Inference on Modelling Cross-Sectional Dependence for a Varying-Coefficient Model

Bin Peng[1]

University of Technology Sydney

Abstract

In this note, I have studied a vary-coefficient model under cross-sectional dependence. The technique of Robinson (2011) is employed to mimic the dependence among cross-sectional data sets. The asymptotic normality is established for the proposed estimator.

Keywords: Asymptotic theory; cross-sectional dependence; varying-coefficient.

JEL classification: C13, C14, C51

[1] *Address for Correspondence*: Bin Peng, Economics Discipline Group, University of Technology Sydney, Australia. Email: Bin.Peng@uts.edu.au.

1 Introduction

The cross sectional dependence has been a hot topic for the past two decades. A dominant branch of modelling the cross-sectional dependence is to use a factor structure in panel data models (c.f. Pesaran (2006), Bai (2009) and so forth). Recently, Robinson (2011) and Lee and Robinson (2016) have employed the time series technique to model the dependence among cross-sectional data sets. Following the spirit of their work, I consider a varying-coefficient model with cross-sectional dependence in this study.

2 Model Specification

The mode is as follows:

$$y_i = x_i' \beta(z_i) + u_i. \tag{2.1}$$

$z_i \in [0, 1]$ is the so-called univariate index variable (Wang and Xia (2009)) and x_i is a $p \times 1$ vector. For simplicity, we consider the scalar case for z_i only and it is straightforward to extend z_i to multivariate case. To distinguish x_i and z_i, they are referred to as regressors and covariates hereafter. In order to impose the cross-sectional dependence, we follow Robinson (2011) and Lee and Robinson (2016) and denote that

$$u_i = \sigma(x_i, z_i) e_i, \quad e_i = \sum_{j=1}^{\infty} b_{ij} \varepsilon_j, \quad b_{ii} \neq 0, \quad B_i = \sum_{j=1}^{\infty} b_{ij}^2 < \infty \quad \text{for } i = 1, \ldots, N, \tag{2.2}$$

where $\sigma : \mathbb{R}^p \times [0, 1] \to \mathbb{R}$, the b_{ij} are real constants, and $\{\varepsilon_j, j \geq 1\}$ is a sequence of independent random variables with zero mean and unit variance, independent of $\{x_j, j \geq 1\}$ and $\{z_j, j \geq 1\}$.

Remark:

> Notice $b_{ii} \neq 0$ rules out the case where the error term e_i does not change across index i. For example, without the restriction of $b_{ii} \neq 0$, one can let $\sigma(x_i, z_i) = 1$, $b_{i1} = 1$, $b_{ij} = 0$ for $i = 1, \ldots, N$ and $j = 2, \ldots, \infty$. Then the model will reduce to $y_i = x_i' \beta(z_i) + \varepsilon_1$. In this case, the consistent estimation cannot be achieved at all.

In this note, our kernel function is denoted as:

$$K_h(z_i - z) = \frac{1}{h} K\left(\frac{z_i - z}{h}\right), \tag{2.3}$$

where $K(w)$ is symmetric denoted on $[-1, 1]$ satisfying $\int_{-1}^{1} K(w) dw = 1$ and h is the bandwidth. In order to facility the development, we adopt the following assumptions.

Assumptions:

1. $\{\varepsilon_j, j \geq 1\}$ is a sequence of independent random variables with zero mean and unit variance, independent of $\{x_j, j \geq 1\}$ and $\{z_j, j \geq 1\}$. $E[\varepsilon_j] = 0$, $E[\varepsilon_j^2] = 1$ and $\max_{j\geq 1} E[\varepsilon_j^{2+\nu}] < \infty$. $\sigma^2(x,z)$ is a uniformly bounded. Moreover, $\max_{1\leq i\leq N} E\|x_i\|^4 < \infty$ and $\max_{z\in[0,1]} \|\beta(z)\| < \infty$.

2. Let $E[x_i x_i' | z_i = z] = \Sigma_{xi}(z)$, where $\|\Sigma_{xi}(z)\|$ is uniformly bounded on $[0,1]$. $\Sigma_{xi}(z)$ has bounded continuous second order derivative with respect to z uniformly in i. Moreover, x_i is the function of z_i and independent of z_j for $i \neq j$.

3. For $1 \leq i \neq j \leq N$, let $f_{ij}(w_1, w_2)$ denote the joint density function for (z_i, z_j) and be bounded uniformly in i, j. For $i = 1, \ldots, N$, let $f_i(w)$ denote the density function for z_i and be bounded uniformly in i. In addition, $f_i(w)$ has uniformly bounded continuous second order derivative with respect to w.

4. (a) $Nh \to \infty$, $h \to 0$;

   (b) $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N B_i = B$ and $\max_{1\leq i\leq N} |B_i| \leq C_1$, where C_1 is a constant. Also, for $\forall z \in [0,1]$, let $V_2(z) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \Sigma_{xi}(z) f_i(z)$ be positive definite uniformly in z.

   (c) $\max_{1\leq j\leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^N |b_{ij}| \to 0$;

   (d) $\frac{\Delta_{2N}}{N^2} \to 0$ and $\frac{\sqrt{\Delta_{1N}}}{Nh} \to 0$, where

$$\Delta_{1N} = \sum_{i,j=1, i\neq j}^N \iint |f_{ij}(w_1,w_2) - f_i(w_1) f_j(w_2)|\, dw_1 dw_2,$$

$$\Delta_{2N} = \sum_{i,j=1, i\neq j}^N |\gamma_{i,j}|, \quad \gamma_{i,j} = \mathrm{Cov}(e_i, e_j);$$

Assumptions 1-4 are standard in the literature (c.f. Wang and Xia (2009), Lee and Robinson (2016)), so the relevant discussions are omitted. In Assumption 4.c, $\max_{1\leq j\leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^N |b_{ij}| \to 0$ certainly captures the i.i.d. case. For example, let $\sigma(x_i, z_i) = 1$. When u_i is i.i.d., the matrix $\mathcal{B} = \{b_{i,j}\}_{N\times N} = I_N$. Then it is easy to see that $\max_{1\leq j\leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^N |b_{ij}| \to 0$ holds. Notice that if z_i is independent across i, one can easily show that $\Delta_{1N} = 0$ and $\gamma_{i,j} = 0$, so Assumption 4.d holds immediately.

For any given $z \in [0,1]$, we investigate the next estimator.

$$\hat{\beta}(z) = \left(\sum_{i=1}^N x_i x_i' K_h(z_i - z)\right)^{-1} \sum_{i=1}^N x_i y_i K_h(z_i - z). \tag{2.4}$$

Then the next result follows based on the above settings.

Theorem 2.1. *Under Assumptions 1-4,*

$$\sqrt{Nh}\left(\hat{\beta}(z)-\beta(z)-O_P(h^2)\right)\to_D N(0,V_2^{-1}(z)V_1(z)V_2^{-1}(z))$$

where

$$\begin{aligned}V_1(z)=&\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}f_i(z)B_i\int\Sigma_{xx\sigma}(w)K^2(w)\,dw\\&+\lim_{N\to\infty}\frac{h}{N}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\gamma_{i_1,i_2}f_{i_1}(z)f_{i_2}(z)\tilde{\eta}\tilde{\eta}',\end{aligned}$$

where $\tilde{\eta}=\int\eta(w)K(w)dw$, $\eta(z_i)=E[x_i\sigma(x_i,z_i)|z_i]$, $\Sigma_{xx\sigma}(z_i)=E[x_ix_i'\sigma^2(x_i,z_i)|z_i]$ *and* $V_2(z)$ *is denoted in Assumption 4.*

3 Conclusion

In this note, I have studied a vary-coefficient model under cross-sectional dependence. The technique of Robinson (2011) and Lee and Robinson (2016) is employed to mimic the dependence among cross-sectional data sets. The asymptotic normality is established for the proposed estimator. The optimal bandwidth selection has been achieved under i.i.d. case in Li and Racine (2010), but what the optimal bandwidth looks like under cross-sectional dependence remains unsolved.

Appendix

Lemma A.1. *Let* $\zeta_i=(x_i,z_i)$ *and Assumption 3 hold. For any bounded function* $g(w)$ *with* $w=(w_1,w_2)\in\mathbb{R}^p\times[0,1]$ *having that* $E[g(\zeta_i)g(\zeta_j)]$ *with* $i\neq j$ *and* $E[g(\zeta_i)]$ *exist uniformly in* $1\leq i,j\leq N$, *we obtain that*

$$\left|\sum_{i,j=1,i\neq j}^{\infty}\{E[g(\zeta_i)g(\zeta_j)]-E[g(\zeta_i)]E[g(\zeta_j)]\}\right|=O(\Delta_{1N}).\tag{A.1}$$

Proof of Lemma A.1:

$$\begin{aligned}&\left|\sum_{i,j=1,i\neq j}^{\infty}\{E[g(\zeta_i)g(\zeta_j)]-E[g(\zeta_i)]E[g(\zeta_j)]\}\right|\\=&\left|\sum_{i,j=1,i\neq j}^{\infty}\int g(w_1)g(w_2)\left(f_{ij}(w_1,w_2)-f_i(w_1)f_j(w_2)\right)dw_1dw_2\right|\\=&O(1)\sum_{i,j=1,i\neq j}^{\infty}\int|f_{ij}(w_1,w_2)-f_i(w_1)f_j(w_2)|\,dw_1dw_2=O(\Delta_{1N}).\end{aligned}\tag{A.2}$$

Then the proof is complete. ■

Lemma A.2. *Under Assumptions 1-4, for any given* $z \in [0,1]$

1. $\frac{1}{N}\sum_{i=1}^{N} x_i x_i' K_h(z_i, z) - \frac{1}{N}\sum_{i=1}^{N} \Sigma_{xi}(z) f_i(z) = O_P(h^2) + O_P\left(\frac{\sqrt{\Delta_{1N}}}{Nh}\right)$;
2. $\frac{1}{N}\sum_{i=1}^{N} x_i u_i K_h(z_i, z) = O_P\left(\frac{1}{\sqrt{Nh}}\right) + O_P\left(\frac{\sqrt{\Delta_{1N}}}{Nh}\right) + O_P\left(\frac{\sqrt{\Delta_{2N}}}{N}\right)$;
3. $\frac{1}{N}\sum_{i=1}^{N} x_i x_i' \left(\beta(z_i) - \beta(z)\right) K_h(z_i - z) = O_P(h^2)$;

Proof of Lemma A.2:

1) Write

$$
\begin{aligned}
& E\left[\frac{1}{N}\sum_{i=1}^{N} x_i x_i' K_h(z_i - z)\right] = \frac{1}{N}\sum_{i=1}^{N} E\left[x_i x_i' K_h(z_i - z)\right] \\
&= \frac{1}{N}\sum_{i=1}^{N} \int \Sigma_{xi}(w) K_h(w - z) f_i(w) dw \\
&= \frac{1}{N}\sum_{i=1}^{N} \int \Sigma_{xi}(z + hw) K(w) f_i(z + hw) dw \\
&= \frac{1}{N}\sum_{i=1}^{N} \Sigma_{xi}(z) f_i(z) + O(h^2),
\end{aligned} \tag{A.3}
$$

where the fourth equality follows from using Taylor expansion on each element of $\Sigma_{xi}(w)$ and $f_i(w)$.

For the second moment, write

$$
\begin{aligned}
& E\left\|\frac{1}{N}\sum_{i=1}^{N}\left(x_i x_i' K_h(z_i - z) - \Sigma_{xi}(z) f_i(z)\right)\right\|^2 \\
&= \frac{1}{N^2}\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{i=1}^{N} E\left[x_{i,m} x_{i,n} K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)\right]^2 \\
&\quad + \frac{1}{N^2}\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{i,j=1, i\neq j}^{N} E\Big[\left(x_{i,m} x_{i,n} K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)\right) \\
&\quad \cdot \left(x_{j,m} x_{j,n} K_h(z_j - z) - \Sigma_{xi,mn}(z) f_j(z)\right)\Big] \\
&:= A_1 + A_2,
\end{aligned} \tag{A.4}
$$

where $\Sigma_{xi,mn}(z)$ denotes the $(m,n)^{th}$ element of $\Sigma_{xi}(z)$ for $i = 1, \ldots, N$.

For A_1, write

$$
\begin{aligned}
A_1 &= \frac{1}{N^2}\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{i=1}^{N} E\left[x_{i,m} x_{i,n} K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)\right]^2 \\
&\leq \frac{1}{N^2 h}\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{i=1}^{N} E\left[x_{i,m}^2 x_{i,n}^2 K_h(z_i - z)\right] \\
&\leq \frac{1}{N^2 h}\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{i=1}^{N} E\left[x_{i,m}^2 x_{i,n}^2 K_h(z_i - z)\right] = O\left(\frac{1}{Nh}\right),
\end{aligned} \tag{A.5}
$$

where we have used the uniform boundedness of $K(w)$.

For A_2, write

$$\begin{aligned}
&\sum_{i,j=1,i\neq j}^{N} E\Big[(x_{i,m}x_{i,n}K_h(z_i-z)-\Sigma_{xi,mn}(z)f_i(z)) \\
&\cdot (x_{j,m}x_{j,n}K_h(z_j-z)-\Sigma_{xj,mn}(z)f_j(z)) \Big] \\
=& \sum_{i,j=1,i\neq j}^{N} E\Big[(\Sigma_{xi,mn}(z_i)K_h(z_i-z)-\Sigma_{xi,mn}(z)f_i(z)) \\
&\cdot (\Sigma_{xj,mn}(z_j)K_h(z_j,z)-\Sigma_{xj,mn}(z)f_j(z)) \Big] \\
=& \sum_{i,j=1,i\neq j}^{N} \iint (\Sigma_{xi,mn}(w_1)K_h(w_1-z)-\Sigma_{xi,mn}(z)f_i(z)) \\
&\cdot (\Sigma_{xj,mn}(w_2)K_h(w_2-z)-\Sigma_{xj,mn}(z)f_j(z))\, f_{ij}(w_1,w_2)dw_1dw_2 \\
=& \sum_{i,j=1,i\neq j}^{N} \int (\Sigma_{xi,mn}(w_1)K_h(w_1-z)-\Sigma_{xi,mn}(z)f_i(z))\, f_i(w_1)dw_1 \\
&\cdot \int (\Sigma_{xj,mn}(w_2)K_h(w_2)-\Sigma_{xj,mn}(z)f_j(z))\, f_j(w_2)dw_2 \\
&+ \sum_{i,j=1,i\neq j}^{N} \iint (\Sigma_{xi,mn}(w_1)K_h(w_1-z)-\Sigma_{xi,mn}(z)f_i(z)) \\
&\cdot (\Sigma_{xj,mn}(w_2)K_h(w_2-z)-\Sigma_{xj,mn}(z)f_j(z))\, (f_{ij}(w_1,w_2)-f_i(w_1)f_j(w_2))\, dw_1dw_2 \\
\leq& O(h^4N^2)+\frac{1}{h^2}\sum_{i,j=1,i\neq j}^{N} \iint \Big| f_{ij}(w_1,w_2)-f_i(w_1)f_j(w_2)\Big| dw_1dw_2 \\
\leq& O(h^4N^2)+O\left(\frac{\Delta_{1N}}{h^2}\right),
\end{aligned} \tag{A.6}$$

where the first inequality follows from (A.3), uniform boundedness of $\Sigma_{xi}(w_1,w_2)$ and $K(w)$; the second inequality follows from Assumption 5.

Thus, we have $A_2=O(h^4)+O\left(\frac{\Delta_{1N}}{N^2h^2}\right)$. Based on the above, the first result of this lemma follows.

2) It is easy to know that $E\left[\frac{1}{N}\sum_{i=1}^{N} x_iu_iK_{H,\Theta}(z_i,z)\right]=0$. For the second moment, write

$$\begin{aligned}
&E\left\|\frac{1}{N}\sum_{i=1}^{N} x_iu_iK_{H,\Theta}(z_i,z)\right\|^2 \\
=&\ \frac{1}{N^2}\sum_{i=1}^{N} E\left[\|x_i\|^2\sigma^2(x_i,z_i)e_i^2K_h^2(z_i-z)\right] \\
&+\frac{1}{N^2}\sum_{i,j=1,i\neq j}^{N} E\left[x_i'x_j\sigma(x_i,z_i)\sigma(x_j,z_j)K_h(z_i,-z)K_h(z_j-z)\right]E[e_ie_j] \\
\leq&\ O(1)\frac{1}{N^2h}\sum_{i=1}^{N} E\left[\|x_i\|^2\sigma^2(x_i,z_i)K_h(z_i-z)\right]E[e_i^2] \\
&+\frac{1}{N^2}\sum_{i,j=1,i\neq j}^{N} E\left[x_i'x_j\sigma(x_i,z_i)\sigma(x_j,z_j)K_h(z_i,-z)K_h(z_j-z)\right]E[e_ie_j]
\end{aligned}$$

$$:= A_1 + A_2. \tag{A.7}$$

For A_1, it is easy to show that

$$
\begin{aligned}
A_1 &= O(1)\frac{1}{N^2 h}\sum_{i=1}^{N} E\left[\|x_i\|^2\sigma^2(x_i,z_i)K_h(z_i-z)\right]E[e_i^2] \\
&\leq O(1)\frac{1}{N^2 h}\sum_{i=1}^{N} E\left[\|x_i\|^2\sigma^2(x_i,z_i)K_h(z_i-z)\right] \\
&\leq O(1)\frac{1}{N^2 h}\sum_{i=1}^{N} E\left[\|x_i\|^2 K_h(z_i-z)\right] = O\left(\frac{1}{Nh}\right).
\end{aligned}
$$

For A_2, write

$$
\begin{aligned}
&\sum_{i,j=1,i\neq j}^{N} \left| E\left[x_i' x_j \sigma(x_i,z_i)\sigma(x_j,z_j)K_h(z_i-z)K_h(z_j-z)\right]E[e_i e_j]\right| \\
=& \sum_{i,j=1,i\neq j}^{N} \left| E\left[\eta_i(z_i)'\eta_j(z_j)K_h(z_i-z)K_h(z_j-z)\right]\gamma_{i,j}\right| \\
=& \sum_{i,j=1,i\neq j}^{N} \left| \iint \eta_i(w_1)'K_h(w_1-z_c)\eta_j(w_2)K_h(w_2-z_c)f_{ij}(w_1,w_2)dw_1dw_2\gamma_{i,j}\right| \\
\leq& \sum_{i,j=1,i\neq j}^{N} \left| \int \eta_i(w_1)'K_h(w_1-z_c)f_i(w_1)dw_1 \int \eta(w_2)K_h(w_2-z_c)f_j(w_2)dw_2\cdot\gamma_{i,j}\right| \\
&+ \sum_{i,j=1,i\neq j}^{N} \Big| \iint \eta(w_1)'K_h(w_1-z)\eta(w_2)K_h(w_2-z) \\
&\cdot\Big(f_{ij}(w_1,w_2)-f_i(w_1)f_j(w_2)\Big)dw_1dw_2\cdot\gamma_{i,j}\Big| \\
\leq& O(1)\sum_{i,j=1,i\neq j}^{N}|\gamma_{i,j}| + \frac{1}{h^2}\sum_{i,j=1,i\neq j}^{N}\iint\Big|f_{ij}(w_1,w_2)-f_i(w_1)f_j(w_2)\Big|dw_1dw_2 \\
\leq& O(\Delta_{2N}) + O\left(\frac{\Delta_{1N}}{h^2}\right),
\end{aligned} \tag{A.8}
$$

where the second inequality follows from the uniform boundedness on $\eta(\cdot)$ and $f_i(w)$.

Therefore, for A_2, we obtain $A_2 = O\left(\frac{\Delta_{2N}}{N^2}\right) + O\left(\frac{\Delta_{1N}}{N^2h^2}\right)$. Based on the analysis on A_1 and A_2, the result follows.

3) We then focus on $\frac{1}{N}\sum_{i=1}^{N} x_i x_i'\left(\beta(z_i)-\beta(z)\right)K_h(z_i-z)$.

$$
\begin{aligned}
&E\left\|\frac{1}{N}\sum_{i=1}^{N} x_i x_i'\left(\beta(z_i)-\beta(z)\right)K_h(z_i-z)\right\| \\
\leq& \frac{1}{N}\sum_{i=1}^{N} E\left[\|x_i x_i'\left(\beta(z_i)-\beta(z)\right)\|K_h(z_i-z)\right] \\
\leq& O(1)\frac{1}{N}\sum_{i=1}^{N}\int K_h(w-z)f_i(w)dw = O(h^2),
\end{aligned}
$$

where the last line follows from (A.3). Then, the result follows immediately. ■

Proof of Theorem 2.1:

We now focus on the normality.

$$\begin{aligned}
&\sqrt{Nh}\left(\hat{\beta}(z)-\beta(z)\right)\\
&=\left(\frac{1}{N}\sum_{i=1}^{N}x_ix_i'K_h(z_i-z)\right)^{-1}\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_ix_i'\left(\beta(z_i)-\beta(z)\right)K_h(z_i-z)\\
&\quad+\left(\frac{1}{N}\sum_{i=1}^{N}x_ix_i'K_h(z_i-z)\right)^{-1}\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_iu_iK_h(z_i-z)\\
&:=A_1+A_2. \qquad\qquad (A.9)
\end{aligned}$$

By Lemma A.2, we just need to focus on $\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_iu_iK_h(z_i-z)$. Notice that by the proof for (2) of Lemma 2.2

$$\begin{aligned}
&\mathrm{Var}\left[\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_iu_iK_h(z_i-z)\right]\\
&=\frac{h}{N}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}E[x_{i_1}x_{i_2}'\sigma(x_{i_1},z_{i_1})\sigma(x_{i_2},z_{i_2})K_h(z_{i_1}-z)K_h(z_{i_2}-z)]\gamma_{i_1,i_2}\\
&=\frac{1}{N}\sum_{i_1=1}^{N}E\left[\Sigma_{xx\sigma}(z)\frac{1}{h}K^2\left(\frac{z_i-z}{h}\right)\right]B_{i_1}\\
&\quad+\frac{h}{N}\sum_{i_1,i_2=1,i_1\neq i_2}^{N}E[\eta(z_{i_1})K_h(z_{i_1}-z)]E[\eta(z_{i_2})'K_h(z_{i_2}-z)]\gamma_{i_1,i_2}\\
&=\frac{1}{N}\sum_{i_1=1}^{N}f_i(z)B_{i_1}\int\Sigma_{xx\sigma}(w)K^2\left(w\right)dw\\
&\quad+\frac{h}{N}\sum_{i_1,i_2=1,i_1\neq i_2}^{N}f_{i_1}(z)f_{i_2}(z)\gamma_{i_1,i_2}\int\eta(w)K(w)dw\int\eta(w)'K(w)dw\\
&\quad+O(h^2)+O\left(\frac{\Delta_{2N}}{N^2}\right)+O\left(\frac{\Delta_{1N}}{Nh^2}\right)\\
&=V_1(z)+o(1), \qquad\qquad (A.10)
\end{aligned}$$

where $\Sigma_{xx\sigma}(z)=E[x_ix_i'\sigma(x_i,z_i)|z_i=z]$; the third equality follows from the procedure similar to (A.3) and (A.8).

Further write

$$\begin{aligned}
&\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_iu_iK_h(z_i-z)=\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_i\sigma(x_i,z_i)e_iK_h(z_i-z)\\
&=\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_i\sigma(x_i,z_i)K_h(z_i-z)\sum_{j=1}^{\infty}b_{ij}\varepsilon_j=\sum_{j=1}^{N}w_{jN}\varepsilon_j+\sum_{j=N+1}^{\infty}w_{jN}\varepsilon_j, \qquad (A.11)
\end{aligned}$$

where $w_{jN}=\sqrt{\frac{h}{N}}\sum_{i=1}^{N}x_i\sigma(x_i,z_i)K_h(z_i-z)b_{ij}$. By the Cramer-Wold device, in order to derive asymptotic normality of the vector, we consider

$$\sqrt{\frac{h}{N}}\sum_{i=1}^{N} c'x_i u_i K_h(z_i - z) = \sum_{j=1}^{N} c' w_{jN}\varepsilon_j + \sum_{j=N+1}^{\infty} c' w_{jN}\varepsilon_j, \tag{A.12}$$

where $c \in \mathbb{R}^p$ is a fixed vector satisfying $\|c\| = 1$.

By (A.10), there must a sufficiently large M satisfying $E\left[\sum_{j=N+1}^{\infty} c' w_{jN}\varepsilon_j\right]^2 = o(1)$ for $N > M$. Since $c' w_{jN}\varepsilon_j$ is martingale difference, we just need to focus on verifying the next two terms

$$\sum_{j=1}^{N} E\left[c' w_{jN}\varepsilon_j\right]^2 \to 1, \tag{A.13}$$

$$\sum_{j=1}^{N} E\left[\left(c' w_{jN}\varepsilon_j\right)^2 1(|c' w_{jN}\varepsilon_j| > \epsilon)\right] \to_P 0, \text{ for } \epsilon > 0. \tag{A.14}$$

For (A.13), write

$$\sum_{j=1}^{N} E\left[c' w_{jN}\varepsilon_j\right]^2 = \sum_{j=1}^{N} (c' w_{jN})^2 - \sum_{j=N+1}^{\infty} (c' w_{jN})^2 = c' V_1(z_c, z_d) c + o(1).$$

Next let ν be as in Assumption 1. Since $\{x_i, i \geq 1\}$ and $\{z_i, i \geq 1\}$ are independent of $\{\varepsilon_j, j \geq 1\}$, we then proceed further by conditional on $\{x_i, i \geq 1\}$ and $\{z_i, i \geq 1\}$. Then, unconditionally, the results automatically hold. Conditional on $\{x_i, i \geq 1\}$ and $\{z_i, i \geq 1\}$, we have

$$\begin{aligned}
&\sum_{j=1}^{N} E\left[(c' w_{jN}\varepsilon_j)^2 1(|c' w_{jN}\varepsilon_j| > \epsilon)\right] = \sum_{j=1}^{N} (c' w_{jN})^2 E\left[\varepsilon_j^2 1(|c' w_{jN}\varepsilon_j| > \epsilon)\right] \\
&\leq \sum_{j=1}^{N} (c' w_{jN})^2 \left\{E\left[|\varepsilon_j|^{2+\nu}\right]\right\}^{\frac{2}{2+\nu}} \left\{E\left[1(|c' w_{jN}\varepsilon_j| > \epsilon)\right]\right\}^{\frac{\nu}{2+\nu}} \\
&\leq \sum_{j=1}^{N} (c' w_{jN})^2 \left\{E\left[|\varepsilon_j|^{2+\nu}\right]\right\}^{\frac{2}{2+\nu}} \left\{\frac{E[|c' w_{jN}\varepsilon_j|]}{\epsilon}\right\}^{\frac{\nu}{2+\nu}} \\
&\leq \sum_{j=1}^{N} (c' w_{jN})^2 \left\{E\left[|\varepsilon_j|^{2+\nu}\right]\right\}^{\frac{2}{2+\nu}} \left\{\frac{|c' w_{jN}|}{\epsilon}\right\}^{\frac{\nu}{2+\nu}} \left\{E|\varepsilon_j|\right\}^{\frac{\nu}{2+\nu}} \\
&= \sum_{j=1}^{N} |c' w_{jN}|^{2+\frac{\nu}{2+\nu}} \epsilon^{-\frac{\nu}{2+\nu}} \left\{E\left[|\varepsilon_j|^{2+\nu}\right]\right\}^{\frac{2}{2+\nu}} \left\{E|\varepsilon_j|\right\}^{\frac{\nu}{2+\nu}} \\
&\leq O(1)\left\{\max_{1\leq j\leq N} |c' w_{jN}|^{\frac{\nu}{2+\nu}}\right\} \sum_{j=1}^{N} |c' w_{jN}|^2 \epsilon^{-\frac{\nu}{2+\nu}}.
\end{aligned}$$

We then just need to verify that $\max_{1\leq j\leq n(N)} |c' w_{jN}|^{\frac{\nu}{2+\nu}} \to 0$. We then obtain

$$\begin{aligned}
|c' w_{jN}| &= \left|c' \sqrt{\frac{h}{N}} \sum_{i=1}^{N} x_i \sigma(x_i, z_i) K_h(z_i - z) b_{ij}\right| \\
&\leq \|c\| \sqrt{\frac{h}{N}} \sum_{i=1}^{N} \|x_i \sigma(x_i, z_i) K_h(z_i - z) b_{ij}\| \\
&\leq O(1)\sqrt{\frac{1}{Nh}} \sum_{i=1}^{N} |b_{ij}| \leq O(1) \max_{1\leq j\leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^{N} |b_{ij}| \to 0.
\end{aligned}$$

Then the proof is complete. ■

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