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Transfer Functions of Infinite-Dimensional Systems: Positive Realness and Stabilization

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Abstract We consider a general class of operator-valued irrational positive-real functions with an emphasis on their frequency-domain properties and the relation with stabilization by output feedback. Such functions arise naturally as the transfer functions of numerous infinite-dimensional control systems, including examples specified by PDEs. Our results include characterizations of positive realness in terms of imaginary axis conditions, as well as characterizations in terms of stabilizing output feedback, where both static and dynamic output feedback are considered. In particular, it is shown that stabilizability by all static output feedback operators belonging to a sector can be characterized in terms of a natural positive-real condition and, furthermore, we derive a characterization of positive realness in terms of a mixture of imaginary axis and stabilization conditions. Finally, we introduce concepts of strict and strong positive realness, prove results which relate these notions and analyze the relationship between the strong positive realness property and stabilization by feedback. The theory is illustrated by examples, some arising from controlled and observed partial differential equations.

Keywords Infinite-dimensional system · positive-real function · stabilization by feedback · system node · transfer function

Mathematics Subject Classification (2000) 47A56 · 47N70 · 93B28 · 93C05 · 93C20 · 93D15 · 93D25.

1 Introduction

The concept of a positive-real function seems to originate in Brune's 1931 paper [11] and underlies the realization theory of electrical networks [1,6,32,48,52]. The appeal of the frequency-domain notion of positive realness in circuits and networks stems from the physical insight it provides together with the compactness and elegance of the mathematical formulation. Positive realness and the associated time-domain concept of passivity are not only pivotal in the theory of circuits and networks, but also play a key role in systems and control theory (see, for example [1,9,10,13,22,25,27,28,47,53]), with the positive-real lemma (or Kalman-Yakubovich-Popov lemma) being perhaps the best known result of the field. Equally important concepts in control theory are stability and stabilization to which positive realness is closely related via absolute stability theory [10,22,25,27,47,53] and positive-real characterizations of sets of stabilizing output feedback gains [9,25]. Whilst classical absolute stability theory focuses on global asymptotic stability and L^p-stability (where, usually, $p = 2$ or $p = \infty$), more recent work [2,25,36,37] shows that the circle criterion extends to an input-to-state stability setting and positive realness continues to play a key role in this context. Furthermore, certain numerical methods (for example, linear multistep methods) can be interpreted as (discrete-time) Lur'e systems and positive-real functions can be used in the stability analysis of these methods [12,17,31].

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The purpose of this paper is to derive basic properties of irrational operator-valued positive-real functions and to study the relationship between positive realness and stabilization by output feedback. Irrational operator-valued positive-real functions arise naturally in the analysis and synthesis of infinite-dimensional control systems. Whilst transfer functions (and, more generally, frequency-domain methods) for infinite-dimensional systems have received considerable attention in the last 25 years [14,16,38,43,44,50,51, 57], we feel that frequency-domain properties of irrational positive-real functions and their relation to stabilization by output feedback have not been sufficiently studied in the literature. Topics which we do not discuss are time-domain characterizations of positive-real functions (such as Poisson integral representations or characterizations in terms of inverse Laplace transforms [52,54,55]) and state-space characterizations (the Kalman-Yakubovich-Popov lemma [3,4,5,41,42]); though we do consider sufficient conditions for positive realness in state-space terms (see Theorem 7.8).

One motivation for the present study is to provide the necessary results on irrational operator-valued positive-real functions needed for the development of satisfactory versions of the circle and Popov criteria for infinite-dimensional systems. Positive realness plays a crucial role in this development (see, for example, [22,25,27,47] for the finite-dimensional case). In order to keep the paper at a reasonable length and with a single focus, we however do not present absolute stability results here (but instead refer to [20]).

We now briefly highlight the main results obtained in this paper. We work in a fairly general stetting given by the class of all operator-valued transfer functions which are holomorphic on a half-plane $\operatorname{Re} s > \alpha$ with the exception of (possibly infinitely many) isolated points, either poles or essential singularities, where α is a real number (usually, $\alpha \leq 0$). We obtain a characterization of positive realness in terms of imaginary axis conditions and a condition at infinity in Theorem 3.7 (which extends a well-known result for rational matrix-valued functions to our general setting). Following the literature on finite-dimensional systems, we introduce concepts of strict and strong positive realness and present a result (Theorem 4.4) which relates these concepts. Furthermore, Theorem 6.3 provides a necessary and sufficient condition for positive realness in terms of a mixture of imaginary axis and stabilizability properties, and we prove that positive realness of a transfer function is equivalent to every strictly dissipative static output feedback being stabilizing (Theorem 6.4). Invoking the concept of strong positive realness, the latter results is extended to include dynamic output feedback (see Theorem 6.16). Our analysis of the relationship between positive realness and stabilization by static output feedback culminates in Theorem 6.8 which is reminiscent of the circle criterion: this result shows that, given a transfer function $\mathbf{H}$ and feedback operators K_1 and K_2, the function $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is positive real if, and only if, every operator K in the "sector" defined by K_1 and K_2 is a stabilizing feedback operator for $\mathbf{H}$.

The paper is organized as follows. Section 2 collects relevant notation, terminology and operator theory preliminaries. Sections 3 and 4, respectively, discuss positive realness and strict/strong positive realness in some detail and contain two of the main results mentioned above (namely, Theorems 3.7 and 4.4). In Section 5 we provide a careful treatment of static output feedback for irrational operator-valued transfer functions: in particular, we introduce the concepts of admissible and stabilizing feedback operators (extending concepts in [51] to our setting) and investigate their properties. Section 6 focuses on the relationship between positive-real concepts and stabilization properties: Theorems 6.4, 6.8 and 6.16, already mentioned above, are the key results in this context. In Section 7, the penultimate section, we discuss links between positive realness and state-space systems (in form of system nodes [43]): in particular, we present a sufficient condition for (strict, strong) positive realness of the transfer function of a system node S in terms of certain dissipativity properties of S (see Theorem 7.8). These properties can frequently be checked in a PDE context and this we do in a number of examples which serve to illustrate some of the main results of the paper. Section 8 contains some summarizing comments and potential future lines of enquiry. Finally, to avoid disruptions to the flow of the presentation, the proofs of some technical results are relegated to the Appendix.

2 Notation and Preliminaries

For $\alpha \in \mathbb{R}$ and $r > 0$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ and $\mathbb{A}_r := \{s \in \mathbb{C} : |s| > r\}$. Let U be a complex Hilbert space. The unit sphere in U will be denoted by $\mathbb{E}_U$, that is, $\mathbb{E}_U := \{u \in U : \|u\| = 1\}$. The Banach space of all linear bounded operators $U \to Y$, where Y is another complex Hilbert space, will be denoted by $\mathcal{L}(U,Y)$. We set $\mathcal{L}(U) := \mathcal{L}(U,U)$. Furthermore, if $K \in \mathcal{L}(Y,U)$ and $r > 0$, we set

$$\mathbb{B}(K,r) := \{L \in \mathcal{L}(Y,U) : \|L - K\| < r\}.$$

The following result is well known (as is the fact that it is not valid for real Hilbert spaces). A proof of statements (1) and (2) may be found in [29, Lemma 3.9-3 (b)] and [29, Theorem 3.10-3 (b)], respectively.

Lemma 2.1 *Let $S \in \mathcal{L}(U)$. The following statements hold.*

(1) *If $\langle Su, u\rangle = 0$ for all $u \in U$, then $S = 0$.*
(2) *If $\langle Su, u\rangle \in \mathbb{R}$ for all $u \in U$, then $S = S^*$.*

For self-adjoint operators S and T in $\mathcal{L}(U)$ we define:

$$\begin{aligned} S \succeq T \quad &\text{if } \langle (S-T)u, u\rangle \geq 0 \text{ for all } u \in \mathbb{E}_U,\\ S \succ T \quad &\text{if } \langle (S-T)u, u\rangle > 0 \text{ for all } u \in \mathbb{E}_U. \end{aligned}$$

For $S \in \mathcal{L}(U)$, we define the self-adjoint operator

$$\operatorname{Re} S := \frac{1}{2}(S + S^*) \in \mathcal{L}(U),$$

the *real part* of S. The proof of the following lemma can be found in the Appendix.

Lemma 2.2 *Let $S \in \mathcal{L}(U)$ and $0 \leq \delta \leq 1$. The following statements are equivalent.*

(1) $2\operatorname{Re} S \succeq (1-\delta^2)(1+\delta^2)^{-1}\big(I + S^*S\big)$.
(2) *$I + S$ is invertible and $\|(I-S)(I+S)^{-1}\| \leq \delta$.*

As an immediate consequence of Lemma 2.2, we obtain the following corollary.

Corollary 2.3 *Let $S \in \mathcal{L}(U)$ and let $\mathcal{B} \subset \mathcal{L}(U)$ be bounded. The following statements hold.*

(1) *$\operatorname{Re} S \succeq 0$ if, and only if, $I + S$ is invertible and $\|(I-S)(I+S)^{-1}\| \leq 1$.*
(2) *There exists $\varepsilon > 0$ such that $\operatorname{Re} T \succeq \varepsilon I$ for all $T \in \mathcal{B}$ if, and only if, $I+T$ is invertible for all $T \in \mathcal{B}$ and $\sup_{T\in\mathcal{B}} \|(I-T)(I+T)^{-1}\| < 1$.*

The mapping $S \mapsto (I-S)(I+S)^{-1}$ is often called the Cayley transform. Corollary 2.3 is well-known, but it is difficult to find a reference where a proof is given; this applies particularly to statement (2). Statement (1) is a special case of a more general result for accretive operators; see, for example, [23, Proposition C.7.2].

Recall that a linear operator $S : D(S) \to U$ with domain $D(S) \subset U$ is said to be dissipative if $\operatorname{Re}\langle Su, u\rangle \leq 0$ for all $u \in D(S)$ and strictly dissipative if there exists $\varepsilon > 0$ such $\operatorname{Re}\langle Su, u\rangle \leq -\varepsilon\|u\|^2$ for all $u \in D(S)$. Note that if $S \in \mathcal{L}(U)$, then dissipativity of S can be expressed as $-\operatorname{Re} S \succeq 0$ and S is strictly dissipative if, and only if, $-\operatorname{Re} S \succeq \varepsilon I$ for some $\varepsilon > 0$.

Lemma 2.4 *An operator $S \in \mathcal{L}(U)$ is strictly dissipative if, and only if, there exists $r > 0$ such that $\|S + rI\| < r$.*

Proof Let $S \in \mathcal{L}(U)$ and $r > 0$ and note that

$$\|(S + rI)u\|^2 = \|Su + ru\|^2 = \|Su\|^2 + 2r\langle \operatorname{Re} S u, u\rangle + r^2 \quad \forall\, u \in \mathbb{E}_U. \tag{2.1}$$

Assume that S is strictly dissipative. Then there exists $\varepsilon > 0$ such that

$$\langle \operatorname{Re} S u, u\rangle \leq -\varepsilon \quad \forall\, u \in \mathbb{E}_U.$$

Consequently, by (2.1),

$$\|S + rI\|^2 = \sup_{u\in\mathbb{E}_U} \|(S+rI)u\|^2 \leq r^2 + \|S\|^2 - 2r\varepsilon.$$

Choosing $r > \|S\|^2/(2\varepsilon)$, we see that $\|S + rI\| < r$.

Conversely, assume that there exists $r > 0$ such that $\|S+rI\| < r$. Setting $\varepsilon := r^2 - \|S+rI\|^2$, it follows from (2.1) that

$$-\varepsilon = \|S + rI\|^2 - r^2 \geq \|Su\|^2 + 2r\langle \mathrm{Re}\, Su, u\rangle \geq 2r\langle \mathrm{Re}\, Su, u\rangle \quad \forall\, u \in \mathbb{E}_U,$$

implying strict dissipativity of S. □

Lemma 2.4 says in particular that every operator in $\mathbb{B}(-rI, r)$ is strictly dissipative. The next result shows that the ball $\mathbb{B}(-rI, r)$ has a straightforward parametrization in terms of all strictly dissipative operators.

Lemma 2.5 *Let $r > 0$. An operator $S \in \mathcal{L}(U)$ satisfies $\|S+rI\| < r$ if, and only if, there exists a strictly dissipative operator $K \in \mathcal{L}(U)$ such that $S = (I - (1/2r)K)^{-1}K$.*

Proof Let $r > 0$ and set $\sigma := 1/(2r)$. If $S \in \mathbb{B}(-rI, r)$, then $\|(1/2)I + \sigma S\| < 1/2$ so that $\|\sigma S\| < 1$, and hence, $I + \sigma S$ is invertible. Defining $K := S(I + \sigma S)^{-1}$, we have that $I - \sigma K = (I + \sigma S)^{-1}$ and so $S = (I - \sigma K)^{-1}K$. It remains to show that K is strictly dissipative. A straightforward calculation yields that

$$\|(I + \sigma K)(I - \sigma K)^{-1}\| = 2\sigma\|S + 1/(2\sigma)I\| = \|S + rI\|/r < 1\,,$$

whence, by Corollary 2.3, it follows that there exists $\varepsilon > 0$ such that $\mathrm{Re}\,(-\sigma K) \succeq \varepsilon I$. We conclude that K is strictly dissipative.

Conversely, let K be strictly dissipative. Then $\mathrm{Re}\,(-\sigma K) \succeq \varepsilon I$ for some $\varepsilon > 0$ and we may use Corollary 2.3 to conclude

$$\|(I - \sigma K)^{-1}K + 1/(2\sigma)I\| = 1/(2\sigma)\|(I - \sigma K)^{-1}(I + \sigma K)\| < 1/(2\sigma)\,.$$

Now $\sigma = 1/(2r)$ and so $(I - (1/2r)K)^{-1}K \in \mathbb{B}(-rI, r)$, as required. □

Next, we introduce notation and terminology for various classes of operator-valued functions that we shall make extensive use of. For open $\Omega \subset \mathbb{C}$, the set of all holomorphic functions $\Omega \to \mathcal{L}(U, Y)$ is denoted by $\mathcal{H}(\Omega, \mathcal{L}(U, Y))$. For $\mathbf{H} \in \mathcal{H}(\Omega, \mathcal{L}(U, Y))$, we set $\mathbf{H}^*(s) := [\mathbf{H}(s)]^*$ (the adjoint of $\mathbf{H}(s)$) for all $s \in \Omega$, and, if $Y = U$, $\mathbf{H}^{-1}(s) := [\mathbf{H}(s)]^{-1}$ for all $s \in \Omega$ for which $\mathbf{H}(s)$ is invertible. Assume that the subset $\Pi \subset \Omega$ does not have any accumulation points in Ω. A function $\mathbf{H} \in \mathcal{H}(\Omega\backslash\Pi, \mathcal{L}(U, Y))$ is said to be meromorphic if all points in Π are poles of $\mathbf{H}$, that is, for every point $p \in \Pi$, the principal part of the Laurent expansion of $\mathbf{H}$ about p is a finite sum.

It is convenient to set

$$\mathcal{H}_\alpha(\mathcal{L}(U, Y)) := \mathcal{H}(\mathbb{C}_\alpha, \mathcal{L}(U, Y)).$$

Furthermore, $\mathcal{H}^*_\alpha(\mathcal{L}(U, Y))$ denotes the set of all $\mathcal{L}(U, Y)$-valued functions which are holomorphic on $\mathbb{C}_\alpha$, with the exception of isolated points, namely poles and essential singularities.[1] This means, $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U, Y))$ if, and only if, there exists a set $\Sigma_{\mathbf{H}} \subset \mathbb{C}_\alpha$ such that $\Sigma_{\mathbf{H}}$ does not have any accumulation points in $\mathbb{C}_\alpha$ (or, equivalently, $\Sigma_{\mathbf{H}} \cap K$ is finite for every compact subset $K \subset \mathbb{C}_\alpha$) and $\mathbf{H} \in \mathcal{H}(\mathbb{C}_\alpha\backslash\Sigma_{\mathbf{H}}, \mathcal{L}(U, Y))$. Every point in $\Sigma_{\mathbf{H}}$ is a pole or essential singularity of $\mathbf{H}$. Trivially, every $\mathcal{L}(U, Y)$-valued function which is meromorphic on $\mathbb{C}_\alpha$ is an element of $\mathcal{H}^*_\alpha(\mathcal{L}(U, Y))$. In particular,

$$\mathcal{H}_\alpha(\mathcal{L}(U, Y)) \subset \mathcal{H}^*_\alpha(\mathcal{L}(U, Y)).$$

Note that $\mathcal{H}_\alpha(\mathcal{L}(U, Y))$ and $\mathcal{H}^*_\alpha(\mathcal{L}(U, Y))$ are vector spaces and, if $U = Y$, these spaces form (non-commutative) algebras with identity $I := I_U$. We refer to [19, Chapter 9] for a treatment of holomorphic and meromorphic functions and isolated singularities in the vector-valued case.

Let $\mathcal{H}^\infty_\alpha(\mathcal{L}(U, Y))$ denote the space of all bounded holomorphic functions $\mathbb{C}_\alpha \to \mathcal{L}(U, Y)$. Obviously, $\mathcal{H}^\infty_\alpha(\mathcal{L}(U, Y)) \subset \mathcal{H}_\alpha(\mathcal{L}(U, Y))$ and, endowed with the norm

$$\|\mathbf{H}\|_{\mathcal{H}^\infty_\alpha} := \sup_{s \in \mathbb{C}_\alpha} \|\mathbf{H}(s)\|,$$

$\mathcal{H}^\infty_\alpha(\mathcal{L}(U, Y))$ is a Banach space. Furthermore, $\mathcal{H}^\infty_\alpha(\mathcal{L}(U))$ is a Banach algebra.

In the scalar-valued case $U = Y = \mathbb{C}$, we simply write $\mathcal{H}^*_\alpha$ and $\mathcal{H}^\infty_\alpha$ for $\mathcal{H}^*_\alpha(\mathcal{L}(\mathbb{C}))$ and $\mathcal{H}^\infty_\alpha(\mathcal{L}(\mathbb{C}))$, respectively.

[1] We do not consider removable singularities: it is understood that they have been removed by holomorphic extension.

3 Positive Realness

We start with the definition of positive-real functions with values in $\mathcal{L}(U)$.

Definition 3.1 *A function* $\mathbf{H} \in \mathcal{H}_\alpha^*(\mathcal{L}(U))$*, where* $\alpha \leq 0$*, is said to be* positive real *if*

$$\operatorname{Re}\langle \mathbf{H}(s)u, u\rangle \geq 0 \quad \forall\, u \in U,\ \forall\, s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}}. \tag{3.1}$$

Alternatively, (3.1) can be expressed in the form,

$$\mathbf{H}(s) + \mathbf{H}^*(s) \succeq 0 \quad \forall\, s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}},$$

or, equivalently,

$$\operatorname{Re}\mathbf{H}(s) \succeq 0 \quad \forall\, s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}}.$$

In Brune's paper [11], a (scalar and rational) *positive-real* function is assumed to be real on the real axis, and the term *positive* is used for the functions that satisfy (3.1). Although many physically motivated transfer functions enjoy a realness property on the real axis, we do not impose it in Definition 3.1 for the simple reason that it is not needed in the present paper. Nevertheless, we still use the terminology *positive real* since it captures that the real part of the function under consideration is positive (non-negative, to be precise).

We provide a number of examples of positive-real functions, each of which has properties which cannot occur in the rational case. These examples show that positive-real functions may have infinitely many simple poles on the imaginary axis and that absence of essential singularities or branch points on the imaginary axis is not necessary for positive realness.

Example 3.2 (a) The hyperbolic tangent function tanh, given by

$$\tanh(s) = \frac{1 - e^{-2s}}{1 + e^{-2s}},$$

is meromorphic on the whole complex plane and hence is in $\mathcal{H}_\alpha^*$ for every $\alpha \in \mathbb{R}$. The function tanh has infinitely many simple poles and infinitely many simple zeros, all of which are on the imaginary axis and are located at $(k + 1/2)\pi i$ and $k\pi i$, respectively, where $k \in \mathbb{Z}$. Since

$$\operatorname{Re}\,\tanh(s) = \frac{1 - e^{-4\operatorname{Re} s}}{|1 + e^{-2s}|^2},$$

for all $s \in \mathbb{C}$ which are not poles of tanh, it is clear that tanh is positive real.

(b) Consider $\mathbf{H}$ defined by

$$\mathbf{H}(s) := \sum_{k=0}^{\infty} \frac{C_k}{s - ik^2},$$

where (C_k) is a bounded sequence of self-adjoint positive-semidefinite operators in $\mathcal{L}(U)$, none of which is the zero operator. The above series converges in the uniform operator topology. It is not difficult to show that the function $\mathbf{H}$ is meromorphic on $\mathbb{C}$, it has simple poles at ik^2 for every $k \in \mathbb{N}_0$, and it is positive real.

(c) Define $\mathbf{H}(s) := 1 + e^{-1/s}$. Then $\mathbf{H} \in \mathcal{H}_\alpha^*$ for every $\alpha \in \mathbb{R}$ and $\mathbf{H} \in \mathcal{H}_0^\infty$. It is clear that $\mathbf{H}$ has precisely one singularity, namely an essential singularity at $s = 0$. A straightforward calculation shows that $\operatorname{Re}\mathbf{H}(s) > 0$ for all $s \in \mathbb{C}_0$, and so $\mathbf{H}$ is positive real.

(d) Let $\mathbf{H}$ be the principal branch of the function $s \mapsto s^q$ defined on the slit plane $\mathbb{C}\backslash(-\infty, 0]$, where $0 < q < 1$. Then $\mathbf{H} \in \mathcal{H}_0^*$ (but $\mathbf{H} \notin \mathcal{H}_\alpha^*$ for any $\alpha < 0$) and $\mathbf{H}$ has a branch point at 0. The function $\mathbf{H}$ maps $\mathbb{C}_0$ onto the sector $\{s \in \mathbb{C}_0 : -q\pi/2 < \arg s < q\pi/2\}$ and hence is positive real. $\Diamond$

It is clear that the sum of two positive-real functions is positive real and that the positive-real property is retained under multiplication with non-negative scalars. Below we will derive a number of important properties of positive-real functions, see Propositions 3.3, 3.4 and 3.5.

Proposition 3.3 *If a function* $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$*, where* $\alpha \le 0$*, is positive real, then* $\mathbf{H}$ *does not have any singularities in* $\mathbb{C}_0$ *(or, equivalently,* $\Sigma_{\mathbf{H}} \cap \mathbb{C}_0 = \emptyset$*), and so* $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$*.*

Proof We prove the claim by contraposition. To this end, suppose that $\Sigma_{\mathbf{H}} \cap \mathbb{C}_0 \ne \emptyset$ and let $s_0 \in \Sigma_{\mathbf{H}} \cap \mathbb{C}_0$. Then, in a sufficiently small punctured open disc $\Delta \subset \mathbb{C}_0$ centered at s_0, $\mathbf{H}$ has a convergent Laurent expansion:

$$\mathbf{H}(s) = \sum_{j=1}^{\infty} H_{-j}(s - s_0)^{-j} + \sum_{j=0}^{\infty} H_j (s - s_0)^j \quad \forall\, s \in \Delta\,,$$

where $H_j \in \mathcal{L}(U)$ for all $j \in \mathbb{Z}$. Let $u \in U$ and define

$$J_u := \{j > 0 : \langle H_{-j}u, u\rangle \ne 0\}.$$

Then there exists $v \in U$ such that $J_v \ne \emptyset$ (otherwise, by Lemma 2.1, $H_{-j} = 0$ for every $j > 1$ and s_0 would not be a singularity). Define the scalar-valued function $h \in \mathcal{H}^*_\alpha$ by $h(s) := \langle \mathbf{H}(s)v, v\rangle$ for all $s \in \mathbb{C}_\alpha$. If J_v is infinite, then h has an essential singularity at s_0 and it follows from the Casorati-Weierstraß theorem [30, Theorem 4, p. 43] that there exists $z \in \Delta$ such that

$$\langle \operatorname{Re} \mathbf{H}(z)v, v\rangle = \operatorname{Re} h(z) < 0,$$

showing that $\mathbf{H}$ is not positive real.

Assume now that J_v is finite and set $k := \max J_v$. Then h has a pole of order k at s_0 and hence, on Δ, the function h is of the form

$$h(s) = \frac{h_0 + g(s)}{(s - s_0)^k} \quad \forall\, s \in \Delta,$$

where $h_0 \ne 0$, g is holomorphic on $\Delta \cup \{s_0\}$ and $g(s_0) = 0$. For sufficiently small $r > 0$ we have

$$h(s_0 + re^{i\theta}) = r^{-k} e^{-ik\theta}\big(h_0 + g(s_0 + re^{i\theta})\big) \quad \forall\, \theta \in (-\pi, \pi].$$

Choosing $\theta_0 \in (-\pi, \pi]$ such that $\operatorname{Re}(e^{-ik\theta_0} h_0) < 0$ and using the fact that $g(s_0) = 0$, it follows, that, for all sufficiently small $r > 0$,

$$\langle \operatorname{Re} \mathbf{H}(s_0 + re^{i\theta_0})v, v\rangle = \operatorname{Re} h(s_0 + re^{i\theta_0}) < 0.$$

This shows that $\mathbf{H}$ is not positive real, completing the proof. □

We remark that, in contrast to Definition 3.1, in the literature analyticity on $\mathbb{C}_0$ is usually included in the definition of the positive-real property, see, for example, [1,10,32,52,54,55]. Proposition 3.3 shows that this is not necessary and that the positive-real property (in the sense of Definition 3.1) implies the absence of any singularities in $\mathbb{C}_0$. The next result is concerned with right-half plane "zeros" (in some sense) of the real part of a positive-real function.

Proposition 3.4 *Assume that* $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$*, where* $\alpha \le 0$*, is positive real, let* $s_0 \in \mathbb{C}_0$ *and let* $u \in U$*. The following statements hold.*

(1) *If* $\langle \operatorname{Re} \mathbf{H}(s_0)u, u\rangle = 0$*, then* $\langle \mathbf{H}(s)u, u\rangle = i\eta$ *for all* $s \in \mathbb{C}_0$*, where* $\eta := \operatorname{Im}\langle \mathbf{H}(s_0)u, u\rangle \in \mathbb{R}$*. In particular,* $\langle \operatorname{Re} \mathbf{H}(s)u, u\rangle = 0$ *for all* $s \in \mathbb{C}_0$*.*

(2) *If* $\langle \operatorname{Re} \mathbf{H}(s_0)u, u\rangle > 0$*, then* $\langle \operatorname{Re} \mathbf{H}(s)u, u\rangle > 0$ *for all* $s \in \mathbb{C}_0$*.*

(3) *If* $\operatorname{Re} \mathbf{H}(s_0) \succ 0$*, then* $\operatorname{Re} \mathbf{H}(s) \succ 0$ *for all* $s \in \mathbb{C}_0$*.*

(4) *If* $\operatorname{Re} \mathbf{H}(s_0) \ne 0$*, then* $\operatorname{Re} \mathbf{H}(s) \ne 0$ *for all* $s \in \mathbb{C}_0$*.*

(5) *If* $\operatorname{Re} \mathbf{H}(s_0) = 0$*, then* $\mathbf{H}(s) = \mathbf{H}(s_0)$ *for all* $s \in \mathbb{C}_0$*.*

Note that the condition $\operatorname{Re} \mathbf{H}(s_0) = 0$ in statement (5) means that $\mathbf{H}(s_0)$ is skew-adjoint.

Proof of Proposition 3.4 Let $s_0 \in \mathbb{C}_0$ and $u \in U$. Define a scalar-valued positive-real function h on $\mathbb{C}_0$ by setting $h(s) := \langle \mathbf{H}(s)u, u\rangle$ for all $s \in \mathbb{C}_0$. To prove statement (1), let $\Omega \subset \mathbb{C}_0$ be a neighborhood of s_0. By the positive realness of h, the set $h(\Omega)$ is not a neighborhood of $h(s_0) = i\eta \in i\mathbb{R}$, and so h must be constant as follows from the open mapping theorem. Consequently, $h(s) = h(s_0) = i\eta$ for all $s \in \mathbb{C}_0$.

Statement (2) is an immediate consequence of statement (1). The proofs of statements (3)–(5) follow from routine arguments based on Lemma 2.1 and statements (1) and (2). □

We remark that statements (1) and (2) of Proposition 3.4 are known for matrix-valued functions with rational components, see [32, Theorem 5.6] (the proof in [32] is based on the maximum modulus principle and not on the open mapping theorem).

The following result shows that the positive-real property is preserved under inversion. Not surprisingly, this fact is well-known for rational positive-real matrices, see, for example, [32, Theorem 5.8].

Proposition 3.5 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \leq 0$, be positive real and assume that $\mathbf{H}(s)$ is invertible for all $s \in \mathbb{C}_0$. Then $\mathbf{H}^{-1}$ is positive real.*

Proof Let $u \in U$ and $s \in \mathbb{C}_0$ and set $v := \mathbf{H}^{-1}(s)u$. Then

$$\langle \operatorname{Re} \mathbf{H}^{-1}(s)u, u\rangle = \operatorname{Re} \langle \mathbf{H}^{-1}(s)u, u\rangle = \operatorname{Re} \langle \mathbf{H}(s)v, v\rangle = \langle \operatorname{Re} \mathbf{H}(s)v, v\rangle \geq 0.$$

Since $u \in U$ and $s \in \mathbb{C}_0$ are arbitrary, we conclude that $\mathbf{H}^{-1}$ is positive real. □

The next result, which is an immediate consequence of Corollary 2.3 and Proposition 3.3, relates positive realness to a certain contraction property (sometimes referred to as "bounded real").

Corollary 3.6 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \leq 0$. The following statements hold.*

(1) *If $\mathbf{H}$ is positive real, then $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$, $I + \mathbf{H}(s)$ is invertible for $s \in \mathbb{C}_0$ and $\|(I - \mathbf{H})(I + \mathbf{H})^{-1}\|_{\mathcal{H}^\infty_0} \leq 1$.*

(2) *If, for every $s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}}$, the operator $I + \mathbf{H}(s)$ is invertible and*

$$\|(I - \mathbf{H}(s))(I + \mathbf{H}(s))^{-1}\| \leq 1\,,$$

then $\mathbf{H}$ is positive real.

Note that if the hypothesis of statement (2) of Corollary 3.6 holds, then $(I - \mathbf{H}(s))(I + \mathbf{H}(s))^{-1}$ extends holomorphically to $\mathbb{C}_0$ and the extension is a contraction (that is, its $\mathcal{H}^\infty_0$-norm is less than or equal to 1). The mapping $\mathbf{H} \mapsto (I - \mathbf{H})(I + \mathbf{H})^{-1}$ is often referred to in the systems and control theory literature as an external Cayley transform, or a diagonal transform, see [32, Theorem 5.13] or [42].

In Section 6, we provide a number of characterizations of positive realness in terms of stabilizing feedback operators. The remainder of the present section considers necessary and sufficient conditions for positive realness in terms of analyticity on $\mathbb{C}_0$, the behavior on the imaginary axis and conditions at ∞.

Theorem 3.7 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha < 0$. The function $\mathbf{H}$ is positive real if, and only if, the following conditions hold.*

(a) $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$.

(b) $\operatorname{Re} \mathbf{H}(i\omega) \succeq 0$ *for all $\omega \in \mathbb{R}$ such that $i\omega \notin \Sigma_{\mathbf{H}}$.*

(c) *If $i\omega_0$ is a pole of $\mathbf{H}$, where $\omega_0 \in \mathbb{R}$, then it is simple and the residue operator*

$$R := \lim_{s \to i\omega_0} (s - i\omega_0)\mathbf{H}(s)\,,$$

is self-adjoint and positive semi-definite, that is, $R = R^* \succeq 0$.

(d) *If $i\omega_0$ is an essential singularity of $\mathbf{H}$, where $\omega_0 \in \mathbb{R}$, then*

$$\liminf_{s \to i\omega_0,\, s \in \mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle \geq 0 \quad \forall\, u \in U\,.$$

(e) $\liminf_{|s| \to \infty,\, s \in \mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle \geq 0 \quad \forall\, u \in U.$

As will follow from an inspection of the proof, Theorem 3.7 remains valid if the inequality in statement (e) is replaced by the condition

$$\liminf_{|s| \to \infty,\, s \in \mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle > -\infty \quad \forall\, u \in U.$$

The proof of Theorem 3.7 is facilitated by the following proposition which we shall prove first, before we proceed to prove Theorem 3.7.

Proposition 3.8 *Let* $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$*, where* $\alpha \leq 0$*, and let* U_0 *be a dense subset of* U*. The function* $\mathbf{H}$ *is positive real if, and only if,* $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$*,*

$$\liminf_{s\to\xi,\, s\in\mathbb{C}_0} \langle \operatorname{Re}\mathbf{H}(s)u, u\rangle \geq 0 \quad \forall\, u \in U_0,\ \forall\, \xi \in i\mathbb{R}, \tag{3.2}$$

and

$$\liminf_{|s|\to\infty,\, s\in\mathbb{C}_0} \langle \operatorname{Re}\mathbf{H}(s)u, u\rangle > -\infty \quad \forall\, u \in U_0. \tag{3.3}$$

Proof Assume that $\mathbf{H}$ is positive real. It is a consequence of Proposition 3.3 that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$, whilst (3.2) and (3.3) follow immediately from the positive realness of $\mathbf{H}$.

Conversely, assume that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ and that (3.2) and (3.3) hold. Let $u \in U_0$ and define $g(s) := -\langle \operatorname{Re}\mathbf{H}(s)u, u\rangle$. Since $\mathbf{H}$ is holomorphic on $\mathbb{C}_0$, we obtain that g is harmonic and, a fortiori, subharmonic on $\mathbb{C}_0$. By (3.2) and (3.3),

$$\limsup_{s\to\xi,\, s\in\mathbb{C}_0} g(s) \leq 0 \quad \forall\, \xi \in i\mathbb{R} \quad \text{and} \quad \limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} g(s) < \infty,$$

and an application of the Phragmén-Lindelöf theorem for subharmonic functions defined in the complex plane (see [34, Theorem 2.3.2 and Corollary 2.3.3]) yields that $g(s) \leq 0$ for all $s \in \mathbb{C}_0$. Since u was an arbitrary element in U_0 and U_0 is dense in U, it follows that $\langle \operatorname{Re}\mathbf{H}(s)u, u\rangle \geq 0$ for all $s \in \mathbb{C}_0$ and all $u \in U$, showing that $\mathbf{H}$ is positive real. □

Proof of Theorem 3.7 Assume that $\mathbf{H}$ is positive real. Then, by Proposition 3.3, $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$, and so condition (a) is satisfied. Conditions (d) and (e) follow trivially from the positive realness of $\mathbf{H}$.

It remains to show that conditions (b) and (c) hold, for which we shall make use of the assumption that $\alpha < 0$. Consequently, $\mathbf{H}(s)$ is well defined for every $s \in i\mathbb{R}\backslash\Sigma_{\mathbf{H}}$. Let $\omega \in \mathbb{R}$ be such that $i\omega \notin \Sigma_{\mathbf{H}}$ and let (s_n) be a sequence in $\mathbb{C}_0$ such that $s_n \to i\omega$ as $n \to \infty$. Then $\mathbf{H}(s_n) \to \mathbf{H}(i\omega)$ (in the uniform operator topology) as $n \to \infty$ and hence $\operatorname{Re}\mathbf{H}(i\omega) \succeq 0$, showing that condition (b) holds.

We proceed to show that condition (c) is satisfied. To this end, let $\omega_0 \in \mathbb{R}$ and assume that $i\omega_0$ is a pole of $\mathbf{H}$. Then, in a sufficiently small punctured open disc $\varDelta \subset \mathbb{C}_\alpha$ centered at $i\omega_0$, $\mathbf{H}$ has a convergent Laurent expansion of the form

$$\mathbf{H}(s) = \sum_{j=1}^{k} H_{-j}(s - i\omega_0)^{-j} + \sum_{j=0}^{\infty} H_j(s - i\omega_0)^j \quad \forall\, s \in \varDelta.$$

where $k \geq 1$ and $H_{-k}, H_{-k+1}, H_{-k+2}, \ldots$ are operators in $\mathcal{L}(U)$ with $H_{-k} \neq 0$. For $u \in U$ define $h_u(s) := \langle \mathbf{H}(s)u, u\rangle$ and $\rho(u) := \langle H_{-k}u, u\rangle$. It is clear that h_u is holomorphic on $\varDelta$ and, by Lemma 2.1, there exists $v \in U$ such that $\rho(v) \neq 0$, implying that h_v has a pole of order k at $i\omega_0$. As in the proof of Proposition 3.3, it can be shown that, for all sufficiently small $r > 0$ and all $u \in U$,

$$h_u(i\omega_0 + re^{i\theta}) = r^{-k}e^{-ik\theta}\big(\rho(u) + g_u(i\omega_0 + re^{i\theta})\big) \quad \forall\, \theta \in (-\pi, \pi], \tag{3.4}$$

where g_u is holomorphic on $\varDelta \cup \{i\omega_0\}$ and $g_u(i\omega_0) = 0$.

Seeking a contradiction, assume that $k \geq 2$. Invoking (3.4) with $u = v$ and using that $\rho(v) \neq 0$, it is clear that there exists $\theta_0 \in (-\pi/2, \pi/2)$ such that $\operatorname{Re}(e^{-ik\theta_0}\rho(v)) < 0$. Since $g_v(i\omega_0) = 0$, it follows that, for all sufficiently small $r > 0$,

$$\langle \operatorname{Re}\mathbf{H}(i\omega_0 + re^{i\theta_0})v, v\rangle = \operatorname{Re} h_v(i\omega_0 + re^{i\theta_0}) < 0.$$

But $i\omega_0 + re^{i\theta_0} \in \mathbb{C}_0$, and thus, we obtain a contradiction to the positive realness of $\mathbf{H}$. Consequently $k = 1$ and the pole at $i\omega_0$ is simple. Furthermore,

$$\rho(u) = \langle H_{-1}u, u\rangle = \langle Ru, u\rangle \quad \forall\, u \in U,$$

where $R := \lim_{s\to i\omega_0}(s - i\omega_0)\mathbf{H}(s)$ is the residue operator. We now use (3.4) to obtain that, for all sufficiently small $r > 0$,

$$0 \leq r \operatorname{Re} h_u(i\omega_0 + re^{i\theta}) = \operatorname{Re}\rho(u)\cos\theta + \operatorname{Im}\rho(u)\sin\theta + O(r) \qquad \forall\, \theta \in (-\pi/2, \pi/2), \tag{3.5}$$

where the term $O(r)$ is real and $O(r) \to 0$ as $r \to 0$. Setting $\theta = 0$ and letting $r \to 0$, it follows from (3.5) that $\operatorname{Re} \rho(u) \geq 0$. Furthermore, letting $r \to 0$ and $\theta \to \pm\pi/2$ in (3.5), we see that $\operatorname{Im} \rho(u) = 0$. Since $u \in U$ is arbitrary, we conclude that

$$\langle Ru, u \rangle = \rho(u) \geq 0 \quad \forall\, u \in U. \tag{3.6}$$

It is a consequence of Lemma 2.1 and (3.6) that R is self-adjoint.

To prove the converse, assume that conditions (a)–(e) are satisfied. By Proposition 3.8 it is sufficient to show that (3.2) and (3.3) hold. Obviously (3.3) is implied by condition (e). Let $\xi \in i\mathbb{R}$. If ξ is not a pole of $\mathbf{H}$, then it follows trivially from conditions (b) and (d) that

$$\liminf_{s \to \xi,\, s \in \mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u \rangle \geq 0 \quad \forall\, u \in U. \tag{3.7}$$

It therefore only remains to show that (3.7) holds if $\xi = i\omega_0$ is a pole of $\mathbf{H}$. To this end, note that, by condition (c), the function $\mathbf{G}$ defined by $\mathbf{G}(s) := \mathbf{H}(s) - R/(s - i\omega_0)$ is holomorphic in $i\omega_0$. Since $R = R^* \succeq 0$ we have

$$\operatorname{Re} \langle \mathbf{G}(s)u, u \rangle \leq \operatorname{Re} \langle \mathbf{H}(s)u, u \rangle \quad \forall\, s \in \mathbb{C}_0,\ \forall\, u \in U,$$

implying that

$$\liminf_{s \to i\omega_0,\, s \in \mathbb{C}_0} \operatorname{Re} \langle \mathbf{G}(s)u, u \rangle \leq \liminf_{s \to i\omega_0,\, s \in \mathbb{C}_0} \operatorname{Re} \langle \mathbf{H}(s)u, u \rangle \quad \forall\, u \in U. \tag{3.8}$$

Choose a sequence $(\omega_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that $\omega_n \to \omega_0$ as $n \to \infty$ and $i\omega_n \notin \Sigma_{\mathbf{H}}$ for all $n \in \mathbb{N}$. We have

$$\operatorname{Re} \langle \mathbf{G}(i\omega_n)u, u \rangle = \operatorname{Re} \langle \mathbf{H}(i\omega_n)u, u \rangle \geq 0 \quad \forall\, n \in \mathbb{N},\ \forall\, u \in U,$$

where the inequality follows from condition (b). Using that $\mathbf{G}$ is holomorphic at $i\omega_0$, we conclude that

$$\operatorname{Re} \langle \mathbf{G}(i\omega_0)u, u \rangle \geq 0 \quad \forall\, u \in U,$$

and so

$$\liminf_{s \to i\omega_0,\, s \in \mathbb{C}_0} \operatorname{Re} \langle \mathbf{G}(s)u, u \rangle = \operatorname{Re} \langle \mathbf{G}(i\omega_0)u, u \rangle \geq 0 \quad \forall\, u \in U.$$

Combining the above equality with (3.8) gives

$$\liminf_{s \to i\omega_0,\, s \in \mathbb{C}_0} \operatorname{Re} \langle \mathbf{H}(s)u, u \rangle \geq 0 \quad \forall\, u \in U,$$

as desired. □

We remark that, in the above proof, the argument which shows that condition (c) is necessary for positive realness is well known and has been used in less general contexts, see, for example, [1, Theorem 2.7.2] or [32, Theorem 5.1]. An inspection of the proof of Theorem 3.7 shows that Theorem 3.7 remains valid if the liminf inequalities in conditions (d) and (e) are assumed to hold only for all u in a dense subset of U.

The conditions (a)–(c) of Theorem 3.7 are familiar from finite-dimensional positive realness theory, in the context of which condition (d) is irrelevant. In certain situations, condition (e) in Theorem 3.7 can be expressed in a different, perhaps more appealing way. To this end, it is useful to introduce the following terminology: a function $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ is said to be *meromorphic at* ∞ if there exists $r > 0$ such that $\mathbf{H}$ is defined on $\mathbb{A}_r$ and meromorphic on $\mathbb{A}_r \cup \{\infty\}$, that is, the function $s \mapsto \mathbf{H}(1/s)$ is meromorphic on the disc with center 0 and radius $1/r$. Note that if $\mathbf{H}$ is meromorphic at ∞, then there exists exists $r > 0$ such that $\mathbf{H}$ is holomorphic on $\mathbb{A}_r$ and, furthermore, there exist $k \in \mathbb{Z}$ and operators $H_j \in \mathcal{L}(U)$, where $j = k, k+1, k+2 \ldots$, such that

$$\mathbf{H}(s) = \sum_{j=k}^{\infty} H_j s^{-j} \quad \forall\, s \in \mathbb{A}_r. \tag{3.9}$$

The operator H_{-1} is said to be the residue of $\mathbf{H}$ at ∞. If $k = 0$, then $\mathbf{H}$ is said to be holomorphic at ∞ and we set

$$\mathbf{H}(\infty) := \lim_{|s| \to \infty} \mathbf{H}(s) = H_0\,.$$

Obvious examples of functions which are meromorphic at ∞ are the so-called rational functions, that is, $\mathcal{L}(U)$-valued functions which are meromorphic on $\mathbb{C} \cup \{\infty\}$, see [35]. It is straightforward to show that a $\mathcal{L}(U)$-valued function $\mathbf{H}$ is rational if, and only if, $\mathbf{H} = (1/p)\mathbf{P}$, where p and $\mathbf{P}$ are scalar-valued and $\mathcal{L}(U)$-valued polynomials, respectively. A class of functions which are holomorphic at ∞ are the resolvents of operators in $\mathcal{L}(U)$.

The next result contains a characterization of positive realness for transfer functions which are meromorphic at ∞.

Corollary 3.9 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha < 0$. Assume that $\mathbf{H}$ is meromorphic at ∞. Under these conditions, $\mathbf{H}$ is positive real, if and only if, conditions* (a), (b) *and* (d) *of* Theorem 3.7 *hold and, further, any pole ξ of $\mathbf{H}$ in $i\mathbb{R} \cup \{\infty\}$ is simple and the residue operator given by*

$$R_\xi := \begin{cases} \lim_{s \to \xi}(s-\xi)\mathbf{H}(s), & \text{if } \xi \in i\mathbb{R}, \\ \lim_{|s|\to\infty}(1/s)\mathbf{H}(s), & \text{if } \xi = \infty. \end{cases} \tag{3.10}$$

satisfies $R_\xi = R_\xi^* \succeq 0$.

Proof By assumption, $\mathbf{H}$ is meromorphic on $\mathbb{C}_\alpha \cup \mathbb{A}_r$ for some $r > 0$. Define $\mathbf{G}$ by $\mathbf{G}(s) = \mathbf{H}(1/s)$ and let $\beta < 0$. We note that the function $s \mapsto 1/s$ maps $\mathbb{C}_\beta$ onto the exterior of the closed disc $\{s \in \mathbb{C} : |s - 1/(2\beta)| \leq 1/(2|\beta|)\}$. By choosing $\beta := \alpha/(r^2) < 0$, it is guaranteed that any point z with $|z - 1/(2\beta)| > 1/(2|\beta|)$ and $\operatorname{Re} z \leq \alpha$ satisfies $|z| > r$, implying that, with this choice of β, the function $s \mapsto 1/s$ maps $\mathbb{C}_\beta$ into $\mathbb{C}_\alpha \cup \mathbb{A}_r$. Consequently, the function $\mathbf{G}$ is defined on (at least) $\mathbb{C}_\beta$, $\mathbf{G} \in \mathcal{H}^*_\beta(\mathcal{L}(U))$ and $\mathbf{G}$ is positive real if, and only if $\mathbf{H}$ is positive real.

Assume that $\mathbf{H}$ is positive real. From Theorem 3.7 we immediately obtain all the desired properties except for the condition at ∞. An application of Theorem 3.7 to the positive-real function $\mathbf{G}$ shows that $\mathbf{G}$ is either holomorphic at 0 or has a simple pole at 0 with residue operator S satisfying $S = S^* \succeq 0$. It follows that $\mathbf{H}$ is either holomorphic at ∞ or has a simple pole at ∞ with residue operator $R_\infty = S$, showing that $\mathbf{H}$ satisfies the desired condition at ∞.

Conversely, assume that $\mathbf{H}$ satisfies conditions (a), (b) and (d) of Theorem 3.7 and that any pole $\xi \in i\mathbb{R} \cup \{\infty\}$ of $\mathbf{H}$ is simple with self-adjoint and positive semi-definite residue operator R_ξ. By Theorem 3.7, it is sufficient to show that

$$\liminf_{|s|\to\infty,\, s \in \mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle \geq 0 \quad \forall\, u \in U. \tag{3.11}$$

Setting

$$\mathbf{H}_0(s) := \mathbf{H}(s) - sR, \quad \text{where} \quad R := \begin{cases} 0, & \text{if } \mathbf{H} \text{ is holomorphic at } \infty \\ R_\infty, & \text{if } \mathbf{H} \text{ has a simple pole at } \infty, \end{cases}$$

it is clear that $\mathbf{H}_0 \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ and $\mathbf{H}_0$ is holomorphic at ∞. There exists $\omega_0 > 0$ such that for all $\omega > \omega_0$ the function $\mathbf{H}$ (and hence $\mathbf{H}_0$) is holomorphic at $i\omega$ and

$$\operatorname{Re} \mathbf{H}_0(i\omega) = \operatorname{Re} \mathbf{H}(i\omega) \succeq 0 \quad \forall\, \omega > \omega_0.$$

Consequently,

$$\operatorname{Re} \mathbf{H}_0(\infty) = \lim_{|s|\to\infty} \operatorname{Re} \mathbf{H}_0(s) = \lim_{\omega\to\infty,\, \omega\in\mathbb{R}} \operatorname{Re} \mathbf{H}_0(i\omega) \succeq 0,$$

and thus,

$$\begin{aligned} \liminf_{|s|\to\infty,\, s\in\mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle &= \langle \operatorname{Re} \mathbf{H}_0(\infty)u, u\rangle + \liminf_{|s|\to\infty,\, s\in\mathbb{C}_0} (\operatorname{Re} s)\langle Ru, u\rangle \\ &= \langle \operatorname{Re} \mathbf{H}_0(\infty)u, u\rangle \geq 0, \end{aligned}$$

establishing (3.11) and completing the proof. □

As an immediate consequence of Corollary 3.9, we obtain the following criterion for positive realness of $\mathcal{L}(U)$-valued rational functions (which, of course, is well known, at least in the case of finite-dimensional U).

Corollary 3.10 *A $\mathcal{L}(U)$-valued rational function $\mathbf{H}$ is positive real if, and only if, conditions* (a) *and* (b) *of* Theorem 3.7 *hold and, further, any pole ξ of $\mathbf{H}$ in $i\mathbb{R} \cup \{\infty\}$ is simple and the residue operator R_ξ given by* (3.10) *satisfies $R_\xi = R_\xi^* \succeq 0$.*

Finally, we replace condition (e) in Theorem 3.7 by a certain boundedness property at ∞, to obtain a result which provides a sufficient condition for positive realness.

Corollary 3.11 *Let* $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$*, where* $\alpha < 0$*, and let* $U_0 \subset U$ *be a dense subset. If conditions* (a)–(d) *of* Theorem 3.7 *are satisfied and*

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} |\langle \mathbf{H}(s)u, u\rangle| < \infty \quad \forall\, u \in U_0, \tag{3.12}$$

holds, then the function $\mathbf{H}$ *is positive real.*

Before proving Corollary 3.11, we discuss an example which illustrates condition (3.12) and demonstrates that (3.12) does not rule out the possibility of infinitely many imaginary axis poles.

Example 3.12 Assume that U is separable and that $(u_n)_{n\in\mathbb{N}}$ is an orthonormal basis of U. Let U_0 be the set of all finite linear combinations of the u_n, which is known to be dense in U. For every $s \in \mathbb{C}\backslash i\mathbb{N}$, define $\mathbf{H}(s) \in \mathcal{L}(U)$ by

$$\mathbf{H}(s)u := \sum_{n=1}^{\infty} \frac{\langle u, u_n\rangle}{s - in} u_n \quad \forall\, u \in U.$$

Defining $R_n \in \mathcal{L}(U)$ by

$$R_n u := \langle u, u_n\rangle u_n \quad \forall\, u \in U,$$

then $R_n = R_n^* \succeq 0$ and the operator $\mathbf{H}(s)$ can be written in the form

$$\mathbf{H}(s) = \sum_{n=1}^{\infty} \frac{1}{s - in} R_n \quad \forall\, s \in \mathbb{C}\backslash i\mathbb{N},$$

where the series on the right-hand side converges in the strong operator topology. A routine argument shows that $\mathbf{H}$ is holomorphic at s for every $s \in \mathbb{C}\backslash i\mathbb{N}$. It is clear that $\mathbf{H}$ is meromorphic on $\mathbb{C}$, and so, in particular, $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ for every $\alpha < 0$. Moreover, $\Sigma_{\mathbf{H}} = i\mathbb{N}$ and $\mathbf{H}$ has a simple pole at in for every $n \in \mathbb{N}$.

Whilst it is obvious that $\mathbf{H}$ is positive real, we nevertheless show that the assumptions of Corollary 3.11 hold (implying that $\mathbf{H}$ is positive real). To this end, it is clear that the conditions (a)–(d) of Theorem 3.7 are satisfied. Further, note that, for every $u \in U_0$, there exists $k_u \in \mathbb{N}$ such that $R_n u = 0$ for all $n \geq k_u$ and so

$$\langle \mathbf{H}(s)u, u\rangle = \sum_{n=1}^{k_u} \frac{\langle R_n u, u\rangle}{s - in} = \sum_{n=1}^{k_u} \frac{|\langle u, u_n\rangle|^2}{s - in} \quad \forall\, u \in U_0.$$

Consequently,

$$\lim_{|s|\to\infty,\, s\in\mathbb{C}_0} \langle \mathbf{H}(s)u, u\rangle = 0 \quad \forall\, u \in U_0,$$

showing that (3.12) holds. Finally, we note that, for any $u \in U\backslash U_0$, the set $J_u := \{n \in \mathbb{N} : \langle u, u_n\rangle \neq 0\}$ is infinite, and therefore,

$$\langle \mathbf{H}(s)u, u\rangle = \sum_{n\in J_u} \frac{|\langle u, u_n\rangle|^2}{s - in}$$

has infinitely many poles on the imaginary axis, showing that (3.12) fails to hold for every $u \in U\backslash U_0$. $\diamondsuit$

Proof of Corollary 3.11 Condition (3.12) implies that

$$\liminf_{|s|\to\infty,\, s\in\mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle > -\infty \quad \forall\, u \in U_0.$$

Since conditions (a)–(d) of Theorem 3.7 are satisfied ($\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$, in particular), arguments very similar to those used in the proof of Theorem 3.7 show that

$$\liminf_{s\to\xi,\, s\in\mathbb{C}_0} \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle \geq 0 \quad \forall\, u \in U_0,\ \forall\, \xi \in i\mathbb{R}.$$

It follows now from Proposition 3.8 that $\mathbf{H}$ is positive real. $\square$

4 Strict and Strong Positive Realness

The present section considers two stronger notions of positive realness and the relationships between them. We start with a definition.

Definition 4.1 *Let* $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$*, where* $\alpha \leq 0$*.*

(1) *The function* $\mathbf{H}$ *is said to be* strictly positive real *if* $\alpha < 0$ *and there exists* $\varepsilon \in (0, -\alpha)$ *such that the function* $s \mapsto \mathbf{H}(s - \varepsilon)$ *is positive real.*
(2) *The function* $\mathbf{H}$ *is said to be* strongly positive real *if there exists* $\delta > 0$ *such that*

$$\operatorname{Re} \mathbf{H}(s) \geq \delta I \quad \forall\, s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}}.$$

Trivially, strictly or strongly positive-real functions are positive real. By Proposition 3.3, strongly (strictly) positive-real functions are holomorphic on $\mathbb{C}_0$ ($\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$). The sum of a positive-real and a strongly positive-real function is strongly positive real, but the sum of a positive-real and a strictly positive-real function is in general not strictly positive real. The sum of two strictly positive-real functions is strictly positive real.

In some papers, such as [45], the term *strictly positive real* is used for positive-real functions $\mathbf{H}$ that satisfy the strict inequality $\operatorname{Re} \mathbf{H}(s) \succ 0$ for all $s \in \mathbb{C}_0$ and the term *extended strictly positive real* is used for what we have termed strongly positive real.

Example 4.2 The function $s \mapsto 2 + e^{-1/s}$ is strongly, but not strictly, positive real. Furthermore, the function $s \mapsto 1/(s + 1)$ is strictly, but not strongly, positive real. Each of the functions $s \mapsto s + 1$ and $s \mapsto 2 + e^{-s}$ are both strictly and strongly positive real. $\Diamond$

The next result is reminiscent of Corollary 3.6 and links strong positive realness to a certain "strict" contraction property.

Corollary 4.3 *Let* $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$*, where* $\alpha \leq 0$*. The following statements hold.*

(1) *If* $\mathbf{H}$ *is strongly positive real and* $\mathbf{H} \in \mathcal{H}^\infty_0(\mathcal{L}(U))$*, then* $I + \mathbf{H}(s)$ *is invertible for* $s \in \mathbb{C}_0$ *and* $\|(I - \mathbf{H})(I + \mathbf{H})^{-1}\|_{\mathcal{H}^\infty_0} < 1$*.*
(2) *If* $I + \mathbf{H}(s)$ *is invertible for all* $s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}}$ *and* $\|(I - \mathbf{H})(I + \mathbf{H})^{-1}\|_{\mathcal{H}^\infty_0} < 1$*, then* $\mathbf{H}$ *is strongly positive real and* $\mathbf{H} \in \mathcal{H}^\infty_0(\mathcal{L}(U))$*.*

Proof Statement (1) is a consequence of Corollary 2.3 and so is statement (2), with the exception of the claim that $\mathbf{H} \in \mathcal{H}^\infty_0(\mathcal{L}(U))$. To prove this claim, set $\mathbf{G} := (I - \mathbf{H})(I + \mathbf{H})^{-1}$. Then, by hypothesis, $\|\mathbf{G}\|_{\mathcal{H}^\infty_0} < 1$. Consequently, $(I + \mathbf{G})^{-1} \in \mathcal{H}^\infty_0(\mathcal{L}(U))$, and thus, $\mathbf{H} = 2(I + \mathbf{G})^{-1} - I \in \mathcal{H}^\infty_0(\mathcal{L}(U))$. $\square$

For a function $\mathbf{H} \in \mathcal{H}_\alpha(\mathcal{L}(U))$, where $\alpha < 0$, we consider the following two conditions:

$$\limsup_{|s| \to \infty,\, s \in \mathbb{C}_\eta} \|\mathbf{H}(s)\| < \infty \quad \text{for some } \eta \in [\alpha, 0) \tag{4.1}$$

and

$$\liminf_{|\omega| \to \infty,\, \omega \in \mathbb{R}} \Big(\inf_{u \in \mathbb{E}_U} \langle \operatorname{Re} \mathbf{H}(i\omega) u, u \rangle \Big) > 0. \tag{4.2}$$

Several results relating strict and strong positive realness are given in the next theorem.

Theorem 4.4 *Let* $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$*, where* $\alpha < 0$*.*

(1) *If* $\mathbf{H}$ *is strictly positive real and conditions* (4.1) *and* (4.2) *hold, then there exist* $\beta \in (\alpha, 0)$ *and* $\delta > 0$ *such that* $\mathbf{H} \in \mathcal{H}^\infty_\beta(\mathcal{L}(U))$ *and*

$$\operatorname{Re} \mathbf{H}(s) \succeq \delta I \quad \forall\, s \in \mathbb{C}_\beta; \tag{4.3}$$

in particular, $\mathbf{H}$ *is strongly positive real.*

(2) *If there exist* $\beta \in (\alpha, 0)$ *and* $\delta > 0$ *such that* $\mathbf{H} \in \mathcal{H}^\infty_\beta(\mathcal{L}(U))$ *and*

$$\operatorname{Re}\mathbf{H}(i\omega) \succeq \delta I \quad \forall\, \omega \in \mathbb{R}, \tag{4.4}$$

then $\mathbf{H}$ *is strictly and strongly positive real.*
(3) *If* $\mathbf{H}$ *is strongly positive real,* $\mathbf{H} \in \mathcal{H}_\beta(\mathcal{L}(U))$ *for some* $\beta \in (\alpha, 0)$ *and* (4.1) *holds, then* $\mathbf{H}$ *is strictly positive real.*

We state an immediate corollary of Theorem 4.4.

Corollary 4.5 *Let* $\mathbf{H} \in \mathcal{H}^\infty_\alpha(\mathcal{L}(U))$ *for some* $\alpha < 0$*. Then* $\mathbf{H}$ *is strongly positive real if, and only if,* $\mathbf{H}$ *is strictly positive real and* (4.2) *holds.*

Proof of Theorem 4.4 To prove statement (1), assume that $\mathbf{H}$ is strictly positive real and conditions (4.1) and (4.2) hold. By strict positive realness there exists $\varphi \in (\alpha, 0)$ such that

$$\operatorname{Re}\mathbf{H}(s) \succeq 0 \quad \forall\, s \in \mathbb{C}_\varphi \backslash \Sigma_{\mathbf{H}}.$$

Invoking Proposition 3.3 shows that $\mathbf{H}$ is holomorphic in $\mathbb{C}_\varphi$. Appealing to (4.1) and choosing $\gamma \in \mathbb{R}$ such that $\max\{\eta, \varphi\} < \gamma < 0$, it follows that $\mathbf{H} \in \mathcal{H}^\infty_\gamma(\mathcal{L}(U))$.
Next we show that there exists $\delta > 0$ such that

$$\operatorname{Re}\mathbf{H}(i\omega) \succeq 2\delta I \quad \forall\, \omega \in \mathbb{R}\,. \tag{4.5}$$

To this end, for every $u \in \mathbb{E}_U$, define $h_u : \mathbb{C}_\gamma \to \mathbb{R}_+$ by

$$h_u(s) = \langle \operatorname{Re}\mathbf{H}(s)u, u\rangle = \operatorname{Re}\langle \mathbf{H}(s)u, u\rangle \geq 0, \quad \forall\, s \in \mathbb{C}_\gamma.$$

Moreover, we define $h : \mathbb{C}_\gamma \to \mathbb{R}_+$ by

$$h(s) = \inf_{\|u\|=1} h_u(s) \geq 0, \quad \forall\, s \in \mathbb{C}_\gamma. \tag{4.6}$$

It is not difficult to show that h is continuous (see Appendix). Let $s_0 \in \mathbb{C}_\gamma$ and $\rho > 0$ such that $\{s \in \mathbb{C} : |s - s_0| \leq \rho\} \subset \mathbb{C}_\gamma$. Then, since h_u is harmonic (being the real part of a holomorphic function), the mean-value property holds, that is,

$$h_u(s_0) = \frac{1}{2\pi}\int_0^{2\pi} h_u(s_0 + \rho e^{i\theta})d\theta,$$

see, for example, [30, Chapter 11] or [34, Chapter 1]. Now, for every $u \in \mathbb{E}_U$, $h_u(s) \geq h(s)$ for all $s \in \mathbb{C}_\gamma$ and so

$$h_u(s_0) \geq \frac{1}{2\pi}\int_0^{2\pi} h(s_0 + \rho e^{i\theta})d\theta \quad \forall\, u \in \mathbb{E}_U.$$

Consequently,

$$h(s_0) \geq \frac{1}{2\pi}\int_0^{2\pi} h(s_0 + \rho e^{i\theta})d\theta,$$

showing that $-h$ is subharmonic (see [30, Chapter 11] or [34, Chapter 2]). It follows from (4.2) that there exists $\omega \in \mathbb{R}$ such that $h(i\omega) > 0$. There are now two possibilities: h is constant or h is not constant. If h is constant, then $h(s) \equiv h(i\omega) > 0$ and (4.5) holds with $\delta = h(i\omega)/2 > 0$. Assume now that h is not constant, and let $s \in \mathbb{C}_\gamma$. Then, for sufficiently small $\rho > 0$, it follows from the maximum principle for subharmonic functions (see [30, Chapter 11] or [34, Chapter 2]) that

$$-h(s) < \sup_{|\zeta - s| = \rho}(-h(\zeta)) \leq 0.$$

Now $s \in \mathbb{C}_\gamma$ was arbitrary and so, $h(s) > 0$ for all $s \in \mathbb{C}_\gamma$. Combining this with (4.2), it follows that there exists $\delta > 0$ such that (4.5) holds.[2] Next we show that

$$\operatorname{Re}\mathbf{H}(s) \succeq 2\delta I \quad \forall\, s \in \mathbb{C}_0\,. \tag{4.7}$$

[2] In the case that U is finite dimensional, the use of subharmonic functions can be avoided by exploiting the compactness of $\mathbb{E}_U$.

Let $u \in \mathbb{E}_U$ and set $g_u(s) := 2\delta - \operatorname{Re}\langle \mathbf{H}(s)u, u\rangle$ for all $s \in \mathbb{C}_\gamma$. Then $g_u(i\omega) \le 0$ for all $\omega \in \mathbb{R}$, and, since $\mathbf{H}$ is bounded on $\mathbb{C}_0$,

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} g_u(s) < \infty.$$

An application of the Phragmén-Lindelöf Theorem for subharmonic functions defined in the complex plane (see [34, Theorem 2.3.2 and Corollary 2.3.3]) shows that $g_u(s) \le 0$ for all $s \in \mathbb{C}_0$. Consequently,

$$\langle \operatorname{Re}\mathbf{H}(s)u, u\rangle = \operatorname{Re}\langle \mathbf{H}(s)u, u\rangle \ge 2\delta \quad \forall\, s \in \mathbb{C}_0.$$

This holds for every $u \in \mathbb{E}_U$, establishing (4.7). In view of (4.5) and (4.7), it only remains to show that there exists $\beta \in (\gamma, 0)$ such that

$$\operatorname{Re}\mathbf{H}(s) \succeq \delta I \quad \text{for all } s \in \mathbb{C} \text{ such that } \beta < \operatorname{Re} s < 0. \tag{4.8}$$

Since $\mathbf{H} \in \mathcal{H}^\infty_\gamma(\mathcal{L}(U))$, $\mathbf{H}$ is uniformly continuous on every vertical strip $a \le \operatorname{Re} s \le b$, where $\gamma < a < b$. Hence there exists $\beta \in (\gamma, 0)$ such that

$$\|\mathbf{H}(r + i\omega) - \mathbf{H}(i\omega)\| \le \delta \quad \forall\, r \in [\beta, 0],\ \forall\, \omega \in \mathbb{R}$$

Consequently,

$$\langle \operatorname{Re}\mathbf{H}(r + i\omega)u, u\rangle \ge \langle \operatorname{Re}\mathbf{H}(i\omega)u, u\rangle - \delta \quad \forall\, r \in [\beta, 0],\, \forall\, \omega \in \mathbb{R},\, \forall\, u \in \mathbb{E}_U,$$

and so, (4.8) follows from (4.5), establishing (4.3).

To prove statement (2), assume that there exist $\beta \in (\alpha, 0)$ and $\delta > 0$ such that $\mathbf{H} \in \mathcal{H}^\infty_\beta(\mathcal{L}(U))$ and (4.4) holds. As in the proof of statement (1), it can be shown that there exist $\gamma \in (\beta, 0)$ and $\varepsilon \in (0, \delta)$ such that

$$\operatorname{Re}\mathbf{H}(s) \succeq \varepsilon I \quad \forall\, s \in \mathbb{C}_\gamma.$$

It now follows that $\mathbf{H}$ is strictly and strongly positive real.

Finally, to establish statement (3), assume that $\mathbf{H}$ is strongly positive real, $\mathbf{H} \in \mathcal{H}_\beta(\mathcal{L}(U))$ for some $\beta \in (\alpha, 0)$ and (4.1) holds. By the two latter assumptions, there exists $\gamma \in (\beta, 0)$ such that $\mathbf{H} \in \mathcal{H}^\infty_\gamma(\mathcal{L}(U))$. Moreover, the strong positive realness together with continuity on $\mathbb{C}_\beta$ implies that (4.4) holds. It follows now from statement (2) that $\mathbf{H}$ is strictly positive real. □

5 Admissible and Stabilizing Feedback Operators

The present section introduces the notions of admissible and stabilizing feedback operators for the class of transfer functions given by

$$\mathcal{H}^*(\mathcal{L}(U, Y)) := \bigcup_{\alpha\in\mathbb{R}} \mathcal{H}^*_\alpha(\mathcal{L}(U, Y)),$$

and provides the required ingredients for the next section which contains results relating positive realness and stabilization. To that end, we define an equivalence relation $\sim$ on $\mathcal{H}^*(\mathcal{L}(U, Y))$ by setting

$$\mathbf{G} \sim \mathbf{H} \text{ if } \mathbf{G} \text{ is a restriction of } \mathbf{H} \text{ or } \mathbf{H} \text{ is a restriction of } \mathbf{G}.$$

The corresponding equivalence classes form a vector space in a natural way and this space is denoted by $\mathcal{H}^*_\sim(\mathcal{L}(U, Y))$. If $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U, Y))$ for some $\alpha \in \mathbb{R}$, then $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U, Y))$ and we will usually identify $\mathbf{H}$ and the corresponding equivalence class $[\mathbf{H}]$ and write $\mathbf{H} \in \mathcal{H}^*_\sim(\mathcal{L}(U, Y))$. We note that in the case wherein $U = Y$, the vector space $\mathcal{H}^*_\sim(\mathcal{L}(U, Y)) = \mathcal{H}^*_\sim(\mathcal{L}(U))$ is a (non-commutative) algebra with identity $I := I_U$ and the concept of an inverse is well defined: if $\mathbf{H} \in \mathcal{H}^*_\sim(\mathcal{L}(U))$, then $\mathbf{G} \in \mathcal{H}^*_\sim(\mathcal{L}(U))$ is said to be an inverse of $\mathbf{H}$ if

$$\mathbf{GH} = \mathbf{HG} = I.$$

If an inverse $\mathbf{G}$ exists, then it is unique and we write $\mathbf{G} = \mathbf{H}^{-1}$. It is clear that a function $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ is invertible in $\mathcal{H}^*_\sim(\mathcal{L}(U))$ if, and only if, there exists $\beta \ge \alpha$ and $\mathbf{G} \in \mathcal{H}^*_\beta(\mathcal{L}(U))$ such that $\mathbf{GH} = \mathbf{HG} = I$ on $\mathbb{C}_\beta$.

Definition 5.1 *An operator* $K \in \mathcal{L}(Y,U)$ *is said to be an* admissible feedback operator *for* $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$ *if* $I - K\mathbf{H}$ *is invertible in* $\mathcal{H}^*_\sim(\mathcal{L}(U))$.

Note that if $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$, then $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U,Y))$ for some $\alpha \in \mathbb{R}$, and $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator for $\mathbf{H}$ if, and only if, there exists $\beta \geq \alpha$ such that $I - K\mathbf{H}$ is invertible in $\mathcal{H}^*_\beta(\mathcal{L}(U))$. If $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator for $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$, then we define

$$\mathbf{H}^K := \mathbf{H}(I - K\mathbf{H})^{-1} \in \mathcal{H}^*(\mathcal{L}(U,Y)). \tag{5.1}$$

The concept of an admissible feedback operator presented in Definition 5.1 is similar, but not the same, as that given in [51, Section 3] where the concept is defined in the context of so called well-posed transfer functions, that is, functions which belong to $\mathcal{H}^\infty_\alpha(\mathcal{L}(U,Y))$ for some $\alpha \in \mathbb{R}$. Proposition 5.2 extends results in [51] to the current setting.

Proposition 5.2 *Let* $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$ *and* $K \in \mathcal{L}(Y,U)$. *The following statements hold.*

(1) *K is an admissible feedback operator for $\mathbf{H}$ if, and only if, $I - \mathbf{H}K$ is invertible in $\mathcal{H}^*_\sim(\mathcal{L}(Y))$. Furthermore, if K is an admissible feedback operator for $\mathbf{H}$, then*

$$\mathbf{H}^K = (I - \mathbf{H}K)^{-1}\mathbf{H}.$$

(2) *Assume that K is an admissible feedback operator for $\mathbf{H}$. Then $L \in \mathcal{L}(Y,U)$ is an admissible feedback operator for $\mathbf{H}^K$ if, and only if, $K + L$ is an admissible feedback operator for $\mathbf{H}$, in which case,*

$$(\mathbf{H}^K)^L = \mathbf{H}^{K+L}.$$

Proof (1) Assume that K is an admissible feedback operator for $\mathbf{H}$ and set $\mathbf{G} := \mathbf{H}^K K + I \in \mathcal{H}^*_\sim(\mathcal{L}(Y))$. Then

$$\mathbf{G}(I - \mathbf{H}K) = \mathbf{H}^K K(I - \mathbf{H}K) + I - \mathbf{H}K = \mathbf{H}^K(I - K\mathbf{H})K + I - \mathbf{H}K = I,$$

and similarly, $(I - \mathbf{H}K)\mathbf{G} = I$, showing that $I - \mathbf{H}K$ is invertible in $\mathcal{H}^*_\sim(\mathcal{L}(Y))$. Furthermore, $(I - \mathbf{H}K)\mathbf{H} = \mathbf{H}(I - K\mathbf{H})$, and consequently,

$$\mathbf{H}^K = \mathbf{H}(I - K\mathbf{H})^{-1} = (I - \mathbf{H}K)^{-1}\mathbf{H}.$$

Conversely, if $I - \mathbf{H}K$ is invertible in $\mathcal{H}^*_\sim(\mathcal{L}(Y))$, then it is easy to show that $K(I - \mathbf{H}K)^{-1}\mathbf{H} + I$ is the inverse of $I - K\mathbf{H}$, implying that K is an admissible feedback operator for $\mathbf{H}$.
(2) Assume that $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator for $\mathbf{H}$ and let $L \in \mathcal{L}(Y,U)$. Noting that

$$(I - L\mathbf{H}^K)(I - K\mathbf{H}) = I - (K + L)\mathbf{H}, \tag{5.2}$$

we conclude that $I - L\mathbf{H}^K$ is invertible if, and only if, $I - (K + L)\mathbf{H}$ is invertible. Consequently, L is an admissible feedback operator for $\mathbf{H}^K$ if, and only if, $K + L$ is an admissible feedback operator for $\mathbf{H}$. Finally, if, say, $K + L$ is admissible for $\mathbf{H}$, then, by (5.2),

$$(I - K\mathbf{H})^{-1}(I - L\mathbf{H}^K)^{-1} = (I - (K + L)\mathbf{H})^{-1}.$$

Multiplying from the left by $\mathbf{H}$ shows that $(\mathbf{H}^K)^L = \mathbf{H}^{K+L}$. □

Definition 5.3 *We say that* $K \in \mathcal{L}(Y,U)$ *is a* stabilizing feedback operator *for* $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$ *if* K *is an admissible feedback operator for* $\mathbf{H}$ *and the intersection* $[\mathbf{H}^K] \cap \mathcal{H}^\infty_0(\mathcal{L}(U,Y))$ *is non-empty.*

If K is a stabilizing feedback operator for $\mathbf{H}$, then, by the identity theorem, there exists a unique function $\mathbb{C}_0 \to \mathcal{L}(U,Y)$ in the intersection $[\mathbf{H}^K] \cap \mathcal{H}^\infty_0(\mathcal{L}(U,Y))$ which will be denoted by $\mathbf{H}^K_{\mathrm{e}}$. Note that if $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$ and K is a stabilizing feedback operator for $\mathbf{H}$, then $\mathbf{H}^K = \mathbf{H}(I - K\mathbf{H})^{-1}$ is defined on $\mathbb{C}_\beta \setminus \Sigma_{\mathbf{H}}$ for some $\beta \in \mathbb{R}$. If $\beta \leq 0$, then the restriction of $\mathbf{H}^K$ to $\mathbb{C}_0 \setminus \Sigma_{\mathbf{H}}$ is bounded and extends holomorphically to $\mathbb{C}_0$. If $\beta > 0$, then $\mathbf{H}^K_{\mathrm{e}} \in \mathcal{H}^\infty_0(\mathcal{L}(U,Y))$ is a bounded holomorphic extension of $\mathbf{H}^K$ to $\mathbb{C}_0$.

Proposition 5.4 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U,Y))$ for some $\alpha \geq 0$ and let $\Delta \subset \mathbb{C}_\alpha$ be the set of points at which $\mathbf{H}$ is holomorphic. Assume that $K \in \mathcal{L}(Y,U)$ is a stabilizing feedback operator for $\mathbf{H}$. The following statements hold.*

(1) *The operator $I - K\mathbf{H}(s)$ is invertible for every $s \in \Delta$ and $\mathbf{H}^K_{\mathrm{e}}(s) = \mathbf{H}(s)(I - K\mathbf{H}(s))^{-1}$ for all $s \in \Delta$.*
(2) *Let (s_n) be a sequence in Δ such that $\lim_{n\to\infty} s_n = \alpha + i\omega$ for some $\omega \in \mathbb{R}$. If the strong limits of $\mathbf{H}(s_n)$ and $\mathbf{H}^*(s_n)$ exist as $n \to \infty$, then $\mathbf{H}^K_{\mathrm{e}}(s_n)$ has a strong limit, and, denoting the strong limits of $\mathbf{H}(s_n)$ and $\mathbf{H}^K_{\mathrm{e}}(s_n)$ by $\mathbf{H}(\alpha+i\omega)$ and $\mathbf{H}^K_{\mathrm{e}}(\alpha+i\omega)$, respectively, we have that $I - K\mathbf{H}(\alpha+i\omega)$ is invertible and $\mathbf{H}^K_{\mathrm{e}}(\alpha+i\omega) = \mathbf{H}(\alpha+i\omega)(I - K\mathbf{H}(\alpha+i\omega))^{-1}$.*

If, in the above proposition, $\alpha > 0$, then, trivially, $\mathbf{H}^K_{\mathrm{e}}(s_n)$ converges in the uniform operator topology as $n \to \infty$ (since $\mathbf{H}^K_{\mathrm{e}}$ is holomorphic on $\mathbb{C}_0$). Statement (1) says that if K is a stabilizing feedback operator for $\mathbf{H}$, then $I - K\mathbf{H}(s)$ is invertible for every $s \in \mathbb{C}_\alpha$ for which $\mathbf{H}(s)$ "makes sense".

The following simple lemma will facilitate the proof of statement (2) of Proposition 5.4. For an invertible operator $S \in \mathcal{L}(U)$, we will use the notation S^{-*} to denote the inverse of S^*.

Lemma 5.5 *Let (S_n) be a sequence of invertible operators in $\mathcal{L}(U)$. Assume that there exists $S \in \mathcal{L}(U)$ such that S and S^* are the strong limits of S_n and S_n^*, respectively. If $\sup_{n\in\mathbb{N}} \|S_n^{-1}\| < \infty$, then S is invertible and S_n^{-1} and S_n^{-*} converge strongly to S^{-1} and S^{-*}, respectively.*

A proof of Lemma 5.5 can be found in the Appendix.

Proof of Proposition 5.4 (1) Note that $(I - K\mathbf{H})^{-1} = K\mathbf{H}^K + I$ in $\mathcal{H}^*_\sim(\mathcal{L}(U,Y))$, that is, there exists $\beta \geq \alpha$ such that

$$(I - K\mathbf{H}(s))^{-1} = K\mathbf{H}^K_{\mathrm{e}}(s) + I \quad \forall\, s \in \mathbb{C}_\beta \backslash \Sigma_{\mathbf{H}},$$

and thus

$$(I - K\mathbf{H}(s))(K\mathbf{H}^K_{\mathrm{e}}(s) + I) = (K\mathbf{H}^K_{\mathrm{e}}(s) + I)(I - K\mathbf{H}(s)) = I \quad \forall\, s \in \mathbb{C}_\beta \backslash \Sigma_{\mathbf{H}}.$$

The identity theorem implies that the above equation holds for all $s \in \mathbb{C}_\alpha \backslash \Sigma_{\mathbf{H}} = \Delta$, showing that, for all $s \in \Delta$, $I - K\mathbf{H}(s)$ is invertible and $\mathbf{H}^K_{\mathrm{e}}(s) = \mathbf{H}(s)(I - K\mathbf{H}(s))^{-1}$.

(2) By statement (1), $I - K\mathbf{H}(s_n)$ is invertible for all $n \in \mathbb{N}$. Since $(I - K\mathbf{H})^{-1} = I + K\mathbf{H}^K$, it follows that

$$\|\big(I - K\mathbf{H}(s_n)\big)^{-1}\| \leq \rho \quad \forall\, n \in \mathbb{N},$$

where $\rho := 1 + \|K\|\|\mathbf{H}^K_{\mathrm{e}}\|_{\mathcal{H}^\infty_0}$. Furthermore, by hypothesis, $I - K\mathbf{H}(s_n)$ and $I - \mathbf{H}^*(s_n)K^*$ have strong limits $I - K\mathbf{H}(\alpha + i\omega)$ and $I - \mathbf{H}^*(\alpha + i\omega)K^*$ as $n \to \infty$. An application of Lemma 5.5 (with $S_n = I - K\mathbf{H}(s_n)$) shows that $I - K\mathbf{H}(\alpha + i\omega)$ is invertible and

$$\mathbf{H}(s_n)\big(I - K\mathbf{H}(s_n)\big)^{-1} \to \mathbf{H}(\alpha + i\omega)\big(I - K\mathbf{H}(\alpha + i\omega)\big)^{-1} \quad \text{strongly,} \quad \text{as } n \to \infty.$$

Finally, since, by statement (1), $\mathbf{H}^K_{\mathrm{e}}(s_n) = \mathbf{H}(s_n)(I - K\mathbf{H}(s_n))^{-1}$ for all $n \in \mathbb{N}$, we see that $\mathbf{H}(\alpha + i\omega)(I - K\mathbf{H}(\alpha + i\omega))^{-1}$ is the strong limit of $\mathbf{H}^K_{\mathrm{e}}(s_n)$ as $n \to \infty$. □

For every $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$, we define

$$\mathbb{S}(\mathbf{H}) := \{K \in \mathcal{L}(Y,U) : \, K \text{ is a stabilizing feedback operator for } \mathbf{H}\}.$$

The following result will be used extensively in Section 6.

Proposition 5.6 *Let $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$, $K \in \mathbb{S}(\mathbf{H})$ and $r > 0$. Then, $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{H})$ if, and only if, $\|\mathbf{H}^K_{\mathrm{e}}\|_{\mathcal{H}^\infty_0} \leq 1/r$.*

It is an immediate consequence of Proposition 5.6 that $\mathbb{S}(\mathbf{H})$ is an open subset of $\mathcal{L}(Y,U)$.

Proof of Proposition 5.6 Assume that $\|\mathbf{H}^K_{\mathrm{e}}\|_{\mathcal{H}^\infty_0} \leq 1/r$. Let $L \in \mathbb{B}(K,r)$. Then L is of the form $L = K + D$ with $D \in \mathcal{L}(Y,U)$ such that $\|D\| < r$. Consequently, $\|\mathbf{H}^K_{\mathrm{e}}\|_{\mathcal{H}^\infty_0}\|D\| < 1$, and so $D \in \mathbb{S}(\mathbf{H}^K)$. Proposition 5.2 guarantees that $L = K + D$ is an admissible feedback operator for $\mathbf{H}$ and, furthermore,

$$\mathbf{H}^L = \mathbf{H}^{K+D} = (\mathbf{H}^K)^D \quad \text{in } \mathcal{H}^*_\sim(\mathcal{L}(U,Y)).$$

Since $[(\mathbf{H}^K)^D] \cap \mathcal{H}_0^\infty(\mathcal{L}(U,Y)) \neq \emptyset$, it follows that $L \in \mathbb{S}(\mathbf{H})$ and hence $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{H})$.

Conversely, assume that $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{H})$. Then, making use of Proposition 5.2, $\mathbb{B}(0,r) \subset \mathbb{S}(\mathbf{H}^K)$ and

$$(I - L\mathbf{H}_{\mathrm{e}}^K)^{-1} = I + L(\mathbf{H}_{\mathrm{e}}^K)^L \in \mathcal{H}_0^\infty(\mathcal{L}(U)) \quad \forall\, L \in \mathbb{B}(0,r). \tag{5.3}$$

Seeking a contradiction, suppose that $\|\mathbf{H}_{\mathrm{e}}^K\|_{\mathcal{H}_0^\infty} > 1/r$. Then there exists $z \in \mathbb{C}_0$ such that $\|\mathbf{H}_{\mathrm{e}}^K(z)\| > 1/r$. It is sufficient to show that there exist operators $L_n \in \mathcal{L}(Y,U)$ such that $I - L_n\mathbf{H}_{\mathrm{e}}^K(z)$ is not invertible and $\|L_n\| \to 1/\|\mathbf{H}_{\mathrm{e}}^K(z)\|$ as $n \to \infty$. Indeed, if this is the case, then, for all sufficiently large n, $L_n \in \mathbb{B}(0,r)$, but $(I - L_n\mathbf{H}_{\mathrm{e}}^K)^{-1} \notin \mathcal{H}_0^\infty(\mathcal{L}(U))$, contradicting (5.3). We proceed to construct a sequence (L_n) of operators with the required properties. To this end, set $M := \mathbf{H}_{\mathrm{e}}^K(z)$ and choose $v_n \in U$ such that $\|v_n\| = 1$ and $\|Mv_n\| \to \|M\|$ as $n \to \infty$. Setting

$$w_n := \frac{1}{\|Mv_n\|} Mv_n \in Y$$

and defining $L_n : Y \to U$ by

$$L_n y := \frac{\langle y, w_n\rangle}{\|Mv_n\|} v_n \quad \forall\, y \in Y,$$

we have $\|L_n\| = 1/\|Mv_n\| \to 1/\|M\|$ as $n \to \infty$ and, furthermore, $(I - L_nM)v_n = 0$, showing that $I - L_nM$ is not invertible and completing the proof. $\square$

The next theorem shows that, if $\mathbf{H}$ has essential singularities in $\mathbb{C}_0$, then there does not exist a stabilizing compact feedback operator for $\mathbf{H}$.

Theorem 5.7 *Let $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U,Y))$ and assume that $\mathbb{S}(\mathbf{H})$ contains a compact operator. Then there exists a meromorphic function $\mathbf{H}_{\mathrm{e}} : \mathbb{C}_0 \to \mathcal{L}(U,Y)$ such that $\mathbf{H} = \mathbf{H}_{\mathrm{e}}$ in $\mathcal{H}_\sim^*(\mathcal{L}(U,Y))$.*

The following lemma is a key tool for the proof of Theorem 5.7.

Lemma 5.8 *Let $\Omega \subset \mathbb{C}$ be open and connected and let $\mathbf{F} \in \mathcal{H}(\Omega, \mathcal{L}(U))$ be such that $\mathbf{F}(s)$ is compact for every $s \in \Omega$. Assume that there exists $z \in \Omega$ such that $I - \mathbf{F}(z)$ is invertible. Then the set*

$$\Delta := \{s \in \Omega :\ I - \mathbf{F}(s) \text{ is not invertible}\}$$

does not have any accumulation points in Ω and, if Δ is non-empty, every $s \in \Delta$ is a pole of $(I - \mathbf{F})^{-1}$. In particular, $(I - \mathbf{F})^{-1}$ is meromorphic on Ω.

A proof of the lemma is given in the Appendix.

Proof of Theorem 5.7 By hypothesis there exists a compact operator $K \in \mathbb{S}(\mathbf{H})$. It follows from Proposition 5.2 that $-K$ is an admissible feedback operator for $\mathbf{H}^K$ and hence $I + K\mathbf{H}^K$ is invertible in $\mathcal{H}_\sim^*(\mathcal{L}(U))$. In particular, there exists $\alpha \geq 0$ such that $I + K\mathbf{H}^K(s)$ is invertible for all $s \in \mathbb{C}_\alpha$. Furthermore,

$$\mathbf{H}^K(I + K\mathbf{H}^K)^{-1} = (\mathbf{H}^K)^{-K} = \mathbf{H} \quad \text{ in } \mathcal{H}_\sim^*(\mathcal{L}(U,Y)). \tag{5.4}$$

An application of Lemma 5.8 with $\Omega = \mathbb{C}_0$ and $\mathbf{F} = -K\mathbf{H}_{\mathrm{e}}^K$ shows that $(I + K\mathbf{H}_{\mathrm{e}}^K)^{-1}$ is meromorphic on $\mathbb{C}_0$. Consequently, $\mathbf{H}_{\mathrm{e}}^K(I + K\mathbf{H}_{\mathrm{e}}^K)^{-1}$ is meromorphic on $\mathbb{C}_0$ and it now follows from (5.4) that the claim holds with $\mathbf{H}_{\mathrm{e}} = \mathbf{H}_{\mathrm{e}}^K(I + K\mathbf{H}_{\mathrm{e}}^K)^{-1}$. $\square$

The next example shows that the compactness assumption in Theorem 5.7 is essential: a function $\mathbf{H} \in \mathcal{H}_0^*(\mathcal{L}(U))$ is constructed such that $\mathbf{H}$ is holomorphic in $\mathbb{C}_0\backslash\{1\}$, has an essential singularity at $s = 1$ and $\mathbb{S}(\mathbf{H})$ is non-empty.

Example 5.9 Let $N \in \mathcal{L}(U)$ be quasi-nilpotent, but not nilpotent, that is,

$$\text{spectral radius of } N = \lim_{q\to\infty} \|N^q\|^{1/q} = 0$$

and $N^q \neq 0$ for every $q \in \mathbb{N}$ (of course, the existence of such an operator N requires U to be infinite dimensional). The function $\mathbf{G} : s \mapsto sI - N$ is holomorphic on $\mathbb{C}$ and $\mathbf{G}(s)$ is invertible for every non-zero

$s \in \mathbb{C}$. Consequently, $\mathbf{G}^{-1}$ is holomorphic on $\mathbb{C}\backslash\{0\}$ and 0 is an isolated singularity of $\mathbf{G}^{-1}$. The Laurent expansion of $\mathbf{G}^{-1}$ about $s = 0$ is given by

$$\mathbf{G}^{-1}(s) = \sum_{q=1}^{\infty} s^{-q} N^{q-1} \quad \forall\, s \in \mathbb{C}\backslash\{0\},$$

showing that 0 is an essential singularity of $\mathbf{G}^{-1}$.

Defining $\mathbf{H}$ by $\mathbf{H}(s) := \mathbf{G}^{-1}(s-1)$, it is clear that $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ for every $\alpha \in \mathbb{R}$ and $\mathbf{H}$ has an essential singularity at $s = 1$. Setting $K := -2I \in \mathcal{L}(U)$, it is obvious that K is not compact and we have that

$$I - K\mathbf{H}(s) = I + 2\big((s-1)I - N\big)^{-1} = \big((s+1)I - N\big)\big((s-1)I - N\big)^{-1} \quad \forall\, s \in \mathbb{C}_1.$$

We conclude that K is an admissible feedback operator for $\mathbf{H}$, and, furthermore,

$$(I - K\mathbf{H}(s))^{-1} = \big((s-1)I - N\big)\big((s+1)I - N\big)^{-1} \quad \forall\, s \in \mathbb{C}_1.$$

Thus,

$$\mathbf{H}(I - K\mathbf{H}(s))^{-1} = \big((s+1)I - N\big)^{-1} = \big(sI - (N - I)\big)^{-1} \quad \forall\, s \in \mathbb{C}_1.$$

The function $s \mapsto \big(sI - (N-I)\big)^{-1}$ is in $\mathcal{H}^\infty_\alpha(\mathcal{L}(U))$ for every $\alpha > -1$. Noting that $\mathbf{H}^K_{\mathrm{e}}(s) = \big(sI - (N - I)\big)^{-1}$ for all $s \in \mathbb{C}_0$, it follows that K is a stabilizing feedback for $\mathbf{H}$. ◊

6 Positive-real functions and stabilization by feedback

In the present section we analyze connections between positive realness and stabilization by feedback by invoking material from Sections 3, 4 and 5. We derive two characterizations of positive realness, Theorems 6.3 and 6.4, in terms of dissipative stabilizing feedback operators, from which we obtain a number of corollaries.

First, we demonstrate that, under mild assumptions, positive realness is preserved under dissipative feedback.

Proposition 6.1 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \leq 0$, be positive real and let $K \in \mathcal{L}(U)$ be dissipative. Assume that*

$$I - K\mathbf{H}(s) \text{ is invertible for all } s \in \mathbb{C}_0. \tag{6.1}$$

Then $\mathbf{H}^K$ is positive real.

The trivial example wherein

$$\mathbf{H}(s) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

shows that positive realness and dissipativity of $\mathbf{H}$ and K, respectively, do not guarantee that (6.1) is satisfied.

Before we prove the above proposition, we state a lemma which shows that (6.1) holds in a number of important situations.

Lemma 6.2 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \leq 0$, be positive real and let $K \in \mathcal{L}(U)$. The following statements hold.*

(1) *If K is strictly dissipative, then* (6.1) *holds.*
(2) *If K is dissipative and $K = K^*$ (meaning that K is negative semi-definite), then* (6.1) *holds.*
(3) *If K is dissipative and an admissible feedback operator for $\mathbf{H}$ and $\dim U < \infty$, then* (6.1) *holds.*

The proof of the above lemma can be found in the Appendix.

Proof of Proposition 6.1 We start by noting that

$$\langle(\mathbf{H}(s)+\mathbf{H}^*(s))u,u\rangle - \langle(K+K^*)\mathbf{H}(s)u,\mathbf{H}(s)u\rangle \geq 0 \quad \forall\, s\in\mathbb{C}_0,\ \forall\, u\in U.$$

Rearrangement of the left-hand side leads to

$$\langle(I-\mathbf{H}^*(s)K^*)\mathbf{H}(s)u+\mathbf{H}^*(s)(I-K\mathbf{H}(s))u,u\rangle \geq 0 \quad \forall\, s\in\mathbb{C}_0,\ \forall\, u\in U.$$

Thus, for arbitrary invertible $T\in\mathcal{L}(U)$, we have that, for all $s\in\mathbb{C}_0$ and all $u\in U$,

$$\langle T(I-\mathbf{H}^*(s)K^*)\mathbf{H}(s)T^*u+T\mathbf{H}^*(s)(I-K\mathbf{H}(s))T^*u,u\rangle \geq 0. \tag{6.2}$$

By (6.1), $(I-\mathbf{H}^*(s)K^*)^{-1}$ is well defined for every $s\in\mathbb{C}_0$ and it follows from (6.2) with $T=(I-\mathbf{H}^*(s)K^*)^{-1}$ that

$$\langle\mathbf{H}(s)(I-K\mathbf{H}(s))^{-1}u+(I-\mathbf{H}^*(s)K^*)^{-1}\mathbf{H}^*(s)u,u\rangle \geq 0 \quad \forall\, s\in\mathbb{C}_0,\ \forall\, u\in U,$$

showing that $\mathbf{H}^K=\mathbf{H}(I-K\mathbf{H})^{-1}$ is positive real. □

Theorem 6.3 *Let $\mathbf{H}\in\mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha\leq 0$. Assume that, for almost every $\omega\in\mathbb{R}$, $\mathbf{H}$ and $\mathbf{H}^*$ have strong non-tangential limits at $i\omega$, denoted by $\mathbf{H}(i\omega)$ and $\mathbf{H}^*(i\omega)$, respectively. The following statements are equivalent.*

(1) $\mathbf{H}$ *is positive real.*
(2) $\operatorname{Re}\mathbf{H}(i\omega)\succeq 0$ *for almost every $\omega\in\mathbb{R}$ and $-I\in\mathbb{S}(\mathbf{H})$.*

Note that if $\alpha<0$, then $\mathbf{H}$ is holomorphic at every point in $i\mathbb{R}\backslash\Sigma_{\mathbf{H}}$ and the assumption on the existence of strong non-tangential limits is trivially satisfied.

For scalar-valued rational functions the above result can be found in [48,49]. In some publications, the conditions in statement (2) are used as defining properties for positive realness of scalar-valued rational functions, see [9,40]. For matrix-valued rational functions, the implication (1) ⇒ (2) appears in [32, Theorem 5.10], albeit with slightly different terminology.

Proof of Theorem 6.3 Assume that statement (1) holds, that is, $\mathbf{H}$ is positive real. Invoking the assumption on the existence of strong non-tangential limits, we obtain that $\operatorname{Re}\mathbf{H}(i\omega)\succeq 0$ for almost every $\omega\in\mathbb{R}$. Moreover, by statement (1) of Corollary 3.6, $(I-\mathbf{H})(I+\mathbf{H})^{-1}\in\mathcal{H}^\infty_0(\mathcal{L}(U))$ and so,

$$2\mathbf{H}(I+\mathbf{H})^{-1}=I-(I-\mathbf{H})(I+\mathbf{H})^{-1}\in\mathcal{H}^\infty_0(\mathcal{L}(U)),$$

showing that $-I$ is a stabilizing feedback operator for $\mathbf{H}$.

Conversely, assume that statement (2) holds. Then $\mathbf{H}_{\mathrm{e}}^{-I}\in\mathcal{H}^\infty_0(\mathcal{L}(U))$, and, by Proposition 5.4, $I+\mathbf{H}(s)$ is invertible for all $s\in\mathbb{C}_0\backslash\Sigma_{\mathbf{H}}$. Furthermore,

$$I-2\mathbf{H}_{\mathrm{e}}^{-I}(s)=I-2\mathbf{H}(s)\big(I+\mathbf{H}(s)\big)^{-1}=\big(I-\mathbf{H}(s)\big)\big(I+\mathbf{H}(s)\big)^{-1} \quad \forall\, s\in\mathbb{C}_0\backslash\Sigma_{\mathbf{H}}.$$

Appealing to statement (1) of Corollary 2.3, it suffices to show that $\|I-2\mathbf{H}_{\mathrm{e}}^{-I}\|_{\mathcal{H}^\infty_0}\leq 1$, or, equivalently,

$$\operatorname{ess\,sup}\{\|I-2\mathbf{H}_{\mathrm{e}}^{-I}(i\omega)\|:\omega\in\mathbb{R}\}\leq 1. \tag{6.3}$$

Invoking the hypothesis on the existence of strong non-tangential limits, it follows from Proposition 5.4 that $I+\mathbf{H}(i\omega)$ is invertible for almost every $\omega\in\mathbb{R}$ and

$$\big(I-\mathbf{H}(i\omega)\big)\big(I+\mathbf{H}(i\omega)\big)^{-1}=I-2\mathbf{H}_{\mathrm{e}}^{-I}(i\omega) \quad \text{for a.e. } \omega\in\mathbb{R}. \tag{6.4}$$

By hypothesis, $\operatorname{Re}\mathbf{H}(i\omega)\succeq 0$ for almost every $\omega\in\mathbb{R}$ and so, another application of statement (1) of Corollary 2.3 yields

$$\|(I-\mathbf{H}(i\omega))(I+\mathbf{H}(i\omega))^{-1}\|\leq 1 \quad \text{for a.e. } \omega\in\mathbb{R}. \tag{6.5}$$

The contraction condition (6.3) follows now from (6.4) and (6.5), completing the proof. □

Theorem 6.4 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \le 0$. The following statements are equivalent.*

(1) $\mathbf{H}$ *is positive real.*
(2) *Every strictly dissipative $K \in \mathcal{L}(U)$ is a stabilizing feedback operator for $\mathbf{H}$.*

Proof Assume that statement (1) holds, that is, $\mathbf{H}$ is positive real. Then, $I + 2r\mathbf{H}$ is positive real for every $r > 0$, and hence, by statement (1) of Corollary 3.6,

$$\|\mathbf{H}(I + r\mathbf{H})^{-1}\|_{\mathcal{H}^\infty_0} = \big(1/r\big)\big\|\big[I - (I + 2r\mathbf{H}(s))\big]\big[I + (I + 2r\mathbf{H}(s))\big]^{-1}\big\|_{\mathcal{H}^\infty_0} \le 1/r \;\forall\, r > 0.$$

Consequently, appealing to Proposition 5.6, every $K \in \mathcal{L}(U)$ for which there exists $r > 0$ such that $\|K + rI\| < r$ is a stabilizing feedback operator for $\mathbf{H}$. It now follows from Lemma 2.4 that every strictly dissipative $K \in \mathcal{L}(U)$ is a stabilizing feedback operator for $\mathbf{H}$.

Conversely, assume that statement (2) holds. Let $r > 0$ be arbitrary. By Lemma 2.4, every $L \in \mathcal{L}(U)$ satisfying $\|L + rI\| < r$ is strictly dissipative and hence is a stabilizing feedback operator for $\mathbf{H}$, that is, $\mathbb{B}(-rI, r) \subset \mathbb{S}(\mathbf{H})$. Setting $K := -rI$, it follows from Proposition 5.6 that $\|\mathbf{H}^K_{\mathrm{e}}\|_{\mathcal{H}^\infty_0} \le 1/r$. Invoking Proposition 5.4, we see that, for every $s \in \mathbb{C}_0\backslash\Sigma_{\mathbf{H}}$, the operator $I + r\mathbf{H}(s)$ is invertible and $\mathbf{H}^K_{\mathrm{e}}(s) = \mathbf{H}(s)(I + r\mathbf{H}(s))^{-1}$. Thus,

$$\|r\mathbf{H}(s)(I + r\mathbf{H}(s))^{-1}\| \le 1 \quad \forall\, s \in \mathbb{C}_0\backslash\Sigma_{\mathbf{H}}. \tag{6.6}$$

Now

$$-r\mathbf{H}(s)(I + r\mathbf{H}(s))^{-1} = \big[I - (I + 2r\mathbf{H}(s))\big]\big[I + (I + 2r\mathbf{H}(s))\big]^{-1} \quad \forall\, s \in \mathbb{C}_0\backslash\Sigma_{\mathbf{H}},$$

and so it follows from statement (2) of Corollary 3.6 and (6.6) that $I + 2r\mathbf{H}$ is positive real. Since $r > 0$ is arbitrary, we see that $I + 2r\mathbf{H}$ is positive real for every $r > 0$, which implies the positive realness of $\mathbf{H}$. □

We next present two corollaries of Theorem 6.4 which identify sets of stabilizing feedback operators for functions $\mathbf{H}$ which have the property that there exists $L \in \mathcal{L}(U)$ such that $-L$ is dissipative and $\mathbf{H} + L$ is positive real.

Corollary 6.5 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \le 0$, and let $L \in \mathcal{L}(U)$ with $-L$ dissipative. The following statements are equivalent.*

(1) $\mathbf{H} + L$ *is positive real.*
(2) *Every strictly dissipative $K \in \mathcal{L}(U)$ is a stabilizing feedback operator for $\mathbf{H} + L$.*
(3) *For every strictly dissipative $K \in \mathcal{L}(U)$, $(I - KL)^{-1}K$ is a stabilizing feedback operator for $\mathbf{H}$.*

The proof of Corollary 6.5 uses the following simple lemma which relates stabilizing feedback operators of $\mathbf{H}$ to those of $\mathbf{H} + L$.

Lemma 6.6 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \le 0$ and let $K, L \in \mathcal{L}(U)$ with $I - KL$ invertible. The feedback operator K is stabilizing for $\mathbf{H} + L$ if, and only if, $(I - KL)^{-1}K$ is a stabilizing feedback operator for $\mathbf{H}$.*

Proof Setting $M := (I - KL)^{-1}K$ and noting that

$$I - M\mathbf{H} = (I - KL)^{-1}(I - K(\mathbf{H} + L)), \tag{6.7}$$

we see that K is admissible for $\mathbf{H} + L$ if, and only if, M is admissible for $\mathbf{H}$.

Assume that M is a stabilizing feedback operator for $\mathbf{H}$. In particular, M is admissible for $\mathbf{H}$, and so K is admissible for $\mathbf{H} + L$. Appealing to (6.7), we obtain that

$$(\mathbf{H} + L)(I - M\mathbf{H})^{-1} = (\mathbf{H} + L)(I - K(\mathbf{H} + L))^{-1}(I - KL),$$

which shows that K is stabilizing for $\mathbf{H} + L$.

The converse follows for symmetry reasons. Indeed, assume that K is a stabilizing feedback operator for $\mathbf{H} + L$. Noting that $I + ML = (I - KL)^{-1}$, it follows that $(I + ML)^{-1}M = K$. Consequently, $(I - M(-L))^{-1}M$ is stabilizing for $\mathbf{H} + L$, and by what has already been proved, we conclude that M is a stabilizing feedback operator for $(\mathbf{H} + L) + (-L) = \mathbf{H}$. □

Proof of Corollary 6.5 The equivalence of statements (1) and (2) follows from Theorem 6.4. Using that, by hypothesis, L is a (constant) positive-real function, Lemma 6.2 guarantees that $I - KL$ is invertible for all strictly dissipative $K \in \mathcal{L}(U)$. The equivalence of (2) and (3) now follows from Lemma 6.6. ☐

In the special case wherein $L = cI$, for positive scalar c, we obtain another corollary, which is an immediate consequence of Lemma 2.5 and Corollary 6.5.

Corollary 6.7 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \leq 0$, let $c > 0$ and define $r := 1/(2c)$. The function $\mathbf{H} + cI$ is positive real if, and only if, $\mathbb{B}(-rI, r) \subseteq \mathbb{S}(\mathbf{H})$.*

The above corollary extends and generalizes [15, Proposition 2.1] where, in the context of well-posed linear systems, it is shown that positive realness of $\mathbf{H} + cI$ implies that, for every $k \in (-2r, 0)$, the "scalar" feedback operator $K = kI$ is stabilizing for $\mathbf{H}$.

Next we will show that, given a transfer function $\mathbf{H}$ and feedback operators $K_1, K_2 \in \mathcal{L}(Y, U)$ with K_1 admissible for $\mathbf{H}$, positive realness of $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is necessary and sufficient for all linear feedback operators in the "sector" $\sec(K_1, K_2)$ to be stabilizing, where $\sec(K_1, K_2)$ is the set of all operators $K \in \mathcal{L}(Y, U)$ satisfying the following "strict" sector condition

$$\sup_{y \in \mathbb{E}_Y} \operatorname{Re}\langle (K - K_1)y, (K - K_2)y\rangle < 0\,, \tag{6.8}$$

where we remind the reader that $\mathbb{E}_Y$ denotes the unit sphere in Y. Recall that an operator $S \in \mathcal{L}(Y, U)$ is said to be left invertible if there exists $T \in \mathcal{L}(U, Y)$ such that $TS = I$. It is well-known (and easy to show) that $S \in \mathcal{L}(Y, U)$ is left invertible if, and only if, S is bounded away from 0, that is,

$$\inf_{y \in \mathbb{E}_Y} \|Sy\| > 0.$$

Theorem 6.8 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U, Y))$ for $\alpha \leq 0$. Let $K_1, K_2 \in \mathcal{L}(Y, U)$ and assume that K_1 is an admissible feedback operator for $\mathbf{H}$. The following statements are equivalent.*

(1) $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ *is positive real and* $K_1 - K_2$ *is left invertible.*
(2) *The set* $\sec(K_1, K_2)$ *is non-empty and every* $K \in \sec(K_1, K_2)$ *is a stabilizing feedback operator for* $\mathbf{H}$.

The implication (1) ⇒ (2) is a linear version of the circle criterion, a well-known absolute stability result (see, for instance, [22,27,47]). Note in this context that, in the literature on absolute stability, it is usually assumed that $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is strictly or strongly positive real and, instead of (6.8), the "weak" sector condition

$$\operatorname{Re}\langle (K - K_1)y, (K - K_2)y\rangle \leq 0 \quad \forall\, y \in Y$$

is imposed. We will analyze this scenario further below by using Theorem 6.8 (see Theorem 6.11). Whilst the implication (1) ⇒ (2) continues to hold for real[3] Hilbert spaces U and Y, this is not the case for the implication (2) ⇒ (1).

Proof of Theorem 6.8 Setting

$$N := \frac{1}{2}(K_1 - K_2) \quad \text{and} \quad M := \frac{1}{2}(K_1 + K_2), \tag{6.9}$$

it is clear that $K_1 = M + N$ and $K_2 = M - N$ and so, for $K \in \mathcal{L}(Y, U)$,

$$\begin{aligned} \operatorname{Re}\langle (K - K_1)y, (K - K_2)y\rangle &= \operatorname{Re}\langle (K - M)y - Ny, (K - M)y + Ny\rangle \\ &= \|Ky - My\|^2 - \|Ny\|^2 \quad \forall\, y \in Y. \end{aligned} \tag{6.10}$$

As a consequence, for every $K \in \sec(K_1, K_2)$, there exists $\delta > 0$ such that

$$-\|Ny\|^2 + \|Ky - My\|^2 \leq -\delta \quad \forall\, y \in \mathbb{E}_Y. \tag{6.11}$$

[3] Note that if U and Y are real, then, since s is a complex variable, the complexifications U_c and Y_c of U and Y will still need to be considered to formulate the positive real condition, in particular, $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U_\mathrm{c}, Y_\mathrm{c}))$.

Assume that statement (2) holds. Let $K \in \sec(K_1, K_2)$ (such an operator K exists, since $\sec(K_1, K_2) \neq \emptyset$ by hypothesis). Therefore, (6.11) holds for some $\delta > 0$, which implies that

$$\|(K_1 - K_2)y\| \geq 2\sqrt{\delta} \quad \forall\, y \in \mathbb{E}_Y.$$

Consequently, $K_1 - K_2$ is left invertible. Next, let $X \in \mathcal{L}(U)$ be strictly dissipative and define $F := K_1 + (I - (1/2)X)^{-1}XN$. Then with $Z := (1/2)X$, which is strictly dissipative, there exists $\varepsilon > 0$ such that

$$\begin{aligned}\mathrm{Re}\langle (F - K_1)y, (F - K_2)y\rangle &= \mathrm{Re}\langle Z(I - Z)^{-1}(K_1 - K_2)y, (I - Z)^{-1}(K_1 - K_2)y\rangle \\ &\leq -\varepsilon\|(I - Z)^{-1}(K_1 - K_2)y\|^2 \\ &\leq \frac{-4\varepsilon\delta}{\|I - Z\|^2} \quad \forall\, y \in \mathbb{E}_Y.\end{aligned}$$

Hence $F \in \sec(K_1, K_2)$, and thus, F is stabilizing for $\mathbf{H}$. From statement (3) of Proposition 5.2 it then follows that $F - K_1 = (I - (1/2)X)^{-1}XN$ is stabilizing for $\mathbf{H}^{K_1}$, and so with $S := (I - (1/2)X)^{-1}X$, we have

$$\mathbf{H}^{K_1}(I - SN\mathbf{H}^{K_1})^{-1} \in \mathcal{H}^\infty_0(\mathcal{L}(U, Y)).$$

Since this implies that $N\mathbf{H}^{K_1}(I - SN\mathbf{H}^{K_1})^{-1} \in \mathcal{H}^\infty_0(\mathcal{L}(U))$, we see that S is stabilizing for $N\mathbf{H}^{K_1}$. From Corollary 6.5 with $L = (1/2)I$, we then obtain that

$$I + 2N\mathbf{H}^{K_1} = (I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}.$$

is positive real.

Conversely, assume that statement (1) holds. Since $K_1 - K_2$ is left invertible,

$$\inf_{y \in \mathbb{E}_Y} \|(K_1 - K_2)y\| > 0,$$

and it follows that $M \in \sec(K_1, K_2)$, showing that $\sec(K_1, K_2) \neq \emptyset$. A further consequence of the left invertibility of $K_1 - K_2$ is that N^*N is invertible; indeed,

$$\|N^*Ny\| \geq |\langle N^*Ny, y\rangle| = \|Ny\|^2, \quad \forall\, y \in \mathbb{E}_Y,$$

showing that N^*N is bounded away from 0 (because N is bounded away from 0) and thus, since N^*N is also self-adjoint, invertibility of N^*N follows. Consequently, $N^\dagger := (N^*N)^{-1}N^*$ is well-defined and a left inverse of N. Obviously, boundedness of N implies that $N^\dagger$ is bounded away from 0 on $\mathrm{im}\, N$, that is, there exists $\nu > 0$ such that

$$\|N^\dagger w\| \geq \nu\|w\| \quad \forall\, w \in \mathrm{im}\, N.$$

We note that the operator $P := NN^\dagger \in \mathcal{L}(U)$ is the orthogonal projection onto $\mathrm{im}\, N = (\ker N^*)^\perp = (\ker N^\dagger)^\perp$.

Let K be an arbitrary element in $\sec(K_1, K_2)$. Then K satisfies (6.11) for some $\delta > 0$. Setting $\varepsilon := \delta/(2\|N\|)$, a routine calculation gives

$$\|Ky - My\| \leq \|Ny\| - \varepsilon\|y\| \quad \forall\, y \in Y.$$

Hence

$$\|KN^\dagger y - MN^\dagger y\| \leq \|Py\| - \varepsilon\|N^\dagger y\| \leq \|y\| - \varepsilon\|N^\dagger y\| \quad \forall\, y \in Y,$$

and thus,

$$\|KN^\dagger y - MN^\dagger y\| = \|KN^\dagger Py - MN^\dagger Py\| \leq \|Py\| - \varepsilon\nu\|Py\| \leq (1 - \varepsilon\nu)\|y\| \quad \forall\, y \in Y,$$

which in turn implies that

$$\sec(K_1, K_2)N^\dagger := \{KN^\dagger : K \in \sec(K_1, K_2)\} \subset \mathbb{B}(MN^\dagger, 1). \tag{6.12}$$

Furthermore, invoking Corollary 3.6, a straightforward calculation shows that the positive realness of $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is equivalent to the following contraction property

$$\|N\mathbf{H}(I - M\mathbf{H})^{-1}\|_{\mathcal{H}^\infty_0} \leq 1. \tag{6.13}$$

Now

$$N\mathbf{H}(I - M\mathbf{H})^{-1} = N\mathbf{H}(I - MN^\dagger N\mathbf{H})^{-1} = (N\mathbf{H})^{MN^\dagger},$$

and thus, appealing to Proposition 5.6 and (6.13), we obtain

$$\mathbb{B}(MN^\dagger, 1) \subset \mathbb{S}(N\mathbf{H}). \tag{6.14}$$

Together with (6.12) this shows that

$$\sec(K_1, K_2)N^\dagger \subset \mathbb{S}(N\mathbf{H}),$$

and therefore,

$$\mathbf{H}(I - K\mathbf{H})^{-1} = N^\dagger N\mathbf{H}(I - KN^\dagger N\mathbf{H})^{-1} \in \mathcal{H}_0^\infty(\mathcal{L}(U, Y)) \quad \forall\, K \in \sec(K_1, K_2),$$

showing that every $K \in \sec(K_1, K_2)$ is a stabilizing feedback for $\mathbf{H}$. □

In the following two corollaries of Theorem 6.8 the sector condition (6.8) is replaced by certain norm conditions.

Corollary 6.9 *Let $\mathbf{H} \in \mathcal{H}_\alpha^*(\mathcal{L}(U, Y))$ for $\alpha \leq 0$ and let $K_1, K_2 \in \mathcal{L}(Y, U)$. Assume that K_1 is an admissible feedback operator for $\mathbf{H}$ and that $K_1 - K_2$ is left invertible. Let $N, M \in \mathcal{L}(Y, U)$ be given by* (6.9) *and set $N^\dagger := (N^*N)^{-1}N$. Under these conditions, the function $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is positive real if, and only if, every $K \in \mathcal{L}(Y, U)$ such that $KN^\dagger \in \mathbb{B}(MN^\dagger, 1)$ is a stabilizing feedback operator for $\mathbf{H}$.*

As an immediate consequence of Corollary 6.9 we have that if $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is positive real, then every $K \in \mathbb{B}(M, 1/\|N^\dagger\|)$ is stabilizing.

Proof of Corollary 6.9 By Theorem 6.8, the claim is equivalent to

$$K \in \sec(K_1, K_2) \iff KN^\dagger \in \mathbb{B}(MN^\dagger, 1). \tag{6.15}$$

It is therefore sufficient to establish (6.15). To this end recall that it has been shown in the proof of Theorem 6.8 that if $K \in \sec(K_1, K_2)$, then $KN^\dagger \in \mathbb{B}(MN^\dagger, 1)$, see (6.12). Conversely, let $K \in \mathcal{L}(Y, U)$ be such that $KN^\dagger \in \mathbb{B}(MN^\dagger, 1)$. Then there exists $\rho \in (0, 1)$ such that

$$\|Ky - My\|^2 = \|KN^\dagger Ny - MN^\dagger Ny\|^2 \leq \rho\|Ny\|^2 \quad \forall\, y \in Y.$$

Consequently,

$$-\|Ny\|^2 + \|Ky - My\|^2 \leq -(1 - \rho)\lambda\|y\|^2 \quad \forall\, y \in Y, \tag{6.16}$$

where the constant $\lambda > 0$ is such that $\|Ny\|^2 \geq \lambda\|y\|^2$ for all $y \in Y$ (such a constant exists by the left invertibility of N). It now follows from (6.10) and (6.16) that $K \in \sec(K_1, K_2)$. □

Corollary 6.10 *Let $\mathbf{H} \in \mathcal{H}_\alpha^*(\mathcal{L}(U, Y))$ for $\alpha \leq 0$, let $K_1, K_2 \in \mathcal{L}(Y, U)$ and let M be as in* (6.9)*. Assume that K_1 is an admissible feedback operator for $\mathbf{H}$ and $K_1 - K_2 = cJ$ for some isometry $J \in \mathcal{L}(Y, U)$ and some non-zero $c \in \mathbb{C}$. The function $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is positive real if, and only if, every $K \in \mathbb{B}(M, |c|/2)$ is a stabilizing feedback operator for $\mathbf{H}$.*

In the the single-input single-output case (that is, $U = Y = \mathbb{C}$), the assumption on $K_1 - K_2$ is trivially satisfied and, furthermore, the condition that every $K \in \mathbb{B}(M, |c|/2)$ is a stabilizing feedback for $\mathbf{H}$ can be checked by using the Nyquist criterion (which applies provided that $\mathbf{H}$ satisfies suitable assumptions, see [9, Theorem 2 in Section 34] for the finite-dimensional and [39] for the infinite-dimensional case).

An application of Corollary 6.10 with $K_1 = 0$ and $K_2 = -2rI$, where $r > 0$, yields that $I + 2r\mathbf{H}$ is positive real if, and only if, $\mathbb{B}(-rI, r) \subset \mathbb{S}(\mathbf{H})$. Obviously, $\mathbf{H}$ is positive real if, and only if, $I + 2r\mathbf{H}$ is positive real for every $r > 0$, and thus, invoking Lemma 2.4, we recover Theorem 6.4, showing that Corollary 6.10 (and hence, Theorem 6.8) can be considered as generalizations of Theorem 6.4.

Proof of Corollary 6.10 Let N be as in (6.9), that is, $N = (c/2)J$. Since J is an isometry, we have that $J^*J = I$ and so $N^\dagger := (N^*N)^{-1}N^* = (2/c)J^*$. Invoking Corollary 6.9, it is sufficient to show that

$$K \in \mathbb{B}(M, |c|/2) \iff KN^\dagger \in \mathbb{B}(MN^\dagger, 1).$$

Let $K \in \mathcal{L}(Y,U)$ and assume that $KN^\dagger \in \mathbb{B}(MN^\dagger, 1)$. Then $\|KJ^* - MJ^*\| < |c|/2$ and so

$$\|Ky - My\| \leq \|KJ^* - MJ^*\|\|Jy\| = \|KJ^* - MJ^*\|\|y\| \quad \forall\, y \in Y,$$

showing that $\|K - M\| \leq \|KJ^* - MJ^*\| < |c|/2$.

Conversely, assume that $K \in \mathbb{B}(M, |c|/2)$. Then there exists $\varepsilon \in (0, |c|/2)$ such that

$$\|Ky - My\| \leq (|c|/2 - \varepsilon)\|y\| = (|c|/2)\|Jy\| - \varepsilon\|y\| = \|Ny\| - \varepsilon\|y\| \quad \forall\, y \in Y.$$

Setting $P := NN^\dagger = JJ^*$, the orthogonal projection onto $\operatorname{im} J = (\ker J^*)^\perp$, it follows that

$$\|KN^\dagger y - MN^\dagger y\| = \|KN^\dagger Py - MN^\dagger Py\| \leq \|NN^\dagger Py\| - \varepsilon\|N^\dagger Py\| \quad \forall\, y \in Y,$$

and therefore,

$$\|KN^\dagger y - MN^\dagger y\| \leq \|Py\| - \varepsilon\|N^\dagger Py\| \leq (1 - \varepsilon\mu)\|Py\| \leq (1 - \varepsilon\mu)\|y\| \quad \forall\, y \in Y,$$

where the constant $\mu > 0$ is such that $\|N^\dagger w\| \geq \mu\|w\|$ for all $w \in \operatorname{im} N = \operatorname{im} J$. We conclude that $KN^\dagger \in \mathbb{B}(MN^\dagger, 1)$. ☐

Next we modify the scenario considered in Theorem 6.8 by replacing positive realness with strong positive realness and the strict sector condition (6.8) with the "weak" sector property

$$\operatorname{Re}\langle (K - K_1)y, (K - K_2)y\rangle \leq 0 \quad \forall\, y \in Y. \tag{6.17}$$

Theorem 6.11 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U,Y))$ for $\alpha \leq 0$ and let $K_1, K_2 \in \mathcal{L}(Y,U)$. Assume that K_1 is an admissible feedback operator for $\mathbf{H}$ and that $K_1 - K_2$ is left invertible. The following statements hold.*

(1) *If the function $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is in $\mathcal{H}^\infty_0(\mathcal{L}(U))$ and is strongly positive real, then every $K \in \mathcal{L}(Y,U)$ satisfying (6.17) is a stabilizing feedback operator for $\mathbf{H}$.*

(2) *If $\dim U < \infty$ and every $K \in \mathcal{L}(Y,U)$ satisfying (6.17) is a stabilizing feedback operator for $\mathbf{H}$, then $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is in $\mathcal{H}^\infty_0(\mathcal{L}(U))$ and is strongly positive real.*

Proof To prove statement (1), we define N by (6.9) and observe that

$$(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1} = I + 2N\mathbf{H}(I - K_1\mathbf{H})^{-1} \in \mathcal{H}^\infty_0(\mathcal{L}(U)).$$

Since N is left invertible, we conclude that $K_1 \in \mathbb{S}(\mathbf{H})$. Together with the openness of $\mathbb{S}(\mathbf{H})$, this yields the existence of a number $\nu^* > 0$ such that $K_1 + \nu N \in \mathbb{S}(\mathbf{H})$ for all $\nu \in [0, \nu^*]$. Defining

$$\mathbf{G}_\nu := \big(I - (K_2 - \nu N)\mathbf{H}\big)\big(I - (K_1 + \nu N)\mathbf{H}\big)^{-1}, \tag{6.18}$$

it is clear that the map

$$[0, \nu^*] \to \mathcal{H}^\infty_0(\mathcal{L}(U)), \quad \nu \mapsto \mathbf{G}_\nu$$

is continuous. Combined with the strong positive realness of $\mathbf{G}_0$, this shows that there exists $\nu^{**} \in (0, \nu^*]$ such that

$$\operatorname{Re}\mathbf{G}_\nu(s) \succeq 0 \quad \forall\, s \in \mathbb{C}_0,\ \forall\, \nu \in [0, \nu^{**}]. \tag{6.19}$$

Let $K \in \mathcal{L}(Y,U)$ be such that (6.17) holds. A straightforward calculation then shows that

$$\operatorname{Re}\langle Ky - (K_1 + \nu N)y, Ky - (K_2 - \nu N)y\rangle \leq -\nu(\nu + 2)\|Ny\|^2 \quad \forall\, y \in Y.$$

Since N is left invertible, there exists $\mu > 0$ such that $\|Ny\|^2 \geq \mu\|y\|^2$ for all $y \in Y$, and thus

$$\operatorname{Re}\langle Ky - (K_1 + \nu N)y, Ky - (K_2 - \nu N)y\rangle \leq -\mu\nu(\nu + 2)\|y\|^2 \quad \forall\, y \in Y.$$

Hence, for every $\nu \in (0, \nu^{**}]$, the operator K is in $\sec(K_1 + \nu N, K_2 - \nu N)$. Trivially, $(K_1 + \nu N) - (K_2 - \nu N) = 2(1+\nu)N$ is left-invertible, and, by (6.19), $\mathbf{G}_\nu$ is positive real for every $\nu \in (0, \nu^{**}]$. An application of Theorem 6.8 now shows that K is a stabilizing feedback operator for $\mathbf{H}$.

To prove statement (2), note that, since K_1 satisfies (6.17), K_1 is a stabilizing feedback operator for $\mathbf{H}$, and thus, $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is in $\mathcal{H}_0^\infty(\mathcal{L}(U))$. We proceed to establish strong positive realness of $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$. By the left invertibility of $K_1 - K_2$, we have that $\dim Y \le \dim U < \infty$. Defining N and M by (6.9), N is left invertible and the function $y \mapsto \|Ny\|$ defines a norm on Y. We denote the corresponding norm on $\mathcal{L}(Y,U)$ by $\|\cdot\|_N$, that is, for $T \in \mathcal{L}(Y,U)$,

$$\|T\|_N := \sup_{y\neq 0} \frac{\|Ty\|}{\|Ny\|}\,.$$

Let $\mathcal{S}$ denote the set of all $K \in \mathcal{L}(Y,U)$ satisfying (6.17). It follows from (6.10) that $K \in \mathcal{S}$ if, and only if,

$$\|Ky - My\| \le \|Ny\| \quad \forall\, y \in Y\,.$$

Consequently,

$$\mathcal{S} = \{K \in \mathcal{L}(Y,U) : \|K - M\|_N \le 1\}\,. \tag{6.20}$$

By hypothesis, $\mathcal{S} \subset \mathbb{S}(\mathbf{H})$, and so, invoking the openness of $\mathbb{S}(\mathbf{H})$, it follows from (6.20) and a routine compactness argument that there exists $\nu > 0$ such that

$$\mathcal{B}_\nu := \{K \in \mathcal{L}(Y,U) : \|K - M\|_N \le 1 + \nu\} \subset \mathbb{S}(\mathbf{H})\,. \tag{6.21}$$

Furthermore,

$$\begin{aligned}\mathrm{Re}\langle Ky - (K_1{+}\nu N)y, Ky - (K_2 - \nu N)y\rangle \\ &= \mathrm{Re}\langle (K-M)y - (1+\nu)Ny, (K-M)y + (1+\nu)Ny\rangle \\ &= \|Ky - My\|^2 - (1+\nu)^2\|Ny\|^2 \quad \forall\, y \in Y,\end{aligned}$$

and therefore,

$$\sec\,(K_1 + \nu N, K_2 - \nu N) \subset \mathcal{B}_\nu.$$

It now follows from (6.21) that every operator in $\sec\,(K_1 + \nu N, K_2 - \nu N)$ is stabilizing for $\mathbf{H}$ and an application of Theorem 6.8 yields that the function $\mathbf{G}_\nu$ defined by (6.18) is positive real. A routine calculation shows that

$$(I - \mathbf{G}_\nu)(I + \mathbf{G}_\nu)^{-1} = (1+\nu)N\mathbf{H}(I - M\mathbf{H})^{-1} = (1+\nu)(I - \mathbf{G}_0)(I + \mathbf{G}_0)^{-1},$$

and consequently, by Corollary 3.6,

$$\|(I - \mathbf{G}_0)(I + \mathbf{G}_0)^{-1}\|_{\mathcal{H}_0^\infty} \le \frac{1}{1+\nu} < 1.$$

Appealing to Corollary 4.3, it follows that $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1} = \mathbf{G}_0$ is strongly positive real, completing the proof. □

The following result is, in a sense, a special case of statement (1) of Theorem 6.11.

Corollary 6.12 *If $\mathbf{H} \in \mathcal{H}_0^\infty(\mathcal{L}(U))$ is strongly positive real, then every dissipative $K \in \mathcal{L}(U)$ is in $\mathbb{S}(\mathbf{H})$.*

Note that in the above corollary, it is assumed that $\mathbf{H} \in \mathcal{H}_0^\infty(\mathcal{L}(U))$, that is, $\mathbf{H}$ is stable. Therefore, Corollary 6.12 should not be viewed as a stabilization result, but as a necessary condition for a function in $\mathcal{H}_0^\infty(\mathcal{L}(U))$ to be strongly positive real or as a robustness result in the sense that stability is retained under perturbations induced by dissipative static feedback. A similar comment applies to Proposition 6.13 below.

Unlike Theorem 6.4, the converse of the conclusion of Corollary 6.12 is false, which is easily seen by considering the scalar positive-real function $\mathbf{H}(s) = 1/(s+1)$. Every $K = k \in \mathbb{C}$ with $\mathrm{Re}\,k \le 0$ (precisely the set of dissipative operators $\mathbb{C} \to \mathbb{C}$) is stabilizing for $\mathbf{H}$, but $\mathbf{H}$ is not strongly positive real.

Proof of Corollary 6.12 Let $K \in \mathcal{L}(U)$ be dissipative and $r > 0$. Since $\mathbf{H}$ is strongly positive real and in $\mathcal{H}_0^\infty(\mathcal{L}(U))$, it is clear that $I - (2K - rI)\mathbf{H} = (I - 2K\mathbf{H}) + r\mathbf{H}$ is in $\mathcal{H}_0^\infty(\mathcal{L}(U))$ for every $r > 0$ and strongly positive real for all sufficiently large $r > 0$. Noting that

$$\mathrm{Re}\,\langle Ku, Ku - (2K - rI)u\rangle = -\|Ku\|^2 + r\mathrm{Re}\,\langle Ku, u\rangle \le 0 \quad \forall\, u \in U,$$

as K is dissipative, an application of statement (1) of Theorem 6.11 with $K_1 := 0$ and $K_2 := 2K - rI$ shows that $K \in \mathbb{S}(\mathbf{H})$. ☐

The next result identifies sets of stabilizing feedback operators for $\mathbf{H}$ under the assumption that $\mathbf{H} + L$ is strongly positive real, where the operator $-L$ is strictly dissipative.

Proposition 6.13 *Let $\mathbf{H} \in \mathcal{H}_0^\infty(\mathcal{L}(U))$ and let $L \in \mathcal{L}(U)$ be such that $-L$ is strictly dissipative and $\mathbf{H} + L$ is strongly positive real. The following statements hold.*

(1) *For every dissipative $K \in \mathcal{L}(U)$, $(I - KL)^{-1}K$ is a stabilizing feedback operator for $\mathbf{H}$.*
(2) *If, additionally, the operator L is self-adjoint, then every $K \in \mathcal{L}(U)$ such that $K = K^*$ and $0 \succeq K \succeq -L^{-1}$ is a stabilizing feedback operator for $\mathbf{H}$.*

Note that in statement (2) of Proposition 6.13, the existence of the inverse of L is guaranteed, since, by self-adjointness of L and strict dissipativity of $-L$, both L and L^* are bounded away from 0. For matrix-valued rational functions, statement (2) can be found in [8, Corollary 2.1] (the proof in [8] relies on finite-dimensional state space realizations).

Proof of Proposition 6.13 Statement (1) can be proved by arguments similar to those used in the proof of Corollary 6.5, where references to Theorem 6.4 should be replaced by references to Corollary 6.12.

To prove statement (2), we assume that $L = L^*$. As $\mathbf{H} + L$ is strongly positive real, there exists $\varepsilon \in (0, 1)$ such that $\mathbf{H} + \varepsilon L = \mathbf{H} + M$ is also strongly positive real, where $M := \varepsilon L$. Let $K \in \mathcal{L}(U)$ be such that $K = K^*$ and $0 \succeq K \succeq -L^{-1}$. The aim is to prove that K is stabilizing for $\mathbf{H}$. By statement (2), it is sufficient to show that there exists a dissipative operator $F \in \mathcal{L}(U)$ such that $K = (I - FM)^{-1}F$. Solving this equation for F gives

$$F = K(I + MK)^{-1}\,.$$

We now have to show that: (i) $I + MK$ is invertible (to make sure that F is well defined), and; (ii) F is dissipative. To establish the invertibility of $I + MK$, note that

$$I + MK = I + M(K + L^{-1}) - \varepsilon I = (1 - \varepsilon)\big[(I - (\varepsilon - 1)^{-1}M(K + L^{-1})\big]\,.$$

Thus, the invertibility of $I + MK$ is equivalent to that of $I - (\varepsilon - 1)^{-1}M(K + L^{-1})$. The invertibility of the latter follows from Lemma 6.2 since $(\varepsilon - 1)^{-1}M = -\varepsilon(1 - \varepsilon)^{-1}L$ is strictly dissipative and $\operatorname{Re}(K + L^{-1}) \succeq 0$.

It remains to show that F is dissipative. To this end, let P and Q denote the square roots of $-K$ and L^{-1}, respectively. Then $P = P^*$, $Q = Q^*$, $K = -P^2$, $L^{-1} = Q^2$ and Q is invertible. Since $0 \succeq K \succeq -L^{-1}$, we have that

$$\langle -L^{-1}v, v\rangle \le \langle Kv, v\rangle \le 0 \quad \forall\, v \in U\,,$$

and so, taking $v = Q^{-1}u$ yields that

$$-\|u\|^2 \le \langle KQ^{-1}u, Q^{-1}u\rangle = -\langle P^*PQ^{-1}u, Q^{-1}u\rangle = -\|PQ^{-1}u\|^2 \quad \forall\, u \in U\,.$$

We conclude that $\|PQ^{-1}\| \le 1$, and thus, $\|Q^{-1}P\| = \|(PQ^{-1})^*\| = \|PQ^{-1}\| \le 1$. Therefore, for $u \in U$ and $v = Q^{-1}u$,

$$\begin{aligned}\langle KLKu, u\rangle &= \langle KLKQv, Qv\rangle = \langle Q^{-1}KQv, Q^{-1}KQv\rangle \\ &= \|Q^{-1}KQv\|^2 = \|Q^{-1}PPQv\|^2 \le \|Q^{-1}P\|^2\|PQv\|^2 \\ &\le \|PQv\|^2 = \langle PQv, PQv\rangle = \langle Pu, Pu\rangle \\ &= -\langle Ku, u\rangle\,. \end{aligned} \tag{6.22}$$

Furthermore, we note that, for $u \in U$ and $v = (I + MK)^{-1}u$,

$$\begin{aligned}\langle Fu, u\rangle &= \langle K(I + MK)^{-1}u, u\rangle = \langle Kv, (I + MK)v\rangle \\ &= \langle Kv, v\rangle + \langle v, KMKv\rangle. \end{aligned}$$

Invoking (6.22), it follows that

$$\langle v, KMKv\rangle = \varepsilon\,\langle v, KLKv\rangle \le -\varepsilon\,\langle v, Kv\rangle\,,$$

whence

$$\langle Fu, u\rangle \le (1-\varepsilon)\langle Kv, v\rangle \le 0,$$

showing that F is dissipative and completing the proof. □

Corollary 6.14 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ for $\alpha \le 0$ and let $K_1, K_2 \in \mathcal{L}(U)$ be self-adjoint and such that $K_2 - K_1$ is strictly dissipative. If K_1 is a stabilizing feedback operator for $\mathbf{H}$, and $\mathbf{H}^{K_1} + (K_1 - K_2)^{-1}$ is strongly positive real, then every self-adjoint $K \in \mathcal{L}(U)$ such that $K_1 \succeq K \succeq K_2$ is a stabilizing feedback operator for $\mathbf{H}$.*

Note that the existence of the inverse of $K_1 - K_2$ is guaranteed, since, by self-adjointness and strict dissipativity of $K_2 - K_1$, both $K_2 - K_1$ and $(K_2 - K_1)^*$ are bounded away from 0. The above corollary is a generalization of statement (2) of Proposition 6.13: indeed, if the assumptions of Corollary 6.14 hold with $K_1 = 0$, then we recover statement (2) of Proposition 6.13 with $L = -K_2^{-1}$.

Proof of Corollary 6.14 Let $K \in \mathcal{L}(U)$ be self-adjoint and such that $K_1 \succeq K \succeq K_2$. Then, obviously, the operator $K - K_1$ is self-adjoint and $0 \succeq K - K_1 \succeq K_2 - K_1$. The hypotheses of statement (2) of Proposition 6.13 hold with $L = (K_1 - K_2)^{-1}$ and $\mathbf{H}$ replaced by $\mathbf{H}^{K_1}$. Hence,

$$\left\{F \in \mathcal{L}(U) \;:\; F \text{ is self-adjoint and } 0 \succeq F \succeq K_2 - K_1\right\} \subseteq \mathbb{S}(\mathbf{H}^{K_1}).$$

Consequently, $K - K_1$ is a stabilizing feedback operator for $\mathbf{H}^{K_1}$, and it follows from statement (2) of Proposition 5.2 that K is a stabilizing feedback operator for $\mathbf{H}$, completing the proof. □

We conclude this section, by turning our attention briefly to feedback with dynamic compensators by defining the notion of a (stabilizing) feedback interconnection of two transfer functions. The following definition generalizes Definitions 5.1 and 5.3 to the dynamic feedback case. Theorem 6.16 below then considers the feedback interconnection of two transfer functions $\mathbf{H}$ and $\mathbf{K}$ with $\mathbf{H}$ positive real and $-\mathbf{K}$ strongly positive real.

Definition 6.15 *Let $\mathbf{H} \in \mathcal{H}^*(\mathcal{L}(U, Y))$ and $\mathbf{K} \in \mathcal{H}^*(\mathcal{L}(Y, U))$. Then $\mathbf{K}$ is called an* admissible feedback *for $\mathbf{H}$ if*

$$\mathbf{F} := \begin{pmatrix} I & -\mathbf{K} \\ -\mathbf{H} & I \end{pmatrix},$$

is invertible in $\mathcal{H}^*_\sim(\mathcal{L}(U \times Y))$. An admissible feedback is called stabilizing *if $[\mathbf{F}^{-1}] \cap \mathcal{H}^\infty_0(\mathcal{L}(U \times Y)) \neq \emptyset$.*

Note that $\mathbf{K}$ is an admissible feedback for $\mathbf{H}$ if, and only if, $I - \mathbf{KH}$ is invertible in $\mathcal{H}^*_\sim(\mathcal{L}(U))$, or, equivalently, if $I - \mathbf{HK}$ is invertible in $\mathcal{H}^*_\sim(\mathcal{L}(Y))$ (cf. Proposition 5.2). Moreover, if $\mathbf{K}$ is an admissible feedback for $\mathbf{H}$, then

$$\begin{pmatrix} I & -\mathbf{K} \\ -\mathbf{H} & I \end{pmatrix}^{-1} = \begin{pmatrix} (I - \mathbf{KH})^{-1} & \mathbf{K}(I - \mathbf{HK})^{-1} \\ \mathbf{H}(I - \mathbf{KH})^{-1} & (I - \mathbf{HK})^{-1} \end{pmatrix}.$$

Since, for admissible $\mathbf{K}$,

$$(I - \mathbf{KH})^{-1} = I + \mathbf{KH}(I - \mathbf{KH})^{-1} \quad \text{and} \quad (I - \mathbf{HK})^{-1} = I + \mathbf{H}(I - \mathbf{KH})^{-1}\mathbf{K},$$

we see that, if $\mathbf{K} \in \mathcal{H}^\infty_0(\mathcal{L}(Y, U))$, then $\mathbf{K}$ is a stabilizing feedback if, and only if, $\mathbf{H}(I - \mathbf{KH})^{-1} \in \mathcal{H}^\infty_0(\mathcal{L}(U, Y))$. In particular, $K \in \mathcal{L}(Y, U)$ is a stabilizing feedback operator according to Definition 5.3 if, and only if, $\mathbf{K}(s) \equiv K$ is a stabilizing feedback in the sense of Definition 6.15.

Theorem 6.16 *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha \le 0$. The following statements are equivalent.*

(1) *$\mathbf{H}$ is positive real.*
(2) *Every $\mathbf{K} \in \mathcal{H}^\infty_0(\mathcal{L}(U))$ such that $-\mathbf{K}$ is strongly positive real is a stabilizing feedback for $\mathbf{H}$.*

Other sufficient conditions for the stability of the feedback interconnection of $\mathbf{H}$ and $\mathbf{K}$ in terms of positive-real properties of $\mathbf{H}$ and $-\mathbf{K}$ have appeared in the literature, we only refer here to the recent result [56, Theorem 4.2].

The proof of Theorem 6.16 is facilitated by two technical results which will be presented next. The proof of the following corollary is similar to that of Lemma 2.4 and is therefore omitted.

Corollary 6.17 *A function* $\mathbf{H} \in \mathcal{H}_0^\infty(\mathcal{L}(U))$ *is strongly positive real if, and only if, there exists an* $r > 0$ *such that* $\|\mathbf{H} - rI\|_{\mathcal{H}_0^\infty} < r$.

The lemma below is a simple small-gain result and is a straightforward generalization of one direction of Proposition 5.6. The proof is left to the interested reader.

Lemma 6.18 *Let* $\mathbf{H} \in \mathcal{H}_\sim^*(\mathcal{L}(U))$, $r > 0$ *and let* $\mathbf{K} \in \mathcal{H}_0^\infty(\mathcal{L}(Y,U))$ *be a stabilizing feedback for* $\mathbf{H}$. *If* $\|\mathbf{H}(I - \mathbf{K}\mathbf{H})^{-1}\|_{\mathcal{H}_0^\infty} \leq 1/r$, *then any* $\mathbf{C} \in \mathcal{H}_0^\infty(\mathcal{L}(U))$ *with* $\|\mathbf{C} - \mathbf{K}\|_{\mathcal{H}_0^\infty} < r$ *is stabilizing for* $\mathbf{H}$.

Proof of Theorem 6.16 The claim that statement (2) implies statement (1) follows from Theorem 6.4 since any strictly dissipative operator $K \in \mathcal{L}(U)$ has the property that $-K$ is a (constant) strongly positive-real function.

The converse direction is proven along the lines of the corresponding part of Theorem 6.4. The references to Lemma 2.4 and Proposition 5.6 should be replaced by references to Corollary 6.17 and Lemma 6.18, respectively. ☐

Example 6.19 We note that in Theorem 6.16, it is crucial that $-\mathbf{K}$ is strongly positive real; it is generally not sufficient to only have $-\mathrm{Re}\,\mathbf{K}(s) \succ 0$ for all s with $\mathrm{Re}(s) \geq 0$. As an example, consider the positive-real function $\mathbf{H}(s) = \tanh(s)$ and the function $\mathbf{K}(s) = -1/(s+1)$. Note that $-\mathrm{Re}\,\mathbf{K}(s) > 0$ for all s with $\mathrm{Re}(s) \geq 0$, but $-\mathbf{K}$ is not strongly positive real. Moreover, we have

$$\mathbf{H}_\mathrm{e}^\mathbf{K}(s) = \frac{\mathbf{H}(s)}{1 - \mathbf{H}(s)\mathbf{K}(s)} = \frac{(s+1)\tanh(s)}{s+1+\tanh(s)}\,.$$

Setting $s_n := (n+1/2)\pi i$ for $n \in \mathbb{Z}$ (the poles of $\mathbf{H}$) we have $\mathbf{H}_\mathrm{e}^\mathbf{K}(s_n) = s_n+1$. In particular, $|\mathbf{H}_\mathrm{e}^\mathbf{K}(s_n)| \to \infty$ as $n \to \infty$. It follows that $\mathbf{H}_\mathrm{e}^\mathbf{K} \notin \mathcal{H}_0^\infty$ and so $\mathbf{K}$ is not stabilizing for $\mathbf{H}$. ♢

7 Connections with Operator Theory and Partial Differential Equations

In this section, we establish some links between the material in Sections 5–6 on the one hand and operator theory and PDEs on the other. In particular, we will provide several examples of positive-real transfer functions arising in PDEs.

For a closed linear operator $A : D(A) \subset U \to U$, we let $\rho(A)$ denote the resolvent set of A. The map $\rho(A) \to \mathcal{L}(U)$, $s \mapsto (sI - A)^{-1}$ is called the resolvent of A.

Proposition 7.1 *Let* $A : D(A) \subset U \to U$ *be densely defined and closed, and let* $\mathbf{H}$ *be the resolvent of* A. *The following statements are equivalent.*

(1) *A is dissipative and $\rho(A) \cap \mathbb{C}_0$ is non-empty.*
(2) *$\mathbf{H}$ belongs to $\mathcal{H}_0^*(\mathcal{L}(U))$ and is positive real.*
(3) *A is the generator of a strongly continuous contraction semigroup.*

Proof The equivalence of statements (1) and (3) is the Lumer–Phillips Theorem (see, for example, [43, Theorem 3.4.8]).

(1)⇒(2): The implication (1)⇒(3) guarantees that A is the generator of a strongly continuous contraction semigroup and so $\mathbb{C}_0 \subset \rho(A)$. Consequently, $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ since the resolvent is holomorphic on $\rho(A)$. Let $s \in \mathbb{C}_0$ and $u \in U$ and set $v := \mathbf{H}(s)u$. Then $v = (sI - A)^{-1}u \in D(A)$ and

$$\mathrm{Re}\langle \mathbf{H}(s)u, u\rangle = \mathrm{Re}\langle v, (sI - A)v\rangle = (\mathrm{Re}\,s)\|v\|^2 - \mathrm{Re}\langle v, Av\rangle \geq 0,$$

which shows that $\mathbf{H}$ is positive real.

(2)⇒(1): Since $\mathbf{H}$ belongs to $\mathcal{H}_0^*(\mathcal{L}(U))$, the resolvent set includes $\mathbb{C}_0 \backslash \Sigma_\mathbf{H}$. Let $s \in \mathbb{C}_0 \backslash \Sigma_\mathbf{H} = \rho(A) \cap \mathbb{C}_0$, let $u \in D(A)$ and set $v := (sI - A)u$. Then $u = \mathbf{H}(s)v$ and

$$(\mathrm{Re}\,s)\|u\|^2 - \mathrm{Re}\langle Au, u\rangle = \mathrm{Re}\langle (sI - A)u, u\rangle = \mathrm{Re}\langle v, \mathbf{H}(s)v\rangle \geq 0 \quad \forall\, s \in \mathbb{C}_0 \backslash \Sigma_\mathbf{H}.$$

Since $\Sigma_\mathbf{H}$ has no accumulation points in $\mathbb{C}_0$, there exists a sequence $s_n \in \mathbb{C}_0 \backslash \Sigma_\mathbf{H}$ with $\mathrm{Re}\,s_n \to 0$ and $(\mathrm{Re}\,s_n)\|u\|^2 - \mathrm{Re}\langle Au, u\rangle \geq 0$ for every $n \in \mathbb{N}$. Letting $n \to \infty$, this gives $-\mathrm{Re}\langle Au, u\rangle \geq 0$, which shows that A is dissipative. ☐

We remark that the equivalence of statements (1) and (2) in Proposition 7.1 is known, see [7, Theorem 4.2]. The above proof simplifies that given in [7] which rests on a result on distributional boundary values of positive-real functions.

Corollary 7.2 *Let $A : D(A) \subset U \to U$ be densely defined and closed, let $\mathbf{H}$ be the resolvent of A and let $\omega > 0$. The following statements are equivalent.*

(1) $\operatorname{Re}\langle Au, u\rangle \le -\omega\|u\|^2$ *for all* $u \in D(A)$ *and* $\rho(A) \cap \mathbb{C}_{-\omega} \neq \emptyset$.
(2) $\mathbf{H}$ *belongs to* $\mathcal{H}^*_{-\omega}(\mathcal{L}(U))$ *and the function* $s \mapsto \mathbf{H}(s-\omega)$ *is positive real.*
(3) A *generates a strongly continuous semigroup* T *which satisfies* $\|T(t)\| \le e^{-\omega t}$ *for all* $t \ge 0$.

Proof Obviously, $\rho(\omega I + A) = \rho(A) + \omega$ and the resolvent of $\omega I + A$ is the function $s \mapsto \mathbf{H}(s-\omega)$. Furthermore, if $\omega I + A$ generates a semigroup T_ω, then A generates the semigroup T given by $T(t) = T_\omega(t)e^{-\omega t}$. The result now follows from an application of Proposition 7.1 to $\omega I + A$. □

The next corollary is an immediate consequence of Corollary 7.2.

Corollary 7.3 *Let $A : D(A) \subset U \to U$ be densely defined and closed, and let $\mathbf{H}$ be the resolvent of A. The following statements are equivalent.*

(1) A *is strictly dissipative and* $\rho(A) \cap \mathbb{C}_0$ *is non-empty.*
(2) $\mathbf{H}$ *is strictly positive real.*
(3) A *is the generator of an exponentially stable strongly continuous semigroup which, for some* $\omega > 0$, *satisfies* $\|T(t)\| \le e^{-\omega t}$ *for all* $t \ge 0$.

From Proposition 7.1 and Corollary 7.3 we obtain the following result.

Corollary 7.4 *Let $A : D(A) \subset X \to X$ be densely defined and closed, and let $B \in \mathcal{L}(U, X)$, where X is a complex Hilbert space. Define $\mathbf{H} : \rho(A) \to \mathcal{L}(U)$ by $\mathbf{H}(s) := B^*(sI - A)^{-1}B$. The following statements hold.*

(1) *If A is dissipative and $\rho(A) \cap \mathbb{C}_0$ is non-empty, then $\mathbf{H}$ belongs to $\mathcal{H}_0(\mathcal{L}(U))$ and is positive real.*
(2) *If A is strictly dissipative and $\rho(A) \cap \mathbb{C}_0$ is non-empty, then there exists an $\alpha < 0$ such that $\mathbf{H}$ belongs to $\mathcal{H}_\alpha(\mathcal{L}(U))$ and is strictly positive real.*

In the following, we will derive a suitable generalization of Corollary 7.4 which allows for unbounded control operators B. To this end, we need the concept of a system node. We note that by [43, Lemma 4.7.7], the definition below is equivalent to that given in [43].

Definition 7.5 *Let U, X and Y be complex Hilbert spaces and let $S : D(S) \subset \left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right) \to \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ be a linear operator. We write S in the form*

$$S = \begin{pmatrix} A\&B \\ C\&D \end{pmatrix}, \quad \text{where } \ A\&B : D(S) \to X \ \text{ and } C\&D : D(S) \to Y$$

and define an operator A by

$$A : D(A) \subset X \to X, \ \ x \mapsto A\&B \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right), \quad \text{where } \ D(A) := \{x \in X : \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \in D(S)\}.$$

We say that S is system node *on the triple of Hilbert spaces (U, X, Y) if the following conditions hold:*

(a) S *is closed;*
(b) $A\&B$ *is closed;*
(c) A *generates a strongly continuous semigroup;*
(d) *for every* $u \in U$, *there exists* $x \in X$ *such that* $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in D(S)$.

Given a system node S on (U, X, Y) and using the notation of Definition 7.5, we denote the usual interpolation and extrapolation spaces associated with A and X by X_1 and X_{-1}, respectively (see, for example, [43, Section 3.6]): letting $\beta \in \rho(A)$, the space X_1 is $D(A)$ endowed with the norm $\|x\|_1 := \|(\beta I - A)x\|_X$ and X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|(\beta I - A)^{-1}x\|_X$. The operator A extends to an operator $A|_X \in \mathcal{L}(X, X_{-1})$ (which generates a strongly continuous semigroup on X_{-1} with the same growth bound as the strongly continuous semigroup generated by A) and $A\&B$ has an extension to an operator $A\&B|_X \in \mathcal{L}(\left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right), X_{-1})$. The *control operator* $B \in \mathcal{L}(U, X_{-1})$ is defined by $Bu := A\&B|_X \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right)$. The *observation operator* $C \in \mathcal{L}(X_1, Y)$ is defined by $Cx = C\&D \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$. For every $s \in \rho(A)$ the operator $\left(\begin{smallmatrix} (sI - A|_X)^{-1}B \\ I \end{smallmatrix}\right)$ maps U into $D(S)$. Let ω be the growth bound of the semigroup generated by A. Then the *transfer function* $\mathbf{H} : \mathbb{C}_\omega \to \mathcal{L}(U, Y)$ of the system node S is defined by

$$\mathbf{H}(s) := C\&D \begin{pmatrix} (sI - A|_X)^{-1}B \\ I \end{pmatrix}. \tag{7.1}$$

The system node S is called *compatible* if there exists a Hilbert space W with $X_1 \subset W \subset X$ (with continuous embeddings) and an operator $C|_W \in \mathcal{L}(W, Y)$ such that

(1) $C|_W z = Cz$ for all $z \in X_1$, and
(2) there exists $s \in \rho(A)$ such that $(sI - A|_X)^{-1}B$ maps U into W.

It can be shown [43, Lemma 5.1.4] that for a compatible system node the operator $(sI - A|_X)^{-1}B$ maps U into W for all $s \in \rho(A)$. Moreover, the operator $D := \mathbf{H}(s) - C|_W(sI - A|_X)^{-1}B$ is independent of $s \in \rho(A)$. This operator D is called the *feedthrough operator* induced by S and $C|_W$ and, from [43, Lemma 5.1.4], we have that

$$C\&D \begin{pmatrix} z \\ v \end{pmatrix} = C|_W z + Dv \quad \text{for all } \begin{pmatrix} z \\ v \end{pmatrix} \in D(S).$$

We note that, in general, W is not unique and moreover that the operator $C|_W$ is generally not uniquely determined by S and W. However, for any compatible system node there is a canonical minimal space $W_{\min}$ (minimal in the sense that $W_{\min} \subset W$ for any compatibility space W), namely (see [43, Theorem 5.1.8 and Lemma 4.3.12])

$$W_{\min} := \{w \in X : \exists\, v \in U \text{ such that } A|_X w + Bv \in X\},$$

(with an inner-product as given in [43, Lemma 4.3.12]). Not every system node is compatible, but those that arise in applications typically are (as they are mixed boundary/distributed control nodes as defined in [43, Definition 5.2.14], which by [43, Theorem 5.2.15] are compatible).

A function $\mathbf{H}$, which belongs to $\mathcal{H}_\alpha(\mathcal{L}(U, Y))$ for some $\alpha \in \mathbb{R}$, is said to have *feedthrough* $D_{\mathbf{H}} : U \to Y$ if, for every every $u \in U$, $\mathbf{H}(s)u$ converges weakly to $D_{\mathbf{H}}u$ as $s \to \infty$ on the real axis, that is

$$\lim_{s \to \infty,\, s \in \mathbb{R}} \langle \mathbf{H}(s)u, y \rangle = \langle D_{\mathbf{H}}u, y \rangle \quad \forall\, u \in U,\ \forall\, y \in Y. \tag{7.2}$$

It is clear that $D_{\mathbf{H}}$ is linear and, by the uniform boundedness principle, the operator $D_{\mathbf{H}}$ is bounded (note that we assume only $\mathbf{H} \in \mathcal{H}_\alpha(\mathcal{L}(U, Y))$ rather than $\mathbf{H} \in \mathcal{H}^\infty_\alpha(\mathcal{L}(U, Y))$ for some $\alpha \in \mathbb{R}$, which is usually done in the literature, for instance, in the context of regular transfer functions, see [50]).

If a system node is compatible and has a transfer function which has feedthrough $D_{\mathbf{H}}$ (defined by (7.2)), then it is natural to demand that $D_{\mathbf{H}}$ is the feedthrough operator of the system node. By [43, Lemma 5.1.10], this choice is always possible and moreover, by fixing the feedthrough operator, we also fix $C|_{W_{\min}}$. Hence for a compatible system node which has transfer function $\mathbf{H}$ with feedthrough $D_{\mathbf{H}}$, we have a canonical extension of the operator $C\&D$ to $W_{\min} \times U$ and with this extension we have

$$\mathbf{H}(s) = D_{\mathbf{H}} + C|_{W_{\min}}(sI - A|_X)^{-1}B.$$

Remark 7.6 We comment that there are alternative, but equivalent, methods of defining transfer functions: transfer functions can be defined via Laplace transforms [14,16] or they can be defined via exponential trajectories (see [24, Chapter 12] and [57]). The equivalence of these definitions is shown in [57] (in a setting less general than that provided by the system node concept). As will be illustrated by several examples later in this section, many physically meaningful control systems may be realized as system nodes, with corresponding transfer functions given by (7.1). A simple example which cannot be represented as a system node is the differentiator $y = \dot{u}$, and thus, the transfer function $\mathbf{H}(s) = s$ cannot be written in the form (7.1) (see [42, Theorem 7.4]). $\diamondsuit$

The following example (adapted from [15, Example 5.6]) shows that Corollary 7.4 is in general not valid for unbounded B, that is, in the system node context, dissipativity of A together with the conditions $C = B^*$ and $D_{\mathbf{H}} = 0$ is not sufficient for the positive realness of $\mathbf{H}$.

Example 7.7 Consider the first order hyperbolic PDE:

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial w}{\partial x}(x,t), \quad w(1,t) - w(0,t) = u(t), \quad y(t) = w(0,t), \quad t > 0, \quad x \in (0,1).$$

The operator S given by

$$D(S) = \left\{ \begin{pmatrix} z \\ v \end{pmatrix} \in \begin{pmatrix} H^1(0,1) \\ \mathbb{C} \end{pmatrix} : z(1) - z(0) = v \right\}, \quad S \begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} z' \\ z(0) \end{pmatrix}.$$

corresponds to the above PDE in a natural way. Using standard PDE techniques, it can be shown that S is a system node on (U, X, Y), where $X = L^2(0,1)$ and $U = Y = \mathbb{C}$. We can calculate $z := (sI - A|_X)^{-1} Bv$ from

$$sz(x) = z'(x), \quad z(1) - z(0) = v\,,$$

which gives $z(x) = e^{sx} v/(e^s - 1)$. Since $C\&D \left(\begin{smallmatrix} z \\ v \end{smallmatrix} \right) = z(0)$, we obtain $\mathbf{H}(s) = 1/(e^s - 1)$. Obviously, $D_{\mathbf{H}} = \lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{H}(s) = 0$, showing that $\mathbf{H}$ has zero feedthrough. Furthermore, $\mathbf{H}(1 + i\pi) = -1/(e+1) < 0$, and thus, $\mathbf{H}$ is not positive real. We show that however A is dissipative and $C = B^*$. We have

$$D(A) = \{z \in H^1(0,1) : z(0) = z(1)\}, \quad Az = z',$$

and so,

$$\mathrm{Re}\langle Az, z\rangle = \mathrm{Re}\langle z', z\rangle = \frac{1}{2}|z(1)|^2 - \frac{1}{2}|z(0)|^2 = 0 \quad \forall\, z \in D(A),$$

showing that A is dissipative. The observation operator is given by $Cz = z(0)$. By [43, Lemma 6.2.14], the adjoint of the control operator can be calculated as the observation operator of S^*. A routine calculation shows that

$$D(S^*) = \left\{ \begin{pmatrix} z \\ v \end{pmatrix} \in \begin{pmatrix} H^1(0,1) \\ \mathbb{C} \end{pmatrix} : z(0) - z(1) = v \right\}, \quad S^* \begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} -z' \\ z(1) \end{pmatrix}.$$

The observation operator of S^* equals $B^* z = z(1)$. We also see that $D(A) = D(A^*)$ and that $A^* = -A$. For $z \in D(A)$ we have $z(0) = z(1)$ and therefore $B^* = C$ as operators on $D(A) = D(A^*)$.

For the minimal compatibility space we have (see [43, Section 5.2]) $W_{\min} = H^1(0,1)$. We can choose $C|_W$ as $C|_W z = z(0)$. As calculated above, we have that $(sI - A|_X)^{-1} Bv$ equals $x \mapsto e^{sx} v/(e^s - 1)$, so that $(sI - A|_X)^{-1} B$ maps U into $W_{\min}$, showing that the system node is indeed compatible. The corresponding feedthrough operator D satisfies $D = 0 = D_{\mathbf{H}}$, as desired. $\diamondsuit$

Notwithstanding the above example, the next result shows that Corollary 7.4 may be generalized to system nodes. Note that Corollary 7.4 can be obtained as a special case of statements (1) and (2) of the theorem below. It is convenient to define

$$J := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \in \mathcal{L}\left(\begin{pmatrix} X \\ U \end{pmatrix} \right),$$

a self-adjoint operator (sometimes referred to as signature operator).

Theorem 7.8 *Let S be a system node on (U, X, U) with transfer function $\mathbf{H}$. The following statements hold.*

(1) *If* $\operatorname{Re}\langle J\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right), S\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right)\rangle \leq 0$ *for all* $\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S)$, *then* $\mathbf{H}$ *is positive real.*
(2) *If there exists* $\varepsilon > 0$ *such that* $\operatorname{Re}\langle J\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right), S\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right)\rangle \leq -\varepsilon\|z\|^2$ *for all* $\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S)$, *then* $\mathbf{H}$ *is strictly positive real.*
(3) *If there exists* $\varepsilon > 0$ *such that* $\operatorname{Re}\langle J\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right), S\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right)\rangle \leq -\varepsilon\|v\|^2$ *for all* $\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S)$, *then* $\mathbf{H}$ *is strongly positive real.*
(4) *If there exists* $\varepsilon > 0$ *such that* $\operatorname{Re}\langle J\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right), S\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right)\rangle \leq -\varepsilon(\|z\|^2 + \|v\|^2)$ *for all* $\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S)$, *then* $\mathbf{H}$ *is strictly and strongly positive real.*

Proof Let $\gamma, \delta \geq 0$ and assume that

$$\operatorname{Re}\left\langle J\begin{pmatrix} z \\ v \end{pmatrix}, S\begin{pmatrix} z \\ v \end{pmatrix}\right\rangle \leq -\gamma\|z\|^2 - \delta\|v\|^2 \quad \text{for all } \begin{pmatrix} z \\ v \end{pmatrix} \in D(S). \tag{7.3}$$

Note that the assumptions imposed in statements (1)–(4) are all special cases of (7.3): for example, the assumptions imposed in statements (2) and (3) correspond to the cases wherein $(\gamma, \delta) = (\varepsilon, 0)$ and $(\gamma, \delta) = (0, \varepsilon)$, respectively.

Let $z \in D(A)$. Evaluating (7.3) with $v = 0$ shows that $\operatorname{Re}\langle Az, z\rangle \leq -\gamma\|z\|^2$. From this we obtain that $\gamma I + A$ is dissipative which implies that $\mathbb{C}_{-\gamma} \subset \rho(A)$ and that $\mathbf{H} \in \mathcal{H}_{-\gamma}(\mathcal{L}(U))$.

Let $v \in U$ and $s \in \mathbb{C}_{-\gamma}$. Define $z := (sI - A)^{-1}Bv$. Then $\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S)$ and

$$\operatorname{Re}\left\langle J\begin{pmatrix} z \\ v \end{pmatrix}, S\begin{pmatrix} z \\ v \end{pmatrix}\right\rangle = \operatorname{Re}(s)\,\|(sI - A)^{-1}Bv\|^2 - \operatorname{Re}\langle\mathbf{H}(s)v, v\rangle.$$

From (7.3) we then obtain, for all $v \in U$ and $s \in \mathbb{C}_{-\gamma}$,

$$\operatorname{Re}(s)\,\|(sI - A)^{-1}Bv\|^2 - \operatorname{Re}\langle\mathbf{H}(s)v, v\rangle \leq -\gamma\|z\|^2 - \delta\|v\|^2,$$

which can be re-arranged to arrive at

$$\operatorname{Re}\langle\mathbf{H}(s)v, v\rangle \geq (\operatorname{Re}(s) + \gamma)\,\|(sI - A)^{-1}Bv\|^2 + \delta\|v\|^2 \quad \forall\, v \in U, \forall\, s \in \mathbb{C}_{-\gamma}.$$

Statements (1)–(4) now follow by choosing, respectively, $\gamma = \delta = 0$, $(\gamma, \delta) = (\varepsilon, 0)$, $(\gamma, \delta) = (0, \varepsilon)$ and $\gamma = \delta = \varepsilon$. $\square$

We provide some commentary on the above theorem.

Remark 7.9 (a) Theorem 7.8 shows that certain dissipativity properties of JS guarantee positive realness properties of the transfer function $\mathbf{H}$ of the system node S. Statement (1) of Theorem 7.8, which also appears in [41, Theorem 4.2], is reminiscent of one direction of what is known in the finite-dimensional setting as the Kalman-Yakubovich-Popov (or positive real) lemma, see, for instance, [1,10,13,21,22]. Whilst Theorem 7.8 is not deep, it is nevertheless useful because it provides a sufficient condition for (strict, strong) positive realness which may be checked in the context of physically motivated PDE examples. Indeed, the analysis of such systems often benefits from dissipativity properties with respect to "energy" norms and, in this context, (7.3) should be interpreted accordingly.

By way of comparison, if a finite-dimensional, continuous-time, linear control system is specified by the operators (A, B, C, D) with A dissipative, $B = C^*$ and $-D$ dissipative, then the dissipativity assumption in statement (1) holds, as

$$0 \succeq \operatorname{Re}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A + A^* & B - C^* \\ B^* - C & -(D + D^*) \end{pmatrix}.$$

(b) Partial converses of statement (1) of Theorem 7.8 have appeared in [41, Theorem 4.2] and [42, Theorem 4.1]. These references address the problem from a time-domain perspective, and [5, Theorem 5.4] focuses on contractive transfer functions. We emphasize that the dissipativity of JS guaranteed by these results is with respect to an inner product which is not necessarily equivalent to the "natural" inner product on $X \times U$. $\Diamond$

We illustrate Theorem 7.8 by a modified version of Example 7.7.

Example 7.10 Consider the first order hyperbolic PDE from Example 7.7, but now with point observation at the right end:

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial w}{\partial x}(x,t), \quad w(1,t) - w(0,t) = u(t), \quad y(t) = w(1,t), \quad t > 0, \quad x \in (0,1).$$

The corresponding system node on (U, X, U), where $X = L^2(0,1)$ and $U = \mathbb{C}$, is given by

$$D(S) = \left\{ \begin{pmatrix} z \\ v \end{pmatrix} \in \begin{pmatrix} H^1(0,1) \\ \mathbb{C} \end{pmatrix} : z(1) - z(0) = v \right\}, \quad S \begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} z' \\ z(1) \end{pmatrix},$$

and so,

$$\operatorname{Re} \left\langle J \begin{pmatrix} z \\ v \end{pmatrix}, S \begin{pmatrix} z \\ v \end{pmatrix} \right\rangle = \operatorname{Re}\langle z, z' \rangle - \operatorname{Re}\langle z(1) - z(0), z(1) \rangle. \tag{7.4}$$

The right-hand side of (7.4) equals

$$\frac{1}{2}\left(|z(1)|^2 - |z(0)|^2\right) - \operatorname{Re}\langle z(1) - z(0), z(1)\rangle = -\frac{1}{2}|z(1) - z(0)|^2 = -\frac{1}{2}|v|^2.$$

Theorem 7.8 shows that the transfer function $\mathbf{H}$ is strongly positive real.

To calculate the transfer function, we consider $z := (sI - A|_X)^{-1}Bv$. Then

$$sz(x) = z'(x), \quad z(1) - z(0) = v,$$

which gives $z(x) = e^{sx}v/(e^s - 1)$. Since $C\&D \left(\begin{smallmatrix} z \\ u \end{smallmatrix} \right) = z(1)$, we have $\mathbf{H}(s) = e^s/(e^s - 1)$. This transfer function has feedthrough $D_{\mathbf{H}} = 1$.

As in Example 7.7, we have $W_{\min} := H^1(0,1)$ and similarly as in that example we see that the system node is compatible. With the choice $C|_{W_{\min}} z = z(0)$, we have that the corresponding feedthrough operator D equals one, as desired.

We note that A and $W_{\min}$ are the same in this example and Example 7.7 and that the observation operators from the two examples are the same on $W_{\min}$. However, the feedthrough operators are different.

Alternatively, in the present example, we could have chosen the feedthrough D to be zero (and therefore not equal to $D_{\mathbf{H}}$). The corresponding extended observation operator is given by $C|_{W_{\min}} z = z(1)$. Then we have a feedthrough operator which is the same as in Example 7.7, an observation operator which (on $D(A)$) is the same as is Example 7.7, but an extended observation operator $C|_{W_{\min}}$ which is not the same as in Example 7.7. $\diamondsuit$

We give several more examples of partial differential equations with positive-real transfer functions to further illustrate some of the results from Sections 3–6.

Example 7.11 Consider the heat equation from [16, Example 4.3.12]:

$$\left. \begin{array}{l} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), \quad \dfrac{\partial w}{\partial x}(1,t) = u(t), \quad \dfrac{\partial w}{\partial x}(0,t) = 0 \\ y(t) = w(1,t) \end{array} \right\} \quad t > 0, \; x \in (0,1).$$

Routine arguments show that the corresponding operator S given by

$$D(S) = \left\{ \begin{pmatrix} z \\ v \end{pmatrix} \in \begin{pmatrix} H^2(0,1) \\ \mathbb{C} \end{pmatrix} : z'(1) = v, z'(0) = 0 \right\}, \quad S \begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} z'' \\ z(1) \end{pmatrix}$$

is a system node on (U, X, U), where $X = L^2(0,1)$ and $U = \mathbb{C}$. We have

$$\operatorname{Re} \left\langle J \begin{pmatrix} z \\ v \end{pmatrix}, S \begin{pmatrix} z \\ v \end{pmatrix} \right\rangle = \operatorname{Re}\langle z'', z \rangle - \operatorname{Re}\langle z'(1), z(1) \rangle \quad \forall \left(\begin{smallmatrix} z \\ v \end{smallmatrix} \right) \in D(S),$$

and integration by parts shows that the right-hand side of the above identity is equal to

$$\operatorname{Re}\langle z(1), z'(1)\rangle - \operatorname{Re}\langle z(0), z'(0)\rangle - \|z'\|^2 - \operatorname{Re}\langle z'(1), z(1)\rangle = -\|z'\|^2 \leq 0.$$

Theorem 7.8 guarantees that the transfer function $\mathbf{H}$ is positive real. To calculate $\mathbf{H}$, set $z := (sI - A|_X)^{-1}Bv$ and note that

$$sz(x) = z''(x), \quad z'(1) = v, \quad z'(0) = 0\,,$$

which gives

$$z(x) = \frac{v\cosh(\sqrt{s}x)}{\sqrt{s}\sinh(\sqrt{s})}.$$

Since $C\&D\left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) = z(1)$, it follows that

$$\mathbf{H}(s) = \frac{1}{\sqrt{s}\tanh\left(\sqrt{s}\right)}.$$

It may appear that $\mathbf{H}$ has a branch point at 0; however, this is not the case. Indeed, consider the power series expansion of $\tanh(z)$ at zero which converges for $|z| < \frac{\pi}{2}$ and has only odd powers of z. It follows from this that $\sqrt{s}\tanh\sqrt{s}$ has a power series expansion at zero which converges for $|s| < \frac{\pi^2}{4}$. Therefore, $\mathbf{H}$ is meromorphic in the whole of $\mathbb{C}$, with poles at $-n^2\pi^2$ for $n \in \mathbb{N}_0$. It is clear that $\mathbf{H}$ is neither strictly nor strongly positive real. ♢

Example 7.12 Consider the following heat equation with control in a Robin boundary condition:

$$\left.\begin{aligned} \frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t), \quad \frac{\partial w}{\partial x}(1,t) + kw(1,t) = u(t), \quad \frac{\partial w}{\partial x}(0,t) = 0 \\ y(t) = w(1,t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \end{aligned}\right\} \quad t > 0,\ x \in (0,1)\,,$$

where $k > 0$. This controlled heat equation is obtained from the system in Example 7.11 by application of feedback with the operator $K \in \mathcal{L}(\mathbb{C})$ defined by $Ku := -ku$. Since K is strictly dissipative, we obtain from Proposition 6.1, Lemma 6.2 and Example 7.11 that the transfer function of the above Robin controlled heat equation is positive real. Moreover, we obtain from Theorem 6.4 that its transfer function is stable (that is, it belongs to $\mathcal{H}_0^\infty$). ♢

Example 7.13 Here we revisit Example 7.11, but now with a Dirichlet rather than Neumann boundary condition at zero and with non-negative feedthrough $D_{\mathbf{H}}$:

$$\left.\begin{aligned} \frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t), \quad \frac{\partial w}{\partial x}(1,t) = u(t), \quad w(0,t) = 0 \\ y(t) = w(1,t) + \kappa\frac{\partial w}{\partial x}(1,t) \qquad\qquad\qquad\qquad \end{aligned}\right\} \quad t > 0,\ x \in (0,1)\,,$$

where $\kappa \geq 0$. With $X = L^2(0,1)$ and $U = \mathbb{C}$, the above system corresponds to a system node S on (U, X, U) given by

$$D(S) = \left\{ \begin{pmatrix} z \\ v \end{pmatrix} \in \begin{pmatrix} H^2(0,1) \\ \mathbb{C} \end{pmatrix} : z'(1) = v, z(0) = 0 \right\}, \quad S\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} z'' \\ z(1) + \kappa z'(1) \end{pmatrix}.$$

Calculations similar to those in Example 7.11 lead to

$$\operatorname{Re}\left\langle J\begin{pmatrix} z \\ v \end{pmatrix}, S\begin{pmatrix} z \\ v \end{pmatrix} \right\rangle = -\|z'\|^2 - \kappa|v|^2 \quad \forall \left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S).$$

Since $z(0) = 0$ we have $\|z\|^2 \leq \|z'\|^2$, and so,

$$\operatorname{Re}\left\langle J\begin{pmatrix} z \\ v \end{pmatrix}, S\begin{pmatrix} z \\ v \end{pmatrix} \right\rangle \leq -\|z\|^2 - \kappa|v|^2 \quad \forall \left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S).$$

From Theorem 7.8 we now obtain that the transfer function $\mathbf{H}$ is strictly positive real, and furthermore, $\mathbf{H}$ is strongly positive real if $\kappa > 0$. Routine calculations give

$$\mathbf{H}(s) = \kappa + \frac{\tanh(\sqrt{s})}{\sqrt{s}},$$

showing in particular that $\mathbf{H}$ has feedthrough $D_{\mathbf{H}} = \kappa$. We see that $\mathbf{H}$ is in fact strongly positive real if, and only if, $\kappa > 0$. ♢

The next example involves an operator-valued transfer function.

Example 7.14 Consider the following heat equation on the square $\Omega := (0,1) \times (0,1)$:

$$\begin{aligned}
&\frac{\partial w}{\partial t}(x_1, x_2, t) = \frac{\partial^2 w}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 w}{\partial x_2^2}(x_1, x_2, t), \\
&w(0, x_2, t) = 0, \quad w(1, x_2, t) = 0, \\
&\frac{\partial w}{\partial x_2}(x_1, 0, t) = 0, \quad \frac{\partial w}{\partial x_2}(x_1, 1, t) = u(x_1, t), \\
&y(x_1, t) = w(x_1, 1, t).
\end{aligned}$$

Setting $X = L^2(\Omega)$ and $U = L^2(0,1)$, it follows from the standard theory of elliptic boundary value problems that this PDE system corresponds to the following system node S on (U, X, U):

$$\begin{aligned}
D(S) &= \left\{ \begin{pmatrix} z \\ v \end{pmatrix} \in \begin{pmatrix} H^2(\Omega) \\ L^2(0,1) \end{pmatrix} : \begin{matrix} z(0, x_2) = 0, & z(1, x_2) = 0 \\ \frac{\partial z}{\partial x_2}(x_1, 0) = 0, & \frac{\partial z}{\partial x_2}(x_1, 1) = v(x_1) \end{matrix} \right\}, \\
S \begin{pmatrix} z \\ v \end{pmatrix} &= \begin{pmatrix} \Delta z \\ z(\cdot, 1) \end{pmatrix}, \quad \text{where } \Delta \text{ is the Laplacian.}
\end{aligned}$$

Invoking Green's identity, we obtain

$$\begin{aligned}
\operatorname{Re} \left\langle J \begin{pmatrix} z \\ v \end{pmatrix}, S \begin{pmatrix} z \\ v \end{pmatrix} \right\rangle &= \operatorname{Re} \langle z, \Delta z \rangle_{L^2(\Omega)} - \operatorname{Re} \int_0^1 \frac{\partial z}{\partial x_2}(x_1, 1) \overline{z(x_1, 1)} \, dx_1 \\
&= -\|\nabla z\|^2_{L^2(\Omega)} \leq 0,
\end{aligned}$$

which holds for all $\left(\begin{smallmatrix} z \\ v \end{smallmatrix} \right) \in D(S)$. As a consequence, the transfer function $\mathbf{H}$ is positive real (where we have used, once again, Theorem 7.8).

We calculate $z := (sI - A|_X)^{-1} B v$ from

$$\begin{aligned}
sz(x_1, x_2) &= \frac{\partial^2 z}{\partial x_1^2}(x_1, x_2) + \frac{\partial^2 z}{\partial x_2^2}(x_1, x_2), \\
z(0, x_2) = 0, \quad z(1, x_2) = 0, \quad &\frac{\partial z}{\partial x_2}(x_1, 0) = 0, \quad \frac{\partial z}{\partial x_2}(x_1, 1) = v(x_1).
\end{aligned}$$

This problem can be solved by separation of variables, so we substitute $\varphi(x_1)\psi(x_2)$ for $z(x_1, x_2)$ and re-arrange to obtain

$$\frac{\varphi''}{\varphi} = s - \frac{\psi''}{\psi} = c,$$

for some constant c (which may depend on s). Solving

$$\varphi''(x_1) = c\varphi(x_1), \quad \varphi(0) = \varphi(1) = 0,$$

we see that $c = -n^2\pi^2$ with $n \in \mathbb{N}$ and $\varphi(x_1) = \sqrt{2} \sin(n\pi x_1)$. Next we solve

$$\psi''(x_2) = (s + n^2\pi^2)\psi(x_2), \quad \psi'(0) = 0,$$

to obtain $\psi_n(x_2) = c_n \cosh(x_2 \sqrt{s + n^2\pi^2})$, where c_n is a constant (depending on n and s). We then have

$$z(x, y) = \sum_{n=1}^{\infty} c_n \cosh(x_2 \sqrt{s + n^2\pi^2}) \sqrt{2} \sin(n\pi x_1)$$

and still need to satisfy the boundary condition $(\partial z / \partial x_2)(x_1, 1) = v(x_1)$. This leads to

$$\sum_{n=1}^{\infty} c_n \sqrt{s + n^2\pi^2} \sinh(\sqrt{s + n^2\pi^2}) \sqrt{2} \sin(n\pi x_1) = v(x_1).$$

We infer that $c_n\sqrt{s+n^2\pi^2}\,\sinh(\sqrt{s+n^2\pi^2})$ must equal the n-th Fourier sine coefficient of v. Therefore

$$c_n = \frac{\gamma_n(v)}{\sqrt{s+n^2\pi^2}\,\sinh(\sqrt{s+n^2\pi^2})},$$

where the linear functional γ_n is given by

$$\gamma_n(v) = \sqrt{2}\langle v, \sin(n\pi\,\cdot)\rangle_{L^2(0,1)} = \sqrt{2}\int_0^1 v(x_1)\sin(n\pi x_1)dx_1.$$

As consequence, we obtain for the transfer function $\mathbf{H}$,

$$\mathbf{H}(s)v = \sum_{n=1}^{\infty} \frac{\cosh(\sqrt{s+n^2\pi^2})\gamma_n(v)}{\sqrt{s+n^2\pi^2}\,\sinh(\sqrt{s+n^2\pi^2})}\,\sqrt{2}\sin(n\pi\,\cdot).$$

Note that this means that the Fourier sine coefficients of $\mathbf{H}(s)v$ are obtained by multiplication of the Fourier sine coefficients $\gamma_n(v)$ of v with

$$h_n(s) := \frac{1}{\sqrt{s+n^2\pi^2}\,\tanh(\sqrt{s+n^2\pi^2})}\,,$$

which itself is a positive-real function. $\Diamond$

Next we will be considering a wave equation equation example.

Example 7.15 Consider the wave equation

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial t^2}(x,t) &= \frac{\partial^2 w}{\partial x^2}(x,t), \quad w(0,t)=0, \quad \frac{\partial w}{\partial x}(1,t) = u(t) \\ y(t) &= \frac{\partial w}{\partial t}(1,t) \end{aligned}\right\} \quad t>0,\ x\in(0,1)\,.$$

Define $X := Z \times L^2(0,1)$, where $Z := \{z \in H^1(0,1) : z(0)=0\}$ with inner product

$$\langle z_1, z_2\rangle_Z := \langle z_1', z_2'\rangle_{L^2}.$$

Note that this inner product is equivalent to the standard inner product which Z inherits from $H^1(0,1)$. Setting $U=\mathbb{C}$, the above wave equation is described by the system node on (U,X,U) given by

$$\begin{aligned} D(S) &= \left\{ \begin{pmatrix} z_1 \\ z_2 \\ v \end{pmatrix} \in \begin{pmatrix} H^2(0,1) \\ H^1(0,1) \\ \mathbb{C} \end{pmatrix} : z_1(0)=0, z_1'(1)=v, z_2(0)=0 \right\}, \\ S\begin{pmatrix} z_1 \\ z_2 \\ v \end{pmatrix} &= \begin{pmatrix} z_2 \\ z_1'' \\ z_2(1) \end{pmatrix}. \end{aligned}$$

We note that the condition $z_2(0)=0$ which appears in the definition of the domain $D(S)$ corresponds to the additional boundary condition $(\partial w/\partial t)(0,t)=0$, which is a compatibility condition.

It follows from the definition of S that

$$\mathrm{Re}\left\langle J\begin{pmatrix} z \\ v \end{pmatrix}, S\begin{pmatrix} z \\ v \end{pmatrix}\right\rangle = \mathrm{Re}\langle z_1, z_2\rangle_Z + \mathrm{Re}\langle z_2, z_1''\rangle_{L^2} - \mathrm{Re}\langle z_1'(1), z_2(1)\rangle \ \forall \left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S).$$

Integration by parts shows that the right-hand side of the above equation is equal to

$$\mathrm{Re}\langle z_1, z_2\rangle_Z - \mathrm{Re}\langle z_2', z_1'\rangle_{L^2} + \mathrm{Re}\langle z_2(1), z_1'(1)\rangle - \mathrm{Re}\langle z_2(0), z_1'(0)\rangle - \mathrm{Re}\langle z_1'(1), z_2(1)\rangle.$$

Using that $\langle z_1, z_2\rangle_{X_1} = \langle z_1', z_2'\rangle_{L^2}$, we see that the above expression equals zero, and hence,

$$\mathrm{Re}\left\langle J\begin{pmatrix} z \\ v \end{pmatrix}, S\begin{pmatrix} z \\ v \end{pmatrix}\right\rangle = 0.$$

Consequently, by Theorem 7.8, the transfer function $\mathbf{H}$ is positive real.

To obtain a formula for $\mathbf{H}$, we calculate $z := (sI - A|_X)^{-1}Bv$ from

$$s^2 z_1(x) = z_1''(x), \quad z_1(0) = 0, \quad z_1'(1) = v, \quad z_2(x) = s z_1(x)\,,$$

which gives

$$z_1(x) = \frac{v \sinh(sx)}{s \cosh(s)}.$$

Since $C\&D \left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) = z_2(1)$, we conclude that $\mathbf{H}(s) = \tanh(s)$. Noting that $\mathbf{H}(0) = 0$ and that $\mathbf{H}$ has poles at $i\pi(2n+1)/2$ for $n \in \mathbb{Z}$, we see that $\mathbf{H}$ is neither strongly nor strictly positive real. $\Diamond$

Example 7.16 Consider the following wave equation

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial t^2}(x,t) &= \frac{\partial^2 w}{\partial x^2}(x,t), \quad w(0,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) + k\frac{\partial w}{\partial t}(1,t) = u(t) \\ y(t) &= \frac{\partial w}{\partial t}(1,t) \end{aligned}\right\} \quad t > 0,\ x \in (0,1)\,.$$

where $k > 0$. This system is obtained from that in Example 7.15 by feedback with the operator $K \in \mathcal{L}(\mathbb{C})$ defined by $Ku := -ku$. Since K is strictly dissipative, we obtain from Proposition 6.1 and Lemma 6.2 that the transfer function of the above wave equation is positive real. Moreover, Theorem 6.4 ensures that its transfer function is stable (that is, belongs to $\mathcal{H}_0^\infty$). $\Diamond$

Example 7.17 Consider the overdamped wave equation

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial t^2}(x,t) &= \frac{\partial^2 w}{\partial x^2}(x,t) + \frac{\partial^3 w}{\partial x^2 \partial t}(x,t), \quad w(0,t) = 0 \\ &\frac{\partial w}{\partial x}(1,t) + \frac{\partial^2 w}{\partial x \partial t}(1,t) = u(t) \\ y(t) &= \frac{\partial w}{\partial t}(1,t) \end{aligned}\right\} \quad t > 0,\ x \in (0,1)\,.$$

Set $U := \mathbb{C}$ and $X := Z \times L^2(0,1)$, where Z is defined as in Example 7.15. The above wave equation is described by the following system node on (U, X, U):

$$D(S) = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ v \end{pmatrix} \in \begin{pmatrix} H^1(0,1) \\ H^1(0,1) \\ \mathbb{C} \end{pmatrix} : \begin{array}{l} z_1 + z_2 \in H^2(0,1), \\ z_1(0) = 0,\ z_1'(1) + z_2'(1) = v, \\ z_2(0) = 0 \end{array} \right\},$$

$$S \begin{pmatrix} z_1 \\ z_2 \\ v \end{pmatrix} = \begin{pmatrix} z_2 \\ z_1'' + z_2'' \\ z_2(1) \end{pmatrix}.$$

Calculations similar to those in Example 7.15 yield

$$\operatorname{Re} \left\langle J \begin{pmatrix} z \\ v \end{pmatrix}, S \begin{pmatrix} z \\ v \end{pmatrix} \right\rangle = -\|z_2'\|_{L^2}^2 \le 0 \quad \forall \left(\begin{smallmatrix} z \\ v \end{smallmatrix}\right) \in D(S),$$

and hence, by Theorem 7.8, the transfer function $\mathbf{H}$ is positive real. As before, $\mathbf{H}$ can be determined via calculation of $z = (sI - A|_X)^{-1}Bv$ and we obtain

$$\mathbf{H}(s) = \frac{\tanh(\frac{s}{\sqrt{s+1}})}{\sqrt{s+1}}.$$

As in Example 7.11, $\mathbf{H}$ does not have a branch point at -1. Points for which $\frac{s}{\sqrt{s+1}} = i\pi\frac{2n-1}{2}$ with $n \in \mathbb{Z}$ (the poles of tanh) are the poles of $\mathbf{H}$. Consequently,

$$\frac{-\pi \pm i\sqrt{16\pi - \pi^2}}{8}, \quad \frac{-\pi(2n-1)^2 - \sqrt{\pi^2(2n-1)^4 - 16\pi(2n-1)}}{8}, \ n \ge 2$$

are poles of $\mathbf{H}$, as are,

$$s_n := \frac{-2\pi(2n-1)^2}{\pi(2n-1)^2 + \sqrt{\pi^2(2n-1)^4 - 16(2n-1)^2}}, \quad n \geq 2.$$

Since $\lim_{n\to\infty} s_n = -1$, we see that -1 is an accumulation point of poles. Furthermore, $s_n < -1$ for all $n \geq 2$, and so $\mathbf{H} \in \mathcal{H}^*_{-1}$, but $\mathbf{H} \notin \mathcal{H}^*_\alpha$ for any $\alpha < -1$. In particular, the transfer function $\mathbf{H}$ is not meromorphic on $\mathbb{C}$, but it is meromorphic on $\mathbb{C}_{-1}$. $\Diamond$

Finally, we consider an example of the feedback interconnection of a heat and a wave equation, both of which have positive-real transfer functions.

Example 7.18 Consider the following coupled heat-wave equation:

$$\left.\begin{array}{l} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial^2 z}{\partial x^2}(x,t), \quad \dfrac{\partial^2 w}{\partial t^2}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), \quad z(0,t) = 0, \quad w(0,t) = 0, \\ y_1(t) = -z(1,t) - \kappa \dfrac{\partial z}{\partial x}(1,t), \quad y_2(t) = \dfrac{\partial w}{\partial t}(1,t), \\ \dfrac{\partial z}{\partial x}(1,t) = y_2(t) + v_1(t), \quad \dfrac{\partial w}{\partial x}(1,t) = y_1(t) + v_2(t), \end{array}\right\} \quad t > 0,\ x \in (0,1)\,,$$

where $\kappa \geq 0$. This is the feedback interconnection of the heat equation from Example 7.13 (with output multiplied by -1) and the wave equation from Example 7.15, with respective input/output pairs (v_1, y_1) and (v_2, y_2). The respective transfer functions are given by

$$\mathbf{K}_\kappa(s) := -\kappa - \frac{\tanh(\sqrt{s})}{\sqrt{s}} \quad \text{and} \quad \mathbf{H}(s) := \tanh(s)\,.$$

Since $\mathbf{H}$ is positive real, $\mathbf{K}_\kappa \in \mathcal{H}^\infty_0$ and $-\mathbf{K}_\kappa$ is strongly positive real for every $\kappa > 0$, it follows from Theorem 6.16 that the transfer function

$$\mathbf{H}(s)(I - \mathbf{K}_\kappa(s)\mathbf{H}(s))^{-1} = \frac{\tanh(s)}{1 + \frac{\tanh(\sqrt{s})}{\sqrt{s}}\tanh(s) + \kappa\tanh(s)}\,,$$

is in $\mathcal{H}^\infty_0$, provided that $\kappa > 0$. Since $\mathbf{K}_\kappa \in \mathcal{H}^\infty_0$, the coupled heat-wave equation is stable for $\kappa > 0$ (in the sense that all four transfer functions are in $\mathcal{H}^\infty_0$). If $\kappa = 0$, then, akin to Example 6.19, it can be shown that the feedback interconnection is not stable. $\Diamond$

8 Conclusion

A general class of irrational and operator-valued transfer functions has been considered, with a particular focus on the positive-realness property, and its relation to stabilization by output feedback. The main result in Section 3, Theorem 3.7, gives a characterization of positive realness in terms of imaginary axis data and provides a clear-cut generalization of the well-known rational case. Section 4 introduces stronger notions of positive realness, namely strict and strong positive realness, and Theorem 4.4 describes relationships between the two. In Section 5, we discuss admissible and stabilizing feedback operators, generalizing the formulation of [51] and "preparing the way" for our main results in Section 6 — relationships between positive realness and stabilization by output feedback. We highlight here Theorem 6.3, which contains a characterization of positive realness in terms of mixed imaginary axis conditions and stabilizability properties, and Theorem 6.4, which states that $\mathbf{H}$ being positive real is equivalent to the condition that every strictly dissipative output feedback operator is a stabilizing feedback operator for $\mathbf{H}$. Furthermore, given a transfer function $\mathbf{H}$ and feedback operators K_1 and K_2, Theorem 6.8 shows that the function $(I - K_2\mathbf{H})(I - K_1\mathbf{H})^{-1}$ is positive real if, and only if, every operator K in the sector defined by K_1 and K_2 is a stabilizing feedback operator for $\mathbf{H}$. The necessity direction of this result is reminiscent of the well-known circle criterion. We note that in less general contexts, the circle criterion is usually formulated for nonlinear control systems (guaranteeing global stability for all nonlinear static locally Lipschitz functions satisfying a sector condition determined by K_1 and K_2) and we remark that suitable extensions of Theorem 6.8 and its corollaries to the nonlinear case are in preparation [20]. Finally, we would like to highlight Theorem 6.16 which shows that positive realness of a transfer function $\mathbf{H}$ is equivalent to $\mathbf{H}$ being stabilized by every stable transfer function $\mathbf{K}$ such that $-\mathbf{K}$ is strongly positive real. Establishing alternative sufficient conditions for the stability of the feedback interconnection of two transfer functions $\mathbf{H}$ and $\mathbf{K}$ in $\mathcal{H}_0(\mathcal{L}(U))$ in terms of positive-real type properties of $\mathbf{H}$ and $-\mathbf{K}$ seems an interesting problem for future work (see also [56, Theorem 4.2] which has some overlap with Theorem 6.16). We feel that the theory developed in the current paper is likely to be useful in this context.

9 Appendix

Proof of Lemma 2.2. We first prove the following lemma.

Lemma 9.1 *If* $S \in \mathcal{L}(U)$ *is such that* $\operatorname{Re} S \succeq 0$*, then* $I + S$ *is invertible.*

Proof Assume that $\operatorname{Re} S \succeq 0$. Noting that

$$\|(I+S)u\|^2 = \|u\|^2 + 2\langle \operatorname{Re} Su, u\rangle + \|Su\|^2 \geq 1 \quad \forall\, u \in \mathbb{E}_U,$$

it follows that $I + S$ is bounded away from 0. Now, since $\operatorname{Re} S^* = \operatorname{Re} S$, we have $\operatorname{Re} S^* \succeq 0$, and, by replacing in the above argument S by S^*, we see that $I + S^*$ is bounded away from 0. Consequently, $I + S$ and $(I + S)^*$ are both bounded away from 0 and therefore $I + S$ is invertible (see [33, Proposition 3.2.6]). □

Proof of Lemma 2.2 We start by noting that

$$2 \operatorname{Re} S \succeq (1-\delta^2)(1+\delta^2)^{-1}\big(I + S^*S\big) \tag{9.1}$$

is equivalent to

$$(1+\delta^2)(S+S^*) - (1-\delta^2)(I+S^*S) \succeq 0, \tag{9.2}$$

which in turn is equivalent to

$$\delta^2(I+S^*)(I+S) - (I-S^*)(I-S) \succeq 0. \tag{9.3}$$

Obviously, (9.1) implies that $\operatorname{Re} S \succeq 0$, and so, by Lemma 9.1, $I+S$ is invertible, and hence, the operator $I + S^*$ is also invertible. Consequently, (9.3) is equivalent to

$$\delta^2 I - (I+S^*)^{-1}(I-S^*)(I-S)(I+S)^{-1} \succeq 0. \tag{9.4}$$

Moreover, (9.4) is equivalent to

$$\begin{aligned}\delta^2 &\geq \langle (I+S^*)^{-1}(I-S^*)(I-S)(I+S)^{-1}u, u\rangle \\ &= \langle (I-S)(I+S)^{-1}u, (I-S)(I+S)^{-1}u\rangle \quad \forall\, u \in \mathbb{E}_U,\end{aligned} \tag{9.5}$$

or, equivalently,

$$\|(I-S)(I+S)^{-1}\| \leq \delta. \tag{9.6}$$

The claim now follows since the inequalities (9.1)–(9.6) are all equivalent. □

Continuity of the function defined by (4.6). Here we show that the function $h : \mathbb{C}_\gamma \to \mathbb{R}_+$ defined by (4.6) is continuous.[4] To this end, let $s \in \mathbb{C}_\gamma$ and let (s_n) be sequence in $\mathbb{C}_\gamma$ such that $s_n \to s$ as $n \to \infty$. We note that

$$|\langle \operatorname{Re} \mathbf{H}(s_n)u, u\rangle - \langle \operatorname{Re} \mathbf{H}(s)u, u\rangle| \leq \|\operatorname{Re} \mathbf{H}(s_n) - \operatorname{Re} \mathbf{H}(s)\| \;\; \forall\, n \in \mathbb{N}, \forall\, u \in \mathbb{E}_U. \tag{9.7}$$

[4] Actually, for the purpose of proving Theorem 4.4, it would be sufficient to show that h is lower semicontinuous (and hence, $-h$ is upper semicontinuous), since that is all that is needed for the mean-value characterization of subharmonic functions.

We now choose $u_n \in \mathbb{E}_U$ such that

$$\delta_n := \langle \operatorname{Re} \mathbf{H}(s_n) u_n, u_n \rangle - h(s_n) \to 0 \quad \text{as } n \to \infty.$$

By (9.7),

$$r_n := \langle \operatorname{Re} \mathbf{H}(s_n) u_n, u_n \rangle - \langle \operatorname{Re} \mathbf{H}(s) u_n, u_n \rangle \to 0 \quad \text{as } n \to \infty.$$

Now, for all $n \in \mathbb{N}$,

$$h(s_n) = \langle \operatorname{Re} \mathbf{H}(s_n) u_n, u_n \rangle - \delta_n = \langle \operatorname{Re} \mathbf{H}(s) u_n, u_n \rangle + r_n - \delta_n,$$

and so, $h(s_n) \geq h(s) + r_n - \delta_n$, showing that

$$\liminf_{n\to\infty} h(s_n) \geq h(s). \tag{9.8}$$

Furthermore, we choose $v_n \in \mathbb{E}_U$ such that

$$\varepsilon_n := \langle \operatorname{Re} \mathbf{H}(s) v_n, v_n \rangle - h(s) \to 0 \quad \text{as } n \to \infty.$$

By (9.7),

$$q_n := \langle \operatorname{Re} \mathbf{H}(s_n) v_n, v_n \rangle - \langle \operatorname{Re} \mathbf{H}(s) v_n, v_n \rangle \to 0 \quad \text{as } n \to \infty.$$

Noting that, for all $n \in \mathbb{N}$,

$$h(s_n) \leq \langle \operatorname{Re} \mathbf{H}(s_n) v_n, v_n \rangle = \langle \operatorname{Re} \mathbf{H}(s) v_n, v_n \rangle + q_n,$$

we conclude that $h(s_n) \leq h(s) + q_n + \varepsilon_n$, which in turn implies

$$\limsup_{n\to\infty} h(s_n) \leq h(s). \tag{9.9}$$

Finally, we obtain from (9.8) and (9.9) that

$$\lim_{n\to\infty} h(s_n) = h(s),$$

proving the continuity of h. $\square$

Proof of Lemma 5.5. By hypothesis, there exists $\mu > 0$ such that

$$\|S_n^{-1}\| = \|S_n^{-*}\| \leq \frac{1}{\mu} \quad \forall\, u \in U,\ \forall\, n \in \mathbb{N}.$$

Furthermore, $\langle S_n^{-*} u, S_n u \rangle = \|u\|^2$ for all $u \in U$ and all $n \in \mathbb{N}$, and thus,

$$\|S_n u\| \geq \mu \|u\| \quad \forall\, u \in U,\ \forall\, n \in \mathbb{N},$$

which in turn implies that

$$\|Su\| \geq \mu \|u\| \quad \forall\, u \in U.$$

Similarly, the identity

$$\langle S_n^{-1} u, S_n^* u \rangle = \|u\|^2 \quad \forall\, u \in U,\ \forall\, n \in \mathbb{N}$$

can be used to show that

$$\|S^* u\| \geq \mu \|u\| \quad \forall\, u \in U.$$

Hence, S and S^* are both bounded away from 0 and therefore S is invertible (see [33, Proposition 3.2.6]). Finally, for every $u \in U$,

$$\|S_n^{-1} u - S^{-1} u\| \leq \frac{1}{\mu} \|u - S_n S^{-1} u\| \to 0 \quad \text{as } n \to \infty$$

and

$$\|S_n^{-*} u - S^{-*} u\| \leq \frac{1}{\mu} \|u - S_n^* S^{-*} u\| \to 0 \quad \text{as } n \to \infty,$$

completing the proof. $\square$

Proof of Lemma 5.8. Under the conditions of the lemma, it follows from [26, Theorem 1.9, Chapter VII] that the set Δ does not have any accumulation points in Ω. It remains to show that, if Δ is non-empty, then every point in Δ is a pole of $(I-\mathbf{F})^{-1}$ (and not an essential singularity). To this end, assume that $\Delta \neq \emptyset$ and let $p \in \Delta$. By [18, Lemma 4.3.3], $I-\mathbf{F}(s)$ is a Fredholm operator for all $s \in \Omega$. Choose an open neighborhood $\Pi \subset \Omega$ of p such that $I-\mathbf{F}(s)$ is invertible for all $s \in \Pi\backslash\{p\} =: \Pi^*$. Obviously, $\operatorname{index}(I-\mathbf{F}(s)) = 0$ for all $s \in \Pi^*$, and so, invoking [18, Theorem 4.3.11], we conclude that $\operatorname{index}(I-\mathbf{F}(p)) = 0$. An application of [18, Theorem 4.3.5] now shows that there exist closed subspaces U_0, U_1, V_0 and V_1 of U such that $\dim U_1 = \dim V_1 < \infty$, $U = U_0 \oplus U_1 = V_0 \oplus V_1$ and $I-\mathbf{F}(p)$ is of the form

$$I-\mathbf{F}(p) = \begin{pmatrix} F_0 & 0 \\ 0 & 0 \end{pmatrix} : U_0 \oplus U_1 \to V_0 \oplus V_1,$$

where $F_0 \in \mathcal{L}(U_0, V_0)$ is an isomorphism. Let $F_1 \in \mathcal{L}(U_1, V_1)$ be an isomorphism and define

$$T := \begin{pmatrix} F_0^{-1} & 0 \\ 0 & F_1^{-1} \end{pmatrix} : V_0 \oplus V_1 \to U_0 \oplus U_1.$$

Trivially, T is an isomorphism and

$$T(I-\mathbf{F}(p)) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

We now partition $T(I-\mathbf{F})$ as follows:

$$T(I-\mathbf{F}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are holomorphic functions defined on Ω with values in $\mathcal{L}(U_0)$, $\mathcal{L}(U_1, U_0)$, $\mathcal{L}(U_0, U_1)$ and $\mathcal{L}(U_1)$, respectively. It is clear that $\mathbf{A}(s)$ is invertible for all s in a sufficiently small neighborhood of p. Therefore, by suitably "shrinking" Π if necessary, we may assume that $\mathbf{A}(s)$ is invertible for all $s \in \Pi$. The Schur complement

$$\mathbf{S}(s) := \mathbf{D}(s) - \mathbf{C}(s)\mathbf{A}^{-1}(s)\mathbf{B}(s)$$

of $T(I-\mathbf{F}(s))$ is holomorphic on Π with values in $\mathcal{L}(U_1)$. Since $T(I-\mathbf{F}(s))$ is invertible for all $s \in \Pi^*$, the Schur complement $\mathbf{S}(s)$ is invertible for all $s \in \Pi^*$ and

$$\big[T(I-\mathbf{F}(s))\big]^{-1} = \begin{pmatrix} \mathbf{A}^{-1}(s)\big(I + \mathbf{B}(s)\mathbf{S}^{-1}(s)\mathbf{C}(s)\mathbf{A}^{-1}(s)\big) & -\mathbf{A}^{-1}(s)\mathbf{B}(s)\mathbf{S}^{-1}(s) \\ -\mathbf{S}^{-1}(s)\mathbf{C}(s)\mathbf{A}^{-1}(s) & \mathbf{S}^{-1}(s) \end{pmatrix} \quad \forall\, s \in \Pi^*, \qquad (9.10)$$

see [46, Proposition 1.6.2]. Since $\mathbf{S}(s)$ is invertible for all $s \in \Pi^*$, $\mathbf{S}$ is holomorphic on Π with values in $\mathcal{L}(U_1)$ and U_1 is finite dimensional, the function $\mathbf{S}^{-1}$ holomorphic on Π^* and the singularity at p is a pole. Finally, each of the functions $\mathbf{A}^{-1}$, $\mathbf{B}$ and $\mathbf{C}$ is holomorphic on Π, and thus it follows from (9.10) that p is a pole of $T(I-\mathbf{F})^{-1}$ and hence of $(I-\mathbf{F})^{-1}$. ☐

Proof of Lemma 6.2. Statement (1) follows immediately from Theorem 6.4. We proceed to establish statement (2). By hypothesis we have that $-K = -K^* \succeq 0$ and hence there exists a unique operator $S \in \mathcal{L}(U)$ such that $S = S^* \succeq 0$ and $S^2 = -K$ (that is, S is the square root of $-K$). Then, trivially, $I - K\mathbf{H} = I + S^2\mathbf{H}$ and thus, for every $s \in \mathbb{C}_0$, the operator $I - K\mathbf{H}(s)$ is invertible if, and only if, $I + S\mathbf{H}(s)S$ is invertible. But the function $s \mapsto S\mathbf{H}(s)S$ is positive real and the invertibility of $I + S\mathbf{H}(s)S$ for every $s \in \mathbb{C}_0$ is a consequence of statement (1) of Corollary 2.3.

Finally, to prove statement (3), assume that K is dissipative and an admissible feedback operator for $\mathbf{H}$ and $\dim U < \infty$. Let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$ as $n \to \infty$ and set $K_n := K - \varepsilon_n I$. Then K_n is strictly dissipative and Theorem 6.4 implies that

$$g_n(s) := \det(I - K_n\mathbf{H}(s)) \neq 0 \quad \forall\, s \in \mathbb{C}_0,\ \forall\, n \in \mathbb{N}. \qquad (9.11)$$

Seeking a contradiction, suppose that (6.1) does not hold. Then, setting $g(s) := \det(I - K\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$, there exists $s_0 \in \mathbb{C}_0$ such that $g(s_0) = 0$. Let $\Delta \subset \mathbb{C}_0$ be a closed disc centered at s_0 such that $g(s) \neq 0$ for all $s \in \Delta$, $s \neq s_0$ (the existence of such a disc follows from the admissibility of K).

The boundary of Δ is denoted by $\partial\Delta$ and we set $\mu := \inf_{s\in\partial\Delta} |g(s)| > 0$. The sequence of holomorphic functions g_n converges locally uniformly to g and so there exists $N \in \mathbb{N}$ such that

$$\sup_{s\in\partial\Delta} |g(s) - g_n(s)| < \mu \quad \forall\, n \geq N.$$

It now follows from Rouché's theorem [30, Theorem 5 in Chapter 5] that, for every $n \geq N$, the function g_n has a zero in the interior of Δ, contradicting (9.11). □

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