Bubbling solutions for Moser-Trudinger type equations on compact Riemann surfaces

Pablo Figueroa* and Monica Musso†

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Abstract

We study an elliptic equation related to the Moser-Trudinger inequality on a compact Riemann surface (S, g),

$$\Delta_g u + \lambda \left(ue^{u^2} - \frac{1}{|S|} \int_S ue^{u^2} dv_g \right) = 0, \quad \text{in } S, \qquad \int_S u \, dv_g = 0,$$

where $\lambda > 0$ is a small parameter, $|S|$ is the area of S, Δ_g is the Laplace-Beltrami operator and dv_g is the area element. Given any integer $k \geq 1$, under general conditions on S we find a bubbling solution u_λ which blows up at exactly k points in S, as $\lambda \to 0$. When S is a flat two-torus in rectangular form, we find that either seven or nine families of such solutions do exist for $k = 2$. In particular, in any square flat two-torus actually nine families of bubbling solutions with two bubbling points do exist. If S is a Riemann surface with non-constant Robin's function then at least two bubbling solutions with $k = 1$ exists.

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1 Introduction

Let (S, g) be a compact, orientable Riemann surface. We denote by $|S|$ the area of S, Δ_g the Laplace-Beltrami operator on S and dv_g the area element. This paper is devoted to the construction of solutions to the problem

$$\begin{cases} \Delta_g u + \lambda \left(ue^{u^2} - \dfrac{1}{|S|} \displaystyle\int_S ue^{u^2} dv_g \right) = 0, & \text{in } S, \\ \displaystyle\int_S u \, dv_g = 0, \end{cases} \tag{1.1}$$

for any values of the small parameter $\lambda > 0$. These solutions turn out to blow-up, as the parameter $\lambda \to 0^+$ at very specific points of S.

Problem (1.1) is related to the Trudinger-Moser inequality [29] over a compact Riemann surface (S, g), which can be stated as follows

$$\sup \left\{ \int_S e^{4\pi u^2} \, dv_g \, : \, u \in H^1(S), \int_S u \, dv_g = 0 \text{ and } \int_S |\nabla u|^2 \, dv_g = 1 \right\} < +\infty.$$

*Escuela de Educación de Matemática e Informática Educativa, Universidad Católica Silva Henríquez, General Jofré 462, Santiago, Chile. E-mail: pfigueroas@ucsh.cl. Author supported by grants Fondecyt Postdoctorado 3120039 and Fondecyt Iniciación 11130517, Chile.

†Departamento de Matemática, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Santiago, Chile. E-mail: mmusso@mat.uc.cl. Author supported by grants FONDECYT Grant 1160135 and Millennium Nucleus Center for Analysis of PDE, NC130017

This type of inequality was first proved in [22] on compact Riemannian manifolds of any dimension n. When the dimension is two, this inequality was proved in [24] on manifolds with and without boundary, and the existence of extremal functions was established. We refer also to [25, 26, 34, 35] for related results and generalizations.

It is simple to see that critical points of the above constrained variational problem satisfy, after a simple scaling, an equation of the form (1.1). Our purpose then is to study the existence of solutions to (1.1) for λ positive and small and to describe their asymptotic behavior as $\lambda \to 0^+$.

Weak solutions of (1.1) are critical points of the following energy functional

$$J_\lambda(u) = \frac{1}{2}\int_S |\nabla u|_g^2 \, dv_g - \frac{\lambda}{2}\int_S e^{u^2} \, dv_g, \quad u \in \bar{H}, \tag{1.2}$$

where $\bar{H} = \{u \in H^1(S) : \int_S u dv_g = 0\}$, which corresponds to the free energy associated to the critical Trudinger embedding in the sense of Orlicz spaces [30, 32, 33]

$$\bar{H} \ni u \mapsto e^{u^2} \in L^p(S) \;\; \forall p \geq 1.$$

The energy functional (1.2) is thus well defined and it has a Mountain Pass geometric structure. Nevertheless, it is characterized by lack of compactness, which makes it impossible to search for critical points of (1.2) using the classical tools of the Calculus of Variations or of the Critical Point Theory. Indeed, loss of compactness translates into the presence of non-convergent Palais-Smale sequences for the corresponding functional and space of functions.

To better understand this, let us consider the flat case, namely, when $S \equiv \Omega \subset \mathbb{R}^2$ is a bounded domain. The Trudinger-Moser inequality concerns the limiting case $p = 2$ of the Sobolev embeddings $W^{1,p}(\Omega) \subset L^{\frac{2p}{2-p}}(\Omega)$. It states that there exists $C_2 > 0$ such that

$$\sup_{\left\{v \in W_0^{1,2}(\Omega),\ \|\nabla v\|_{L^2(\Omega)} = 1\right\}} \int_\Omega e^{\alpha |v|^2} \, dx \; \begin{cases} \leq C_2|\Omega|, & \text{if } \alpha \leq \alpha_2 \\ = +\infty, & \text{if } \alpha > \alpha_2 \end{cases},$$

where $|\Omega|$ is the area of Ω and $\alpha_2 = 4\pi$. After a simple scaling, critical points of the above constrained variational problem satisfy the equation

$$\Delta u + \lambda u e^{u^2} = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\lambda > 0$, whose associated energy functional is

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega e^{u^2}, \quad u \in H_0^1(\Omega).$$

For the functional I_λ a precise classification of all Palais-Smale sequences does not seem possible after the results in [2]. Some information is available for sequences of solutions to (1.3), thanks to the result in [17], that states

Assume that u_n solves problem (1.3) for $\lambda = \lambda_n$, with $I_{\lambda_n}(u_n)$ bounded and $\lambda_n \to 0$ as $n \to +\infty$. Then, passing to a subsequence if necessary, there is an integer $k \geq 0$ such that as $n \to +\infty$

$$I_{\lambda_n}(u_n) = 2\pi k + o(1). \tag{1.4}$$

A more precise characterization of the sequence of solutions $(u_n)_n$ is known when $k = 1$, see [1]: for all large n, the solution u_n has only one isolated maximum, whose value diverges to $+\infty$ as $\lambda_n \to 0$, which is attained around a very specific point $x_0 \in \Omega$. In fact, x_0 is a critical point of Robin's function, defined as $x \mapsto H_\Omega(x, x)$, where H_Ω is the regular part of the corresponding Green's function for the homogeneous Dirichlet problem in Ω.

Concerning existence of solutions to (1.3) satisfying (1.4), in [2], it is proven that there is a $\lambda_0 > 0$ such that a solution to (1.3) exists whenever $0 < \lambda < \lambda_0$ (this is in fact true for a larger class of nonlinearities with *critical exponential growth*). By construction this solution falls into the bubbling category (1.4) with $k = 1$ as $\lambda \to 0$. If Ω has a sufficiently small hole, Struwe in [31] built a solution taking advantage of the presence of topology. This solution exists for a class of nonlinearities, perturbation of the Trudinger-Moser one, that also include $\lambda u e^{u^2-u}$ for which no solution exists for small λ, in a disk, see [3, 12]. It is reasonable to believe that the construction of Struwe in reality produces a second solution of equation (1.3), but this is not known yet. Similar results to [4, 17] on compact Riemann surfaces are obtained in [34].

In [15] authors addressed the existence of bubbling solutions for (1.3) as $\lambda \to 0$ when Ω is not contractible to a point, and k in (1.4) is any integer number. They provide sufficient conditions for the existence of solutions to (1.3) for small λ, which satisfy the bubbling condition (1.4) and give a precise characterization of its bubbling location. In particular, they show that if Ω has a hole of any size, namely, Ω is not simple connected then at least one of such a solutions exists with $k = 2$ and if Ω has $d \geq 1$ holes, then $d + 1$ solutions with $k = 1$ exist.

The question we address in this paper is whether it is possible to construct a family of solutions u_λ to problem (1.1), for any $\lambda > 0$ small, whose energy $J_\lambda(u_\lambda)$, defined in (1.2), is quantized in the sense of (1.4), and whose asymptotic behavior resembles a bubbling phenomena at points, for $\lambda \to 0$.

In order to state our general result, let us introduce some notations. For a given Riemann surface (S, g), we introduce the Green's function $G(x, p)$ with pole at $p \in S$ as the solution of

$$\begin{cases} -\Delta_g G(\cdot, p) = \delta_p - \dfrac{1}{|S|} \quad \text{in } S \\ \displaystyle\int_S G(x, p)\, dv_g = 0. \end{cases} \tag{1.5}$$

Let $k \geq 1$ be an integer, $\xi_1, \xi_2, \dots, \xi_k \in S$ be k distinct points and $m_1, m_2, \dots, m_k$ be k positive numbers. We define the following functional

$$\begin{aligned} \psi_k(\xi, m) = (\log 16 - 2) \sum_{j=1}^k m_j^2 + \sum_{j=1}^k m_j^2 \log m_j^2 - 4\pi \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j) \\ - 4\pi \sum_{i=1}^k \sum_{j=1, j\neq i}^k m_i m_j G(\xi_i, \xi_j), \end{aligned} \tag{1.6}$$

where $\xi = (\xi_1, \dots, \xi_k)$ and $m = (m_1, \dots, m_k)$. Here, G is the Green's function for the Laplace-Beltrami operator on S given by (1.5) and H is its regular part. Let us consider an open set $\mathcal{D}$ compactly contained in the domain of the functional ψ_k, namely

$$\bar{\mathcal{D}} \subset \{(\xi, m) \in S^k \times \mathbb{R}_+^k \ \mid \ \xi_i \neq \xi_j \ \forall\, i \neq j\ \}.$$

We say that ψ_k has a *stable critical point situation* if there exists a $\delta > 0$ such that for any $g \in C^1(\bar{\mathcal{D}})$ with $\|g\|_{C^1(\bar{\mathcal{D}})} < \delta$, the perturbed functional $\psi_k + g$ has a critical point in $\mathcal{D}$.

We can now state our general result.

Theorem 1.1. *Let (S, g) be a compact, orientable Riemann surface. Let $k \geq 1$ and assume that there is an open set $\mathcal{D}$ where ψ_k has a stable critical point situation. Then, for all small $\lambda > 0$ there exists a family of solutions u_λ of problem* (1.1) *such that as $\lambda \to 0$*

$$\frac{1}{2} \int_S |\nabla u_\lambda|_g^2\, dv_g - \frac{\lambda}{2} \int_S e^{u_\lambda^2}\, dv_g \ = \ 2k\pi + O(\lambda). \tag{1.7}$$

Moreover, there exists $(\xi_\lambda, m_\lambda) \in \mathcal{D}$, with $\xi_\lambda = \big(\xi_\lambda^{(1)}, \dots, \xi_\lambda^{(k)}\big)$ and $m_\lambda = \big(m_\lambda^{(1)}, \dots, m_\lambda^{(k)}\big)$ such that, passing to a subsequence, $(\xi_\lambda, m_\lambda) \to (\xi_0, m_0)$ with $\nabla\psi_k(\xi_0, m_0) = 0$ and

$$u_\lambda(x) \;=\; \sqrt{\lambda}\left(8\pi\sum_{j=1}^{k} m_\lambda^{(j)} G(x, \xi_\lambda^{(j)}) \;+\; O(\lambda)\right) \tag{1.8}$$

as $\lambda \to 0$, uniformly on compact subsets of $S \setminus \{\xi_1, \dots, \xi_k\}$.

Concrete examples of surfaces S on which problem (1.1) has solutions satisfying (1.7)-(1.8) depends on the possibility to ensure the existence of special critical points for the function ψ_k defined in (1.6).

To start with, we observe that if S is a compact Riemann surface with non-constant Robin's function then at least two bubbling solutions with $k = 1$ exists. Indeed, since S is compact then $H(\xi, \xi)$ attains its minimum and its maximum. In this case, it is easy to show that ψ_1 has two stable critical point situations. Thus, problem (1.1) has one solution which is bubbling near the global minimizer of $H(\xi, \xi)$, and another solution which is bubbling near the global maximizer, as $\lambda \to 0^+$. Unfortunately, this kind of solutions are hopeless to be found for instance when S is the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ or when S is the flat two-torus T, since, in these examples, the function $H(\xi, \xi)$ is constant. Nevertheless, in these two examples, we can prove the existence of solutions with $k = 2$ bubbling points. The case of the flat torus is particularly surprising.

If $k = 2$, the functional ψ_k in (1.6) takes the simplified form

$$\psi_2(\xi, m) = \; A\sum_{j=1}^{2} m_j^2 + \sum_{j=1}^{k} m_j^2 \log m_j^2 - 8\pi m_1 m_2 G(\xi_1, \xi_2)$$

where $A = \log 16 - 2 - 8\pi c$, where c is the constant value of $H(\xi, \xi)$ in the case of the sphere and the flat torus.

Let us start with $S = \mathbb{S}^2$. Since problem (1.1) is invariant under rotations, it is not restrictive to look for solutions with one bubbling point to be a fixed point on $\mathbb{S}^2$, say ξ_2. Indeed, by a rotation, one can get another solution bubbling at any other point of $\mathbb{S}^2$, just rotating ξ_2 up to this other point. Thus, we fix $\xi_2 \in \mathbb{S}^2$ and we are reduced to study the existence of critical points for $\xi_1 \in \mathbb{S}^2 \mapsto G(\xi_1, \xi_2)$. Notice that $\xi_1 \mapsto G(\xi_1, \xi_2)$ has a global minimum. Then from simple arguments, one sees that ψ_2 has a stable critical point situation. We thus get the validity of

Theorem 1.2. *Assume that $S = \mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ and fix $\xi_2 \in \mathbb{S}^2$. Then there exists a family of solutions u_λ to problem (1.1) with two bubbling points such that as $\lambda \to 0$ the two bubbling points converge to (ξ_1, ξ_2) with ξ_1 the global minimum of $G(\cdot, \xi_2)$ and*

$$\frac{1}{2}\int_{\mathbb{S}^2} |\nabla u_\lambda|_{g_0}^2 \, dv_{g_0} - \frac{\lambda}{2}\int_{\mathbb{S}^2} e^{u_\lambda^2}\, dv_{g_0} = 4\pi + o(1), \quad \text{as} \quad \lambda \to 0$$

where g_0 is the standard round metric on $\mathbb{S}^2$.

Let us now discuss the case when $S = T$ is a rectangle and we look for solutions u to (1.1) that are doubly periodic functions on ∂T. The surprising fact of this case is the multiplicity of solutions.

Without loss of generality, assume that, in complex notation,

$$T = \Big\{ z = s\frac{a}{2} + t\frac{\mathrm{i}b}{2} \;:\; s, t \in \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big\}, \tag{1.9}$$

with i the imaginary unity. Since the equation is invariant under translations, if u is a solution to (1.1) then $u(\cdot + p)$ is also a solution to (1.1) for any $p \in T$. In this setting, our next result states the existence of families of solutions of the form (1.8), and satisfying (1.7), with $k = 2$. The exact number of such solutions can be 7 or 9, depending on the value of

$$\tau = b/a.$$

Our result states as follows.

Theorem 1.3. *Assume that T is a rectangle in $\mathbb{R}^2$ given by (1.9) and $\tau = b/a$. Then there are $\tau_0 < 1 < \tau_1$ such that if either $\tau \in (0, \tau_0] \cup [\tau_1, +\infty)$ or $\tau \in (\tau_0, \tau_1)$ then there is $\lambda_0 > 0$ such that for any $0 < \lambda \leq \lambda_0$ there exist either seven or nine different families of doubly periodic on ∂T bubbling solutions $u_{\lambda,i}$ respectively to problem (1.1). These solutions safisfy*

$$J_\lambda(u_{\lambda,i}) = \frac{1}{2}\int_T |\nabla u_{\lambda,i}|^2\,dx - \frac{\lambda}{2}\int_T e^{u^2_{\lambda,i}}\,dx = 4\pi + O(\lambda), \tag{1.10}$$

as $\lambda \to 0$. Moreover, there exist bubbling points $\xi_{\lambda,i} = (\xi^{(1)}_{\lambda,i}, \xi^{(2)}_{\lambda,i}) \in T^2$ and weights $m_{\lambda,i} = (m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i}) \in \mathbb{R}^2_+$ such that, passing to a subsequence, $(\xi_{\lambda,i}, m_{\lambda,i}) \to (\xi_{0,i}, m_{0,i})$ with $\nabla\psi_2(\xi_{0,i}, m_{0,i}) = 0$ and

$$u_{\lambda,i}(x) \;=\; \sqrt{\lambda}\,\Big(8\pi m^{(1)}_{\lambda,i} G\big(x, \xi^{(1)}_{\lambda,i}\big) + 8\pi m^{(2)}_{\lambda,i} G\big(x, \xi^{(2)}_{\lambda,i}\big) \;+\; O(\lambda)\Big) \tag{1.11}$$

as $\lambda \to 0$, uniformly on compact subsets of $T \setminus \{\xi^{(1)}_{0,i}, \xi^{(2)}_{0,i}\}$, where $\xi_{0,i} = (\xi^{(1)}_{0,i}, \xi^{(2)}_{0,i})$. Here we intend that $i \in \{1, \ldots, 7\}$ when $\tau \in (0, \tau_0] \cup [\tau_1, +\infty)$, and $i \in \{1, \ldots, 9\}$ when $\tau \in (\tau_0, \tau_1)$.

The location of the two bubbling points for the solutions in this result is completely determined. Indeed, fixing $i \in \{1, \ldots, 7\}$ when $\tau \in (0, \tau_0] \cup [\tau_1, +\infty)$ or $i \in \{1, \ldots, 9\}$ when $\tau \in (\tau_0, \tau_1)$ the bubbling points $\xi^{(1)}_{\lambda,i}$ and $\xi^{(2)}_{\lambda,i}$ satisfy $\xi^{(1)}_{\lambda,i} - \xi^{(2)}_{\lambda,i} \to p_j$ for some $j \in \{1, 2, 3\}$ as $\lambda \to 0$ where p_1, p_2 and p_3 are the half periods of T:

$$p_1 = \frac{a}{2}, \quad p_2 = \frac{\mathrm{i}b}{2}, \quad p_3 = \frac{a + \mathrm{i}b}{2}. \tag{1.12}$$

Moreover, we can show some properties of the weights m_i's. If $0 < \tau \leq \tau_0$ then

- there are two bubbling points $\big(\xi^{(1)}_\lambda, \xi^{(2)}_\lambda\big)$ satisfying $\xi^{(1)}_\lambda - \xi^{(2)}_\lambda \to p_1$, and three different pairs of weights $(m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i})$ converging to either (m_0, m_0), (m_1, m_2) or (m_2, m_1) as $\lambda \to 0$ for some m_0, m_1 and m_2 with $m_0 \neq m_1$, $m_0 \neq m_2$ and $m_1 \neq m_2$ such that they give rise to three bubbling solutions;
- there are two bubbling points $\big(\xi^{(1)}_\lambda, \xi^{(2)}_\lambda\big)$ satisfying $\xi^{(1)}_\lambda - \xi^{(2)}_\lambda \to p_2$, and only a pair of weights $(m_{\lambda,1}, m_{\lambda,2})$ converging to (m_3, m_3) as $\lambda \to 0$ for some m_3 such that it gives rise to a bubbling solution; and
- there are two bubbling points $(\xi^{(1)}_\lambda, \xi^{(2)}_\lambda)$ satisfying $\xi^{(1)}_\lambda - \xi^{(2)}_\lambda \to p_3$, and three different pairs of weights $(m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i})$ converging to either (m_4, m_4), (m_5, m_6) or (m_6, m_5) as $\lambda \to 0$ for some m_4, m_5 and m_6 with $m_4 \neq m_5$, $m_4 \neq m_6$ and $m_5 \neq m_6$ such that they give rise to three bubbling solutions.

If $\tau \geq \tau_1$ then

- there are two bubbling points $\big(\xi^{(1)}_\lambda, \xi^{(2)}_\lambda\big)$ satisfying $\xi^{(1)}_\lambda - \xi^{(2)}_\lambda \to p_1$, and only a pair of weights $(m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i})$ converging to either (m_0, m_0) as $\lambda \to 0$ for some m_0 such that it gives rise to only a bubbling solution;

- there are two bubbling points $\left(\xi_\lambda^{(1)}, \xi_\lambda^{(2)}\right)$ satisfying $\xi_\lambda^{(1)} - \xi_\lambda^{(2)} \to p_2$, and three different pairs of weights $(m_{\lambda,1}, m_{\lambda,2})$ converging to either (m_1, m_1), (m_2, m_3) or (m_3, m_2) as $\lambda \to 0$ for some m_1, m_2 and m_3 with $m_1 \neq m_2$, $m_2 \neq m_3$ and $m_1 \neq m_3$ such that they give rise to three bubbling solution; and

- there are two bubbling points $(\xi_\lambda^{(1)}, \xi_\lambda^{(2)})$ satisfying $\xi_\lambda^{(1)} - \xi_\lambda^{(2)} \to p_3$, and three different pairs of weights $(m_{\lambda,i}^{(1)}, m_{\lambda,i}^{(2)})$ converging to either (m_4, m_4), (m_5, m_6) or (m_6, m_5) as $\lambda \to 0$ for some m_4, m_5 and m_6 with $m_4 \neq m_5$, $m_4 \neq m_6$ and $m_5 \neq m_6$ such that they give rise to three bubbling solutions.

If $\tau \in (\tau_0, \tau_1)$ then for every $j = 1, 2, 3$

- there are two bubbling points $\left(\xi_{\lambda,j}^{(1)}, \xi_{\lambda,j}^{(2)}\right)$ satisfying $\xi_{\lambda,j}^{(1)} - \xi_{\lambda,j}^{(2)} \to p_j$, and three different pairs of weights $(m_{\lambda,i,j}^{(1)}, m_{\lambda,i,j}^{(2)})$ converging to either $(m_{0,j}, m_{0,j})$, $(m_{1,j}, m_{2,j})$ or $(m_{2,j}, m_{1,j})$ as $\lambda \to 0$ for some differents $m_{0,j}$, $m_{1,j}$ and $m_{2,j}$ such that they give rise to three bubbling solutions.

A very interesting situation in the rectangular case is when $a = b$ (or $\tau = 1$), namely, in case of a square. Recall that $\tau_0 < 1 < \tau_1$. This fact follows from the analysis in the proof of the previous result, but we highlight it due to the multiplicity of bubbling solutions we obtain: *there exist* **nine** *different families of doubly periodic on ∂T bubbling solutions to problem* (1.1) *in any square.*

Theorem 1.4. *Assume that T is a square in $\mathbb{R}^2$. Then for any λ small enough and for every half period p_j, $j \in \{1, 2, 3\}$, see* (1.12)*, there exist bubbling points $\xi_{\lambda,j} = (\xi_{\lambda,j}^{(1)}, \xi_{\lambda,j}^{(2)}) \in T^2$ and three different pairs of weights $m_{\lambda,i,j} = (m_{\lambda,i,j}^{(1)}, m_{\lambda,i,j}^{(2)}) \in \mathbb{R}_+^2$, $i = 1, 2, 3$ giving rise to nine bubbling solutions $u_{\lambda,i,j}$, satisfying* (1.10) *as $\lambda \to 0$ for $i = 1, 2, 3$ such that, passing to a subsequence, $(\xi_{\lambda,j}, m_{\lambda,i,j}) \to (\xi_{0,j}, m_{0,i,j})$, $\xi_{\lambda,j}^{(1)} - \xi_{\lambda,j}^{(2)} \to p_j$ and the property* (1.11) *holds as $\lambda \to 0$, uniformly on compact subsets of $T \setminus \{\xi_{0,j}^{(1)}, \xi_{0,j}^{(2)}\}$, where $\xi_{0,j} = (\xi_{0,j}^{(1)}, \xi_{0,j}^{(2)})$.*

Theorems 1.3 and 1.4 follow from the fact that the existence of nondegenerate critical points of ψ_2 is a stable critical point situation. From similar ideas follows Theorem 1.2, studying first critical points of ξ_1's and then the weights m_i's .

For the case $k \geq 3$, or the case $k = 2$ on a surface S where the function $H(\xi, \xi)$ is not constant, the analysis of the map $(\xi, m) \mapsto \psi_k(\xi, m)$ is much harder.

We conclude our introduction mentioning the link between the theorems 1.3 - 1.2 and the results contained in [15, 16] on concentration phenomena for the Liouville-type problem

$$\Delta u + \varepsilon^2 e^u = 0, \text{in } \Omega, \quad u = 0, \text{on } \partial\Omega, \tag{1.13}$$

where Ω is bounded smooth domain in $\mathbb{R}^2$, see [5, 14, 19] and references therein. Our results are also connected to those for Liouville-type equations on compact Riemann surfaces, see [6, 7, 8, 13, 18, 21]. The fine blow-up structure for Liouville-type equations on domains in $\mathbb{R}^2$ or on manifolds very close to the bubbling points is similar to that we found in the problems we are discussing in this paper, nevetheless scalings and intermediate regimes are much more subtle for doubly exponential nonlinearities. Even though the choice of our first approximation to our bubbling solutions is inspired by the discovery of the blow-up shapes which was obtained first in [4] and then in [1, 17], in our problem, more accurate information is needed, due to the role of the distinct weights m_j's, which were discovered in [15]. In fact, the presence of the weights m_j marks a strong difference between double exponential nonlinearity and Liouville type nonlinearity.

As in the usual Lyapunov-Schmidt scheme, the strategy of the proof involves linearization about a first approximation, to later reduce the problem to a finite dimensional variational one of adjusting the bubbling centers and the corresponding weights. The critical character of this nonlinearity is very much reflected in the delicate error terms left by the first approximation, which makes the linear elliptic theory needed fairly subtle because of the multiple-regime in the error size and adapted to the Riemann surface S through the use isothermal coordinates.

The paper is organized as follows: in Section 2, we construct a first approximation to a solution to (1.1) with the required properties and we estimate the size of the error of approximation with appropriate norms. In Section 3 we describe the scheme of our proofs, by stating the principal results we need, and we give the proof of our Theorems. Section 4 is devoted to the computation of the expansion of the energy functional on the first approximation we constructed in Section 2. Sections A, B and C are devoted to rigorously prove the intermediate results we state in Section 3.

2 Approximation of the solution

It is convenient for our purposes to rewrite problem (1.1) by replacing $u = \sqrt{\lambda} v$, so that the problem becomes

$$\Delta_g v + \lambda \left(v e^{\lambda v^2} - \frac{1}{|S|} \int_S v e^{\lambda v^2} \, dv_g \right) = 0, \tag{2.1}$$

with $\int_S v \, dv_g = 0$. Following [15], to construct approximating solutions of (2.1), the main idea is to use as "basic cells" the functions

$$u_{\delta,\xi}(x) = w_\delta(x - \xi) \qquad \delta > 0, \; \xi \in \mathbb{R}^2,$$

where

$$w_\mu(y) := \log \frac{8\mu^2}{(\mu^2 + |y|^2)^2}, \quad \mu > 0. \tag{2.2}$$

They are all the solutions of

$$\begin{cases} \Delta u + e^u = 0 & \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} e^u < +\infty, \end{cases}$$

and do satisfy the following concentration property:

$$e^{u_{\delta,\xi}} \rightharpoonup 8\pi \delta_\xi \quad \text{in measure sense}$$

as $\delta \to 0$. We will use now isothermal coordinates to pull-back $u_{\delta,\xi}$ in S as in [18].

Let us recall that every Riemann surface (S, g) is locally conformally flat, and the local coordinates in which g is conformal to the Euclidean metric are referred to as isothermal coordinates (see for example the simple existence proof provided by Chern [11]). For every $\xi \in S$ it amounts to find a local chart y_ξ, with $y_\xi(\xi) = 0$, from a neighborhood of ξ onto $B_{2r_0}(0)$ (the choice of r_0 is independent of ξ) in which $g = e^{\hat{\varphi}_\xi(y_\xi(x))} dx$, where $\hat{\varphi}_\xi \in C^\infty(B_{2r_0}(0), \mathbb{R})$. In particular, $\hat{\varphi}_\xi$ relates with the Gaussian curvature K of (S, g) through the relation:

$$\Delta \hat{\varphi}_\xi(y) = -2K(y_\xi^{-1}(y)) e^{\hat{\varphi}_\xi(y)} \qquad \text{for } y \in B_{2r_0}(0). \tag{2.3}$$

We can also assume that y_ξ, $\hat{\varphi}_\xi$ depends smoothly in ξ and that $\hat{\varphi}_\xi(0) = 0$, $\nabla \hat{\varphi}_\xi(0) = 0$.

We now pull-back $u_{\delta,0} = w_\delta$ in $\xi \in S$, for $\delta > 0$, by simply setting

$$U_{\delta,\xi}(x) = w_\delta(y_\xi(x)) = \log \frac{8\delta^2}{(\delta^2 + |y_\xi(x)|^2)^2}$$

for $x \in y_\xi^{-1}(B_{2r_0}(0))$. Letting $\chi \in C_0^\infty(B_{2r_0}(0))$ be a radial cut-off function so that $0 \le \chi \le 1$, $\chi \equiv 1$ in $B_{r_0}(0)$, we introduce the function $PU_{\delta,\xi}$ as the unique solution of

$$\begin{cases} -\Delta_g PU_{\delta,\xi}(x) = \chi_\xi(x) e^{-\varphi_\xi(x)} e^{U_{\delta,\xi}(x)} - \dfrac{1}{|S|} \displaystyle\int_S \chi_\xi e^{-\varphi_\xi} e^{U_{\delta,\xi}} dv_g & \text{in } S \\ \displaystyle\int_S PU_{\delta,\xi} dv_g = 0, \end{cases} \tag{2.4}$$

where $\chi_\xi(x) = \chi\big(y_\xi(x)\big)$ and $\varphi_\xi(x) = \hat{\varphi}_\xi\big(y_\xi(x)\big)$. Notice that the R.H.S. in (2.4) has zero average and smoothly depends in x, and then (2.4) is uniquely solvable by a smooth solution $PU_{\delta,\xi}$.

Let us recall the transformation law for Δ_g under conformal changes: if $\tilde{g} = e^\varphi g$, then

$$\Delta_{\tilde{g}} = e^{-\varphi} \Delta_g. \tag{2.5}$$

Decompose now the Green function $G(x,\xi)$, $\xi \in S$, as

$$G(x,\xi) = -\frac{1}{2\pi} \chi_\xi(x) \log|y_\xi(x)| + H(x,\xi),$$

and by (1.5) then deduce that

$$\begin{cases} -\Delta_g H = -\dfrac{1}{2\pi} \Delta_g \chi_\xi \, \log|y_\xi(x)| - \dfrac{1}{\pi} \langle \nabla \chi_\xi, \nabla \log|y_\xi(x)| \rangle_g - \dfrac{1}{|S|} & \text{in } S \\ \displaystyle\int_S H(\cdot,\xi)\, dv_g = \dfrac{1}{2\pi} \displaystyle\int_S \chi_\xi \log|y_\xi(\cdot)| dv_g. \end{cases}$$

We have used that

$$\Delta_g \log|y_\xi(x)| = e^{-\hat{\varphi}_\xi(y)} \Delta \log|y| \Big|_{y = y_\xi(x)} = 2\pi \delta_\xi$$

in view of (2.5).

For $r \le 2r_0$ define $B_r(\xi) = y_\xi^{-1}(B_r(0))$ and $A_{r_1,r_2}(\xi) = B_{r_1}(\xi) \setminus B_{r_2}(\xi)$, for $r_2 < r_1 \le 2r_0$. Setting

$$\Psi_{\delta,\xi}(x) = PU_{\delta,\xi}(x) - \chi_\xi[U_{\delta,\xi} - \log(8\delta^2)] - 8\pi H(x,\xi),$$

we have the following asymptotic expansion of $PU_{\delta,\xi}$ as $\delta \to 0$, as shown in [18]:

Lemma 2.1. *The function $PU_{\delta,\xi}$ satisfies*

$$PU_{\delta,\xi} = \chi_\xi \left[U_{\delta,\xi} - \log(8\delta^2) \right] + 8\pi H(x,\xi) + O(\delta^2 |\log \delta|) \tag{2.6}$$

uniformly in S. In particular, there holds

$$PU_{\delta,\xi} = 8\pi G(x,\xi) + O(\delta^2 |\log \delta|)$$

locally uniformly in $S \setminus \{\xi\}$.

The ansatz will be constructed as follows. Given $k \in \mathbb{N}$, let us consider distinct points $\xi_j \in S$, $m_j > 0$ and $\delta_j = \mu_j \varepsilon_j > 0$, $j = 1, \dots, k$. In order to have a good approximation, we will assume that the parameters μ_j's and ε_j's are given by

$$\log(8\mu_j^2) = -2\log(2m_j^2) + 8\pi H(\xi_j,\xi_j) + 8\pi \sum_{i=1, i\neq j}^{k} m_i m_j^{-1} G(\xi_i,\xi_j), \quad \text{for all } j = 1, \dots, k \tag{2.7}$$

and

$$\log \frac{1}{\varepsilon_j^4} = \frac{1}{2\lambda m_j^2} - 2\log(2m_j^2), \quad \text{for all } j = 1, \dots, k. \tag{2.8}$$

Up to take r_0 smaller, we assume that the points ξ_j's are well separated and $m_j's$ are in a compact subset of $(0,+\infty)$, namely, we choose $\xi=(\xi_1,\dots,\xi_k)\in\Xi$ and $m=(m_1,\dots,m_k)\in\mathcal{M}$, where

$$\Xi=\{(\xi_1,\dots,\xi_k)\in S^k \mid d_g(\xi_i,\xi_j)\ge 4r_0 \ \forall\, i,j=1,\dots,k,\ i\neq j\} \quad \text{and} \quad \mathcal{M}=\left[\delta_0,\frac{1}{\delta_0}\right]^k,$$

for some small fixed constant $\delta_0>0$. Denote $U_j:=U_{\mu_j\varepsilon_j,\xi_j}$, $j=1,\dots,k$. Thus, our approximating solution is

$$V(x)=\sum_{j=1}^k m_j PU_j(x), \quad x\in S, \tag{2.9}$$

where P is the projection operator defined by (2.4). Notice that $\lambda\to 0$ if and only if $\varepsilon_j\to 0$ for each $j=1,\dots,k$. The idea is that the choice of the numbers μ_j,ε_j makes the error of approximation for V small around each point ξ_j. Let us estimate the error which by definition is

$$R\,=\,\Delta_g V+\lambda\left(Ve^{\lambda V^2}-\frac{1}{|S|}\int_S Ve^{\lambda V^2}\,dv_g\right). \tag{2.10}$$

Setting $w_j(x)=w_{\mu_j}\Big(\frac{y_{\xi_j}(x)}{\varepsilon_j}\Big)$ for $x\in B_{2r_0}(\xi_j)$, introduce the following L^∞-weighted norm for bounded functions defined in S

$$\|h\|_*=\sup_{x\in S}\rho(x)^{-1}|h(x)|, \tag{2.11}$$

where

$$\rho(x):=\sum_{j=1}^k \chi_{B_{r_0}(\xi_j)}(x)\rho_j(x)+1,$$

with

$$\begin{aligned}\rho_j(x)=&\chi\Big(\frac{r_0|y_{\xi_j}(x)|}{\delta\varepsilon_j|\log\varepsilon_j|^2}\Big)(1+|w_j|+w_j^2)\varepsilon_j^{-2}e^{w_j(x)}\\ &+\left[1-\chi\Big(\frac{2r_0|y_{\xi_j}(x)|}{\delta\varepsilon_j|\log\varepsilon_j|^2}\Big)\right]\left[\{1+|\log|y_{\xi_j}(x)|\}e^{\lambda m_j^2w_j^2(x)}+\lambda^{-1}\right]\varepsilon_j^{-2}e^{w_j(x)},\end{aligned} \tag{2.12}$$

$\delta>0$ a large fixed constant and χ_A is the characteristic function of the set A. Thus, we have proven the following fact.

Lemma 2.2. *Assume* (2.7)-(2.8). *There exists a constant $C>0$, independent of $\lambda>0$ small, such that*

$$\|R\|_*\le C\lambda \tag{2.13}$$

for all $\xi\in\Xi$, and $m\in\mathcal{M}$.

Proof: First, notice that for $x\in B_{2r_0}(\xi_j)$ $U_{\mu\varepsilon,\xi}(x)-\log(8\mu^2\varepsilon^2)=w_\mu\Big(\frac{y_\xi(x)}{\varepsilon}\Big)-\log(8\mu^2)+\log\frac{1}{\varepsilon^4}$. By (2.7)-(2.8) we find that in $B_{r_0}(\xi_j)$

$$V(x)=m_j\left[w_j(x)+\frac{1}{2\lambda m_j^2}+\theta_j(x)\right] \tag{2.14}$$

where

$$\begin{aligned}\theta_j(x)=&8\pi\Big\langle\nabla(H(\cdot,\xi_j)\circ y_{\xi_j}^{-1})(0)+\sum_{i\neq j}m_im_j^{-1}\nabla(G(\cdot,\xi_i)\circ y_{\xi_j}^{-1})(0),y_{\xi_j}(x)\Big\rangle\\ &+O(|y_{\xi_j}(x)|^2)+\sum_{i=1}^k O(\varepsilon_i^2|\log\varepsilon_i|).\end{aligned} \tag{2.15}$$

Hence, we obtain that in $B_{r_0}(\xi_j)$

$$\lambda V = \frac{1}{2m_j} + \lambda m_j(w_j + \theta_j) \qquad \text{and} \qquad \lambda V^2 = w_j + \theta_j + \lambda m_j^2(w_j + \theta_j)^2 + \frac{1}{4\lambda m_j^2}. \tag{2.16}$$

Thus, from (2.16) we have that in $B_{r_0}(\xi_j)$

$$\begin{aligned} \lambda V e^{\lambda V^2} &= \frac{1}{2m_j} e^{1/(4\lambda m_j^2)} [1 + 2\lambda m_j^2(w_j + \theta_j)] e^{w_j + \theta_j + \lambda m_j^2 (w_j+\theta_j)^2} \\ &= m_j [1 + 2\lambda m_j^2 (w_j + \theta_j)] \varepsilon_j^{-2} e^{w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2}, \end{aligned} \tag{2.17}$$

in view of $\frac{1}{4\lambda m_j^2} = \log \frac{2m_j^2}{\varepsilon_j^2}$. Furthermore, in $S \setminus \cup_{j=1}^k B_{r_0}(\xi_j)$ we have that

$$V(x) = \sum_{j=1}^k m_j \left[8\pi G(x, \xi_j) + O(\varepsilon_j^2 |\log \varepsilon_j|) \right] = O(1), \tag{2.18}$$

so that $\lambda V e^{\lambda V^2} = O(\lambda)$ in $S \setminus \cup_{j=1}^k B_{r_0}(\xi_j)$.

On the other hand, from the definition of V it is readily checked that

$$\Delta_g V = -\sum_{j=1}^k m_j \chi_j e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} + \frac{1}{|S|} \sum_{j=1}^k m_j [8\pi + O(\varepsilon_j^2)], \tag{2.19}$$

where $\chi_j = \chi_{\xi_j}$, $\varphi_j = \varphi_{\xi_j}$ for $j = 1, \ldots, k$ and in view of $e^{U_j} = \varepsilon_j^{-2} e^{w_j}$ and

$$\int_S \chi_j e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} dv_g = \int_{B_{r_0}(0)} \frac{8\mu_j^2 \varepsilon_j^2}{(\mu_j^2 \varepsilon_j^2 + |y|^2)^2} dy + O(\mu_j^2 \varepsilon_j^2) = 8\pi + O(\varepsilon_j^2).$$

Now, let us estimate the integral term. By using (2.18) we find that

$$\lambda \int_S V e^{\lambda V^2} = \sum_{j=1}^k \int_{B_{r_0}(\xi_j)} \lambda V e^{\lambda V^2} + O(\lambda) \tag{2.20}$$

Now, we write as follows for $\delta > 0$ large enough and fixed (the same as in the definition of ρ_j in (2.12))

$$\lambda \int_{B_{r_0}(\xi_j)} V e^{\lambda V^2} dv_g = \left[\int_{A_{r_0, \delta\sqrt{\varepsilon_j}}(\xi_j)} + \int_{A_{\delta\sqrt{\varepsilon_j}, \delta\varepsilon_j |\log \varepsilon_j|}(\xi_j)} + \int_{B_{\delta\varepsilon_j |\log \varepsilon_j|}(\xi_j)} \right] \lambda V e^{\lambda V^2} dv_g.$$

In $A_{r_0,\delta\sqrt{\varepsilon_j}}(\xi_j)$, we have that uniformly $V(x) = -4m_j \log |y_{\xi_j}(x)| + O(1)$, in view of the expansion in $S \setminus \cup_{j=1}^k B_{\delta\sqrt{\varepsilon_j}}(\xi_j)$

$$PU_j(x) = -4\chi_j(x) \log |y_{\xi_j}(x)| + 8\pi H(x, \xi_j) + \chi_j(x) \log \left(1 + \frac{\mu_j^2 \varepsilon_j^2}{|y_{\xi_j}(x)|^2} \right) + O(\varepsilon_j^2 |\log \varepsilon_j|),$$

and for $i \neq j$ and $x \in A_{r_0, \delta\sqrt{\varepsilon_j}}(\xi_j)$, $PU_i(x) = 8\pi G(x, \xi_i) + O(\varepsilon_i^2 |\log \varepsilon_i|) = O(1)$. Hence, we find that

$$\begin{aligned} \int_{A_{r_0,\delta\sqrt{\varepsilon_j}}(\xi_j)} V e^{\lambda V^2} dv_g &= m_j \int_{A_{r_0,\delta\sqrt{\varepsilon_j}}(\xi_j)} [-4 \log |y_{\xi_j}(x)| + O(1)] e^{\lambda m_j^2 [16 \log^2 |y_{\xi_j}(x)| + O(|\log |y_{\xi_j}(x)||)]} dv_g \\ &= m_j \int_{B_{r_0}(0) \setminus B_{\delta\sqrt{\varepsilon_j}}(0)} [-4 \log |y| + O(1)] e^{\lambda m_j^2 [16 \log^2 |y| + O(|\log |y||)]} e^{\hat{\varphi}_j(y)} dy \\ &= O\left(\int_{B_{r_0}(0) \setminus B_{\delta\sqrt{\varepsilon_j}}(0)} \big| \log |y| \big| \, e^{16\lambda m_j^2 \log^2 |y|} dy \right) = O(1), \end{aligned}$$

in view of $y = y_{\xi_j}(x)$, $e^{\hat\varphi_j} = O(1)$, $\lambda m_j^2 |\log|y|| = O(1)$ in the considered region and

$$\begin{aligned}\int_{B_{r_0}(0)\setminus B_{\delta\sqrt{\varepsilon_j}}(0)} \big|\log|y|\big| e^{16\lambda m_j^2 \log^2|y|} dy &= 2\pi \int_{\delta\sqrt{\varepsilon_j}}^{r_0} |\log s| e^{16\lambda m_j^2 \log^2 s} s\, ds \\ &= 2\pi \int_{\log(\delta\sqrt{\varepsilon_j})}^{\log r_0} |t| e^{2t+16\lambda m_j^2 t^2}\, dt \quad (t = \log s) \\ &= O\left(\int_{\log(\delta\sqrt{\varepsilon_j})}^{\log r_0} |t| e^{t}\, dt\right) = O(1),\end{aligned}$$

since for $r_0 < 1$ (if necessary), $\frac{1}{2}\log\varepsilon_j + \log\delta \le t \le \log r_0 < 0$ and (2.8) implies that

$$(16\lambda m_j^2 t + 2)t \le (1 + 4\lambda m_j^2 \log(2m_j^2) + 16\lambda m_j^2 \log\delta)t \le t + \alpha \qquad \text{for some constant } \alpha.$$

Now, we get that $\delta\varepsilon_j|\log\varepsilon_j| \le |y_{\xi_j}(x)| \le \delta\sqrt{\varepsilon_j}$ implies that

$$2\lambda\log\varepsilon_j + \lambda\log\frac{8\mu_j^2}{(\mu_j^2\varepsilon_j^2 + \delta^2)^2} \le \lambda w_j(x) \le -4\lambda\log|\log\varepsilon_j| + \lambda\log\frac{8\mu_j^2}{(\frac{\mu_j^2}{|\log\varepsilon_j|^2} + \delta^2)^2} < 0,$$

for λ small enough, so that, we find that $\lambda w_j = O(1)$ uniformly in $A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)$. Furthermore, it follows that

$$w_j(1 + \lambda m_j^2 w_j) \le w_j\left(1 + 2\lambda m_j^2\log\varepsilon_j + \lambda\log\frac{8\mu_j^2}{(\mu_j^2\varepsilon_j^2 + \delta^2)^2}\right) \le \frac{3}{4}w_j + \beta,$$

for some constant β in $A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}$, in view of $2\lambda m_j^2\log\varepsilon_j = \lambda m_j^2\log(2m_j^2) - \frac{1}{4}$. Hence, by using (2.17), $\theta_j = O(1)$ and scaling $\varepsilon_j z = y_{\xi_j}(x)$, we obtain that

$$\begin{aligned}\int_{A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}} \lambda V e^{\lambda V^2} &= O\Big(\int_{A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}} \varepsilon_j^{-2} e^{w_j + \lambda m_j^2 w_j^2}\Big) = O\Big(\int_{A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}} \varepsilon_j^{-2} e^{\frac{3}{4}w_j}\Big) \\ &= O\Big(\int_{B_{\delta/\sqrt{\varepsilon_j}}(0)\setminus B_{\delta|\log\varepsilon_j|}(0)} \exp\Big(\frac{3}{4}\log\frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\Big)\, dz\Big) \\ &= O\Big(\int_{\delta|\log\varepsilon_j|}^{\delta/\sqrt{\varepsilon_j}} \Big(\frac{8\mu_j^2}{(\mu_j^2 + s^2)^2}\Big)^{3/4} s\, ds\Big) = O\Big(\int_{\delta|\log\varepsilon_j|}^{\delta/\sqrt{\varepsilon_j}} \frac{ds}{s^2}\Big) = O(\lambda).\end{aligned}$$

In the ball $B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)$, we have that

$$\begin{aligned}\lambda\int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} V e^{\lambda V^2} &= \int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} m_j[1 + 2\lambda m_j^2(w_j + \theta_j)]\varepsilon_j^{-2} e^{w_j + \theta_j + \lambda m_j^2(w_j+\theta_j)^2} \\ &= m_j\Big[\int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} \varepsilon_j^{-2} e^{w_j+\theta_j}(1 + O(\lambda w_j^2 + \lambda|w_j| + \lambda))\, dv_g \\ &\quad + 2\lambda m_j^2\int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} \varepsilon_j^{-2} e^{w_j} O\left([1 + |w_j|][1 + \lambda w_j^2 + \lambda|w_j|]\right)\, dv_g\Big] \\ &= m_j[8\pi + O(\lambda)]\end{aligned} \tag{2.21}$$

in view of $\theta_j = O(1)$, $\lambda(w_j + w_j^2) = O(1)$,

$$\int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} \varepsilon_j^{-2} e^{w_j+\theta_j} = 8\pi + O\Big(\lambda + \sum_{j=1}^k \varepsilon_j^2|\log\varepsilon_j|\Big)$$

by using (2.15) (and similar expansion for e^{θ_j}), scaling $\varepsilon_j z = y_{\xi_j}(x)$ so that $\varepsilon_j^2 dy = e^{-\varphi_j(x)} dv_g$ and

$$\int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} \varepsilon_j^{-2} e^{w_j}(|w_j| + w_j^2 + |w_j|^3) dv_g = O(1),$$

since $0 \le |y_{\xi_j}(x)| \le \delta\varepsilon_j|\log\varepsilon_j|$ implies that

$$-4\log|\log\varepsilon_j| + \log\frac{8\mu_j^2}{(\mu_j^2|\log\varepsilon_j|^{-2} + \delta^2)^2} \le w_j(x) \le \log\frac{8}{\mu_j^2},$$

namely, $w_j = O(|\log\lambda|)$ in $B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)$. Therefore, we conclude that

$$\lambda\int_S V e^{\lambda V^2} dv_g = 8\pi\sum_{j=1}^k m_j + O(\lambda). \tag{2.22}$$

Notice that from (2.8), we find that $\varepsilon_j^2|\log\varepsilon_j| = O(\lambda)$, for all $j = 1, \dots, k$.

From (2.19), (2.17), (2.18) and (2.21) it follows that

- in $S \setminus \cup_{j=1}^m B_{r_0}(\xi_j)$ there holds $R = O(\lambda)$;
- in $B_{r_0}(\xi_j)$, $j \in \{1, \dots, k\}$, there holds

$$R = m_j\varepsilon_j^{-2}e^{w_j}\left(\left[1 + 2\lambda m_j^2(w_j + \theta_j)\right] e^{\lambda m_j^2(w_j+\theta_j)^2+\theta_j} - e^{-\varphi_j}\right) + O(\lambda).$$

Observe that for $x \in B_{\varepsilon_j r_0}(\xi_j)$ we have that $R = O(\lambda\varepsilon_j^{-2}e^{w_j} + \lambda)$, since $w_j = O(1)$ uniformly in $B_{\varepsilon_j r_0}(\xi_j)$. Moreover, $w_j = O(|\log\lambda|)$ in $B_{\delta\varepsilon_j|\log\varepsilon_j|^2}(\xi_j)$ and hence,

$$R = \varepsilon_j^{-2}e^{w_j} O\left(\lambda|w_j| + \lambda w_j^2 + \sum_{j=1}^k \varepsilon_j|\log\varepsilon_j|^2\right) + O(\lambda)$$

in view of $\lambda(w_j + \theta_j)^2 = O(\lambda|\log\lambda|^2)$ and $\theta_j(x) = O(|y_{\xi_j}(x)| + \sum_{j=1}^k \varepsilon_j|\log\varepsilon_j|^2)$. Furthermore, from the choice of ε_j, $j = 1, \dots, k$ (2.8) it follows that

$$\begin{aligned} R = {} & m_j\varepsilon_j^{-2}e^{w_j}\Big(\left[-4\log(\mu_j^2\varepsilon_j^2 + |y_{\xi_j}(x)|^2) + 4\log(2m_j^2) + 2\log(8\mu_j^2) + 2\theta_j\right] \\ & \times \lambda m_j^2 e^{\lambda m_j^2 w_j^2 + \lambda m_j^2\theta_j^2 + \theta_j(1+2\lambda m_j^2 w_j)} - e^{-\varphi_j}\Big) + O(\lambda) \\ = {} & \lambda O\left(\left[\left|\log|y_{\xi_j}(x)|\right| + 1\right]\varepsilon_j^{-2}e^{w_j + \lambda m_j^2 w_j^2} + \lambda^{-1}\varepsilon_j^{-2}e^{w_j}\right) + O(\lambda), \end{aligned}$$

in $A_{r_0,\delta\varepsilon_j|\log\varepsilon_j|^2}(\xi_j)$, in view of $\log(\mu_j^2\varepsilon_j^2 + |y_{\xi_j}(x)|^2) = 2\log|y_{\xi_j}(x)| + O(1)$. In particular, $w_j + \frac{1}{2\lambda m_j^2} = -4\log|y_{\xi_j}(x)| + O(1)$, so that, in $A_{r_0,\delta\sqrt{\varepsilon_j}}(\xi_j)$ it holds $\varepsilon_j^{-2}e^{w_j+\lambda m_j^2 w_j^2} = O(e^{16\lambda m_j^2\log^2|y_{\xi_j}(x)|})$, in view of $\lambda w_j = O(1)$ uniformly in $B_{2r_0}(\xi_j)$. Hence, the error of approximation satisfies the global bound

$$|R(x)| \le C\lambda\rho(x).$$

This completes the proof. □

For simplicity, here and in what follows f designates the nonlinearity

$$f(v) = \lambda v e^{\lambda v^2}. \tag{2.23}$$

Now, we will look for a solution v of (2.1) in the form $v = V + \phi$, for some small remainder term ϕ. In terms of ϕ, the problem (2.1) is equivalent to find $\phi \in \bar{H}$ so that

$$\begin{aligned} \Delta_g \phi + f'(V)\phi - \frac{1}{|S|}\int_S f'(V)\phi \, dv_g = & - R - \Big[f(V+\phi) - f(V) - f'(V)\phi \\ & - \frac{1}{|S|}\int_S [f(V+\phi) - f(V) - f'(V)\phi] \, dv_g \Big] \end{aligned} \tag{2.24}$$

Here, it is clear that $f'(V) = \lambda e^{\lambda V^2}(1 + 2\lambda V^2)$. However, instead of solving directly the problem (2.24) we shall study a different problem. To this purpose we need to estimate $f'(V)$. Thus, denoting

$$K := \sum_{j=1}^{k} \chi_j e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} \tag{2.25}$$

we have the following result.

Lemma 2.3. *Assume* (2.7)-(2.8). *There exists a constant $C > 0$, independent of $\lambda > 0$ small, such that*

$$\|f'(V) - K\|_* \leq C\lambda \tag{2.26}$$

for all $\xi \in \Xi$, and $m \in \mathcal{M}$.

Proof: From (2.16) it follows that in $B_{r_0}(\xi_j)$

$$e^{\lambda V^2} = 2m_j^2 \varepsilon_j^{-2} e^{w_j + \theta_j + \lambda m_j^2 (w_j+\theta_j)^2} \quad \text{and} \quad 2\lambda V^2 = 2(w_j + \theta_j) + 2\lambda m_j^2 (w_j + \theta_j)^2 + \frac{1}{2\lambda m_j^2},$$

so that, in $B_{r_0}(\xi_j)$

$$f'(V) = (1 + 2\lambda m_j^2 + 4\lambda m_j^2 (w_j + \theta_j) + 4\lambda^2 m_j^4 (w_j + \theta_j)^2) \varepsilon_j^{-2} e^{w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2}.$$

Thus, it is clear that $f'(V) = \varepsilon_j^{-2} e^{w_j}(1 + O(\lambda))$ uniformly in $B_{\delta\varepsilon_j}(\xi_j)$ and $f'(V) = O(\lambda)$ in $S \setminus \cup_{j=1}^k B_{r_0}(\xi_j)$. Furthermore, we have that

$$\begin{aligned} f'(V) - K = \varepsilon_j^{-2} e^{w_j} \Big[& \big(1 + 2\lambda m_j^2 + 4\lambda m_j^2 (w_j + \theta_j) + 4\lambda^2 m_j^4 (w_j + \theta_j)^2\big) \\ & \times \varepsilon_j^{-2} e^{w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2} - e^{-\varphi_j} \Big] \end{aligned}$$

uniformly in $B_{r_0}(\xi_j)$ and $f'(V) - K = O(\lambda)$ in $S \setminus \cup_{j=1}^k B_{r_0}(\xi_j)$. Similar to the estimate (2.13), we conclude (2.26). □

In order to simplify the arguments, in view of (2.26), we write (2.24) in the form

$$L(\phi) = -[R + N(\phi)] \qquad \text{in } S, \tag{2.27}$$

where the linear operator L is defined as

$$L(\phi) = \Delta_g \phi + K\phi - \frac{1}{|S|}\int_S K\phi \, dv_g, \tag{2.28}$$

and the nonlinear part N is given by

$$\begin{aligned} N(\phi) = & f(V+\phi) - f(V) - f'(V)\phi - \frac{1}{|S|}\int_S [f(V+\phi) - f(V) - f'(V)\phi] \\ & + [f'(V) - K]\phi - \frac{1}{|S|}\int_S [f'(V) - K]\phi \, dv_g. \end{aligned} \tag{2.29}$$

Notice that for all $\phi \in \bar{H}$

$$\int_S L(\phi) dv_g = \int_S N(\phi) dv_g = \int_S R dv_g = 0.$$

3 Variational reduction and proof of main results

In the so-called nonlinear Lyapunov-Schimdt reduction, an important step is the solvability theory for the linear operator, obtained as the linearization of (2.1) at the approximating solution V, namely, (2.24). In our approach, in order to simplify the arguments we will study the operator L given in (2.28) under suitable orthogonality conditions. Let us observe that $L(\phi) = \tilde{L}(\phi)+c(\phi)$, for functions ϕ defined on S, with

$$\tilde{L}(\phi) = \Delta_g \phi + K\phi \tag{3.1}$$

and $c(\phi) := -\frac{1}{|S|}\int_S K\phi\, dv_g$, where K is given by (2.25). Observe that, as $\lambda \to 0$, formally the operator $\tilde{L}$, scaled and centered at 0 by setting $y = y_{\xi_j}(x)/\varepsilon_j$ for $x \in B_{\varepsilon_j r_0}(\xi_j)$, approaches $\hat{L}_j$ defined in $\mathbb{R}^2$ as

$$\hat{L}_j(\phi) = \Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |y|^2)^2}\phi.$$

Due to the intrinsic invariances, the kernel of $\hat{L}_j$ in $L^\infty(\mathbb{R}^2)$ is non-empty and is spanned by Y_{ij}, $i = 0, 1, 2$, where

$$Y_{ij}(y) = \frac{4\mu_j y_i}{\mu_j^2 + |y|^2}, \qquad i = 1, 2, \qquad \text{and} \qquad Y_{0j}(y) = 2\,\frac{\mu_j^2 - |y|^2}{\mu_j^2 + |y|^2}.$$

Since [14, 19, 15] it is by now rather standard to show the invertibility of L in a suitable "orthogonal" space, and a sketched proof of it will be given in Appendix A. See also [18] for an extension to a Riemann surface. Furthermore, an important goal in the study of the L operator is to get rid of the presence of the term $c(\phi)$.

To be more precise, for $i = 0, 1, 2$ and $j = 1, \dots, k$ introduce the functions

$$Z_{ij}(x) = Y_{ij}\left(\frac{y_{\xi_j}(x)}{\varepsilon_j}\right) = \begin{cases} 2\dfrac{\mu_j^2\varepsilon_j^2 - |y_{\xi_j}(x)|^2}{\mu_j^2\varepsilon_j^2 + |y_{\xi_j}(x)|^2} & \text{for } i = 0 \\[2ex] \dfrac{4\mu_j\varepsilon_j (y_{\xi_j}(x))_i}{\mu_j^2\varepsilon_j^2 + |y_{\xi_j}(x)|^2} & \text{for } i = 1, 2. \end{cases}$$

and let PZ_{ij} be the projections of Z_{ij} as the solutions in $\bar{H}$ of

$$\begin{cases} \Delta_g PZ_{ij} = \chi_j \Delta_g Z_{ij} - \dfrac{1}{|S|}\displaystyle\int_S \chi_j \Delta_g Z_{ij} dv_g & \text{in } S \\ \displaystyle\int_S PZ_{ij} dv_g = 0, \end{cases} . \tag{3.2}$$

Notice that $-\Delta_g Z_{ij} = e^{-\varphi_j}\varepsilon_j^{-2} e^{w_j} Z_{ij}$ in $B_{2r_0}(\xi_j)$ for all $i = 0, 1, 2$ and $j = 1, \dots, k$. In Appendix A we will prove the following result:

Proposition 3.1. *There exists $\lambda_0 > 0$ so that for any points $\xi = (\xi_1, \dots, \xi_k) \in \Xi$ and $m = (m_1, \dots, m_k) \in \mathcal{M}$, there is a unique solution $\phi \in \bar{H}(S) \cap W^{2,2}(S)$ and coefficients $c_{ij} \in \mathbb{R}$ of*

$$\begin{cases} L(\phi) = h + \displaystyle\sum_{i=0}^{2}\sum_{j=1}^{m} c_{ij}\Delta_g PZ_{ij} & \text{in } S \\ \displaystyle\int_S \phi \Delta_g PZ_{ij} dv_g = 0 & \forall\, i = 0, 1, 2,\ j = 1, \dots, m \end{cases} \tag{3.3}$$

for all $0 < \lambda < \lambda_0$, $h \in C(S)$ *with* $\|h\|_* < +\infty$ *and* $\displaystyle\int_S h\, dv_g = 0$. *Moreover, the map* $(\xi, m) \mapsto (\phi, c_{ij})$ *is differentiable in* (ξ, m) *with*

$$\|\phi\|_\infty \le C\|h\|_* \,, \qquad \sum_{i=0}^{2}\sum_{j=1}^{k} |c_{ij}| \le C\|h\|_* \tag{3.4}$$

$$\sum_{j=1}^{k}\left(\sum_{i=1}^{2} \varepsilon_j \|\partial_{(\xi_j)_i}\phi\|_\infty + \frac{1}{|\log \varepsilon_j|}\|\partial_{m_j}\phi\|_\infty\right) \le C\|h\|_* \tag{3.5}$$

for some $C > 0$.

Let us stress that the right hand side of the equation (3.3) of $L(\phi)$ integrates zero.

Let us recall that $v = V + \phi$ solves (2.1) if $\phi \in \bar{H}$ does satisfy (2.27). Since the operator L is not fully invertible, in view of Proposition 3.1 one can solve the nonlinear problem (2.27) just up to a linear combination of $\Delta_g PZ_{ij}$'s, as explained in the following (see Appendix B for a proof):

Proposition 3.2. *Let* $\delta_0, r_0 > 0$ *small and fixed. Then there exist* $\lambda_0 > 0$, $C > 0$ *such that for* $0 < \lambda < \lambda_0$, *for any* $\xi = (\xi_1, \dots, \xi_k) \in \Xi$ *and* $m = (m_1, \dots, m_k) \in \mathcal{M}$, *problem*

$$\begin{cases} L(\phi) = -[R + N(\phi)] + \displaystyle\sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij}\Delta_g PZ_{ij} & \text{in } S \\ \displaystyle\int_S \phi \Delta_g PZ_{ij} dv_g = 0 \quad \text{for all } i = 0,1,2,\ j = 1,\dots,k \end{cases} \tag{3.6}$$

admits a unique solution $\phi(\xi, m) \in \bar{H} \cap W^{2,2}(S)$ *and* $c_{ij} = c_{ij}(\xi, m)$, $i = 0, 1, 2$, $j = 1, \dots, k$ *such that*

$$\|\phi\|_\infty \le C\lambda, \qquad \sum_{i=0}^{2}\sum_{j=1}^{k} |c_{ij}| \le C\lambda \tag{3.7}$$

where R, N *are given by* (2.10) *and* (2.29), *respectively. Furthermore, the map* $(\xi, m) \mapsto \phi(\xi, m) \in C(S)$ *is* C^1 *and for* $l = 1, \dots, k$ *we have*

$$\|\partial_{\xi_l}\phi\|_\infty \le \frac{C\lambda}{\varepsilon_l} \quad \text{and} \quad \|\partial_{m_l}\phi\|_\infty \le C. \tag{3.8}$$

The function $\phi(\xi, m)$ obtained in Proposition 3.2 will be a true solution of (2.27) if ξ and m are such that $c_{ij}(\xi, m) = 0$ for all $i = 0, 1, 2$, and $j = 1, \dots, k$. This problem is equivalent to finding critical points of the reduced energy

$$\mathcal{F}_\lambda(\xi, m) = J_\lambda\big(U(\xi, m) + \tilde{\phi}(\xi, m)\big), \tag{3.9}$$

where J_λ is given by (1.2), $U(\xi, m) = \sqrt{\lambda}\, V(\xi, m)$ and $\tilde{\phi}(\xi, m) = \sqrt{\lambda}\, \phi(\xi, m)$, as stated in (See Appendix C)

Lemma 3.3. *There exists* λ_0 *such that, if* $(\xi, m) \in \Xi \times \mathcal{M}$ *is a critical point of* $\mathcal{F}_\lambda$ *with* $0 < \lambda < \lambda_0$, *then* $v = V(\xi, m) + \phi(\xi, m)$ *is a solution of* (2.1), *i.e.,* $c_{ij}(\xi, m) = 0$ *for all* $i = 0, 1, 2$, *and* $j = 1, \dots, k$.

Once equation (2.1) has been reduced to the search of c.p.'s for $\mathcal{F}_\lambda$, it becomes crucial to show that the main asymptotic term of $\mathcal{F}_\lambda$ is given by $J_\lambda(U)$, for which we also have an expansion. More precisely, in section 5 we will prove

Proposition 3.4. *Assume* (2.7)-(2.8). *The following expansion does hold*

$$\mathcal{F}_\lambda(\xi, m) \quad = \quad 2\pi k - \frac{\lambda|S|}{2} + 8\pi\lambda\psi_k(\xi, m) + \theta_\lambda(\xi, m)$$

in $C^1(\Xi \times \mathcal{M})$ *as* $\lambda \to 0^+$, *where* $\psi_k(\xi, m) = \psi_k(\xi_1, \ldots, \xi_k, m_1, \ldots, m_k)$ *is given by* (1.6) *and the term* $\theta_\lambda(\xi, m)$ *satisfies*

$$|\theta_\lambda(\xi, m)| + \sum_{l=1}^{k} \left[\sum_{i=1}^{2} \left|\partial_{(\xi_l)_i} \theta_\lambda(\xi, m)\right| + |\partial_{m_l} \theta_\lambda(\xi, m)| \right] = O(\lambda^2 |\log \lambda|) \tag{3.10}$$

uniformly for points $(\xi, m) \in \Xi \times \mathcal{M}$.

We are now in position to prove the main results stated in the Introduction.

Proof (of Theorem 1.1): According to Lemma 3.3, we have a solution of problem (1.1) if we adjust (ξ, m) so that it is a critical point of $\mathcal{F}_\lambda$ defined by (3.9). This is equivalent to finding a critical point of

$$\tilde{\mathcal{F}}_\lambda(\xi, m) = \frac{1}{8\pi\lambda} \left[\mathcal{F}_\lambda(\xi, m) - 2\pi k + \frac{\lambda|S|}{2} \right].$$

Thanks to Proposition 3.4, we have that the function $\tilde{\mathcal{F}}_\lambda(\xi, m)$ is C^1-close to $\psi_k(\xi, m)$ in $\Xi \times \mathcal{M}$ when λ is small enough. Now, let $\mathcal{D}$ be the open set such that

$$\bar{\mathcal{D}} \subset \{(\xi, m) \in S^k \times \mathbb{R}_+^k \,:\, \xi_i \neq \xi_j, \forall i \neq j\},$$

where ψ_k has a stable critical point situation. Then any C^1-perturbation of ψ_k has a critical point in $\mathcal{D}$. Thus, choosing r_0 and δ_0 smaller if necessary so that $\bar{\mathcal{D}} \subset \Xi \times \mathcal{M}$, we conclude that $\tilde{\mathcal{F}}_\lambda$ has a critical point (ξ_λ, m_λ) in $\mathcal{D}$, for all such small λ. Therefore

$$u_\lambda(x) = U(\xi_\lambda, m_\lambda)(x) + \tilde{\phi}(\xi_\lambda, m_\lambda)(x) = \sqrt{\lambda}\,[V(\xi_\lambda, m_\lambda)(x) + \phi(\xi_\lambda, m_\lambda)(x)]$$

is a solution to our problem (1.1). The qualitative properties of this solution predicted by Theorem 1.1 are direct consequence of our construction. This concludes the proof. □

Proof (of Theorem 1.3): We shall apply the result of Theorem 1.1 for the case $k = 2$ with $S = T$ a flat two-torus in rectangular form given by (1.9). In this case, it holds that the function $H(\xi, \xi)$ is constant. Notice that on T we have invariance under translations, in other words, if u is a solution to (1.1) then $u(\cdot + p)$ is also a solution to (1.1) for any $p \in T$. Furthermore, by the same property, it is know that the Green's function satisfies $G(\xi_1, \xi_2) = G(\xi_1 - \xi_2, 0)$. Hence, with a slightly abuse of notation we denote $G(z)$ by the Green's function $G(\cdot, 0)$ and we make the change of variables $z = \xi_1 - \xi_2$. Thus, we are reduced to look for critical points of the functional

$$\mathcal{F}_\lambda(z, m) = 4\pi - \frac{\lambda|T|}{2} + 8\pi\lambda\psi_2(z, m) + \theta_\lambda(z, m),$$

with

$$\psi_2(z, m) = A \sum_{j=1}^{2} m_j^2 + 2 \sum_{j=1}^{2} m_j^2 \log m_j - 8\pi m_1 m_2 G(z). \tag{3.11}$$

where A is an absolute constant $A = \log 16 - 2 - 8\pi H(\xi, \xi)$, and, it is sufficient first to find nondegenerate critical points z of ψ_2 (for any $m = (m_1, m_2)$) and hence, to look for nondegenerate critical points m of $\psi_2(z, \cdot)$ (for the latter z), so that (z, m) are nondegenerate critical points of ψ_2, since this is a stable critical point situation. Therefore, there exist $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ there exist critical points (ξ_λ, m_λ) of $\mathcal{F}_\lambda$. Let us stress that we can find critical points z of ψ_2 independent of m, in view of (3.11).

Claim 3.5. *If T is a rectangle then there exist exactly three nondegenerate critical points of $\psi_2(\cdot, m)$ for any $m = (m_1, m_2)$. They are the half periods of T: $p_1 = \frac{a}{2}$, $p_2 = \frac{\mathrm{i}b}{2}$ (saddle points) and $p_3 = \frac{a+\mathrm{i}b}{2}$ (minimum point), with* i the imaginary unity.

Proof: It is know that the Green's function G has exactly three nondegenerate critical points p_1, p_2 and p_3, which are the half periods of T given by (1.12), see [8]. Hence, choosing $z = p_i$ for some $i = 1, 2, 3$ we have nondegenerate critical points of $\psi_2(\cdot, m)$ in points z for any m. $\square$

Now, let us look for critical points in $m = (m_1, m_2)$. Thus, we get

$$\partial_{m_i}\psi_2(z, m) = 2(A+1)m_i + 4m_i \log m_i - 8\pi m_j G(z), \qquad i, j = 1, 2, \quad j \neq i$$

and we have to find points $m_1, m_2 \in (0, +\infty)$ solutions to the system

$$\begin{cases} (A+1)m_1 + 2m_1 \log m_1 &= 4\pi m_2 G(z) \\ (A+1)m_2 + 2m_2 \log m_2 &= 4\pi m_1 G(z) \end{cases}. \tag{3.12}$$

Let us stress that for each $z = p_i$, $i = 1, 2, 3$ we look for m a critical point of $\psi_2(p_i, \cdot)$.

Claim 3.6. *If $G(z) \geq 0$ then there exists an only solution (m_1, m_2) to the system* (3.12) *and it satifies $m_1 = m_2$. If $G(z) < 0$ then there exist exactly three different pairs of solutions (m_1, m_2) to the system* (3.12) *in the form (m_1, m_2) or (m_0, m_0) or (m_2, m_1) with some positive real numbers satisfying $m_1 < m_0 < m_2$.*

Proof: To this aim, denote $B = 4\pi G(z)$ and first assume that $B \neq 0$. Consider the function

$$f_0(t) = \frac{A+1}{B}t + \frac{2}{B}t\log t \tag{3.13}$$

so that we re-write (3.12) in the form

$$\begin{cases} f_0(t) &= s \\ f_0(s) &= t \end{cases}. \tag{3.14}$$

Thus, we look for the intersection points between the two curves $s = f_0(t)$ and $t = f_0(s)$ in the plane ts. Note that f_0 satisfies $f_0(0) = 0$, its derivative is $f_0'(t) = \frac{A+1}{B} + \frac{2}{B}\log t + \frac{2}{B}$ and hence, f_0' is strictly increasing if $B > 0$ and strictly decreasing if $B < 0$, so that f_0 is strictly convex if $B > 0$ and strictly concave in $[0, +\infty)$ if $B > 0$ and

$$f_0'(t) \to \begin{cases} -\infty, & B > 0 \\ +\infty, & B < 0 \end{cases} \quad \text{as } t \to 0^+ \quad \text{and} \quad f_0'(t) \to \begin{cases} +\infty, & B > 0 \\ -\infty, & B < 0 \end{cases} \quad \text{as } t \to +\infty.$$

Now, assume that $B > 0$, namely, $G(z) > 0$. From the previous analysis, f_0 satisfies that $f(0) = 0$, f_0 is strictly convex in $[0, +\infty)$, $f_0'(t) \to -\infty$ as $t \to 0^+$ and $f_0'(t) \to +\infty$ as $t \to +\infty$. Furthermore, it is readily check that f_0 is strictly decreasing in $[0, e^{-(A+3)/2}]$, its graph is below the axis t in $[0, e^{-(A+3)/2}]$ (f_0 is negative), it is strictly increasing in $[e^{-(A+3)/2}, +\infty)$ and it has a minimum at $t = e^{-(A+3)/2}$. Therefore, there is a unique $m_0 = m_0(B) > 0$ such that $f_0(m_0) = m_0$, namely, m_0 is the only fixed point of f_0, it satisfies

$$(A+1)m_0 + 2m_0 \log m_0 = Bm_0$$

and (m_0, m_0) is the only solution to system (3.12), since reflecting the curve $s = f_0(t)$ with respect to the line $s = t$ we obtain the curve $t = f_0(s)$ and (m_0, m_0) is the only point of intersection of the curves.

On the other hand, assume that $B<0$, namely, $G(z)<0$. Then, f_0 satisfies that $f(0)=0$, f_0 is strictly increasing in $[0,e^{-(A+3)/2}]$, its graph is above the diagonal $s=t$ in $[0,e^{-(A+3)/2}]$, it is strictly decreasing in $[e^{-(A+3)/2},+\infty)$, it has a maximum at $t=e^{-(A+3)/2}$, it is strictly concave in $(0,+\infty)$, $f_0(t)\to-\infty$ as $t\to+\infty$, $f_0'(t)\to+\infty$ as $t\to0^+$ and $f_0'(t)\to-\infty$ as $t\to+\infty$. Notice that reflecting the curve $s=f_0(t)$ with respect to the line $s=t$ we obtain the curve $t=f_0(s)$. Hence, we deduce that there are three distinct points of intersection between the curves $s=f_0(t)$ and $t=f_0(s)$. Therefore, there is a unique $m_0=m_0(B)>0$ such that $f_0(m_0)=m_0$ and two more distinct solutions (m_1,m_2) and (m_2,m_1) to the system (3.12) with $m_1<m_0<m_2$ and satisfying $f_0(f_0(m_1))=m_1$ and $f_0(f_0(m_2))=m_2$, namely, with m_1 and m_2 are fixed points of $f_0\circ f_0$ (but $f_0(m_1)\neq m_1$ and $f_0(m_2)\neq m_2$). Also, it follows that $m_1<e^{-(A+3)/2}<m_0$. Let us stress that m_0 is a fixed point of f_0 and m_1 and m_2 are fixed points of $f_0\circ f_0$ but not of f_0.

Now, for the case $B=0$, system (3.12) is reduced to solve the equation

$$(A+1)t+2t\log t=0\iff t\,[A+1+2\log t]=0.$$

Since we look for solutions $m_1,m_2>0$, we get that $m_1=m_2=e^{-\frac{A+1}{2}}$ is the only solution to the system (3.12) with $G(z)=0$.

The proof of the claim is finished. □

To conclude that a critical point (m_1,m_2) of $\psi_2(z,\cdot)$, namely, a solution to the system (3.12) is nondegenerate we study its Hessian matrix. Notice that if $G(z)=-\dfrac{1}{4\pi}$ then the system (3.12) has three pairs of solutions in the form (m_1,m_2) or (m_0,m_0) or (m_2,m_1) with some positive real numbers $m_1<m_0<m_2$.

Claim 3.7. *If* $G(z)\neq-\dfrac{1}{4\pi}$ *then all the solutions* (m_1,m_2) *to the system* (3.12) *are nondegenerate critical points of* $\psi_2(z,\cdot)$*. If* $G(z)=-\dfrac{1}{4\pi}$ *then the pairs of solutions* (m_1,m_2) *and* (m_2,m_1) *to the system* (3.12) *are nondegenerate critical points of* $\psi_2(z,\cdot)$ *and* (m_0,m_0) *is degenerate.*

Proof: Direct computations lead us to find the determinant of the Hessian matrix as

$$\begin{aligned}|H_m\psi_2(z,m)|&=\Big(8\pi\frac{m_2}{m_1}G(z)+4\Big)\Big(8\pi\frac{m_1}{m_2}G(z)+4\Big)-64[\pi G(z)]^2\\&=32\pi\Big(\frac{m_2}{m_1}+\frac{m_1}{m_2}\Big)G(z)+16\end{aligned}$$

by using that (m_1,m_2) satisfies (3.12). Therefore, if $2\pi\Big(\dfrac{m_2}{m_1}+\dfrac{m_1}{m_2}\Big)G(z)\neq-1$ then $m=(m_1,m_2)$ is a nondegenerate critical point of $\psi_2(z,\cdot)$ (for fixed z). In particular, if $G(z)\geq0$ then we have an only nondegenerate critical point $m=(m_0,m_0)$, with m_0 the only fixed point of f_0 defined by (3.13), when $B>0$ or $m_0=e^{-\frac{A+1}{2}}$ when $B=0$, since $|H_m\psi_2(z,m)|\geq16$. In case $G(z)<0$ we have at most three nondegenerate critical points. More precisely, we have several cases readily checked:

- $m=(m_0,m_0)$, with m_0 the only fixed point of f_0 defined by (3.13), is a nondegenerate critical point of $\psi_2(z,\cdot)$ only if $G(z)\neq-\dfrac{1}{4\pi}$ in view of $|H_m\psi_2(z,m)|=64\pi G(z)+16\neq0$; in other words, if $G(z)<-\dfrac{1}{4\pi}$ then $|H_m\psi_2(z,m)|=64\pi G(z)+16<0$ and if $-\dfrac{1}{4\pi}<G(z)<0$ then $|H_m\psi_2(z,m)|=64\pi G(z)+16>0$;
- if $G(z)=-\dfrac{1}{4\pi}$ then on one hand, (m_0,m_0), with m_0 the only fixed point of f_0 defined by (3.13), is a degenerate critical point of $\psi_2(z,\cdot)$, in view of $|H_m\psi_2(z,m)|=64\pi G(z)+16=0$, and on the other hand, both (m_1,m_2) and (m_2,m_1), where m_1 and m_2 are the fixed points

of $f_0 \circ f_0$ different than m_0 the fixed point of f_0, with $m_1 < m_0 < m_2$, are nondegenerate critical points of $\psi_2(z,\cdot)$ in view of $m_1 \neq m_2$, so that,

$$2\pi\Big(\frac{m_2}{m_1}+\frac{m_1}{m_2}\Big)G(z) = -\frac{1}{2}\Big(\frac{m_2}{m_1}+\frac{m_1}{m_2}\Big) \neq -1; \quad \text{and}$$

- either (m_1, m_2) or (m_2, m_1), with m_1 and m_2 the fixed points of $f_0 \circ f_0$ different than m_0 the fixed point of f_0 defined by (3.13), with $m_1 < m_0 < m_2$, are nondegenerate critical points of $\psi_2(z,\cdot)$ if $G(z) < -\dfrac{1}{4\pi}$ in view of $|H_m\psi_2(z,m)| \le 64\pi G(z) + 16 < 0$, since $\dfrac{m_2}{m_1}+\dfrac{m_1}{m_2} \ge 2$.

Notice that it remains to analyze the critical points (m_1, m_2) or (m_2, m_1), with m_1 and m_2 the fixed points of $f_0 \circ f_0$ different than m_0 the fixed point of f_0 defined by (3.13) when $-\dfrac{1}{4\pi} < G(z) < 0$. To this aim, we shall use the equation that must satisfy degenerate critical point of $\psi_2(z,\cdot)$. Thus, we get that

$$2\pi\Big(\frac{m_2}{m_1}+\frac{m_1}{m_2}\Big)G(z) + 1 = 0 \iff 4\pi G(z) m_1^2 + 2m_1 m_2 + 4\pi G(z) m_2^2 = 0.$$

Recall that $B = 4\pi G(z)$, so that, $-1 < B < 0$ when $-\dfrac{1}{4\pi} < G(z) < 0$. So, we re-write the latter equality in the form $Bt^2 + 2ts + Bs^2 = 0$ and we consider the system

$$\left\{\begin{array}{rcl} (A+1)t + 2t\log t &=& Bs \\ (A+1)s + 2s\log s &=& Bt \\ Bt^2 + 2ts + Bs^2 &=& 0 \end{array}\right. . \tag{3.15}$$

Let us show that this system does not have any solution, so that the critical points (m_1, m_2) or (m_2, m_1) are nondegenerate. Assume that (t,s) is a solution of the system (3.15). From the third equation it follows that either $Bt + \left(1+\sqrt{1-B^2}\right)s = 0$ or $Bt + \left(1-\sqrt{1-B^2}\right)s = 0$. First, assume that $Bt + \left(1-\sqrt{1-B^2}\right)s = 0$. Hence, we have that

$$Bt = -\left(1-\sqrt{1-B^2}\right)s \iff Bs = -\left(1+\sqrt{1-B^2}\right)t.$$

So, replacing Bt in the second equation of the system (3.15) and since $s \neq 0$, we get that

$$s = \exp\left(-\left[A+2-\sqrt{1-B^2}\right]/2\right) \quad \text{and} \quad t = \frac{-1+\sqrt{1-B^2}}{B}\exp\left(-\left[A+2-\sqrt{1-B^2}\right]/2\right).$$

On the other hand, similarly as above replacing Bs in the first equation of the system (3.15) we get that

$$t = \exp\left(-\left[A+2+\sqrt{1-B^2}\right]/2\right) \quad \text{and} \quad s = \frac{-1-\sqrt{1-B^2}}{B}\exp\left(-\left[A+2+\sqrt{1-B^2}\right]/2\right).$$

If (t,s) is a solution of the system (3.15) then necessarily

$$s = \exp\left(-\left[A+2-\sqrt{1-B^2}\right]/2\right) = \frac{-1-\sqrt{1-B^2}}{B}\exp\left(-\left[A+2+\sqrt{1-B^2}\right]/2\right).$$

From this equality we obtain that

$$B\exp\left(\sqrt{1-B^2}\right) + \sqrt{1-B^2} + 1 = 0. \tag{3.16}$$

Performing the change of variable $t = \sqrt{1-B^2}$, we have that $0 < t < 1$, $B = -\sqrt{1-t^2}$ and t satisfies the equation

$$-\sqrt{1-t^2}e^t + t + 1 = 0.$$

It turns out that the function $g(t) = \sqrt{1-t^2}e^t$ satisfies $g(0) = 1$, $g'(0) = 1$, g is strictly concave and its tangent line at $t = 0$ is exactly $s = t + 1$. Therefore, $g(t) < t + 1$ for all $0 < t < 1$ and the equality is attained at $t = 0$. In other words, there is no B with $-1 < B < 0$ satisfying the equation (3.16). In case $Bt + (1 + \sqrt{1-B^2})s = 0$, a similar analysis lead us to find the equation (3.16). Thus, in any case we arrive at a contradiction and the system (3.15) has no solutions. Therefore, if $G(z) \neq -\dfrac{1}{4\pi}$ then all the critical points are nondegenerate. This finishes the proof of the claim. □

From the previous result it remains to study the case $G(z) = -\dfrac{1}{4\pi}$ for some $z \in \{p_1, p_2, p_3\}$, since we have that there exist three critical points of ψ_2 but one of them is degenerate. We shall address this difficulty by study directly the functional $\mathcal{F}_\lambda$ in that case.

Claim 3.8. *Assume that $G(z) = -\dfrac{1}{4\pi}$ for some fixed $z \in \{p_1, p_2, p_3\}$, then there exist three critical points $(z_{\lambda,i}, m_{\lambda,i})$ $i = 1, 2, 3$, of $\mathcal{F}_\lambda(z, m)$ such that, up to subsequences, as $\lambda \to 0$ $z_{\lambda,i} \to z$ and $m_{\lambda,i} \to m_{0,i}$ with $m_{0,1} = (m_1, m_2)$, $m_{0,2} = (m_0, m_0)$ and $m_{0,3} = (m_2, m_1)$ the three different pairs of solutions to the system (3.12) obtained in Claim 3.6.*

Proof: We know that z is a nondegenerate critical point of $\psi_2(\cdot, m)$. This is a stable critical point situation so that for each $m \in (0, +\infty) \times (0, +\infty)$ there exist a critical point $z_\lambda(m) \in T$ of $\tilde{\mathcal{F}}_\lambda(\cdot, m)$ with

$$\tilde{\mathcal{F}}_\lambda(z, m) = \frac{1}{8\pi\lambda}\left[\mathcal{F}_\lambda(z, m) - 4\pi + \frac{\lambda|T|}{2}\right],$$

so that $z_\lambda(m) \to z$ as $\lambda \to 0$. Let us stress that $\tilde{\mathcal{F}}_\lambda(z, m) = \psi_2(z, m) + O(\lambda|\log\lambda|)$, where O is uniformly in C^1-sense for points in $T \times \Xi$. Moreover, by IFT the map $m \in \Xi \mapsto z_\lambda(m)$ is a C^1-function in m. Now, let us define $\mathcal{E}_\lambda(m) = \tilde{\mathcal{F}}_\lambda\big(z_\lambda(m), m\big)$. Then, it readily follows that

$$\nabla_m \mathcal{E}_\lambda(m) = \nabla_m \tilde{\mathcal{F}}_\lambda\big(z_\lambda(m), m\big) = \nabla_m \psi_2\big(z_\lambda(m), m\big) + O(\lambda|\log\lambda|),$$

since $\nabla_z \tilde{\mathcal{F}}_\lambda\big(z_\lambda(m), m\big) = 0$. Hence, we look for critical points of $\mathcal{E}$ by study the system $\nabla_m \mathcal{E}(m) = 0$. This is equivalent to finding solutions to the perturbation of the system (3.12) given by

$$\left\{\begin{array}{rcl} (A+1)m_1 + 2m_1\log m_1 - 4\pi m_2 G\big(z_\lambda(m_1, m_2)\big) + \lambda|\log\lambda|\,\tilde{\theta}_{\lambda,1}(m_1, m_2) & = & 0 \\ (A+1)m_2 + 2m_2\log m_2 - 4\pi m_1 G\big(z_\lambda(m_1, m_2)\big) + \lambda|\log\lambda|\,\tilde{\theta}_{\lambda,2}(m_1, m_2) & = & 0 \end{array}\right., \qquad (3.17)$$

where it holds $\tilde{\theta}_{\lambda,i} = O(1)$, $i = 1, 2$ uniformly for $m \in \Xi$ and $4\pi G\big(z_\lambda(m_1, m_2)\big) \to -1$ as $\lambda \to 0$. Since system (3.12) with $4\pi G(z) = -1$ has exactly three different pairs of solutions, as proved in Claim 3.6, it follows that for λ small enough there are at least three different pairs $m_{\lambda,i}$ of solutions to (3.17) such that as $\lambda \to 0$ converge, up to a subsequence, to a solution to (3.12). Let us stress that we can consider the curves (or the implicit functions)

$$4\pi m_j G(z_\lambda(m_1, m_2)) = (A+1)m_i + 2m_i\log m_i + \lambda|\log\lambda|\,\tilde{\theta}_{\lambda,i}(m_1, m_2)$$

for $i \neq j$ converging uniformly to the curves (or the implicit functions)

$$-m_j = (A+1)m_i + 2m_i\log m_i$$

for $i \neq j$ locally around each m_0, m_1 and m_2 in order to obtain the existence of the pairs $m_{\lambda,i}$, $i = 1, 2, 3$. Thus, we conclude that $m_{\lambda,i}$ $i = 1, 2, 3$ are critical points of $\mathcal{E}_\lambda$, namely, $\nabla_m \mathcal{E}(m_{\lambda,i}) = 0$ and $m_{\lambda,i} \to m_{0,i}$ with $m_{0,1} = (m_1, m_2)$, $m_{0,2} = (m_0, m_0)$ and $m_{0,3} = (m_2, m_1)$ the three different pairs of solutions to the system (3.12) obtained in Claim 3.6. By the procedure it follows that $\big(z_\lambda(m_{\lambda,i}), m_{\lambda,i}\big)$, $i = 1, 2, 3$ are critical points of $\tilde{\mathcal{F}}_\lambda$, in view, of

$$\nabla_z \tilde{\mathcal{F}}_\lambda\big(z_\lambda(m_{\lambda,i}), m_{\lambda,i}\big) = 0 \qquad \text{and} \qquad \nabla_m \tilde{\mathcal{F}}_\lambda\big(z_\lambda(m_{\lambda,i}), m_{\lambda,i}\big) = \nabla_m \mathcal{E}_\lambda\big(m_{\lambda,i}\big) = 0.$$

The proof of the Claim is finished. □

In order to complete the study of existence of two bubbling solutions to (1.1) in the flat two-torus in rectangular form T, we shall show that the multiplicity depends on the values $G(z)$ with $z \in \{p_1, p_2, p_3\}$, precisely, depends on the form of T. Recall, G has three nondegenerate critical points: $p_1 = \frac{a}{2}$, $p_2 = \frac{\mathrm{i}b}{2}$ (saddle points) and $p_3 = \frac{a+\mathrm{i}b}{2}$ (minimum point). Notice that since G has zero average and p_3 is a minimum point we have that $G(p_3) < 0$ for any $a, b > 0$. In other words, $G\Big(\frac{a+\mathrm{i}b}{2}\Big) \le G\Big(\frac{a}{2}\Big)$ and $G\Big(\frac{a+\mathrm{i}b}{2}\Big) \le G\Big(\frac{\mathrm{i}b}{2}\Big)$. From an explicit formula for G shown in [10], direct computations lead us to get that

$$G\left(p_i\right) = f_i\Big(\frac{b}{a}\Big), \quad i = 1, 2, 3$$

where f_i, $i = 1, 2, 3$ are given by

$$f_1(\tau) = \frac{\tau}{12} - \frac{1}{2\pi}\log 2 - \frac{1}{\pi}\sum_{n=1}^{+\infty}\log\left(1 + e^{-2\pi n\tau}\right),$$

$$f_2(\tau) = -\frac{\tau}{24} - \frac{1}{\pi}\sum_{n=0}^{+\infty}\log\left(1 - e^{-\pi(2n+1)\tau}\right)$$

and

$$f_3(\tau) = -\frac{\tau}{24} - \frac{1}{\pi}\sum_{n=0}^{+\infty}\log\left(1 + e^{-\pi(2n+1)\tau}\right),$$

so that we can study them in terms of $\tau = \frac{b}{a}$. By symmetry arguments it follows that in case $a = b$, namely, $\tau = 1$, it holds $G\Big(\frac{a}{2}\Big) = G\Big(\frac{\mathrm{i}a}{2}\Big)$, so, equivalently $f_1(1) = f_2(1) \approx -0.03$. By studying f_i, $i = 1, 2, 3$ we obtain the following fact.

Claim 3.9. *There exist $\tau_0 < 1 < \tau_1$ such that $f_1(\tau_1) = 0$, $f_2(\tau_0) = 0$. If $\tau \in (0, \tau_0] \cup [\tau_1, +\infty)$ then there exist seven different critical points $(\xi_{\lambda,i}, m_{\lambda,i})$ $i = 1, \dots, 7$ of $\mathcal{F}_\lambda$. If $\tau \in (\tau_0, \tau_1)$ then there exist nine different critical points $(\xi_{\lambda,i}, m_{\lambda,i})$ $i = 1, \dots, 9$ of $\mathcal{F}_\lambda$.*

Proof: From the definition of f_1 it follows that f_1 is a continuous function, strictly increasing and strictly concave, $f_1(0) \approx -1.43$ and $f_1(\tau) \to +\infty$ as $\tau \to +\infty$, so that, there is $\tau_1 > 1$ such that $f_1(\tau_1) = 0$. Also, f_2 is a continuous function, strictly decreasing and strictly convex, with $f_2(\tau) \to +\infty$ as $\tau \to 0^+$ and $f_2(\tau) \to -\infty$ as $\tau \to +\infty$ so that there is $\tau_0 < 1$ such that $f_2(\tau_0) = 0$. For f_3 we obtain that it is a continuous function, strictly concave, $\tau = 1$ is a maximum with $f_3(1) \approx -0.06$, $f_3(0) \approx -1.32$ and $f_3(\tau) \to -\infty$ as $\tau \to +\infty$.

Hence, depending on the value of $\tau = \frac{b}{a}$ we have three cases.

1. If $\tau \in (0, \tau_0]$ then $f_1(\tau) = G\Big(\frac{a}{2}\Big) < 0$, $f_2(\tau) = G\Big(\frac{\mathrm{i}b}{2}\Big) \ge 0$ and $f_3(\tau) = G\Big(\frac{a+\mathrm{i}b}{2}\Big) < 0$, so that we have seven critical points of $\tilde{\mathcal{F}}_\lambda$. Precisely, $G\Big(\frac{a}{2}\Big) < 0$, $G\Big(\frac{\mathrm{i}b}{2}\Big) \ge 0$ and $G\Big(\frac{a+\mathrm{i}b}{2}\Big) < 0$ gives rise to three, one and three three critical points respectively.

2. If $\tau \in (\tau_0, \tau_1)$ then $G\Big(\frac{a}{2}\Big) < 0$, $G\Big(\frac{\mathrm{i}b}{2}\Big) < 0$ and $G\Big(\frac{a+\mathrm{i}b}{2}\Big) < 0$, so that we have nine critical points of $\tilde{\mathcal{F}}_\lambda$. Precisely, $G\Big(\frac{a}{2}\Big), G\Big(\frac{\mathrm{i}b}{2}\Big), G\Big(\frac{a+\mathrm{i}b}{2}\Big) < 0$ gives rise to three critical points each one.

3. If $\tau \in (\tau_1, +\infty)$ then $G\Big(\dfrac{a}{2}\Big) \geq 0$, $G\Big(\dfrac{ib}{2}\Big) < 0$ and $G\Big(\dfrac{a+ib}{2}\Big) < 0$, so that we have seven critical points of $\tilde{\mathcal{F}}_\lambda$. Precisely, $G\Big(\dfrac{a}{2}\Big) \geq 0$, $G\Big(\dfrac{ib}{2}\Big), G\Big(\dfrac{a+ib}{2}\Big) < 0$ gives rise to one, three and three three critical points respectively.

The proof of the Claim is finished □

Therefore, given a rectangle we can obtain exactly either seven or nine different family of solutions. This completes the proof. □

Proof (of Theorem 1.2): Assume that $S = \mathbb{S}^2$. By invariance under rotations it follows that $H(\xi,\xi)$ is constant. Furthermore, as it was said in the introduction problem (1.1) is invariant under rotations, so we look for solutions with one bubbling point fixed. Thus, with a slightly abuse of notation, we are reduced to look for critical points of the functional

$$\mathcal{F}_\lambda(\xi_1, m) = 4\pi - \frac{\lambda|T|}{2} + 8\pi\lambda\left[A\sum_{j=1}^{2} m_j^2 + 2\sum_{j=1}^{2} m_j^2 \log m_j - 8\pi m_1 m_2 G(\xi_1,\xi_2)\right] + \theta_\lambda(\xi_1, m),$$

where $A = \log 16 - 2 - 8\pi H(\xi,\xi)$. In other words, we fix $\xi_2 \in \mathbb{S}^2$ and look for critical points on ξ_1 of $G(\cdot,\xi_2)$. Since $G(\cdot,\xi_2)$ has a global minimum, for any m there exist $\xi_{1,\lambda}(m) \in \mathbb{S}^2$ such that $\mathcal{F}_\lambda$ attains its minimum at $\xi_1 = \xi_{1,\lambda}(m)$. Hence, from the same procedure used in Claim 3.8 it follows that for λ small enough there exists m_λ such that $\big(\xi_{1,\lambda}(m_\lambda), m_\lambda\big)$ is actually a critical point of $\mathcal{F}_\lambda(\xi_1, m)$. This finishes the proof. □

4 Proof of Proposition 3.4

The purpose of this section is to give a proof of the Proposition 3.4, namely, an asymptotic expansion of the "reduced energy" $\mathcal{F}_\lambda(\xi, m) = J_\lambda\big(U(\xi,m) + \tilde{\phi}(\xi,m)\big)$, where J_λ is the energy functional given by (1.2), $U = \sqrt{\lambda}V$ with V defined by (2.9) and $\tilde{\phi} = \sqrt{\lambda}\phi$ with ϕ the solution given by Proposition 3.2. The proof will be divided into several steps. To this aim, we recall the following result. See [18] for a proof.

Lemma 4.1. *Letting $\bar{f} \in C^{2,\gamma}(S)$ (possibly depending in ξ), $0 < \gamma < 1$. The following expansions do hold as $\delta \to 0$:*

$$\int_S \chi_\xi e^{-\varphi_\xi} \bar{f}(x) e^{U_{\delta,\xi}} dv_g = 8\pi \bar{f}(\xi) - 4\pi\delta^2 \log\delta \Delta_g \bar{f}(\xi) + O(\delta^2),$$

$$\int_S \chi_\xi e^{-\varphi_\xi} \bar{f}(x) e^{U_{\delta,\xi}} \frac{dv_g}{\delta^2 + |y_\xi(x)|^2} = \frac{4\pi}{\delta^2}\bar{f}(\xi) + \pi\Delta_g \bar{f}(\xi) + O(\delta^\gamma)$$

and

$$\int_S \chi_\xi e^{-\varphi_\xi} \bar{f}(x) e^{U_{\delta,\xi}} \frac{a\delta^2 - |y_\xi(x)|^2}{(\delta^2 + |y_\xi(x)|^2)^2} dv_g = \frac{4\pi}{3\delta^2}(2a-1)\bar{f}(\xi) + (a-2)\frac{\pi}{3}\Delta_g \bar{f}(\xi) + O(\delta^\gamma)$$

for $a \in \mathbb{R}$.

We are now ready to establish the expansion of $J_\lambda(U)$:

Claim 4.2. *The following expansion does hold*

$$J_\lambda(U) = 2\pi k - \frac{\lambda|S|}{2} + 8\pi\lambda\psi_k(\xi, m) + O(\lambda^2|\log\lambda|^2) \tag{4.1}$$

in $C(\Xi \times \mathcal{M})$ as $\lambda \to 0^+$.

Proof: First, let us consider the term

$$\int_S |\nabla V|_g^2 dv_g = \int_S V(-\Delta_g V) dv_g = \sum_{j,l=1}^k m_j m_l \int_S \chi_j e^{-\varphi_j} e^{U_j} PU_l dv_g$$

in view of $\int_S V dv_g = 0$. Since by (1.5) and (2.4)

$$\int_S \chi_j e^{-\varphi_j} e^{U_j} G(x,\xi_l) dv_g = \int_S (-\Delta_g PU_j) G(x,\xi_l) dv_g = PU_j(\xi_l) \tag{4.2}$$

for all $j, l = 1, \dots, m$, by Lemmata 2.1, 4.1 and (4.2) we have that for $l = j$

$$\begin{aligned}
&\int_S \chi_j e^{-\varphi_j} e^{U_j} PU_j dv_g \\
&= \int_S \chi_j e^{-\varphi_j} e^{U_j} \left[\chi_j (U_j - \log(8\mu_j^2 \varepsilon_j^2)) + 8\pi H(x,\xi_j) + O(\varepsilon_j^2 |\log \varepsilon_j|)\right] dv_g \\
&= \int_S \chi_j e^{-\varphi_j} e^{U_j} \left[\chi_j \log \frac{|y_{\xi_j}(x)|^4}{(\mu_j^2 \varepsilon_j^2 + |y_{\xi_j}(x)|^2)^2} + 8\pi G(x,\xi_j)\right] dv_g + O(\varepsilon_j^2 |\log \varepsilon_j|) \\
&= 8 \int_{B_{2r_0/\mu_j \varepsilon_j}(0)} \frac{\chi^2(\mu_j \varepsilon_j |y|)}{(1+|y|^2)^2} \log \frac{|y|^4}{(1+|y|^2)^2} dy + 8\pi PU_j(\xi_j) + O(\varepsilon_j^2 |\log \varepsilon_j|) \\
&= -16\pi - 32\pi \log \mu_j \varepsilon_j + 64\pi^2 H(\xi_j,\xi_j) + O(\varepsilon_j^2 |\log \varepsilon_j|)
\end{aligned}$$

in view of

$$\int_{\mathbb{R}^2} \frac{dy}{(1+|y|^2)^2} \log \frac{|y|^4}{(1+|y|^2)^2} = 2\pi \int_0^\infty \frac{ds}{(1+s)^2} \log \frac{s}{1+s} = -2\pi \int_0^\infty \frac{ds}{(1+s)^2} = -2\pi$$

by means of an integration by parts. Similarly, by Lemmata 2.1, 4.1 and (4.2) we have that for $l \neq j$

$$\begin{aligned}
\int_S \chi_j e^{-\varphi_j} e^{U_j} PU_l dv_g &= \int_S \chi_j e^{-\varphi_j} e^{U_j} \left[8\pi G(x,\xi_l) + O(\varepsilon_l^2 |\log \varepsilon_l|)\right] dv_g \\
&= 64\pi^2 G(\xi_l,\xi_j) + O(\varepsilon_j^2 |\log \varepsilon_j| + \varepsilon_l^2 |\log \varepsilon_l|).
\end{aligned}$$

By using the definition of μ_j and ε_j and summing up the two previous expansions, for the gradient term we get that

$$\begin{aligned}
&\frac{1}{2} \int_S |\nabla U|_g^2 dv_g = \frac{\lambda}{2} \int_S |\nabla V|_g^2 \, dv_g \\
&= 2\pi k - 8\pi\lambda(1 - \log 8) \sum_{j=1}^k m_j^2 + 8\pi\lambda \sum_{j=1}^k m_j^2 \log(2m_j^2) - 32\pi^2 \lambda \sum_{j=1}^k m_j^2 H(\xi_j,\xi_j) \\
&\quad -32\pi^2 \lambda \sum_{i \neq j} m_i m_j G(\xi_i,\xi_j) + \lambda \sum_{j=1}^k O(\varepsilon_j^2 |\log \varepsilon_j|).
\end{aligned}$$

Let us now expand the potential term in $J_\lambda(U)$. By Lemma 2.1 for any $j = 1, \dots, k$ we find that $PU_j = O(\big|\log|\log \varepsilon_j|\big|) = O(|\log \lambda|)$, in $B_{r_0}(\xi) \setminus B_{\frac{\delta}{|\log \varepsilon_j|}}(\xi_j)$. Recall that $PU_j = O(1)$ in $S \setminus \cup_{j=1}^k B_{r_0}(\xi_j)$ for each $j = 1, \dots, k$. Hence, we have that $V = O(|\log \lambda|)$, in $S \setminus \cup_{j=1}^k B_{\frac{\delta}{|\log \varepsilon_j|}}(\xi_j)$ and also,

$$\int_S e^{\lambda V^2} \, dv_g = \sum_{j=1}^k \int_{B_{\delta |\log \varepsilon_j|^{-1}}(\xi_j)} e^{\lambda V^2} dv_g + |S| + O(\lambda |\log \lambda|^2).$$

Now, we write

$$\int_{B_{\delta|\log\varepsilon_j|^{-1}}(\xi_j)} e^{\lambda V^2} dv_g = \left[\int_{A_{\delta|\log\varepsilon_j|^{-1},\delta\sqrt{\varepsilon_j}}(\xi_j)} + \int_{A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} + \int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)}\right] e^{\lambda V^2} dv_g$$

In $A_{\delta|\log\varepsilon_j|^{-1},\delta\sqrt{\varepsilon_j}}(\xi_j)$, we know that uniformly $V(x) = -4m_j \log|y_{\xi_j}(x)| + O(1)$. Thus, we find that

$$\begin{aligned}
\int_{A_{\delta|\log\varepsilon_j|^{-1},\delta\sqrt{\varepsilon_j}}(\xi_j)} e^{\lambda V^2}\, dv_g &= \int_{A_{\delta|\log\varepsilon_j|^{-1},\delta\sqrt{\varepsilon_j}}(\xi_j)} e^{\lambda m_j^2[16\log^2|y_{\xi_j}(x)| + O(|\log|y_{\xi_j}(x)||)]} dv_g \\
&= O\left(\int_{B_{\delta|\log\varepsilon_j|^{-1}}(0)\setminus B_{\delta\sqrt{\varepsilon_j}}(0)} e^{16\lambda m_j^2 \log|y|} dy\right) = O(\lambda),
\end{aligned}$$

in view of $e^{\hat\varphi_j} = O(1)$, $\lambda m_j^2 |\log|y|| = O(1)$ in the considered region and

$$\begin{aligned}
\int_{B_{\delta|\log\varepsilon_j|^{-1}}(0)\setminus B_{\delta\sqrt{\varepsilon_j}}(0)} e^{16\lambda m_j^2\log|y|} dy &= 2\pi \int_{\delta\sqrt{\varepsilon_j}}^{\delta|\log\varepsilon_j|^{-1}} e^{16\lambda m_j^2 \log^2 s} s\, ds \\
&= 2\pi \int_{\log(\delta\sqrt{\varepsilon_j})}^{\log(\delta|\log\varepsilon_j|^{-1})} e^{2t + 16\lambda m_j^2 t^2}\, dt \quad (t = \log s) \\
&= O\left(\int_{\log(\delta\sqrt{\varepsilon_j})}^{\log(\delta|\log\varepsilon_j|^{-1})} e^t\, dt\right) = O(\lambda).
\end{aligned}$$

Now, we shall use that in $B_{r_0}(\xi_j)$ it holds that $e^{\lambda V^2} = 2m_j^2 \varepsilon_j^{-2} e^{w_j + \theta_j + \lambda m_j^2 (w_j+\theta_j)^2}$ and $\lambda w_j = O(1)$ uniformly. Furthermore, we know that $w_j(1 + \lambda m_j^2 w_j) \le \frac{3}{4} w_j + \beta$, where β is a constant in $A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}$. Hence, we obtain that

$$\int_{A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}} e^{\lambda V^2} = O\Big(\int_{A_{\delta\sqrt{\varepsilon_j},\delta\varepsilon_j|\log\varepsilon_j|}} \varepsilon_j^{-2} e^{w_j + \lambda m_j^2 w_j^2}\Big) = O(\lambda).$$

Also, it follows that

$$\begin{aligned}
\int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} e^{\lambda V^2}\, dv_g &= 2m_j^2 \int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} \varepsilon_j^{-2} e^{w_j + \theta_j + O(\lambda|\log\lambda|^2)} dv_g \\
&= 2m_j^2 [8\pi + O(\lambda|\log\lambda|^2),
\end{aligned}$$

in view of $w_j = O(|\log\lambda|)$ in $B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)$ and

$$\int_{B_{\delta\varepsilon_j|\log\varepsilon_j|}(\xi_j)} \varepsilon_j^{-2} e^{w_j+\theta_j} = 8\pi + O\Big(\lambda + \sum_{j=1}^k \varepsilon_j^2 |\log\varepsilon_j|\Big).$$

Therefore, we conclude that

$$\int_S e^{\lambda V^2}\, dv_g = 16\pi \sum_{j=1}^k m_j^2 + |S| + O(\lambda|\log\lambda|^2) \tag{4.3}$$

and consequently,

$$\begin{aligned}
J_\lambda(U) = 2\pi k &+ 8\pi\lambda(\log 16 - 1)\sum_{j=1}^k m_j^2 + 8\pi\lambda \sum_{j=1}^k m_j^2 \log(m_j^2) - 32\pi^2\lambda \sum_{j=1}^k m_j^2 H(\xi_j,\xi_j) \\
&- 32\pi^2\lambda \sum_{i\neq j} m_i m_j G(\xi_i,\xi_j) + \lambda \sum_{j=1}^k O(\varepsilon_j^2|\log\varepsilon_j|) - \frac{\lambda}{2}\left[16\pi \sum_{j=1}^k m_j^2 + |S| + O(\lambda|\log\lambda|^2)\right].
\end{aligned}$$

This completes the proof. □

In order to expand the derivatives $\partial_{(\xi_l)_q}\mathcal{F}_\lambda$ and $\partial_{m_l}\mathcal{F}_\lambda$, and get some estimates for $\partial_{(\xi_l)_q}\phi$ and $\partial_{m_l}\phi$, we have to expand $\partial_{(\xi_l)_q}V$ and $\partial_{m_l}V$ for $q=1,2$ and $l=1,\dots,k$. Let us notice that from the definition of PU_j and similar ideas to obtain the expansion (2.6), we have that the derivatives $\partial_{(\xi_l)_q}PU_j$, for $q=1,2$ and $\partial_{m_l}PU_j$ for $j,\,l=1,\dots,k$ expand as follows

$$\begin{aligned}\partial_{(\xi_l)_q}PU_j(x) =& \,\delta_{jl}\partial_{(\xi_l)_q}\chi_j[U_j-\log(8\mu_j^2\varepsilon_j^2)]+\chi_j\partial_{(\xi_l)_q}[U_j-\log(8\mu_j^2\varepsilon_j^2)]\\ &+\delta_{jl}8\pi\partial_{(\xi_l)_q}H(x,\xi_j)+O\left(\varepsilon_j^2|\log\varepsilon_j|\right)\end{aligned}\tag{4.4}$$

and

$$\partial_{m_l}PU_j(x)=\chi_j\partial_{m_l}[U_j-\log(8\mu_j^2\varepsilon_j^2)]+O\left(\varepsilon_j^2|\log\varepsilon_j|^2\right),\tag{4.5}$$

uniformly in S. Let us stress that $\mu_j=\mu_j(\xi,m)$ and $\varepsilon_j=\varepsilon_j(m_j)$. Furthermore, from the definition of ε_l we get

$$\partial_{m_l}\varepsilon_l=\frac{\varepsilon_l}{4}\left(\frac{1}{\lambda m_l^3}+\frac{4}{m_l}\right)=\frac{\varepsilon_l}{m_l}\left[-2\log\varepsilon_l+\log(2m_l^2)+1\right].\tag{4.6}$$

Hence, we have that uniformly in S

$$\begin{aligned}\partial_{(\xi_l)_q}V(x)=&-2m_l\partial_{(\xi_l)_q}\chi_l\log(\mu_l^2\varepsilon_l^2+|y_{\xi_l}(x)|^2)-m_l\chi_l\frac{4\mu_l\partial_{(\xi_l)_q}\mu_l\varepsilon_l^2+2\partial_{(\xi_l)_q}(|y_{\xi_l}(x)|^2)}{\mu_l^2\varepsilon_l^2+|y_{\xi_l}(x)|^2}\\ &+8\pi m_l\partial_{(\xi_l)_q}H(x,\xi_l)-\sum_{j\neq l}m_j\chi_j\frac{4\mu_j\partial_{(\xi_l)_q}\mu_j\varepsilon_j^2}{\mu_j^2\varepsilon_j^2+|y_{\xi_j}(x)|^2}+\sum_{j=1}^kO\left(\varepsilon_j^2|\log\varepsilon_j|\right)\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}\partial_{m_l}V(x)=&-2\chi_l\log(\mu_l^2\varepsilon_l^2+|y_{\xi_l}(x)|^2)+8\pi H(x,\xi_l)-m_l\chi_l\frac{4\mu_l\partial_{m_l}\mu_l\varepsilon_l^2+4\mu_l^2\varepsilon_l\partial_{m_l}\varepsilon_l}{\mu_l^2\varepsilon_l^2+|y_{\xi_l}(x)|^2}\\ &-\sum_{j\neq l}m_j\chi_j\frac{4\mu_j\partial_{m_l}\mu_j\varepsilon_j^2}{\mu_j^2\varepsilon_j^2+|y_{\xi_j}(x)|^2}+\sum_{j=1}^kO\left(\varepsilon_j^2|\log\varepsilon_j|^2\right).\end{aligned}\tag{4.8}$$

Claim 4.3. *The following expansion does hold*

$$\partial_{m_l}[J_\lambda(U)]=8\pi\lambda\partial_{m_l}\psi_k(\xi,m)+O(\lambda^2|\log\lambda|^2)\tag{4.9}$$

in $C(\Xi\times\mathcal{M})$ *as* $\lambda\to0^+$.

Proof: Notice that

$$\partial_{m_l}[J_\lambda(U)]=-\lambda\int_S\left[\Delta_gV+\lambda Ve^{\lambda V^2}\right]\partial_{m_l}V\,dv_g.$$

Hence, we have that

$$\int_S\Delta_gV\,\partial_{m_l}V\,dv_g=-\sum_{i=1}^km_i\int_S\chi_ie^{-\varphi_i}e^{U_i}\partial_{m_l}V$$

in view of $\displaystyle\int_S\partial_{m_l}V=0$. For $i\neq l$, by using Lemma 4.1 and (4.8), we find that

$$\begin{aligned}\int_S\chi_ie^{-\varphi_i}e^{U_i}\,\partial_{m_l}V\,dv_g=&\int_S\chi_ie^{-\varphi_i}e^{U_i}\left[8\pi G(x,\xi_l)-m_i\chi_i\frac{4\mu_i\partial_{m_l}\mu_i\varepsilon_i^2}{\mu_i^2\varepsilon_i^2+|y_{\xi_i}(x)|^2}+\sum_{j=1}^kO\left(\varepsilon_j^2|\log\varepsilon_j|^2\right)\right]\\ =&\,64\pi^2G(\xi_i,\xi_l)-16\pi m_i\frac{\partial_{m_l}\mu_i}{\mu_i}+\sum_{j=1}^kO\left(\varepsilon_j^2|\log\varepsilon_j|^2\right)=\sum_{j=1}^kO\left(\varepsilon_j^2|\log\varepsilon_j|^2\right),\end{aligned}$$

in view of $\chi_j \equiv 0$ in $B_{2r_0}(\xi_i)$ with $j \neq i$ and from (2.7)

$$2\frac{\partial_{m_l}\mu_j}{\mu_j} = 8\pi m_j^{-1} G(\xi_l,\xi_j) \quad \text{with } j \neq l. \tag{4.10}$$

Also, using again (4.8), we find that

$$\begin{aligned}\int_S \chi_l e^{-\varphi_l} e^{U_l}\, \partial_{m_l} V\, dv_g = \int_S \chi_l e^{-\varphi_l} e^{U_l} \bigg[& -2\chi_l \log(\mu_l^2\varepsilon_l^2 + |y_{\xi_l}(x)|^2) + 8\pi H(x,\xi_l) \\ & - m_l\chi_l \frac{2\partial_{m_l}(\mu_l^2\varepsilon_l^2)}{\mu_l^2\varepsilon_l^2 + |y_{\xi_l}(x)|^2} + \sum_{j=1}^k O\left(\varepsilon_j^2 |\log\varepsilon_j|^2\right) \bigg] \\ = & -16\pi - 32\pi\log(\mu_l\varepsilon_l) + 64\pi^2 H(\xi_l,\xi_l) - 8\pi m_l \pi \frac{\partial_{m_l}(\mu_l^2\varepsilon_l^2)}{\mu_l^2\varepsilon_l^2} \\ & + \sum_{j=1}^k O\left(\varepsilon_j^2|\log\varepsilon_j|^2\right).\end{aligned}$$

From the definition (2.7)-(2.8), we get that

$$2\frac{\partial_{m_l}\mu_l}{\mu_l} = -\frac{4}{m_l} - \frac{8\pi}{m_l^2}\sum_{i\neq l} m_i G(\xi_l,\xi_i) \qquad \text{and} \qquad -\frac{4}{\varepsilon_l}\partial_{m_l}\varepsilon_l = -\frac{1}{\lambda m_j^3} - \frac{4}{m_j} \tag{4.11}$$

so that, we conclude that

$$\begin{aligned}\int_S \Delta_g V\, \partial_{m_l} V\, dv_g = & -16\pi\log 8m_l - 16\pi m_l \log(2m_l^2) + 64\pi^2 m_l H(\xi_l,\xi_l) \\ & + 64\pi^2 \sum_{j=1, j\neq l}^k m_j G(\xi_j,\xi_l) + O(\lambda|\log\lambda|).\end{aligned}$$

On the other hand, combining arguments to deduce (2.22) and (4.3), using Lemma 4.1 and the derivatives (4.8) and (4.10)-(4.11), we find that

$$\int_S \lambda V e^{\lambda V^2} \partial_{m_l} V\, dv_g = 16\pi m_l + O(\lambda|\log\lambda|).$$

Thus, we conclude (4.9). □

Claim 4.4. *The following expansion does hold*

$$\mathcal{F}_\lambda(\xi,m) = J_\lambda(U) + \tilde\theta_\lambda(\xi,m) \tag{4.12}$$

in $C(\Xi\times\mathcal{M})$ and $C^1(\mathcal{M})$ as $\lambda\to 0^+$, where the term $\tilde\theta_\lambda(\xi,m)$ satisfies

$$|\tilde\theta_\lambda(\xi,m)| + \sum_{l=1}^k \lambda\left|\partial_{m_l}\tilde\theta_\lambda(\xi,m)\right| = O(\lambda^3) \tag{4.13}$$

as $\lambda\to 0^+$ uniformly for points $(\xi,m)\in\Xi\times\mathcal{M}$.

Proof: Taking into account that $DJ_\lambda(U+\tilde\phi)[\tilde\phi] = 0$, a Taylor expansion, $\sqrt{\lambda}V = U$ and $\sqrt{\lambda}\,\phi(\xi,m) = \tilde\phi(\xi,m)$, the definition of K and (3.6) gives us

$$\begin{aligned}J_\lambda(U+\tilde\phi) - J_\lambda(U) &= -\int_0^1 D^2 J_\lambda\big(U + t\tilde\phi\big)\big[\tilde\phi,\tilde\phi\big]\, t\, dt, \\ &= -\lambda\int_0^1 \left(\int_S [R + N(\phi)]\,\phi + \int_S [K - f'(V+t\phi)]\,\phi^2\right) t\, dt.\end{aligned} \tag{4.14}$$

Therefore, we get

$$J_\lambda(U+\tilde\phi)-J_\lambda(U)=O(\lambda^3),$$

since $\|R\|_*\le C\lambda$, $\|N(\phi)\|_*\le C[\lambda\|\phi\|_\infty+\|\phi\|_\infty^2]$, for some $s\in(0,t)$

$$\|K-f'(V+t\phi)\|_*\le\|f''(V+s\phi)\,\phi\|_*+\|K-f'(V)\|_*\le C[\|\phi\|_\infty+\lambda]$$

and $\|\phi\|_\infty\le C\lambda$.

Let us differentiate with respect to $\beta=m_l$. We use representation (4.14) and differentiate directly under the integral sign, thus obtaining, for each $l=1,\dots,k$

$$\begin{aligned}\partial_\beta\big[J_\lambda(U+\tilde\phi)-J_\lambda(U)\big]=&-\lambda\int_0^1\left(\int_S\{[\partial_\beta R+\partial_\beta N(\phi)]\,\phi+[R+N(\phi)]\,\partial_\beta\phi\}\right)t\,dt\\&-\lambda\int_0^1\left(\int_S\{\partial_\beta K-\partial_\beta[f'(V+t\phi)]\}\,\phi^2+[K-f'(V+t\phi)]\,2\phi\,\partial_\beta\phi\right)t\,dt.\end{aligned}$$

Using Proposition 3.2 and the computations in the Appendix B, we get that

$$\begin{aligned}\big|\partial_{m_l}\big[J_\lambda(U+\tilde\phi)-J_\lambda(U)\big]\big|\le C\lambda\Big(&\big[\lambda\log\varepsilon_l+\lambda^2\log\varepsilon_l+\lambda\|\partial_{m_l}\phi\|_\infty\big]\;\|\phi\|_\infty\\&+\big[\lambda+\lambda\|\phi\|_\infty+\|\phi\|_\infty^2\big]\;\|\partial_{m_l}\phi\|_\infty+\lambda\log\varepsilon_l\|\phi\|_\infty^2+[\|\phi\|_\infty+\lambda]\;\|\phi\|_\infty\,\|\partial_{m_l}\phi\|_\infty\Big).\end{aligned}$$

Thus, we conclude

$$\partial_{m_l}\Big[J_\lambda(U+\tilde\phi)-J_\lambda(U)\Big]=O\big(\lambda^3\,|\log\varepsilon_l|\big)=O(\lambda^2),\qquad l=1,\dots,k$$

Now, taking $\tilde\theta_\lambda(\xi,m)=\mathcal{F}_\lambda(\xi,m)-J_\lambda(U)$, we have shown (4.13) as $\lambda\to0^+$. The continuity in (ξ,m) of all these expressions is inherited from that of ϕ and its derivatives in ξ and m in the L^∞ norm. □

Now, we are going to study the derivatives of $\mathcal{F}_\lambda$ with respect to $\beta=(\xi_l)_q$ with $q=1,2$ and $l=1,\dots,k$. Due to the estimates (3.8) given in Proposition 3.2 we have to address this expansion in a different way. We shall use similar ideas first presented in [20] and also used in [9].

Claim 4.5. *The following expansion does hold*

$$\partial_{(\xi_l)_q}\mathcal{F}_\lambda(\xi,m)=8\pi\lambda\partial_{(\xi_l)_q}\psi_k(\xi,m)+O(\lambda^2|\log\lambda|^2)\tag{4.15}$$

in $C(\Xi\times\mathcal{M})$ *as* $\lambda\to0^+$.

Proof: Let us differentiate the function $\mathcal{F}_\lambda(\xi,m)$ with respect to $(\xi_l)_q$ with $q=1,2$ and $l=1,\dots,k$. Since $\sqrt\lambda\,V(\xi,m)=U(\xi,m)$ and $\sqrt\lambda\,\phi(\xi,m)=\tilde\phi(\xi,m)$, we can differentiate directly $J_\lambda\big(\sqrt\lambda[V+\phi]\big)$ (under the integral sign), so that integrating by parts we get

$$\begin{aligned}\partial_{(\xi_l)_q}\mathcal{F}_\lambda(\xi,m)=&-\lambda\int_S\Big[\Delta_g(V+\phi)+\lambda(V+\phi)e^{\lambda(V+\phi)^2}\Big]\;\big[\partial_{(\xi_l)_q}V+\partial_{(\xi_l)_q}\phi\big]\;dv_g\\=&-\lambda\sum_{i=0}^{2}\sum_{j=1}^{k}c_{ij}\int_S\Delta_gPZ_{ij}\;\big[\partial_{(\xi_l)_q}V+\partial_{(\xi_l)_q}\phi\big]\,dv_g,\end{aligned}$$

since $\int_S\big(\partial_{(\xi_l)_q}V+\partial_{(\xi_l)_q}\phi\big)dv_g=0$. From the orthogonal conditions we find that for $j\ne l$

$$c_{ij}\int_S\Delta_gPZ_{ij}\,\partial_{(\xi_l)_q}\phi\,dv_g=-c_{ij}\int_S\partial_{(\xi_l)_q}\left[\Delta_gPZ_{ij}\right]\,\phi\,dv_g=O\left(\max_{ij}|c_{ij}|\,\|\phi\|_\infty\right)=O(\lambda^2)$$

in view of $\int_S \rho(x)\,dv_g \le C$, $\|\partial_{(\xi_l)_q}\Delta_g PZ_{ij}\|_* \le C$ for $j \ne l$ and (3.7). For $j = l$, we compare $\partial_{(\xi_l)_q}\Delta_g PZ_{il}$ with derivatives $\partial_{x_q}\Delta_g PZ_{il}$ to get

$$\int_S \Delta_g PZ_{il}\,\partial_{(\xi_l)_q}\phi\,dv_g = -\int_S \partial_{(\xi_l)_q}\left[\Delta_g PZ_{il}\right]\,\phi\,dv_g = \int_S \partial_{x_q}\left[\Delta_g PZ_{il}\right]\,\phi\,dv_g + O(\lambda)$$

Thus, integrating by parts we deduce that

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij}\int_S \Delta_g PZ_{ij}\,\partial_{(\xi_l)_q}\phi\,dv_g &= -\sum_{i=0}^{2} c_{il}\int_S \Delta_g PZ_{il}\,\partial_{x_q}\phi\,dv_g + O\left(\lambda^2\right)\\ &= -\int_{B_{2r_0}(\xi_l)}\left[\Delta_g(V+\phi) + \lambda(V+\phi)e^{\lambda(V+\phi)^2}\right]\,\partial_{x_q}\phi\,dv_g + O(\lambda^2),\end{aligned}$$

On the other hand, from (4.7), we obtain that for $j \ne l$ in $B_{2r_0}(\xi_j)$

$$\partial_{(\xi_l)_q}V(x) = 8\pi m_l \partial_{(\xi_l)_q}G(x,\xi_l) - m_j\chi_j \frac{4\mu_j\partial_{(\xi_l)_q}\mu_j\varepsilon_j^2}{\mu_j^2\varepsilon_j^2 + |y_{\xi_j}(x)|^2} + \sum_{i=1}^{k} O\left(\varepsilon_i^2|\log\varepsilon_i|\right)$$

and consequently

$$\begin{aligned}\int_S \Delta_g PZ_{ij}\,\partial_{(\xi_l)_q}V\,dv_g = -\int_{B_{2r_0}(\xi_j)} \chi_j e^{-\varphi_j}\varepsilon_j^{-2}e^{w_j}Z_{ij}\Bigg[&8\pi m_l\partial_{(\xi_l)_q}G(x,\xi_l) - m_j\chi_j\frac{4\mu_j\partial_{(\xi_l)_q}\mu_j\varepsilon_j^2}{\mu_j^2\varepsilon_j^2 + |y_{\xi_j}(x)|^2}\\ &+ \sum_{i=1}^{k} O\left(\varepsilon_i^2|\log\varepsilon_i|\right)\Bigg]\,dv_g\end{aligned}$$

From the definition of Z_{ij} it follows that for $j \ne l$

$$\int_S \Delta_g PZ_{ij}\,\partial_{(\xi_l)_q}V\,dv_g = \begin{cases} O(1) & \text{if } i = 0,\\ O(\varepsilon_j) & \text{if } i = 1,2.\end{cases}$$

For $j = l$, we compare $\partial_{(\xi_l)_q}V$ with derivatives $\partial_{x_q}V$ to get

$$\int_S \Delta_g PZ_{il}\,\partial_{(\xi_l)_q}V\,dv_g = -\int_S \Delta_g PZ_{il}\,\partial_{x_q}V\,dv_g + O(\lambda).$$

Hence, taking into account that $|c_{ij}| \le C\lambda$ for all $i = 0,1,2$ and $j = 1,\dots,k$, we obtain that

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij}\int_S \Delta_g PZ_{ij}\,\partial_{(\xi_l)_q}V\,dv_g &= \sum_{i=0}^{2} c_{il}\int_S \Delta_g PZ_{il}\,\partial_{(\xi_l)_q}V\,dv_g + O(\lambda)\\ &= -\sum_{i=0}^{2} c_{il}\int_S \Delta_g PZ_{il}\,\partial_{x_q}V\,dv_g + O(\lambda)\\ &= -\int_{B_{2r_0}(\xi_l)}\left[\Delta_g(V+\phi) + \lambda(V+\phi)e^{\lambda(V+\phi)^2}\right]\partial_{x_q}V + O(\lambda).\end{aligned}$$

Therefore, denoting $v_\xi = V + \phi$ we get that

$$\partial_{(\xi_l)_q}\mathcal{F}_\lambda(\xi,m) = \lambda\int_{B_{2r_0}(\xi_l)}\left[\Delta_g v_\xi + \lambda v_\xi e^{\lambda v_\xi^2}\right]\,\partial_{x_q}v_\xi\,dv_g + O(\lambda)$$

Hence, using a Pohozaev type of identity as used in [20, Proof of Lemma 5.3] or [9, Proof of Proposition 3.2] and the expansion

$$V(x) + \phi(x) = m_l \left[-4\chi_l(x)\log|y_{\xi_l}(x)| + 8\pi H(x,\xi_l)\right] + \sum_{j=1, j\neq l}^{k} 8\pi G(x,\xi_j) + O(\lambda)$$

uniformly on compact subsets of $\bar{B}_{2r_0}(\xi_l)\setminus\{\xi_l\}$ in C^1-sense we obtain the following expansion

$$\int_{B_{2r_0}(\xi_l)} \left[\Delta_g v_\xi + \lambda v_\xi e^{\lambda v_\xi^2}\right] \nabla_g v_\xi \, dv_g = 8\pi \nabla_{\xi_l}\psi(\xi,m) + O(\lambda|\log\lambda|).$$

This finishes the proof. □

Therefore, taking into account the expansions (4.1), (4.9), (4.12) and (4.15) we conclude the proof of Proposition 3.4.

Appendix A: The linear theory

In this section, we will study the linearized operator under suitable orthogonality conditions. Throughout the main part of this section we only assume that the numbers μ_j, $j = 1,\ldots,k$ satisfy $C_0^{-1} \le \mu_j \le C_0$ for all $j = 1,\ldots,k$ independently of λ and that the points $\xi_j \in S$, $j = 1,\ldots,k$ are uniformly separated from each other, namely, $\xi = (\xi_1,\ldots,\xi_k) \in \Xi$.

For $\mu,\varepsilon,m > 0$ define the function

$$\begin{aligned}\rho_{\mu,\varepsilon,m}(y) = &\chi\Big(\frac{r_0|y|}{\delta\varepsilon|\log\varepsilon|^2}\Big)\Big(1 + \big|w_\mu\big(\frac{y}{\varepsilon}\big)\big| + \big|w_\mu\big(\frac{y}{\varepsilon}\big)\big|^2\Big)\varepsilon^{-2}e^{w_\mu\left(\frac{y}{\varepsilon}\right)}\\ &+ \Big[1 - \chi\Big(\frac{2r_0|y|}{\delta\varepsilon|\log\varepsilon|^2}\Big)\Big]\Big[\big\{1 + |\log|y||\big\}e^{\lambda m^2 w_\mu\left(\frac{y}{\varepsilon}\right)} + \lambda^{-1}\Big]\varepsilon^{-2}e^{w_\mu\left(\frac{y}{\varepsilon}\right)},\end{aligned} \tag{A.1}$$

with $y \in \mathbb{R}^2$, so that, $\rho_j(x) = \rho_{\mu_j,\varepsilon_j,m_j}(y_{\xi_j}(x))$ for $x \in B_{2r_0}(\xi_j)$, with ρ_j defined in (2.12). First, we will prove the following result.

Lemma A.1. *There exist positive constants $C, \varepsilon_0 > 0$ with $C = C(\delta, r)$ such that for all $0 < \varepsilon < \varepsilon_0$ the solution ψ to problem*

$$-\Delta\psi = \rho_{\mu,\varepsilon,m}, \quad \delta\varepsilon < |y| < r,$$

$$\psi(y) = 0 \text{ for } |y| = \delta\varepsilon, \quad |y| = r$$

satisfies the estimate $\|\psi\|_\infty \le C$. Furthermore, $C(\delta,r)$ could be smaller if we choose δ large and r small.

To be more precise, we will need to take δ large and r small enough so that $2C(\delta,r) + r^2 < 1$.

Proof: Since $\rho_{\mu,\varepsilon,m}$ is radial, $\psi(y) = \psi(|y|)$. If $\varphi(t) = \psi(e^t)$ then for $t \in [\log(\delta\varepsilon), \log r]$ we study

$$-\varphi''(t) = e^{2t}\rho_{\mu,\varepsilon,m}(e^t) := g_0(t), \qquad \varphi(\log(\delta\varepsilon)) = 0, \qquad \varphi(\log r) = 0.$$

Direct computations shows that

$$\varphi(t) = \frac{t - \log(\delta\varepsilon)}{\log\frac{r}{\delta\varepsilon}} \int_{\log(\delta\varepsilon)}^{\log r}\int_{\log(\delta\varepsilon)}^{s} g_0(\tau)\,d\tau\,ds - \int_{\log(\delta\varepsilon)}^{t}\int_{\log(\delta\varepsilon)}^{s} g(\tau)\,d\tau\,ds.$$

Notice that φ is concave and its maximum t_0 satisfies

$$\varphi'(t_0) = \frac{1}{\log\frac{r}{\delta\varepsilon}} \int_{\log(\delta\varepsilon)}^{\log r}\int_{\log(\delta\varepsilon)}^{s} g_0(\tau)\,d\tau\,ds - \int_{\log(\delta\varepsilon)}^{t_0} g(\tau)\,d\tau = 0$$

and hence,

$$\varphi(t_0) = \int_{\log(\delta\varepsilon)}^{t_0} g_0(s)(s - \log(\delta\varepsilon))\, ds = \int_{t_0}^{\log r} g(s)(\log r - s)\, ds.$$

Thus, we deduce that for all $t \in [\log(\delta\varepsilon), \log r]$

$$0 \le \varphi(t) \le \varphi(t_0) \le \max\left\{ \int_{\log(\delta\varepsilon)}^{\log(\delta\sqrt{\varepsilon})} g_0(s)(s - \log(\delta\varepsilon))\, ds, \int_{\log(\delta\sqrt{\varepsilon})}^{\log r} g(s)(\log r - s)\, ds \right\}.$$

We estimate every integral in the following way

$$\begin{aligned}
\int_{\log(\delta\varepsilon)}^{\log(\delta\sqrt{\varepsilon})} g_0(s)(s - \log(\delta\varepsilon))\, ds &\le C\Big[\int_{\log(\delta\varepsilon)}^{\log(2\delta\varepsilon|\log\varepsilon|^2)} \varepsilon^2 e^{-2s}(1 + |\log\varepsilon - s| + |\log\varepsilon - s|^2)(s - \log(\delta\varepsilon))\, ds \\
&\qquad + \int_{\log(\frac{\delta}{2}\varepsilon|\log\varepsilon|^2)}^{\log(\delta\sqrt{\varepsilon})} [\varepsilon\{1+s\}e^{-s} + \lambda^{-1}\varepsilon^2 e^{-2s}](s - \log(\delta\varepsilon))\, ds \Big] \\
&\le C\Big[\int_{\log\delta}^{\log(2\delta|\log\varepsilon|^2)} e^{-2t}(1 + |t| + |t|^2)(t - \log\delta)\, dt \\
&\qquad + \int_{\log(\frac{\delta}{2}|\log\varepsilon|^2)}^{\log(\frac{\delta}{\sqrt{\varepsilon}})} (\{1 + t + \log\varepsilon\}e^{-t} + \lambda^{-1}e^{-2t})(t - \log\delta)\, dt \Big] \\
&\le C_1(\delta)[C_2(\varepsilon) + 1] \qquad \text{(with } t = s - \log\varepsilon)
\end{aligned}$$

with $C_1(\delta), C_2(\varepsilon) \to 0$ as $\delta \to +\infty$ and $\varepsilon \to 0$, in view of (A.1),

$$\left| w_\mu\left(\frac{e^s}{\varepsilon}\right) \right| = |\log(8\mu^2) - 2\log(1 + \mu^2\varepsilon^2 e^{-2s}) + 4(\log\varepsilon - s)| = O(1 + |\log\varepsilon - s|),$$

for $s \in [\log(\delta\varepsilon), \log(2\delta\varepsilon|\log\varepsilon|^2)]$, $\varepsilon^{-2}e^{w_\mu(\frac{e^s}{\varepsilon})} = O(\varepsilon^2 e^{-4s})$, for $s \in [\log(\delta\varepsilon), \log(\delta\sqrt{\varepsilon})]$ and $e^{w_\mu(\frac{e^s}{\varepsilon}) + \lambda m^2 w_\mu^2(\frac{e^s}{\varepsilon})} = O(e^{\frac{3}{4}w_\mu(\frac{e^s}{\varepsilon})}) = O(\varepsilon e^{-3s})$, for $s \in [\log(\delta\varepsilon|\log\varepsilon|^2), \log(2\delta\sqrt{\varepsilon})]$. And for the second integral

$$\begin{aligned}
\int_{\log(\delta\sqrt{\varepsilon})}^{\log r} g_0(s)(\log r - s)\, ds &\le C\int_{\log(\delta\sqrt{\varepsilon})}^{\log r} \left[(1+|s|)e^{2s + 16\lambda m^2 s^2} + \lambda^{-1}\varepsilon^2 e^{-2s}\right][\log r - s]\, ds \\
&\le C\int_{\log(\delta\sqrt{\varepsilon})}^{\log r} \left[e^s(1 + |s| + |s|^2) + \lambda^{-1}\varepsilon^2 e^{-2s}(\log r - s)\right] ds \\
&\le C_3(r)[1 + C_2(\varepsilon)] + C_1(\delta)C_2(\varepsilon)[1 + |\log r|]
\end{aligned}$$

with $C_3(r) \to 0$ as $r \to 0$, in view of (A.1), $\varepsilon^{-2}e^{w_\mu(\frac{e^s}{\varepsilon})} = O(\varepsilon^2 e^{-4s})$, for $s \in [\log(\delta\sqrt{\varepsilon}), \log r]$ and $\varepsilon^{-2}e^{w_\mu(\frac{e^s}{\varepsilon}) + \lambda m^2 w_\mu^2(\frac{e^s}{\varepsilon})} = O(e^{16\lambda m^2 s^2})$, and $e^{2s + 16\lambda m^2 s^2} = O(e^s)$, for $s \in [\log(\delta\sqrt{\varepsilon}), \log r]$. □

We are now ready for

Proof (of Proposition 3.1): The proof of estimate (3.4) consists of several steps. Let assume the opposite, namely, the existence of sequences $\lambda_n \to 0$, points $\xi^n = (\xi_1^n, \dots, \xi_k^n) \in \Xi$, numbers m_j^n with $m^n = (m_1^n, \dots, m_k^n) \in \mathcal{M}$, μ_j^n, ε_j^n and c_{ij}^n, functions h_n with $\|h_n\|_* \to 0$ as $n \to +\infty$, ϕ_n with $\|\phi_n\|_\infty = 1$, and

$$\begin{cases} L(\phi_n) = h_n + \sum_{i=0}^2 \sum_{j=1}^k c_{ij}^n \Delta_g PZ_{ij}^n, & \text{in } S, \\ \int_S \phi_n \Delta_g PZ_{ij}^n\, dv_g = 0, \text{ for all} & i = 0,1,2, j = 1, \dots, m, \quad \int_S \phi_n\, dv_g = 0, \end{cases} \tag{A.2}$$

Without loss of generality, we assume that $\xi_j^n \to \xi_j^*$, $m_j^n \to m_j^*$, $\mu_j^n \to \mu_j^*$ as $n \to +\infty$ and $\xi^* = (\xi_1^*, \dots, \xi_k^*) \in \Xi$ $m^* = (m_1^*, \dots, m_k^*) \in \mathcal{M}$. First, we have the following fact.

Claim A.2. *There exists a constant $C > 0$ independent of n such that for all $i = 0, 1, 2$, $j = 1, \dots, k$ it holds $|c_{ij}^n| \le C\,[\|h_n\|_* + \|\phi_n\|_\infty]$ as $n \to +\infty$.*

Proof: For notational purpose we omit the index n. To estimate the values of the c_{ij}'s, test equation (A.2) against PZ_{ij}, $i = 0, 1, 2$ and $j = 1, \dots, m$:

$$\int_S \phi(\Delta_g PZ_{ij} + KPZ_{ij})\,dv_g = \int_S hPZ_{ij}\,dv_g + \sum_{p=0}^{2}\sum_{q=1}^{k} c_{pq}\int_S \Delta_g PZ_{pq}PZ_{ij}\,dv_g.$$

Since for $j = 1, \dots, k$ we have the following estimates in $C(S)$

$$PZ_{ij} = \chi_j Z_{ij} + O(\varepsilon_j)\,, \quad i = 1, 2\,, \qquad PZ_{0j} = \chi_j(Z_{0j} + 2) + O(\varepsilon_j^2|\log\varepsilon_j|), \tag{A.3}$$

it readily follows that

$$\int_S \Delta_g PZ_{pq}PZ_{ij}dv_g = -\frac{32\pi}{3}\delta_{pi}\delta_{qj} + O(\varepsilon_j), \tag{A.4}$$

where the δ_{ij}'s are the Kronecker's symbols. Furthermore, we find that

$$\begin{aligned}\int_S \phi\tilde{L}(PZ_{0j})\,dv_g &= \int_S \phi\Big(\chi_j\Delta_g Z_{0j} + \sum_{l=1}^{k}\chi_l e^{-\varphi_l}\varepsilon_l^{-2}e^{w_l}\left[\chi_j(Z_{0j}+2) + O(\varepsilon_j^2|\log\varepsilon_j|)\right]\Big)\,dv_g\\ &= \int_S \phi\chi_j e^{-\varphi_j}\varepsilon_j^{-2}e^{w_j}\Big(-Z_{0j} + \chi_j Z_{0j} + 2\chi_j\Big)\,dv_g + O(\varepsilon_j^2|\log\varepsilon_j|\|\phi\|_\infty)\\ &= 2\int_S \phi\chi_j^2 e^{-\varphi_j}\varepsilon_j^{-2}e^{w_j}\,dv_g + O(\varepsilon_j^2|\log\varepsilon_j|\,\|\phi\|_\infty)\end{aligned}$$

and similarly for $i = 1, 2$

$$\begin{aligned}\int_S \phi\tilde{L}(PZ_{ij})\,dv_g &= \int_S \phi\Big(\chi_j\Delta_g Z_{ij} + \sum_{l=1}^{k}\chi_l e^{-\varphi_l}\varepsilon_l^{-2}e^{w_l}\left[\chi_j Z_{ij} + O(\varepsilon_j)\right]\Big)\,dv_g\\ &= \int_S \phi\chi_j e^{-\varphi_j}\varepsilon_j^{-2}e^{w_j}\big(-Z_{ij} + \chi_j Z_{ij}\big)\,dv_g + O(\varepsilon_j\|\phi\|_\infty)\\ &= O(\varepsilon_j\|\phi\|_\infty)\end{aligned}$$

Hence, we get the estimates $|c_{0j}| \le C\Big[\|h\|_* + \|\phi\|_\infty + \varepsilon_j\sum_{p=0}^{2}\sum_{q=1}^{k}|c_{pq}|\Big]$ and

$$|c_{ij}| \le C\Big[\|h\|_* + \varepsilon_j\|\phi\|_\infty + \varepsilon_j\sum_{p=0}^{2}\sum_{q=1}^{k}|c_{pq}|\Big], \qquad \text{for } i = 1, 2. \tag{A.5}$$

Thus, the claim follows. □

Now, we prove the asymptotic behavior of ϕ_n on compact subsets of $S \setminus \{\xi_1^*, \dots, \xi_m^*\}$.

Claim A.3. *There holds $\phi_n \to 0$ as $n \to +\infty$ in C^1 uniformly over compact subsets of $S \setminus \{\xi_1^*, \dots, \xi_m^*\}$. In particular, given any $2r_0 > r > 0$ we have*

$$\|\phi_n\|_{L^\infty(S\setminus\cup_{j=1}^m B_r(\xi_j^n))} \to 0 \qquad \text{as } n \to +\infty. \tag{A.6}$$

Proof: Note that for any $0 < r < r_0$ it holds that up to a subsequence as $n \to +\infty$

$$c(\phi_n) = -\frac{1}{|S|}\int_S K(x)\phi_n(x)\,dv_g = -\frac{1}{|S|}\sum_{j=1}^{k}\int_{B_r(\xi_j)} e^{-\varphi_j}(\varepsilon_j^n)^{-2}e^{w_j}\phi_n\,dv_g + O([\varepsilon_j^n]^2) = c_0 + o(1).$$

From Claim A.2, it readily follows that $\sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij}^n \Delta_g PZ_{ij}^n = o(1)$ as $n \to +\infty$ in $S \setminus \cup_{j=1}^m B_r(\xi_j^n)$ for a given $r > 0$, in view of $\Delta_g PZ_{ij}^n = O([\varepsilon_j^n]^2) - \frac{1}{|S|}\int_S \chi_j \Delta_g Z_{ij}^n dv_g = O([\varepsilon_j^n]^2)$ in $S \setminus \cup_{j=1}^m B_r(\xi_j^n)$. Thus, we get

$$\Delta_g \phi_n(x) = \sum_{j=1}^{k} O((\varepsilon_j^n)^2) + c_0 + o(1) \qquad \text{uniformly for} \quad x \in S \setminus \cup_{j=1}^m B_r(\xi_j^n).$$

Therefore, passing to a subsequence $\phi_n \to \phi^*$ as $n \to +\infty$ in C^1 sense over compact subsets of $S \setminus \{\xi_1^*, \ldots, \xi_m^*\}$. Since $|\phi^*(x)| \le 1$ for all $x \in S \setminus \{\xi_1^*, \ldots, \xi_m^*\}$, it follows that ϕ^* can be extended continuously to S so that ϕ^* satisfies $\Delta_g \phi^* = c_0$, in S and $\int_S \phi^* \, dv_g = 0$ using dominated convergence. By, $\int_S \Delta_g \phi^* = 0$ we get that $c_0 = 0$. Therefore, $\phi^* \equiv 0$, and the claim follows. □

Claim A.4. *For all $i = 0, 1, 2$, $j = 1, \ldots, k$ it holds that $c_{ij}^n \to 0$ as $n \to +\infty$.*

Proof: The claim readily follows for $i = 1, 2$ and $j = 1, \ldots, k$ from Claim A.2 and the estimate in (A.5) in view of $\|h\|_* = o(1)$. Let us refine the estimate for c_{0j}, $j = 1, \ldots, k$. It is clear that in $B_{r_0}(\xi_j)$

$$\Delta_g \phi + e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} \phi + c(\phi) = h + \sum_{p=0}^{2} c_{pj} \Delta_g Z_{pj} - \sum_{p=0}^{2}\sum_{q=1}^{k} \frac{c_{pq}}{|S|} \int_S \chi_q \Delta_g Z_{pq} \, dv_g.$$

Hence, integrating on $B_r(\xi_j)$ with $0 < r < r_0$ we find that as $n \to +\infty$

$$\begin{aligned} \int_{B_r(\xi_j)} e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} \phi \, dv_g &= -\int_{B_r(\xi_j)} \Delta_g \phi \, dv_g - c(\phi)|B_r(\xi_j)| + \int_{B_r(\xi_j)} h \, dv_g \\ &\quad + \sum_{p=0}^{2} c_{pj} \int_{B_r(\xi_j)} \Delta_g Z_{pj} \, dv_g - \sum_{p=0}^{2}\sum_{q=1}^{k} \frac{c_{pq}}{|S|} |B_r(\xi_j)| \int_S \chi_p \Delta_g Z_{pq} \, dv_g \\ &= -\int_{\partial B_r(0)} \partial_\nu (\phi \circ y_{\xi_j}^{-1}) \, d\sigma + O\bigg(|c(\phi)| + \|h\|_* + \sum_{q=1}^{k} \varepsilon_q^2 \bigg) = o(1), \end{aligned}$$

in view of Claims A.2 and A.3 (the convergence $\phi \to 0$ in C^1 sense). Thus, we obtain that

$$c_{0j} = O\bigg(\bigg| \int_{B_r(\xi_j)} e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} \phi \, dv_g \bigg| + \varepsilon_j^2 |\log \varepsilon_j| \, \|\phi\|_\infty + \|h\|_* + \varepsilon_j \sum_{p=0}^{2}\sum_{q=1}^{k} |c_{pq}| \bigg) = o(1)$$

and the claim follows. □

We follow ideas shows in [14] to prove an estimate for ϕ_n. For $y \in B_{r_0}(0)$, define $\hat\phi_{n,j}(y) = \phi_n(y_{\xi_j}^{-1}(y))$ and

$$\hat h_n(y) = e^{\hat\varphi_j(y)} \Big[-c(\phi_n) + h_n(y_{\xi_j^n}^{-1}(y)) + \sum_{p=0}^{2}\sum_{q=1}^{k} c_{pq} \Delta_g PZ_{pq}^n (y_{\xi_j}^{-1}(y)) \Big],$$

so that,

$$\hat L_j(\hat\phi_{n,j}) := \Delta \hat\phi_{n,j} + \frac{8\mu_j^2 \varepsilon_j^2}{(\mu_j^2 \varepsilon_j^2 + |y|^2)^2} \hat\phi_{n,j} = \hat h_n.$$

Let us fix such a number $\delta > 0$ which we may take larger whenever it is needed and a small $0 < r < r_0$. Now, let us consider the "annulus norm" and "boundary annulus norm"

$$\|\phi\|_a = \|\phi\|_{L^\infty(\Omega_{\delta,r})} \qquad \text{and} \qquad \|\phi\|_b = \|\phi\|_{L^\infty(\partial\Omega_{\delta,r})},$$

where $\Omega_{\delta,r} := B_r(0) \setminus \bar{B}_{\delta\varepsilon_j}(0)$. Note that $\partial\Omega_{\delta,r} = \cup_{j=1}^m [\partial B_{\delta\varepsilon_j}(0) \cup \partial B_r(0)]$. By now it is rather standard to show that for functions on $\Omega_{\delta,r}$ the operator $\hat{L}_j$ satisfies the maximum principle in $\Omega_{\delta,r}$ for δ large and $r > 0$ small enough, see for example [14]. In fact, the function $g(y) := Y_{0j}(a\varepsilon_j^{-1}y) = 2\frac{a^2|y|^2-\mu_j^2\varepsilon_j^2}{\mu_j^2\varepsilon_j^2+a^2|y|^2}$ with $0 < 4a < 1$ and $\sqrt{\frac{5}{3}}a < \delta$ satisfies $\hat{L}_j(g) < 0$ and $g > \frac{1}{2} > 0$ in $\Omega_{\delta,r}$. As a consequence, we get that

Claim A.5. *There is a constant $C > 0$ such that if $\hat{L}_j(\phi) = h$ in $\Omega_{\delta,r}$ then*

$$\|\phi\|_a \leq C[\|\phi\|_b + \|h\|_{**}], \tag{A.7}$$

where

$$\|h\|_{**} = \sup_{y\in B_r(0)} \frac{|h(y)|}{\rho_{\mu,\varepsilon,m}(y)+1}.$$

Proof: We shall omit the subscript n in the quantities involved. We will establish this inequality with the use of suitable barriers. Consider now the solution of the problem

$$-\Delta\psi = 1, \qquad \text{for } \delta\varepsilon_j < |y| < r,$$

$$\psi(y) = 0 \quad \text{for } |y| = \delta\varepsilon_j,\ |y| = r.$$

Direct computation shows that

$$\psi_{j,1}(y) = \frac{\delta^2\varepsilon_j^2 - r^2}{4} + \frac{\delta^2\varepsilon_j^2 - r^2}{4}\frac{\log\frac{|y|}{\delta\varepsilon_j}}{\log\frac{\delta\varepsilon_j}{r}},$$

Note that $0 \leq \psi_{j,1} \leq \dfrac{r^2-\delta^2\varepsilon_j^2}{4} \leq \dfrac{r^2}{4}$ hence these functions $\psi_{j,1}$ have a uniform bound independent of ε_j.

On the other hand, let us consider the function g defined above, and let us set

$$\psi(y) = 2\|\phi\|_b\, g(y) + \|h\|_{**}[\psi_{j,1}(y) + \psi_{j,2}(y)],$$

where $\psi_{j,2}$ is the solution to

$$-\Delta\psi = 2\rho_{\mu_j,\varepsilon_j,m_j}, \quad \delta\varepsilon_j < |y| < r,$$

$$\psi(y) = 0 \text{ for } |y| = \delta\varepsilon, \quad |y| = r.$$

Then, it is easily checked that, choosing δ larger if necessary, $\hat{L}_j(\psi) \leq h$ and $\psi \geq |\phi|$ on $\partial\Omega_{\delta,r}$. Hence $|\phi| \leq \psi$ in $\Omega_{\delta,r}$. In fact, we have that for all $y \in \partial\Omega_{\delta,r}$

$$\psi(y) \geq 2\|\phi\|_b\, g(y) \geq \|\phi\|_b \geq |\phi(y)|.$$

Also, we have that choosing $2C(\delta,r) + r^2 < 1$ (for δ large enough and r small enough)

$$\begin{aligned}
\hat{L}_j(\psi) &< \|h\|_{**}\left(\Delta[\psi_{j,1}+\psi_{j,2}] + \frac{8\mu_j^2\varepsilon_j^2}{(\mu_j^2\varepsilon_j^2+|y|^2)^2}[\psi_{j,1}+\psi_{j,2}]\right)\\
&\leq \|h\|_{**}\left(-1 - 2\rho_{\mu_j,\varepsilon_j,m_j}(y) + \frac{8\mu_j^2\varepsilon_j^2}{(\mu_j^2\varepsilon_j^2+|y|^2)^2}[2C(\delta,r)+r^2]\right)\\
&\leq -\|h\|_{**}[\rho_{\mu_j,\varepsilon_j,m_j}+1] \leq h,
\end{aligned}$$

in view of $\rho_{\mu_j,\varepsilon_j,m_j}(y) \geq \frac{8\mu_j^2\varepsilon_j^2}{(\mu_j^2\varepsilon_j^2+|y|^2)^2}$ Hence, we conclude that $|\phi(y)| \leq \psi(y)$ for all $\delta\varepsilon_j < |y| < r$, for every $j = 1, \ldots, k$ and the claim follows. □

The following intermediate result provides another estimate. Again, for notational simplicity *we omit* the subscript n in the quantities involved.

Claim A.6. *There exist constants $C > 0$ such that for large n*

$$\|\phi\|_{L^\infty(\cup_{j=1}^m B_{r_0}(\xi_j))} \le C\left\{\|\phi\|_{L^\infty(\cup_{j=1}^m B_{\delta\varepsilon_j}(\xi_j))} + o(1)\right\}. \tag{A.8}$$

Proof: First, note that

$$\|\hat{h}\|_{**} \le C\left[|c(\phi)| + \|h\|_* + \sum_{p=0}^{2}\sum_{q=1}^{k}|c_{pq}|\right]$$

in view of the definition of $\|\cdot\|_*$ and $\|\cdot\|_{**}$ and $\|\Delta_g PZ_{pq}\|_* \le C$. From estimate (A.7) we deduce that there is a constant $C > 0$ such that

$$\begin{aligned}\|\phi\|_{L^\infty(B_r(\xi_j)\setminus\bar{B}_{\delta\varepsilon_j}(\xi_j))} = \|\hat{\phi}\|_a &\le C\left[\|\hat{\phi}\|_b + \|\hat{h}\|_{**}\right]\\ &\le C\left[\|\phi\|_{L^\infty(\partial[B_r(\xi_j)\setminus\bar{B}_{\delta\varepsilon_j}(\xi_j)])} + |c(\phi)| + \|h\|_* + \sum_{p=0}^{2}\sum_{q=1}^{k}|c_{pq}|\right]\end{aligned} \tag{A.9}$$

From (A.6) we find that for large n

$$\|\phi\|_{L^\infty(S\setminus\cup_{l=1}^m B_r(\xi_j))} = o(1). \tag{A.10}$$

Furthermore, we have that $c(\phi) = o(1)$, since $c_0 = 0$. By the assumption, we know that $\|h\|_* = o(1)$. Now, from (A.9) it is clear that

$$\|\phi\|_{L^\infty(\cup_{j=1}^m B_r(\xi_j))} \le \|\phi\|_{L^\infty(\cup_{j=1}^m B_{\delta\varepsilon_j}(\xi_j))} + C\left[\|\hat{\phi}\|_b + \|\hat{h}\|_*\right]$$

and the conclusion follows by (A.10). □

We continue with the proof of Proposition 3.1 and we get the following fact.

Claim A.7. *There exists an index $j \in \{1, \dots, k\}$ such that passing to a subsequence if necessary,*

$$\liminf_{n\to\infty}\|\phi_n\|_{L^\infty(B_{\delta\varepsilon_j}(\xi_j^n))} \ge \kappa > 0. \tag{A.11}$$

Proof: Arguing by contradiction, if for all $j = 1, \dots, k$

$$\liminf_{n\to\infty}\|\phi_n\|_{L^\infty(B_{\delta\varepsilon_j}(\xi_j^n))} = 0,$$

then (A.8) and (A.10) implies that, passing to a subsequence if necessary, $\|\phi_n\|_\infty \to 0$ as $n \to +\infty$. On the other hand, we know that $\|\phi\|_\infty = 1$ for all $n \in \mathbb{N}$. This conclude (A.11). □

Let us set $\psi_{n,j}(z) = \phi_n\big(y_{\xi_j^n}^{-1}(\varepsilon_j^n z)\big) = \hat{\phi}_{n,j}(\varepsilon_j^n z)$ for any j and $z \in B_{r_0/\varepsilon_j^n}(0)$. We notice that $\psi_{n,j}$ satisfies

$$\Delta\psi_{n,j} + \frac{8[\mu_j^n]^2}{([\mu_j^n]^2 + |z|^2)^2}\,\psi_{n,j} = [\varepsilon_j^n]^2\hat{h}_{n,j}(\varepsilon_j^n z) \quad \text{in } B_{r_0/\varepsilon_j^n}(0).$$

Elliptic estimates and (A.11) readily imply that $\psi_{n,j}$ converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded, non-zero solution ψ_j^* of

$$\Delta\psi + \frac{8\mu^2}{(\mu_j^2 + |z|^2)^2}\psi = 0, \qquad \mu_j = \mu_j^*$$

in view of $\left|[\varepsilon_j^n]^2 \hat{h}_{n,j}(\varepsilon_j^n z)\right| \le C\|h_n\|_*$ for z in compact subsets of $\mathbb{R}^2$. This implies that ψ_j^* is a linear combination of the functions Y_{ij}, $i = 0, 1, 2$. Thus, we have that for some constants a_{ij}, $i = 0, 1, 2$, $\psi_j^* = a_{0j}Y_{0j} + a_{1j}Y_{1j} + a_{2j}Y_{2j}$. See [5] for a proof. But, from (A.2), orthogonality conditions over $\psi_{n,j}$ pass to the limit thanks to $\|\psi_{n,j}\|_\infty \le C$ and dominated convergence, namely,

$$\int_{\mathbb{R}^2} \Delta Y_{ij}\, \psi_j^* = 0, \qquad \text{for} \quad i = 0, 1, 2.$$

A contradiction with (A.11) arises since this implies that $a_{0j} = a_{1j} = a_{2j} = 0$. Thus, we get the estimate $\|\phi\|_\infty \le C\|h\|_*$. Hence, from the same argument shown in the proof of the Claim A.2, we conclude the estimates (3.4).

Now, let us prove the solvability assertion. To this purpose we consider the space

$$H = \left\{ \phi \in H_0^1(S) \ : \ \int_S \Delta_g PZ_{ij}\, \phi \, = \, 0 \quad \text{for} \ \ i = 0, 1, 2, \ j = 1, \ldots, k \right\},$$

endowed with the usual inner product $[\phi, \psi] = \displaystyle\int_S \langle \nabla\phi, \nabla\psi\rangle_g \, dv_g$. Problem (3.3) expressed in weak form is equivalent to that of finding a $\phi \in H$, such that

$$[\phi, \psi] = \int_S [K\phi - h]\, \psi \; dv_g, \qquad \text{for all} \ \ \psi \in H.$$

Recall that $\displaystyle\int_S h\, dv_g = 0$. With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operator form $\phi = \mathcal{K}(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where $\mathcal{K}$ is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = \mathcal{K}(\phi)$ has only the zero solution in H. This last equation is equivalent to (3.3) with $h \equiv 0$. Thus existence of a unique solution follows from the a priori estimate (3.4).

We have just proven that the unique solution $\phi = T_\lambda(h)$ of (3.3) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in $L^\infty(S)$ for which $\|h\|_* < +\infty$ and $\displaystyle\int_S h = 0$, into L^∞, with bounded norm.

It is important to understand the differentiability of the operator T with respect to the variable either $\beta = \xi_j$ or $\beta = m_j$. Fix $h \in \mathcal{C}_*$ and let $\phi = T_\lambda(h)$. Let us recall that ϕ satisfies (3.3), for some (uniquely determined) constants c_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, k$. We want to compute derivatives of ϕ with respect to the parameters $\beta = \xi_l$ or $\beta = m_l$. Formally $X = \partial_\beta \phi$ should satisfy

$$L(X) = -\partial_\beta K\, \phi + \frac{1}{|S|} \int_S \partial_\beta K \phi + \sum_{i=0}^{2} \sum_{j=1}^{k} c_{ij}\, \partial_\beta(\Delta_g PZ_{ij}) + \sum_{i=0}^{2} \sum_{j=1}^{k} d_{ij} \Delta_g PZ_{ij}\,,$$

where (still formally) $d_{ij} = \partial_\beta(c_{ij})$, $i = 0, 1, 2$, $j = 1, \ldots, k$. The orthogonality conditions now become

$$\int_S \Delta_g PZ_{ij} X = -\int_S \partial_\beta(\Delta_g PZ_{ij})\phi, \qquad i = 0, 1, 2, j = 1, \ldots, k.$$

We will recast X as follows. Consider the $Y = X + \sum_{i,j} b_{ij} PZ_{ij}$, where the coefficients b_{ij} are chosen so that Y satisfies the orthogonality conditions $\displaystyle\int_S Y \Delta_g PZ_{ij} = 0$ for all i, j. The coefficients b_{ij} are well-defined since they satisfy an almost diagonal system in view of (A.4).

Furthermore, it holds that for $\beta = \xi_l$, $|b_{ij}| \leq C\|h\|_*$ if $j \neq l$ and $|b_{il}| \leq \dfrac{C}{\varepsilon_l}\|h\|_*$; and for $\beta = m_l$, $|b_{ij}| \leq C\|h\|_*$ if $j \neq l$ and $|b_{il}| \leq C|\log \varepsilon_l|\|h\|_*$, in view of $\|\partial_{\xi_l}(\Delta_g PZ_{ij})\|_* \leq C$ if $j \neq l$ and $\|\partial_{\xi_l}(\Delta_g PZ_{il})\|_* \leq \dfrac{C}{\varepsilon_l}$; and $\|\partial_{m_l}(\Delta_g PZ_{ij})\|_* \leq C$ if $j \neq l$ and $\|\partial_{m_l}(\Delta_g PZ_{il})\|_* \leq C|\log \varepsilon_l|$. Then the function X above can be uniquely expressed as

$$X = T_\lambda(f) - \sum_{i=0}^{2} \sum_{j=1}^{k} b_{ij}\, PZ_{ij}.$$

where the function

$$f := -\partial_\beta K\, \phi + \frac{1}{|S|}\int_S \partial_\beta K \phi + \sum_{i=0}^{2}\sum_{j=1}^{k} \Big[b_{ij}\, L(PZ_{ij}) + c_{ij}\, \partial_\beta (\Delta_g PZ_{ij}) \Big].$$

This computation is not just formal. Arguing directly by definition it shows that indeed $\partial_\beta \phi = X$. Also, we find that $\|f\|_* \leq \dfrac{C}{\varepsilon_l}\|h\|_*$, for $\beta = \xi_l$ and $\|f\|_* \leq C|\log \varepsilon_l|\|h\|_*$, for $\beta = m_l$, in view of

$$\|f\|_* \leq \|\partial_\beta K\|_*\, \|\phi\|_\infty \left[1 + \frac{1}{|S|}\int_S \rho(x)\, dv_g \right] + \sum_{i=0}^{2}\sum_{j=1}^{k} \Big[|b_{ij}|\, \|L(PZ_{ij})\|_* + |c_{ij}|\, \|\partial_\beta(\Delta_g PZ_{ij})\|_* \Big].$$

Indeed, it is easy to check that $\displaystyle\int_S \rho(x)\, dv_g \leq C$, $\|L(PZ_{ij})\|_* \leq C\varepsilon_j$ for $i = 1, 2$ and $\|L(PZ_{0j})\|_* \leq C$. Furthermore, from the definition of K it follows that $\|\partial_{\xi_l} K\|_* \leq \dfrac{C}{\varepsilon_l}$ and $\|\partial_{m_l} K\|_* \leq C|\log \varepsilon_l|$. Moreover, using estimate (3.4) applied with R.H.S. f, we find that

$$\|\partial_{\xi_l}\phi\|_\infty \leq C\left[\|f\|_* + \sum_{i=0}^{2}\sum_{j=1}^{k} |b_{ij}| \right],$$

so that, $\|\partial_\beta \phi\|_* \leq \dfrac{C}{\varepsilon_l}\|h\|_*$ and $\|\partial_{m_l}\phi\|_\infty \leq C|\log \varepsilon_l|\|h\|_*$. Finally, we conclude that

$$\|\partial_{(\xi_l)_i} T_\lambda(h)\|_\infty \leq \frac{C}{\varepsilon_l}\, \|h\|_* \quad \text{for} \;\; i = 1, 2, l = 1, \dots, k \tag{A.12}$$

and

$$\|\partial_{m_l} T_\lambda(h)\|_\infty \leq C\, |\log \varepsilon_l|\, \|h\|_* \quad \text{for} \;\; l = 1, \dots, k. \tag{A.13}$$

This finishes the proof of proposition 3.1. □

Appendix B : The nonlinear problem

By Proposition 3.1 we now deduce the following.

Proof (of Proposition 3.2). First, note that $R \in L^\infty(S)$, $\|R\|_* < +\infty$, $\displaystyle\int_S R = 0$ and $\displaystyle\int_S N(\phi) = 0$ for any $\phi \in C(S)$. Next, we observe that in terms of the operator T defined in Proposition 3.1, the latter problem becomes

$$\phi = -T\,(R + N(\phi)) := \mathcal{A}(\phi). \tag{B.1}$$

For a given number $\nu > 0$, let us consider

$$\mathcal{F}_\nu = \{\phi \in C(S) : \|\phi\|_\infty \le \nu\lambda\}$$

From the Proposition 3.1, we get

$$\|\mathcal{A}(\phi)\|_\infty \le C\left[\|R\|_* + \|N(\phi)\|_*\right].$$

From (2.13) we know that $\|R\|_* \le C\lambda$. Furthermore, it follows that for certain $0 < s < s^* < 1$

$$f(V+\phi)-f(V)-f'(V)\phi = \int_0^1 [f'(V+t\phi)-f'(V)]\,dt\,\phi = [f'(V+s^*\phi)-f'(V)]\phi = f''(V+s\phi)\,s^*\phi^2.$$

From the definition of f in (2.23) and the estimates used to prove Lemma 2.2, it follows that

$$f''(V+s\phi) = 2\lambda^2(V+s\phi)e^{\lambda(V+s\phi)^2}\left[3+2\lambda(V+\phi)^2\right] = \lambda V e^{\lambda V^2}O(1) + \lambda^2 e^{\lambda V^2}O(1)$$

so that, from the definition of $\|\cdot\|_*$ we obtain that $\|f''(V+s\phi)\|_* \le C$. Thus, we find that

$$\begin{aligned}\|N(\phi)\|_* &\le \Big[\|f(V+\phi)-f(V)-f'(V)\phi\|_* + \|f'(V)-K\|_*\|\phi\|_\infty\Big]\Big[1+\frac{1}{|S|}\int_S \rho(x)\,dv_g\Big]\\ &\le C\big[\|\phi\|_\infty^2 + \lambda\|\phi\|_\infty\big] \le C\lambda^2[\nu^2+\nu],\end{aligned}$$

in view of (2.26) and $\displaystyle\int_S \rho(x)\,dv_g \le C$. Hence, we get for any $\phi \in \mathcal{F}_\nu$,

$$\|\mathcal{A}(\phi)\|_\infty \le C\lambda\left[1+(\nu+\nu^2)\lambda\right].$$

On the other hand, for ϕ_1 and ϕ_2 and certain $0 < s, t^* < 1$ we have that

$$\begin{aligned}&f(V+\phi_1)-f(V+\phi_2)-f'(V)(\phi_1-\phi_2)\\ &\qquad= \int_0^1 [f'\big(V+\phi_2+t\{\phi_1-\phi_2\}\big)-f'(V)]\,dt\,[\phi_1-\phi_2]\\ &\qquad= \left[f'\big(V+t^*\phi_1+\{1-t^*\}\phi_2\big)-f'(V)\right][\phi_1-\phi_2]\\ &\qquad= f''\big(V+s[t^*\phi_1+\{1-t^*\}\phi_2]\big)\,[t^*\phi_1+\{1-t^*\}\phi_2]\,[\phi_1-\phi_2],\end{aligned}$$

so that,

$$\|f(V+\phi_1)-f(V+\phi_2)-f'(V)(\phi_1-\phi_2)\|_* \le C\left[\|\phi_1\|_\infty+\|\phi_2\|_\infty\right]\|\phi_1-\phi_2\|_\infty,$$

in view of $\|f''\big(V+s[t^*\phi_1+\{1-t^*\}\phi_2]\big)\|_* \le C$. Hence, given any $\phi_1, \phi_2 \in \mathcal{F}_\nu$, we have that

$$\begin{aligned}\|N(\phi_1)-N(\phi_2)\|_* &\le C(\|\phi_1\|_\infty+\|\phi_2\|_\infty)\|\phi_1-\phi_2\|_\infty + C\lambda\|\phi_1-\phi_2\|_\infty\\ &\le C\lambda[\nu+1]\,\|\phi_1-\phi_2\|_\infty\end{aligned}$$

with C independent of ν. Therefore, from the Proposition 3.1

$$\|\mathcal{A}(\phi_1)-\mathcal{A}(\phi_2)\|_\infty \le C\|N(\phi_1)-N(\phi_2)\|_* \le C\lambda[\nu+1]\,\|\phi_1-\phi_2\|_\infty$$

It follows that for all ε sufficiently small $\mathcal{A}$ is a contraction mapping of $\mathcal{F}_\nu$ (for ν large enough), and therefore a unique fixed point of $\mathcal{A}$ exists in $\mathcal{F}_\nu$.

Let us now discuss the differentiability of ϕ depending on (ξ, m), i.e., $(\xi, m) \mapsto \phi(\xi, m) \in C(S)$ is C^1. Since R depends continuously (in the $*$-norm) on (ξ, m), using the fixed point characterization (B.1), we deduce that the mapping $(\xi, m) \mapsto \phi$ is also continuous. Then, formally

$$\begin{aligned}\partial_\beta N(\phi) = & [f'(V+\phi) - f'(V) - f''(V)\phi]\partial_\beta V + [f'(V+\phi) - f'(V)]\partial_\beta \phi \\ & - \frac{1}{|S|}\int_S \left\{[f'(V+\phi) - f'(V) - f''(V)\phi]\partial_\beta V + [f'(V+\phi) - f'(V)]\partial_\beta \phi\right\} \\ & + [f''(V)\partial_\beta V - \partial_\beta K]\phi + [f'(V) - K]\partial_\beta \phi \\ & - \frac{1}{|S|}\int_S \big([f''(V)\partial_\beta V - \partial_\beta K]\phi + [f'(V) - K]\partial_\beta \phi\big).\end{aligned}$$

so that, we estimate as follows

$$\begin{aligned}\|\partial_\beta N(\phi)\|_* \le C\Big[& \|f''(V+s'\phi) - f''(V)\|_* \, \|\phi\|_\infty \|\partial_\beta V\|_\infty + \|f''(V+s'\phi)\|_* \|\phi\|_\infty \|\partial_\beta \phi\|_\infty \\ & + \|f''(V)\partial_\beta V - \partial_\beta K]\|_* \|\phi\|_\infty + \|[f'(V) - K]\|_* \|\partial_\beta \phi\|_\infty\Big],\end{aligned}$$

for some $s' \in (0,1)$. In particular, precisely for $\beta = \xi_l$ we obtain

$$\begin{aligned}\|\partial_{\xi_l} N(\phi)\|_* & \le C\Big[\|\phi\|_\infty^2 \|\partial_{\xi_l} V\|_\infty + \|\phi\|_\infty \|\partial_{\xi_l}\phi\|_\infty + \frac{\lambda}{\varepsilon_l}\|\phi\|_\infty + \lambda\|\partial_{\xi_l}\phi\|_\infty\Big] \\ & \le C\Big[\frac{\lambda^2}{\varepsilon_l} + \lambda\|\partial_{\xi_l}\phi\|_\infty\Big],\end{aligned}$$

in view of $\|f''(V+s'\phi) - f''(V)\|_* \le C\|\phi\|_\infty$, $\|\partial_{\xi_l} V\|_\infty \le \dfrac{C}{\varepsilon_l}$, $\|f''(V)\partial_{\xi_l} V - \partial_{\xi_l} K\|_* \le \dfrac{C\lambda}{\varepsilon_l}$ and estimate (2.26), and for $\beta = m_l$ we obtain

$$\begin{aligned}\|\partial_{m_l} N(\phi)\|_* & \le C\Big[\|\phi\|_\infty^2 \|\partial_{m_l} V\|_\infty + \|\phi\|_\infty \|\partial_{m_l}\phi\|_\infty + \lambda|\log\varepsilon_l|\, \|\phi\|_\infty + \lambda\|\partial_{m_l}\phi\|_\infty\Big] \\ & \le C\Big[\lambda^2|\log\varepsilon_l| + \lambda\|\partial_{m_l}\phi\|_\infty\Big],\end{aligned}$$

in view of $\|\partial_{m_l} V\|_\infty \le C|\log\varepsilon_l|$, $\|f''(V)\partial_{m_l} V - \partial_{m_l} K\|_* \le C\lambda|\log\varepsilon_l| \le C$. Also, observe that we have

$$\partial_\beta \phi = -(\partial_\beta T)\,(R + N(\phi)) - T\,(\partial_\beta\,[R + N(\phi)])\,.$$

So, using (A.12) and previous estimates, we get

$$\begin{aligned}\|\partial_{\xi_l}\phi\|_\infty & \le \frac{C}{\varepsilon_l}\,\|R + N(\phi)\|_* + C\,\|\partial_{\xi_l}(R + N(\phi))\|_* \\ & \le \frac{C}{\varepsilon_l}\,\left[\lambda + \lambda\|\phi\|_\infty + \|\phi\|_\infty^2\right] + C\left[\frac{\lambda}{\varepsilon_l} + \frac{\lambda^2}{\varepsilon_l} + \lambda\|\partial_{\xi_l}\phi\|_\infty\right] \\ & \le \frac{C\lambda}{\varepsilon_l} + C\lambda\|\partial_{\xi_l}\phi\|_\infty.\end{aligned}$$

and similarly, using (A.13) and previous estimates, we get

$$\begin{aligned}\|\partial_{m_l}\phi\|_\infty & \le C|\log\varepsilon_l|\,\|R + N(\phi)\|_* + C\,\|\partial_{m_l}(R + N(\phi))\|_* \\ & \le C|\log\varepsilon_l|\,\left[\lambda + \lambda\|\phi\|_\infty + \|\phi\|_\infty^2\right] + C\left[\lambda|\log\varepsilon_l| + \lambda^2|\log\varepsilon_l| + \lambda\|\partial_{\xi_l}\phi\|_\infty\right] \\ & \le C\lambda|\log\varepsilon_l| + C\lambda\|\partial_{\xi_l}\phi\|_\infty.\end{aligned}$$

We have used an estimate for $\|\partial_\beta R\|_*$. From the definition of V, the definition of $\partial_\beta R$

$$\partial_\beta R(y) = \Delta_g \partial_\beta V(y) + f'(V)\partial_\beta V - \frac{1}{|S|}\int_S f'(V)\partial_\beta V\, dv_g,$$

similar computations to deduce (2.13) and from the definition of *-norm it follows that

$$\|\partial_{\xi_l} R\|_* \le \frac{C\lambda}{\varepsilon_l} \qquad \text{and} \qquad \|\partial_{m_l} R\|_* \le C\lambda |\log \varepsilon_l|.$$

Thus, we conclude (3.8).

The above computations can be made rigorous by using the implicit function theorem and the fixed point representation (B.1) which guarantees C^1 regularity in (ξ, m). □

Appendix C: Proof of Lemma 3.3

Proof: Let us differentiate the function $\mathcal{F}_\lambda(\xi, m)$ with respect to either $\beta = (\xi_l)_q$ or $\beta = m_l$, with $q = 1, 2$ and $l = 1, \dots, k$. Since $\sqrt{\lambda}\, V(\xi, m) = U(\xi, m)$ and $\sqrt{\lambda}\, \phi(\xi, m) = \tilde{\phi}(\xi, m)$, we can differentiate directly $J_\lambda\big(\sqrt{\lambda}[V + \phi]\big)$ (under the integral sign), so that integrating by parts we get

$$\begin{aligned} \partial_\beta \mathcal{F}_\lambda(\xi, m) &= \sqrt{\lambda}\, DJ_\lambda \left(\sqrt{\lambda}[V + \phi]\right) [\partial_\beta V + \partial_\beta \phi] \\ &= -\lambda \int_S \left[\Delta_g (V + \phi) + \lambda (V + \phi) e^{\lambda (V+\phi)^2}\right] [\partial_\beta V + \partial_\beta \phi]\, dv_g \\ &= -\lambda \sum_{i=0}^{2} \sum_{j=1}^{k} c_{ij} \int_S \Delta_g P Z_{ij}\, [\partial_\beta V + \partial_\beta \phi], \end{aligned}$$

since $\displaystyle\int_S \big(\partial_\beta V + \partial_\beta \phi\big) = 0$. From the result of 3.2, this expression defines a continuous function of (ξ, m). Let us stress that $\mu_j = \mu_j(\xi, m)$ and $\varepsilon_j = \varepsilon_j(m_j)$. Hence, from (4.7) we have that uniformly in S

$$\partial_{(\xi_l)_q} V(x) = -\frac{m_l}{\mu_l \varepsilon_l} \chi_l \frac{2\mu_l \varepsilon_l \partial_{(\xi_l)_q}(|y_{\xi_l}(x)|^2)}{\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2} + O(1)$$

and from (4.6) and (4.8)

$$\partial_{m_l} V(x) = -2\chi_l \log(\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2) + \chi_l \frac{8\mu_l^2 \varepsilon_l^2 \log \varepsilon_l}{\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2} + O(1),$$

in view of $\varepsilon_l \partial_{m_l} \varepsilon_l = -\dfrac{2\varepsilon_l^2}{m_l} \log \varepsilon_l + O(1)$. Let us assume that $D_\xi \mathcal{F}_\lambda(\xi, m) = 0$ and $D_m \mathcal{F}_\lambda(\xi, m) = 0$. Then, from the latter equality and the estimates (3.8) we get

$$\sum_{i=0}^{2} \sum_{j=1}^{k} c_{ij} \int_S \Delta_g P Z_{ij}\, \left[\varepsilon_l \partial_{(\xi_l)q} V + O(\lambda)\right] = 0, \qquad q = 0, 1, 2,\ l = 1, \dots, k$$

and also,

$$\sum_{i=0}^{2} \sum_{j=1}^{k} c_{ij} \int_S \Delta_g P Z_{ij}\, \left[\frac{\partial_{m_l} V}{\log \varepsilon_l} + O(\lambda)\right] = 0, \qquad l = 1, \dots, k.$$

Using

$$\varepsilon_l \partial_{(\xi_l)_q} V = -\frac{m_l}{\mu_l} \chi_l \frac{2\mu_l \varepsilon_l \partial_{(\xi_l)_q}(|y_{\xi_l}(x)|^2)}{\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2} + O(\varepsilon_l)$$

and

$$\frac{\partial_{m_l} V}{\log \varepsilon_l} = \frac{1}{\log \varepsilon_l} \chi_l [U_l - \log(8\mu_l^2 \varepsilon_l^2)] + 2\chi_l [Z_{0l} + 2] + O\Big(\frac{1}{|\log \varepsilon_l|}\Big),$$

where $O(\varepsilon_l)$ and $O(|\log\varepsilon_l|^{-1})$ are in the L^∞ norm as $\lambda\to 0$, it follows

$$\sum_{i=0}^{2}\sum_{j=1}^{m} c_{ij}\int_S \Delta_g PZ_{ij}\,\left[\chi_l\frac{2\mu_l\varepsilon_l\partial_{(\xi_l)_q}(|y_{\xi_l}(x)|^2)}{\mu_l^2\varepsilon_l^2+|y_{\xi_l}(x)|^2}+o(1)\right]=0,\qquad q=0,1,2,\ l=1,\dots,m.$$

$$\sum_{i=0}^{2}\sum_{j=1}^{m} c_{ij}\int_S \Delta_g PZ_{ij}\,\left[\chi_l(Z_{0l}+2)+o(1)\right]=0,\qquad l=1,\dots,m.$$

with $o(1)$ small in the sense of the L^∞ norm as $\lambda\to 0$. The above system is diagonal dominant and we thus get $c_{ij}=0$ for $i=0,1,2,\ j=1,\dots,k$. We have used that

$$\int_S \Delta_g PZ_{ij}\,\chi_l[U_l-\log(8\mu_l^2\varepsilon_l^2)]=O(1).$$

The proof of Lemma 3.3 is finished. □

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