numerical simulations; the agreement is excellent for values of ϕ far from the fragmentation transition. This result validates the heuristic scaling argument presented in Ref. [7].

If $\rho < 1/2$ then ε is negative and the large N asymptotic of (7) can be computed by Laplace's method as

$$T_C \approx \frac{e^{|\varepsilon| N}}{4(\phi_* - \phi)}\sqrt{\frac{\pi}{|\varepsilon|^3 N}}. \quad (9)$$

Alternatively, for $\rho > 1/2$ the system (6) is deterministically unstable and thus the main contribution to T_C comes from the initial symmetry breaking perturbation. In the neighborhood of the initial condition $x = 0$, we have the linearized equation

$$\frac{dx}{dt} = 2\varepsilon(\phi_* - \phi)x + \sqrt{\frac{2(\phi_* - \phi)}{N}}\,\xi(t). \quad (10)$$

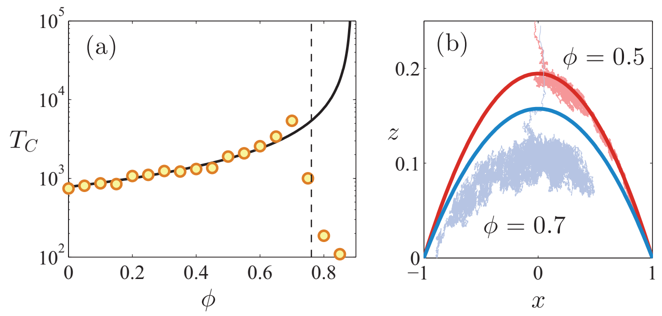

FIG. 2. (Color online) (a) Dependence of consensus time T_C on the rewiring rate ϕ, when $\varepsilon = 0$ and $k = 10$. Orange circles show the average of 100 samples, the solid black line is the theoretical prediction (8), while the dashed line indicates the point of the fragmentation transition, as derived in Ref. [8]. (b) Comparison between the slow manifold (4) and typical simulation trajectories for $\phi = 0.5$ (red) and $\phi = 0.7$ (blue). The discrepancy between simulations and theory in the case $\phi = 0.7$ illustrates the breakdown of the pair-level approximation close to the fragmentation transition.

Following Ref. [23] again we find $\langle x^2 \rangle = (e^{4\varepsilon(\phi_* - \phi)t} - 1)/2\varepsilon N$, and thus the time taken for x to reach a magnitude of order one is

$$T_C \sim \frac{\log N}{4\varepsilon(\phi_* - \phi)}. \quad (11)$$

A comparison of these predictions with numerical results (Fig. 3) shows excellent agreement. These results establish a trichotomy between exponential, linear, and logarithmic scaling laws, dependent on the parameter ρ. Note that the original adaptive voter model lies on the critical boundary between scaling regimes.

The above result suggests linear scaling of consensus time to be the exception rather than the rule, and likely to be destroyed by small changes in model specification. This is indeed the case, as can be seen by considering some other variants of the adaptive voter model. In some studies the target nodes in rewiring events are chosen randomly without regard to their opinion [9,13]. We refer to this as the *rewire-to-random* scheme, as opposed to the *rewire-to-same* scheme we considered above.

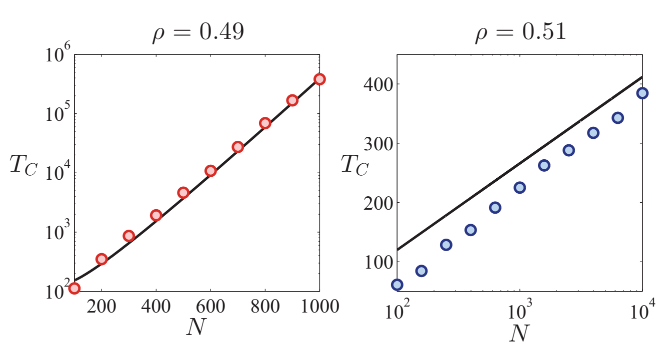

FIG. 3. (Color online) Large N scaling for ρ either side of the critical value $\rho_c = 1/2$. On both plots circles show the average consensus time for 100 simulation runs with $k = 10$ and $\phi = 0.1$. On the left the black line is given by Eq. (9), while on the right the slope is given by Eq. (11), whereas the intercept has been chosen manually for comparison.

TABLE I. Summary of the scaling laws (in large N and k) found for the mean time to consensus in various specifications of the adaptive voter model.

	Rewire to same	Rewire to random
Edge	$T_C \sim N$	$T_C \sim N$
Node direct	$T_C \sim N$	$T_C \sim k\log(Nk)$
Node reverse	$T_C \sim N$	$T_C \sim e^{N/k}\sqrt{k^3/N}$

The mechanism for choosing nodes to update may also be altered from the *link update* rules we have used so far. Alternative model formulations use node update rules [13], where one first chooses a node i before selecting one of its neighbors j, and then in *direct* node update i copies j's opinion, whereas in *reverse* node update j copies i. The corresponding models are the classical adaptive voter model (direct node update) and the adaptive invasion model (reverse node update).

These changes in model specification result in different expressions for the event probability vector $\boldsymbol{r}$, however, the rest of the analysis may be repeated analogously. The results are summarized in Table I (see the Supplemental Material [24] for details). For link update rules and reasonably large k, we find that the choice of rewiring rule does not change the typical time to reach consensus. This effect is demonstrated numerically in Fig. 4(a). However, by using node update rules a range of scaling behaviors is possible. For example, the growth of T_C is slightly slower than exponential in the case of node reverse and rewire to random. We can test this prediction by considering dense networks in which the average degree scales with the number of nodes according to $k = cN$. Here the theory predicts a return to linear growth, which is confirmed numerically in Fig. 4(b).

In summary, we have formulated an analytical theory for the emergence of consensus in an extension of the adaptive voter model. By including the simple and sociologically plausible conflict resolution mechanism of seeking another opinion, we have shown how the formation of consensus may be enhanced or suppressed. This effect is manifested in a trichotomy of scaling laws for consensus time, between exponential, linear, and logarithmic. We also applied the proposed method to several other specifications of the model, showing how the previously observed sensitivity to model specification breaks down for highly connected networks. In all of the models investigated, consensus formation is driven by the intrinsic noise arising from the microscopic dynamical rules of the system. This noise is a universal feature of network models, being a result of the discrete nature of the individual interacting nodes and edges. We expect that the methods developed in this work will provide insights into emergent phenomena in other network systems.

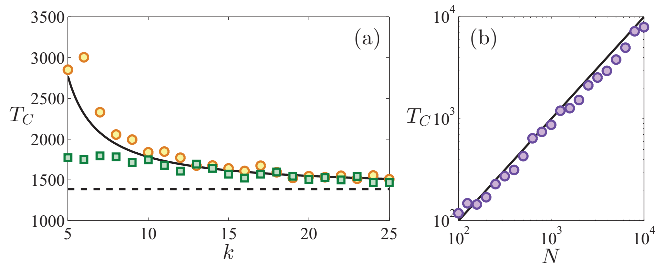

FIG. 4. (Color online) Consensus time in other model specifications. (a) For link update rules with moderate k, consensus time does not depend strongly on the rewiring rule. Theoretical predictions of Eq. (8) (solid line) and the large k limit (dashed line) are compared to simulation results averaged over 1000 networks for *rewire to same* (orange circles) and *rewire to random* (green squares). (b) Linear scaling for *node direct* updates in dense networks with $k = N/10$. The theoretical prediction of $T_C \sim N$ (solid line) is compared to simulation results averaged over 100 samples (purple circles). Parameters: $\phi = 0.5$, $N = 1000$.

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[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.88.030102 for further details on the mathematical derivation of the main results discussed in the article.