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# Discrete linear Weingarten surfaces

(F Burstall, U Hertrich-Jeromin & W Rossman)

**Abstract.** Discrete linear Weingarten surfaces in space forms are characterized as special discrete  $\Omega$ -nets, a discrete analogue of Demoulin’s  $\Omega$ -surfaces. It is shown that the Lie-geometric deformation of  $\Omega$ -nets descends to a Lawson transformation for discrete linear Weingarten surfaces, which coincides with the well-known Lawson correspondence in the constant mean curvature case.

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**Keywords.** linear Weingarten surface; hyperbolic space; de Sitter space; Lie sphere geometry; Legendre map; Demoulin’s surface;  $\Omega$ -surface; deformation; isothermic surface; Calapso transformation; Lawson correspondence; discrete isothermic net; mixed area; constant mean curvature; constant Gauss curvature.

## 1. Introduction

A Lie geometric approach to flat fronts in hyperbolic space and, more generally, (smooth) linear Weingarten surfaces in (Riemannian and Lorentzian) space forms was outlined in two short notes [7] and [8]. Apart from providing a unified treatment and a natural realm for a transparent analysis of the singularities of fronts, this Lie geometric approach also revealed a close relationship to the theory of isothermic surfaces: linear Weingarten surfaces in space forms are Lie-applicable,<sup>1)</sup> cf [1, §85] or [16]. In particular, non-tubular linear Weingarten surfaces envelop a pair of isothermic sphere congruences that separate the curvature sphere congruences harmonically, see [10; 11] and [1, §85], where each isothermic sphere congruence takes values in a linear sphere complex. Up to a mild genericity assumption this yields a characterization of linear Weingarten surfaces or, more generally, fronts, see [8].

As a main result of the present text we shall provide a similar characterization in the discrete case, see Thm 2.8, where discrete “linear Weingarten nets” are defined in terms of mixed areas, see Defs 2.3 and 2.4. This generalizes and unifies the rather different approaches to constant mean curvature nets of [6] and [4].

In the process, see Def 3.1, we introduce discrete  $\Omega$ -nets as a new class of integrable discrete surfaces. For this definition, we employ the new and geometrically somewhat obscure idea of Königs duality of suitable *homogeneous coordinates* of Königs nets in a projective space: classically, *affine projections* of Königs nets or surfaces in projective geometry admit duals — and are characterized by their existence, cf [2, Def 2.22] or [4, Sect 2]. However, this idea is motivated by observations in the smooth case.

A fact that sets our notion of discrete  $\Omega$ -nets apart from their smooth analogues is the existence of multiple pairs of enveloped isothermic sphere congruences, see Lemma 3.3. This hints strongly at the non-existence of a sensible notion of vertex curvature spheres for a discrete Legendre map (Def 2.1) or “principal contact element net” [2, Def 3.23]: as initial spheres for a pair of isothermic sphere congruences of an  $\Omega$ -net can be chosen arbitrarily in one contact element, no pair of geometrically defined sphere congruences will satisfy the aforementioned property of harmonic separation. Nevertheless, all isothermic sphere congruences of the family given in Lemma 3.3 are conformal in the sense that they share the same cross ratio function on faces — in the smooth case conformality of the induced metrics is intimately related to the harmonic separation property.

One merit of describing linear Weingarten surfaces or nets in the Lie geometric realm is the natural description of their transformations in terms of the transformations of their Legendre lifts:  $\Omega$ -nets come with their Lie geometric deformation, the Calapso deformation of Def 3.9, as well as with Darboux transformations, inherited by the corresponding transformations of the enveloped isothermic sphere congruences. These transformations give rise to Lawson transformations, see Def 4.1, and Bianchi-Bäcklund transformations of linear Weingarten nets.

The Lawson transformation will be discussed in detail in Sect 4 of the present text. In particular, we shall justify our terminology by showing that the Lawson transformation becomes the well-known Lawson correspondence in the case of constant mean curvature nets, see Example 4.2 and

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1) In fact, linear Weingarten surfaces in Riemannian space forms are special Guichard surfaces.

[6, Sect 5]. The Bianchi-Bäcklund transformation shall be discussed in a forthcoming paper.

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## 2. Discrete linear Weingarten surfaces in space forms

We aim to describe discrete linear Weingarten surfaces, defined in terms of mixed areas, cf [3, Def 8] and [4, Def 3.1], in Riemannian and Lorentzian space forms in a unified manner. To this end we consider the space form geometries as subgeometries of Lie sphere geometry: fix orthogonal vectors  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{4,2} \setminus \{0\}$  and let

$$\mathfrak{Q}^3 := \{\boldsymbol{\eta} \in \mathbb{R}^{4,2} \mid (\boldsymbol{\eta}\boldsymbol{\eta}) = 0, (\boldsymbol{\eta}\mathbf{q}) = -1, (\boldsymbol{\eta}\mathbf{p}) = 0\} \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $\mathbb{R}^{4,2}$ ;  $\langle \cdot, \dots, \cdot \rangle$  will denote the linear span of vectors. If  $(\mathbf{p}\mathbf{p}) \neq 0$  then  $\mathfrak{Q}^3$  is a 3-dimensional quadric of constant sectional curvature  $-(\mathbf{q}\mathbf{q})$ .

In this setting, the projective light cone or *Lie quadric*  $\mathcal{L}^4 := \{\langle \boldsymbol{\eta} \rangle \mid \boldsymbol{\eta} \in \mathbb{R}^{4,2}, (\boldsymbol{\eta}\boldsymbol{\eta}) = 0\} \subset \mathbb{P}(\mathbb{R}^{4,2})$  parametrises the set of oriented 2-spheres (thus, complete, totally umbilic hypersurfaces) in  $\mathfrak{Q}^3$  via

$$s \mapsto \mathfrak{Q}^3 \cap s^\perp.$$

In particular, for  $\boldsymbol{\eta} \in \mathfrak{Q}^3$ ,  $s = \langle \boldsymbol{\eta} \rangle$  corresponds to the point-sphere  $\{\boldsymbol{\eta}\}$  while, when  $s \in \mathcal{L}^4$  differs from its reflection  $s'$  in the hyperplane orthogonal to  $\mathbf{p}$ ,  $s, s'$  correspond to the same sphere but with opposite orientations.

In general, a non-zero point  $\boldsymbol{\xi} \in \mathbb{R}^{4,2}$  (or, more properly, a point  $\langle \boldsymbol{\xi} \rangle \in \mathbb{P}(\mathbb{R}^{4,2})$ ), defines the *linear sphere complex*  $\mathcal{L}^4 \cap \boldsymbol{\xi}^\perp$ , a 3-dimensional family of 2-spheres. In particular, taking  $\boldsymbol{\xi} = \mathbf{q}$  yields

$$\mathfrak{P}^3 := \{\boldsymbol{\eta} \in \mathbb{R}^{4,2} \mid (\boldsymbol{\eta}\boldsymbol{\eta}) = 0, (\boldsymbol{\eta}\mathbf{q}) = 0, (\boldsymbol{\eta}\mathbf{p}) = -1\}, \quad (2.2)$$

the complex of (spacelike if  $(\mathbf{p}\mathbf{p}) > 0$ ) hyperplanes (thus, complete, totally geodesic hypersurfaces) in the space form  $\mathfrak{Q}^3$ , cf [14, Sect 1.4].

Two oriented 2-spheres are in oriented contact if and only if the corresponding points of  $\mathcal{L}^4$  are orthogonal. It follows that lines in  $\mathcal{L}^4$  correspond to pencils of 2-spheres sharing a common contact element and so parametrise those contact elements. For more details, see Cecil [9, Chapter 1].

To make this approach more tangible, assume that  $(\mathbf{p}\mathbf{p}) = \mp 1$  and  $(\mathbf{q}\mathbf{q}) \neq 0$ . Now the constant offset

$$\{\mathbf{x} = \mathbf{f} + \frac{\mathbf{q}}{(\mathbf{q}\mathbf{q})} \mid \mathbf{f} \in \mathfrak{Q}^3\} \subset \langle \mathbf{p}, \mathbf{q} \rangle^\perp$$

yields the standard model of a space form as a (connected component of a) quadric in a 4-dimensional linear space with non-degenerate inner product, and the unit (timelike if  $(\mathbf{p}\mathbf{p}) = +1$ ) tangent space of the space form at  $\mathbf{x} = \mathbf{f} + \frac{\mathbf{q}}{(\mathbf{q}\mathbf{q})}$  becomes the constant offset

$$\{\mathbf{n} = \mathbf{t} + \frac{\mathbf{p}}{(\mathbf{p}\mathbf{p})} \mid \mathbf{t} \in \mathfrak{P}^3 \cap \langle \mathbf{f} \rangle^\perp\} \subset \langle \mathbf{p}, \mathbf{q} \rangle^\perp.$$

In the case  $(\mathbf{q}\mathbf{q}) = 0$  of a flat ambient space form geometry, the situation becomes slightly less obvious: here a choice of origin  $\mathbf{o} \in \mathfrak{Q}^3$  yields an identification via inverse stereographic projection,

$$\mathfrak{Q}^3 \ni \mathbf{f} = \mathbf{o} + \mathbf{x} + \frac{1}{2}(\mathbf{x}\mathbf{x})\mathbf{q} \leftrightarrow \mathbf{x} \in \langle \mathbf{o}, \mathbf{p}, \mathbf{q} \rangle^\perp \cong \begin{cases} \mathbb{R}^3 & \text{if } (\mathbf{p}\mathbf{p}) = -1, \\ \mathbb{R}^{2,1} & \text{if } (\mathbf{p}\mathbf{p}) = +1, \end{cases} \quad (2.3)$$

and  $\mathfrak{P}^3 \cap \langle \mathbf{f} \rangle^\perp$  becomes the unit (timelike in the Lorentzian case) tangent space of  $\mathbb{R}^3$  or  $\mathbb{R}^{2,1}$ , respectively, via

$$\mathfrak{P}^3 \cap \langle \mathbf{f} \rangle^\perp \ni \mathbf{t} = -\frac{\mathbf{p}}{(\mathbf{p}\mathbf{p})} + \mathbf{n} + (\mathbf{x}\mathbf{n})\mathbf{q} \leftrightarrow \mathbf{n} \in \langle \mathbf{o}, \mathbf{p}, \mathbf{q} \rangle^\perp. \quad (2.4)$$

Now consider a discrete principal (circular) net<sup>2)</sup>  $\mathbf{f} : \mathbb{Z}^2 \rightarrow \mathfrak{Q}^3$ , that is,  $\mathbf{f}$  has planar faces in  $\mathbb{R}^{4,2}$ , cf [2, Thm 3.9]. For non-degeneracy we assume that neither edges nor diagonals of  $\mathbf{f}$  be isotropic: if  $(ijkl)$  denotes an elementary quadrilateral of  $\mathbb{Z}^2$  then the vectors

$$d\mathbf{f}_{ij} := \mathbf{f}_j - \mathbf{f}_i \quad \text{and} \quad \delta\mathbf{f}_{ik} := \mathbf{f}_k - \mathbf{f}_i$$

2) For simplicity we restrict to  $\mathbb{Z}^2$  as a domain; throughout,  $\mathbb{Z}^2$  may be replaced by a (simply connected) quad-graph.

will be assumed to be non-null. In particular, any two or three vertices of a face of  $\mathfrak{f}$  span a 2- or 3-dimensional subspace of  $\mathbb{R}^{4,2}$ , respectively, with a non-degenerate induced inner product. Further, in order to be able to define the Gauß and mean curvatures via mixed areas below, we shall assume that the faces of  $\mathfrak{f}$  have non-parallel diagonals, so that their areas do not vanish.

Such a principal net admits a 2-parameter family of Gauß maps, that is, unit (timelike in the Lorentzian case) “normal” vector fields along  $\mathfrak{f}$  so that, for each edge  $(ij)$ , there is an edge curvature sphere  $\kappa_{ij}$  that is orthogonal to the “normal” vectors at the endpoints<sup>3)</sup>  $\mathfrak{f}_i$  and  $\mathfrak{f}_j$ , cf [2, Thm 3.36]. In our Lie geometric setup, a choice of Gauß map for the principal net  $\mathfrak{f}$  amounts to a choice of a “tangent plane” congruence  $\mathfrak{t} : \mathbb{Z}^2 \rightarrow \mathfrak{P}^3$  with  $\mathfrak{t} \perp \mathfrak{f}$ . This pair of maps gives rise to the Legendre lift (principal contact element net, cf [2, Def 3.23]) of a principal net in a space form with Gauß map:

**2.1 Def.** Let  $\mathfrak{f} : \mathbb{Z}^2 \rightarrow \Omega^3$  be a principal net in a quadric  $\Omega^3$  of constant sectional curvature with tangent plane congruence  $\mathfrak{t} : \mathbb{Z}^2 \rightarrow \mathfrak{P}^3$ ,  $\mathfrak{t} \perp \mathfrak{f}$ . The line congruence<sup>4)</sup>

$$\mathbb{Z}^2 \ni i \mapsto f_i := \langle \mathfrak{f}_i, \mathfrak{t}_i \rangle \subset \mathcal{L}^4$$

will be called the Legendre lift of the pair  $(\mathfrak{f}, \mathfrak{t})$  if adjacent lines  $f_i$  and  $f_j$  intersect;  $\kappa_{ij} := f_i \cap f_j$  is called the curvature sphere of  $f$  on the edge  $(ij)$ .

The pair  $(\mathfrak{f}, \mathfrak{t})$  will be called the space form projection of the Legendre map  $f$ .

We shall exclusively deal with pairs  $(\mathfrak{f}, \mathfrak{t}) : \mathbb{Z}^2 \rightarrow \Omega^3 \times \mathfrak{P}^3$  occurring as space form projections of Legendre maps. Note that, generically,<sup>5)</sup> any choice of a point sphere complex  $\mathfrak{p} \in \mathbb{R}^{4,2}$ ,  $(\mathfrak{p}\mathfrak{p}) \neq 0$ , and a space form vector  $\mathfrak{q} \in \mathbb{R}^{4,2} \setminus \{0\}$ ,  $(\mathfrak{q}\mathfrak{p}) = 0$ , gives rise to a space form projection  $(\mathfrak{f}, \mathfrak{t})$  of a given Legendre map  $f$ .

As the edge curvature sphere  $\kappa_{ij} \in \mathcal{L}^4$  is obtained as the intersection  $\kappa_{ij} = f_i \cap f_j$  of the lines of the Legendre lift of a principal net  $\mathfrak{f}$  with tangent plane congruence  $\mathfrak{t}$  at the endpoints of an edge  $(ij)$ , we may write

$$\kappa_{ij} = \mathfrak{t}_i + k_{ji}\mathfrak{f}_i = \mathfrak{t}_j + k_{ij}\mathfrak{f}_j \quad (2.5)$$

for (a lift of) the curvature sphere with suitable coefficients  $k_{ij}$  and  $k_{ji}$ . Now

$$k_{ji} = \frac{(\kappa_{ij}\mathfrak{q})}{(\kappa_{ij}\mathfrak{p})} = k_{ij},$$

showing that  $(ij) \mapsto k_{ij}$  is an *edge function*, that is, takes equal values for opposite orientations of an edge. This yields a notion of a *principal curvature function* on the edges of a principal net  $\mathfrak{f}$  in  $\Omega^3$  with tangent plane congruence  $\mathfrak{t}$ . Rewriting (2.5) as

$$0 = d\mathfrak{t}_{ij} + k_{ij}d\mathfrak{f}_{ij} \quad (2.6)$$

we obtain a Rodrigues’ type formula. Conversely, (2.6) implies that  $\mathfrak{f}$ , hence  $\mathfrak{t}$ , are conjugate nets in  $\mathbb{R}^{4,2}$  as long as the principal curvature function  $k$  is not constant around an elementary quadrilateral, that is, away from *umbilical faces*, where the curvature spheres of the four edges of a face coincide. Thus we obtain the following characterization:

**2.2 Lemma.** A space form projection  $(\mathfrak{f}, \mathfrak{t}) : \mathbb{Z}^2 \rightarrow \Omega^3 \times \mathfrak{P}^3$  of a Legendre map  $f$  is a pair of edge-parallel nets in  $\mathbb{R}^{4,2}$ . Conversely, if  $\mathfrak{f} : \mathbb{Z}^2 \rightarrow \Omega^3$  and  $\mathfrak{t} : \mathbb{Z}^2 \rightarrow \mathfrak{P}^3$  satisfy (2.6), then any non-umbilical face of  $\mathfrak{f}$  is planar and, away from umbilical faces,  $(\mathfrak{f}, \mathfrak{t})$  is the space form projection of a Legendre map  $f$ .

In particular, the faces of  $\mathfrak{f}$  and  $\mathfrak{t}$  lie in parallel planes so that the  $\Lambda^2\mathbb{R}^{4,2}$ -valued (mixed) area functions

$$A(\mathfrak{t}, \mathfrak{t})_{ijkl} = \frac{1}{2}\delta\mathfrak{t}_{ik} \wedge \delta\mathfrak{t}_{jl} \quad \text{and} \quad A(\mathfrak{f}, \mathfrak{t})_{ijkl} = \frac{1}{4}\{\delta\mathfrak{f}_{ik} \wedge \delta\mathfrak{t}_{jl} + \delta\mathfrak{t}_{ik} \wedge \delta\mathfrak{f}_{jl}\}$$

are multiples of  $A(\mathfrak{f}, \mathfrak{f})_{ijkl} = \frac{1}{2}\delta\mathfrak{f}_{ik} \wedge \delta\mathfrak{f}_{jl}$ ; note that  $A(\mathfrak{f}, \mathfrak{f}) \neq 0$  by our regularity assumption on  $\mathfrak{f}$ .

3) Note that, in contrast to the Euclidean case, the normal lines in  $\Omega^3$  defined by the Gauß map do not necessarily intersect: this is the case when the “curvature sphere”  $\kappa_{ij}$  is not a distance sphere in the ambient space form geometry.

4) Here, a “line congruence” just means a map into the space of lines in a projective space.

5) In the definite case  $(\mathfrak{p}\mathfrak{p}) < 0$ , the only obstruction is that the point sphere map may hit the infinity boundary of the space form; in the Lorentzian case, additional obstructions occur as a contact element may consist entirely of “point spheres”.

**2.3 Lemma & Def.** *There are two functions,  $H$  and  $K$ , defined on the faces<sup>6)</sup> of a space form projection of a Legendre map so that*

$$0 \equiv A(\mathfrak{f}, \mathfrak{t}) + H A(\mathfrak{f}, \mathfrak{f}) = A(\mathfrak{t}, \mathfrak{t}) - K A(\mathfrak{f}, \mathfrak{f}).$$

*These will be called the mean curvature and Gauß curvature of the pair  $(\mathfrak{f}, \mathfrak{t})$ , respectively.*

As the mixed areas are invariant under translation, the mean and Gauß curvatures defined here clearly coincide with those of [4, Def 3.1] in the case of a Riemannian ambient geometry.

To see that they coincide with the ones of [2, Def 4.45] and [3, Def 8] in the case of a principal net  $\mathfrak{r}$  in  $\mathbb{R}^3$  with (unit) Gauß map  $\mathfrak{n}$  we employ (2.3) and (2.4) to observe that the mixed areas of

$$\mathfrak{f} = \mathfrak{o} + \mathfrak{r} + \frac{1}{2}(\mathfrak{r}\mathfrak{r})\mathfrak{q} \quad \text{and} \quad \mathfrak{t} = \mathfrak{p} + \mathfrak{n} + (\mathfrak{r}\mathfrak{n})\mathfrak{q}$$

take values in  $\Lambda^2\langle\mathfrak{o}, \mathfrak{p}, \mathfrak{q}\rangle^\perp \oplus (\langle\mathfrak{o}, \mathfrak{p}, \mathfrak{q}\rangle^\perp \wedge \langle\mathfrak{q}\rangle)$  and that the mean and Gauß curvatures  $H$  and  $K$  are therefore determined by the  $\Lambda^2\langle\mathfrak{o}, \mathfrak{p}, \mathfrak{q}\rangle^\perp$ -parts  $A(\mathfrak{n}, \mathfrak{n})$ ,  $A(\mathfrak{r}, \mathfrak{n})$  and  $A(\mathfrak{r}, \mathfrak{r})$  of the mixed areas  $A(\mathfrak{t}, \mathfrak{t})$ ,  $A(\mathfrak{t}, \mathfrak{f})$  and  $A(\mathfrak{f}, \mathfrak{f})$ .

**2.4 Def.** *The space form projection  $(\mathfrak{f}, \mathfrak{t}) : \mathbb{Z}^2 \rightarrow \Omega^3 \times \mathfrak{P}^3$  of a Legendre map is called a linear Weingarten net if its mean and Gauß curvatures satisfy a non-trivial affine relation*

$$0 = \alpha K + 2\beta H + \gamma. \quad (2.7)$$

Note the symmetry of the situation: in case  $(\mathfrak{q}\mathfrak{q}) \neq 0$  we may interchange the geometric interpretations of  $\mathfrak{p}$  and  $\mathfrak{q}$ , thus swapping the roles of  $\mathfrak{f}$  and  $\mathfrak{t}$ . That is,  $\mathfrak{t}$  is interpreted as the principal net and  $\mathfrak{f}$  as its tangent plane congruence. As long as  $A(\mathfrak{t}, \mathfrak{t}) \neq 0$ , that is,  $K \neq 0$ , we obtain  $\frac{H}{K}$  and  $\frac{1}{K}$  as the mean and Gauß curvatures of the pair  $(\mathfrak{t}, \mathfrak{f})$ , which is therefore a linear Weingarten net also. In particular, if  $(\mathfrak{f}, \mathfrak{t})$  is a minimal net,  $H \equiv 0$ , then so is  $(\mathfrak{t}, \mathfrak{f})$ . In this case  $A(\mathfrak{f}, \mathfrak{t}) \equiv 0$  so that  $\mathfrak{f}$  and  $\mathfrak{t}$  are Königs dual nets in  $\mathbb{R}^{4,2}$ , cf [2, Def 2.22]:

**2.5 Def.** *Two discrete maps  $\sigma^\pm$  into an affine space are called Königs dual if they are edge-parallel and their opposite diagonals are parallel: for any edge  $(ij)$  and any elementary quadrilateral  $(ijkl)$*

$$d\sigma_{ij}^+ \parallel d\sigma_{ij}^- \quad \text{and} \quad \delta\sigma_{ik}^\pm \parallel \delta\sigma_{jl}^\mp.$$

Indeed,  $\delta\sigma_{ik}^+ \wedge \delta\sigma_{jl}^- = \delta\sigma_{ik}^- \wedge \delta\sigma_{jl}^+$  for any edge-parallel nets  $\sigma^+$  and  $\sigma^-$ , so that the vanishing of their mixed area,  $A(\sigma^+, \sigma^-)_{ijkl} \equiv 0$ , is readily seen to be equivalent to their opposite diagonals being parallel. Thus we obtain a characterization of minimal nets in space forms via Königs duality, cf [4]:

**2.6 Thm.** *The space form projection  $(\mathfrak{f}, \mathfrak{t}) : \mathbb{Z}^2 \rightarrow \Omega^3 \times \mathfrak{P}^3$  of a Legendre map is minimal if and only if  $\mathfrak{f}$  and  $\mathfrak{t}$  are Königs dual lifts in  $\mathbb{R}^{4,2}$  of nets in the Lie quadric.*

We aim to generalize this description for linear Weingarten nets. To this end, suppose that  $\sigma^\pm$  are Königs dual lifts of sphere congruences<sup>7)</sup>  $s^\pm$  spanning a Legendre map  $f$ ; further suppose that each sphere congruence takes values in a linear sphere complex  $\mathfrak{k}^\pm$ . Since  $\sigma^+$  and  $\sigma^-$  are edge-parallel nets and  $\sigma^\pm \perp \mathfrak{k}^\pm$ , the inner products  $(\sigma^\pm \mathfrak{k}^\mp) \equiv \text{const}$ . As long as these inner products do not vanish we may, without loss of generality, assume the same relative normalizations as for space form projections:<sup>8)</sup>

$$(\sigma^\pm \mathfrak{k}^\pm) = 0 \quad \text{and} \quad (\sigma^\pm \mathfrak{k}^\mp) = -1.$$

6) Note that these Gauß and mean curvatures do not depend on the orientation of an elementary quadrilateral  $(ijkl)$ : reversing the orientation, all (mixed) areas change sign so that the curvatures remain unaffected.

7) A sphere congruence is simply a map  $s : \mathbb{Z}^2 \rightarrow \mathcal{L}^4$  to the space of 2-spheres: the components  $\mathfrak{f}, \mathfrak{t}$  of a space-form projection of a Legendre map are examples.

8) We shall see below that we need to allow  $\sigma^\pm$  to become complex conjugate in order to capture general linear Weingarten nets: in this case  $\mathfrak{k}^\pm$  can be assumed to be complex conjugate as well and this relative normalization can be achieved while maintaining complex conjugacy since  $(\sigma^\pm \mathfrak{k}^\mp)$  are complex conjugate also.

Now  $\mathfrak{k}^\pm$  span a plane and choosing a point sphere complex  $\mathfrak{p}$  and a space form vector  $\mathfrak{q}$  for a space form projection  $(\mathfrak{f}, \mathfrak{t})$  of  $f$  in this plane,  $\langle \mathfrak{q}, \mathfrak{p} \rangle = \langle \mathfrak{k}^+, \mathfrak{k}^- \rangle$ , our relative normalizations control the relation between basis transformations: with  $B \in \text{Gl}(2)$  a change of basis

$$\langle \mathfrak{q}, \mathfrak{p} \rangle = \langle \mathfrak{k}^+, \mathfrak{k}^- \rangle B \text{ yields } (\sigma^-, \sigma^+) = (\mathfrak{f}, \mathfrak{t}) B^t. \quad (2.8)$$

As both the symmetric products on  $\langle \mathfrak{k}^-, \mathfrak{k}^+ \rangle$  and the mixed areas of pairs of edge-parallel nets spanning  $f$ , are symmetric bilinear forms they change in a similar way:

$$\begin{pmatrix} \mathfrak{q} \odot \mathfrak{q} & \mathfrak{q} \odot \mathfrak{p} \\ \mathfrak{p} \odot \mathfrak{q} & \mathfrak{p} \odot \mathfrak{p} \end{pmatrix} = B^t \begin{pmatrix} \mathfrak{k}^+ \odot \mathfrak{k}^+ & \mathfrak{k}^+ \odot \mathfrak{k}^- \\ \mathfrak{k}^- \odot \mathfrak{k}^+ & \mathfrak{k}^- \odot \mathfrak{k}^- \end{pmatrix} B \text{ and } \begin{pmatrix} A(\sigma^-, \sigma^-) & A(\sigma^-, \sigma^+) \\ A(\sigma^+, \sigma^-) & A(\sigma^+, \sigma^+) \end{pmatrix} = B \begin{pmatrix} A(\mathfrak{f}, \mathfrak{f}) & A(\mathfrak{f}, \mathfrak{t}) \\ A(\mathfrak{t}, \mathfrak{f}) & A(\mathfrak{t}, \mathfrak{t}) \end{pmatrix} B^t. \quad (2.9)$$

Thus if  $\sigma^\pm$  are Königs dual,  $A(\sigma^+, \sigma^-) \equiv 0$ , then the constructed space form projection  $(\mathfrak{f}, \mathfrak{t})$  is a linear Weingarten net:

$$\alpha A(\mathfrak{t}, \mathfrak{t}) - 2\beta A(\mathfrak{t}, \mathfrak{f}) + \gamma A(\mathfrak{f}, \mathfrak{f}) \equiv 0 \quad (2.10)$$

for suitable constants  $\alpha, \beta, \gamma \in \mathbb{R}$  which are determined from the basis representation of the symmetric bilinear form<sup>9)</sup>

$$W := 2\mathfrak{k}^- \odot \mathfrak{k}^+ = \frac{1}{2(\alpha\gamma - \beta^2)} \{ \alpha \mathfrak{q} \odot \mathfrak{q} + 2\beta \mathfrak{q} \odot \mathfrak{p} + \gamma \mathfrak{p} \odot \mathfrak{p} \}. \quad (2.11)$$

To see the converse we merely reverse this line of argument. Let  $(\mathfrak{f}, \mathfrak{t}) : \mathbb{Z}^2 \rightarrow \mathfrak{Q}^3 \times \mathfrak{P}^3$  be a linear Weingarten net, that is, its mixed areas satisfy a linear relation (2.10). We seek  $\mathfrak{k}^\pm$  satisfying (2.11), that is, factorizing

$$W := \alpha \mathfrak{q} \odot \mathfrak{q} + 2\beta \mathfrak{q} \odot \mathfrak{p} + \gamma \mathfrak{p} \odot \mathfrak{p}.$$

Clearly, this ambition is in vain if  $W$  does not have full rank, that is, if  $\alpha\gamma - \beta^2 = 0$ : as the sought-after  $\mathfrak{k}^\pm$  are linearly independent  $\mathfrak{k}^+ \odot \mathfrak{k}^-$  has rank 2. Thus we shall exclude this case from the investigation. The following terminology is chosen in analogy to the smooth case, where the linear Weingarten surfaces with  $\alpha\gamma = \beta^2$  are those with a constant principal curvature:

**2.7 Def.** A linear Weingarten net  $(\mathfrak{f}, \mathfrak{t}) : \mathbb{Z}^2 \rightarrow \mathfrak{Q}^3 \times \mathfrak{P}^3$  with  $\alpha\gamma - \beta^2 = 0$  will be called *tubular*.

In the non-tubular case  $\delta^2 := \beta^2 - \alpha\gamma \neq 0$  we can now solve the factorization problem up to order and (geometrically irrelevant) scaling, hence obtaining a pair of Königs dual lifts  $\sigma^\pm$  of sphere congruences  $s^\pm$ :

If  $\alpha \neq 0$  then  $\mathfrak{k}^+ \odot \mathfrak{k}^- = \frac{1}{4\alpha} W$  with  $\mathfrak{k}^\pm := \frac{1}{2\alpha} \{ (\alpha\mathfrak{q} + \beta\mathfrak{p}) \pm \delta\mathfrak{p} \}$  and  $\sigma^\pm = \mathfrak{f} \pm \frac{1}{\delta} (\beta\mathfrak{f} - \alpha\mathfrak{t})$  yield the sought-after Königs dual lifts of sphere congruences  $s^\pm := \langle \sigma^\pm \rangle : \mathbb{Z}^2 \rightarrow \mathcal{L}^4$ . By construction, the sphere congruences  $s^\pm$  take values in (different) linear sphere complexes  $\mathfrak{k}^\pm$ ,  $s^\pm \perp \mathfrak{k}^\pm$ . Note that  $\mathfrak{k}^\pm$  and  $\sigma^\pm$  become complex conjugate when  $\alpha\gamma - \beta^2 > 0$ .

If  $\alpha = 0$  then  $\beta \neq 0$  and  $\mathfrak{k}^+ \odot \mathfrak{k}^- = \frac{1}{2\beta} W$  with  $\mathfrak{k}^- := \mathfrak{p}$  and  $\mathfrak{k}^+ := \mathfrak{q} + \frac{\gamma}{2\beta} \mathfrak{p}$ . In this case, not too surprisingly, we recover the constant mean curvature net  $\sigma^- = \mathfrak{f}$  together with its *mean curvature sphere congruence*  $\sigma^+ = \mathfrak{t} + H\mathfrak{f}$ , cf [6, Def 5.1] or [4, Def 4.1], as a pair of enveloped sphere congruences with Königs dual lifts. Again,  $s^\pm$  take values in the linear sphere complexes given by  $\mathfrak{k}^\pm$ .

Thus we have proved:

**2.8 Thm.** The Legendre lift of a non-tubular linear Weingarten net is spanned by a pair of (possibly complex conjugate) sphere congruences  $s^\pm$  that admit Königs dual lifts. The sphere congruences take values in different linear sphere complexes  $\mathfrak{k}^\pm$ ,  $s^\pm \perp \mathfrak{k}^\pm$ .

Conversely, if  $f$  is a Legendre map spanned by a pair of sphere congruences  $s^\pm$  that admit Königs dual lifts  $\sigma^\pm$  and take values in different linear sphere complexes  $\mathfrak{k}^\pm$ , then any space form projection  $(\mathfrak{f}, \mathfrak{t})$  of  $f$  with  $\langle \mathfrak{q}, \mathfrak{p} \rangle = \langle \mathfrak{k}^+, \mathfrak{k}^- \rangle$  is a non-tubular linear Weingarten net.

In particular, as parallel nets in a space form are obtained from space form projections  $(\mathfrak{f}, \mathfrak{t})$  and  $(\tilde{\mathfrak{f}}, \tilde{\mathfrak{t}})$  of the same Legendre map  $f$  with respect to bases  $(\mathfrak{q}, \mathfrak{p})$  and  $(\tilde{\mathfrak{q}}, \tilde{\mathfrak{p}})$  that are related by an orthogonal transformation of their common plane, we have not too surprisingly also learned:

**2.9 Cor.** The parallel nets of a linear Weingarten net in a space form are linear Weingarten.

<sup>9)</sup> In the smooth case,  $W$  realizes the linear Weingarten condition as an orthogonality condition for the curvature spheres. Note that, when  $\mathfrak{k}^\pm$  are complex conjugate,  $W$  is real.

### 3. Discrete $\Omega$ -surfaces and their Calapso deformation

Recall that a discrete *Legendre map* is a line congruence  $\mathbb{Z}^2 \ni i \mapsto f_i \subset \mathcal{L}^4 \subset \mathbb{P}(\mathbb{R}^{4,2})$  so that adjacent lines share a (unique) *curvature sphere*,  $f_i \cap f_j = \kappa_{ij} \in \mathcal{L}^4$ , cf [2, Def 3.23]. In Thm 2.8 we have seen that non-tubular linear Weingarten nets lift to Legendre maps that are spanned by pairs of sphere congruences admitting Königs dual lifts, that is, they lift to  $\Omega$ -nets of Lie sphere geometry:

**3.1 Def.** *A discrete Legendre map is called a discrete  $\Omega$ -net if it is spanned by a pair of sphere congruences  $s^\pm : \mathbb{Z}^2 \rightarrow \mathcal{L}^4$  that admit Königs dual lifts  $\sigma^\pm : \mathbb{Z}^2 \rightarrow \mathbb{R}^{4,2}$ .*

For regularity we assume, as for the principal net  $\mathfrak{f}$  of a space form projection, that the spheres at different vertices of an elementary quadrilateral do not touch, that is,  $d\sigma^\pm$  and  $\delta\sigma^\pm$  never become isotropic. Hence the endpoints of an edge of  $s^\pm$  span a 2-dimensional Minkowski space and the vertices of any face of  $s^\pm$  span a 3-dimensional space with non-degenerate induced inner product. We shall also exclude umbilical faces, where the curvature spheres of the incident edges all coincide.

As an immediate consequence of this definition, the sphere congruences  $s^\pm$  are *Ribaucour sphere congruences* in the sense of [2, Def 3.27]: both sphere congruences  $s^\pm$  have planar faces.

Moreover,  $s^\pm : \mathbb{Z}^2 \rightarrow \mathcal{L}^4$  are *isothermic sphere congruences* as Königs nets in the Lie quadric.<sup>10)</sup> To see that  $s^\pm$  are indeed Königs nets in  $\mathbb{P}(\mathbb{R}^{4,2})$  we employ a characterization of Königs nets in terms of their diagonal vertex stars,<sup>11)</sup> cf [2, Thm 2.27]. Since the Königs dual lifts  $\sigma^\pm$  of  $s^\pm$  are Königs nets (in  $\mathbb{R}^{4,2}$ ) their diagonal vertex stars lie in 3-dimensional (affine) subspaces of  $\mathbb{R}^{4,2}$ . Consequently, the diagonal vertex stars of  $s^\pm$  span 4-dimensional (linear) subspaces of  $\mathbb{R}^{4,2}$ , hence lie in 3-dimensional (projective) subspaces of  $\mathbb{P}(\mathbb{R}^{4,2})$ . To summarise:

**3.2 Lemma.** *If  $f = s^- \oplus s^+$  is an  $\Omega$ -net spanned by a pair of sphere congruences  $s^\pm$  that admit Königs dual lifts, then  $s^\pm$  are isothermic and, in particular, Ribaucour sphere congruences.*

Below we shall see that the sphere congruences  $s^\pm$  come with a cross ratio factorizing edge-labelling as well as their respective isothermic loops of flat connections, cf [12, Def 4 and Prop 10] and [6, Def 2.1 and Lemma 2.5].

The definition of an  $\Omega$ -net aims to provide a discrete analogue of smooth  $\Omega$ -surfaces, the generic<sup>12)</sup> deformable surfaces of Lie geometry, see [1, §85] or [16]. In the smooth case, these come in two classes: the ones originally investigated by Demoulin [10; 11], which are given by a real pair of isothermic sphere congruences, and the ones where the enveloped isothermic sphere congruences become complex conjugate. In the discrete case, these two classes merge and Def 3.1 captures the entire class without the need to allow for complex conjugate pairs of sphere congruences:

**3.3 Lemma.** *Let  $f$  be an  $\Omega$ -net. Then, for any given pair of spanning spheres  $s_0^\pm \in f_0$  at an initial point  $0 \in \mathbb{Z}^2$ , there is a pair of isothermic sphere congruences  $s^\pm$  through  $s_0^\pm$  that admit Königs dual lifts and span  $f$ .*

To prove this lemma we investigate how to construct a new pair of isothermic sphere congruences from a given one: thus let  $f = s^+ \oplus s^-$  with a pair of isothermic sphere congruences  $s^\pm$  that have Königs dual lifts  $\sigma^\pm$ . Hence there is a real function  $r$  so that  $\sigma^\pm$  satisfy the Christoffel formula

$$d\sigma_{ij}^- = r_i r_j d\sigma_{ij}^+ \Leftrightarrow d\sigma_{ij}^+ = \frac{1}{r_i r_j} d\sigma_{ij}^- \quad (3.1)$$

and  $\mu^\pm = r^{\pm 1} \sigma^\pm$  are Moutard lifts of  $s^\pm$ , see [2, Thms 2.31 and 2.32].<sup>13)</sup> note that the Moutard

10) Note our somewhat unusual point of view: naturally, Königs nets form a class of nets in projective geometry as they can be characterized in terms of incidence relations, while the notion of their duality belongs to an affine subgeometry of the projective ambient geometry as it relies on a notion of parallelity. In contrast, we consider Königs duality of (not necessarily affine) lifts of Königs nets in the linear space of homogeneous coordinates of their ambient projective geometry.

11) Alternatively, planarity of intersection points of diagonals of adjacent faces could be employed, cf [2, Thm 2.26].

12) Excluding “ $\Omega_0$ -surfaces”, where the two isothermic sphere congruences coincide with one of the two curvature sphere congruences.

13) By the  $\pm$ -symmetry of (3.1) the two functions  $r^\pm$  obtained from [2, Thms 2.31 and 2.32] can be chosen to be reciprocal,  $r^\pm = r^{\pm 1}$  with a single function  $r$ .

equations [2, (2.44)] for  $\mu^\pm$  are nothing but the integrability conditions<sup>14)</sup> for (3.1),

$$(r_k^{\pm 1} - r_i^{\pm 1})\{r_l^{\pm 1}\sigma_l^\pm - r_j^{\pm 1}\sigma_j^\pm\} = (r_l^{\pm 1} - r_j^{\pm 1})\{r_k^{\pm 1}\sigma_k^\pm - r_i^{\pm 1}\sigma_i^\pm\}.$$

Now, for any two constants  $c^+ \neq c^-$ ,

$$\tilde{\sigma}^\pm := \frac{1}{r+c^\mp}\{\sigma^- + c^\pm r\sigma^+\} \quad (3.2)$$

yields Königs dual lifts of another pair of isothermic sphere congruences spanning the same  $\Omega$ -net: the fact that  $\tilde{\sigma}^\pm$  are edge-parallel,

$$(r_i + c^-)(r_j + c^-)d\tilde{\sigma}_{ij}^+ = (r_i + c^+)(r_j + c^+)d\tilde{\sigma}_{ij}^-,$$

hinges on the Christoffel equation (3.1), while the Königs duality of  $\tilde{\sigma}^\pm$ ,

$$\frac{(r_i+c^-)(r_k+c^-)}{r_k-r_i}\delta\tilde{\sigma}_{ik}^+ = \frac{(r_j+c^+)(r_l+c^+)}{r_l-r_j}\delta\tilde{\sigma}_{jl}^-,$$

follows from the Königs duality [2, (2.40)] of  $\sigma^\pm$  and the Moutard equations [2, (2.44)] for  $\mu^\pm$ ,

$$\frac{r_i^{\pm 1}r_k^{\pm 1}}{r_k^{\pm 1}-r_i^{\pm 1}}\delta\sigma_{ik}^\pm = \frac{1}{r_l^{\pm 1}-r_j^{\pm 1}}\delta\sigma_{jl}^\mp \quad \text{and} \quad \frac{1}{r_k^{\pm 1}-r_i^{\pm 1}}\delta\mu_{ik}^\pm = \frac{1}{r_l^{\pm 1}-r_j^{\pm 1}}\delta\mu_{jl}^\pm. \quad (3.3)$$

These computations also yield  $\tilde{r} := \frac{r+c^-}{r+c^+}$  as a rescaling for Moutard lifts  $\tilde{\mu}^\pm = \tilde{r}^{\pm 1}\tilde{\sigma}^\pm$  of the new isothermic sphere congruences  $\tilde{s}^\pm = \langle \tilde{\sigma}^\pm \rangle$ .

Thus, given an  $\Omega$ -net  $f = s^+ \oplus s^-$  in terms of a pair of isothermic sphere congruences  $s^\pm$  with Königs dual lifts  $\sigma^\pm$ , another such pair  $\tilde{s}^\pm$  can be constructed to pass through any two spanning spheres at a given initial point  $0 \in \mathbb{Z}^2$  by choosing the constants  $c^\pm$  appropriately. We have therefore proved Lemma 3.3.

Clearly, choosing  $c^\pm$  complex conjugate in (3.2), say  $c^\pm = \pm i$ , a complex conjugate pair

$$\tilde{\sigma}^\pm = \frac{1}{r \mp i}\{\sigma^- \pm i r \sigma^+\}$$

of Königs dual lifts in  $\mathbb{R}^{4,2} \otimes \mathbb{C} \cong \mathcal{C}^6$  is obtained from a real pair  $\sigma^\pm$ . To see that, conversely, a real pair can be obtained from a complex conjugate pair,  $\sigma^+ = \overline{\sigma^-}$ , first note that we can, without loss of generality, assume that  $|r|^2 \equiv 1$ : on any edge  $|r_i|^2|r_j|^2 = 1$  by (3.1) when  $\sigma^\pm$  are complex conjugate; hence  $|r|^2 \equiv 1$  as soon as the scaling of  $r$  is chosen so that  $|r_0|^2 = 1$  at some initial point  $0 \in \mathbb{Z}^2$ . Now we obtain a purely imaginary pair

$$\tilde{\sigma}^\pm = \frac{\pm i}{|r \mp i|^2}\{(\sigma^+ + \sigma^-) \pm i(r\sigma^+ - \bar{r}\sigma^-)\},$$

by choosing  $c^\pm = \pm i$  again. Hence  $\mp i\tilde{\sigma}^\pm$  define Königs dual lifts<sup>15)</sup> of a real  $\Omega$ -net and we conclude:

**3.4 Cor.** *Any  $\Omega$ -net can be spanned by pairs of isothermic sphere congruences with complex conjugate Königs dual lifts and, conversely, any such pair gives rise to an  $\Omega$ -net.*

Thus our Def 3.1 encompasses also the linear Weingarten nets with  $\alpha\gamma - \beta^2 > 0$  by Thm 2.8.

The transformations of the enveloped isothermic sphere congruences of an  $\Omega$ -net give rise to transformations of the net. As the isothermic transformation theory of discrete isothermic nets hinges on the isothermic loop of flat connections of the net which, in turn, depends on the cross ratio factorizing function of the net, we shall start by getting our hands on this function.

First observe that rewriting (3.1) we obtain (a lift of) the edge curvature sphere

$$\kappa_{ij} = r_i r_j \sigma_i^+ - \sigma_i^- = r_i r_j \sigma_j^+ - \sigma_j^- \in f_i \cap f_j \quad (3.4)$$

14) Having excluded umbilical faces we must have  $(r_k - r_i)(r_l - r_j) \neq 0$ .

15) Note that reciprocal constant rescaling of  $\sigma^\mp$  demands rescaling of  $r$  in (3.1) by the same factor. In the case at hand, the purely imaginary function  $\tilde{r}$  is turned into the real function  $i\tilde{r}$ .



of the Legendre map  $f$  on an edge  $(ij)$ , cf (2.5). Using that  $\kappa_{ij} \perp \sigma_i^\pm, \sigma_j^\pm$  we learn that

$$(\sigma_i^- \sigma_j^+) = (\sigma_i^+ \sigma_j^-) = (\mu_i^\pm \mu_j^\pm) =: a_{ij}. \quad (3.5)$$

Clearly,  $a$  is an edge function,  $a_{ij} = a_{ji}$ . Rearranging the Moutard equation for  $\mu^+$  from (3.3) suitably and taking norm squares,

$$\frac{\mu_k^+}{r_k - r_i} - \frac{\mu_l^+}{r_l - r_j} = \frac{\mu_i^+}{r_k - r_i} - \frac{\mu_j^+}{r_l - r_j} \Rightarrow \frac{a_{ij} - a_{kl}}{(r_k - r_i)(r_l - r_j)} = 0,$$

we also learn that  $a$  is an *edge-labelling* in the sense of [2, Def 4.4], that is, it is constant across opposite edges of faces,  $a_{ij} = a_{kl}$ , cf [2, Thm 4.29].

To see that  $a$  is indeed a cross ratio factorizing function, cf [14, §5.7.2] or [12, Prop 10], first note that the vertices of a face of either isothermic sphere congruence  $s^\pm$  lie on a conic in a projective plane since  $s^\pm$  are Ribaucour sphere congruences in the sense of [2, Def 3.27]. Fixing three points  $s_i^\pm, s_j^\pm$  and  $s_l^\pm$  of a face, the cross ratio  $q = [s_i^\pm, s_j^\pm, s_k^\pm, s_l^\pm] \in \mathbb{R} \cup \{\infty\}$  bijectively parametrizes the conic via

$$s_k^\pm = \langle \sigma_i^\pm + \frac{1}{(\sigma_j^\pm \sigma_l^\pm)} \{ (q-1)(\sigma_i^\pm \sigma_l^\pm) \sigma_j^\pm + (\frac{1}{q}-1)(\sigma_i^\pm \sigma_j^\pm) \sigma_l^\pm \} \rangle, \quad (3.6)$$

where  $\sigma^\pm$  is any lift of  $s^\pm$ , cf [12, (B.7)]<sup>16</sup> or [6, Sect 2.1]. It is now straightforward to verify that  $q = \frac{a_{ij}}{a_{jk}}$  as, for Moutard lifts  $\mu^\pm$  of  $s^\pm$  and taking inner products with  $\mu_j^\pm$  in (3.3),

$$\mu_k^\pm = \mu_i^\pm - \frac{a_{ij} - a_{jk}}{(\mu_j^\pm \mu_l^\pm)} \delta \mu_{jl}^\pm = \mu_i^\pm + \frac{r_k^{\pm 1} - r_i^{\pm 1}}{r_i^{\pm 1} - r_j^{\pm 1}} \delta \mu_{jl}^\pm. \quad (3.7)$$

In summary:

**3.5 Lemma.** *The edge-labelling  $a$  of (3.5) factorizes the cross ratios of faces of either isothermic sphere congruence,*

$$[s_i^\pm, s_j^\pm, s_k^\pm, s_l^\pm] = \frac{a_{ij}}{a_{jk}}.$$

*In particular, the edge-labelling  $a$  of (3.5) is, up to constant rescaling, a well defined Lie geometric invariant of each isothermic sphere congruence,  $s^+$  and  $s^-$ .*

We are now in a position to exploit the zero-curvature representation of discrete isothermic nets. We begin with a rapid review of the formalism of metric connections on the discrete vector bundle  $\mathbb{Z}^2 \times \mathbb{R}^{4,2} \rightarrow \mathbb{Z}^2$  cf [6, Def 2.4]: a metric connection  $\Gamma$  on  $\mathbb{Z}^2 \times \mathbb{R}^{4,2}$  assigns to each oriented edge  $(ij)$ , a linear isometry  $\Gamma_{ij} : \{j\} \times \mathbb{R}^{4,2} \rightarrow \{i\} \times \mathbb{R}^{4,2}$  such that  $\Gamma_{ji} = \Gamma_{ij}^{-1}$ , for all edges  $(ij)$ . In this context, a *gauge transformation* is a map  $i \mapsto T_i : \mathbb{Z}^2 \rightarrow \text{SO}(4,2)$  where we view  $T_i$  as a linear isometry of  $\{i\} \times \mathbb{R}^{4,2}$ . Gauge transformations  $T$  act on connections  $\Gamma$  by

$$(T\Gamma)_{ij} = T_i \Gamma_{ij} T_j^{-1}.$$

A connection  $\Gamma$  is *flat* if, on every elementary quadrilateral  $(ijkl)$ , we have

$$\Gamma_{ij} \Gamma_{jk} \Gamma_{kl} \Gamma_{li} = id \text{ or, equivalently, } \Gamma_{ij} \Gamma_{jk} = \Gamma_{il} \Gamma_{lk}.$$

In this case, we can trivialise the connection, that is, there is a gauge transformation  $T$  with  $T\Gamma = id$ :

$$\Gamma_{ij} = T_i^{-1} T_j,$$

for all edges  $(ij)$ . Clearly, any gauge transform of a flat connection is also flat.

With this understood, we are able to introduce the isothermic loop of connections of an isothermic sphere congruence,

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16) Note that  $q = [s_i^\pm, s_j^\pm, s_k^\pm, s_l^\pm] = cr(s_j^\pm, s_l^\pm, s_i^\pm, s_k^\pm)$  as our definition of the cross ratio differs from the classical one used in [12] by the order of points.

**3.6 Def.** Let  $s : \mathbb{Z}^2 \rightarrow \mathcal{L}^4$  be an isothermic sphere congruence with cross ratio factorizing edge-labelling  $a$ . The isothermic loop of connections  $(\Gamma(t))_{t \in \mathbb{R}}$  of  $s$  is a 1-parameter family of connections given by

$$\Gamma_{ij}(t)x := \begin{cases} (1 - ta_{ij})x & \text{if } x \in s_i, \\ x & \text{if } x \in (s_i \oplus s_j)^\perp, \\ \frac{1}{1 - ta_{ij}}x & \text{if } x \in s_j. \end{cases}$$

Clearly  $\Gamma_{ji}(t)\Gamma_{ij}(t) = id$  away from the singularity  $t = \frac{1}{a_{ij}}$ , so that  $\Gamma(t)$  defines indeed a connection on the discrete vector bundle  $\mathbb{Z}^2 \times \mathbb{R}^{4,2}$ . When  $\sigma$  denotes any lift of the isothermic sphere congruence  $s$  then

$$\Gamma_{ij}(t)x = x + \frac{ta_{ij}}{(\sigma_i \sigma_j)} \left\{ \frac{1}{1 - ta_{ij}} (x\sigma_i)\sigma_j - (x\sigma_j)\sigma_i \right\}.$$

Note the structural similarity to (3.6) — indeed parallel sections of  $\Gamma(t)$  in the Lie quadric yield Darboux transforms of  $s$ : the corresponding condition on edges realizes a propagation by cross ratio  $ta$ , thus yields a discrete version of *Darboux's linear system*, cf [6, Def 4.1]. For both, Darboux and Calapso transformations of an isothermic sphere congruence, flatness of the connections  $\Gamma(t)$  is paramount. Thus, returning to our context of an  $\Omega$ -net  $f$  enveloped by a pair  $s^\pm$  of isothermic sphere congruences, we aim to convince ourselves that the connections  $\Gamma^+(t)$  of  $s^+$  are flat, cf [6, Lemma 2.5]:

**3.7 Lemma.** Given an  $\Omega$ -net  $f = s^+ \oplus s^-$  in terms of a pair of isothermic sphere congruences  $s^\pm$  that admit Königs dual lifts  $\sigma^\pm$  the isothermic loop of connections of  $s^+$  consists of flat connections.

Thus we wish to show that, on an elementary quadrilateral  $(ijkl)$  and as long as  $t \neq \frac{1}{a_{ij}}, \frac{1}{a_{jk}}$ ,

$$\Gamma_{ij}^+(t)\Gamma_{jk}^+(t) = \Gamma_{il}^+(t)\Gamma_{lk}^+(t).$$

Having obtained the cross ratio factorizing property of the edge-labelling  $a$  in Lemma 3.5 above, the relevant part of the proof of [6, Lemma 2.5] applies, asserting correctness of the claim.

For autonomy we outline a simple algebraic proof here: first observe that, clearly,

$$\Gamma_{ij}^+(t)\Gamma_{jk}^+(t)x = \frac{1 - ta_{jk}}{1 - ta_{ij}}x \text{ for } x \in s_j^+$$

and a straightforward computation, using the Moutard lift  $\mu^+ = r\sigma^+$  of  $s^+$  and (3.7), shows that

$$\Gamma_{ij}^+(t)\Gamma_{jk}^+(t)x = \frac{1 - ta_{ij}}{1 - ta_{jk}}x \text{ for } x \in s_l^+.$$

Since  $\Gamma_{ij}^+(t)\Gamma_{jk}^+(t) \in \text{SO}(\mathbb{R}^{4,2})$  acts trivially on  $(s_i^+ \oplus s_j^+ \oplus s_l^+)^\perp$  it must act trivially on  $(s_j^+ \oplus s_l^+)^\perp$  and flatness of  $\Gamma^+(t)$  follows by symmetry and the fact that  $a$  is an edge-labelling.<sup>17)</sup>

Instead of using a symmetry argument to deduce the flatness of the connections  $\Gamma^-(t)$  we employ a gauge theoretic argument: we shall see that  $\Gamma^-(t)$  and  $\Gamma^+(t)$  are gauge equivalent, hence the flatness of  $\Gamma^+(t)$  from Lemma 3.7 implies flatness of  $\Gamma^-(t)$ .

To this end let  $g$  be any function on  $\mathbb{Z}^2$  and consider the following gauge transform of  $\Gamma^-(t)$ :

$$\Gamma_{ij}^g(t) := (A^g \Gamma^-)_{ij}(t) = A_i^g(t)\Gamma_{ij}^-(t)A_j^g(-t), \text{ for } A^g(t) := 1 - tg(\sigma^+ \wedge \sigma^-) = \exp(-tg\sigma^+ \wedge \sigma^-), \quad (3.8)$$

where we identify  $\mathfrak{so}(4,1) \cong \bigwedge^2 \mathbb{R}^{4,2}$  via  $(x \wedge y)z = (xz)y - (yz)x$ . Then any of the  $\Gamma^g(t)$  is a metric connection on the discrete vector bundle  $\mathbb{Z}^2 \times \mathbb{R}^{4,2}$ .

Next note that the connections  $\Gamma^g(t)$  have the same shape as the connections  $\Gamma^\pm(t)$  of the isothermic loops of connections of  $s^\pm$ : firstly,  $\Gamma_{ij}^g(t)$  acts trivially on the curvature sphere  $f_i \cap f_j$ ; secondly, using the lift (3.4) of the curvature sphere  $\langle \kappa_{ij} \rangle = f_i \cap f_j$ , we learn that  $\Gamma_{ij}^g(t)$  has eigenspaces<sup>18)</sup>

$$x_{ij} := A_j^g\left(\frac{1}{a_{ij}}\right)s_i^- = \langle \sigma_i^- + g_j \kappa_{ij} \rangle \text{ and } x_{ji} := A_i^g\left(\frac{1}{a_{ij}}\right)s_j^- = \langle \sigma_j^- + g_i \kappa_{ij} \rangle$$

17) Note how, conversely, the limit  $t \rightarrow \infty$  yields the cross ratio factorizing nature of the edge-labelling  $a$  of Lemma 3.5.

18) At this point we see that the  $\Gamma^g$  generally do not come from an isothermic sphere congruence: the condition that the eigenspaces  $x_{ij} \subset f_i$  coincide for all incident edges imposes a restriction on the function  $g$ .

with eigenvalues  $(1 - ta_{ij})^{\pm 1}$ , respectively, since  $(\sigma^+ \wedge \sigma^-)_j \sigma_i^- = (\sigma^+ \wedge \sigma^-)_i \sigma_j^- = -a_{ij} \kappa_{ij}$ ; and, finally,  $\Gamma_{ij}^g(t)$  acts trivially on  $(f_i + f_j)^\perp$  — hence

$$\Gamma_{ij}^g(t)x = \begin{cases} (1 - ta_{ij})x & \text{if } x \in x_{ij} \subset f_i, \\ x & \text{if } x \in (x_{ij} \oplus x_{ji})^\perp, \\ \frac{1}{1-ta_{ij}}x & \text{if } x \in x_{ji} \subset f_j. \end{cases}$$

In particular,  $x_{ij} = s_i^+$  and  $x_{ji} = s_j^+$  for  $g \equiv 1$ , showing that  $\Gamma^+ = \Gamma^1$  so that flatness of  $\Gamma^+(t)$  from Lemma 3.7 yields flatness of  $\Gamma^-(t) = \Gamma^0(t)$ , hence of all connections  $\Gamma^g(t)$ :

**3.8 Cor.** *All connections  $\Gamma^g(t)$  defined by (3.8) are flat.*

Note that the gauge family of loops of connections  $\Gamma^g$  also comprises the isothermic loops of the sphere congruences  $\tilde{s}^\pm = \langle \tilde{\sigma}^\pm \rangle$  of (3.2):  $\tilde{s}^\pm$  and  $s^\pm$  share the same edge-labelling  $a$  and

$$x_{ij} = \langle \sigma_i^- + c^\pm r_i \sigma_i^+ \rangle = \tilde{s}_i^\pm \quad \text{and} \quad x_{ji} = \langle \sigma_j^- + c^\pm r_j \sigma_j^+ \rangle = \tilde{s}_j^\pm \quad \text{for } g = \frac{c^\pm}{r \pm c^\pm}.$$

In particular, when  $s^\pm$  have a complex conjugate pair of Königs dual lifts, the gauge family contains real connections. A simple choice is given by  $g \equiv \frac{1}{2}$ : using that  $r_i r_j$  is unitary,  $r_i r_j = e^{-2i\alpha}$ , we find that  $\Gamma_{ij}^g(t)$  has real eigenspaces

$$x_{ij} = \langle e^{i\alpha}(r_i r_j \sigma_i^+ + \sigma_i^-) \rangle \quad \text{and} \quad x_{ji} = \langle e^{i\alpha}(r_i r_j \sigma_j^+ + \sigma_j^-) \rangle.$$

By their flatness all connections  $\Gamma^g(t)$  can be trivialized: there are (away from singularities) maps

$$T^g(t) : \mathbb{Z}^2 \rightarrow \text{SO}(\mathbb{R}^{4,2}) \quad \text{so that} \quad (T^g \Gamma^g)_{ij}(t) = T_i^g(t) \Gamma_{ij}^g(t) (T_j^g(t))^{-1} = id \quad (3.9)$$

on every edge  $(ij)$  of  $\mathbb{Z}^2$ . And, as the connections are gauge equivalent via (3.8), the gauge transformations  $T^g(t)$  are, up to constants of integration, related by  $T^-(t) = (T^g A^g)(t)$ . In particular, recall that the *Calapso transformation*  $T$  of an isothermic surface (in a quadric of any signature) is obtained by trivialising the loop of isothermic connections, see [6, Def 2.7] or [5, Thm 4.15] where this is treated in a rather general setting. In the present case, the Calapso transformations  $T^\pm$  of the pair  $s^\pm$  of isothermic sphere congruences are related by  $T^- = T^+ A^1$ . Thus, as  $A_i^g(t)$  acts trivially on the contact element  $f_i$ , any of the gauge transformations  $T^g$  realizes the *Calapso transforms*  $T^\pm(t)s^\pm = T^g(t)s^\pm$  of both isothermic sphere congruences  $s^\pm$  spanning an  $\Omega$ -net. This motivates the following definition, cf [7, Sect 3] and [6, Lemma 2.7]:

**3.9 Thm & Def.** *Let  $f = s^+ \oplus s^-$  be an  $\Omega$ -net spanned by a pair of isothermic sphere congruences  $s^\pm$  admitting Königs dual lifts  $\sigma^\pm$  and let  $T^g(t)$  be trivializing gauge transformations (3.9) of the connections  $\Gamma^g(t)$  of (3.8). Then, the deformation*

$$t \mapsto f(t) := T^g(t)f = T^\pm(t)f$$

*does not depend on the choice of gauge function  $g$ . It will be called the Calapso deformation of the  $\Omega$ -net  $f$ . The Calapso transforms  $f(t)$  of  $f$  are  $\Omega$ -nets with enveloped isothermic sphere congruences  $s^\pm(t) = T^g(t)s^\pm = T^\pm(t)s^\pm$  and Königs dual lifts  $\sigma^\pm(t) = T^g(t)\sigma^\pm$ .*

Only the last claim of the theorem — that  $f(t) = (s^+ \oplus s^-)(t)$  is an  $\Omega$ -net with  $s^\pm(t)$  admitting Königs dual lifts — requires further thought. As the Calapso transform does not depend on the choice of  $g$  we may, without loss of generality, assume  $g \equiv 0$ , that is,  $T^g = T^-$ .

Since  $T^-(t) : \mathbb{Z}^2 \rightarrow \vee(\mathbb{R}^{4,2})$  the Calapso transforms  $f(t)$  of  $f$  take values in the space of contact elements. To prove that  $\sigma^\pm(t)$  are edge-parallel, hence  $f(t)$  satisfies the contact condition, we employ the lift (3.4) of the curvature spheres: then, on any edge  $(ij)$ ,

$$r_i r_j d\sigma_{ij}^+(t) = T_i^-(t) \{ \Gamma_{ij}^-(t) (\sigma_j^- + \kappa_{ij}) - (\sigma_i^- + \kappa_{ij}) \} = T_i^-(t) \{ \Gamma_{ij}^-(t) \sigma_j^- - \sigma_i^- \} = d\sigma_{ij}^-(t). \quad (3.10)$$

Thus  $\sigma^\pm(t)$  are edge-parallel and  $f(t)$  is a Legendre map with curvature spheres

$$\langle \kappa_{ij}(t) \rangle = \langle r_i r_j \sigma_i^+(t) - \sigma_i^-(t) \rangle = \langle r_i r_j \sigma_j^+(t) - \sigma_j^-(t) \rangle = f_i(t) \cap f_j(t).$$

This also teaches us that  $\mu^\pm(t) := r^{\pm 1} \sigma^\pm(t)$  with  $r(t) = r$  yields Moutard lifts of  $s^\pm(t)$  since the Moutard equations (3.3) were just the integrability conditions of (3.1). Königs duality (away from umbilical faces) of  $\sigma^\pm(t)$ , see (3.3), now also follows directly from (3.10):

$$\begin{aligned} r_i r_k (r_l - r_j) \delta \sigma_{ik}^+(t) &= r_i r_k r_l (d\sigma_{il}^+(t) + d\sigma_{lk}^+(t)) - r_i r_j r_k (d\sigma_{ij}^+(t) + d\sigma_{jk}^+(t)) \\ &= r_k (d\sigma_{ji}^-(t) + d\sigma_{il}^-(t)) - r_i (d\sigma_{jk}^-(t) + d\sigma_{kl}^-(t)) \\ &= (r_k - r_i) \delta \sigma_{jl}^-(t). \end{aligned}$$

This concludes the proof of Thm 3.9.

Note that, in contrast to the function  $r$  relating Königs dual and Moutard lifts that is invariant under the Calapso deformation, the edge-labelling  $a$  changes, cf [14, §5.7.16] or [6, Lemma 2.7]:

$$a_{ij}(t) = (\mu_i^\pm(t) \mu_j^\pm(t)) = (\mu_i^\pm \Gamma_{ij}^\pm \mu_j^\pm) = \frac{1}{1 - t a_{ij}} (\mu_i^\pm \mu_j^\pm) = \frac{a_{ij}}{1 - t a_{ij}}.$$

## 4. Lawson transformation of linear Weingarten surfaces

In this section, we shall see how the Calapso deformation of Thm 3.9 for  $\Omega$ -nets descends to a “Lawson transformation” for linear Weingarten nets. To this end we need to investigate the effect of the deformation on the two linear sphere complexes that come with a linear Weingarten net. Thus let  $f = s^+ \oplus s^-$  be an  $\Omega$ -net so that  $s^\pm$  have Königs dual lifts  $\sigma^\pm$  and take values in linear sphere complexes  $\mathfrak{k}^\pm$ , where we assume the above relative normalizations  $(\mathfrak{k}^\pm \sigma^\mp) = -1$ .

For symmetry we base our analysis on the “middle connection”  $\Gamma^g$  with  $g \equiv \frac{1}{2}$ . Recall that  $\Gamma^g$  is then real in the complex conjugate case and, consequently, so is  $T^g$  when fixing  $T^g$  to be real at an initial point, say  $T_0^g = id$  at some point  $0 \in \mathbb{Z}^2$ . Now

$$\mathfrak{k}^\pm(t) := T^g(t) \{ \mathfrak{k}^\pm + \frac{t}{2} \sigma^\pm \} = T^\pm(t) \mathfrak{k}^\pm \equiv \text{const} \quad (4.1)$$

for any fixed  $t$  since  $\mathfrak{k}^\pm \perp s_i^\pm \oplus s_j^\pm$  so that  $\Gamma_{ij}^\pm(t) \mathfrak{k}^\pm = \mathfrak{k}^\pm$  on any edge  $(ij)$ . Moreover,

$$(\mathfrak{k}^\pm(t) s^\pm(t)) \equiv 0 \text{ for } s^\pm(t) = T^g(t) s^\pm$$

since  $T^g : \mathbb{Z}^2 \rightarrow \vee(\mathbb{R}^{4,2})$ . Thus  $\mathfrak{k}^\pm(t)$  define linear sphere complexes that the isothermic sphere congruences  $s^\pm(t)$  spanning the Calapso transform  $f(t) = (s^+ \oplus s^-)(t)$  of  $f$  take values in.

Note that, with the Königs dual lifts  $\sigma^\pm(t) = T^g(t) \sigma^\pm$  of  $s^\pm(t)$ , see Thm 3.9, the deformation preserves the relative scaling  $(\mathfrak{k}^\pm(t) \sigma^\mp(t)) \equiv -1$  and, in the complex conjugate case,  $\sigma^\pm(t)$  as well as  $\mathfrak{k}^\pm(t)$  are complex conjugate again.

Consequently, we have proved that the Calapso deformation yields a transformation for linear Weingarten nets: given a linear Weingarten net, its Legendre lift is an  $\Omega$ -net (by Thm 2.8) admitting (see Thm 3.9) Calapso deformation into a new  $\Omega$ -net, which has the characteristics of the Legendre lift of a linear Weingarten net (cf Thm 2.8). The only potential issue is that the resulting  $\Omega$ -net may not admit an appropriate space form projection — if the plane spanned by  $\mathfrak{k}^+(t)$  and  $\mathfrak{k}^-(t)$  becomes null then it does not contain a point sphere complex  $\mathfrak{p}(t)$ . Computing inner products we find

$$(\mathfrak{k}^\pm(t) \mathfrak{k}^\pm(t)) = (\mathfrak{k}^\pm \mathfrak{k}^\pm) \text{ and } (\mathfrak{k}^+(t) \mathfrak{k}^-(t)) = (\mathfrak{k}^+ \mathfrak{k}^-) - t, \quad (4.2)$$

showing that this issue does generically not occur: it only occurs at a single value of  $t$  when  $\mathfrak{k}^\pm$  define a (possibly complex conjugate) pair of spheres. Below we shall discuss these cases.

**4.1 Thm & Def.** *Let  $f$  be the Legendre lift of a linear Weingarten net  $(\mathfrak{f}, \mathfrak{t}) : \mathbb{Z}^2 \rightarrow \mathfrak{Q}^3 \times \mathfrak{P}^3$ . The Calapso transforms  $f(t)$  of  $f$  are generically<sup>19)</sup> the Legendre lifts of suitable linear Weingarten nets  $(\mathfrak{f}, \mathfrak{t})(t) : \mathbb{Z}^2 \rightarrow \mathfrak{Q}^3(t) \times \mathfrak{P}^3(t)$ . These will be called *Lawson transforms* of  $(\mathfrak{f}, \mathfrak{t})$ .*

To justify the terminology we consider constant mean curvature nets, cf [6, Sect 5].

<sup>19)</sup> That is, as long as the sphere complexes  $\mathfrak{k}^\pm(t)$  from (4.1) do not span a contact element.

**4.2 Example.** Let  $(f, t) : \mathbb{Z}^2 \rightarrow \Omega^3 \times \mathfrak{P}^3$  be a constant mean curvature net, that is, a linear Weingarten net with

$$0 = \alpha K + 2\beta H + \gamma, \quad \text{where } \alpha = 0. \quad (4.3)$$

In the discussion leading up to Thm 2.8 we already saw, cf [4, Lemma 4.1], that

$$\sigma^- = f \quad \text{and} \quad \sigma^+ = t + H f$$

yield a Königs dual pair of isothermic sphere congruences  $s^\pm$  that take values in linear sphere complexes

$$\mathfrak{k}^- = p \quad \text{and} \quad \mathfrak{k}^+ = q - H p.$$

To recover the Lawson correspondence of [6, Def 5.2] we follow the arguments that proved Thm 4.1 but now base our analysis on  $T^-$  instead of  $T^{\frac{1}{2}}$ . Since  $\Gamma_{ij}^-(t)\mathfrak{k}^- = \mathfrak{k}^-$  for all  $t$  and all edges  $(ij)$  we may assume that  $T^-(t)p = p$  for all  $t$ , that is,  $T^-(t)$  is a Möbius geometric Calapso transformation of the discrete isothermic net  $f : \mathbb{Z}^2 \rightarrow \Omega^3$  when  $(pp) = -1$ , see [14, §5.7.16]. Hence, for all  $t$ ,

$$\mathfrak{k}^-(t) = \mathfrak{k}^- = p$$

provides a canonical point sphere complex for space form projection of the Calapso transform  $f(t)$  of the Legendre lift  $f$  of  $(f, t)$ . Further

$$\mathfrak{k}^+(t) = T^-(t)\{\mathfrak{k}^+ + t\sigma^+\}$$

is obtained as the image of (the Lie geometric lift of) the *linear conserved quantity* of [6, Def 5.1]. Consequently,

$$q(t) := \mathfrak{k}^+(t) - \frac{(\mathfrak{k}^+(t)p)}{(pp)}p = T^-(t)\{q + t(\sigma^+ + \frac{p}{(pp)})\}$$

yields a canonical space form vector for the space form projection.

Hence, as  $\sigma^\pm(t) = T^-(t)\sigma^\pm$ , the corresponding space form projection is given by

$$f(t) = T^-(t)f : \mathbb{Z}^2 \rightarrow \Omega^3(t) \quad \text{and} \quad t(t) = T^-(t)\{t - \frac{t}{(pp)}f\} : \mathbb{Z}^2 \rightarrow \mathfrak{P}^3(t).$$

Now  $\sigma^+(t) = t(t) + (H + \frac{t}{(pp)})f(t)$ , showing that the linear Weingarten net  $(f, t)(t)$  has constant mean curvature

$$H(t) = H + \frac{t}{(pp)}.$$

Note that we also recover the *Lawson invariant*, relating the mean curvature of the constant mean curvature net and its ambient constant sectional curvature, cf (4.2):

$$(pp)H^2(t) + (q(t)q(t)) = (\mathfrak{k}^+(t)\mathfrak{k}^+(t)) = (\mathfrak{k}^+\mathfrak{k}^+) = (pp)H^2 + (qq).$$

Thus our Lawson transformation of Def 4.1 does indeed generalize the Lawson correspondence of [6, Sect 5] for discrete constant mean curvature nets.

Coming back to the genericity issue of the Lawson transformation from Thm 4.1 we consider the discrete analogue of (intrinsically) flat surfaces in hyperbolic space, cf [15, Sect 4.3] and [7].

**4.3 Example.** Thus let  $(f, t) : \mathbb{Z}^2 \rightarrow \Omega^3 \times \mathfrak{P}^3$ , where  $(pp) = -1$  and  $(qq) = +1$  so that  $\Omega^3$  becomes hyperbolic space, satisfy  $A(t, t) = A(f, f)$ , cf [15, Lemma 6.5], that is,  $(f, t)$  is linear Weingarten with

$$0 = \alpha K + 2\beta H + \gamma, \quad \text{where } \alpha + \gamma = \beta = 0. \quad (4.4)$$

Now  $A(f + t, f - t) = 0$  so that a Königs dual pair of enveloped isothermic sphere congruences is given by

$$\sigma^\pm := f \pm t.$$

As  $f$  and  $t$  take values in  $\Omega^3$  and  $\mathfrak{P}^3$ , respectively,  $s^\pm$  take values in linear sphere complexes

$$\mathfrak{k}^\pm := \frac{1}{2}(q \mp p).$$

These define two oriented spheres since  $(\mathfrak{k}^\pm \mathfrak{k}^\pm) = 0$ : they are the two orientations of the infinity boundary of the ambient hyperbolic space, cf [7, Sect 2], and the fact that  $\sigma^\pm$  take values in the sphere complexes defined by  $\mathfrak{k}^\pm$  teaches us that  $\sigma^\pm$  touch the infinity boundary (with opposite orientations), that is,  $\sigma^\pm$  are *horosphere congruences*.

Before proceeding to the Lawson transformation, note how the parallel nets of  $(\mathfrak{f}, \mathfrak{t})$  are obtained by simultaneous reciprocal rescaling of  $\mathfrak{k}^\pm$  and  $\sigma^\pm$ , cf (2.8) and Cor 2.9: with

$$\tilde{\mathfrak{k}}^\pm = e^{\pm\rho} \mathfrak{k}^\pm \quad \text{and} \quad \tilde{\sigma}^\pm = e^{\pm\rho} \sigma^\pm$$

the relative scalings are preserved and a new choice of point sphere complex and space form vector

$$\left. \begin{aligned} \tilde{\mathfrak{p}} &:= \tilde{\mathfrak{k}}^- - \tilde{\mathfrak{k}}^+ = \mathfrak{p} \cosh \rho - \mathfrak{q} \sinh \rho \\ \tilde{\mathfrak{q}} &:= \tilde{\mathfrak{k}}^- + \tilde{\mathfrak{k}}^+ = \mathfrak{q} \cosh \rho - \mathfrak{p} \sinh \rho \end{aligned} \right\} \text{ yields } \left\{ \begin{aligned} \tilde{\mathfrak{f}} &= \mathfrak{f} \cosh \rho + \mathfrak{t} \sinh \rho, \\ \tilde{\mathfrak{t}} &= \mathfrak{f} \sinh \rho + \mathfrak{t} \cosh \rho. \end{aligned} \right.$$

The linear Weingarten condition (4.4) is preserved by this change of space form projection by parallel transformation. Thus the parallel nets of a flat net in hyperbolic space are flat. For the analysis of the Lawson transformation below we shall disregard this freedom.

To honour the symmetry of the situation we again use the Calapso transformations  $T^g(t)$  of the ‘‘middle connection’’  $\Gamma^g(t)$  with  $g \equiv \frac{1}{2}$ , as above and in [7, Sect 3]. Thus, cf (4.1), we obtain

$$\sigma^\pm(t) = T^g(t) \sigma^\pm = T^g(t) \{ \mathfrak{f} \pm \mathfrak{t} \} \quad \text{and} \quad \mathfrak{k}^\pm(t) = T^g(t) \{ \mathfrak{k}^\pm + \frac{t}{2} \sigma^\pm \} = T^g(t) \{ \frac{\mathfrak{q} + t\mathfrak{f}}{2} \mp \frac{\mathfrak{p} - t\mathfrak{t}}{2} \}$$

as the Königs dual pair spanning the Calapso transform  $f(t)$  of the original  $\Omega$ -net and the sphere complexes the enveloped isothermic sphere congruences take values in. By (4.2) the new sphere complexes still define two oriented spheres,  $(\mathfrak{k}^\pm \mathfrak{k}^\pm) = 0$ , which can be interpreted as the two orientations of the infinity boundary of a hyperbolic space as long as they do not touch,

$$(\mathfrak{k}^+(t) \mathfrak{k}^-(t)) = (\mathfrak{k}^+ \mathfrak{k}^-) - t = \frac{1}{2}(1 - 2t) \neq 0,$$

that is, as long as the genericity condition of Thm 4.1 is satisfied.<sup>20)</sup> Then a choice of

$$\mathfrak{p}(t) := \mathfrak{k}^-(t) - \mathfrak{k}^+(t) = T^g(t)(\mathfrak{p} - t\mathfrak{t}) \quad \text{and} \quad \mathfrak{q}(t) := \mathfrak{k}^-(t) + \mathfrak{k}^+(t) = T^g(t)(\mathfrak{q} + t\mathfrak{f})$$

for the point sphere complex and space form vector yields a projection to a hyperbolic space and a de Sitter space as its (unit) tangent bundle, or vice versa, depending on the sign of  $1 - 2t$ :

$$(\mathfrak{p}(t)\mathfrak{p}(t)) = -(1 - 2t) \quad \text{and} \quad (\mathfrak{q}(t)\mathfrak{q}(t)) = +(1 - 2t).$$

The corresponding space form projection

$$(\mathfrak{f}, \mathfrak{t})(t) = (T^g(t)\mathfrak{f}, T^g(t)\mathfrak{t}) : \mathbb{Z}^2 \rightarrow \mathfrak{Q}^3(t) \times \mathfrak{P}^3(t)$$

is a linear Weingarten net satisfying the same linear Weingarten condition (4.4) since, as before,

$$W(t) := 2 \mathfrak{k}^+(t) \odot \mathfrak{k}^-(t) = \frac{1}{2} \{ \mathfrak{q}(t) \odot \mathfrak{q}(t) - \mathfrak{p}(t) \odot \mathfrak{p}(t) \}.$$

In particular, as long as  $t < \frac{1}{2}$ , the Lawson transforms  $(\mathfrak{f}, \mathfrak{t})(t)$  of  $(\mathfrak{f}, \mathfrak{t})$  remain discrete analogues of (intrinsically) flat surfaces in hyperbolic space.<sup>21)</sup> Beyond the singularity of the Lawson transformation, when  $t > \frac{1}{2}$ , we obtain linear Weingarten nets  $(\mathfrak{f}, \mathfrak{t})(t)$  with constant (extrinsic) Gauss curvature  $K = 1$  in de Sitter space.<sup>22)</sup>

Thus we obtained a case, where the genericity issue for the Lawson transformation does occur and, in particular, we have seen how the two conserved quantities  $\mathfrak{k}^\pm(t)$  in this case become spheres —

20) When  $t = \frac{1}{2}$  the plane  $\langle \mathfrak{k}^+, \mathfrak{k}^- \rangle$  is isotropic hence does not contain a point sphere complex.

21) Rescaling  $\mathfrak{p}(t)$  and  $\mathfrak{q}(t)$  by  $\frac{1}{\sqrt{1-2t}}$  yields the standard model of hyperbolic space with de Sitter space as its unit tangent bundle.

22) Note that, swapping the roles of  $\mathfrak{f}(t)$  and  $\mathfrak{t}(t)$  in this case, yields linear Weingarten nets in hyperbolic space again.

the two orientations of the infinity spheres of the ambient hyperbolic geometries — as predicted from (4.2). Below we shall give a more exhaustive discussion of the genericity phenomenon.

Generalizing Example 4.3 we next discuss discrete nets of (arbitrary) constant Gauss curvature in (possibly Lorentzian) space forms:

**4.4 Example.** Let  $(\mathbf{f}, \mathbf{t}) : \mathbb{Z}^2 \rightarrow \Omega^3 \times \mathfrak{P}^3$ , where  $\varepsilon := -(\mathbf{p}\mathbf{p}) = \pm 1$  and  $\kappa = -(\mathbf{q}\mathbf{q})$ , be a space form projection into a quadric  $\Omega^3$  of constant curvature  $\kappa$  satisfying

$$0 = \alpha K + 2\beta H + \gamma \quad \text{with } \beta = 0. \quad (4.5)$$

Clearly,<sup>23)</sup>  $\alpha \neq 0$  so that we may, without loss of generality, assume that  $\alpha = 1$ ; excluding tubular linear Weingarten surfaces we have  $K = -\gamma \neq 0$ . Now

$$2W = 4\mathfrak{k}^+ \odot \mathfrak{k}^- = \mathbf{p} \odot \mathbf{p} - \frac{1}{K} \mathbf{q} \odot \mathbf{q} \quad \text{with } \mathfrak{k}^\pm := \frac{1}{2} \left\{ \mathbf{p} \mp \frac{1}{\sqrt{K}} \mathbf{q} \right\},$$

showing that  $\mathfrak{k}^\pm$  and the corresponding Königs dual lifts  $\sigma^\pm = \mathbf{t} \pm \sqrt{K} \mathbf{f}$  of enveloped isothermic sphere congruences  $s^\pm$  become complex conjugate when  $K < 0$ .

To investigate the Lawson transformation, we use the “middle connection”  $\Gamma^g$  with  $g \equiv \frac{1}{2}$  again, as in the case of flat fronts in hyperbolic space: recall that this connection is real in the complex conjugate case so that the Calapso transformations  $T^g(t)$  can be assumed to be real as well. Now, cf (4.1),

$$\sigma^\pm(t) = T^g(t) \{ \mathbf{t} \pm \sqrt{K} \mathbf{f} \} \quad \text{and} \quad \mathfrak{k}^\pm(t) = T^g(t) \left\{ \frac{\mathbf{p} + t\mathbf{t}}{2} \mp \frac{\mathbf{q} - tK\mathbf{f}}{2\sqrt{K}} \right\}$$

yields the Calapso transform  $f(t)$  of the original  $\Omega$ -net  $f = (\mathbf{f}, \mathbf{t})$  and the two sphere complexes that its pair of enveloped isothermic sphere congruences  $s^\pm(t) = \langle \sigma^\pm(t) \rangle$  take values in. Thus, as long as  $\mathbf{p} + t\mathbf{t}$  does not become isotropic,  $\varepsilon + 2t \neq 0$ , we may choose

$$\mathbf{p}(t) := \frac{1}{\sqrt{|\varepsilon + 2t|}} T^g(t) \{ \mathbf{p} + t\mathbf{t} \} \quad \text{and} \quad \mathbf{q}(t) := T^g(t) \{ \mathbf{q} - tK\mathbf{f} \}$$

as the new point sphere complex and space form vector for the space form projection  $(\mathbf{f}, \mathbf{t})(t)$  of  $f(t)$ , so that

$$\mathbf{f}(t) = T^g(t)\mathbf{f} \quad \text{and} \quad \mathbf{t}(t) = \sqrt{|\varepsilon + 2t|} T^g(t)\mathbf{t}.$$

Note that  $\varepsilon(t) = -(\mathbf{p}(t)\mathbf{p}(t))$  changes sign at  $t = -\frac{\varepsilon}{2}$ , hence the ambient geometry of  $(\mathbf{f}, \mathbf{t})(t)$  changes signature, as in the case of flat nets in hyperbolic space.

Now

$$2W(t) = 4\mathfrak{k}^+(t) \odot \mathfrak{k}^-(t) = |\varepsilon + 2t| \mathbf{p}(t) \odot \mathbf{p}(t) - \frac{1}{K} \mathbf{q}(t) \odot \mathbf{q}(t)$$

encodes the linear Weingarten condition for  $(\mathbf{f}, \mathbf{t})(t)$ : the new (constant) Gauss and ambient curvatures become

$$K(t) = K |\varepsilon + 2t| \quad \text{and} \quad \kappa(t) = -(\mathbf{q}(t)\mathbf{q}(t)) = \kappa - 2tK.$$

As a consequence the “intrinsic Gauss curvatures”

$$K_{int}(t) = \varepsilon(t)K(t) + \kappa(t) \equiv K_{int}$$

of the nets remain unchanged by the Lawson transformation. Of course, this fact depends on the chosen normalization of the space form vectors  $\mathbf{q}(t)$ : a rescaling of  $\mathbf{q}(t)$  results in a rescaling of both the extrinsic and intrinsic Gauss curvatures by the square of the factor — in particular, if  $\frac{1}{\sqrt{|\varepsilon + 2t|}} \mathbf{q}(t)$  had been chosen for the space form projections instead, the extrinsic Gauss curvatures would remain unchanged while the intrinsic Gauss curvatures would get scaled in the family.

Similar thoughts show that the apparent problem of the space form projection when  $t = -\frac{\varepsilon}{2}$  is easily resolved as long as the intrinsic Gauss curvature of the original net did not vanish: as long as neither  $\mathbf{p}(t)$  nor  $\mathbf{q}(t)$  become isotropic their roles can be interchanged after suitable rescalings. In particular, a common rescaling by  $\frac{1}{\sqrt{|\kappa(t)|}}$  and re-interpretation of  $\frac{1}{\sqrt{|\kappa(t)|}} \mathbf{q}(t)$  as the point sphere complex results in a linear Weingarten net  $\sqrt{|\kappa(t)|}(\mathbf{t}, \mathbf{f})(t)$  of constant Gauss curvature  $\frac{1}{K(t)}$ . Hence

<sup>23)</sup> Otherwise  $\gamma = 0$  as well and the linear Weingarten condition would become trivial.

the Lawson transformation is well defined as long as  $\mathbf{p}(t)$  and  $\mathbf{q}(t)$  do not simultaneously become isotropic, that is, as long as

$$\kappa(-\frac{\varepsilon}{2}) = K_{int} \neq 0.$$

As a third special class of linear Weingarten surfaces, obtained by the vanishing of the third coefficient in the linear Weingarten condition, we discuss nets of constant harmonic mean curvature as a discrete analogue of the surfaces with constant average of their curvature radii:

**4.5 Example.** Fix a space form projection  $(\mathbf{f}, \mathbf{t}) : \mathbb{Z}^2 \rightarrow \mathfrak{Q}^3 \times \mathfrak{P}^3$ , where  $-(\mathbf{p}\mathbf{p}) =: \varepsilon = \pm 1$  and the ambient curvature is given by  $-(\mathbf{q}\mathbf{q}) = \kappa$  as before, and suppose that

$$0 = \alpha K + 2\beta H + \gamma, \quad \text{where } \gamma = 0. \quad (4.6)$$

As we exclude tubular linear Weingarten nets we must have  $\beta \neq 0$ . Hence, without loss of generality,  $\beta = 1$  and  $\alpha = -2\frac{H}{K}$  is given by the constant harmonic mean curvature of the net. The enveloped isothermic sphere congruences of the constant harmonic mean curvature net then turn out to be its tangent plane congruence and its “middle sphere congruence”,

$$\sigma^- = \mathbf{t} \quad \text{and} \quad \sigma^+ = \mathbf{f} + \frac{H}{K} \mathbf{t},$$

which take values in the linear sphere complexes

$$\mathfrak{k}^- = \mathbf{q} \quad \text{and} \quad \mathfrak{k}^+ = \mathbf{p} - \frac{H}{K} \mathbf{q}.$$

Note the similarity to the constant mean curvature nets of Example 4.2. Writing (4.6) in the more symmetric form (2.10),

$$0 = \alpha A(\mathbf{t}, \mathbf{t}) - 2\beta A(\mathbf{t}, \mathbf{f}) + \gamma A(\mathbf{f}, \mathbf{f}),$$

makes this similarity more tangible: as long as  $\kappa \neq 0$  the aforementioned “duality” for linear Weingarten nets relates constant harmonic mean curvature nets and constant mean curvature nets. In particular, a common rescaling of the point sphere complex and the space form vector,

$$(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = \frac{1}{\sqrt{|\kappa|}}(\mathbf{p}, \mathbf{q}), \quad \text{yields } (\tilde{\mathbf{f}}, \tilde{\mathbf{t}}) = \sqrt{|\kappa|}(\mathbf{t}, \mathbf{f})$$

as a constant mean curvature  $\tilde{H} = \frac{H}{K}$  net in a quadric of constant curvature  $\tilde{\kappa} = \frac{\varepsilon}{|\kappa|}$ , whose signature is given by  $\tilde{\varepsilon} = \frac{\kappa}{|\kappa|}$ . The Lawson transformation of Example 4.2 then yields a transformation into constant mean curvature nets with<sup>24)</sup>

$$\tilde{H}(t) = \tilde{H} - \tilde{\varepsilon} \frac{t}{|\kappa|} \quad \text{and} \quad \tilde{\kappa}(t) = \tilde{\kappa} + 2\tilde{H} \frac{t}{|\kappa|} - \tilde{\varepsilon} \left(\frac{t}{|\kappa|}\right)^2.$$

As long as  $\tilde{\kappa}(t) \neq 0$  the same “duality” can then be used to obtain constant harmonic mean curvature nets  $(\tilde{\mathbf{f}}(t), \tilde{\mathbf{t}}(t)) = \sqrt{|\tilde{\kappa}(t)|}(\tilde{\mathbf{t}}(t), \tilde{\mathbf{f}}(t))$  as Lawson transforms of the original net  $(\mathbf{f}, \mathbf{t})$ .

Aiming to obtain the Lawson transformation for nets of constant harmonic mean curvature directly, we recover the same regularity issues as outlined above. Motivated by the observation that the characterizing feature of constant harmonic mean curvature nets in our setup is its tangent plane congruence  $\mathbf{t}$  being one of the enveloped isothermic sphere congruences, we base our analysis on  $T^-$ , where constants of integration are adjusted so that  $T^-(t)\mathbf{q} = \mathbf{q}$ . Then we aim to obtain

$$\mathfrak{k}^+(t) = T^-(t)\{\mathfrak{k}^+ + t\sigma^+\} \parallel \mathbf{p}(t) - \frac{H(t)}{K(t)} \mathbf{q}$$

with a normalized point sphere complex  $\mathbf{p}(t) \perp \mathbf{q}$  in order to recover the linear Weingarten condition of a constant harmonic mean curvature net. Orthogonalization then requires  $\kappa \neq 0$  and normalization requires also  $\tilde{\kappa}(t) \neq 0$ : when both are satisfied<sup>25)</sup>

$$\mathbf{p}(t) = \frac{1}{\sqrt{|\kappa\tilde{\kappa}(t)|}} T^-(t)\{\mathbf{p} + t(\sigma^+ - \frac{1}{\kappa}\mathbf{q})\} \quad \text{yields} \quad \frac{1}{\sqrt{|\kappa\tilde{\kappa}(t)|}} \mathfrak{k}^+(t) = \mathbf{p}(t) - \frac{1}{\sqrt{|\kappa\tilde{\kappa}(t)|}} \left(\frac{H}{K} - \frac{1}{\kappa} t\right) \mathbf{q}.$$

24) We use the edge-labelling  $a$  of the original Königs dual pair  $(\sigma^+, \sigma^-)$  for the Calapso transformation here, which results in a rescaling of the spectral parameter of Example 4.2.

25) Note that  $\tilde{\kappa}(t) \neq 0$  encodes the non-degeneracy of the induced metric of the plane spanned by  $\mathfrak{k}^\pm(t)$ .



Starting from a constant harmonic mean curvature net in a flat ambient geometry the Lawson transformation is still well defined but the above approach, following Example 4.2, does not yield a suitable space form projection. However, exploiting the fact that constant harmonic mean curvature nets in flat ambient geometries arise as parallel nets of minimal nets, leads to an alternative approach: choosing

$$\tilde{\mathbf{p}}(t) := \mathbf{k}^+(t) \quad \text{and} \quad \tilde{\mathbf{q}}(t) := \mathbf{k}^-(t) - \varepsilon t \mathbf{k}^+(t)$$

for a space form projection<sup>26)</sup> we obtain a constant mean curvature net

$$(\tilde{\mathbf{f}}(t), \tilde{\mathbf{t}}(t)) = (\sigma^+(t), \sigma^-(t) + \varepsilon t \sigma^+(t))$$

as, clearly,  $A(\tilde{\mathbf{f}}(t), \tilde{\mathbf{t}}(t)) = \varepsilon t A(\mathbf{f}(t), \mathbf{t}(t))$ . Note that, in particular, this choice of projection yields a minimal net  $(\tilde{\mathbf{f}}(0), \tilde{\mathbf{t}}(0))$  for  $t = 0$ . When  $t \neq 0$  the ambient curvature of the constant mean curvature net  $(\tilde{\mathbf{f}}(t), \tilde{\mathbf{t}}(t))$  does not vanish,  $\tilde{\kappa}(t) = -\varepsilon t^2 \neq 0$ , so that a choice

$$\mathbf{p}(t) := \frac{1}{t} \tilde{\mathbf{q}}(t) \quad \text{and} \quad \mathbf{q}(t) := \tilde{\mathbf{p}}(t)$$

now yields a net  $(\mathbf{f}(t), \mathbf{t}(t)) = (\tilde{\mathbf{f}}(t), t\tilde{\mathbf{t}}(t))$  of constant harmonic mean curvature  $\frac{H(t)}{K(t)} = -\varepsilon$  in a (Lorentzian if  $\varepsilon = 1$ ) quadric of constant curvature  $\kappa(t) = \varepsilon$ .

Indeed, as our discussion of the Lawson transformation for constant harmonic mean curvature nets hints at, the Lawson transformation is a transformation for parallel families of linear Weingarten nets rather than for individual nets: it involves a choice of space form projection that, essentially, is a choice of a net in a parallel family. We shall see that every parallel family of discrete linear Weingarten nets contains nets of at least one of the particular types discussed in the preceding examples.

**4.6 Parallel families.** In Cor 2.9 we already saw that parallel nets of a linear Weingarten net are linear Weingarten, cf [14, Sect 2.7]. Using the same setup as above, let  $(\mathbf{f}, \mathbf{t})$  denote a linear Weingarten net in a space form given by a point sphere complex  $\mathbf{p}$  and a space form vector  $\mathbf{q} \perp \mathbf{p}$ . A change of basis

$$(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = (\mathbf{q}, \mathbf{p})B,$$

where  $B \in \text{Gl}(2)$  is chosen to preserve inner products of the basis, yields a parallel linear Weingarten net  $(\tilde{\mathbf{f}}, \tilde{\mathbf{t}})$ . The coefficients of the linear Weingarten relations (2.7) of  $(\mathbf{f}, \mathbf{t})$  and  $(\tilde{\mathbf{f}}, \tilde{\mathbf{t}})$  are then related by<sup>27)</sup>

$$\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\beta} & \tilde{\gamma} \end{pmatrix} = B \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} B^t.$$

As the shape of the basis transformations  $B$  depends on the signature of the plane  $\langle \mathbf{q}, \mathbf{p} \rangle$  we discuss the cases that occur in turn, cf [17, Sect 3.4] or [13, Sect II.5].

- (1) In the definite case we assume  $(\mathbf{q}, \mathbf{p})$  to be an orthonormal basis so that the parallel family of nets is parametrized by

$$B = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}.$$

Writing  $\frac{\alpha+\gamma}{2} = \mu$ ,  $\frac{\alpha-\gamma}{2} = \varrho \cos 2\omega$  and  $\beta = \varrho \sin 2\omega$ , the coefficients of the linear Weingarten condition (2.7) of the parallel nets  $(\mathbf{f}, \mathbf{t})(\vartheta)$  become

$$\alpha(\vartheta) = \mu + \varrho \cos 2(\vartheta + \omega), \quad \beta(\vartheta) = \varrho \sin 2(\vartheta + \omega) \quad \text{and} \quad \gamma(\vartheta) = \mu - \varrho \cos 2(\vartheta + \omega).$$

Thus, if  $\varrho \neq 0$ , then  $\sin 2(\vartheta + \omega) = 0$  yields two pairs of antipodal constant Gauss curvature nets and, if also

$$\mu^2 - \varrho^2 = \alpha\gamma - \beta^2 < 0,$$

then  $\cos 2(\vartheta + \omega) = \mp \frac{\mu}{\varrho}$  yields two pairs of antipodal constant mean curvature nets and of constant harmonic mean curvature nets, respectively. When  $\mu = 0$  these coincide and yield

26) Recall (4.2): when  $\kappa = 0$  we have  $(\mathbf{k}^+(t)\mathbf{k}^+(t)) = -\varepsilon$  and  $(\mathbf{k}^+(t)\mathbf{k}^-(t)) = -t$ .

27) This follows easily by interpreting the linear Weingarten condition (2.7) as an orthogonality condition with respect to the inner product on symmetric  $2 \times 2$ -matrices given by the determinant as its quadratic form.

minimal nets. Note the symmetric spacing of the twelve or eight nets that appear in this case.

If, on the other hand,  $\varrho = 0$  then all  $(\mathfrak{f}, \mathfrak{t})(\vartheta)$  are intrinsically flat, cf Example 4.4.

- (2) In the degenerate case we assume that  $(\mathfrak{q}\mathfrak{q}) = 0$  and  $(\mathfrak{p}\mathfrak{p}) = \pm 1$  so that the parallel family is parametrized by

$$B = \begin{pmatrix} 1 & \vartheta \\ 0 & 1 \end{pmatrix}$$

and the linear Weingarten coefficients of (2.7) of the parallel nets  $(\mathfrak{f}, \mathfrak{t})(\vartheta)$  become

$$\alpha(\vartheta) = \alpha + 2\beta\vartheta + \gamma\vartheta^2, \quad \beta(\vartheta) = \beta + \gamma\vartheta \quad \text{and} \quad \gamma(\vartheta) = \gamma.$$

Thus, in the generic case,  $\vartheta = -\frac{\beta}{\gamma}$  yields a constant Gauss curvature net, which has two parallel constant mean curvature nets at  $\vartheta = -\frac{\beta}{\gamma} \pm \frac{1}{\gamma}\sqrt{\beta^2 - \alpha\gamma}$  as soon as the root is real: we recover the classical Bonnet theorem in this case, cf [14, §2.7.4].

If however  $\gamma = 0$  then the parallel family consists of the parallel constant harmonic mean curvature nets of a minimal net at  $\vartheta = -\frac{\alpha}{2\beta}$ , as discussed in Example 4.5.

- (3) In the indefinite case of hyperbolic and de Sitter spaces we base the analysis on an orthonormal basis again, so that

$$B = \begin{pmatrix} \cosh \vartheta & \sinh \vartheta \\ \sinh \vartheta & \cosh \vartheta \end{pmatrix}$$

now parametrizes the parallel family. To obtain a convenient representation of the linear Weingarten coefficients of (2.7) for the parallel nets  $(\mathfrak{f}, \mathfrak{t})(\vartheta)$  we distinguish three cases.

- (i) If  $|\frac{\alpha+\gamma}{2}| > |\beta|$  we write  $\frac{\alpha-\gamma}{2} = \mu$ ,  $\frac{\alpha+\gamma}{2} = \varrho \cosh 2\omega$  and  $\beta = \varrho \sinh 2\omega$  to obtain

$$\alpha(\vartheta) = \mu + \varrho \cosh 2(\vartheta + \omega), \quad \beta(\vartheta) = \varrho \sinh 2(\vartheta + \omega) \quad \text{and} \quad \gamma(\vartheta) = -\mu + \varrho \cosh 2(\vartheta + \omega).$$

Thus  $\vartheta = -\omega$  yields one constant Gauss curvature net, which has two parallel constant mean or constant harmonic mean curvature nets if  $\varrho^2 - \mu^2 = \alpha\gamma - \beta^2 < 0$ .

- (ii) If  $\frac{\alpha+\gamma}{2} = \pm\beta$  we find, with  $\frac{\alpha-\gamma}{2} = \mu \neq 0$ ,

$$\alpha(\vartheta) = \mu \pm \beta e^{\pm 2\vartheta}, \quad \beta(\vartheta) = \beta e^{\pm 2\vartheta} \quad \text{and} \quad \gamma(\vartheta) = -\mu \pm \beta e^{\pm 2\vartheta}.$$

Thus, as long as  $\beta \neq 0$ , the parallel family contains no constant Gauss curvature net, but either one constant mean or one constant harmonic mean curvature net, depending on the sign of  $\frac{\alpha-\gamma}{\alpha+\gamma}$ . On the other hand,  $\beta = 0$  yields a parallel family of flat fronts, see Example 4.3.

Note that we obtain discrete analogues of *linear Weingarten surfaces of Bryant type* in hyperbolic or de Sitter space here: with  $\varepsilon = -(\mathfrak{p}\mathfrak{p})$  and  $\kappa = -(\mathfrak{q}\mathfrak{q}) = -\varepsilon$  and assuming, without loss of generality,<sup>28)</sup> that  $\frac{\alpha+\gamma}{2} = -\beta$ , the linear Weingarten condition reads

$$0 = (\mu - \beta)(K - 1) + 2\beta(H - 1) = (\mu - \beta)\varepsilon K_{int} + 2\beta(H - 1).$$

As in the smooth case, see [8, Sect 4], a characteristic feature of these nets is that one of the linear sphere complexes  $\mathfrak{k}^\pm$  consists of spheres touching the infinity sphere  $\mathfrak{p} - \mathfrak{q}$ :

$$W = (\mu - \beta)\mathfrak{q} \odot \mathfrak{q} + 2\beta\mathfrak{q} \odot \mathfrak{p} - (\mu + \beta)\mathfrak{p} \odot \mathfrak{p} = \mathfrak{k}^+ \odot \mathfrak{k}^-$$

with  $\mathfrak{k}^+ = \mathfrak{p} - \mathfrak{q}$  and  $\mathfrak{k}^- = (\beta - \mu)\mathfrak{q} - (\beta + \mu)\mathfrak{p}$ . This observation leads to a geometric Weierstrass type representation for these linear Weingarten nets, cf [15].

- (iii) If  $|\frac{\alpha+\gamma}{2}| < |\beta|$  we write  $\frac{\alpha-\gamma}{2} = \mu$ ,  $\frac{\alpha+\gamma}{2} = \varrho \sinh 2\omega$  and  $\beta = \varrho \cosh 2\omega$  to find

$$\alpha(\vartheta) = \mu + \varrho \sinh 2(\vartheta + \omega), \quad \beta(\vartheta) = \varrho \cosh 2(\vartheta + \omega) \quad \text{and} \quad \gamma(\vartheta) = -\mu + \varrho \sinh 2(\vartheta + \omega).$$

Thus the family does not contain any nets of constant Gauss curvature but a constant mean and a constant harmonic mean curvature net are obtained when  $\sinh 2(\vartheta + \omega) = \mp \frac{\mu}{\varrho}$ . When  $\mu = 0$  these coincide and the net is minimal.

Thus in any family of parallel (non-tubular) linear Weingarten nets occurs at least one of the special nets that the Lawson transformation was discussed for in the above examples. In particular, our analysis shows that the genericity issue of the Lawson transformation only occurs in the cases of parallel families of intrinsically flat surfaces in non-zero ambient curvature.<sup>29)</sup>

28) A change  $\mathfrak{p} \rightarrow -\mathfrak{p}$  of orientation reverses the sign of  $H$ .

29) Note that ‘‘intrinsically flat nets’’ in a flat ambient geometry would be tubular.

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F Burstall  
Department of Mathematical Sciences  
University of Bath  
Bath, BA2 7AY (United Kingdom)  
Email: f.e.burstall@bath.ac.uk

U Hertrich-Jeromin  
Technische Universität Wien  
Wiedner Hauptstraße 8-10/104  
1040 Wien (Austria)  
Email: udo.hertrich-geromin@tuwien.ac.at

W Rossman  
Department of Mathematics  
Kobe University  
Rokko, Kobe 657-8501 (Japan)  
Email: wayne@math.kobe-u.ac.jp