THE INFLUENCE OF OSCILLATORY CORRELATION ON THE ZERO CROSSINGS OF GAUSSIAN PROCESSES

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ABSTRACT

The problem of zero-crossings is of great historical prevalence and promises extensive application. The challenge is to identify the Probability Density Function (PDF) for the times between successive zero-crossings of a stochastic process. In this paper we address the zero-crossing problem for a Gaussian process and investigate the effect of introducing oscillations into the prescribed auto-correlation function. Statistics for the number of zero-crossings occurring within a set time period are calculated and verified by simulations of the process. We find that highly oscillatory auto-correlation functions cause realizations of the stochastic process to become increacingly 'regular' or 'deterministic'. Zeros occur at more regular intervals, implying that the inter-event PDF has an exponential tail with large persistence exponent. The persistence exponent exhibits a complex phenomenology that is strongly influenced by the oscillatory nature of the auto-correlation function. Comparison is made between the theoretical predictions and numerical simulation results.

INTRODUCTION

Continuous random processes are used to mathematically model a huge variety of real world phenomena. In particular, the zero-crossings of such processes are relevant to physical problems such as diffusion (Majumdar et al. 1996) and signal processing (Davenport and Root, 1987). There are further applications in hydrology (Salas et al. 2005), meteorology (Edwards and Hurst, 2001), genetics (Poland, 2004) and finance (Wu et al. 2006). Zero-crossings have also impacted on queuing theory (Brill, 1979), reliability theory (Rychlik, 2000) and applied probability (Malevich, 1969).

Often, the random processes of interest are Gaussian processes, where random samples from the process are Gaussian distributed. Not only do zero-crossings of Gaussian processes inform as to return times and threshold crossings which may be of interest but this zero crossing problem is also relevant to extremal problems. This is because the turning points of one process occur at the same time as the zeros of the derivative of that process (where it exists), which is itself a Gaussian process (Jakeman and Ridley, 2006).

As a result of its far reaching relevance, the zero crossing problem is historically rich and has generated a vast pool of literature. In their review of the zero-crossing problem, Blake and Lindsey (1973) comment that 'the ultimate goal of such an investigation would be to determine the probability density of the lengths of the intervals between zeros of the process' and noted that 'very little success has been achieved in finding this function'. Forty years on, despite various attempts, this is still very much the case. The only analytical result for the pdf of interval times is particular to a Gaussian process with a specific auto-correlation function (Wong, 1966).

The auto-correlation function strongly influences the behaviour of the process and is essential to the determination of the statistical properties of the process and its zero-crossings. If the process is stationary, the auto-correlation function is independent of the origin of time and given by

$$\rho(\tau') = \frac{\langle X(0)X(\tau')\rangle}{\sigma_G^2}.$$

where σ_G^2 is the variance of $X(t)$.

This paper considers a stationary Gaussian process $X(t)$ with zero mean and unit variance and examines anti-correlation through adopting the function:

$$\rho(\tau') = \cos\left(\frac{a\tau'}{L}\right)\exp\left(-\frac{\tau'^2}{2L^2}\right)$$

with L set equal to 1 (equivalent to considering the dimensionless variable $\hat{\tau}' = \tau'/L$) so that

$$\rho(\tau') = \cos(a\tau')\exp\left(-\frac{\tau'^2}{2}\right). \tag{1}$$

This auto-correlation function has two competing length-scales $\ell_1 = 1/a$ from the cosine and $\ell_2 = \sqrt{2}$ from the Gaussian function. This is a smooth process and all derivatives exist meaning there are no sub-fractal effects and the density of zero-crossings is finite (Rice, 1954).

THE FANO FACTOR

The mean zero-crossings rate $\bar{r}$ for the process $X(t)$ is calculated via Rice (1954):

$$\bar{r} = \frac{(-\rho''(0))^{1/2}}{\pi} = \frac{(1+a^2)^{1/2}}{\pi}. \tag{2}$$

As the auto-correlation function gets more oscillatory with increasing a, the rate of zero-crossings increases. For sufficiently large a, the increase in $\bar{r}$ is linear.

The Fano Factor describes the fluctuations in N, the number of zero-crossings, in an interval of fixed length T, and is given by (Fano, 1947)

$$F(T) := \frac{Var(N)}{\langle N\rangle} = 1 + \frac{\langle N(N-1)\rangle}{\langle N\rangle} - \langle N\rangle. \tag{3}$$

When $Var(N) = \langle N\rangle$, $F(T) = 1$ and there are Poisson number fluctuations. $F > 1$ corresponds to super-Poissonian behaviour and zeros occur in clusters, while $F < 1$ corresponds to sub-Poissonian behaviour and zeros are repelled from each other. The Fano Factor is calculated via Rice's result for the mean number of crossings (2) and the following result by Middleton et al. (1996):

$$\begin{aligned}
\langle N(N-1)\rangle &= \frac{2T^2}{\pi^2}\int_0^1 \frac{\mathrm{d}y(1-y)}{(1-\rho^2(yT))^{3/2}}\left[|A^2-B^2|^{1/2} + B\arctan\left(\frac{B}{|A^2-B^2|^{1/2}}\right)\right] \\
A &= -\rho''(0)\left[1-\rho^2(yT)\right] - \rho'^2(yT) \\
B &= \rho''(yT)\left[1-\rho^2(yT)\right] + \rho(yT)\rho'^2(yT)
\end{aligned} \tag{4}$$

In Figure 1, $F(T)$ is plotted as a function of a for various values of T . The overriding trend is that as a increases, the Fano Factor gets small. This means that, for large a (i.e. when $\ell_1 << \ell_2$), there is little fluctuation in the number of zero-crossings in an interval, i.e. zeros occur at more regular intervals and the process appears increasingly 'deterministic' or 'smooth'. Simulations of the process confirm this. The simulations are obtained by multiplying an Gaussian random noise with the autocorrelation function in the frequency domain having Fourier transformed them both, and then Fourier inverting the result back into 'real' space (this is equivalent to forming a convolution of Gaussian random noise and the auto-correlation function).

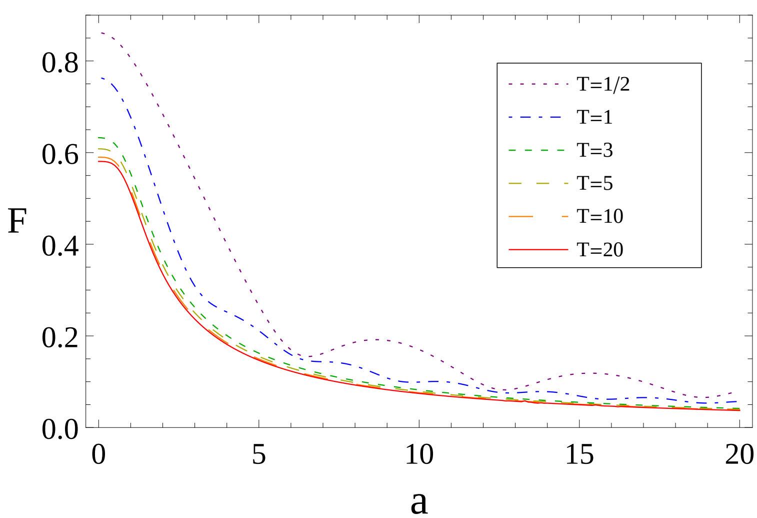

Figure 1. $F(a)$ **for various values of** T **using the full expression (3) via (2) and (4).**

For the smaller values of T there are fluctuations or periodicities in $F(a)$, these arise due to the oscillatory nature of the integrand in the equation for the second factorial moment (4). In order to understand how the periodicities in the Fano Factor relate to the behaviour of $X(t)$ itself, it is helpful to consider it as a function of T as in Figure 2. This reveals that the first minimum of F occurs when $T = \langle\tau\rangle$ the mean interval length, such that $\langle\tau\rangle = 1/\bar{r} = \pi/(1+a^2)^{1/2}$. Subsequent minima occur at $T = n\langle\tau\rangle$ until, with sufficiently large T, the periodicities are damped. For large a the zeros occur regularly and so, when $T = n\langle\tau\rangle$ there will almost always be n zeros in an interval, hence there is very little fluctuation in the number of zeros and so F is small. If $T = \frac{1}{2}(n+1)\langle\tau\rangle$ there are equally likely to be n or $n+1$ zeros in an interval, this leads to greater fluctuation in the number of zeros and so F is larger. This is plainly visibly in the RHS of Figure 2 where the gridlines at n and correspond to the minima in F. As a gets smaller, zeros do not occur regularly and we see that the periodicities in F get much less prominent.

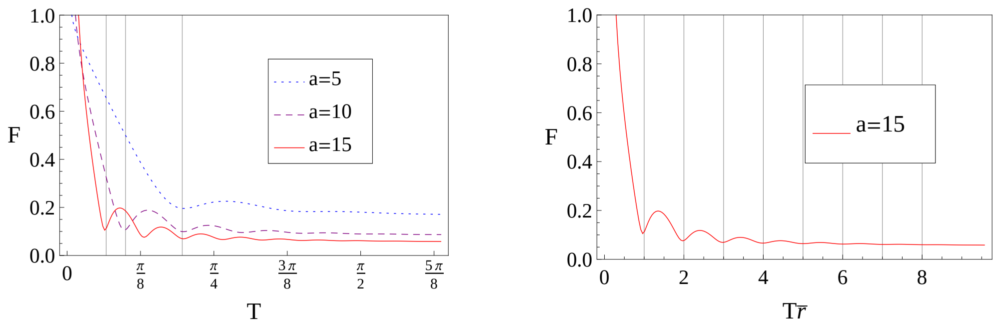

Figure 2. LHS: $F(T)$ for $a = 5, 10$ and 15 with gridlines at $T = \langle\tau\rangle = \pi/(1+a^2)^{1/2}$. RHS: F for $a = 15$ plotted against $\bar{r}T$ so that gridlines correspond to $T = n\langle\tau\rangle = n/\bar{r}$ for $n = 1..8$.

THE INTER-EVENT DENSITY FUNCTION AND ITS ASYMPTOTIC BEHAVIOUR

Theory The time between zero-crossing events is a *continuous* random variable τ with $P_n(\tau)\mathrm{d}\tau$ denoting the probability that the $n+1$th event occurring after time t_0 falls within the interval $t_0 + \tau$ to $t_0 + \tau + \mathrm{d}\tau$ and $P_0(\tau)$ is the density of the interval length between successive events.

Following McFadden (1958), the clipped process derived from $X(t)$

$$\xi(t) = \begin{cases} 1 & X(t) \geq 0 \\ -1 & X(t) < 0 \end{cases}$$

identifies the zero-crossings of X(t). The auto-correlation function of $\xi(t)$ is $R(\tau') =$

$\langle \xi(t)\xi(t+\tau')\rangle$, from which it can be shown that

$$\frac{R''(\tau)}{4\bar{r}} = \sum_{n=0}^{\infty}(-1)^n P_n(\tau).$$

where $\bar{r}$ is the crossing rate (2). Taking the Laplace transformations of the above quantities

$$r(s) = \mathcal{L}\left(\frac{R''(\tau)}{4\bar{r}}\right) \quad p_n(s) = \mathcal{L}\left(P_n(\tau)\right),$$

obtains

$$r(s) \;=\; \sum_{n=0}^{\infty}(-1)^n p_n(s) \tag{5}$$

Assuming the intervals are statistically independent, so that $p_n(s) = p_0^{n+1}(s)$, enables (5) to be summed and solved to give:

$$p_0(s) \;=\; \frac{r(s)}{1-r(s)} \tag{6}$$

as obtained by heuristic arguments in (Derrida et al. 1996), and indicates that $p_0(s)$ has a pole at the (negative) value of $s = -\theta$ which solves

$$1 - r(-\theta) = 0. \tag{7}$$

The asymptotic form of the inter-event distribution will then be $P_0(\tau) \sim \exp(-\theta\tau)$ where the persistence exponent θ describes the rate of decay. Relaxing the assumption of statistical independence modifies the location of this pole and consequently the asymptotic form of the inter-event density function. This is achieved by assuming the model

$$p_n(s) = a_n(s)p_0^{n+1}(s) \tag{8}$$

where $a_n(s)$ embodies information about correlation. Noting that $p_n(s)$ is a moment generating function enables the function $a_n(s)$ to be determined as

$$a_n(s) = \frac{p_n(s)}{p_0(s)^{n+1}} = 1 + \frac{s^2}{2}\left(\left\langle\left(\sum_{j=1}^{n+1}\tau_j\right)^2\right\rangle - (n+1)\left(n\langle\tau\rangle^2 + \langle\tau^2\rangle\right)\right) + O(s^3)$$

where τ_i is the interval between the ith and $i+1th$ crossings. The sumnation term contains correlation terms of the form $\langle\tau_i\tau_j\rangle$ in addition to terms involving moments of the single inter-event τ. Defining the correlation coefficient to be

$$\kappa_{ij} = \left(\langle\tau_i\tau_j\rangle - \langle\tau\rangle^2\right)/\sigma^2 \qquad i \neq j$$

with $\sigma^2 \equiv \langle\tau^2\rangle - \langle\tau\rangle^2$, and on assuming stationarity so that $\kappa_{i,i+j} = \kappa_j$, gives

$$a_n(s) = 1 + \sigma^2 s^2 \sum_{j=1}^{n+1}(n+1-j)\kappa_j + O(s^3).$$

A closed form expression for this involving only $\kappa = \kappa_1$ can be obtained on assuming a suitable closure model. Assuming that $\kappa_j = 0$ for $j \geq 2$ gives

$$a_n(s) = 1 + n\sigma^2 s^2\kappa + O(s^3), \tag{9}$$

Whereupon, inserting (8-9) in (5) yields

$$r(s) \;=\; \frac{p_0}{1+p_0} - \frac{\kappa(\sigma s p_0)^2}{(1+p_0)^2}$$

revealing their relation with (6) if $\kappa = 0$. Moreover, solving in favour of $p_0(s)$, gives

$$p_0(s) = \frac{(1-2r(s)) + \left(1 - 4\kappa\sigma^2 s^2 r(s)\right)^{1/2}}{2\left(r(s) - 1 + \kappa\sigma^2 s^2\right)}$$

The pole of $p_0(s)$ occurs where the denominator of vanishes when $s \to -\theta$, which now becomes the solution of

$$r(-\theta) - 1 = -\kappa\sigma^2\theta^2. \tag{10}$$

Note that this is not a linear perturbation of the uncorrelated assumption result in (7), but retrieves it if $\kappa = 0$.

The intervals between crossings can also be related to the conditional probability that a zero falls between $t_o + \tau$ and $t_0 + \tau + \mathrm{d}\tau$ given that a zero occurs at t_0 (McFadden, 1958). Denoting this probability by $U(\tau)\mathrm{d}\tau$ it follows that

$$U(\tau) = \sum_{n=0}^{\infty} P_n(\tau).$$

It can be shown that $\langle\tau\rangle = 1/\bar{r}$ as expected and σ^2 and κ are found to be

$$\sigma^2 = \frac{1}{2}\left(\frac{2I}{\bar{r}} + \frac{(1+2J)}{\bar{r}^2}\right), \quad \text{and} \quad \kappa = \frac{1}{2}\left(\frac{1+2J-2\bar{r}I}{1+2J+2\bar{r}I}\right), \tag{11}$$

with

$$I = \int_0^\infty R(\tau)\mathrm{d}\tau, \qquad J = \int_0^\infty (U(\tau) - \bar{r})\mathrm{d}\tau.$$

The function $U(\tau)$ has been determined for a Gaussian process to be (Rice, 1954),

$$U(\tau) = \frac{1}{\bar{r}\pi^2 \left(1 - \rho^2(yT)\right)^{3/2}} \left[|A^2 - B^2|^{1/2} + B \arctan\left(\frac{B}{|A^2 - B^2|^{1/2}} \right) \right]$$

with A and B as in (4). Evaluating the clipped auto-correlation function requires an integral of the bivariate density function for X and the result for a Gaussian process is (Van Vleck and Middleton, 1966)

$$R(\tau) = \frac{2}{\pi} \arcsin\left(\rho(\tau)\right)$$

and these expressions permit $\bar{r}$ and σ^2 to be determined self-consistently. Also,

$$r(s) = L\left(\frac{R''(\tau)}{4\bar{r}} \right) = \frac{1}{2} + \frac{s}{2\pi\bar{r}} \int_0^\infty \exp(-s\tau) \frac{\rho'(\tau)}{(1-\rho^2)^{1/2}} \mathrm{d}\tau.$$

enabling the poles $-\theta_j$ to be evaluated by solving (10). The tail of the pdf is then determined from

$$\lim_{\tau\to\infty} P_0(\tau) \sim \mathcal{L}^{-1} \left(\sum_j \frac{1}{r'(-\theta_j)(s+\theta_j)} \right)$$

and the persistence exponent is obtained from a linear fit to

$$\theta = -\frac{\mathrm{d}}{\mathrm{d}\tau} \ln \left| \mathcal{L}^{-1} \left(\sum_j \frac{1}{r'(-\theta_j)(s+\theta_j)} \right) \right|. \tag{12}$$

Results The mean, variance and correlation coefficient are shown in Figure 3. Note that the variance declines rapidly with increasing a and becomes less than the mean at $a \sim 1.2$, indicating the distribution is narrowing. The graph of κ shows that for small a, the interval lengths are anti-correlated. This indicates that zeros of the random process are repelled from each other, being principally affected by the exponential factor in $\rho(\tau')$ which primarily affects the stochastic properties of the process . The intervals become positively correlated once the oscillations in $\rho(\tau')$ begin to occur within the characteristic width of the exponential function (i.e. for $\ell_1 << \ell_2$). This indicates that the process is becoming more regular, in accord with the behaviour found for $F(T)$.

The dispersion relation (10) has a particularly rich structure for the oscillatory autocorrelation functions considered here. Its solution reveals that $p_0(s)$ possesses multiple real-valued poles at the (negative) values of $s = -\theta_j$ which can coalesce with increasing a to form a complex pole $s = -\theta_R - \theta_I i$. The topology of the poles as a function of a is shown in Figure 4. For small a, the magnitude of the least negative pole (solid line) coincides with the value of θ obtained from (12) (solid orange line with circles). As the second real pole (solid line) nears the first, its contribution in (12) becomes significant (note that $r'(-\theta_1)$ and $r'(-\theta_2)$ are necessarily of opposite sign) creating logarithmic modifications to the tail of the pdf which persist even for

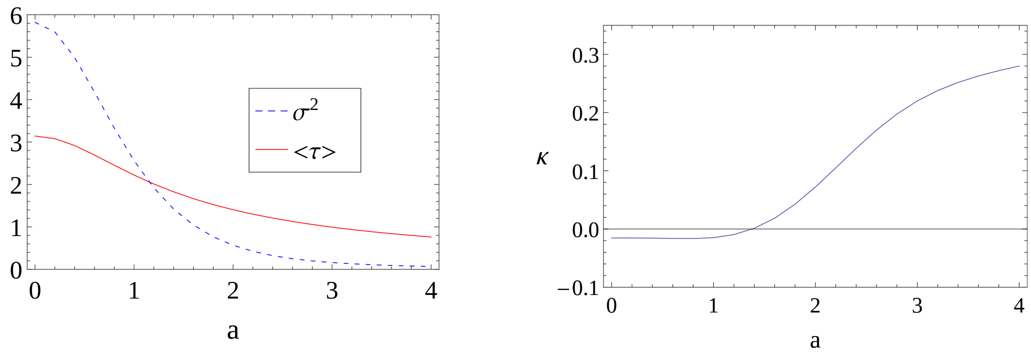

Figure 3. LHS: Mean and variance of interval between crossings as a function of a calculated via equations 2 and 11. RHS the correlation coefficient as a function of a calculated via equation 11

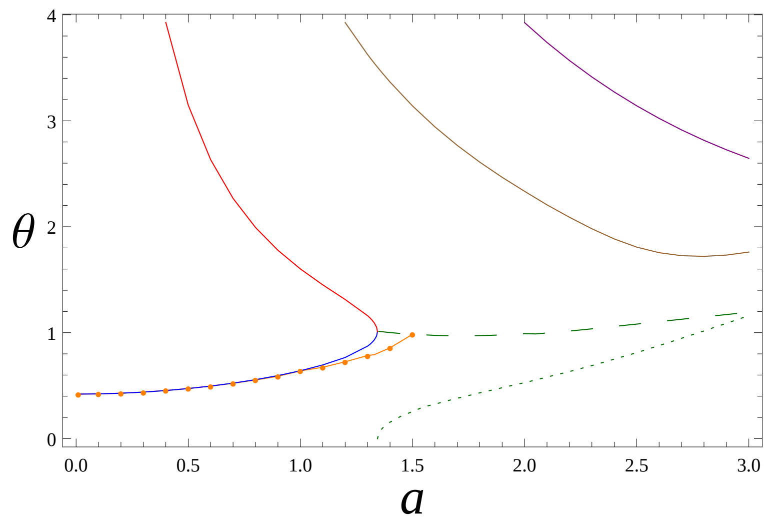

Figure 4. The real (solid lines) and complex (real part: dashed line, imaginary part: dotted line) poles of (10), as well as the calculated value of θ (solid line with circles) as a solution to (12) for $a \leq \sqrt{2}$

$\tau/\langle\tau\rangle > 1$. The effect of these adjustments is that the persistence exponent is *reduced* from that value predicted by the first pole regarded in isolation. The first two real poles merge at $a \sim 1.34$, whereupon the pole becomes complex, the real part being shown in Figure 4 by the dashed curve, the imaginary part by the dotted curve. This confluence leads to a structural change in the asymptotic form of the pdf and as a consequence the corresponding persistence exponent. Figure 5 compares the calculated θ with that obtained from simulations. The value of θ increases with a. This is consistent with the analysis of the Fano factor which revealed that, as a increases intervals become more regular and hence, the tail of the distribution must decay faster as a increases.

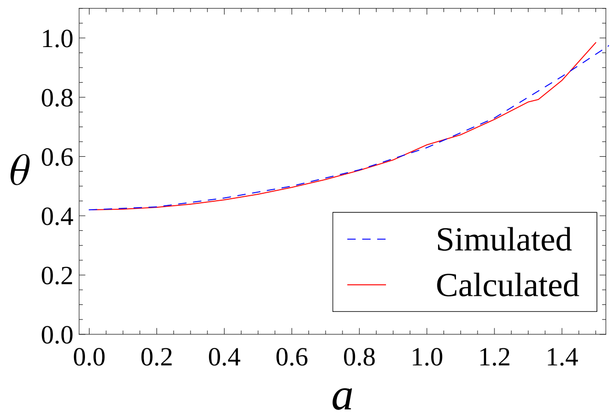

Figure 5. The persistence exponent θ, from calculations and from simulations.

CONCLUSIONS

Analysis of the Fano Factor reveals that, by adjusting the anti-correlation scale size a in the autocorrelation function ρ, the process can be made 'more deterministic' so that the intervals between zero crossings are more correlated with each other and have a steeper pdf. Due to the inticate structure of the dispersion relation (10) in this case, calculating the persistence exponent requires careful attention as to how many poles contribute to the asymptotic behaviour of the tail of the inter-event distribution. If the poles are sufficiently well separated, then the least negative, or principal pole provides an accurate estimate of the persistence exponent θ. But as the scale-size of the oscillations in ρ become comparable with that for the exponential fall-off in ρ, the poles move closer to each other and eventually merge to form a complex pole. The nature of θ is affected by all the poles in the proximity of the principal pole. This causes θ to be less than that predicted by the principal pole alone, and this has been verified quantitatively by the simulation results. This work shows that the structure of the auto-correlation function of the random process is the principal cause of the rich phenomenology exhibited by the inter-event PDF. In particular, if the scale-size of the oscillations in ρ is smaller than that associated with the characteristic decay in ρ, then care must be exercised in determination of θ. The implications for this on auto-correlation functions with power-law decay will be treated elsewhere.

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