INFINITE-DIMENSIONAL LUR'E SYSTEMS: INPUT-TO-STATE STABILITY AND CONVERGENCE PROPERTIES*

CHRIS GUIVER†, HARTMUT LOGEMANN†, AND MARK R. OPMEER†

Abstract. We consider forced Lur'e systems in which the linear dynamic component is an infinite-dimensional well-posed system. Numerous physically motivated delay- and partial-differential equations are known to belong to this class of infinite-dimensional systems. We investigate input-to-state stability (ISS) and incremental ISS properties: our results are reminiscent of well-known absolute stability criteria such as the complex Aizerman conjecture and the circle criterion. The incremental ISS results are used to derive certain convergence properties, namely the convergent-input convergent-state (CICS) property and asymptotic periodicity of the state and output under periodic forcing. In particular, we provide sufficient conditions for ISS and incremental ISS. The theory is illustrated with examples.

Keywords. Absolute stability, converging-input converging-state property, incremental stability, input-to-state stability, Lur'e systems, infinite-dimensional well-posed linear systems

MSC(2010). 93C10, 93C25, 93C35, 93C80, 93D05, 93D09, 93D10, 93D20, 93D25.

1. Introduction.

We consider stability and convergence properties of the feedback interconnection between a forced, infinite-dimensional well-posed linear system Σ and a static nonlinear output feedback f, see Figure 1.1. Such feedback interconnections are often called Lur'e systems. Note that, in Figure 1.1, the signals v, y and u are given by

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \quad u = \begin{pmatrix} v^1 \\ f(y^2) + v^2 \end{pmatrix}. \tag{1.1}$$

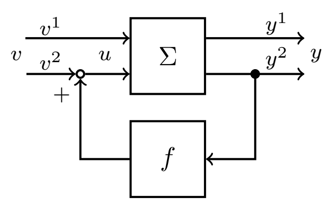


FIGURE 1.1. *Block diagram of forced Lur'e system: the feedback interconnection of the well-posed linear system Σ and the static nonlinearity f.*

Lur'e systems are a common and important class of nonlinear control systems, and the study of their stability properties has been termed *absolute stability theory* (see, for example, [10, 11, 12, 18, 19, 42, 46]). Classical absolute stability theory comes in two flavours: in a state-space setting, unforced ($v = 0$) finite-dimensional systems are considered and the emphasis is on global asymptotic stability, whilst the input-output approach (initiated by Sandberg and Zames in the 1960s) focusses on L^2-stability and, to a lesser extent, on L^∞-stability, see [7, 42]. A more recent

*Submitted to the editors in October 2017, revised May 2018 and re-revised August 2018.

†Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom (c.guiver@bath.ac.uk, h.logemann@bath.ac.uk, m.opmeer@bath.ac.uk). Corresponding author: C. Guiver

development is the analysis of state-space systems of Lur'e format in an *input-to-state stability* (ISS) context, thereby, in a sense, merging the two strands of the earlier theory [2, 14, 15, 16, 32]. The ISS concept was introduced (for general nonlinear control systems) in [33] and further developed across a range of papers, including [1, 17, 34, 35, 36] (see also the tutorial papers [5, 37]).

So far, the ISS approach to Lur'e systems is very much restricted to finite-dimensional systems (with [14] being an exception, see the commentary after Corollary 4.7). The aim of the present paper is to analyze ISS and incremental ISS properties of infinite-dimensional Lur'e systems. Incremental ISS [1] is a stronger property than ISS and enables us to obtain convergence properties such as the *converging-input converging-state* (CICS) property and the asymptotic periodicity of the state and output trajectories under periodic forcing. Our results show that certain classical sufficient conditions for absolute stability such as the complex Aizerman conjecture [11, 12] and the circle criterion [16, 18] (or variations thereof) in fact guarantee (incremental) ISS. A major novelty in this article is that we consider a general four-block Lur'e system, the linear component of which is a well-posed infinite-dimensional system. Systems in this class allow for considerable unboundedness of the control and observation operators and they encompass many of the most commonly studied partial differential equations with boundary control and observation, and a large class of functional differential equations of retarded and neutral type with delays in the inputs and outputs. There exists a highly developed state-space and frequency-domain theory for well-posed infinite-dimensional systems; see, for example, [27, 28, 38, 39, 40, 43, 44]. Throughout the paper we mean well-posedness in the L^2 sense, which is natural to assume as frequency-domain methods, familiar from classical absolute stability theory, generalize nicely in this infinite-dimensional framework.

Under suitable incremental conditions of small-gain type, the incremental stability estimates which we obtain are of the form

$$\|x_1(t) - x_2(t)\| \le \Gamma_q\big(e^{-\gamma t}\|x_1(0) - x_2(0)\| + \|v_1 - v_2\|_{L^q(0,t)}\big)\,, \tag{1.2}$$

and

$$\|x_1 - x_2\|_{L^2_\alpha(0,t)} + \|y_1 - y_2\|_{L^2_\alpha(0,t)} \le \Gamma\big(\|x_1(0) - x_2(0)\| + \|v_1 - v_2\|_{L^2_\alpha(0,t)}\big)\,. \tag{1.3}$$

These inequalities hold for all $2 \le q \le \infty$ and all $t \ge 0$, the constants γ, Γ and Γ_q are positive, α is non-negative and (v_k, x_k, y_k), for $k \in \{1,2\}$, are certain input/state/output trajectories of the Lur'e system[1] (one of which may, for instance, be an equilibrium solution). As usual, L^q_α denotes an exponentially weighted L^q space and $\|x\|^2_{L^2_\alpha(0,t)} = \int_0^t e^{2\alpha\sigma}\|x(\sigma)\|^2 d\sigma$. If (1.2) holds for all trajectories (v_k, x_k, y_k), then the Lur'e system is incrementally ISS. We note that at the level of generality which we consider, the outputs y_k need not have well-defined point evaluations everywhere and thus we should not expect pointwise estimates for $\|y_1(t) - y_2(t)\|$ to hold. Instead, the estimate (1.3) provides an upper bound for the difference of the outputs in a weighted L^q norm. In particular, if $v_1 - v_2 \in L^2_\alpha(0,\infty)$, then (1.3) guarantees that $y_1 - y_2 \in L^2_\alpha(0,\infty)$ which in turn yields convergence of $y_1(t) - y_2(t)$ to zero in the sense that

$$e^{2\alpha t}\int_t^\infty \|y_1(\sigma) - y_2(\sigma)\|^2 d\sigma \to 0 \quad \text{as } t \to \infty.$$

[1] Here v_k and y_k should not be confused with v^i and y^i, $i \in \{1,2\}$, which appear in (1.1) and Figure 1.1.

We proceed to give a more systematic overview of our results. With regards to stability properties, our main result is Theorem 4.1, which is reminiscent of the complex Aizerman conjecture [11, 12], familiar from finite-dimensional control theory. We emphasize that the conclusions of Theorem 4.1 relate to (incremental) ISS properties, in contrast to the complex Aizerman conjecture results in [11, 12] which guarantee global asymptotic stability of unforced Lur'e systems. Loosely speaking, our Theorem 4.1 states that if a ball of *complex linear* static output feedback gains is stabilizing, then the forced Lur'e system is (incrementally) ISS for all *nonlinear* feedbacks which satisfy, in a natural (incremental) sense, the same "ball" condition. As corollaries, we obtain a small-gain formulation in Corollary 4.3 and various generalizations of the circle criterion in Corollaries 4.5, 4.7 and 4.9. The proof of Theorem 4.1 rests on a combination of small-gain and exponential weighting techniques.

With regard to convergence properties, our main results are Theorems 5.2 and 5.4, the first of which provides a sufficient condition for the CICS property to hold, whilst the second result ensures that, under the same conditions that are sufficient for CICS, periodic forcing of the Lur'e system generates state and output trajectories which, in a certain sense, asymptotically approach periodic functions with the same period. The proofs of these results rely on incremental ISS properties of Lur'e systems established in Section 4. We comment that our main CICS result (Theorem 5.2) extends recent work by the authors [3] to an infinite-dimensional setting, whilst the result on asymptotic periodicity under periodic forcing (Theorem 5.4) was inspired by [1, Proposition 4.4] and provides a far-reaching generalization of results in [29, 30, 45].

To relate the present paper to the wider context, we comment briefly on some of the literature on ISS theory in infinite dimensions, a relatively new area of research. The article most closely related to the present paper is [14], which we discuss in more detail after the statement of Corollary 4.7. The papers [13, 26] analyze ISS properties of linear infinite-dimensional systems: whilst a class of linear time-varying hyperbolic PDEs is considered in [26], the authors of [13] investigate the relation between ISS and integral ISS for linear infinite-dimensional systems with an unbounded control operator. The articles [23, 25] consider certain specific infinite-dimensional systems, namely classes of semilinear parabolic PDEs with boundary disturbances [23] and time-delay systems [25]. The former is based on a bespoke approach relying on results from the theory of monotone control systems theory and the latter uses ISS Lyapunov theory to establish ISS properties for the systems under consideration. Neither paper focusses on Lur'e systems, and the set-ups and approaches differ substantially from ours: whilst a direct comparison of results is difficult, the overlap with the present work is negligible. The series of articles [6, 22, 24] considers ISS properties in an abstract framework of controlled nonlinear infinite-dimensional systems. A comparison between the results of these papers and the present work is again difficult: the continuity assumptions imposed in [6, 22] are too restrictive to encompass the unboundedness in the control and observation permitted in the linear component of the Lur'e systems studied in the present work. The paper [24] studies characterizations of the ISS property for an abstract class of nonlinear systems which is in some sense more general than the class of systems in the present work (for instance, no Lur'e structure is assumed) and in some sense more restrictive (since existence and uniqueness of solutions is implicitly assumed in [24], and outputs are not considered). None of the articles [6, 22, 24] consider sufficient conditions for the ISS property which are in the spirit of classical absolute stability theory, again making the overlap with the present work minimal.

The paper is organized as follows. Section 2 gathers notation and required material from the theory of well-posed linear systems. Section 3 discusses infinite-dimensional Lur'e systems. Our main results are contained in Sections 4 and 5, which contain stability and convergence results, respectively. The manuscript concludes with two examples in Section 6 which serve to illustrate the theory.

Finally, in order to keep the present text to a reasonable length, applications of the theory developed in Sections 4 and 5 to low-gain integral control in the presence of input nonlinearities will be presented in a future publication.

2. Preliminaries. For real or complex Hilbert spaces U and Y, let $\mathcal{L}(U,Y)$ denote the space of all linear bounded operators mapping U to Y. For $Z \in \mathcal{L}(U,Y)$ and $r > 0$, define

$$\mathbb{B}(Z,r) := \{T \in \mathcal{L}(U,Y) : \|T - Z\| < r\}\,,$$

the open ball in $\mathcal{L}(U,Y)$, with centre Z and radius r. We set $\mathcal{L}(U) := \mathcal{L}(U,U)$ and, for $S,T \in \mathcal{L}(U)$, we write $S \succeq T$ if $S-T$ is positive semi-definite, that is, $\langle Su - Tu, u\rangle \geq 0$ for all $u \in U$. It is well-known that, if U is a complex Hilbert space and $S \succeq 0$, then $S = S^*$.

For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \mathrm{Re}\, s > \alpha\}$. The space of all holomorphic and bounded functions $\mathbb{C}_\alpha \to \mathcal{L}(U,Y)$ is denoted by $H^\infty_\alpha(\mathcal{L}(U,Y))$. Endowed with the norm

$$\|\mathbf{H}\|_{H^\infty_\alpha} := \sup_{s\in\mathbb{C}_\alpha} \|\mathbf{H}(s)\|\,,$$

$H^\infty_\alpha(\mathcal{L}(U,Y))$ is a Banach space. We write $H^\infty(\mathcal{L}(U,Y))$ for $H^\infty_0(\mathcal{L}(U,Y))$. For an arbitrary Banach space W and $t \geq 0$, define the projection operator $\mathbf{P}_t : L^2_{\mathrm{loc}}(\mathbb{R}_+,W) \to L^2(\mathbb{R}_+,W)$ by

$$(\mathbf{P}_t w)(\tau) = \begin{cases} w(\tau), & \forall\, \tau \in [0,t] \\ 0, & \forall\, \tau > t. \end{cases}$$

For $\tau \geq 0$, the left-shift operator $\mathbf{L}_\tau : L^2_{\mathrm{loc}}(\mathbb{R}_+,W) \to L^2_{\mathrm{loc}}(\mathbb{R}_+,W)$ is given by $(\mathbf{L}_\tau w)(t) = w(t+\tau)$ for all $t \geq 0$. For $\alpha \in \mathbb{R}$ and $1 \leq q \leq \infty$, we define the weighted L^q-space

$$L^q_\alpha(\mathbb{R}_+,W) := \{w \in L^q_{\mathrm{loc}}(\mathbb{R}_+,W) : \exp_\alpha w \in L^q(\mathbb{R}_+,W)\},$$

where $\exp_\alpha : \mathbb{R} \to \mathbb{R}$ is given by $\exp_\alpha(t) := e^{\alpha t}$. Endowed with the norm

$$\|w\|_{L^q_\alpha} = \|\exp_\alpha w\|_{L^q}\,,$$

$L^q_\alpha(\mathbb{R}_+,W)$ is a Banach space. The function $\mathbb{R}_+ \to \mathbb{R}_+$ which is constant with value equal to 1 will be denoted by θ, that is, $\theta(t) = 1$ for all $t \geq 0$.

Below we will provide a brief review of some material from the theory of well-posed systems, for more details we refer the reader to [38, 40, 43, 44]. Throughout, we shall be considering a well-posed linear system $\Sigma = (\mathbb{T},\Phi,\Psi,\mathbb{G})$ with state space X, input space U and output space Y. Here X, U and Y are separable complex Hilbert spaces, $\mathbb{T} = (\mathbb{T}_t)_{t\geq0}$ is a strongly continuous semigroup on X, $\Phi = (\Phi_t)_{t\geq0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+,U)$ to X (input-to-state maps), $\Psi = (\Psi_t)_{t\geq0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+,Y)$ (state-to-output maps) and $\mathbb{G} = (\mathbb{G}_t)_{t\geq0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+,U)$ to $L^2(\mathbb{R}_+,Y)$ (input-to-output maps). In order for Σ to qualify as a well-posed system, these families of operators need to satisfy certain natural conditions, see [38, 40, 43]. Particular consequences of these conditions are the following properties:

$$\Phi_t\mathbf{P}_t = \Phi_t, \quad \mathbf{P}_t\Psi_{t+\tau} = \Psi_t, \quad \mathbf{P}_t\mathbb{G}_{t+\tau}\mathbf{P}_t = \mathbf{P}_t\mathbb{G}_{t+\tau} = \mathbb{G}_t \quad \forall\, t,\tau \geq 0\,.$$

It follows that Φ_t extends in a natural way to $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and there exist operators $\Psi_\infty : X \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and $\mathbb{G}_\infty : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ such that

$$\mathbf{P}_t \Psi_\infty = \Psi_t, \quad \mathbf{P}_t \mathbb{G}_\infty = \mathbb{G}_t \quad \forall\, t \geq 0\,.$$

The operator $\mathbb{G}_\infty$ is shift-invariant (and hence causal) and is called the input-output operator of Σ. Given an initial state x^0 and an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the corresponding state and output trajectories x and y of Σ are defined by

$$\left.\begin{aligned} x(t) &= \mathbb{T}_t x^0 + \Phi_t u \\ \mathbf{P}_t y &= \Psi_t x^0 + \mathbb{G}_t u \end{aligned}\right\} \quad \forall\, t \geq 0\,, \tag{2.1}$$

respectively.

Let (A, B, C) denote the generating operators of Σ. The operator A is the generator of the strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ and the operators $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$ are the unique operators satisfying

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\tau} B u(\tau) \mathrm{d}\tau \quad \forall\, u \in L^2(\mathbb{R}_+, U),\ \forall\, t \geq 0\,,$$

and

$$(\Psi_\infty x^0)(t) = C \mathbb{T}_t x^0 \quad \forall\, x^0 \in X_1,\ \forall\, t \geq 0,$$

where the spaces X_1 and X_{-1}, respectively, are the usual interpolation and extrapolation spaces associated with A and X.

The transfer function $\mathbf{G}$ of Σ has the property that $\mathbf{G} \in H^\infty_\alpha(\mathcal{L}(U, Y))$ for every $\alpha > \omega(\mathbb{T})$, where $\omega(\mathbb{T})$ denotes the exponential growth constant of $\mathbb{T}$. The relationship between $\mathbf{G}$ and the operators (A, B, C) is expressed by the formula

$$\frac{1}{s-z}\big(\mathbf{G}(s) - \mathbf{G}(z)\big) = -C(sI - A)^{-1}(zI - A_{-1})^{-1}B \quad \forall\, s, z \in \mathbb{C}_{\omega(\mathbb{T})},\ s \neq z,$$

see [38, equation (4.6.9)], where $A_{-1} \in \mathcal{L}(X, X_{-1})$ extends A to X and, considered as an unbounded operator on X_{-1}, generates a semigroup on X_{-1} which extends $\mathbb{T}$ to X_{-1}. Furthermore, for $\beta \in \mathbb{R}$, the operator $\mathbb{G}_\infty$ is in $\mathcal{L}(L^2_\beta(\mathbb{R}_+, U), L^2_\beta(\mathbb{R}_+, Y))$ if, and only if, $\mathbf{G} \in H^\infty_{-\beta}(\mathcal{L}(U, Y))$, in which case

$$\|\mathbb{G}_\infty\|_\beta = \|\mathbf{G}\|_{H^\infty_{-\beta}}, \tag{2.2}$$

where $\|\cdot\|_\beta$ denotes the L^2_β-induced operator norm. We remark that $\beta < -\omega(\mathbb{T})$ is sufficient for $\mathbb{G}_\infty$ to be in $\mathcal{L}(L^2_\beta(\mathbb{R}_+, U), L^2_\beta(\mathbb{R}_+, Y))$. We also record that, for every $\beta < -\omega(\mathbb{T})$, there exist positive constants φ and ψ such that

$$\|e^{\beta t} \Phi_t u\| \leq \varphi \|\mathbf{P}_t u\|_{L^2_\beta} \ \forall\, u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U),\ \forall\, t \geq 0\,,$$

and

$$\|\Psi_\infty x^0\|_{L^2_\beta} \leq \psi \|x^0\| \quad \forall\, x^0 \in X\,.$$

The system (2.1) is said to be *optimizable* if, for every $x^0 \in X$, there exists $u \in L^2(\mathbb{R}_+, U)$, such that $x \in L^2(\mathbb{R}_+, X)$. Furthermore, we say that (2.1) is *estimatable* if, the "dual" system is optimizable, that is, for every $z^0 \in X$, there exists $v \in L^2(\mathbb{R}_+, Y)$ such that the function $t \mapsto \mathbb{T}^*_t z^0 + \Psi^*_t v$ is in $L^2(\mathbb{R}_+, X)$. We note that, by [21], optimizability is equivalent to exponential stabilizability and estimatability is equivalent to exponential detectability (where exponential stabilizability and detectability are understood in the sense of [38]).

An operator $K \in \mathcal{L}(Y,U)$ is said to be an *admissible feedback operator* for Σ (or for $\mathbf{G}$) if there exists $\alpha \in \mathbb{R}$ such that $I - \mathbf{G}K$ is invertible in $H^\infty_\alpha(\mathcal{L}(Y))$. If $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator, then, for every $t \geq 0$, the operator $I - \mathbb{G}_t K$ is invertible in $\mathcal{L}(L^2(\mathbb{R}_+, Y))$, and, $I - \mathbb{G}_\infty K$ has a causal inverse $(I - \mathbb{G}_\infty K)^{-1}$ (mapping $L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ into itself). Furthermore, if $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator for Σ, then there exists a unique well-posed system $\Sigma^K = (\mathbb{T}^K, \Phi^K, \Psi^K, \mathbb{G}^K)$ such that

$$\Sigma^K_t = \Sigma_t + \Sigma_t \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \Sigma^K_t \qquad \forall\, t \geq 0, \tag{2.3}$$

where

$$\Sigma_t := \begin{pmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{G}_t \end{pmatrix}, \quad \Sigma^K_t := \begin{pmatrix} \mathbb{T}^K_t & \Phi^K_t \\ \Psi^K_t & \mathbb{G}^K_t \end{pmatrix}.$$

It follows from (2.3) that, for all $t \geq 0$,

$$\mathbb{T}^K_t = \mathbb{T}_t + \Phi_t K \Psi^K_t, \quad \Phi^K_t = \Phi_t(I + K\mathbb{G}^K_t), \quad \Psi^K_t = (I + \mathbb{G}^K_t K)\Psi_t\,,$$

and,

$$(I - \mathbb{G}_t K)^{-1} = I + \mathbb{G}^K_t K, \quad \mathbb{G}^K_t = (I - \mathbb{G}_t K)^{-1}\mathbb{G}_t.$$

Moreover,

$$\Psi^K_\infty = (I + \mathbb{G}^K_\infty K)\Psi_\infty, \quad (I - \mathbb{G}_\infty K)^{-1} = I + \mathbb{G}^K_\infty K, \quad \mathbb{G}^K_\infty = (I - \mathbb{G}_\infty K)^{-1}\mathbb{G}_\infty.$$

The transfer function $\mathbf{G}^K$ of Σ^K is given by $\mathbf{G}^K = (I - \mathbf{G}K)^{-1}\mathbf{G}$.

The interpretation of (2.3) is that Σ^K is the closed-loop system shown in Figure 2.1.

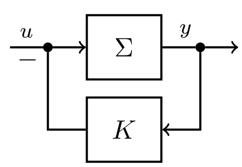


FIGURE 2.1. *Block diagram of closed-loop feedback system of* Σ *in connection with output feedback* K.

It follows from (2.3) that if x and y are the state and output trajectories of Σ associated with the initial state x^0 and input u, then x and y are the state and output trajectories of Σ^K associated with the initial state x^0 and input $u - Ky$. We state this fact, somewhat more precisely, in the form of a lemma.

LEMMA 2.1. *Let* $(u,x,y) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \times C(\mathbb{R}_+, X) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ *and let* $K \in \mathcal{L}(Y,U)$ *be an admissible feedback operator for* Σ. *The triple* (u,x,y) *satisfies* (2.1) *with* $x^0 = x(0)$ *if, and only if,*

$$\left.\begin{aligned} x(t) &= \mathbb{T}^K_t x(0) + \Phi^K_t(u - Ky) \\ \mathbf{P}_t y &= \Psi^K_t x(0) + \mathbb{G}^K_t(u - Ky) \end{aligned}\right\} \quad \forall\, t \geq 0\,.$$

We say that an operator $K \in \mathcal{L}(Y,U)$ *stabilizes* $\mathbf{G}$ (or stabilizes Σ in the input-output sense) if $(I - \mathbf{G}K)^{-1}\mathbf{G} \in H^\infty(\mathcal{L}(U,Y))$. The set of all operators stabilizing $\mathbf{G}$ is denoted by $\mathbb{S}(\mathbf{G})$. Trivially, every element in $\mathbb{S}(\mathbf{G})$ is an admissible feedback operator for $\mathbf{G}$.

The following lemma is a special case of [8, Proposition 5.6].

LEMMA 2.2. *For $K \in \mathcal{L}(Y,U)$ and $r > 0$, $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{G})$ if, and only if, $\|(I - \mathbf{G}K)^{-1}\mathbf{G}\|_{H^\infty} \leq 1/r$.*

An immediate consequence of the sufficiency part of Lemma 2.2 is that $\mathbb{S}(\mathbf{G})$ is an open subset of $\mathcal{L}(Y,U)$. Note that the sufficiency part is simply a version of the small-gain theorem. The assumption that the Hilbert spaces U and Y are complex plays an important role in the necessity part which in general does not hold for real Hilbert spaces.

In the following, the input and output spaces U and Y will be of the form $U = U^1 \times U^2$ and $Y = Y^1 \times Y^2$, where U^i and Y^i are complex Hilbert spaces, $i = 1,2$. It is convenient to introduce the following maps

$$P^i : Y \to Y^i,\ \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \mapsto y^i, \quad i = 1,2\,,$$

and

$$E^1 : U^1 \to U,\ u \mapsto \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad E^2 : U^2 \to U,\ u \mapsto \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

If $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, then $P^i y$ is the function in $L^2_{\mathrm{loc}}(\mathbb{R}_+, Y^i)$ given by $(P^i y)(t) = P^i y(t)$. Similarly, for $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U^i)$, the symbol $E^i u$ denotes the function in $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ given by $(E^i u)(t) = E^i u(t)$. The decompositions of the input and output spaces, $U = U^1 \times U^2$ and $Y = Y^1 \times Y^2$, respectively, induces four well-posed systems, namely,

$$\Sigma^{ij} := (\mathbb{T}, \Phi E^j, P^i \Psi, P^i \mathbb{G} E^j), \quad i,j = 1,2\,.$$

Obviously, the state, input and output spaces of Σ^{ij} are given by X, U^j and Y^i, respectively. For $K^{ij} \in \mathcal{L}(Y^j, U^i)$, let $K \in \mathcal{L}(Y,U)$ be defined by

$$Ky = E^i K^{ij} P^j y \quad \forall\, y \in Y\,. \tag{2.4}$$

For example, if $i = 1$ and $j = 2$, then

$$K = \begin{pmatrix} 0 & K^{12} \\ 0 & 0 \end{pmatrix}.$$

The next result compares the feedback systems obtained by applying the feedback operators K^{ij} and K to Σ^{ji} and Σ, respectively.

PROPOSITION 2.3. *Let $i,j \in \{1,2\}$ and let $K^{ij} \in \mathcal{L}(Y^j, U^i)$ be an admissible feedback operator for Σ^{ji}. Then $K \in \mathcal{L}(Y,U)$ given by (2.4) is an admissible feedback operator for Σ and the following identities hold:*

$$\begin{cases} \mathbb{T}^K = \mathbb{T}^{K^{ij}}, & \Phi^K E^i = (\Phi E^i)^{K^{ij}}, \\ P^j \Psi^K = (P^j \Psi)^{K^{ij}}, & P^j \mathbb{G}^K E^i = (P^j \mathbb{G} E^i)^{K^{ij}}, \end{cases} \tag{2.5}$$

where the last identity can be formulated in terms of transfer functions as follows:

$$P^j \mathbf{G}^K E^i = (P^j \mathbf{G} E^i)^{K^{ij}}\,.$$

Proof. Admissibility is most easily shown on a case-by-case basis. Setting $\mathbf{G}^{ji} := P^j \mathbf{G} E^i$, we have that, for $(i,j) = (1,1)$, $(i,j) = (1,2)$ $(i,j) = (2,1)$ and $(i,j) = (2,2)$, the function $I - \mathbf{G}K$ equals

$$\begin{pmatrix} I - \mathbf{G}^{11}K^{11} & 0 \\ -\mathbf{G}^{21}K^{11} & I \end{pmatrix}, \quad \begin{pmatrix} I & -\mathbf{G}^{11}K^{12} \\ 0 & I - \mathbf{G}^{21}K^{12} \end{pmatrix}, \quad \begin{pmatrix} I - \mathbf{G}^{12}K^{21} & 0 \\ -\mathbf{G}^{22}K^{21} & I \end{pmatrix}, \quad \begin{pmatrix} I & -\mathbf{G}^{12}K^{22} \\ 0 & I - \mathbf{G}^{22}K^{22} \end{pmatrix},$$

respectively. From this we see that K is admissible for $\mathbf{G}$ if, and only if, K^{ij} is admissible for $\mathbf{G}^{ji}$.

Introducing the well-posed system

$$\Sigma_t^{ji} := \begin{pmatrix} \mathbb{T}_t & \Phi_t E^i \\ P^j \Psi_t & P^j \mathbb{G}_t E^i \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix}, \tag{2.6}$$

it is clear that the identities (2.5) are equivalent to

$$\left(\Sigma_t^{ij}\right)^{K^{ij}} = \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t^K \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix}. \tag{2.7}$$

It is convenient to denote the right-hand side of (2.7) by $\tilde{\Sigma}_t$, that is,

$$\tilde{\Sigma}_t := \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t^K \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix}. \tag{2.8}$$

It is sufficient to prove that

$$\tilde{\Sigma}_t = \Sigma_t^{ji} + \Sigma_t^{ji} \begin{pmatrix} 0 & 0 \\ 0 & K^{ij} \end{pmatrix} \tilde{\Sigma}_t. \tag{2.9}$$

Indeed, by the uniqueness in (2.3), it follows from (2.9) that $\tilde{\Sigma}$ is the closed-loop system obtained by applying the feedback K^{ij} to Σ^{ji}, and so (2.7) holds.

To establish (2.9), we first substitute (2.3) into (2.8) to give

$$\tilde{\Sigma}_t = \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \Sigma_t^K \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix}. \tag{2.10}$$

Using that

$$\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & K^{ij} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix},$$

we see that the right-hand side of (2.10) equals

$$\begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & K^{ij} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P^j \end{pmatrix} \Sigma_t^K \begin{pmatrix} I & 0 \\ 0 & E^i \end{pmatrix},$$

which, by (2.6) and (2.8), is identical to

$$\Sigma_t^{ji} + \Sigma_t^{ji} \begin{pmatrix} 0 & 0 \\ 0 & K^{ij} \end{pmatrix} \tilde{\Sigma}_t.$$

Hence we have shown that (2.9) holds, completing the proof. ∎

3. Infinite-dimensional Lur'e systems. Here we define precisely the class of Lur'e systems, the stability and convergence properties of which we shall study, thereby formalizing the arrangement depicted in Figure 1.1. Given an initial state x^0 and an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the corresponding state and output trajectories of Σ are given by (2.1). Let $i, j \in \{1, 2\}$ and let $f : Y^j \to U^i$ be a nonlinearity. The closed-loop system obtained by applying the feedback

$$u = E^i(f \circ P^j y) + v, \quad \text{where } v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U),$$

is then given by

$$\left.\begin{aligned} x(t) &= \mathbb{T}_t x^0 + \Phi_t\big(E^i(f \circ P^j y) + v\big) \\ \mathbf{P}_t y &= \Psi_t x^0 + \mathbb{G}_t\big(E^i(f \circ P^j y) + v\big) \end{aligned}\right\} \quad \forall\, t \geq 0 . \tag{3.1}$$

As an illustration, Figure 1.1 corresponds to the case $i = j = 2$. Given $x^0 \in X$ and $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, a *solution* of the Lur'e system (3.1) on $[0, \sigma)$, where $0 < \sigma \leq \infty$, is a pair $(x, y) \in C([0, \sigma), X) \times L^2_{\mathrm{loc}}([0, \sigma), Y)$ such that $f \circ P^j y \in L^2_{\mathrm{loc}}([0, \sigma), U^i)$ and (3.1) holds for all $t \in [0, \sigma)$. If $\sigma = \infty$, the solution is called *global*. Obviously, if (x, y) is a solution of (3.1), then $x(0) = x^0$.

It can be shown (by invoking Zorn's lemma) that, for every solution of (3.1) on $[0, \sigma)$, there exists a *maximally defined* solution (3.1) defined on $[0, \tau)$ with $\sigma \leq \tau \leq \infty$ which cannot be extended any further (that is, τ is maximal). System (3.1) is said to have the *blow-up property* if, for every maximally defined solution (x, y) with finite interval of existence $[0, \sigma)$,

$$\max\left\{\limsup_{t \uparrow \sigma} \|x(t)\|,\ \lim_{t \uparrow \sigma} \int_0^t \|y(\tau)\|^2 \mathrm{d}\tau\right\} = \infty . \tag{3.2}$$

We remark that if the blow-up property holds and (x, y) is a solution of (3.1) on $[0, \sigma)$ with $0 < \sigma < \infty$ and the left-hand side of (3.2) is finite, then the solution (x, y) can be extended to the right beyond σ.

The set of all triples (v, x, y) in $L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \times C(\mathbb{R}_+, X) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ such that (3.1) holds with $x^0 = x(0)$ is said to be the behaviour of (3.1) and is denoted by $\mathcal{B}$. In particular, if $(v, x, y) \in \mathcal{B}$, then (x, y) is a global solution of (3.1) with $x^0 = x(0)$. In an ISS context, we consider external inputs v which belong to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$. More generally, for $2 \leq q \leq \infty$, we may wish to consider inputs v in $L^q_{\mathrm{loc}}(\mathbb{R}_+, U) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$. It is therefore convenient to define the following "sub-behaviour" of $\mathcal{B}$:

$$\mathcal{B}^q := \{(v, x, y) \in \mathcal{B} : v \in L^q_{\mathrm{loc}}(\mathbb{R}_+, U)\} .$$

Obviously, we have $\mathcal{B}^2 = \mathcal{B}$. In this paper, we are mainly concerned with stability and convergence properties of (3.1): existence and/or uniqueness of solutions is not our main concern. The question of existence requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration. However, we state a simple, but important, existence and uniqueness result from [40].

PROPOSITION 3.1. *If $f : Y^j \to U^i$ is globally Lipschitz with Lipschitz constant $\lambda \geq 0$ and*

$$\lambda \liminf_{\alpha \to \infty} \|P^j \mathbf{G} E^i\|_{H^\infty_\alpha} < 1 ,$$

then, for all $x^0 \in X$ and $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the Lur'e system (3.1) has a unique global solution.

The next result ("loop shifting" in control engineering jargon) shows that the behaviour $\mathcal{B}$ of (3.1) is identical to the behaviour of the feedback interconnection obtained when the linear system Σ^K is subjected to the feedback law $u = E^i f(P^j y) - Ky + v$, where $K \in \mathcal{L}(Y, U)$ is an admissible feedback operator for Σ.

COROLLARY 3.2. *Let $K \in \mathcal{L}(Y, U)$ be an admissible feedback operator for Σ and let $(v, x, y) \in L^2_{\rm loc}(\mathbb{R}_+, U) \times C(\mathbb{R}_+, X) \times L^2_{\rm loc}(\mathbb{R}_+, Y)$. The triple (v, x, y) is in $\mathcal{B}$ if, and only if,*

$$\left.\begin{aligned} x(t) &= \mathbb{T}^K_t x(0) + \Phi^K_t\big(E^i(f \circ P^j y) + v - Ky\big) \\ \mathbf{P}_t y &= \Psi^K_t x(0) + \mathbb{G}^K_t\big(E^i(f \circ P^j y) + v - Ky\big) \end{aligned}\right\} \quad \forall\, t \geq 0\,.$$

The above corollary is an immediate consequence of Lemma 2.1.

A triple $(v^{\rm e}, x^{\rm e}, y^{\rm e}) \in U \times X \times Y$ is said to be an *equilibrium* or *equilibrium triple* of the Lur'e system (3.1) if the constant trajectory $t \mapsto (v^{\rm e}, x^{\rm e}, y^{\rm e})$ belongs to $\mathcal{B}$. The next result provides formulas relating the components of an equilibrium triple $(v^{\rm e}, x^{\rm e}, y^{\rm e})$.

PROPOSITION 3.3. *Let $(v^{\rm e}, x^{\rm e}, y^{\rm e}) \in U \times X \times Y$, let $\eta \in \mathbb{C}$ such that $\operatorname{Re}\eta > \omega(\mathbb{T})$ and set $u^{\rm e} := E^i f(P^j y^{\rm e}) + v^{\rm e}$. The triple $(v^{\rm e}, x^{\rm e}, y^{\rm e})$ is an equilibrium of (3.1) if, and only if,*

$$Ax^{\rm e} + Bu^{\rm e} = 0 \quad \textit{and} \quad y^{\rm e} = C\big(x^{\rm e} - (\eta I - A)^{-1} B u^{\rm e}\big) + \mathbf{G}(\eta) u^{\rm e}\,.$$

Note that the identity $Ax^{\rm e} + Bu^{\rm e} = 0$ implies that $x^{\rm e} - (\eta I - A)^{-1} B u^{\rm e} \in X_1$ and thus, the expression $C\big(x^{\rm e} - (\eta I - A)^{-1} B u^{\rm e}\big)$ is well defined.

Proof of Proposition 3.3. It is clear that $(v^{\rm e}, x^{\rm e}, y^{\rm e})$ is an equilibrium of (3.1) if, and only if, for all $t \geq 0$,

$$x^{\rm e} = \mathbb{T}_t x^{\rm e} + \Phi_t(u^{\rm e}\theta), \quad \mathbf{P}_t(y^{\rm e}\theta) = \Psi_t x^{\rm e} + \mathbb{G}_t(u^{\rm e}\theta),$$

where we recall that the function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\theta(t) = 1$ for all $t \geq 0$. The claim now follows from [39, Proposition 2.6] and [39, Theorem 3.2]. □

4. Input-to-state stability properties. The current section contains our main results pertaining to stability properties of (3.1), namely Theorem 4.1 from which we derive a number of corollaries. We start by introducing some terminology. An equilibrium triple $(v^{\rm e}, x^{\rm e}, y^{\rm e})$ of (3.1) is said to be *exponentially input-to-state stable* (exponentially ISS) if there exist positive constants Γ and γ such that

$$\|x(t) - x^{\rm e}\| \leq \Gamma\big(e^{-\gamma t}\|x(0) - x^{\rm e}\| + \|\mathbf{P}_t(v - v^{\rm e}\theta)\|_{L^\infty}\big) \quad \forall\, t \geq 0,\ \forall\,(v, x, y) \in \mathcal{B}^\infty\,.$$

Furthermore, (3.1) is said to be *exponentially incrementally input-to-state stable* (*exponentially δISS*) if there exist positive constants Γ and γ such that, for all (v_1, x_1, y_1), $(v_2, x_2, y_2) \in \mathcal{B}^\infty$ and all $t \geq 0$,

$$\|x_1(t) - x_2(t)\| \leq \Gamma\big(e^{-\gamma t}\|x_1(0) - x_2(0)\| + \|\mathbf{P}_t(v_1 - v_2)\|_{L^\infty}\big)\,.$$

We introduce a further type of "sub-behaviour" which shall be useful in formulating our stability results. For a non-empty subset $Z \subset Y^j$ and $2 \leq q \leq \infty$, we set

$$\mathcal{B}^q_Z := \{(v, x, y) \in \mathcal{B}^q : P^j y(t) \in Z \ \text{ for a.e } t \geq 0\}\,.$$

Furthermore, $\mathcal{B}_Z := \mathcal{B}^2_Z$.

The following theorem is reminiscent of the complex Aizerman conjecture in finite dimensions (which is known to be true, see [11, 12, 16]): (incremental) stability properties of the nonlinear system (3.1) are guaranteed by the assumption that a corresponding linear feedback system is stable for all linear complex feedback operators belonging to a certain ball, provided the nonlinearity satisfies, in a suitable and natural sense, the same ball condition.

THEOREM 4.1. *Let* $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{G})$ *be a well-posed linear system, let* $i, j \in \{1, 2\}$, $K^{ij} \in \mathcal{L}(Y^j, U^i)$, $r > 0$ *and let* $Z_1, Z_2 \subset Y^j$ *be non-empty subsets. Assume that* $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ *is optimizable and estimatable and* $\mathbb{B}(K^{ij}, r) \subset \mathbb{S}(P^j\mathbf{G}E^i)$. *If* $f : Y^j \to U^i$ *satisfies*

$$\nu(f, Z_1, Z_2) := \sup_{(z_1,z_2)\in Z_1\times Z_2,\, z_1\neq z_2} \frac{\|f(z_1) - f(z_2) - K^{ij}(z_1 - z_2)\|}{\|z_1 - z_2\|} < r, \tag{4.1}$$

then the following statements hold.

(1) *There exist constants* $\Gamma > 0$ *and* $\varepsilon > 0$ *such that, for all* $(v_1, x_1, y_1) \in \mathcal{B}_{Z_1}$, *all* $(v_2, x_2, y_2) \in \mathcal{B}_{Z_2}$, *all* $\alpha \in [0, \varepsilon]$ *and all* $t \geq 0$,

$$\|\mathbf{P}_t(x_1 - x_2)\|_{L^2_\alpha} + \|\mathbf{P}_t(y_1 - y_2)\|_{L^2_\alpha} \leq \Gamma\big(\|x_1(0) - x_2(0)\| + \|\mathbf{P}_t(v_1 - v_2)\|_{L^2_\alpha}\big).$$

(2) *Let* $2 \leq q \leq \infty$. *There exist constants* $\Gamma_q > 0$ *and* $\gamma > 0$ *such that, for all* $(v_1, x_1, y_1) \in \mathcal{B}^q_{Z_1}$, *all* $(v_2, x_2, y_2) \in \mathcal{B}^q_{Z_2}$ *and all* $t \geq 0$,

$$\|x_1(t) - x_2(t)\| \leq \Gamma_q\big(e^{-\gamma t}\|x_1(0) - x_2(0)\| + \|\mathbf{P}_t(v_1 - v_2)\|_{L^q}\big).$$

Here Γ_q *depends on* q, *but* γ *does not.*

(3) *Let* $\rho \in (0, r)$. *Then the estimates in statement* (1) *and* (2) *hold uniformly in* f *for all* f *with* $\nu(f, Z_1, Z_2) \leq \rho$, *that is, the constants* Γ, ε, Γ_q *and* γ *do only depend on* ρ, *but not on the specific nonlinearity* f *(satisfying* $\nu(f, Z_1, Z_2) \leq \rho$*).*

(4) *If* $(v_1, x_1, y_1) \in \mathcal{B}^\infty_{Z_1}$, $(v_2, x_2, y_2) \in \mathcal{B}^\infty_{Z_2}$ *and* $e^{\alpha t}(v_1(t) - v_2(t)) \to 0$ *as* $t \to \infty$ *for some* $\alpha \geq 0$, *then* $x_1(t) - x_2(t) \to 0$ *as* $t \to \infty$. *If* $\alpha > 0$, *then the rate of convergence of* $x_1(t) - x_2(t)$ *to* 0 *is exponential.*

(5) *If* $(v_1, x_1, y_1) \in \mathcal{B}^q_{Z_1}$, $(v_2, x_2, y_2) \in \mathcal{B}^q_{Z_2}$, *where* $2 \leq q < \infty$, *and* $v_1 - v_2 \in L^q_\alpha(\mathbb{R}_+, U)$ *for some* $\alpha \geq 0$, *then* $x_1(t) - x_2(t) \to 0$ *as* $t \to \infty$. *If* $\alpha > 0$, *then the rate of convergence of* $x_1(t) - x_2(t)$ *to* 0 *is exponential.*

We provide some commentary before proving Theorem 4.1, in particular highlighting two important special cases.

Special case 1. Assume that $(v^e, x^e, y^e) \in U \times X \times Y$ is an equilibrium triple of the Lur'e system (3.1) and the assumptions of Theorem 4.1 hold with $Z_1 = Y^j$ and $Z_2 = \{P^j y^e\}$. Then the constant trajectory (v^e, x^e, y^e) is in $\mathcal{B}^\infty_{Z_2}$ and statement (2) implies that the equilibrium (v^e, x^e, y^e) is exponentially ISS. Furthermore, by statement (4), if (v, x, y) is in $\mathcal{B}^\infty$ and $v(t) \to v^e$ as $t \to \infty$, then $x(t) \to x^e$ as $t \to \infty$. In particular, if

$$\sup_{z\in Y^j,\, z\neq 0} \frac{\|f(z) - K^{ij}z\|}{\|z\|} < r,$$

and f is continuous, then $f(0) = 0$ and $(0, 0, 0)$ is an exponentially ISS equilibrium of (3.1). In this scenario, (3.1) has the 0-converging-input converging-state property (0-CICS), that is, if (v, x, y) is in $\mathcal{B}^\infty$ and $v(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$.

Special case 2. Assume that the hypotheses of Theorem 4.1 hold with $Z_1 = Z_2 = Y^j$ (and so (4.1) is equivalent to $z \mapsto f(z) - K^{ij}z$ being globally Lipschitz with Lipschitz constant smaller than r). In this case, statement (2) of Theorem 4.1 implies that the Lur'e system (3.1) is exponentially δISS. Furthermore, as a consequence of Proposition 3.1 and Corollary 3.2, for every pair $(x^0, v) \in X \times L^2_{\rm loc}(\mathbb{R}_+, U)$, there exists a unique triple $(v, x, y) \in \mathcal{B}$ such that $x(0) = x^0$.

We now proceed to prove Theorem 4.1. Further special cases and applications of Theorem 4.1 will be presented after the proof (in particular, see Theorem 5.2 and Theorem 5.4).

Proof of Theorem 4.1. By hypothesis, $\mathbb{B}(K^{ij}, r) \subset \mathbb{S}(P^j \mathbf{G} E^i)$, and hence, using Lemma 2.2 and Proposition 2.3, we conclude that

$$\|(P^j \mathbb{G}_\infty E^i)^{K^{ij}}\| = \|(P^j \mathbf{G} E^i)^{K^{ij}}\|_{H^\infty} = \|P^j \mathbb{G}^K_\infty E^i\| = \|P^j \mathbf{G}^K E^i\|_{H^\infty} \le 1/r,$$

where $\|\cdot\|$ is the L^2-induced operator norm and K is given by (2.4). Furthermore, by (4.1), with $\nu := \nu(f, Z_1, Z_2)$, it follows that

$$\nu \|P^j \mathbf{G}^K E^i\|_{H^\infty} = \nu \|P^j \mathbb{G}^K_\infty E^i\| < 1. \tag{4.2}$$

Since Σ^{ji} is optimizable and estimatable and $(P^j \mathbf{G} E^i)^{K^{ij}} \in H^\infty(\mathcal{L}(U^i, Y^j))$, it follows from [44, Theorem 1.1] that $\mathbb{T}^{K^{ij}}$ is exponentially stable. Defining $K \in \mathcal{L}(Y, U)$ by (2.4), we invoke Proposition 2.3, to conclude that $\mathbb{T}^K = \mathbb{T}^{K^{ij}}$ is exponentially stable.

To establish statements (1) and (2), we will make use of an exponential weighting argument. To this end, let $\alpha \in (0, -\omega(\mathbb{T}^K))$ and define shift-invariant operators $H_\alpha : L^2_{\rm loc}(\mathbb{R}_+, U) \to L^2_{\rm loc}(\mathbb{R}_+, Y)$ and $H^{ji}_\alpha : L^2_{\rm loc}(\mathbb{R}_+, U^i) \to L^2_{\rm loc}(\mathbb{R}_+, Y^j)$ as follows:

$$H_\alpha w := \exp_\alpha \mathbb{G}^K_\infty(\exp_{-\alpha} w), \quad H^{ji}_\alpha w := \exp_\alpha P^j \mathbb{G}^K_\infty E^i(\exp_{-\alpha} w).$$

Note that the transfer functions $\mathbf{H}_\alpha$ and $\mathbf{H}^{ji}_\alpha$ of H_α and H^{ji}_α, respectively, are given by

$$\mathbf{H}_\alpha(s) = \mathbf{G}^K(s - \alpha), \quad \mathbf{H}^{ji}_\alpha(s) = P^j \mathbf{G}^K(s - \alpha) E^i.$$

Fixing $\delta \in (0, -\omega(\mathbb{T}^K))$, we have $\mathbf{H}_\alpha \in H^\infty(\mathcal{L}(U, Y))$ and $\mathbf{H}^{ji}_\alpha \in H^\infty(\mathcal{L}(U^i, Y^j))$ for all $\alpha \in [0, \delta]$. It follows that, for every $\alpha \in [0, \delta]$,

$$H_\alpha \in \mathcal{L}(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y)), \;\; H^{ji}_\alpha \in \mathcal{L}(L^2(\mathbb{R}_+, U^i), L^2(\mathbb{R}_+, Y^j)),$$

and

$$\|H_\alpha\| = \|\mathbf{H}_\alpha\|_{H^\infty}, \quad \|H^{ji}_\alpha\| = \|\mathbf{H}^{ji}_\alpha\|_{H^\infty}.$$

Furthermore, since $\mathbf{H}^{ji}_\delta \in H^\infty(\mathcal{L}(U^i, Y^j))$, the transfer function $P^j \mathbf{G}^K E^i$ is uniformly continuous in any vertical strip of the form $\sigma_1 \le \operatorname{Re} s \le \sigma_2$, where $-\delta < \sigma_1 < \sigma_2$, and therefore, invoking (4.2), we conclude there there exists $\varepsilon \in (0, \delta)$ such that

$$\nu \|\mathbf{H}^{ji}_\alpha\|_{H^\infty} = \nu \|H^{ji}_\alpha\| < 1 \quad \forall\, \alpha \in [0, \varepsilon]\,. \tag{4.3}$$

For the following, it is convenient to define

$$g : Y^j \times Y^j \to U^i, \; (z_1, z_2) \mapsto f(z_1) - f(z_2)\,. \tag{4.4}$$

We note that

$$\sup_{(z_1, z_2) \in Z_1 \times Z_2,\, z_1 \neq z_2} \frac{\|g(z_1, z_2) - K^{ij}(z_1 - z_2)\|}{\|z_1 - z_2\|} = \nu < r\,.$$

Consequently, for every $\alpha \in [0,\varepsilon]$,

$$(4.5)\quad \begin{cases} \|\mathbf{P}_t \exp_\alpha(g(w_1,w_2) - K^{ij}(w_1-w_2))\|_{L^2} \le \nu \|\mathbf{P}_t \exp_\alpha(w_1-w_2)\|_{L^2} \ \forall\, t\ge 0, \\ \quad \forall\, w_1,w_2 \in L^2_{\mathrm{loc}}(\mathbb{R}_+,Y^j) \text{ s.t. } (w_1(\tau),w_2(\tau)) \in Z_1\times Z_2 \text{ for a.e. } \tau \ge 0\,. \end{cases}$$

Let $(v_1,x_1,y_1) \in \mathcal{B}_{Z_1}$, $(v_2,x_2,y_2)\in\mathcal{B}_{Z_2}$ and set

$$v := v_1 - v_2, \quad x := x_1 - x_2, \quad \text{and} \quad y := y_1 - y_2\,.$$

To establish statement (1), we invoke Corollary 3.2 and (2.4) to obtain

$$\begin{aligned} y &= \Psi^K_\infty x(0) + \mathbb{G}^K_\infty\big(E^i g(P^j y_1, P^j y_2) + v - Ky\big) \\ &= \Psi^K_\infty x(0) + \mathbb{G}^K_\infty\big(E^i g(P^j y_1, P^j y_2) - E^i K^{ij} P^j y + v\big)\,. \end{aligned}$$

Multiplying the above equation by $\exp_\alpha$, where $\alpha\in[0,\varepsilon]$, leads to

$$(4.6)\qquad \exp_\alpha y = \exp_\alpha \Psi^K_\infty x(0) + H_\alpha\big(\exp_\alpha E^i[g(P^j y_1,P^j y_2) - K^{ij}P^j y] + \exp_\alpha v\big)\,.$$

Applying the operator P^j to both sides of this equation leads to

$$\exp_\alpha P^j y = \exp_\alpha P^j \Psi^K_\infty x(0) + H^{ji}_\alpha\big(\exp_\alpha(g(P^j y_1,P^j y_2) - K^{ij}P^j y)\big) + P^j H_\alpha \exp_\alpha v\,.$$

Taking norms and invoking (4.5) gives, for every $t\ge 0$,

$$(4.7)\quad \|\mathbf{P}_t(P^j y)\|_{L^2_\alpha} \le \|P^j\Psi^K_\infty x(0)\|_{L^2_\alpha} + \|H^{ji}_\alpha\|\nu\|\mathbf{P}_t(P^j y)\|_{L^2_\alpha} + \|P^j H_\alpha\|\|\mathbf{P}_t v\|_{L^2_\alpha}.$$

In light of (4.3), we have that $\|H^{ji}_\alpha\|\nu < 1$, and so, setting

$$\mu_\alpha := \frac{\max\{\|P^j H_\alpha\|, 1\}}{1-\nu\|H^{ji}_\alpha\|} > 0\,,$$

we may rearrange (4.7) to yield

$$(4.8)\qquad \|\mathbf{P}_t(P^j y)\|_{L^2_\alpha} \le \mu_\alpha\big(\|P^j\Psi^K_\infty x(0)\|_{L^2_\alpha} + \|\mathbf{P}_t v\|_{L^2_\alpha}\big) \quad \forall\, t\ge 0\,.$$

By taking norms in (4.6), and inserting the estimates (4.5) and (4.8), it follows that

$$(4.9)\qquad \|\mathbf{P}_t y\|_{L^2_\alpha} \le \Gamma'\big(\|x(0)\| + \|\mathbf{P}_t v\|_{L^2_\alpha}\big) \quad \forall\, t\ge 0,\ \forall\,\alpha\in[0,\varepsilon]\,,$$

with

$$\Gamma' := \max\{(1+\nu\mu_\varepsilon\|H_\varepsilon\|)\|\Psi^K_\infty\|_\varepsilon, (1+\nu\mu_\varepsilon)\|H_\varepsilon\|\}\,.$$

Here $\|\Psi^K_\infty\|_\varepsilon$ denotes the L^2_ε-induced norm of the operator Ψ^K_∞. Now

$$x(t) = \mathbb{T}^K_t x(0) + \Phi^K_t\big(E^i g(P^j y_1, P^j y_2) + v - Ky\big)\,,$$

and so, combining this identity with (4.5) and (4.9), it follows via standard results from well-posed linear systems theory that there exists a constant $\Gamma'' > 0$ such that

$$\|\mathbf{P}_t x\|_{L^2_\alpha} \le \Gamma''\big(\|x(0)\| + \|\mathbf{P}_t v\|_{L^2_\alpha}\big) \quad \forall\, t\ge 0,\ \forall\,\alpha\in[0,\varepsilon]\,.$$

This estimate, together with (4.9), shows that statement (1) holds with $\Gamma := \Gamma' + \Gamma''$.

To prove statement (2), we fix $\gamma \in (0, \varepsilon]$ and invoke Corollary 3.2 to obtain

$$\begin{aligned} e^{\gamma t}\big(x(t) - \mathbb{T}^K_t x(0)\big) &= e^{\gamma t}\Phi^K_t\big(E^i g(P^j y_1, P^j y_2) + v - Ky\big) \\ &= e^{\gamma t}\Phi^K_t\big(E^i(g(P^j y_1, P^j y_2) - K^{ij}P^j y) + v\big) \quad \forall\, t \geq 0\,. \end{aligned} \tag{4.10}$$

Since $\gamma \in (0, -\omega(\mathbb{T}^K))$, there exist a constant $\varphi > 0$ such that

$$\|e^{\gamma t}\Phi^K_t w\| \leq \varphi\|\mathbf{P}_t w\|_{L^2_\gamma} \ \ \forall\, w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U),\ \forall\, t \geq 0.$$

Combining this with (4.10), we have that

$$e^{\gamma t}\|x(t) - \mathbb{T}^K_t x(0)\| \leq \varphi\big[\|\mathbf{P}_t(g(P^j y_1, P^j y_2) - K^{ij}P^j y)\|_{L^2_\gamma} + \|\mathbf{P}_t v\|_{L^2_\gamma}\big] \quad \forall\, t \geq 0.$$

Therefore, by (4.5),

$$e^{\gamma t}\|x(t) - \mathbb{T}^K_t x(0)\| \leq \varphi\big[\nu\|\mathbf{P}_t y\|_{L^2_\gamma} + \|\mathbf{P}_t v\|_{L^2_\gamma}\big] \quad \forall\, t \geq 0.$$

Invoking statement (1), it follows that

$$e^{\gamma t}\|x(t) - \mathbb{T}^K_t x(0)\| \leq \varphi\nu\Gamma\|x(0)\| + (\varphi\nu\Gamma + 1)\|\mathbf{P}_t v\|_{L^2_\gamma} \quad \forall\, t \geq 0.$$

With $M \geq 1$ such $\|\mathbb{T}^K_t\| \leq Me^{-\gamma t}$ for all $t \geq 0$, we obtain

$$\|x(t)\| \leq \Gamma^{\mathrm{s}} e^{-\gamma t}\big[\|x(0)\| + \|\mathbf{P}_t v\|_{L^2_\gamma}\big] \quad \forall\, t \geq 0, \tag{4.11}$$

where $\Gamma^{\mathrm{s}} := \varphi\nu\Gamma + M$.

Finally, if $q \in (2, \infty)$, then there exists $p \in (1, \infty)$ such that $2/q + 1/p = 1$, and, by Hölder's inequality,

$$\|\mathbf{P}_t v\|^2_{L^2_\gamma} \leq \left(\int_0^t \|v(\tau)\|^q \mathrm{d}\tau\right)^{\frac{2}{q}} \left(\int_0^t e^{2p\gamma\tau}\mathrm{d}\tau\right)^{\frac{1}{p}} \leq \frac{e^{2\gamma t}}{(2p\gamma)^{1/p}}\|\mathbf{P}_t v\|^2_{L^q} \quad \forall\, t \geq 0,$$

yielding

$$\|\mathbf{P}_t v\|_{L^2_\gamma} \leq \frac{e^{\gamma t}}{(2p\gamma)^{1/2p}}\|\mathbf{P}_t v\|_{L^q} \quad \forall\, t \geq 0.$$

Trivially, for $q = 2, \infty$,

$$\|\mathbf{P}_t v\|_{L^2_\gamma} \leq e^{\gamma t}\|\mathbf{P}_t v\|_{L^2} \ \ \forall\, t \geq 0 \quad \text{and} \quad \|\mathbf{P}_t v\|_{L^2_\gamma} \leq \frac{e^{\gamma t}}{(2\gamma)^{1/2}}\|\mathbf{P}_t v\|_{L^\infty} \ \ \forall\, t \geq 0.$$

Consequently, for every q with $2 \leq q \leq \infty$, there exists a positive constant N_q such that

$$\|\mathbf{P}_t v\|_{L^2_\gamma} \leq N_q e^{\gamma t}\|\mathbf{P}_t v\|_{L^q} \quad \forall\, t \geq 0,$$

and hence, appealing to (4.11),

$$\|x(t)\| \leq \Gamma^{\mathrm{s}} e^{-\gamma t}\|x(0)\| + \Gamma^{\mathrm{s}} N_q\|\mathbf{P}_t v\|_{L^q} \quad \forall\, t \geq 0.$$

Statement (2) now follows with $\Gamma_q := \Gamma^{\mathrm{s}}\max\{1, N_q\}$.

We proceed to prove statements (3)–(5). Let $\rho \in (0, r)$ and consider all f with $\nu(f, Z_1, Z_2) \leq \rho$. An inspection of the arguments establishing the existence of the constants Γ, ε, Γ_q and γ in the above proofs of statements (1) and (2) shows that statement (3) holds. Finally, to establish statements (4) and (5), let $(v_1, x_1, y_1) \in \mathcal{B}^q_{Z_1}$ and $(v_2, x_2, y_2) \in \mathcal{B}^q_{Z_2}$, where $2 \le q \le \infty$. We consider two cases: $\alpha = 0$ and $\alpha > 0$

CASE 1: $\alpha = 0$. Assume that $v_1 - v_2 \in L^q(\mathbb{R}_+, U)$ if $q < \infty$ and that $v_1(t) - v_2(t) \to 0$ as $t \to \infty$ if $q = \infty$. Setting $u_k := E^i(f \circ P^j y_k) + v$ for $k = 1, 2$, it is clear that

$$x_k(t) = \mathbb{T}_t x_k(0) + \Phi_t u_k, \quad \mathbf{P}_t y = \Psi_t x_k(0) + \mathbb{G}_t u_k; \quad \forall\, t \ge 0,\, k = 1, 2.$$

It follows from the theory of well-posed linear systems that, for every $\tau \ge 0$,

$$(\mathbf{L}_\tau x_k)(t) = \mathbb{T}_t x_k(\tau) + \Phi_t(\mathbf{L}_\tau u_k),\;\; \mathbf{P}_t(\mathbf{L}_\tau y) = \Psi_t x_k(\tau) + \mathbb{G}_t(\mathbf{L}_\tau u_k);\; \forall\, t \ge 0,\, k = 1, 2,$$

where we remind the reader that $\mathbf{L}_\tau$ denotes the left-shift operator. Consequently, the triples $(\mathbf{L}_\tau v_1, \mathbf{L}_\tau x_1, \mathbf{L}_\tau y_1)$ and $(\mathbf{L}_\tau v_2, \mathbf{L}_\tau x_2, \mathbf{L}_\tau y_2)$ are in $\mathcal{B}^q_{Z_1}$ and $\mathcal{B}^q_{Z_2}$, respectively. It follows from statement (2) that

$$\begin{aligned} \|x_1(t+\tau) - x_2(t+\tau)\| \le \Gamma_q\big(&e^{-\gamma t}\|x_1(\tau) - x_2(\tau)\| \\ &+ \|\mathbf{P}_t(\mathbf{L}_\tau v_1 - \mathbf{L}_\tau v_2)\|_{L^q}\big) \quad \forall\, t \ge 0\,. \end{aligned} \tag{4.12}$$

Let $\delta > 0$ and choose $\tau > 0$ and $\sigma > 0$ such that

$$\Gamma_q e^{-\gamma\sigma}\|x_1(\tau) - x_2(\tau)\| \le \delta/2 \quad \text{and} \quad \Gamma_q\|\mathbf{L}_\tau v_1 - \mathbf{L}_\tau v_2\|_{L^q} \le \delta/2\,.$$

Then, $\|x_1(t) - x_2(t)\| \le \delta$ for all $t \ge \tau + \sigma$. Since $\delta > 0$ was arbitrary, this shows that $x_1(t) - x_2(t) \to 0$ as $t \to \infty$.

CASE 2: $\alpha > 0$. Assume that $v_1 - v_2 \in L^q_\alpha(\mathbb{R}_+, U)$ if $q < \infty$ and that $e^{\alpha t}(v_1(t) - v_2(t)) \to 0$ as $t \to \infty$ if $q = \infty$. Let $\beta > 0$ be such that $2\beta \le \min\{\alpha, \gamma\}$. Writing

$$e^{\beta t}\|x_1(t) - x_2(t)\| = e^{2\beta(t/2)}\|x_1(t/2 + t/2) - x_2(t/2 + t/2)\|,$$

and invoking (4.12) with t and τ both replaced by $t/2$, a routine calculation gives

$$e^{\beta t}\|x_1(t) - x_2(t)\| \le \Gamma_q\big(e^{(2\beta-\gamma)t/2}\|x_1(t/2) - x_2(t/2)\| + \|v_1 - v_2\|_{L^q_{2\beta}(t/2,t)}\big) \;\; \forall\, t \ge 0.$$

By Case 1, the function $x_1 - x_2$ is bounded and, so by choice of β, the right-hand side of the above estimate is bounded, showing that $x_1(t) - x_2(t)$ converges to 0 exponentially fast as $t \to \infty$. ☐

Remark 4.2. (1) An inspection of the above proof shows that if X, U and Y are real Hilbert spaces, then Theorem 4.1 remains true, provided that the *complex* ball condition $\mathbb{B}^{\mathrm{c}}(K^{ij}, r) \subset \mathbb{S}^{\mathrm{c}}(P^j \mathbf{G} E^i)$ holds in the context of the complexifications U_i^{c} and Y_j^{c} of U^i and Y^j, respectively. Here $\mathbb{B}_{\mathrm{c}}(K^{ij}, r) := \{F \in \mathcal{L}(Y_j^{\mathrm{c}}, U_i^{\mathrm{c}}) : \|F - K^{ij}\| < r\}$ and $\mathbb{S}^{\mathrm{c}}(P^j \mathbf{G} E^i) := \{F \in \mathcal{L}(Y_j^{\mathrm{c}}, U_i^{\mathrm{c}}) : F \text{ stabilizes } P^j \mathbf{G} E^i\}$. Similar comments apply to the corollaries of Theorem 4.1 which will be presented below.

(2) Assume that $(v^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ is an equilibrium of (3.1), the assumptions of Theorem 4.1 hold with $Z_1 = Y^j$ and $Z_2 = \{P^j y^{\mathrm{e}}\}$ and (3.1) has the blow-up property. An inspection of the proof of Theorem 4.1 reveals that, under these conditions, every maximally defined solution (x, y) of (3.1) is global. ♢

Theorem 4.1 has an obvious small-gain interpretation which we now state in the form of a corollary.

COROLLARY 4.3. *Let Σ, f, Z_1 and Z_2 be as in* Theorem 4.1, *let $i, j \in \{1,2\}$ and $K^{ij} \in \mathbb{S}(P^j \mathbb{G} E^i)$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j \Phi, P^j \mathbb{G} E^i)$ is optimizable and estimatable. If*

$$\sup_{(z_1,z_2)\in Z_1\times Z_2,\, z_1\neq z_2} \frac{\|f(z_1) - f(z_2) - K^{ij}(z_1 - z_2)\|}{\|z_1 - z_2\|} \|(P^j \mathbf{G} E^i)^{K^{ij}}\|_{H^\infty} < 1\,,$$

then statements (1), (2) *and* (4) *of Theorem 4.1 hold.*

Proof. Defining $r := 1/\|P^j \mathbf{G} E^i)^{K^{ij}}\|_{H^\infty}$, an application of Lemma 2.2 yields that $\mathbb{B}(K^{ij}, r) \subset \mathbb{S}(P^j \mathbf{G} E^i)$ and the claim follows from Theorem 4.1. ∎

The remainder of the section is dedicated to proving further corollaries of Theorem 4.1, namely versions of the so-called circle criterion, the first of which is Corollary 4.5 below. For this we need the familiar frequency-domain concept of positive realness, a recent treatment of which in an infinite-dimensional setting may be found in [8].

Let H be a complex Hilbert space. We say that $\mathbf{H} : \mathbb{C}_0 \to \mathcal{L}(H)$ is *positive real* if $\mathbf{H} : \mathbb{C}_0 \to \mathcal{L}(H)$ is holomorphic on $\mathbb{C}_0$ with the exception of isolated singularities and

$$\mathbf{H}(s) + [\mathbf{H}(s)]^* \succeq 0 \text{ for all } s \in \mathbb{C}_0 \text{ which are not singularities of } \mathbf{H}.$$

In fact, it is known that if $\mathbf{H}$ as above is positive real, then $\mathbf{H}$ is holomorphic on $\mathbb{C}_0$, see [8, Proposition 3.3]. The next technical lemma, the proof of which may be found in [8, Corollary 3.7], is well-known in the rational case and demonstrates that the so-called Cayley transform maps positive-real functions to contractive H^∞ functions (also called bounded-real functions in the control theory literature).

LEMMA 4.4. *If $\mathbf{H} : \mathbb{C}_0 \to \mathcal{L}(H)$ is positive real, then $I + \mathbf{H}(s)$ is invertible for every $s \in \mathbb{C}_0$ and*

$$\|(I - \mathbf{H})(I + \mathbf{H})^{-1}\|_{H^\infty} \leq 1\,.$$

Lemma 4.4 will be used in the proof of the following version of the circle criterion.

COROLLARY 4.5. *Let Σ, f, Z_1 and Z_2 be as in* Theorem 4.1, *let $i, j \in \{1,2\}$ and $K_1, K_2 \in \mathcal{L}(Y^j, U^i)$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j \Phi, P^j \mathbb{G} E^i)$ is optimizable and estimatable, K_1 is an admissible feedback operator for Σ^{ji} and $Z_2 = Y^j$. If $(I - K_2 P^j \mathbf{G} E^i)(I - K_1 P^j \mathbf{G} E^i)^{-1}$ is positive real and there exists $\varepsilon > 0$ such that*

$$\text{(4.13)} \quad \begin{cases} \operatorname{Re}\langle f(z_1) - f(z_2) - K_1(z_1 - z_2), f(z_1) - f(z_2) - K_2(z_1 - z_2)\rangle \\ \qquad\qquad \leq -\varepsilon\|z_1 - z_2\|^2 \quad \forall\,(z_1, z_2) \in Z_1 \times Y^j\,, \end{cases}$$

then statements (1), (2) *and* (4) *of* Theorem 4.1 *hold* (*with $Z_2 = Y^j$*).

Proof. The idea underlying the proof is to apply Theorem 4.1 to a certain Lur'e system closely related to (3.1) (see (4.17) below) by suitably exploiting the sector condition (4.13) and obtaining a condition of the form (4.1) for (4.17). From this we will able to deduce the claimed stability properties of the original Lur'e system (3.1). To this end, we define $g : Y^j \times Y^j \to U^i$ as in (4.4), and observe that the sector condition (4.13) can be written in the form

$$\text{(4.14)} \quad \begin{cases} \operatorname{Re}\langle g(z_1, z_2) - K_1(z_1 - z_2), g(z_1, z_2) - K_2(z_1 - z_2)\rangle \leq -\varepsilon\|z_1 - z_2\|^2 \\ \qquad\qquad \forall\,(z_1, z_2) \in Z_1 \times Y^j. \end{cases}$$

Setting

$$L := \frac{1}{2}(K_1 - K_2) \in \mathcal{L}(Y^j, U^i) \quad \text{and} \quad M := \frac{1}{2}(K_1 + K_2) \in \mathcal{L}(Y^j, U^i)\,,$$

we rewrite the left-hand side of the sector condition (4.14) in terms of L and M:

$$\begin{aligned}\operatorname{Re}\langle g(z_1,z_2)-&K_1(z_1-z_2), g(z_1,z_2)-K_2(z_1-z_2)\rangle\\ &= \operatorname{Re}\langle g(z_1,z_2)-(L+M)(z_1-z_2), g(z_1,z_2)+(L-M)(z_1-z_2)\rangle\\ &= -\|L(z_1-z_2)\|^2+\|g(z_1,z_2)-M(z_1-z_2)\|^2 \quad \forall\,(z_1,z_2)\in Z_1\times Y^j\,. \end{aligned} \tag{4.15}$$

It now follows from (4.14) that $\|Lz\|\geq\sqrt{\varepsilon}\|z\|$ for all $z\in Y^j$. Consequently

$$\|L^*Lz\|\|z\|\geq|\langle L^*Lz,z\rangle|=\|Lz\|^2\geq\varepsilon\|z\|^2 \quad \forall\,z\in Y^j\,,$$

and so, $\|L^*Lz\|\geq\varepsilon\|z\|$ for all $z\in Y^j$, showing that L^*L is bounded away from 0. Combining this with the self-adjointness of L^*L, yields that L^*L is invertible. Consequently, $L^\sharp:=(L^*L)^{-1}L^*\in\mathcal{L}(U^i,Y^j)$ is a left inverse of L. Setting $\mathbf{H}:=P^j\mathbf{G}E^i$ and exploiting the positive realness of $(I-K_2\mathbf{H})(I-K_1\mathbf{H})^{-1}$ via Lemma 4.4 yields

$$\|L\mathbf{H}(I-M\mathbf{H})^{-1}\|_{H^\infty}\leq 1.$$

Trivially, $L\mathbf{H}(I-M\mathbf{H})^{-1}=L\mathbf{H}(I-ML^\sharp L\mathbf{H})^{-1}$, and so, appealing to Lemma 2.2,

$$\mathbb{B}(ML^\sharp,1)\subset\mathbb{S}(L\mathbf{H}). \tag{4.16}$$

Let $\sigma:\{1,2\}\to\{1,2\}$ be the permutation $\sigma(1)=2$ and $\sigma(2)=1$, set $\tilde{Y}:=U^i\times Y^{\sigma(j)}$ and introduce the maps

$$\tilde{P}^1:\tilde{Y}\to U^i,\ \begin{pmatrix}u\\y\end{pmatrix}\mapsto u \quad\text{and}\quad \tilde{L}:Y\to\tilde{Y},\ y\mapsto\begin{pmatrix}LP^jy\\P^{\sigma(j)}y\end{pmatrix}.$$

Note that $\tilde{L}$ is left-invertible owing to the left-invertibility of L. Furthermore, we define $\tilde{f}:U^i\to U^i$ by

$$\tilde{f}(z):=f(L^\sharp z) \quad \forall\,z\in U^i\,.$$

Since $\tilde{P}^1\tilde{L}=LP^j$, it follows that

$$\tilde{f}(\tilde{P}^1\tilde{L}z)=f(P^jz) \quad \forall\,z\in Y\,.$$

and thus, for all $(v,x,y)\in\mathcal{B}$,

$$\left.\begin{aligned} x(t)&=\mathbb{T}_tx(0)+\Phi_t\big(E^i[\tilde{f}\circ\tilde{P}^1(\tilde{L}y)]+v\big)\\ \mathbf{P}_t\tilde{L}y&=\tilde{L}\Psi_tx(0)+\tilde{L}\mathbb{G}_t\big(E^i[\tilde{f}\circ\tilde{P}^1(\tilde{L}y)]+v\big)\end{aligned}\right\}\quad\forall\,t\geq 0\,.$$

Therefore, letting $\tilde{\mathcal{B}}$ denote the behaviour of the Lur'e system

$$\left.\begin{aligned} x(t)&=\mathbb{T}_tx(0)+\Phi_t\big(E^i[\tilde{f}\circ\tilde{P}^1w]+v\big)\\ \mathbf{P}_tw&=\tilde{L}\Psi_tx(0)+\tilde{L}\mathbb{G}_t\big(E^i[\tilde{f}\circ\tilde{P}^1w]+v\big)\end{aligned}\right\}\quad\forall\,t\geq 0, \tag{4.17}$$

we conclude that

$$(v,x,\tilde{L}y)\in\tilde{\mathcal{B}} \quad \forall\,(v,x,y)\in\mathcal{B}\,. \tag{4.18}$$

Furthermore, setting $\tilde{Z}_k:=LZ_k\subset U^i$ and

$$\tilde{\mathcal{B}}_{\tilde{Z}_k}:=\{(\tilde{v},\tilde{x},\tilde{y})\in\tilde{\mathcal{B}}:(\tilde{P}^1\tilde{y})(t)\in\tilde{Z}_k \text{ for a.e. } t\geq 0\}\,,$$

where $k = 1, 2$, it follows from the identity $\tilde{P}^1\tilde{L} = LP^j$ that

$$(v, x, \tilde{L}y) \in \tilde{\mathcal{B}}_{\tilde{Z}_k} \quad \forall\, (v, x, y) \in \mathcal{B}_{Z_k}\,.$$

The underlying well-posed linear system of the Lur'e system (4.17) is

$$\tilde{\Sigma} := (\mathbb{T}, \Phi, \tilde{L}\Psi, \tilde{L}\mathbb{G})\,.$$

The system $\tilde{\Sigma}$ has output space $\tilde{Y}$ and transfer function $\tilde{\mathbf{G}} := \tilde{L}\mathbf{G}$. Using the identity $\tilde{P}^1\tilde{L} = LP^j$ again, we obtain

$$\tilde{P}^1\tilde{\mathbf{G}}E^i = LP^j\mathbf{G}E^i = L\mathbf{H}\,. \tag{4.19}$$

Let $\zeta_k \in \tilde{Z}_k$ and let $z_k \in Z_k$ be such that $\zeta_k = Lz_k$, where $k = 1, 2$. Defining

$$\tilde{g} : U^i \times U^i \to U^i,\ (w_1, w_2) \mapsto \tilde{f}(w_1) - \tilde{f}(w_2)\,,$$

it follows that

$$\tilde{g}(\zeta_1, \zeta_2) - ML^\sharp(\zeta_1 - \zeta_2) = f(z_1) - f(z_2) - M(z_1 - z_2) = g(z_1, z_2) - M(z_1 - z_2)\,.$$

Therefore, by (4.14) and (4.15),

$$\begin{aligned} \|\tilde{g}(\zeta_1, \zeta_2) - ML^\sharp(\zeta_1 - \zeta_2)\|^2 &\le \|L(z_1 - z_2)\|^2 - \varepsilon\|z_1 - z_2\|^2 \\ &= \|LL^\sharp(\zeta_1 - \zeta_2)\|^2 - \varepsilon\|L^\sharp(\zeta_1 - \zeta_2)\|^2 \quad \forall\, (\zeta_1, \zeta_2) \in \tilde{Z}_1 \times \tilde{Z}_2. \end{aligned} \tag{4.20}$$

Since L has a left inverse, $\operatorname{im} L$ is closed and so,

$$\operatorname{im} L = (\ker L^*)^\perp = (\ker L^\sharp)^\perp.$$

It is now straightforward to show that $Q := LL^\sharp \in \mathcal{L}(U^i)$ is the orthogonal projection onto $(\ker L^\sharp)^\perp$ along $\ker L^\sharp$. Consequently, invoking (4.20),

$$\|\tilde{g}(\zeta_1, \zeta_2) - ML^\sharp(\zeta_1 - \zeta_2)\|^2 \le \|Q(\zeta_1 - \zeta_2)\|^2 - \varepsilon\|L^\sharp Q(\zeta_1 - \zeta_2)\|^2 \quad \forall\, (\zeta_1, \zeta_2) \in \tilde{Z}_1 \times \tilde{Z}_2.$$

Since $LL^\sharp Qz = Q^2 z = Qz$ for all $z \in U^i$, it follows that there exists $c > 0$ such that

$$\|L^\sharp Qz\| \ge c\|Qz\| \quad \forall\, z \in U^i.$$

Therefore,

$$\|\tilde{g}(\zeta_1, \zeta_2) - ML^\sharp(\zeta_1 - \zeta_2)\|^2 \le (1 - \varepsilon c)\|Q(\zeta_1 - \zeta_2)\|^2 \le (1 - \varepsilon c)\|\zeta_1 - \zeta_2\|^2 \quad \forall\, (\zeta_1, \zeta_2) \in \tilde{Z}_1 \times \tilde{Z}_2.$$

and so,

$$\|\tilde{g}(\zeta_1, \zeta_2) - ML^\sharp(\zeta_1 - \zeta_2)\| \le \delta\|\zeta_1 - \zeta_2\| \quad \forall\, (\zeta_1, \zeta_2) \in \tilde{Z}_1 \times \tilde{Z}_2,$$

where $\delta := \sqrt{1 - \varepsilon c} \in (0, 1)$. Consequently,

$$\sup_{(\zeta_1, \zeta_2) \in \tilde{Z}_1 \times \tilde{Z}_2,\, \zeta_1 \neq \zeta_2} \frac{\|\tilde{f}(\zeta_1) - \tilde{f}(\zeta_2) - ML^\sharp(\zeta_1 - \zeta_2)\|}{\|\zeta_1 - \zeta_2\|} < 1. \tag{4.21}$$

In view of (4.16), (4.18), (4.19) and (4.21) combined with the left-invertibility of $\tilde{L}$ and the fact that

$$\tilde{\Sigma}^{1i} := (\mathbb{T}, \Phi E^i, \tilde{P}^1\tilde{L}\Psi, \tilde{P}^1\tilde{L}\mathbb{G}E^i) = (\mathbb{T}, \Phi E^i, LP^j\Psi, LP^j\mathbb{G}E^i)\,,$$

is optimizable and estimatable (which follows from the optimizability and estimatability of $(\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ and the left-invertibility of L), the claim follows from an application of Theorem 4.1 to (4.17). ∎

Remark 4.6. (1) Corollary 4.5 remains valid if the roles of Z_1 and Z_2 are interchanged, that is, if $Z_1 = Y^j$, Z_2 is a non-empty subset of Y^j and the inequality in (4.13) holds for all $(z_1, z_2) \in Y^j \times Z_2$.

(2) An inspection of the first part of the proof shows that the conclusions of Corollary 4.5 continue to hold if the assumption $Z_2 = Y^j$ is dropped and, instead, it is assumed that $K_1 - K_2$ is left invertible.

(3) As can be seen from an inspection of the proof of [8, Theorem 6.8], the following converse of Corollary 4.5 holds: if the set of functions $f : Y^j \to U^i$ satisfying (4.13) is non-empty and every f in this set is stabilizing in the sense that statements (1) or (2) of Theorem 4.1 hold, then $(I - K_2 P^j \mathbf{G} E^i)(I - K_1 P^j \mathbf{G} E^i)^{-1}$ is positive real. We emphasize that, in this context, it is crucial that U and Y are complex Hilbert spaces, in the case of real Hilbert spaces, the converse is not true in general. ◊

We next present alternative formulations of the circle criterion, seeking to demonstrate the interplay between the various hypotheses made. To do so requires some additional terminology: we say that a positive-real function $\mathbf{H} : \mathbb{C}_0 \to \mathcal{L}(H)$ (H a complex Hilbert space) is *strongly positive real* if there exists $\delta > 0$ such that

$$\mathbf{H}(s) + [\mathbf{H}(s)]^* \succeq \delta I \quad \forall\, s \in \mathbb{C}_0 \,,$$

The next result is a variant of Corollary 4.5 and, loosely, relaxes the sector condition imposed on f at the expense of strengthening the positive-real assumption.

COROLLARY 4.7. *Let* Σ, f, Z_1 *and* Z_2 *be as in* Theorem 4.1*, let* $i, j \in \{1, 2\}$ *and* $K_1, K_2 \in \mathcal{L}(Y^j, U^i)$*. Assume that* $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j \Phi, P^j \mathbb{G} E^i)$ *is optimizable and estimatable,* K_1 *is an admissible feedback operator for* Σ^{ji} *and* $K_1 - K_2$ *is left invertible. If the function* $(I - K_2 P^j \mathbf{G} E^i)(I - K_1 P^j \mathbf{G} E^i)^{-1}$ *is in* $H^\infty(\mathcal{L}(U^i))$ *and is strongly positive real, and*

$$\text{(4.22)} \qquad \begin{cases} \operatorname{Re} \langle f(z_1) - f(z_2) - K_1(z_1 - z_2), f(z_1) - f(z_2) - K_2(z_1 - z_2) \rangle \le 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, (z_1, z_2) \in Z_1 \times Z_2 \,, \end{cases}$$

then statements (1)*,* (2) *and* (4) *of* Theorem 4.1 *hold.*

We comment that Corollary 4.7 above overlaps with the ISS results in [14], particularly [14, Theorem 4.5], and we provide some comparisons before giving the proof. Whilst the feedback configuration considered in [14] is less general then the four-block structure studied in the present paper, the linear component of the Lur'e systems analyzed in [14] is a general well-posed system. However, the main result [14, Theorem 4.5] fails to provide a clear-cut generalization of the circle-criterion, with the exception of the case wherein K_1 and K_2 are scalar multiples of the identity; see [14, Corollary 4.7].

Proof of Corollary 4.7. Set $M := K_1 - K_2 \in \mathcal{L}(Y^j, U^i)$ and with $\mathbf{H}$ given by

$$\text{(4.23)} \qquad \mathbf{H} := (I - K_2 P^j \mathbf{G} E^i)(I - K_1 P^j \mathbf{G} E^i)^{-1} \,,$$

we observe that

$$\mathbf{H} = I + M P^j \mathbf{G} E^i (I - K_1 P^j \mathbf{G} E^i)^{-1} \in H^\infty(\mathcal{L}(U^i)) \,.$$

Since M is left invertible, we conclude that $K_1 \in \mathbb{S}(P^j \mathbf{G} E^i)$. Together with the openness of $\mathbb{S}(P^j \mathbf{G} E^i)$, this yields the existence of a number $\nu^* > 0$ such that $K_1 + \nu M \in \mathbb{S}(P^j \mathbf{G} E^i)$ for all $\nu \in [0, \nu^*]$. Defining

$$\mathbf{H}_\nu := \big(I - (K_2 - \nu M) P^j \mathbf{G} E^i\big)\big(I - (K_1 + \nu M) P^j \mathbf{G} E^i\big)^{-1} \,,$$

it is clear that the map $[0,\nu^*] \to H^\infty(\mathcal{L}(U))$, $\nu \mapsto \mathbf{H}_\nu$ is continuous. Combined with the strong positive realness of $\mathbf{H} = \mathbf{H}_0$, this shows that there exists $\nu^{**} \in (0,\nu^*]$ such that

$$\mathbf{H}_\nu(s) + [\mathbf{H}_\nu(s)]^* \succeq 0 \quad \forall\, s \in \mathbb{C}_0,\ \forall\, \nu \in [0,\nu^{**}]\,. \tag{4.24}$$

It is convenient to define, for all $(z_1,z_2) \in Y^j \times Y^j$,

$$S_\nu(z_1,z_2) := \langle f(z_1) - f(z_2) - (K_1 + \nu M)(z_1 - z_2), f(z_1) - f(z_2) - (K_2 - \nu M)(z_1 - z_2)\rangle\,.$$

In light of (4.24), the claim will follow from Corollary 4.5, provided that we can show that, for $\nu \in (0,\nu^{**}]$, there exists $\varepsilon > 0$ such that

$$\operatorname{Re} S_\nu(z_1,z_2) \le -\varepsilon \|z_1 - z_2\|^2 \quad \forall\,(z_1,z_2) \in Z_1 \times Z_2\,. \tag{4.25}$$

Invoking (4.22), a straightforward calculation shows that

$$\operatorname{Re} S_\nu(z_1,z_2) \le -\nu(\nu+1)\|M(z_1 - z_2)\|^2 \quad \forall\,(z_1,z_2) \in Z_1 \times Z_2\,.$$

By left-invertibility of M, there exists $\mu > 0$ such that $\|Mz\| \ge \mu\|z\|$ for all $z \in Y^j$, and so,

$$\operatorname{Re} S_\nu(z_1,z_2) \le -\mu\nu(\nu+1)\|z\|^2 \quad \forall\,(z_1,z_2) \in Z_1 \times Z_2\,.$$

showing that (4.25) holds with $\varepsilon := \mu\nu(\nu+1)$, completing the proof. □

The "classical" circle criterion which guarantees global asymptotic stability, see, for example, [9, Theorem 5.1], [10, Corollary 5.8] and [18, Theorem 7.1], is typically formulated in terms of the concept of strict positive realness of the function in (4.23). We recall that for a complex Hilbert space H and $\alpha > 0$, a function $\mathbf{H} : \mathbb{C}_{-\alpha} \to \mathcal{L}(H)$ is said to be *strictly positive real* if there exists $\beta \in (0,\alpha]$ such that the function $s \mapsto \mathbf{H}(s - \beta)$ is positive real.

We will show that a faithful infinite-dimensional generalization of the classical circle criterion follows from Corollary 4.7. To this end, we state the following lemma, the proof of which can be found in [8].

Lemma 4.8. *Let $\alpha > 0$, let H be a complex Hilbert space and assume that $\mathbf{H} : \mathbb{C}_{-\alpha} \to \mathcal{L}(H)$ is holomorphic with the exception of isolated singularities. Then the following statements hold.*

(1) *If $\mathbf{H}$ is strictly positive real,*

$$\limsup_{|s|\to\infty,\, s \in \mathbb{C}_{-\beta}} \|\mathbf{H}(s)\| < \infty \quad \text{for some } \beta \in (0,\alpha], \tag{4.26}$$

and

$$\liminf_{|\omega|\to\infty,\, \omega\in\mathbb{R}} \Big[\inf_{\|u\|=1} \operatorname{Re}\langle \mathbf{H}(i\omega)u, u\rangle\Big] > 0, \tag{4.27}$$

then there exist $\varepsilon > 0$ and $\delta > 0$ such that $\mathbf{H} \in H^\infty_{-\varepsilon}(\mathcal{L}(H))$ and

$$\mathbf{H}(s) + [\mathbf{H}(s)]^* \succeq \delta I \quad \forall\, s \in \mathbb{C}_{-\varepsilon}.$$

In particular, $\mathbf{H}$ is strongly positive real.

(2) *If there exist $\varepsilon > 0$ and $\delta > 0$ such that $\mathbf{H} \in H^\infty_{-\varepsilon}(\mathcal{L}(H))$ and*

$$\mathbf{H}(i\omega) + [\mathbf{H}(i\omega)]^* \succeq \delta I \quad \forall\, \omega \in \mathbb{R},$$

then $\mathbf{H}$ is strictly positive real.

By way of commentary, if $\mathbf{H}$ is of the form $\mathbf{H} = (I - K_2 P^j \mathbf{G} E^i)(I - K_1 P^j \mathbf{G} E^i)^{-1}$ (cf. Corollaries 4.5 and 4.7), then a sufficient condition for $\mathbf{H}$ to satisfy (4.26) and (4.27) is, for example, given by

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_{-\varepsilon}} \|K_k P^j \mathbf{G}(s) E^i\| < \sqrt{2} - 1 \quad \text{for some } \varepsilon > 0,\ k = 1, 2.$$

Combining Corollary 4.7 and Lemma 4.8, we obtain the final result of this section, which is a faithful infinite-dimensional generalization of the classical circle criterion.

Corollary 4.9. *Let Σ, f, Z_1 and Z_2 be as in* Theorem 4.1*, let $i, j \in \{1, 2\}$ and $K_1, K_2 \in \mathcal{L}(Y^j, U^i)$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j \Phi, P^j \mathbb{G} E^i)$ is optimizable and estimatable, K_1 is an admissible feedback operator for Σ^{ji} and $K_1 - K_2$ is left invertible. If $\mathbf{H} := (I - K_2 P^j \mathbf{G} E^i)(I - K_1 P^j \mathbf{G} E^i)^{-1}$ is strictly positive real and satisfies conditions* (4.26) *and* (4.27) *and f satisfies the incremental sector condition* (4.22)*, then statements* (1)*,* (2) *and* (4) *of* Theorem 4.1 *hold.*

Interestingly, Corollaries 4.7 and 4.9 show that the conditions of the circle criterion are actually sufficient for ISS. Moreover, it is not difficult to see that if $K_1 - K_2$ is not left-invertible, then, in general, the conclusions of these corollaries do not hold.

5. Convergence properties. The penultimate section concentrates on establishing convergence properties of (3.1). Our main results are Theorems 5.2 and 5.4 which, roughly, state that the hypotheses of the small-gain Corollary 4.3 are sufficient for (3.1) to exhibit the CICS property and convergence to periodic states and outputs when the forcing is periodic, respectively. In this section, if $K \in \mathcal{L}(Y, U)$ is an admissible feedback operator for the well-posed linear system Σ, then we let (A^K, B^K, C^K) denote the generating operators of Σ^K.

The map F_K defined in (5.1) below will play an important role in the following. The next result shows how this map relates to equilibria of (3.1).

Proposition 5.1. *Let $i, j \in \{1, 2\}$, $K^{ij} \in \mathbb{S}(P^j \mathbf{G} E^i)$ and $K \in \mathcal{L}(Y, U)$ be given by* (2.4)*, and define the map*

$$F_K : Y^j \to Y^j,\ z \mapsto z - P^j \mathbf{G}^K(0)(E^i f(z) - E^i K^{ij} z)\,. \tag{5.1}$$

Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j \Psi, P^j \mathbb{G} E^i)$ is optimizable and estimatable. Then $\mathbb{T}^K = \mathbb{T}^{K^{ij}}$ is exponentially stable and the following statements hold.

(1) *If $(v^e, x^e, y^e) \in U \times X \times Y$ is an equilibrium of* (3.1)*, then*

$$x^e = -(A^K)^{-1} B^K u^e,\ y^e = \mathbf{G}^K(0) u^e,\ \text{where } u^e := E^i(f(P^j y^e) - K^{ij} P^j y^e) + v^e,$$

and $F_K(P^j y^e) = P^j \mathbf{G}^K(0) v^e$.

(2) *Let $v^e \in U$, assume that there exists $z^e \in Y^j$ such that $F_K(z^e) = P^j \mathbf{G}^K(0) v^e$, and define*

$$x^e := -(A^K)^{-1} B^K w^e,\ y^e := \mathbf{G}^K(0) w^e,\ \text{where } w^e := E^i(f(z^e) - K^{ij} z^e) + v^e.$$

Then $P^j y^e = z^e$ and the triple (v^e, x^e, y^e) is an equilibrium of (3.1)*.*

Proof. It follows as in the proof of Theorem 4.1 that $\mathbb{T}^K = \mathbb{T}^{K^{ij}}$ is exponentially stable, and so $\omega(\mathbb{T}^K) < 0$. To prove statement (1), let (v^e, x^e, y^e) be an equilibrium of (3.1). By Corollary 3.2, we have that (v^e, x^e, y^e) is also an equilibrium of the Lur'e system

$$
(5.2)\qquad \begin{cases} x(t) = \mathbb{T}^K_t x(0) + \Phi^K_t\big(E^i(f\circ P^j y) - Ky + v\big)\\ \mathbf{P}_t y = \Psi^K_t x(0) + \mathbb{G}^K_t\big(E^i(f\circ P^j y) - Ky + v\big). \end{cases}
$$

Since $\mathbb{T}^K$ is exponentially stable, we may apply Proposition 3.3 with $\eta = 0$ to the Lur'e system (5.2), yielding the formulas for x^{e} and y^{e}. Furthermore,

$$
F_K(P^j y^{\mathrm{e}}) = P^j\mathbf{G}^K(0)u^{\mathrm{e}} - P^j\mathbf{G}^K(0)E^i(f(P^j y^{\mathrm{e}}) - K^{ij}P^j y^{\mathrm{e}}) = P^j\mathbf{G}^K(0)v^{\mathrm{e}},
$$

completing the proof of statement (1).

To prove statement (2), note that

$$
P^j y^{\mathrm{e}} = P^j\mathbf{G}^K(0)\big(E^i(f(z^{\mathrm{e}}) - K^{ij}z^{\mathrm{e}}) + v^{\mathrm{e}}\big) = z^{\mathrm{e}} - F_K(z^{\mathrm{e}}) + P^j\mathbf{G}^K(0)v^{\mathrm{e}} = z^{\mathrm{e}}.
$$

Furthermore,

$$
A^K x^{\mathrm{e}} + B^K w^{\mathrm{e}} = 0,\;\; y^{\mathrm{e}} = C^K(x^{\mathrm{e}} + (A^K)^{-1}B^K w^{\mathrm{e}}) + \mathbf{G}^K(0)w^{\mathrm{e}},
$$

and so it follows from Proposition 3.3 that the constant trajectory $(w(t), x(t), y(t)) \equiv (w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ satisfies

$$
x(t) = \mathbb{T}^K_t x(0) + \Phi^K_t w,\;\; \mathbf{P}_t y = \Psi^K_t x(0) + \mathbb{G}^K_t w, \quad \forall\, t \geq 0.
$$

Since $P^j y^{\mathrm{e}} = z^{\mathrm{e}}$, we have that $w^{\mathrm{e}} = E^i(f(P^j y^{\mathrm{e}}) - K^{ij}P^j y^{\mathrm{e}}) + v^{\mathrm{e}}$ and it follows from another application of Proposition 3.3 that $(v^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ is an equilibrium of (5.2). Finally, invoking Corollary 3.2, we conclude that $(v^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ is also an equilibrium of (3.1). ∎

We say that the Lur'e system (3.1) has the *converging-input converging-state* property (CICS) if, for every $v^\infty \in U$, there exists $x^\infty \in X$ such that, for every $(v, x, y) \in \mathcal{B}^\infty$ with $v(t) \to v^\infty$ as $t \to \infty$, it follows that $x(t) \to x^\infty$ as $t \to \infty$.

THEOREM 5.2. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{G})$ be a well-posed linear system, $i, j \in \{1, 2\}$, $K^{ij} \in \mathbb{S}(P^j\mathbf{G}E^i)$ and let $Z \subset Y^j$ be non-empty. Furthermore, let $K \in \mathcal{L}(Y, U)$ and F_K be given by (2.4) and (5.1), respectively, and let $v^\infty \in U$ be such that $F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty) \cap Z \neq \emptyset$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ is optimizable and estimatable. If $f : Y^j \to U^i$ satisfies*

$$
(5.3)\qquad \sup_{(z_1,z_2)\in Y^j\times Z,\, z_1\neq z_2} \frac{\|f(z_1) - f(z_2) - K^{ij}(z_1 - z_2)\|}{\|z_1 - z_2\|}\|(P^j\mathbf{G}E^i)^{K^{ij}}\|_{H^\infty} < 1,
$$

then the set $F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty)$ is a singleton and there exists unique $(x^\infty, y^\infty) \in X \times Y$ such that $(v^\infty, x^\infty, y^\infty)$ is an equilibrium of (3.1).

The vectors x^∞ and y^∞ are given by

$$
(5.4)\qquad x^\infty := -(A^K)^{-1}B^K w^\infty \quad \text{and} \quad y^\infty := \mathbf{G}^K(0)w^\infty,
$$

where $w^\infty := E^i(f(z^\infty) - K^{ij}z^\infty) + v^\infty$ with $\{z^\infty\} = F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty)$, and the equilibrium $(v^\infty, x^\infty, y^\infty)$ is exponentially ISS. Furthermore, the following statements hold.

(1) *There exist constants* $\Gamma > 0$ *and* $\varepsilon > 0$ *such that, for all* $(v, x, y) \in \mathcal{B}$*, all* $\alpha \in [0, \varepsilon]$ *and all* $t \geq 0$*,*

$$\|\mathbf{P}_t(x - x^\infty \theta)\|_{L^2_\alpha} + \|\mathbf{P}_t(y - y^\infty \theta)\|_{L^2_\alpha} \leq \Gamma\big(\|x(0) - x^\infty\| + \|\mathbf{P}_t(v - v^\infty \theta)\|_{L^2_\alpha}\big).$$

In particular, $x - x^\infty\theta \in L^2_\alpha(\mathbb{R}_+, X)$ *and* $y - y^\infty\theta \in L^2_\alpha(\mathbb{R}_+, Y)$*, provided that* $v - v^\infty\theta \in L^2_\alpha(\mathbb{R}_+, U)$*.*

(2) *For every* $2 \leq q \leq \infty$*, there exist constants* $\Gamma_q > 0$ *and* $\gamma > 0$ *such that, for all* $(v, x, y) \in \mathcal{B}^q$ *and all* $t \geq 0$*,*

$$\|x(t) - x^\infty\| \leq \Gamma_q\big(e^{-\gamma t}\|x(0) - x^\infty\| + \|\mathbf{P}_t(v - v^\infty\theta)\|_{L^q}\big).$$

Here Γ_q *depends on* q*, but* γ *does not.*

(3) *Let* $(v, x, y) \in \mathcal{B}^q$*, where* $2 \leq q \leq \infty$*, and let* $\alpha \geq 0$*. If* $q < \infty$ *and* $v - v^\infty\theta \in L^q_\alpha(\mathbb{R}_+, U)$*, or, if* $q = \infty$ *and* $e^{\alpha t}(v(t) - v^\infty) \to 0$ *as* $t \to \infty$*, then* $x(t) \to x^\infty$ *as* $t \to \infty$ *and, if* $\alpha > 0$*, then the rate of convergence is exponential.*

Proof. Let $v^\infty \in U$, $z^\infty \in F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty) \cap Z$ and $z \in F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty)$. To show that $F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty)$ is a singleton, we need to establish that $z = z^\infty$. Since $F_K(z^\infty) = F_K(z)$, we have

$$z - z^\infty = P^j\mathbf{G}^K(0)E^i\big(f(z) - f(z^\infty) - K^{ij}(z - z^\infty)\big).$$

Thus, if $z \neq z^\infty \in Z$, then it follows from (5.3) that

$$\|z - z^\infty\| \leq \|P^j\mathbf{G}^K(0)E^i\|\|f(z) - f(z^\infty) - K^{ij}(z - z^\infty)\| < \|z - z^\infty\|,$$

which is impossible. Hence, $z = z^\infty$.

It is clear from statement (2) of Proposition 5.1 that with x^∞ and y^∞ given by (5.4), $(v^\infty, x^\infty, y^\infty)$ is an equilibrium of (3.1). To show uniqueness of (x^∞, y^∞), let $(x^*, y^*) \in X \times Y$ and assume that (v^∞, x^*, y^*) is an equilibrium of (3.1). Define

$$u^* := E^i(f(P^jy^*) - K^{ij}P^jy^*) + v^\infty \quad \text{and} \quad u^\infty := E^i(f(P^jy^\infty) - K^{ij}P^jy^\infty) + v^\infty\,.$$

By statement (1) of Proposition 5.1,

$$F_K(P^jy^*) = P^j\mathbf{G}^K(0)v^\infty = F_K(P^jy^\infty),$$

and so, since $F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty) = \{z^\infty\}$, it follows that $P^jy^* = z^\infty = P^jy^\infty$. Consequently, $u^* = u^\infty$. Appealing once more to statement (1) of Proposition 5.1, we obtain $y^* = y^\infty$ and $x^* = x^\infty$.

Finally, since $P^jy^\infty = z^\infty \in Z$, the constant trajectory $(v^\infty, x^\infty, y^\infty)$ is in $\mathcal{B}_Z$, and the remaining claims follow from Theorem 4.1. ∎

COROLLARY 5.3. *Using the notation of Theorem 5.2, assume that* $Z = Y^j$*. Then, under the assumptions of Theorem 5.2, the map* F_K *is a bijection and the Lur'e system* (3.1) *has the CICS property.*

Note that, by (5.3) with $Z = Y^j$, Proposition 3.1 and Corollary 3.2, for every pair $(x^0, v) \in X \times L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, there exists a unique trajectory $(v, x, y) \in \mathcal{B}$ such that $x(0) = x^0$.

Under the assumptions of Corollary 5.3, the map F_K is a bijection and so, setting

$$S_K(w) := E^i\big(f(F_K^{-1}(P^j\mathbf{G}^K(0)w)) - K^{ij}F_K^{-1}(P^j\mathbf{G}^K(0)w)\big) + w \quad \forall\, w \in U\,,$$

results in a well-defined map $S_K : U \to U$. Given $(v, x, y) \in \mathcal{B}$ such that $v(t) \to v^\infty$ as $t \to \infty$ or $v - v^\infty\theta \in L^2_\alpha(\mathbb{R}_+, U)$, we have that

$$\lim_{t\to\infty} x(t) = -(A^K)^{-1}B^K S_K(v^\infty) \quad \text{or} \quad y - \mathbf{G}^K(0)S_K(v^\infty)\theta \in L^2_\alpha(\mathbb{R}_+, U)\,,$$

respectively. The nonlinear maps

$$(5.5) \qquad U \to X,\, w \mapsto -(A^K)^{-1}B^K S_K(w) \quad \text{and} \quad U \to Y,\, w \mapsto \mathbf{G}^K(0)S_K(w)\,,$$

provide natural generalizations of the concept of "steady-state gains' for stable linear systems. Finally, invoking (5.3) with $Z = Y^j$, it is easy to show that F_K^{-1} is globally Lipschitz (with minimal Lipschitz constant less or equal to $1/(1-\mu)$, where μ is equal to the left-hand side of (5.3)). This implies that S_K is globally Lipschitz, and hence, the steady-state gain maps (5.5) are globally Lipschitz.

We mention that Corollary 5.3 could be given a circle-criterion interpretation: in this sense, Corollary 5.3 is reminiscent of the main result in the paper [31] which provides a description of the steady-state error of finite-dimensional single-input single-output Lur'e systems in response to a class of polynomial inputs (including unbounded signals such as ramps) under the assumption that the conditions of the SISO circle criterion are met. Whilst the CICS property is not mentioned in [31], part (1) of [31, (unnumbered) Theorem] can be interpreted in CICS terms. Furthermore, CICS properties of finite-dimensional Lur'e systems have been investigated in some detail in [3]: Theorem 5.2 and Corollary 5.3 can be viewed as partial extensions to infinite dimensions of some of the results in [3].

Proof of Corollary 5.3. Injectivity of F_K can be proved by an argument similar to that used in the proof of Theorem 5.2 to establish that z^∞ is the only element in $F_K^{-1}(P^j\mathbf{G}^K(0)v^\infty)$. To show surjectivity of F_K, let $\zeta \in Y^j$. It follows from (5.3) with $Z = Y^j$ that the map $z \mapsto z - F_K(z) + \zeta$ is a contraction and thus, by the Banach fixed point theorem, there exists $z^* \in Y^j$ such that $F_K(z^*) = \zeta$, showing that F_K is surjective. In particular, we have that $F_K^{-1}(P^j\mathbf{G}^K(0)w) \neq \emptyset$ for every $w \in U$, and the CICS property follows now from an application of Theorem 5.2. ☐

The remainder of the section is devoted to considering convergence properties of the Lur'e system (3.1) when subject to periodic or asymptotically periodic forcing. As usual, for a positive number τ, a function $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is said to be τ-periodic if $\mathbf{L}_\tau v = v$. A trajectory $(v, x, y) \in \mathcal{B}$ is called τ-periodic if $(\mathbf{L}_\tau v, \mathbf{L}_\tau x, \mathbf{L}_\tau y) = (v, x, y)$.

The next result shows that, given an essentially bounded τ-periodic input v^{p}, then, under suitable conditions, there exists a unique τ-periodic trajectory $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}) \in \mathcal{B}$, such that, for every trajectory $(v^{\mathrm{p}}, x, y) \in \mathcal{B}$ generated by v^{p}, the pair (x, y) approaches $(x^{\mathrm{p}}, y^{\mathrm{p}})$ in a certain sense.

THEOREM 5.4. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{G})$ be a well-posed linear system, $i, j \in \{1, 2\}$, $K^{ij} \in \mathbb{S}(P^j\mathbf{G}E^i)$, $Z \subset Y^j$ a non-empty and closed subset, $\tau > 0$ and let $v^{\mathrm{p}} \in L^\infty(\mathbb{R}_+, U)$ be τ-periodic. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ is optimizable and estimatable, $f : Y^j \to U^i$ satisfies the incremental small-gain condition*

$$(5.6) \qquad \sup_{(z_1,z_2)\in Z\times Z,\, z_1\neq z_2} \left(\frac{\|f(z_1) - f(z_2) - K^{ij}(z_1 - z_2)\|}{\|z_1 - z_2\|}\right) \|(P^j\mathbf{G}E^i)^{K^{ij}}\|_{H^\infty} < 1,$$

and there exist $\tilde{v} \in L^\infty(\mathbb{R}_+, U)$, $\tilde{x}, \hat{x} \in C(\mathbb{R}_+, X)$ and $\tilde{y}, \hat{y} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ with $\tilde{x}$ bounded and such that $(\tilde{v}, \tilde{x}, \tilde{y})$ and $(v^{\mathrm{p}}, \hat{x}, \hat{y})$ are in $\mathcal{B}^\infty_Z$.

Under these conditions there exist a unique τ-periodic trajectory $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}) \in \mathcal{B}^\infty_Z$ and $\varepsilon > 0$ such that, for all $(v^{\mathrm{p}}, x, y) \in \mathcal{B}^\infty_Z$,

$$(5.7) \quad \lim_{t\to\infty} \|(x(t) - x^{\mathrm{p}}(t))e^{\varepsilon t}\| = 0, \;\; x - x^{\mathrm{p}} \in L^2_\varepsilon(\mathbb{R}_+, X) \quad \text{and} \quad y - y^{\mathrm{p}} \in L^2_\varepsilon(\mathbb{R}_+, Y)\,.$$

The following remark focusses on the important case wherein $Z = Y^j$.

Remark 5.5. Assume that (5.6) holds with $Z = Y^j$. Then the existence of trajectories $(\tilde{v}, \tilde{x}, \tilde{y})$ and $(v^{\rm p}, \hat{x}, \hat{y})$ with the required properties is guaranteed. Indeed, by Corollary 5.3 and statement (2) of Proposition 5.1, for every $v^{\rm e} \in U$, there exist $x^{\rm e} \in X$ and $y^{\rm e} \in Y$ such that $(v^{\rm e}, x^{\rm e}, y^{\rm e})$ is an equilibrium of (3.1). Furthermore, the existence of a pair $(\hat{x}, \hat{y}) \in C(\mathbb{R}_+) \times L^2_{\rm loc}(\mathbb{R}_+, Y)$ such that $(v^{\rm p}, \hat{x}, \hat{y}) \in \mathcal{B}^\infty$ follows from Proposition 3.1 and Corollary 3.2. ◊

Proof of Theorem 5.4. The assumptions of Theorem 4.1 hold with $Z_1 = Z_2 = Z$ and $r = 1/\|(P^j \mathbf{G} E^i)^{K^{ij}}\|_{H^\infty}$, and thus it follows from Theorem 4.1 that there exist constants $\Gamma > 0$ and $\gamma > \varepsilon > 0$ such that, for all $(v_1, x_1, y_1), (v_2, x_2, y_2) \in \mathcal{B}^\infty_Z$ and all $t \geq 0$,

$$\|x_1(t) - x_2(t)\| \leq \Gamma\big(e^{-\gamma t}\|x_1(0) - x_2(0)\| + \|\mathbf{P}_t(v_1 - v_2)\|_{L^\infty}\big)\,, \tag{5.8}$$

and

$$\|\mathbf{P}_t(x_1 - x_2)\|_{L^2_\varepsilon} + \|\mathbf{P}_t(y_1 - y_2)\|_{L^2_\varepsilon} \leq \Gamma\big(\|x_1(0) - x_2(0)\| + \|\mathbf{P}_t(v_1 - v_2)\|_{L^2_\varepsilon}\big)\,. \tag{5.9}$$

Let $(v^{\rm p}, x, y) \in \mathcal{B}^\infty_Z$ (such a trajectory exists by hypothesis). Invoking (5.8) with $(v_1, x_1, y_1) = (v^{\rm p}, x, y)$ and $(v_2, x_2, y_2) = (\tilde{v}, \tilde{x}, \tilde{y})$ and using the boundedness of $\tilde{v}$ and $\tilde{x}$ shows that x is bounded. We deduce that there exists $\mu > 0$ such that

$$\|x(t)\| \leq \mu \quad \forall\, t \geq 0\,.$$

Furthermore, since $(\mathbf{L}_\sigma v, \mathbf{L}_\sigma x, \mathbf{L}_\sigma y) \in \mathcal{B}^\infty_Z$ for every $\sigma \geq 0$, the inequalities (5.8) and (5.9) yield,

$$\|(\mathbf{L}_\sigma x)(s) - (\mathbf{L}_{\sigma + k\tau} x)(s)\| \leq \Gamma e^{-\gamma s}\|x(\sigma) - x(\sigma + k\tau)\| \quad \forall\, s, \sigma \geq 0,\ \forall\, k \in \mathbb{N}_0\,, \tag{5.10}$$

and

$$\|\mathbf{L}_\sigma y - \mathbf{L}_{\sigma + k\tau} y\|_{L^2_\varepsilon} \leq \Gamma\|x(\sigma) - x(\sigma + k\tau)\| \quad \forall\, \sigma \geq 0,\ \forall\, k \in \mathbb{N}_0\,, \tag{5.11}$$

where we have used that $\mathbf{L}_\sigma v^{\rm p} - \mathbf{L}_{\sigma + k\tau} v^{\rm p} = 0$ since $v^{\rm p}$ is τ-periodic.

To construct the periodic "limit" $x^{\rm p}$ of the state trajectory x, we use an argument from [1, Proof of Proposition 4.4]: for arbitrary $t \geq 0$ and arbitrary positive integers $n \leq m$, it follows from (5.10) that

$$\|(\mathbf{L}_{n\tau} x)(t) - (\mathbf{L}_{m\tau} x)(t)\| = \|(\mathbf{L}_t x)(n\tau) - (\mathbf{L}_{t + (m-n)\tau} x)(n\tau)\| \leq 2\mu\Gamma e^{-\gamma n\tau}.$$

Consequently, $(\mathbf{L}_{n\tau} x)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space of bounded continuous X-valued functions defined on $\mathbb{R}_+$ and hence converges to a bounded continuous function $x^{\rm p}$. Moreover, invoking (5.11) with $\sigma = n\tau$ and $k = m - n$, $n \leq m$ and using that $(x(n\tau))_{n \in \mathbb{N}}$ is a Cauchy sequence in X, yields the existence of a function $y^{\rm p} \in L^2_{\rm loc}(\mathbb{R}_+, Y)$ such that,

$$\lim_{n \to \infty} \|\mathbf{P}_t\big(\mathbf{L}_{n\tau} y - y^{\rm p}\big)\|_{L^2} = 0 \quad \forall\, t \geq 0. \tag{5.12}$$

We proceed to show that $x^{\rm p}$ and $y^{\rm p}$ are τ-periodic. For $t \geq 0$, we have that

$$x^{\rm p}(t) = \lim_{n \to \infty} x(t + n\tau) = \lim_{n \to \infty} x(t + (n+1)\tau) = \lim_{n \to \infty} x(t + \tau + n\tau) = x^{\rm p}(t + \tau),$$

showing that x^{p} is τ-periodic. To establish τ-periodicity of y^{p}, we note that, for arbitrary $\sigma > 0$,

$$\int_0^\sigma \|y^{\mathrm{p}}(t+\tau) - y^{\mathrm{p}}(t)\|^2 dt \le \int_0^\sigma \|y^{\mathrm{p}}(t+\tau) - (\mathbf{L}_{n\tau}y)(t+\tau)\|^2 dt + \int_0^\sigma \|y^{\mathrm{p}}(t) - (\mathbf{L}_{(n+1)\tau}y)(t)\|^2 dt\,.$$

Now the right-hand side of the above estimate converges to 0 as $n \to \infty$, showing that $y^{\mathrm{p}}(t+\tau) = y^{\mathrm{p}}(t)$ for a.e. $t \in [0,\sigma]$. Since $\sigma > 0$ is arbitrary, we conclude that y^{p} is τ-periodic.

The next step is to verify that $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}) \in \mathcal{B}^\infty_Z$. To this end, set $x_n := \mathbf{L}_{n\tau}x$ and $y_n := \mathbf{L}_{n\tau}y$, where $n \in \mathbb{N}$. We start by showing that

$$(5.13) \qquad P^j y^{\mathrm{p}}(t) \in Z \quad \text{for a.e. } t \ge 0.$$

Let $\sigma > 0$ be fixed, but arbitrary, and note that, by (5.12), $\mathbf{P}_\sigma y_n \to \mathbf{P}_\sigma y^{\mathrm{p}}$ in $L^2(\mathbb{R}_+, Y)$ as $n \to \infty$. Consequently, there exists a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $y_{\varphi(n)}(t)$ converges to $y^{\mathrm{p}}(t)$ as $n \to \infty$ for a.e. $t \in [0,\sigma]$. Now $P^j y(t) \in Z$ for a.e. $t \ge 0$, and so, by the closedness of Z, $P^j y^{\mathrm{p}}(t) \in Z$ for a.e. $t \in [0,\sigma]$. Since σ was arbitrary, it follows that (5.13) holds.

We proceed to show that $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}) \in \mathcal{B}^\infty$. Using the periodicity of v^{p}, it is clear that $(v^{\mathrm{p}}, x_n, y_n) \in \mathcal{B}^\infty$, and so, for every $t \ge 0$,

$$x_n(t) = \mathbb{T}_t x_n(0) + \Phi_t u_n, \;\; \mathbf{P}_t y_n = \Psi_t x_n(0) + \mathbb{G}_t u_n, \;\; \text{where } u_n = E^i f(P^j y_n) + v^{\mathrm{p}}.$$

The functions x_n converge uniformly to x^{p} as $n \to \infty$, and, for every $t \ge 0$, u_n converges to $u^{\mathrm{p}} := E^i f(P^j y^{\mathrm{p}}) + v^{\mathrm{p}}$ in $L^2([0,t], U)$ as $n \to \infty$, where we have used that $f|_Z$ is globally Lipschitz (as follows from (5.6)). Consequently, letting $n \to \infty$ in the above equations for x_n and y_n and using the continuity properties of the well-posed linear system Σ, we see that the triple $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}})$ satisfies, for all $t \ge 0$,

$$x^{\mathrm{p}}(t) = \mathbb{T}_t x^{\mathrm{p}}(0) + \Phi_t u^{\mathrm{p}}, \quad \mathbf{P}_t y^{\mathrm{p}} = \Psi_t x^{\mathrm{p}}(0) + \mathbb{G}_t u^{\mathrm{p}}, \quad u^{\mathrm{p}} = E^i f(P^j y^{\mathrm{p}}) + v^{\mathrm{p}},$$

establishing that $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}) \in \mathcal{B}^\infty$. Combining this with (5.13) yields that the trajectory $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}})$ is in $\mathcal{B}^\infty_Z$.

Invoking the estimates (5.8) and (5.9) with

$$(v_1, x_1, y_1) = (v^{\mathrm{p}}, x, y) \quad \text{and} \quad (v_2, x_2, y_2) = (v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}})$$

shows that (5.7) holds. Finally, we note that, by (5.7), if $(v^{\mathrm{p}}, x^\dagger, y^\dagger) \in \mathcal{B}^\infty_Z$ is a τ-periodic trajectory, then $x^\dagger = x^{\mathrm{p}}$ and $y^\dagger = y^{\mathrm{p}}$, showing the uniqueness of $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}})$ and completing the proof. □

Our final result provides information about the response of the Lur'e system (3.1) to inputs which are asymptotically periodic in a certain sense.

COROLLARY 5.6. *Let Σ, Z, τ and v^{p} be as in Theorem 5.4. Then, under the assumptions of Theorem 5.4, there exists a unique τ-periodic trajectory $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}) \in \mathcal{B}^\infty_Z$ such that the following statements hold.*

(1) *For every $(v, x, y) \in \mathcal{B}^\infty_Z$ such that $v - v^{\mathrm{p}} \in L^2(\mathbb{R}_+, U)$, we have that $y - y^{\mathrm{p}} \in L^2(\mathbb{R}_+, Y)$, $x - x^{\mathrm{p}} \in L^2(\mathbb{R}_+, X)$ and $x(t) - x^{\mathrm{p}}(t) \to 0$ as $t \to \infty$.*

(2) *For every* $(v, x, y) \in \mathcal{B}_Z^\infty$ *such that* $v - v^{\mathrm{p}} \in L^q_\alpha(\mathbb{R}_+, U)$*, where* $2 \le q < \infty$ *and* $\alpha \ge 0$*,* $x(t) - x^{\mathrm{p}}(t) \to 0$ *as* $t \to \infty$ *and, if* $\alpha > 0$*, then the rate of the convergence is exponential.*

(3) *For every* $(v, x, y) \in \mathcal{B}_Z^\infty$ *such that* $e^{\alpha t}(v(t) - v^{\mathrm{p}}(t)) \to 0$ *as* $t \to \infty$*, for some* $\alpha \ge 0$*, we have that* $x(t) - x^{\mathrm{p}}(t) \to 0$ *as* $t \to \infty$ *and, if* $\alpha > 0$*, then the rate of the convergence is exponential.*

Proof. By Theorem 5.4 there exists a unique pair $(x^{\mathrm{p}}, y^{\mathrm{p}})$ such that $(v^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}})$ is a τ-periodic trajectory in $\mathcal{B}_Z^\infty$. Statements (1), (2) and (3) now follow from an application of Theorem 4.1 with $Z_1 = Z_2 = Z$. ∎

Earlier papers which study the response of Lur'e systems to (asymptotically) periodic inputs include [29, 30, 45]: whilst [29, 30] adopt an input-output approach, [45] focusses on finite-dimensional state-space systems. Corollary 5.6 can be viewed as a far-reaching generalization of [29, Theorem 4], [30, Theorem 2] and of the first part of [45, Theorem 1].

6. Examples. In this section, we consider two elementary examples which serve to illustrate the theory. Both are examples of controlled and observed heat equations: the first illustrates Theorems 4.1, and the second example serves to illustrate two of the three circle criteria derived in Section 4, namely Corollaries 4.5 and 4.7. Throughout this section, we consider real input and output spaces (so that complex ball conditions are defined in terms of complexifications), see part (1) of Remark 4.2. In what follows, recall that, as in (1.1), superscripts on state, input and output variables refer to components within a decomposition of their respective spaces.

Example 6.1. Consider the following equations modelling the dissipation of heat in a unit rod, with temperature $w(\xi, t)$ at position ξ and time t:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2}, \quad \frac{\partial w}{\partial \xi}(0, t) = 0, \quad \xi \in (0, 1),\ t \ge 0\,. \tag{6.1a}$$

The PDE (6.1a) becomes a controlled and observed system when subject to

$$u(t) = \frac{\partial w}{\partial \xi}(1, t), \quad y(t) = w(1/6, t) \quad t \ge 0\,, \tag{6.1b}$$

with single input u and single output y denoting an applied flux at the right end of rod and an interior point temperature observation, respectively. Note that the above SISO example can be trivially embedded in the four-block framework (by choosing $U^1 = \mathbb{R}$, $U^2 = \{0\}$, $Y^1 = \mathbb{R}$ and $Y^2 = \{0\}$). The transfer function of (6.1) may be calculated similarly as in [4] and is given by

$$\mathbf{G}(s) = \frac{\cosh(\sqrt{s}/6)}{\sqrt{s}\sinh(\sqrt{s})} \quad \forall\, s \in \mathbb{C}_0\,.$$

The controlled and observed PDE (6.1) may be written as a well-posed linear system Σ with state space $X = L^2(0, 1)$, input space $\mathbb{R}$ and output space $\mathbb{R}$ (which follows from [38, Theorem 5.7.3] and standard properties of the Laplacian). We note that $\mathbf{G}$ has a simple pole at 0, and so Σ is not exponentially stable.

Consider the forced nonlinear output feedback

$$u = f(y) + v\,, \tag{6.2}$$

for locally Lipschitz f, where v is a forcing function. We shall identify conditions on f which will guarantee that the zero equilibrium triple of the resulting Lur'e system is exponentially ISS. To this end, we define $\mathbf{L}(s) := s\mathbf{G}(s)$ and note that, for every $\alpha > -\pi^2$, the function $\mathbf{L}$ is holomorphic and bounded on $\mathbb{C}_\alpha$, and, furthermore, $\mathbf{L}(0) = 1$. Setting

$$\lambda := 2 \sup_{\omega \in \mathbb{R}} \left| \mathrm{Re}\, \frac{\mathbf{L}(i\omega) - \mathbf{L}(0)}{i\omega} \right| > 0 \,,$$

an application of [20, Lemma 3.1 and Corollary 3.4] yields that

$$\|\mathbf{G}^{-k}\|_{H^\infty} = 1/k \quad \forall\, k \in (0, 1/\lambda)\,.$$

A numerical computation shows that $\lambda \approx 0.6638$ and so $1/\lambda > 3/2$. Invoking Lemma 2.2, it follows that the disk $\mathbb{B}(-3/2, 3/2)$ is contained in $\mathbb{S}(\mathbf{G})$, and thus, if $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\sup_{z \in \mathbb{R},\, z \neq 0} \frac{|f(z) + 3z/2|}{|z|} < \frac{3}{2}\,,$$

then the hypotheses of Theorem 4.1 are satisfied with $i = j = 1$, $Z_1 = \mathbb{R}$, $Z_2 = \{0\}$, $K^{11} = -3/2$, $r = 3/2$. Statement (2) of Theorem 4.1 now ensures that the zero equilibrium triple of the Lur'e system given by (6.1) and (6.2) is exponentially ISS. Moreover, if f is such that $z \mapsto f(z) + 3z/2$ is Lipschitz with Lipschitz constant less than $3/2$ (which, for example, is the case if f is continuously differentiable with $\sup_{z \in \mathbb{R}} f'(z) < 0$ and $\inf_{z \in \mathbb{R}} f'(z) > -3$), then an application of Theorem 4.1 (now with $Z_2 = \mathbb{R}$) shows that the Lur'e system enjoys various incremental stability properties, including exponential δISS. ♢

Example 6.2. Consider the following equations modelling the dissipation of heat in a unit rod, with temperature $w(\xi, t)$ at position ξ and time t:

$$\text{(6.3a)} \qquad \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2}, \quad w(0, t) = 0, \quad \xi \in (0, 1),\ t \geq 0\,.$$

The PDE (6.3a) becomes a controlled and observed system when subject to

$$\text{(6.3b)} \qquad u^1(t) = \frac{\partial w}{\partial \xi}(1, t), \quad y^1(t) = w(1/4, t), \quad y^2(t) = w(1/2, t) \quad t \geq 0\,,$$

with single input u^1 and two outputs y^j, $j = 1, 2$, denoting an applied flux at the right end of rod and two interior point temperature observations, respectively. The transfer function of (6.3) may be calculated similarly as in [4] and is given by

$$\mathbf{G}(s) = \begin{pmatrix} \frac{\sinh(\sqrt{s}/4)}{\sqrt{s}\cosh\sqrt{s}} \\ \frac{\sinh(\sqrt{s}/2)}{\sqrt{s}\cosh\sqrt{s}} \end{pmatrix} \quad \forall\, s \in \mathbb{C}_0\,.$$

The controlled and observed PDE (6.3) may be written as an exponentially stable well-posed linear system Σ with state space $X = L^2(0, 1)$, input space $\mathbb{R}$ and output space $\mathbb{R}^2$ (which follows from [38, Theorem 5.7.3] and standard properties of the Laplacian). In our four-block framework we choose $U^1 = \mathbb{R}$, $U^2 = \{0\}$, $Y^1 = Y^2 = \mathbb{R}$.

For purposes of illustration, we seek to apply Corollaries 4.5 and 4.7 to the feedback connection of (6.3) and

$$\text{(6.4)} \qquad u = \begin{pmatrix} u^1 \\ 0 \end{pmatrix} = \begin{pmatrix} f(y^2) + v^1 \\ 0 \end{pmatrix}\,,$$

with $Z_1 = \mathbb{R}$, $Z_2 = \{0\}$, $i = 1$ and $j = 2$. To this end, we consider two cases in terms of the gains K_1 and K_2 and provide sufficient conditions for the (strong) positive realness of the function

$$\mathbf{H} := \frac{1 - K_2 P^2 \mathbf{G} E^1}{1 - K_1 P^2 \mathbf{G} E^1}$$

in each of these cases.

(a) $K_1 < 0 < K_2$: $\mathbf{H}$ is positive real if the Nyquist plot of $P^2\mathbf{G}E^1$ is contained in the closed disc with center $\frac{1}{2}\big(\frac{1}{K_1} + \frac{1}{K_2}\big)$ and radius $\frac{1}{2}\big(\frac{1}{K_2} - \frac{1}{K_1}\big)$, see, for example, [16, Lemma 10];

(b) $K_1 < 0 = K_2$: $\mathbf{H}$ is strongly positive real if the Nyquist plot of $P^2\mathbf{G}E^1$ is to the right of, and bounded away from, the vertical line passing through $1/K_1$.

Consequently, in the specific case wherein $K_1 = -2$ and $K_2 = 2$, it follows from Figure 6.1(a) that $\mathbf{H}$ is positive real, and hence, Corollary 4.5 guarantees that the zero equilibrium triple of the feedback interconnection of Σ and (6.4) is exponentially ISS, for any $f : \mathbb{R} \to \mathbb{R}$ for which there exists $\varepsilon > 0$ such that

$$\big(f(\zeta) + 2\zeta\big)\big(f(\zeta) - 2\zeta\big) \leq -\varepsilon\zeta^2, \quad \forall\, \zeta \in \mathbb{R}\,,$$

(see Figure 6.1(b) for an illustration) or, equivalently,

$$(-2 + \varepsilon)\zeta^2 \leq f(\zeta)\zeta \leq (2 - \varepsilon)\zeta^2, \qquad \forall\, \zeta \in \mathbb{R}\,.$$

Alternatively, if $K_1 = -27$ and $K_2 = 0$, then, by Figure 6.2(a), the function $\mathbf{H}$ is strongly positive real, and thus, invoking Corollary 4.7, we conclude that the zero equilibrium triple of the feedback interconnection of Σ and (6.4) is exponentially ISS, for any $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\big(f(\zeta) + 27\zeta\big) f(\zeta) \leq 0, \quad \forall\, \zeta \in \mathbb{R}\,, \text{ or, equivalently, } -27\zeta^2 \leq f(\zeta)\zeta \leq 0, \quad \forall\, \zeta \in \mathbb{R}\,,$$

(see Figure 6.2(b) for an illustration). ◊

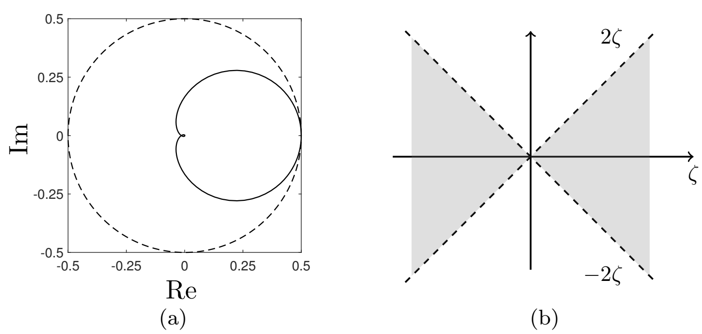
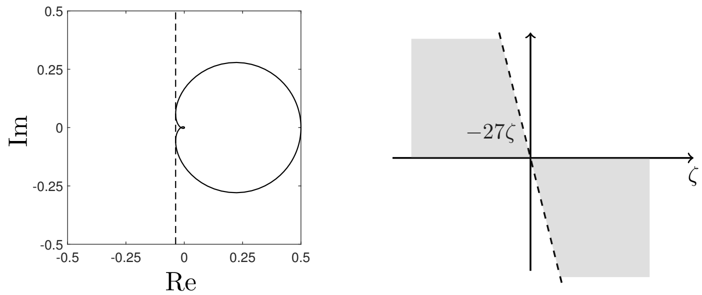


FIGURE 6.1. *Application of Corollary 4.5.* (a) *Nyquist plot of* $P^2\mathbf{G}E^1$ *and circle with center* $\frac{1}{2}\big(\frac{1}{K_1} + \frac{1}{K_2}\big)$ *and radius* $\frac{1}{2}\big(\frac{1}{K_2} - \frac{1}{K_1}\big)$ *for* $K_1 = -2$ *and* $K_2 = 2$. (b) *Accompanying sector.*



(a) (b)

FIGURE 6.2. *Application of Corollary 4.7.* (a) *Nyquist plot of* $P^2\mathbf{G}E^1$ *and vertical line passing through* $1/K_1$ *for* $K_1 = -27$ *and* $K_2 = 0$. (b) *Accompanying sector.*

REFERENCES

[1] D. Angeli. A Lyapunov stability approach to incremental stability properties, *IEEE Trans. Automatic Control*, **37** (2002), 410–421.

[2] M. Arcak and A. Teel. Input-to-state stability for a class of Lurie systems, *Automatica*, **38** (2002), 1945–1949.

[3] A. Bill, C. Guiver, H. Logemann and S. Townley. The converging-input converging-state property for Lur'e systems, *Math. Control Signals Systems*, **29** (2017).

[4] R.F. Curtain and K. Morris. Transfer functions of distributed parameter systems: a tutorial, *Automatica*, **45** (2009), 1101–1116.

[5] S.N. Dashkovskiy, D.V. Efimov and E.D. Sontag. Input-to-state stability and allied system properties, *Autom. Remote Control*, **72** (2011), 1579–1614.

[6] S.N. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinite-dimensional control systems, *Math. Control Signals Systems*, **25** (2013), 1–35.

[7] C.A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[8] C. Guiver, H.Logemann and M.R. Opmeer. Transfer functions of infinite-dimensional systems: positive realness and stabilization, *Math. Control Signals Systems*, **29** (2017), https://doi.org/10.1007/s00498-017-0203-z.

[9] W. M. Haddad and D. S. Bernstein. Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability (part I: continuous-time theory), *Int. J. Robust. Nonlin.*, **3** (1993), 313–339.

[10] W.M. Haddad and V. Chellaboina. *Nonlinear Dynamical Systems and Control*, Princeton University Press, Princeton, 2008.

[11] D. Hinrichsen and A. J. Pritchard. Destabilization by output feedback, *Differ. Integral Eqn*, **5** (1995), 357–386.

[12] D. Hinrichsen and A.J. Pritchard. *Mathematical Systems Theory I,* Springer-Verlag, Berlin, 2005.

[13] B. Jacob, R. Nabiullin, J.R. Partington and F.L. Schwenninger. Infinite-dimensional input-to-state stability and Orlicz Spaces, *SIAM J. Control Optim.*, **56** (2018), 868–889.

[14] B. Jayawardhana, H. Logemann and E.P. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability, *Commun. Inf. Systems*, **8** (2008), 403–434.

[15] B. Jayawardhana, H. Logemann and E.P. Ryan. Input-to-state stability of differential inclusions with applications to hysteretic and quantized feedback systems, *SIAM J. Control Optim.*, **48** (2009), 1031–1054.

[16] B. Jayawardhana, H. Logemann and E.P. Ryan. The circle criterion and input-to-state stability: new perspectives on a classical result, *IEEE Control Systems Mag.*, **31** (2011), 32–67.

[17] Z. P. Jiang, A. R. Teel and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Systems*, **7** (1994), 95–120.

[18] H.K. Khalil. *Nonlinear Systems*, 3rd ed., Prentice-Hall, Upper Saddle River, NJ, 2002.

[19] M.R. Liberzon. Essays on the absolute stability theory, *Autom. Remote Control*, **67** (2006), 1610–1644.

[20] H. Logemann, E. P. Ryan and S. Townley. Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain, *IEEE Trans. Automat. Control*, **44** (1999), 1315–1319.

[21] K.M. Mikkola. State-feedback stabilization of well-posed linear systems, *Integral Equations Ope-*

rator Theory, **55** (2006), 249–271.

[22] A. Mironchenko. Local input-to-state stability: Characterizations and counter examples, *Systems Control Lett.* **87** (2016), 23–28.

[23] A. Mironchenko, I. Karafyllis and M. Krstic. Monotonicity methods for input-to-state stability of nonlinear parabolic PDEs with boundary disturbancess, (2017), arXiv preprint arXiv:1706.07224.

[24] A. Mironchenko and F. Wirth. Characterizations of Input-to-State Stability for Infinite-Dimensional Systems, *IEEE Trans. Automat. Control*, **63** (2018), 1692–1707.

[25] P. Pepe and Z. P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems, *Systems Control Lett.*, **55** (2006), 1006–1014.

[26] C. Prieur and F. Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws, *Math. Control Signals Systems*, **24** (2012), 111–134.

[27] D. Salamon. Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.*, **300** (1987), 383–431.

[28] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, **21** (1989), 147–164.

[29] I.W. Sandberg. Some results on the theory of physical systems governed by nonlinear functional equations, *Bell System Tech. J.*, **44** (1965), 871–898.

[30] I.W. Sandberg. Criteria for the response of nonlinear systems to be L-asymptotically periodic, *Bell System Tech. J.*, **60** (1981), 2359–2371.

[31] I.W. Sandberg and K. Kramer Johnson. Steady state errors in nonlinear control systems, *IEEE Trans. Automat. Control*, **37** (1992), 1985–1989.

[32] E. Sarkans and H. Logemann. Input-to-state stability of Lur'e systems, *Math. Control Signals Systems*, **27** (2015), 439–465.

[33] E.D. Sontag. Smooth stabilization implies coprime factorization, *IEEE Trans. Automat. Control*, **34** (1989), 435–443.

[34] E.D. Sontag and Y. Wang. On characterizations of the input-to-state stability property, *Systems Control Lett.*, **24** (1995), 351–359.

[35] E.D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Trans. Automat. Control*, **41** (1997), 1283–1294.

[36] E.D. Sontag. Comments on integral variants of ISS, *Syst. Control Lett.*, **34** (1998), 93–100.

[37] E.D. Sontag. Input to state stability: basic concepts and results, in P. Nistri and G. Stefani (eds.) *Nonlinear and Optimal Control Theory*, pp. 163-220, Springer Verlag, Berlin, 2006.

[38] O.J. Staffans. *Well-Posed Linear Systems*, Cambridge University Press, Cambridge, 2005.

[39] O.J. Staffans and G. Weiss. Transfer functions of regular linear systems part II: the systems operator and the Lax-Phillips semigroup, *Trans. American Math. Soc.*, **354**, 3229-3262.

[40] M. Tucsnak and G. Weiss. Well-posed systems — the LTI case and beyond, *Automatica*, **50** (2014), 1757–1779.

[41] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*, Birkhäuser, Berlin, 2009.

[42] M. Vidyasagar. *Nonlinear Systems Analysis*, 2nd edition, Prentice-Hall, Englewood Cliffs, NJ,1993.

[43] G. Weiss. Regular linear systems with feedback, *Math. Control Signals Systems*, **7** (1994), 23–57.

[44] G. Weiss and R. Rebarber. Optimizability and estimatability for infinite-dimensional linear systems, *SIAM J. Control Optim.*, **39** (2000), 1204–1232.

[45] V.A. Yakubovich. Matrix inequalities method in stability theory for nonlinear control systems: I. Absolute stability of forced vibrations, *Autom. Remote Control*, **7** (1964), 905–917.

[46] V.A. Yakubovich, G.A. Leonov and A.Kh. Gelig. *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, World Scientific, Singapore, 2004.