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PHD

Non-simple abelian varieties and (1; 3) Theta divisors

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Award date:
2012

Awarding institution:
University of Bath

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Non-simple abelian varieties and (1, 3) Theta divisors

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

August 2012

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Paweł Borówka

Contents

1	Basics	8
1.1	Complex tori and line bundles on them	10
1.1.1	Complex tori	10
1.1.2	Line bundles	11
1.2	Abelian varieties	12
1.2.1	The dual abelian variety and ϕ_L	13
1.2.2	Characteristics	15
1.2.3	Moduli space	16
1.2.4	Theta functions	18
1.3	Symmetric line bundles	21
1.3.1	Symmetric divisors	23
1.3.2	Theta constants	25
1.4	Endomorphisms of abelian varieties	26
1.5	Curves and their Jacobians	30
1.5.1	Moduli of Jacobians	32
1.6	Symplectic forms on finite abelian groups	33
2	Generalised Humbert locus	35
2.1	Preliminaries	35
2.2	Generalised Humbert locus	39
2.2.1	Irreducibility	43
2.3	Special case: $\text{Is}_{(1,p)}^4$	46
3	(1,3) Theta divisors	52
3.1	Construction	52
3.1.1	Smoothness	54
3.1.2	Basic properties of C_A	55
3.1.3	Product of elliptic curves in detail	56

3.1.4	Jacobian of C_A and $\text{Is}_{(1,3)}^4$	57
3.2	Unanswered questions	61
4	Hyperelliptic Jacobians and $\text{Is}_{(1,p)}^4$	63
4.1	Locus of hyperelliptic Jacobians in dimension 4	63
4.2	Transversality	66

Summary

This thesis studies non-simple Jacobians and non-simple abelian varieties. The motivation of the study is a construction which gives a distinguished genus 4 curve in the linear system of a $(1, 3)$ -polarised surface. The main theorem characterises such curves as hyperelliptic genus 4 curves whose Jacobian contains a $(1, 3)$ -polarised surface.

This leads to investigating the locus of non-simple principally polarised abelian g -folds. The main theorem of this part shows that the irreducible components of this locus are $\text{Is}_{\underline{d}}^g$, defined as the locus of principally polarised g -folds having an abelian subvariety with induced polarisation of type $\underline{d} = (d_1, \dots, d_k)$, where $k \leq \frac{g}{2}$. Moreover, there are theorems which characterise the Jacobians of curves that are étale double covers or double covers branched in two points.

There is also a detailed computation showing that, for $p > 1$ an odd number, the hyperelliptic locus meets $\text{Is}_{(1,p)}^4$ transversely in the Siegel upper half space.

Acknowledgements

First of all, I would like to thank my supervisor Gregory Sankaran for his guidance and encouragement to develop and follow my ideas. I would also like to thank all my lecturers and colleagues who helped me to understand Mathematics better, especially Slawomir Cynk, Angela Ortega and participants of COW, Calf and Bath Geometry seminars. I would like to thank University of Bath for financial support. Finally, I would like to thank my wife Ola for numerous moments of support and for motivating me to harder work and my daughter Joanna for the joy she gave me.

Introduction

One of the most interesting questions in the theory of principally polarised abelian varieties, called the Schottky problem, is to determine the locus of Jacobians. There are also other Schottky-type questions like determining the locus of hyperelliptic Jacobians. In this way, people want to understand the geometry of the moduli space of abelian varieties using the geometry of curves. That is possible because of the Torelli theorem, which says that the Jacobian completely characterises the curve. Because of that, many geometric constructions from the theory of curves give rise to interesting constructions in the theory of Jacobians. One remarkable construction is the Prym construction, which gives a subvariety of a Jacobian for any finite cover of curves. More precisely, every cover of curves $f : C \rightarrow C'$ induces a pullback map $f^* : JC' \rightarrow JC$. Therefore JC is a non-simple abelian variety, as it contains $\text{im } f^*$ and the complementary abelian subvariety (Definition 1.72) called the Prym variety of the cover.

The main theme of the thesis is to go the opposite way. The question behind most results of this thesis is:

(\star) *Assume the Jacobian of a curve is non-simple. What can we say about the curve?*

Before trying to answer the question, it is important to understand the locus of non-simple abelian varieties. For abelian surfaces, Humbert in [H] proved that if we assume that the surface is non-simple, hence contains an elliptic curve, the only remaining invariant is the type of induced polarisation. In other words, the locus of non-simple principally polarised abelian surfaces is the union, parametrised by the type of the induced polarisation to a curve, of countably many irreducible components, called Humbert surfaces (see Theorem 2.2 and Remark 2.4).

In Chapter 2, we propose a definition of the generalised Humbert locus, denoted by $\text{Is}_{\underline{d}}^g$, which is the locus of principally polarised abelian g -folds having an abelian subvariety with induced polarisation of type $\underline{d} = (d_1, \dots, d_k)$, where $k < \frac{g}{2}$. The main result of Chapter 2 is Theorem 2.18.

Theorem 2.18. *Let $k \leq \frac{g}{2}$. Then Is_d^g is a non-empty irreducible component of the locus of non-simple abelian g -folds of dimension $\binom{k+1}{2} + \binom{g-k+1}{2}$.*

In other words, the only discrete invariants of the locus of non-simple principally polarised abelian varieties are the dimensions of subvarieties and the type of the induced polarisation on the smaller one. Moreover, all these possibilities can occur. Following Humbert's ideas, we tried to find equations of Is_d^g in the Siegel space \mathfrak{h}_g . Theorem 2.24 provides a set of equations of an irreducible component of the preimage in \mathfrak{h}_4 of $\text{Is}_{(1,p)}^4$, for $p > 1$ odd.

In full generality, the question (\star) is hard. However, in some specific situations, we have a complete answer. When the genus of the curve is 2, the answer can be easily extracted from the work of Humbert [H]: *the Jacobian of a genus 2 curve C is non-simple and contains an elliptic curve E with the induced polarisation of type n if and only if the curve C is an $n : 1$ cover of E .*

In genus 3, the answer is well known and completely analogous. Proposition 2.6 states that *the Jacobian of a genus 3 curve C is non-simple and contains an elliptic curve E with the induced polarisation of type n if and only if the curve C is an $n : 1$ cover of E .*

Moreover, if we restrict our attention to hyperelliptic Jacobians that belong to Is_2^3 then we find a nice characterisation of étale double covers of genus 2 curves: see Remark 2.8.

Remark 2.8 is generalised in Propositions 2.9 and 2.10. Roughly speaking, these say that if the Jacobian of a curve contains an abelian subvariety of half the dimension and the type of the induced polarisation is twice the principal polarisation, then there is a double cover of curves, that yields the Jacobian and the subvariety.

The first example of curves that are not covers of lower positive genus curves, but have non-simple Jacobians arises in dimension 4. It is also the historical motivation for our considerations. There is a construction which produces a canonical curve on a $(1, 3)$ -polarised abelian surface, similar to the Theta curve on a principally polarised abelian surface. It is called a $(1, 3)$ Theta divisor and may be defined as the zero locus of any odd theta function, which is a section of a $(1, 3)$ polarising symmetric line bundle on A of characteristic 0 (with respect to some decomposition) (see Section 3.1)

Chapter 3 is devoted to understanding the construction and its outcome. The main result of this chapter can be summarised as

Theorem 3.14. *Let C be a smooth hyperelliptic genus 4 curve. Then JCS contains a $(1, 3)$ polarised surface M if and only if C can be embedded into \hat{M} , as the $(1, 3)$ Theta divisor.*

Chapter 4 is an attempt to find other loci of genus 4 curves with non-simple Jacobians. Roughly speaking, we proved that the equations of $\text{Is}_{(1,p)}^4$ for $p > 1$ odd, are independent with the equations of the locus hyperelliptic Jacobians. More precisely, Theorems 4.9 and 4.12 provide a direct proof that the locus of hyperelliptic Jacobians meets $\text{Is}_{(1,p)}^4$ transversely, where $p > 1$ is odd. For $p = 3$, the result is weaker than the main result of Chapter 3, because it does not prove that the intersection is irreducible. However, it says that the induced polarisation of type $(1, 3)$ is in some way not special, i.e. the condition that a Jacobian of a hyperelliptic curve of genus 4 is non-simple (with induced polarisation of type $(1, p)$) cuts out a locus which has a 3-dimensional component.

The thesis begins with an introduction to the theory of abelian varieties and theta functions. The theory has a lot of technical background, which is covered in many good books written among others by Mumford ([M1], [M3] and [M4]), Iguassu [I], and Birkenhake and Lange [BL]. This part of the thesis is not self contained, because it is unproductive to rewrite proofs of the well-known facts from the well-known books. To make reading easier, I decided to refer to one book, namely [BL]. Only Section 1.5 contains a few facts which refer to [ACGH], because they are not covered by [BL].

Chapter 1

Basics

Notation

X — a complex torus

$X[n]$ — the group of n -torsion points on X

g — dimension of an abelian variety or genus of a curve

Λ — a lattice

A — an abelian variety

\hat{A} — the dual abelian variety

H — hermitian form

(A, H) — a polarised abelian variety

ϕ_H — the map $A \rightarrow \hat{A}$ defined by H

L — a line bundle

$L_{(H, \chi)}$ — a line bundle constructed from H and χ by $a_{H, \chi}$

D or \underline{d} — a type of polarisation $D = \text{diag } \underline{d} = \text{diag}(d_1, \dots, d_g)$

$\mathfrak{h}_g = \{Z \in M_g(\mathbb{C}) : Z = {}^t Z, \text{Im } Z > 0\}$ — the Siegel upper half space

$\mathcal{A}_D = \mathfrak{h}_g / G_D$ — the moduli space of D -polarised abelian varieties

$\langle Z \ D \rangle = Z\mathbb{Z}^g + D\mathbb{Z}^g$ — the lattice generated by columns of a matrix $[Z \ D]$

θ — theta functions

θ_A — the odd theta function of a symmetric line bundle of characteristic 0

\mathfrak{D} — an ample effective divisor, usually of the form $(\theta = 0)$

$C_A = (\theta_A = 0)$ — the zero curve of the odd theta function

E, F — elliptic curves

JC — Jacobian of a curve C

$\mathcal{J}, \mathcal{JH}$ — the loci of Jacobians and hyperelliptic Jacobians

If $(M, H_M), (N, H_N)$ are polarised abelian varieties then $(M \times N, H_M \boxtimes H_N)$ is a product of abelian varieties with the product polarisation

ρ — a polarised isogeny

$\omega = (0, \frac{1}{3})$

c — characteristics of a line bundle

$\text{Is}_{\underline{d}}^g$ — locus of principally polarised abelian g -folds with polarised subvariety of type \underline{d}

1.1 Complex tori and line bundles on them

1.1.1 Complex tori

Our basic object is a g -dimensional complex torus with a distinguished point 0 .

Definition 1.1. Let Λ be a *lattice* in \mathbb{C}^g , i.e. a discrete subgroup of (maximal) rank $2g$. Then $X = \mathbb{C}^g/\Lambda$ is called a g -dimensional complex torus. By π we denote the quotient map. Any complex torus inherits the structure of a complex Lie group from \mathbb{C}^g .

All complex tori of dimension g are diffeomorphic, as we can always take an \mathbb{R} -linear isomorphism that takes a basis of Λ to a canonical \mathbb{R} -basis of \mathbb{C}^g . However, they are not biholomorphic, which can be easily deduced from Proposition 1.2.

The first class of non-constant holomorphic maps in the category of complex tori is translations $t_c(x) = x + c$. Most of the time, when we think about a problem, we try to rigidify it by thinking “up to translations”. The following proposition tells us that after rigidifying, what are left are holomorphic group homomorphisms.

Proposition 1.2. [BL, Prop 1.2.1] Let $X = \mathbb{C}^g/\Lambda$ and $Y = \mathbb{C}^{g'}/\Lambda'$. Let $h: X \rightarrow Y$ be a holomorphic map. Then there exists a unique group homomorphism $f: X \rightarrow Y$, such that $h(x) = f(x) + h(0)$. Moreover, there exists a unique \mathbb{C} -linear map $F: \mathbb{C}^g \rightarrow \mathbb{C}^{g'}$, with $F(\Lambda) \subset \Lambda'$ inducing f .

□

We will now introduce some more basic definitions.

Definition 1.3. The unique map F from Proposition 1.2 is called the *analytic representation* of f . Its restriction F_Λ to the lattice Λ is called the *rational representation* of f .

Definition 1.4. An *isogeny* ρ is a surjective homomorphism of complex tori with finite kernel. The *degree of the isogeny*, denoted by $\deg \rho$, is the order of its kernel. The *exponent of the isogeny*, denoted by $e(\rho)$, is the smallest number n such that the kernel is contained in the set of n -torsion points.

Proposition 1.5. If f is a homomorphism then $\text{im}(f)$ is a subtorus and $\ker(f)$ is a closed subgroup with finitely many connected components. The connected component $\ker^0(f)$ containing 0 is a complex subtorus. Therefore, for any homomorphism f , we have the following factorisation via projection and isogeny

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & \text{im}(f) & \longrightarrow & Y \\
 & \searrow & \nearrow & & \\
 & & X/\ker^0(f) & &
 \end{array}$$

which is Stein factorisation.

□

Section 2.3 provides a few explicit examples of these definitions.

1.1.2 Line bundles

The first important theorem, which fully describes the Picard and Néron-Severi groups, is the Appel-Humbert Theorem. To state it, we need the following definition.

Definition 1.6. A *Riemann form* H is a hermitian form on \mathbb{C}^g satisfying the condition $\text{Im}(H)(\Lambda, \Lambda) \subset \mathbb{Z}$. Denote by $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$ the circle group. Then $\text{Hom}(\Lambda, \mathbb{C}_1)$ is the group of characters of Λ .

A *semicharacter for H* is a map $\chi: \Lambda \rightarrow \mathbb{C}_1$ satisfying

$$\chi(x + y) = \chi(x)\chi(y) \exp(\pi i \text{Im } H(x, y)) \quad \text{for all } x, y \in \mathbb{C}$$

Denote by $\mathcal{P}(\Lambda)$ the group of all pairs (H, χ) , where H is a Riemann form and χ is a semicharacter for H . The operation is given by $(H_1, \chi_1) \circ (H_2, \chi_2) = (H_1 + H_2, \chi_1 \chi_2)$. Then $\iota: \text{Hom}(\Lambda, \mathbb{C}_1) \rightarrow \mathcal{P}(\Lambda)$ given by $\chi \mapsto (0, \chi)$ is a natural inclusion.

Now we are able to state the Appel-Humbert Theorem.

Theorem 1.7. [BL, Appel-Humbert Theorem 2.2.3] *Let $X = \mathbb{C}^g/\Lambda$ be a complex torus. Then we can identify the group of Riemann forms on \mathbb{C}^g with the Néron-Severi group $\text{NS}(X)$. Therefore we can define a natural projection $\pi: \mathcal{P}(\Lambda) \ni (H, \chi) \mapsto H \in \text{NS}(X)$.*

Moreover, there exists a natural isomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}(\Lambda, \mathbb{C}_1) & \xrightarrow{\iota} & \mathcal{P}(\Lambda) & \xrightarrow{\pi} & \text{NS}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \phi & & \downarrow & & \\ 0 & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & \text{NS}(X) & \longrightarrow & 0 \end{array}$$

Proof. The full proof is given in [BL] in Section 2.2 and uses most results of Section 2.1 and Appendix B. We will give some ideas and important definitions here.

Definition 1.8. A *factor of automorphy* is an element of $Z^1(\Lambda, H^0(\mathcal{O}_{\mathbb{C}^g}^*))$, i.e. a holomorphic map $f: \Lambda \times \mathbb{C}^g \rightarrow \mathbb{C}$ satisfying the cocycle relation

$$f(\lambda\mu, x) = f(\lambda, \mu x)f(\mu, x), \quad \text{for all } \lambda, \mu \in \Lambda, x \in \mathbb{C}^g$$

A factor of automorphy defines a line bundle on X via the following construction. Start with the action of Λ on a trivial line bundle $\mathbb{C}^g \times \mathbb{C}$ by $\lambda: (x, t) \mapsto (x + \lambda, f(\lambda, x)t)$. The action is free and properly discontinuous, so we can define the quotient $L = (\mathbb{C}^g \times \mathbb{C})/\Lambda$. Then L with the projection to X is a holomorphic line bundle on X .

The map ϕ is defined by constructing the canonical factor of automorphy using a Riemann form and a semicharacter.

Definition 1.9. Let $(H, \chi) \in \mathcal{P}(\Lambda)$. The cocycle $a_{(H, \chi)}$ defined by

$$a_{(H, \chi)}(\lambda, x) = \chi(\lambda) \exp(\pi H(x, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))$$

is called the *canonical factor of automorphy*. We denote by $L_{(H, \chi)}$ the line bundle constructed from $a_{(H, \chi)}$.

To finish the proof one shows that this induces an isomorphism between the group of characters and the group of degree 0 line bundles. \square

Remark 1.10. The isomorphism ϕ from the Appel-Humbert Theorem is natural, i.e. for f a homomorphism of complex tori and $L(H, \chi)$ a line bundle, we have

$$f^*(L(H, \chi)) = L(F^*H, (F|_{\Lambda})^*\chi),$$

where F is the analytic representation of f .

Proof. [BL, Appendix B] \square

1.2 Abelian varieties

Let us introduce the notion of an abelian variety. For convenience it is defined as a complex torus with a positive definite Riemann form. It can be proved that such a form is the first Chern class of an ample line bundle and vice versa [BL, Prop 4.5.2]. Therefore abelian varieties are projective complex tori.

Definition 1.11. A complex torus $X = \mathbb{C}^g/\Lambda$ admitting a positive definite Riemann form is called an *abelian variety*.

The hermitian form H is called a polarisation. Its imaginary part, denoted by $\text{Im}(H)$, is an alternating, integer valued form on Λ . By the elementary divisor theorem [Bou, Alg.IX.5.1 Th.1], there exists a symplectic basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ such

that the matrix of $\text{Im}(H)|_\Lambda$ is $\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$, where $D = \text{diag}(d_1, \dots, d_g)$ is the diagonal matrix with positive integers d_1, \dots, d_g , satisfying $d_i | d_{i+1}$ for all i . As numbers, d_1, \dots, d_g are uniquely determined: they are called the type of the polarisation.

Definition 1.12. A *polarised abelian variety of type* (d_1, \dots, d_g) , is a pair (A, H) , where A is an abelian variety and H is a polarisation of type (d_1, \dots, d_g) . If $d_g = 1$, then a polarisation is called principal, and a variety is called *principally polarised*.

To shorten notation, we will talk about D -polarised abelian varieties, where $D = \text{diag}(d_1, \dots, d_g)$. We will denote the set of *polarising line bundles* by

$$\text{Pic}^H(A) = \{L \in \text{Pic}(A) : c_1(L) = H\}.$$

There is an obvious generalisation of morphisms to polarised morphisms.

Definition 1.13. A *polarised homomorphism* $\rho : (A, H) \rightarrow (A', H')$ is a homomorphism with the property that $\rho^*(H') = H$.

Example 1.14. Every complex torus of dimension 1 is an abelian variety with a canonical principal polarisation. This is because we can choose a real basis of \mathbb{C} such that $\Lambda = \tau\mathbb{Z} + \mathbb{Z}$ for some $\tau \in \mathbb{C}$ satisfying $\text{Im } \tau > 0$. Then we define $H_\tau(x, y) = \frac{x\bar{y}}{\text{Im } \tau}$, so that the imaginary part $\text{Im}(H)|_\Lambda$ has matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

1.2.1 The dual abelian variety and ϕ_L

The Appel-Humbert theorem says that $\text{Pic}^0(X)$ is isomorphic to $\text{Hom}(\Lambda, \mathbb{C}_1)$, which is a real torus. Now, we will define a natural complex structure on $\text{Pic}^0(X)$ to obtain the dual complex torus.

Consider the space of antilinear maps $\bar{\Omega} = \text{Hom}_{\bar{\mathbb{C}}}(\mathbb{C}^g, \mathbb{C})$. Define the dual pairing $\bar{\Omega} \times \mathbb{C}^g \rightarrow \mathbb{R}$, given by $(l, x) \mapsto \text{Im } l(x)$. We have the canonical homomorphism

$$\bar{\Omega} \rightarrow \text{Hom}(\Lambda, \mathbb{C}_1), \text{ given by } l \mapsto \exp(2\pi i \text{Im } l(\cdot))$$

The kernel of this map is called the *dual lattice* and is denoted by

$$\hat{\Lambda} = \{l \in \bar{\Omega} : \text{Im } l|_\Lambda \in \mathbb{Z}\}.$$

The map induces an isomorphism $\bar{\Omega}/\hat{\Lambda} \cong \text{Pic}^0(X)$. From now on we will denote $\text{Pic}^0(X)$ by \hat{X} . It is easy to check that $\hat{}$ is an exact functor.

Now we are in position to define another important map, namely ϕ_L . Let L be a line bundle with $c_1(L) = H$. Then for any point $x \in X$, the line bundle $t_x^*L \otimes L^{-1}$ is in $\text{Pic}^0(X)$. Therefore we can define $\phi_L: X \rightarrow \hat{X}$ given by

$$x \mapsto t_x^*L \otimes L^{-1}.$$

The most important properties are listed in the following proposition.

Proposition 1.15. ϕ_L depends only on the first Chern class $H = c_1(L)$ of L and its analytic representation is $\Phi_H(v) = H(v, \cdot)$.

If L is a polarisation, then ϕ_L is an isogeny.

For any homomorphism $f: X \rightarrow Y$ the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi_{f^*L}} & \hat{X} \\ \downarrow f & & \uparrow \hat{f} \\ Y & \xrightarrow{\phi_L} & \hat{Y} \end{array}$$

□

In view of the fact that ϕ_L depends on the class H of L , we will often write ϕ_H instead of ϕ_L .

Definition 1.16. We denote by $K(L)$ the kernel of the map ϕ_L .

Define $\Lambda(L) = \Phi_H^{-1}(\hat{\Lambda}) = \{z \in \mathbb{C}^g : \text{Im } H(z, \Lambda) \subset \mathbb{Z}\}$.

Using this definition, we can state the following easy yet useful fact.

Proposition 1.17. $K(L) = \Lambda(L)/\Lambda$ and $K(L) = \{c \in X : t_c^*L \cong L\}$.

The group $K(L)$ comes with a natural alternating form. It may be defined by the commutator map for the theta group, but we will define it explicitly.

Definition 1.18. [BL, Prop 6.3.1] Let $L \in \text{Pic}^H(A)$ be a polarising line bundle. For all $w_1, w_2 \in \Lambda(L)$, we denote their equivalence classes by $\overline{w_1}, \overline{w_2} \in K(L)$. Then

$$e^L(\overline{w_1}, \overline{w_2}) = \exp(-2\pi i \text{Im}(H)(w_1, w_2)) \in \mathbb{C}^*.$$

Corollary 1.19. e^L is a multiplicative alternating form on $K(L)$ with values in \mathbb{C}^* , because for any $x, y, z \in K(L)$ we have

$$e^L(x + y, z) = e^L(x, z)e^L(y, z), \quad e^L(x, y) = e^L(y, x)^{-1}$$

and in particular $e^L(x, x) = 1$.

Theorem 1.20. [BL, Thm 6.3.4] L is a polarising line bundle if and only if e^L is a nondegenerate form.

□

Most of the time, when we work with polarisations on abelian varieties, we think almost interchangeably about Riemann forms and line bundles. We may do that because the positive definite hermitian form defines a line bundle up to translation. This is the content of the following proposition.

Proposition 1.21. [BL, Prop 2.5.3 and 2.5.4] Let L be a polarising line bundle on A . Let L' be a line bundle. Then the following are equivalent.

1. $L' \otimes L^{-1} \in \text{Pic}^0(A)$
2. $\phi_L = \phi_{L'}$
3. $c_1(L) = c_1(L')$
4. $L = t_c^*(L')$ for some $c \in A$

□

To finish the section on dual abelian varieties, let us recall the fact that the dual abelian variety comes with the dual polarisation.

Proposition 1.22. [BL, Prop 14.4.1] Suppose H is of type (d_1, \dots, d_g) . There is a unique polarisation \hat{H} on \hat{X} characterised by the equivalent properties

$$\phi_H^* \hat{H} \equiv d_1 d_g H \quad \text{or,} \quad \phi_{\hat{H}} \phi_H = d_1 d_g \text{id}_X.$$

The polarisation \hat{H} is of type $(d_1, \frac{d_1 d_g}{d_{g-1}}, \dots, \frac{d_1 d_g}{d_2}, d_g)$.

□

1.2.2 Characteristics

Now, we will define a characteristic of a positive definite line bundle. Start with the positive definite Riemann form H .

Definition 1.23. A decomposition for H is a decomposition of \mathbb{C}^g into two real vector spaces V_1, V_2 such that $\Lambda_1 = V_1 \cap \Lambda$ and $\Lambda_2 = V_2 \cap \Lambda$ is a direct sum decomposition of Λ into isotropic sublattices with respect to $\text{Im}(H)$.

Let $x = (x_1, x_2)$. We define a map $\chi_0(x) = \exp(\pi i \operatorname{Im}(H)(x_1, x_2))$. This is a semicharacter, because

$$\chi_0(x + y) = \chi_0(x)\chi_0(y) \exp(\pi i \operatorname{Im}(H)(x, y)) \exp(-2\pi i \operatorname{Im}(H)(x_2, y_1)).$$

Therefore we can construct a line bundle $L_{(H, \chi_0)}$. We have the following:

Lemma 1.24. [BL, Lemma 3.1.2] *Suppose H is a positive definite hermitian form and $\mathbb{C}^g = V_1 \oplus V_2$ is a decomposition for H . Then*

1. $L_0 = L_{(H, \chi_0)}$ is the unique line bundle whose semicharacter is trivial on Λ_1 and Λ_2 .
2. For every $L = L_{(H, \chi)}$, there exists $c \in \mathbb{C}^g$, such that $L \cong t_c^*(L_0)$. The vector $c \in \mathbb{C}^g$ is uniquely determined up to translations by elements of $\Lambda(L)$. \square

Definition 1.25. A vector $c \in \mathbb{C}^g$, sometimes written as $c_1 + c_2$, such that $L \cong t_c^*(L_0)$ is called a *characteristic* of a line bundle L with respect to a decomposition $V_1 \oplus V_2$.

Let us state a useful lemma, which describes previous notions in the new setting

Lemma 1.26. [BL, Lemma 3.1.4] *Let L be a polarisation of type (d_1, \dots, d_g) and let us consider a decomposition $\mathbb{C}^g = V_1 \oplus V_2$ for L . Then*

1. $\Lambda(L) = \Lambda(L)_1 \oplus \Lambda(L)_2$, with $\Lambda(L)_i = \Lambda(L) \cap V_i$, for $i = 1, 2$.
2. $K(L) = K(L)_1 \oplus K(L)_2$ with $K(L)_i = \Lambda(L)_i / \Lambda_i$, for $i = 1, 2$.
3. $K_i \cong \mathbb{Z}^g / D\mathbb{Z}^g = \bigoplus_{k=1}^g \mathbb{Z} / d_k \mathbb{Z}$, for $i = 1, 2$.

\square

Corollary 1.27. *A principal polarisation H on A gives an isomorphism $\phi_H: A \longrightarrow \hat{A}$.*

1.2.3 Moduli space

In this section we introduce the Siegel space and the moduli space, as a quotient space.

Let (A, H) be a D -polarised abelian variety, where $D = \operatorname{diag}(d_1, \dots, d_g)$. Take a universal cover of A , which is a vector space, say V , with $A = V/\Lambda$, for some lattice Λ . As before, we can take a symplectic basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ for Λ , such that $\operatorname{Im}(H)$ has matrix $\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$.

Lemma 1.28. [BL, Lemma 3.2.1] *With the previous notation, $\frac{\mu_1}{d_1}, \dots, \frac{\mu_g}{d_g}$ is a complex basis for V .* \square

Using this basis, we define an isomorphism between V and \mathbb{C}^g . Then the matrix of the symplectic basis of Λ is of the form $[Z \ D]$, for some matrix Z .

Proposition 1.29. *[BL, Prop 8.1.1] With the previous notation*

1. $Z = Z^t$ and $\text{Im } Z$ is positive definite;
2. $(\text{Im } Z)^{-1}$ is the matrix of the hermitian form H with respect to the basis $\frac{\mu_1}{d_1}, \dots, \frac{\mu_g}{d_g}$.

□

Definition 1.30. The Siegel upper half space is

$$\mathfrak{h}_g = \{Z \in M_g(\mathbb{C}) : Z = {}^t Z, \text{Im } Z > 0\}.$$

It is useful to denote by $\langle Z \ D \rangle$ the lattice generated by column vectors of a matrix $[Z \ D]$. In other words $\langle Z \ D \rangle = Z\mathbb{Z}^g + D\mathbb{Z}^g$.

Proposition 1.31. *[BL, Prop 8.1.2] Given a type D , the Siegel upper half space is the moduli space for D -polarised abelian varieties with symplectic basis.*

Proof. For any D -polarised abelian g -fold with a symplectic basis we constructed $Z \in \mathfrak{h}_g$. Conversely, for $Z \in \mathfrak{h}_g$, we can construct a D -polarised abelian g -fold with the following construction. We start with $\Lambda_Z = \langle Z \ D \rangle$. It is a lattice of rank $2g$, since $\text{Im } Z > 0$. We define $A_Z = \mathbb{C}^g / \Lambda_Z$ and let H_Z be defined by a matrix $(\text{Im } Z)^{-1}$. The columns of $[Z \ D]$ form a symplectic basis for Λ_Z , since $\text{Im } H_Z |_{\Lambda_Z} \times \Lambda_Z$ is given by a matrix

$$\text{Im}({}^t [Z \ D] (\text{Im } Z)^{-1} \overline{[Z \ D]}) = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$$

□

Proposition 1.31 leads us to the conclusion that the moduli space of D -polarised abelian varieties should be the quotient space of \mathfrak{h}_g by an action of a symplectic group. However, making it precise contains some technical difficulties, so we only will make final statements. For proofs, see section [BL, 8.2]

Definition 1.32. Let us define an action of the real symplectic group $\text{Sp}_{2g}(\mathbb{R})$ on \mathfrak{h}_g by $M(Z) = (aZ + b)(cZ + d)^{-1}$, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Z \in \mathfrak{h}_g$.

Before introducing the moduli space of abelian varieties, we need the following notation.

Definition 1.33. $\Lambda_D = \begin{pmatrix} \text{id}_g & 0 \\ 0 & D \end{pmatrix} \mathbb{Z}^{2g}$, $G_D = \{M \in \text{Sp}_{2g}(\mathbb{Q}) : {}^t M \Lambda_D \subset \Lambda_D\}$

Theorem 1.34. [BL, Theorem 8.2.6] *The normal complex analytic space*

$$\mathcal{A}_D = \mathfrak{h}_g / G_D$$

is the moduli space for polarised abelian varieties of type D . If $D = \text{id}_g$ then we write \mathcal{A}_g instead. □

Remark 1.35. [BL, Corollary 8.2.7] There is also another approach to defining \mathcal{A}_D , where the group is a bit easier to handle but the action is twisted. To make it precise, let

$$\Gamma_D = \text{Sp}_{2g}^D(\mathbb{Z}) = \left\{ M \in M_{2g}(\mathbb{Z}) : M \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} {}^t M = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} \right\},$$

with an action defined by $M(Z) = (aZ + bD)(D^{-1}cZ + D^{-1}dD)^{-1}$, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Z \in \mathfrak{h}_g$. Then

$$\mathcal{A}_D \cong \mathfrak{h}_g / \Gamma_D$$
□

It is often easier to work with the Siegel spaces \mathfrak{h}_g than with the moduli spaces \mathcal{A}_D , because, firstly, \mathfrak{h}_g is a subset of the set of matrices and secondly, it does not depend on the polarisation.

With a similar construction, one can choose other subgroups of $\text{Sp}_{2g}(\mathbb{R})$ to define moduli spaces of abelian varieties with various structures.

1.2.4 Theta functions

This section introduces one of the most important notion in the theory of complex abelian varieties, namely theta functions. They allow us not only to understand the sections of line bundles but also, when suitably generalised, to define interesting loci in the Siegel space or the moduli space.

To start, we identify the space of global sections of the line bundle on X , using the factor of automorphy f , as the space of holomorphic maps satisfying the quasi-periodicity condition:

$$H^0(L) \cong \{\theta \in \mathcal{O}(\mathbb{C}^g) \mid \theta(x + \lambda) = f(\lambda, x)\theta(x) \text{ for all } \lambda \in \Lambda, x \in \mathbb{C}^g\}$$

Those maps are called *theta functions*. We have already defined the canonical factor of automorphy and the corresponding theta functions are called canonical theta functions.

Definition 1.36. Let $a_{(H,\chi)}$ be the canonical factor of automorphy. A function satisfying $\theta(x + \lambda) = a_{(H,\chi)}(\lambda, x)\theta(x)$ for all $\lambda \in \Lambda, x \in \mathbb{C}^g$ is called a *canonical (Riemann) theta function*.

More explicitly, let H be a hermitian form and c be a characteristic with respect to some decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$. Then the *canonical theta functions* $\theta^c: \mathbb{C}^g \rightarrow \mathbb{C}$ of characteristic c are given by

$$\theta^c(x) = e(H, B, x, c) \sum_{\lambda \in \Lambda_1} \exp(\pi(H - B)(x + c, \lambda) - \frac{\pi}{2}(H - B)(\lambda, \lambda)),$$

where B is the \mathbb{C} -bilinear extension of $H|_{\Lambda_2 \times \Lambda_2}$ and

$$e(H, B, x, c) = \exp(-\pi H(x, c) - \frac{\pi}{2}H(c, c) + \frac{\pi}{2}B(x + c, x + c)).$$

The proof that θ^c is well defined and indeed defines global sections of a suitable line bundle can be found in [BL, Section 3.2].

The first important theorem is the description of $H^0(L)$, using canonical theta functions.

Theorem 1.37. [BL, Theorem 3.2.7] Suppose $L = L(H, \chi)$ is a positive definite line bundle on A and let c be a characteristic of L with respect to some decomposition $\mathbb{C}^g = V_1 \oplus V_2$. Define $\theta_{\bar{\omega}}^c = a_L(w, \cdot)^{-1}\theta^c(\cdot + \omega)$. Then the set $\{\theta_{\bar{\omega}}^c: \bar{\omega} \in K(L)_1\}$ is a basis of the vector space $H^0(L)$ of canonical theta functions for L . As a consequence, we get $h^0(L) = \det D = d_1 \cdot \dots \cdot d_g$.

Proof. A proof can be found in [BL, 3.2.7]. It is technical and, as a main tool, it uses Fourier expansion of theta functions to prove linear independence. To prove that canonical theta functions generate $H^0(L)$ they translate the problem to classical theta functions, which are defined below, and uses their Fourier expansion. \square

When working in the moduli space, the classical Riemann theta functions have some useful properties. Therefore we define the *classical factor of automorphy*

$$\begin{aligned} e_L: \Lambda \times \mathbb{C}^g &\longrightarrow \mathbb{C}^* \\ e_L(\lambda, x) &= \chi(\lambda)e(\pi(H - B)(x, \lambda) + \frac{\pi}{2}(H - B)(\lambda, \lambda)). \end{aligned}$$

It is equivalent to the canonical factor a_L , because

$$e_L(\lambda, x) = a_L(\lambda, x) \exp(\frac{\pi}{2}B(x, x)) \exp(\frac{\pi}{2}B(x + \lambda, x + \lambda))^{-1}.$$

Therefore we can also define classical theta functions.

Definition 1.38. The functions satisfying $\theta(x+\lambda) = e_{(H,\chi)}(\lambda, x)\theta(x)$ for all $\lambda \in \Lambda$, $x \in \mathbb{C}^g$ are called *classical (Riemann) theta functions*.

More explicitly, for $Z \in \mathfrak{h}_g$ and $c', c'' \in \mathbb{R}^g$

$$\theta_{[c'']}^{[c']}(v, Z) = \sum_{l \in \mathbb{Z}^g} \exp(\pi i {}^t(l + c')Z(l + c') + 2\pi i {}^t(v + c'')(l + c'))$$

is called the *classical theta function of real characteristic* $c = (c', c'') = \begin{bmatrix} c' \\ c'' \end{bmatrix}$.

The following proposition tells us why classical theta functions are global sections of suitable line bundles.

Proposition 1.39. [BL, Remark 8.5.3] Fix $Z \in \mathfrak{h}_g$ and let A_Z be the corresponding principally polarised abelian variety.

1. The factor of automorphy $e_{[0]}^0: \Lambda_Z \times \mathbb{C}^g \rightarrow \mathbb{C}^*$, given by

$$e_{[0]}^0(\lambda_1 Z + \lambda_2, v) = \exp(-\pi i \lambda_1 Z \lambda_1 - 2\pi i {}^t v \lambda_1)$$

is the classical factor of automorphy of the line bundle of characteristic 0 in $\text{Pic}^{\text{Hz}}(A_Z)$ with respect to the decomposition $\Lambda_Z = \mathbb{Z}Z^g \oplus \mathbb{Z}^g$.

2. For every $(c', c'') \in \mathbb{R}^{2g}$ the function $\theta_{[c'']}^{[c']}(\cdot, Z)$ is a theta function with respect to the lattice $\mathbb{Z}Z^g \oplus \mathbb{Z}^g$, with functional equation

$$\theta_{[c'']}^{[c']}(v + \lambda_1 Z + \lambda_2, Z) = \exp(2\pi i ({}^t c' \lambda_2 - {}^t c'' \lambda_1) - \pi i \lambda_1 Z \lambda_1 - 2\pi i {}^t v \lambda_1) \theta_{[c'']}^{[c']}(v, Z),$$

for all $c \in \mathbb{C}^g$ and $\lambda_1, \lambda_2 \in \mathbb{Z}^g$.

3. For all $c', c'' \in \mathbb{R}^g$

$$\theta_{[c'+l_1]}^{[c'+l_2]}(\cdot, Z) = \theta_{[c'']}^{[c']}(\cdot, Z) \Leftrightarrow l_1, l_2 \in \mathbb{Z}^g.$$

This reflects the fact that in the principally polarised abelian variety, the characteristic $c \in \mathbb{C}^g$ is uniquely determined modulo the lattice.

Fix a type D . Then $\theta_{[c'']}^{[c']}(\cdot, Z)$ is a theta function with a factor $e_{[0]}^0$ with respect to the lattice $\Lambda_Z = \mathbb{Z}Z^g \oplus D\mathbb{Z}^g$ if and only if $(c', c'') \in D^{-1}\mathbb{Z}^g \oplus \mathbb{Z}^g$. Moreover, the functions $\theta_{[0]}^{[c_1]}(\cdot, Z), \dots, \theta_{[0]}^{[c_N]}(\cdot, Z)$, where c_0, \dots, c_N is the set of representatives of $D^{-1}\mathbb{Z}^g/\mathbb{Z}^g$, form a basis of the vector space of classical theta functions for the line bundle on $A_{Z,D}$ determined by the factor $e_{[0]}^0$.

□

Moreover, we have

Proposition 1.40. [BL, Prop 8.5.4] *The classical Riemann theta function $\theta_{[c']}$ is holomorphic on $\mathbb{C}^g \times \mathfrak{h}_g$, for any $c', c'' \in \mathbb{R}^g$.*

□

Example 1.41. In Chapter 3, we will be interested in $(1, 3)$ polarised abelian surfaces. Therefore let us write this example explicitly. Take $Z \in \mathfrak{h}_2$. Define $D = \text{diag}(1, 3)$. Define $A_Z \in \mathcal{A}_{(1,3)}$ with the polarisation $H_Z = (\text{Im } Z)^{-1}$. Take the standard decomposition $\mathbb{C}^2 = Z\mathbb{R}^2 + \mathbb{R}^2$. Take $\mathcal{L} = L_{(H_Z, \chi_0)}$ the line bundle of characteristic 0. Denote by $\omega = (0, \frac{1}{3})$. Then $K(\mathcal{L})_1 = D^{-1}\mathbb{Z}^g/\mathbb{Z}^g = \{0, \omega, -\omega\}$ and the space of classical theta functions is spanned by $\{\theta_{[0]}^0, \theta_{[0]}^\omega, \theta_{[0]}^{-\omega}\}$.

We will also use the fact that theta functions are solutions to the heat equation.

Proposition 1.42. [BL, Prop 8.5.5] *For any symmetric matrix (s_{jk}) and any $c', c'' \in \mathbb{R}^g$, the theta function $\theta_{[c']}$ is a solution to a partial differential equation, called the heat equation:*

$$\sum_{j=1}^g \sum_{k=1}^g s_{jk} \frac{\partial^2 \theta_{[c']}}{\partial v_j \partial v_k} = 4\pi i \sum_{j=1}^g \sum_{k=j}^g s_{jk} \frac{\partial \theta_{[c']}}{\partial z_{jk}}.$$

□

1.3 Symmetric line bundles

Every abelian variety A is an abelian group so has an obvious automorphism, namely (-1) . This allows us to distinguish line bundles with extra properties. Those will be crucial in Chapter 3. In principle, we should distinguish between $(-1)_{\mathbb{C}^g}$ and $(-1)_A$ but we will abuse notation by forgetting the subscripts.

The aim of this section is to build induced actions of (-1) on various structures. We start with $(-1)_A$ being induced to the space of line bundles to define a symmetric line bundle L . Then $(-1)_A$ induces the normalised automorphism of a symmetric line bundle $(-1)_L$, which induces an isomorphism on the space of global sections $H^0(L)$. The identification of $H^0(L)$ with the space of theta functions translates the action back to $(-1)_{\mathbb{C}^g}^*$.

The first action of (-1) is an obvious pullback action on $\text{Pic}(A)$. We define

Definition 1.43. A line bundle $L \in \text{Pic}(A)$ is called *symmetric* if $(-1)^*L \cong L$. Denote by $\text{Pic}_s(A)$ the set of all symmetric line bundles and $\text{Pic}_s^H(A)$ the set of symmetric line bundle with the first Chern class equal to H .

An easy corollary says

Corollary 1.44. [BL, Corollary 2.3.7] $L(H, \chi)$ is symmetric if and only if $\text{im } \chi \subset \{\pm 1\}$. □

Example 1.45. In Section 1.2.2, we defined a line bundle L_0 of characteristic 0 with respect to some decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$. Its semicharacter is $\chi_0(\lambda) = \exp(\pi i \text{Im}(H)(\lambda_1, \lambda_2))$, so by Corollary 1.44, L_0 is symmetric.

Lemma 1.46. [BL, Lemma 4.6.2] Let (A, H) be a polarised abelian variety of dimension g .

1. $\text{Pic}_s^0(A) = \frac{1}{2}\Lambda(H) = \hat{A}[2]$ is the group of 2-torsion points on \hat{A} .
2. $\text{Pic}_s^H(A)$ is a principal homogeneous space over $\text{Pic}_s^0(A)$.

□

The action of $\text{Pic}_s^0(A)$ on $\text{Pic}_s^H(A)$ is induced by the map $c \mapsto t_c^* L_0$

We would like to define the induced (-1) action on the space of global sections of L . Before that, we need the following definition and lemma.

Definition 1.47. An *isomorphism of L over (-1)* is a biholomorphic map $\phi: L \rightarrow L$ such that the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L \\ \downarrow & & \downarrow \\ A & \xrightarrow{(-1)} & A \end{array}$$

and the induced map on fibres is \mathbb{C} -linear.

A *normalised isomorphism* has the additional property that it is the identity over $0 \in A$.

Lemma 1.48. [BL, Lemma 4.6.3] Every symmetric line bundle L admits a unique normalised isomorphism over (-1) , denoted by $(-1)_L$. □

Suppose (A, H) is a polarised abelian variety and $L \in \text{Pic}_s^H(A)$. Then $(-1)_L$ induces the involution on the space of global sections

$$\theta \mapsto (-1)^* \theta$$

The obvious question is to understand this involution in terms of canonical theta functions. The Inverse Formula is the answer to this.

Theorem 1.49. [BL, Inverse Formula 4.6.4] Let $\{\theta_\omega^c : \omega \in K(L)_1\}$ denote the basis of the space of canonical theta function stated in Theorem 1.37 and let $c = c_1 + c_2 \in \Lambda(H)$ be the characteristic of L . Then

$$(-1)^*\theta_\omega^c = \exp(4\pi i \operatorname{Im} H(\omega + c_1, c_2))\theta_{-\omega-2c_1}^c$$

□

As (-1) is a linear involution, we can ask for eigenvectors of eigenvalues ± 1 . We define

Definition 1.50. $H^0(L)_+$ is the space of *even theta functions* (i.e. $+1$ eigenspace), whereas $H^0(L)_-$ is the space of *odd theta functions* (-1 eigenspace). By $h^0(L)_\pm$ we denote the dimension of those spaces.

One can compute these dimensions, but in general it involves some combinatorics, as it depends on the type of the polarisation and the characteristic of the line bundle. It will be done for $(1, 3)$ -polarised surfaces in Chapter 3.

1.3.1 Symmetric divisors

Now, we would like to understand the zero locus of theta functions better. As theta functions are quasiperiodic, we can talk about their zeros both in \mathbb{C}^g and in A . It is useful to use the language of divisors although one could translate everything to theta functions and their Taylor series. As the letter D is being used for the type of polarisation, we will use \mathfrak{D} to denote divisors. As we are ultimately interested in the zeros of theta functions, we will always assume that \mathfrak{D} is ample and effective. Let us introduce some notation.

Definition 1.51. Let A be an abelian variety. Let \mathfrak{D} be an ample effective divisor on A . Then $L = \mathcal{O}_A(\mathfrak{D})$ is an ample line bundle. Moreover, there exists $\theta \in H^0(L) \setminus \{0\}$, unique up to multiplication by a constant, such that $\pi^*\mathfrak{D} = (\theta = 0)$, where $\pi: \mathbb{C}^g \rightarrow A$ is the usual projection.

An ample effective divisor \mathfrak{D} is called *symmetric* if $(-1)^*\mathfrak{D} = \mathfrak{D}$.

By $\operatorname{mult}_x \mathfrak{D}$ we denote the multiplicity of \mathfrak{D} in a point $x \in A$. A symmetric divisor is called *even or odd* depending on the parity of $\operatorname{mult}_0 \mathfrak{D}$.

This definition is consistent, as it is obvious that a divisor is symmetric if and only if its theta function is even or odd and in that case, the multiplicity of \mathfrak{D} at 0 is the order of vanishing of the Taylor series of θ .

We are interested in the behaviour of \mathfrak{D} in the set of 2-torsion points. The following proposition is the first step towards it.

Proposition 1.52. [BL, Prop 4.7.2] Let $A = \mathbb{C}^g/\Lambda$ be an abelian variety and $L = L(H, \chi)$ be an ample symmetric line bundle. Let $\lambda \in \frac{1}{2}\Lambda$, so $c = \pi(\lambda)$ is a 2-torsion point on A . For any symmetric divisor \mathfrak{D} with $L = \mathcal{O}(\mathfrak{D})$ we have

$$(-1)^{\text{mult}_c(\mathfrak{D})} = \chi(2\lambda)(-1)^{\text{mult}_0(\mathfrak{D})}$$

□

Before stating our main result in this section, we need to prove two lemmas and introduce one more definition.

Lemma 1.53. Let A, H be a (d_1, \dots, d_g) -polarised abelian variety. Then

1. If d_g is odd, then $2\Lambda(H) \cap \Lambda = 2\Lambda$.
2. Let χ be a semicharacter for H , with $\text{im}(\chi) = \{\pm 1\}$. Then $\chi(2\lambda) = 1$.

Proof. The first part follows from the fact that 2 and d_g are coprime and as $K(H) = \Lambda(H)/\Lambda$ is of odd order, we have $x \in \Lambda(H) \setminus \Lambda$ if and only if $2x \in \Lambda(H) \setminus \Lambda$.

As for the second, it is obvious that $\chi(2x) = \chi(x)^2 = 1$. □

Lemma 1.54. Let $\chi(2\lambda) = \exp(\pi i \text{Im}(H)(2\lambda_1, 2\lambda_2))$ be a semicharacter for H defining L of characteristic 0. There exist exactly $2^{g-1}(2^g - 1)$ points $\lambda \in \frac{1}{2}\Lambda/\Lambda$ such that $\chi(2\lambda) = -1$.

Proof. There are exactly $2^g - 1$ non-zero elements $\lambda_1 \in \frac{1}{2}\Lambda_1/\Lambda_1$. Fixing one, we get a map $\lambda_2 \mapsto \exp(\pi i \text{Im}(H)(2\lambda_1, 2\lambda_2))$. By Lemma 1.53, $\lambda_1 \notin \Lambda(H)$, so the map is a surjective homomorphism to $\{\pm 1\}$. Thus, there exist exactly 2^{g-1} elements λ_2 which map to -1 . From the construction, it is obvious that we get all possible $\lambda = \lambda_1 + \lambda_2$ that satisfy $\chi(2\lambda) = -1$, and there are exactly $2^{g-1}(2^g - 1)$ of them. □

Definition 1.55. Define

$$\begin{aligned} A_2^+(\mathfrak{D}) &= \{c \in A[2]: \text{mult}_c(D) \equiv 0 \pmod{2}\}, \\ A_2^-(\mathfrak{D}) &= \{c \in A[2]: \text{mult}_c(D) \equiv 1 \pmod{2}\}. \end{aligned}$$

The main result of this section is to compute the size of $A_2^\pm(\mathfrak{D})$. It is the subject of [BL, Exercise 4.12.14].

Proposition 1.56. Let $A = \mathbb{C}^g/\Lambda$ be an abelian variety and H a polarisation of type (d_1, \dots, d_g) , with d_g odd. Let $L \in \text{Pic}_s^H(A)$. Then

1. $2\Lambda(H) \cap \Lambda = 2\Lambda$

2. If L is of characteristic 0 and \mathfrak{D} a symmetric divisor then

$$A^\pm(\mathfrak{D}) = \begin{cases} 2^{g-1}(2^g \pm 1) & \text{if } \mathfrak{D} \text{ is even} \\ 2^{g-1}(2^g \mp 1) & \text{if } \mathfrak{D} \text{ is odd} \end{cases}$$

3. There are $2^{g-1}(2^g \pm 1)$ line bundles L such that

$$A^+(D) = 2^{g-1}(2^g \pm 1)$$

for all even symmetric divisors \mathfrak{D} .

Proof. The first part was already proved in Lemma 1.53.

As for the second, the number of λ 's such that $\chi(2\lambda) = -1$ is computed in Lemma 1.54. Then the assertion follows from Proposition 1.52.

The last part follows from the following observation. By changing \mathfrak{D} to $t_\lambda^* \mathfrak{D}$, we translate the line bundle $\mathcal{O}(\mathfrak{D})$ by λ but obviously do not change the size of $A_2^\pm(\mathfrak{D})$. However, translating by λ changes the parity of \mathfrak{D} if and only if $\chi(2\lambda) = -1$. Therefore Lemma 1.54 again gives the assertion. \square

1.3.2 Theta constants

Theta constants play the crucial role in constructing modular forms. We will not go into this beautiful theory, but we will use theta constants in Chapter 4 to find equations of hyperelliptic Jacobians. Therefore, we will state a few definitions and one result.

Definition 1.57. For any numbers $c_1, c_2 \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$, we define

$$e_*(c_1, c_2) = \exp(4\pi i \ {}^t c_1 c_2).$$

Let $\zeta, \xi \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ and $J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$. Then we define

$$e_2(\zeta, \xi) = \exp(4\pi i \ {}^t \zeta J \xi).$$

Let $Z \in \mathfrak{h}_g$. Let A_Z be a principally polarised abelian variety, with the standard decomposition $Z\mathbb{R}^g \oplus \mathbb{R}^g$.

Definition 1.58. A 2-torsion point $c = (c_1, c_2) \in A_Z[2]$ is called *even or odd* depending on the value $e_*(c_1, c_2) = \pm 1$.

Remark 1.59. Note that $e_*(c_1, c_2) = \exp(\pi i \operatorname{Im}(H)(2c_1, 2c_2)) = \chi_0(2c)$, where χ_0 is the semicharacter of a line bundle of characteristic 0.

The Inverse Formula (Theorem 1.49) translates into

$$\theta^c(-v) = (-1)^* \theta^c(v) = e_*(c_1, c_2) \theta^c(v).$$

In particular a characteristic c is even/odd if and only if θ^c and $\theta_{[c']}$ are even/odd functions.

According to Lemma 1.54, on a principally polarised abelian variety there are exactly $2^{g-1}(2^g \pm 1)$ even/odd 2-torsion points.

Definition 1.60. *Theta constants*, sometimes called theta null values, are values of classical theta functions with half integer values $c', c'' \in \frac{1}{2}\mathbb{Z}^g$.

$$\theta_{[c']}(0, Z) = \sum_{l \in \mathbb{Z}^g} \exp(\pi i^t(l + c')Z(l + c') + 2\pi i^t c''(l + c'))$$

Theta constants can be viewed as functions on \mathfrak{h}_g .

Proposition 1.61. $\theta_{[c']}(0, Z) = 0$ for any $Z \in \mathfrak{h}^g$ if and only if (c', c'') is an odd 2-torsion point.

Proof. For c an odd 2-torsion point, $\theta_{[c']}$ is an odd function, so $\theta_{[c']}(0, Z) = 0$. If c is even then $\theta_{[c']}(0, Z) = 0$ gives $\text{mult}_0 \theta_{[c']}(\cdot, Z) \geq 2$, so 0 is a singularity of $\theta_{[c']}$ in A_Z . Proposition 3.3 says that for a general $Z \in \mathfrak{h}_g$, the locus $\theta_{[c']}(\cdot, Z) = 0$ is smooth, which gives the assertion. \square

1.4 Endomorphisms of abelian varieties

The aim of this section is to introduce notions that lead to Poincaré's Complete Reducibility Theorem, which tells us that every abelian variety is isogenous to a product of simple abelian varieties. By simple, we mean an abelian variety of dimension g with no proper abelian subvarieties, i.e. no abelian subvarieties of dimensions $1, \dots, g-1$.

When working with isogenies, it is useful to work with *rational homomorphisms* $\text{Hom}_{\mathbb{Q}}(A, A') = \text{Hom}(A, A') \otimes \mathbb{Q}$ because of the following basic observation.

Proposition 1.62. [BL, 1.2.6 and 1.2.7] *Let $f: A \rightarrow A'$ be an isogeny of exponent e . Then, there exists a unique $g: A' \rightarrow A$, such that $fg = e \text{id}_{A'}$, $gf = e \text{id}_A$. In particular, $h \in \text{End}(A)$ is invertible in $\text{End}_{\mathbb{Q}}(A)$ if and only if it is an isogeny.*

\square

In section 1.2.1, we defined an important isogeny, namely for H a polarisation, we have $\phi_H: A \rightarrow \hat{A}$. The exponent of ϕ_H is denoted by $e(H)$ and called the exponent of

the polarisation. We denote by $\psi_H: A' \rightarrow A$ the isogeny from Proposition 1.62. We also have $\phi_H^{-1} = \frac{1}{e(H)}\psi_H \in \text{Hom}_{\mathbb{Q}}(A, A')$.

Many geometric properties of an abelian variety are encoded in the endomorphism algebra $\text{End}_{\mathbb{Q}}(A) = \text{End}(A) \otimes \mathbb{Q}$. One of the most useful properties of the endomorphism algebra is existence of the involution called the Rosati involution.

Definition 1.63. Let (A, L) be a polarised abelian variety. Then

$$': \text{End}_{\mathbb{Q}}(A) \rightarrow \text{End}_{\mathbb{Q}}(A) \text{ given by } f \mapsto f' = \phi_L^{-1} \hat{f} \phi_L$$

is called the *Rosati involution*.

From now on we will mainly consider *symmetric endomorphisms* and we therefore define $\text{End}^s(A) = \{f \in \text{End}(A) : f = f'\}$.

In the case of a principally polarised abelian variety we usually identify A with \hat{A} , and therefore the Rosati involution is identified with dualisation.

The following lemma implies that a general abelian variety has no non-trivial symmetric endomorphisms.

Lemma 1.64. *Let $Z \in \mathfrak{h}_g$. Let $f \in \text{End}(A_Z)$. Denote by $\rho_r(f)$ and $\rho_a(f)$ the rational and analytic representation of f . Then f is symmetric if and only if $\rho_r(f) = \begin{bmatrix} \alpha & \beta \\ \gamma & {}^t\alpha \end{bmatrix}$, with $\alpha, \beta, \gamma \in M_g(\mathbb{Z})$, with β and γ antisymmetric, satisfying the matrix equation*

$$Z\beta Z + {}^t\alpha Z - Z\alpha - \gamma = 0 \tag{1.1}$$

Proof. As the Rosati involution is adjoint to $\text{Im}(H)$ (see [BL, Prop 5.1.1]), we have

$${}^t\rho_r(f) \begin{bmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{bmatrix} \rho_r(f)$$

if and only if f is symmetric. Therefore $\rho_r(f) = \begin{bmatrix} \alpha & \beta \\ \gamma & {}^t\alpha \end{bmatrix}$, with $\alpha, \beta, \gamma \in M_g(\mathbb{Z})$, with β and γ antisymmetric. As for the equation, let us note that analytic and rational representations are connected by

$$\rho_a(f)[Z \text{ id}] = [Z \text{ id}]\rho_r(f).$$

Thus

$$(\rho_a(f)Z, \rho_a(f)) = (Z\alpha + \gamma, Z\beta + {}^t\alpha),$$

so

$$0 = \rho_a(f)Z - \rho_a(f)Z = Z\alpha + \gamma - Z\beta Z - {}^t\alpha Z.$$

□

Remark 1.65. The above lemma is a generalisation of a well-known lemma in dimension 2, which allowed Humbert to define the so-called singular relation. To see it, we have to write explicitly

$$Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}, \alpha = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \beta = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$$

As the matrices in (1.1) are antisymmetric, it induces only one equation

$$b(z_1 z_3 - z_2^2) + a_1 z_2 + a_3 z_3 - z_1 a_2 - z_2 a_4 - c = 0,$$

which is usually rewritten as

$$az_1 + bz_2 + cz_3 + d \det(Z) + e = 0.$$

In Chapter 2, we will try to understand the locus of non-simple abelian varieties. We will need the following definition.

Definition 1.66. Let $\iota: Y \rightarrow A$ be an abelian subvariety of a principally polarised abelian variety (A, H) . Then ι^*H is a polarisation on Y . Define the *exponent of Y* by $e(Y) = e(\iota^*H)$. Moreover, we define the *norm-endomorphism of A associated to Y* by

$$\text{Nm}_Y = \iota \psi_{\iota^*H} \hat{\iota} \phi_H.$$

$\varepsilon_Y = \frac{1}{e(Y)} \text{Nm}_Y \in \text{End}_{\mathbb{Q}}^s(A)$ is called the *associated symmetric idempotent*.

The name symmetric idempotent is justified by the following.

Lemma 1.67. [BL, 5.3.1] *Let Y be an abelian subvariety of A . Then*

$$\text{Nm}_Y = (\text{Nm}_Y)' \text{ and } \text{Nm}_Y^2 = e(Y) \text{Nm}_Y.$$

□

Definition 1.68. For any symmetric idempotent $\varepsilon \in \text{End}_{\mathbb{Q}}^s(A)$ there exists $n \in \mathbb{N}$ such that $n\varepsilon \in \text{End}^s(A)$, and we can define the abelian subvariety $A^\varepsilon = \text{im}(n\varepsilon)$.

If A is principally polarised, we get a nice characterisation of norm-endomorphisms. Before that, we need one more definition.

Definition 1.69. $f \in \text{End}(A)$ is called *primitive* if it is not a multiple of another endomorphism, i.e. there is no $n > 2$, such that $f = ng, g \in \text{End}(A)$. It is equivalent to say that $A[n] \not\subseteq \ker(f)$ for any $n > 2$.

Lemma 1.70. [BL, Norm-endomorphism Criterion 5.3.4] *Let (A, L) be a principally polarised abelian variety. For $f \in \text{End}(A)$ the following statements are equivalent*

1. $f = \text{Nm}_Y$ for some abelian subvariety Y of A .
2. $f = 0$, or f is primitive symmetric and satisfies $f^2 = ef$ for some positive integer e .

□

The next theorem is the main tool in proving Poincaré's Reducibility Theorems.

Theorem 1.71. [BL, 5.3.2] *The assignments $Y \mapsto \varepsilon_Y$ and $\varepsilon \mapsto A^\varepsilon$ are inverse to each other and give a bijection between the sets of abelian subvarieties of A and symmetric idempotents in $\text{End}_{\mathbb{Q}}(A)$*

□

The main advantage of translating the existence of subvarieties into symmetric idempotents is the fact that the latter have an obvious canonical involution $\varepsilon \mapsto 1 - \varepsilon$. This leads to the following definition.

Definition 1.72. Let A be a polarised abelian variety. Then the polarisation induces a canonical involution on the set of abelian subvarieties of A :

$$Y \mapsto Z = A^{1-\varepsilon_Y}$$

We call Z the *complementary abelian subvariety* of Y in A , and (Y, Z) a *pair of complementary abelian subvarieties*.

Let us state Poincaré's Reducibility Theorems.

Theorem 1.73. [BL, Poincaré's Reducibility Theorem 5.3.5] *Let (A, L) be a polarised abelian variety and (Y, Z) a pair of complementary abelian subvarieties of A . Then the map*

$$(\text{Nm}_Y, \text{Nm}_Z): A \longrightarrow Y \times Z$$

is an isogeny.

Theorem 1.74. [BL, Poincaré’s Complete Reducibility Theorem 5.3.7] For an abelian variety A there is an isogeny

$$A \longrightarrow A_1^{n_1} \times \dots \times A_r^{n_r}$$

with simple abelian varieties A_i not isogenous to each other. Moreover the abelian varieties A_i and integers n_i are uniquely determined up to isogenies and permutations.

1.5 Curves and their Jacobians

Most results from this section can be found in [ACGH, Sections 1.3, 1.5, 6.3].

Let C be a smooth projective curve of genus g . Let us consider $\text{Pic}^0(C)$, the group of line bundles of degree 0 on C . It is also useful to view it as the group of degree 0 divisors $\text{Div}^0(C)$ modulo the subgroup of principal divisors. On the other hand, we have the vector space $H^0(\omega_C)^*$ of holomorphic 1-forms. It is of dimension g . Inside it, there is the first homology group $H_1(C, \mathbb{Z})$, which is a free abelian group of rank $2g$. If we choose a symplectic basis of $H_1(C, \mathbb{Z})$, then the intersection matrix is of the form $\begin{bmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{bmatrix}$. With the help of the Riemann relations and the fact that the intersection product is Poincaré dual to the cup product one can prove the following.

Proposition 1.75. [BL, Prop 11.1.2] There exists a Riemann form H on $H^0(\omega_C)^*$, which makes the quotient $H^0(\omega_C)^*/H_1(C, \mathbb{Z})$ into a canonically principally polarised abelian variety of dimension g .

$\text{Pic}^0(C)$ and $H^0(\omega_C)^*/H_1(C, \mathbb{Z})$ are connected via the *Abel-Jacobi map*:

$$\sum_{k=1}^N (p_k - q_k) \mapsto (\omega \mapsto \sum_{k=1}^N \int_{q_k}^{p_k} \omega) \in H^0(\omega_C)^*/H_1(C, \mathbb{Z})$$

The first important theorem is the Abel-Jacobi Theorem. It is a combination of two theorems: Abel’s Theorem, which says that the map is injective, and Jacobi’s Inversion Theorem, which gives surjectivity.

Theorem 1.76. [BL, Abel-Jacobi Theorem 11.1.3] The Abel-Jacobi map induces a canonical isomorphism $\text{Pic}^0(C) \cong H^0(\omega_C)/H_1(C, \mathbb{Z})$.

Proof. A proof of Abel’s Theorem is the content of [ACGH, Section 1.3]. Jacobi’s Inversion Theorem is proved as a part of Poincaré’s Formula in [ACGH, Section 1.5]. \square

Definition 1.77. Any divisor Θ on $H^0(\omega_C)^*/H_1(C, \mathbb{Z})$ or $\text{Pic}^0(C)$ such that $\mathcal{O}(\Theta)$ defines the principal polarisation from Proposition 1.75 is called a *theta divisor*.

Definition 1.78. Define the *Jacobian* variety of a smooth curve C to be the principally polarised abelian variety $(\text{Pic}^0(C), \Theta)$ and denote it by (JC, Θ) or by JC .

Example 1.79. For $g = 0$, we have $C = \mathbb{P}^1$ and $\text{Pic}^0(\mathbb{P}^1) = 0$ is trivial. For $g = 1$, C is an elliptic curve and $\text{Pic}^0(C) = \hat{C} \cong C$. The theta divisor is a point P (unique up to translation) and its Néron-Severi class is defined in Example 1.14.

We restrict our attention to the case $g > 1$, as we have seen above that cases $g = 0, 1$ are trivial.

There is a natural map from C to its Jacobian, which is called the Abel-Jacobi map.

Definition 1.80. For any $c \in C$, define the *Abel-Jacobi map*

$$\alpha_c: C \longrightarrow JC, \quad \text{given by } x \longmapsto \mathcal{O}(x - c).$$

Remark 1.81. Note that $\alpha_c(c) = 0$. Therefore $c \in C$ is sometimes called the base point of the Abel-Jacobi map and is usually omitted. By Proposition 1.82, it can always be retrieved as the preimage of 0.

Proposition 1.82. [BL, Cor 11.1.5] *For any point in a curve of genus $g \geq 1$ the Abel-Jacobi map is an embedding.*

□

The Jacobian of a curve has a universal property.

Theorem 1.83. [BL, Universal Property of the Jacobian 11.4.1] *Suppose A is an abelian variety and $f: C \longrightarrow A$ a morphism. Then there exists a unique homomorphism $\tilde{f}: JC \longrightarrow A$ such that for every $c \in C$ the following diagram commutes*

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ \downarrow \alpha_c & & \uparrow t_{f(c)} \\ JC & \xrightarrow{\tilde{f}} & A \end{array}$$

□

Remark 1.84. The uniqueness of \tilde{f} applied to the Abel-Jacobi map implies that the image of $\alpha(C)$ generates JC .

Let us state Torelli's theorem, which says that the Jacobian variety completely characterises the curve.

Theorem 1.85. [BL, Torelli's Theorem 11.1.7] Suppose C and C' are smooth curves. If their Jacobians (JC, Θ) and (JC', Θ') are isomorphic as polarised abelian varieties, then C and C' are isomorphic.

Proof. A proof can be found in [ACGH, Section 6.3] □

There is a stronger version of Torelli's Theorem, which constructs an explicit isomorphism of curves. It can be found in [W]. The following theorem follows from the strong Torelli theorem and relates automorphism groups of a curve and its Jacobian. It is also stated in [BL, Exercise 11.12.19].

Theorem 1.86. [W] Let C be a smooth curve and JC its Jacobian. Then

$$\mathrm{Aut}(C) = \begin{cases} \mathrm{Aut}(JC) & \text{if } C \text{ is hyperelliptic} \\ \mathrm{Aut}(JC)/(-1) & \text{if } C \text{ is not hyperelliptic} \end{cases}$$

1.5.1 Moduli of Jacobians

There is an interesting question of distinguishing the locus of Jacobians inside \mathcal{A}_g . It is called the Schottky problem and it is unsolved in general. For small dimensions, however, we can say quite a lot. Let us consider $g = 2$ and $g = 3$. Then we have the following theorems.

Theorem 1.87. [BL, 11.8.2] Let (A, Θ) be a principally polarised surface. We have two cases

1. $\{\Theta = 0\}$ is a smooth curve of genus 2, say C and $A = JC$, or
2. $\{\Theta = 0\}$ is a reducible curve $E \cup F$ of genus 2 and $A = E \times F$ is a canonically polarised product of elliptic curves.

Theorem 1.88. [BL, 11.8.2] A principally polarised threefold is either the Jacobian of a smooth genus 3 curve or the product of a principally polarised abelian surface with an elliptic curve, with the product polarisation.

□

Denote by \mathcal{M}_g the moduli space of smooth genus g curves. For $g > 3$, we have $\dim(\mathcal{M}_g) = 3g - 3 < \binom{g+1}{2} = \dim(\mathcal{A}_g)$, so dimension counting tells us that the locus of Jacobians is a proper subset of \mathcal{A}_g .

1.6 Symplectic forms on finite abelian groups

In Chapter 2, we will be interested in special isotropic subspaces of the kernel of polarisation $K(L)$. We need a result that says that they are equivariant under the action of symplectic group. It will follow from Propositions 1.93 and 1.94.

As $K(L)$ is a finite abelian group, we need some definitions and basic facts from the theory of finite symplectic \mathbb{Z} -modules. We will use the fundamental theorem of finite abelian groups.

Theorem 1.89. *Any finite \mathbb{Z} -module X is of the form $\mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_k}$, where $d_1 > 1$ and $d_i | d_{i+1}$, $i = 1, \dots, k-1$, for a unique (d_1, \dots, d_k) . We will call (d_1, \dots, d_k) the type of the \mathbb{Z} -module.*

Definition 1.90. We say that X is of *dimension* k and a *basis* of X is an image of any basis of \mathbb{Z}^k by an epimorphism

$$\mathbb{Z}^k \longrightarrow \mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_k}.$$

Assume X is of the form $(\mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_k})^2$. A *symplectic form* on X is the image of a symplectic form on $\mathbb{Z}^k \times \mathbb{Z}^k$.

Example 1.91. Let L be a polarisation of type (d_1, \dots, d_k) , with $d_1 = \dots = d_m = 1$, for some m on an abelian variety A and consider a decomposition $V_1 \oplus V_2$ for L . By Lemma 1.26 we have

$$K(L) = K(L)_1 \oplus K(L)_2, \quad \text{with } K(L)_1 \cong K(L)_2 \cong \mathbb{Z}_{d_m} \times \dots \times \mathbb{Z}_{d_k},$$

so both $K(L)_1$ and $K(L)_2$ are of the same type and of dimension $k - m$. Moreover $K(L)$ has symplectic form e^L .

It is convenient to work under the assumption that the domain and codomain are of the same type. Therefore, we define

Definition 1.92. Let (X, ω_X) and (Y, ω_Y) be symplectic \mathbb{Z} -modules of the same type. Then a \mathbb{Z} -linear map $f: X \longrightarrow Y$ is called an *antisymplectic map* if for all $x, y \in X$, we have

$$\omega_X(x, y) = -\omega_Y(f(x), f(y)). \tag{1.2}$$

Proposition 1.93. *Every antisymplectic map is a bijection and the inverse map is also antisymplectic. Moreover, the space of antisymplectic maps is modelled on $\text{Sp}(X, \mathbb{Z})$, i.e. for every antisymplectic $f, g: X \longrightarrow Y$, we have $g^{-1} \circ f \in \text{Sp}(X, \mathbb{Z})$ and for all $s_X \in \text{Sp}(X, \mathbb{Z})$, the maps $f \circ s_X$ are antisymplectic.*

By symmetry it is modelled on $\text{Sp}(Y, \mathbb{Z})$, too.

Proof. By equation (1.2), if $f(x) = 0$, then $\omega_X(x, y) = 0$ for every y , so $x = 0$, which means f is injective. Bijectivity comes from the domain and codomain having the same type. The rest of the proposition comes from the fact that $(-1) \cdot (-1) = 1$. \square

Proposition 1.94. *Let (X, ω_X) and (Y, ω_Y) be symplectic \mathbb{Z} -modules of the same type. Consider $(X \oplus Y, \omega_X + \omega_Y)$. Then the set of graphs of antisymplectic maps is the set of maximal isotropic subspaces of $X \oplus Y$ intersecting X and Y only in $\{0\}$.*

Proof. It is obvious that the graph of an antisymplectic map is an isotropic subspace with the desired properties. For the converse, let Z be a maximal isotropic subspace. We can define projections π_X and π_Y and from the properties of Z , we deduce that π_X and π_Y are bijections. Therefore $\pi_Y \circ \pi_X^{-1}$ is antisymplectic and Z is the graph of $\pi_Y \circ \pi_X^{-1}$. \square

Chapter 2

Generalised Humbert locus

The main motivation for this chapter is to understand all discrete invariants of the locus of non-simple principally polarised abelian varieties. The idea is to make a very natural definition of generalised Humbert locus $\text{Is}_{\underline{d}}^g$ of principally polarised abelian g -folds having a subvariety with induced polarisation of type \underline{d} . In the main result we prove that $\text{Is}_{\underline{d}}^g$ is a non-empty irreducible locus of dimension $\binom{k+1}{2} + \binom{g-k+1}{2}$ in the moduli of principally polarised g -folds. The idea comes from the work of Humbert [H] who found equations of non-simple abelian surfaces. Using the notion of complementary abelian subvarieties, we translate the problem to the setting, which can be easily generalised. The notation comes from the fact that an abelian variety is non-simple if and only if it is an image of so called allowed isogeny.

In this chapter we also try to 'invert the Prym construction' i.e. characterise Jacobians of curves which are double covers branched in at most 2 points. These results are motivated by a few intriguing questions of Ortega and Lange. In [LO] they recently found one case of principally polarised Prym varieties. It turns out that their construction describes the hyperelliptic Jacobians which belong to $\text{Is}_{(3,3)}^4$. As this point of view is a rich source of questions, it may lead to finding a new way of understanding some loci in theory of curves.

In Section 2.4 we find equations of one of preimages of $\text{Is}_{(1,p)}^4$ in \mathfrak{h}_4 similar to the equations found by Humbert.

2.1 Preliminaries

Let us introduce the following notation: k, g are integers such that $0 < k \leq \frac{g}{2}$, and $\underline{d} = (d_1, \dots, d_k)$ and $\underline{d}' = (1, \dots, 1, d_1, \dots, d_k)$ are k - and $(g - k)$ -tuples of positive integers such that $d_i | d_{i+1}$. We will also abuse notation by freely identifying an abstract

abelian variety with its image under inclusions.

We will often consider products of polarised abelian varieties. If (M, H_M) and (N, H_N) are polarised abelian varieties of types \underline{d} and \underline{d}' then, even if not written explicitly, we will treat the product $M \times N$ as the $(\underline{d}, \underline{d}')$ -polarised variety with the canonical polarisation. It is strictly speaking incorrect, but one can deal with it in two ways. The first is to remember to make a correct permutation of coordinates and the second is to allow the less standard matrix of a symplectic form, namely

$$\begin{bmatrix} 0 & \underline{d} & 0 & 0 \\ -\underline{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{d}' \\ 0 & 0 & -\underline{d}' & 0 \end{bmatrix}.$$

In Chapter 1, we introduced the notion of complementary abelian subvarieties. There is some more information that can be extracted in the principally polarised case.

Proposition 2.1. *Let (A, Θ) be a principally polarised abelian variety. The following conditions are equivalent:*

1. *there exists $M \subset A$ such that $\Theta|_M$ is of type \underline{d} .*
2. *there exists $N \subset A$ such that $\Theta|_N$ is of type \underline{d}' .*
3. *there exists a pair (M, N) of complementary abelian subvarieties in A of types \underline{d} and \underline{d}' respectively.*
4. *there exists a polarised isogeny ρ from $M \times N$ with the product polarisation of type $(\underline{d}, \underline{d}')$ to A , such that its kernel intersects $M \times \{0\}$ and $\{0\} \times N$ in $\{0\}$.*

Proof. The equivalence of conditions (1), (2), (3) follows from the definition and [BL, Corollary 12.1.5].

(3) \Rightarrow (4) is a consequence of [BL, Corollary 5.3.6]. The condition on the kernel states that $\rho|_{M \times \{0\}}$ and $\rho|_{\{0\} \times N}$ are inclusions.

(4) \Rightarrow (3) Let us denote the inclusions by $\iota_M = \rho|_{M \times \{0\}}$ and $\iota_N = \rho|_{\{0\} \times N}$ Then $\rho(m, n) = \iota_M(m) + \iota_N(n)$ and so

$$\varepsilon_M + \varepsilon_N = \iota_M \phi_{\iota_M^* \Theta}^{-1} \iota_M \hat{m} \phi_\Theta + \iota_N \phi_{\iota_N^* \Theta}^{-1} \iota_N \hat{n} \phi_\Theta = \rho \phi_{\rho^* \Theta}^{-1} \hat{\rho} \phi_\Theta = \phi_\Theta^{-1} \phi_\Theta = 1$$

□

There are many possible conditions that define Humbert surfaces. The following theorem summarises 6 of them.

Theorem 2.2. *Let $p = 2p_1 + p_2 \in \mathbb{Z}$, where $p_2 = 0$ or 1 depending on the parity of p . Let (A, Θ) be a principally polarised abelian surface. The following conditions are equivalent:*

1. *there exists an elliptic curve $E \subset A$ such that $\Theta|_E$ is of type p ;*
2. *there exists an exact sequence*

$$0 \longrightarrow E \longrightarrow A \longrightarrow F \longrightarrow 0,$$

and therefore its dual

$$0 \longrightarrow F \longrightarrow A \longrightarrow E \longrightarrow 0,$$

such that E, F are elliptic curves and the induced map $E \times F \longrightarrow A$ is an isogeny of degree p^2 ;

3. *there exists a pair (E, F) of complementary elliptic curves in A of type p ;*
4. *$\text{End}(A)$ contains a primitive symmetric endomorphism f with discriminant p^2 ;*
5. *$\text{End}(A)$ contains a symmetric endomorphism f with analytic and rational representations given by*

$$\begin{bmatrix} 0 & 1 \\ p_1^2 + p_1 p_2 & p_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & p_1^2 + p_1 p_2 & 0 & 0 \\ 1 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & p_1^2 + p_1 p_2 & p_2 \end{bmatrix};$$

6. *(A, Θ) is isomorphic to an abelian surface defined by a period matrix*

$$\begin{bmatrix} t_1 & t_2 & 1 & 0 \\ t_2 & (p_1^2 + p_1 p_2)t_1 + p_2 t_2 & 0 & 1 \end{bmatrix},$$

with elliptic curves defined by period matrices $[t_2 + p_1 t_1 \ 1]$ and $[t_2 - (p_1 + p_2)t_1 \ 1]$ embedded as $s \mapsto (s, (p_1 + p_2)s)$ and $s \mapsto (s, -p_1 s)$.

Proof. We have already proved the equivalence of (1), (2) and (3) in Proposition 2.1 in a more general setting. Next, we will show that (3) \implies (4) \implies (5) \implies (6) \implies (1).

For the (3) \implies (4), take f to be the norm-endomorphism associated to either elliptic curve. By Lemma 1.70, f is primitive, symmetric and has characteristic polynomial $f^2 - pf$. So its discriminant equals p^2 .

(4) \implies (5) is the content of [G, Lemma 2.8]. The explicit construction can be found in [BW] and [R].

(5) \implies (6) follows from Remark 1.65, with $a = p_1^2 + p_1p_2, b = p_2, c = -1, d = e = 0$.

For the last implication, to simplify notation, we write $t' = \text{Im}(t)$ for any $t \in \mathbb{C}$. Then

$$\det(\text{Im } Z) = (p_1t'_1 + t'_2)((p_1 + p_2)t'_1 - t'_2),$$

$$H = (\text{Im } Z)^{-1} = \det(\text{Im } Z)^{-1} \begin{bmatrix} (p_1^2 + p_1p_2)t'_1 + p_2t'_2 & -t'_2 \\ -t'_2 & t'_1 \end{bmatrix}.$$

Define $\iota_E: s \mapsto (s, (p_1 + p_2)s)$. Its analytic representation is given by the matrix $\begin{bmatrix} 1 \\ p_1 + p_2 \end{bmatrix}$. Then $\phi_{\iota_E^* \Theta} = \hat{\iota}_E \circ \phi_H \circ \iota_E$ is defined by

$$\begin{bmatrix} 1 & p_1 + p_2 \end{bmatrix} \det(\text{Im } Z)^{-1} \begin{bmatrix} (p_1^2 + p_1p_2)t'_1 + p_2t'_2 & -t'_2 \\ -t'_2 & t'_1 \end{bmatrix} \begin{bmatrix} 1 \\ p_1 + p_2 \end{bmatrix} = p \begin{bmatrix} (p_1t'_1 + t'_2)^{-1} \end{bmatrix},$$

so $\Theta|_E$ is of type p . □

Condition (6) of Theorem 2.2 implies that in \mathcal{A}_2 , the locus of all principally polarised abelian surfaces satisfying the above conditions is the image of the surface given by the equation $t_3 = (p_1^2 + p_1p_2)t_1 + p_2t_2$ in \mathfrak{h}_2 , and therefore it is an irreducible surface in \mathcal{A}_2 .

Definition 2.3. The locus in \mathcal{A}_2 of all principally polarised abelian surfaces that satisfy the above conditions is called the *Humbert surface of discriminant p^2* .

Remark 2.4. Humbert showed more in [H]. He found the equations defining the preimage in \mathfrak{h}_2 of all Humbert surfaces. To be precise, any 5-tuple of integers without common divisor (a, b, c, d, e) with the same discriminant $\Delta = b^2 - 4ac - 4de$ gives us the so-called singular relation

$$at_1 + bt_2 + ct_3 + d(t_2^2 - t_1t_3) + e = 0$$

In other words, the period matrix $Z = \begin{bmatrix} t_1 & t_2 \\ t_2 & t_3 \end{bmatrix} \in \mathfrak{h}_2$ is a solution to a singular relation with $\Delta = p^2$ if and only if the abelian surface $A_Z = \mathbb{C}^2 / (ZZ^2 + \mathbb{Z}^2)$ contains an elliptic curve with induced polarisation of type p .

If we recall that $\mathcal{A}_2 = \mathfrak{h}_2 / \text{Sp}(4, \mathbb{Z})$, then it means that all matrices which satisfy the singular equation for some $\Delta = p^2$ form a symplectic orbit. Then condition 6 of Theorem 2.2 says that there always exists a 'normalised' period matrix i.e. such that

$$a = p_1^2 + p_1p_2, \quad b = p_2, \quad c = -1, \quad d = e = 0.$$

2.2 Generalised Humbert locus

We would like to generalise the notion of Humbert surface to higher dimensions. There are a few immediate problems that arise. Firstly, Humbert surfaces are divisors in \mathcal{A}_2 globally defined by one equation in \mathfrak{h}_2 , whereas in higher dimensions that is not the case. So we cannot hope for a nice symplectic invariant like Δ . Secondly, all elliptic curves are essentially canonically principally polarised whereas in higher dimensions polarisations are much richer. Thirdly, it has to be decided what route should be taken and what is the purpose of the new definition.

A good starting point is Poincaré's Reducibility Theorem. In this view, an Humbert surface just comes from a decomposition of non-simple surfaces into simple factors in terms of the degree of isogeny. As the notion of complementary abelian subvarieties is very natural and useful we make the following definition.

Definition 2.5. The *generalised Humbert locus of type $\underline{d} = (d_1, \dots, d_k)$ in dimension g* , denoted by $\text{Is}_{\underline{d}}^g$, is the locus in \mathcal{A}_g of principally polarised g -folds X such that there exists a k -dimensional subvariety Z of X with the induced polarisation from X to Z being of type \underline{d} . If $d_1 = d_k$ then we say it is of principal type.

Firstly some obvious remarks and connections with previously known notions:

1. Every non-simple principally polarised abelian variety belongs to a generalised Humbert locus for some \underline{d} .
2. The name comes from the fact that for surfaces, it gives back Humbert surfaces of discriminant d^2 .
3. Note that principal type does not mean $d_k = 1$. If $d_k = 1$ then the isogeny from Proposition 2.1 is actually an isomorphism and we get only the locus of products of principally polarised abelian varieties. If we restrict the generalised Humbert locus of principal type to Jacobians of smooth curves then we get Jacobians containing Prym-Tyurin varieties.

The first example of generalised Humbert locus arises in dimension three.

Proposition 2.6. *A variety $A \in \text{Is}_n^3$ is either a product of an elliptic curve with a surface or the Jacobian of a smooth genus 3 curve which is an n -covering of an elliptic curve branched in 4 points and all such Jacobians are contained in Is_n^3 . In other words, the only non-simple Jacobians are Jacobians of covers of elliptic curves.*

Proof. By Proposition 1.88 we can restrict our attention to Jacobians. If $JC \in \text{Is}_n^3$, then it contains an elliptic curve, say E . Taking an Abel-Jacobi map composed with the dual of the inclusion, we get a map $C \rightarrow E$. As $\Theta|_E$ is of type n , it is an n to 1 cover. Using the Hurwitz formula, we get that it has to be branched in 4 points. Conversely, if we have an n to 1 cover $\pi : C \rightarrow E$, then we can have the *norm map* $\text{Nm}_\pi : JC \rightarrow JE = E$, given by $\text{Nm}_\pi(P - Q) = \pi(P) - \pi(Q)$. Moreover $\text{Nm}_\pi(\pi^{-1}(P' - Q')) = n(P' - Q')$ is a multiplication by n , so the induced polarisation is of type n . □

Before stating Propositions 2.7 – 2.11, I would like to note that the results are based on ideas which are probably known to experts, but I was not able to find any exact references in the literature. I assume, the reader is familiar with the theory of norm maps (see [ACGH, Appendix B] for details).

The first result is a characterisation of a 3-dimensional family of Jacobians of étale double covers of genus 2 curves.

Proposition 2.7. *Denote by \mathcal{JH} the locus of hyperelliptic Jacobians. Then $\text{Is}_2^3 \cap \mathcal{JH}$ is exactly the locus of Jacobians of étale double covers of genus 2 curves.*

Proof. Let $f : C \rightarrow C'$ be an étale double cover. It is defined by a 2-torsion point in JC' , say η . Then $\ker(f^*) = \{0, \eta\}$ ([ACGH, Ex. B.14]). Therefore $f^*JC' = JC' / \langle \eta \rangle$ is a $(1, 2)$ -polarised abelian surface, which is an abelian subvariety of JC . Hence $JC \in \text{Is}_2^3$. To finish the implication, let us note that C' , being of genus 2, has to be hyperelliptic and any étale double cover of a hyperelliptic curve is hyperelliptic. This implication can be easily deduced from the proof of part (a) of [M2, Theorem 7.1], or from [O].

As for the other implication, let $JC \in \text{Is}_2^3 \cap \mathcal{JH}$. Denote by E an elliptic curve in JC . Denote by ι_E the involution of C which defines the double cover and by i_E its extension to JC . From construction, $\text{im}(1 - i_E) = E$, and therefore $\epsilon_E = \frac{1 - i_E}{2}$. On the other hand, if we denote by ι the hyperelliptic involution on C , then its extension to JC is (-1) . This is because for a branch point Q , we have $(P - Q) + (\iota(P) - Q) = 0$, being the principal divisor of a pullback of a meromorphic function on \mathbb{P}^1 . Now, $\iota \circ \iota_E$ is an automorphism on C and its extension is $-i_E$. Denoting by $Z = \text{im}(1 - (-i_E))$ and $\epsilon_Z = \frac{1 + i_E}{2}$, we immediately get that $\epsilon_Z + \epsilon_E = 1$ and so (E, Z) is a pair of complementary abelian subvarieties of JC .

Denote by $\iota_Z = \iota \circ \iota_E$ and let $C' = C/\iota_Z$ be the quotient curve with the cover $f : C \rightarrow C'$ given by $P \mapsto \{P, \iota_Z(P)\}$. Then $f^*(\{P, \iota_Z(P)\}) = P + i_E(P)$, so $Z = \text{im}(f^*)$.

It is obvious that $\dim JC' = \dim Z = 2$, so C' is of genus 2 and by the Hurwitz Formula f has to be an étale double cover. \square

Remark 2.8. Proposition 2.7 says that a genus 3 curve is a double cover of a genus 2 curve if and only if it is both hyperelliptic and a double cover of an elliptic curve branched in 4 points.

The above can be generalised further.

Proposition 2.9. *Let $\underline{2} = (2, \dots, 2)$ be a g -tuple of 2's. Then, $\text{Is}_{\underline{2}}^{2g+1} \cap \mathcal{JH}$ is exactly the locus of Jacobians of étale double covers of hyperelliptic genus $g+1$ curves.*

Proof. The proof that the Jacobians of étale double covers belongs to $\text{Is}_{\underline{2}}^{2g+1}$ comes from the Prym construction ([BL, Theorem 12.3.3]).

As for the other direction, the proof is very similar to the previous one, so we will keep the notation. Let E be a subvariety with the induced polarisation of type $(2, \dots, 2)$. Then $\text{Nm}_E^2 = 2\text{Nm}_E$ and therefore $i_E = (1 - \text{Nm}_E)$ is an involution of JC , because

$$(1 - \text{Nm}_E)^2 = 1 - 2\text{Nm}_E + \text{Nm}_E^2 = 1.$$

By Theorem 1.86, there exists an involution ι_E on C which induces i_E . As in the previous proof, we can define the complementary subvariety Z , the involution $\iota_Z = \iota \circ \iota_E$ and C' . As $g+1 = \dim Z = \dim JC'$, the Hurwitz formula tells us that $2(2g+1) - 2 = 2(2(g+1) - 2) + b$, so $b = 0$ and C is an étale double cover of C' . Finally, C' is hyperelliptic because C is hyperelliptic. \square

When the dimension is even, we have a nice symmetry which gives a slightly better result. Denote by \mathcal{J} the locus of Jacobians of smooth curves. Then

Proposition 2.10. *Let $\underline{2} = (2, \dots, 2)$ be a g -tuple of 2's. Then, $\text{Is}_{\underline{2}}^{2g} \cap \mathcal{J}$ is exactly the locus of Jacobians of double covers of genus g curves branched in 2 points.*

Proof. If $C \rightarrow C'$ is a double cover branched in 2 points, then JC' can be embedded in JC with the induced polarisation being twice a principal polarisation ([BL, Prop 11.4.3]).

As for the other direction, let (E, Z) be a pair of complementary subvarieties both of type $\underline{2}$ in JC . As $\text{Nm}_E + \text{Nm}_Z = 2$, we see that $(1 - \text{Nm}_E) = -(1 - \text{Nm}_Z)$ are involutions on JC . By Theorem 1.86, there exists an involution on C which extends to one of those, say $(1 - \text{Nm}_Z)$. Denote it by ι_Z . Again, we can define $C' = C/\iota_Z$ and $f: C \rightarrow C'$ to get $Z = f^*(JC')$.

As $g = \dim JC'$, the Hurwitz formula tells us that $2(2g) - 2 = 2(2g - 2) + b$, so $b = 2$ and C is a double cover of C' branched in 2 points. \square

In particular when C is not hyperelliptic and $JC \in \text{Is}_2^{2g}$, then for a pair of complementary subvarieties (E, Z) of type $\underline{2}$ exactly one of them is the Jacobian JC' of genus g curve such that C is a double cover of C' and the other is the Prym variety of the double covering.

If C is hyperelliptic, then both subvarieties are Jacobians and Pryms for each other.

The idea of the previous proof leads to a very interesting observation.

Proposition 2.11. *Let $\underline{d} = (1, \dots, 2)$ be a g -tuple with a positive number of 1's and 2's. Then*

1. $\text{Is}_{\underline{d}}^{2g} \cap \mathcal{J} = \emptyset$,
2. $\text{Is}_{\underline{d}}^{2g+1} \cap \mathcal{J} = \emptyset$.

Proof. If either of those was non-empty, we would find a pair of complementary subvarieties (E, Z) and an involution on C which induces one of the involutions $(1 - \text{Nm}_E)$ or $(1 - \text{Nm}_Z)$. Taking the quotient curve C' with the quotient map f we would find that $E = f^*(JC')$ or $Z = f^*(JC')$.

In the first case both subvarieties are of dimension g , so the Hurwitz formula tells us that f has to be a double cover branched in 2 points, which means that the induced polarisation on f^*JC' is twice a principal polarisation, a contradiction.

In the second case the Hurwitz formula states that $2(2g+1) - 2 = 2(2g(C') - 2) + b$, where b is the number of branch points, and gives two possibilities. If $g(C') = g + 1$, then $b = 0$ and we get a contradiction because of Proposition 2.9. If $g(C') = g$, then $b = 4$ and again we get a contradiction because the induced polarisation on E has to be twice the principal one ([BL, Prop 11.4.3]). \square

Remark 2.12. When the dimension grows, the codimensions of both Is_2^n and \mathcal{J} (or \mathcal{JH}) are high, so if they were in general position, the intersection would be empty. But we saw in Propositions 2.9 and 2.10, that they do intersect.

There is one more result related to Is_2^3 .

Proposition 2.13. *There is a 1 to 1 correspondence between the set of smooth genus 3 hyperelliptic curves (up to translation) on a general abelian surface A and the set of degree 2 polarised isogenies $A \rightarrow B$, where B is the Jacobian of a smooth genus 2 curve. In particular, there are exactly three hyperelliptic curves in the linear system of a $(1, 2)$ polarising line bundle on a (very) general abelian surface.*

Proof. Let (JC', Θ) be the Jacobian of a smooth genus 2 curve. Let $\rho: A \rightarrow JC'$ be a degree 2 isogeny. Then $\rho^{-1}(\Theta = 0)$ is a genus 3 hyperelliptic curve, which is an étale double cover of C' .

Conversely, let C be a hyperelliptic genus 3 curve on A . Then $\mathcal{O}(C)$ is a $(1, 2)$ polarising line bundle. From the universal property of Jacobians (Theorem 1.83) there exists a surjective map $f: JC \rightarrow A$. By analogous reasoning to the proof of Lemma 3.8, $\ker(f)$ is connected, and by Proposition 3.9, \hat{f} is an embedding of \hat{A} with restricted polarisation of type $(1, 2)$. Therefore $JC \in \text{Is}_2^3$. As C is also hyperelliptic, Proposition 2.7 tells us that there exists an étale double cover $C \rightarrow C'$. It is defined by a 2-torsion point, say η , and there is an embedding of $JC'/\langle \eta \rangle$ to JC with the restricted polarisation of type $(1, 2)$. As A is general, we have $\hat{A} = JC'/\langle \eta \rangle$ and by dualising the quotient map, we obtain a degree 2 polarised isogeny $A \rightarrow JC'$.

The last part follows from the fact that there are exactly three non-zero 2-torsion points in the kernel $K(\mathcal{O}(C))$. A (very) general surface means one for which the resulting principally polarised abelian surface is the Jacobian of a smooth curve. \square

2.2.1 Irreducibility

The aim of this section is to show that $\text{Is}_{\underline{d}}^g$ is irreducible. This will be an indication that the choice of definition is a good one.

Recall the notation: k, g are integers such that $0 < k \leq \frac{g}{2}$, and $\underline{d} = (d_1, \dots, d_k)$ and $\underline{d}' = (1, \dots, 1, d_1, \dots, d_k)$ are k - and $(g - k)$ -tuples of positive integers such that $d_i | d_{i+1}$.

In the proof of this fact we will use condition (4) of Proposition 2.1, so we define

Definition 2.14. Let M, N, A be polarised abelian varieties. An *allowed isogeny* is a polarised isogeny $\rho: M \times N \rightarrow A$, such that its kernel has $\{0\}$ intersection with $M \times \{0\}$ and $\{0\} \times N$.

Definition 2.15. Let $(M, H_M), (N, H_N)$ be polarised abelian varieties of type \underline{d} and \underline{d}' . A subgroup $K \subset M \times N$ is called an *allowed isotropic subgroup* if it is maximal (i.e. $|K| = (d_1 \cdot \dots \cdot d_k)^2$) isotropic subgroup of $K(H_M \boxtimes H_N)$, with respect to $e^{H_M \boxtimes H_N}$, such that $K \cap K(H_M) = K \cap K(H_N) = \{0\}$.

Let us recall

Proposition 2.16. [BL, Corollary 6.3.5] For an isogeny $\rho: Y \rightarrow X$ and $L \in \text{Pic}(Y)$ the following statements are equivalent

1. $L = \rho^*(L')$ for some $L' \in \text{Pic}(X)$.
2. $\ker(\rho)$ is an isotropic subgroup of $K(L)$ with respect to e^L .

\square

This leads to an obvious corollary.

Corollary 2.17. *Let A be a principally polarised abelian variety. Let M, N be polarised abelian varieties of type \underline{d} and \underline{d}' . Then*

1. *If $\rho: M \times N \rightarrow A$ is an allowed isogeny then $\ker(\rho)$ is an allowed subgroup of $M \times N$.*
2. *If K is an allowed subgroup of $M \times N$, then $(M \times N)/K$ is a principally polarised abelian variety and the quotient map $\rho: M \times N \rightarrow (M \times N)/K$ is an allowed isogeny.*

Let us state the main result of this section.

Theorem 2.18. *Let k, g be integers such that $0 < k \leq \frac{g}{2}$, and $\underline{d} = (d_1, \dots, d_k)$ is a k -tuple of positive integers such that $d_i | d_{i+1}$.*

Then $\text{Is}_{\underline{d}}^g$ is a non-empty irreducible subvariety of \mathcal{A}_g of dimension $\binom{k+1}{2} + \binom{g-k+1}{2}$.

Proof. Proposition 2.1 tells us that A belongs to $\text{Is}_{\underline{d}}^g$ if and only if there exists an allowed isogeny to A . Therefore the idea of the proof is to show that there exists one map from $\mathfrak{h}_k \times \mathfrak{h}_{g-k}$ which covers all possible allowed isogenies and so $\text{Is}_{\underline{d}}^g$ is the image of an irreducible variety.

Let us describe the situation. Take polarised abelian varieties (M, H_M) and (N, H_N) of types \underline{d} and \underline{d}' respectively. Take their product with product polarisation $(M \times N, H_M \boxtimes H_N)$. By Lemma 1.26, we have $K(H_M) \cong K(H_N)$ and $K(H_M \boxtimes H_N)$, of order $\prod d_i^4$, is a symplectic \mathbb{Z} -module with the non-degenerate symplectic form $e^{H_M \boxtimes H_N}$. Therefore there exists an allowed isotropic subgroup $G \subset K(H_M \boxtimes H_N)$ and by Propositions 1.93 and 1.94 all such are equivalent under the action of the symplectic group. Hence there exists an allowed isogeny $\rho: M \times N \rightarrow (M \times N)/G$ and so $\text{Is}_{\underline{d}}^g$ is non-empty. Moreover, the action of the symplectic group on $K(H_M \boxtimes H_N)$ is induced by the symplectic action on \mathfrak{h}_g , which gives us irreducibility.

To make this more precise, we need to recall that a period matrix of an abelian variety is a choice of symplectic basis of a lattice in its universal cover.

Let $l \leq k$ be the number of integers bigger than 1 in \underline{d} . For $i = M, N$, by the elementary divisor theorem, let $B^i = \{\lambda_1^i, \dots, \lambda_l^i, \mu_1^i, \dots, \mu_l^i\}$ be symplectic bases of $K(H_M)$ and $K(H_N)$. Let $B = \{B_M, B_N\}$ be a symplectic basis of $K(H_M) \boxtimes (H_N)$ (with the less standard matrix of symplectic form). Let K_B be given by the image of the matrix

$$K = \begin{bmatrix} \text{id}_l & 0 \\ 0 & \text{id}_l \\ \text{id}_l & 0 \\ 0 & -\text{id}_l \end{bmatrix}.$$

Then K_B is an allowed isotropic subgroup. Moreover, if we change bases using the symplectic action, then $\text{im}(K)$ will always define an allowed isotropic subgroup and by Propositions 1.93 and 1.94, every allowed isotropic subgroup arises in this way.

When we take the universal cover V of $M \times N$, in order to write the period matrix, we need to choose a symplectic basis of V . The obvious choice is to enlarge the symplectic basis B to a symplectic basis \overline{B} . We need to enlarge the matrix K by zero-blocks to a matrix \overline{K} , such that its image is equal to K_B .

From this discussion, we have found the period matrix

$$\Lambda = \left\langle \begin{array}{cccc} Z(M) & 0 & \text{diag}(\underline{d}) & 0 \\ 0 & Z(N) & 0 & \text{diag}(\underline{d}') \end{array} \right\rangle, \quad M \times N = \mathbb{C}^g / \Lambda,$$

and a matrix \overline{K} , such that $\text{im}(\overline{K})$ is an allowed isotropic subgroup of $M \times N$.

The data defining \overline{K} is discrete, so $\text{im}(\overline{K})$ will be allowed isotropic for any matrices $Z \in \mathfrak{h}_k, Z' \in \mathfrak{h}_{g-k}$. Moreover, the symplectic action on $\mathfrak{h}_k \times \mathfrak{h}_{g-k}$ gives all possible period matrices hence all possible symplectic bases and therefore all possible allowed isotropic subgroups.

Thus we have proved that there exists a global map

$$\Psi: \mathfrak{h}_k \times \mathfrak{h}_{g-k} \ni Z \times Z' \mapsto (X_{Z'} \times X_{Z''}) / \text{im}(\overline{K}) \in \mathcal{A}_g,$$

which covers all possible allowed isogenies; that is, for any allowed isogeny $M \times N \rightarrow A$, there exist period matrices $Z(M)$ and $Z(N)$ such that $\Psi(Z(M) \times Z(N)) = A$.

From the construction, it is obvious that $\text{Is}_{\underline{d}}^g$ is the image of the above map and as the domain is irreducible, it follows that $\text{Is}_{\underline{d}}^g$ is an irreducible variety. \square

There is a generalisation of Humbert surfaces to the moduli of non-principally polarised abelian surfaces. However, in that case, the generalised Humbert surface is no longer irreducible. For details, see [vdG].

One can also generalise further our result. The idea is to define for any $\underline{d}_1 \in \mathbb{Z}^k, \underline{d}_2 \in \mathbb{Z}^{g-k}$ the locus $\text{Is}_{\underline{d}_1, \underline{d}_2}^D$ of D -polarised varieties which have a pair of complementary subvarieties of types \underline{d}_1 and \underline{d}_2 . The obvious question is whether $\text{Is}_{\underline{d}_1, \underline{d}_2}^D$ is non-empty. Using Proposition 2.16 one can translate the question into one about the existence of isotropic subgroups analogous to the allowed ones. The proof of Theorem 2.18 can be easily generalised, but one must have in mind that the number of irreducible components of $\text{Is}_{\underline{d}_1, \underline{d}_2}^D$ will be equal to the number of orbits of such isotropic subgroups. To sum up, the problem can be solved if one can deal with the combinatorics related to special isotropic subgroups in finite symplectic groups. Certainly this is possible in many cases, such as $(1, p)$ -polarised surfaces.

2.3 Special case: $\text{Is}_{(1,p)}^4$

As in the Humbert surface case, we would like to find equations for a locus in \mathfrak{h}_4 which maps to $\text{Is}_{(1,p)}^4$ in \mathcal{A}_4 . We will assume that p is an odd positive integer.

Let us define

$$\Lambda_M = \left\langle \begin{array}{cccc} a & pb & 1 & 0 \\ pb & p^2c & 0 & p \end{array} \right\rangle, \quad M = \mathbb{C}^2/\Lambda_M,$$

$$\Lambda_N = \left\langle \begin{array}{cccc} d & pe & 1 & 0 \\ pe & p^2f & 0 & p \end{array} \right\rangle, \quad N = \mathbb{C}^2/\Lambda_N,$$

$$\Lambda_A = \left\langle \begin{array}{cccccc} a & b & 2a & 2b & 1 & 0 & 0 & 0 \\ b & (\frac{p-1}{2})^2f + c & 2b - \frac{p-1}{2}e & 2c - \frac{p-1}{2}f & 0 & 1 & 0 & 0 \\ 2a & 2b - \frac{p-1}{2}e & d + 4a & 4b + e & 0 & 0 & 1 & 0 \\ 2b & 2c - \frac{p-1}{2}f & e + 4b & 4c + f & 0 & 0 & 0 & 1 \end{array} \right\rangle, \quad A = \mathbb{C}^4/\Lambda_A,$$

where $a, b, c, d, e, f \in \mathbb{C}$ are such that $\Lambda_M, \Lambda_N, \Lambda_A$ are of maximal rank, so M, N, A are complex tori.

Lemma 2.19. *There exist embeddings*

$$\iota_M: M \longrightarrow A$$

$$(s, t) + \Lambda_M \longmapsto (s, \frac{t}{p}, 2s, 2\frac{t}{p}) + \Lambda_A$$

$$\iota_N: N \longrightarrow A$$

$$(s, t) + \Lambda_N \longmapsto (0, \frac{(1-p)t}{2p}, s, \frac{t}{p}) + \Lambda_A$$

Proof. Let us write a vector (s, t) in terms of columns of M which form a basis of an underlying real vector space \mathbb{C}^2 . We have

$$(s, t) = r_1(a, pb) + r_2(pb, p^2c) + r_3(1, 0) + r_4(0, p) = (r_1a + r_2pb + r_3, r_1pb + r_2p^2c + r_4p)$$

Then

$$\begin{aligned} \iota_M(s, t) &= (r_1a + r_2pb + r_3, r_1b + r_2pc + r_4, 2r_1a + 2r_2pb + 2r_3, 2r_1b + 2r_2pc + 2r_4) \\ &= r_1(a, b, 2a, 2b) + r_2(b, (\frac{p-1}{2})^2f + c, 2b - \frac{p-1}{2}e, 2c - \frac{p-1}{2}f) \\ &\quad + \frac{p-1}{2}r_2(2b, 2c - \frac{p-1}{2}f, e + 4b, f + 4c) + r_3(1, 0, 0, 0) + 2r_3(0, 0, 1, 0) \\ &\quad + r_4(0, 1, 0, 0) + 2r_4(0, 0, 0, 1) \end{aligned}$$

Now it is clear that

$$(s, t) \in \Lambda_M \iff r_1, r_2, r_3, r_4 \in \mathbb{Z} \iff \iota_M(s, t) \in \Lambda_A,$$

which means that ι_M is an embedding.

As for ι_N , the proof is similar and uses the fact that

$$\begin{aligned} \iota_N(r_1(d, pe)) &= r_1(2a, 2b - \frac{p-1}{2}e, d + 4a, e + 4b) - 2r_1(a, b, 2a, 2b) \\ \iota_N(r_2(pe, p^2f)) &= r_2(2b, 2c - \frac{p-1}{2}f, e + 4b, 4c + f) \\ &\quad - 2r_2(b, (\frac{p-1}{2})^2f + c, 2b - \frac{p-1}{2}e, 2c - \frac{p-1}{2}f) \\ \iota_N(r_3(1, 0)) &= r_3(0, 0, 1, 0), \\ \iota_N(r_4(0, p)) &= \frac{1-p}{2}r_4(0, 1, 0, 0) + r_4(0, 0, 0, 1), \end{aligned}$$

which implies

$$(s, t) \in \Lambda_N \iff r_1, r_2, r_3, r_4 \in \mathbb{Z} \iff \iota_N(s, t) \in \Lambda_A.$$

□

Remark 2.20. If A is an abelian variety then M, N , as subvarieties, are also abelian varieties. Conversely, if M and N are abelian surfaces, then A is an abelian variety, as it is isogenous to a product $M \times N$ via

$$\rho = \iota_M + \iota_N: M \times N \longrightarrow A.$$

Proof. We only need to show that ρ is surjective. We can show it using the universal coverings for A, M and N . Then $\iota_M(\mathbb{C}^2)$ and $\iota_N(\mathbb{C}^2)$ are complementary vector spaces in \mathbb{C}^4 defined by $2z_1 = z_3, 2z_2 = z_4$ and $z_1 = 0, z_2 = \frac{1-p}{2}z_4$, so they generate it. Therefore ρ is surjective. □

Lemma 2.21. *The group $\iota_M(M) \cap \iota_N(N)$ is finite and is generated by $(0, \frac{1}{p}, 0, \frac{2}{p})$ and $(b, c, 2b, 2c)$.*

Proof. From the proof of Remark 2.20 we know that locally $(0,0,0,0)$ is the only point of $\iota_M(M) \cap \iota_N(N)$. By translations, the intersection is discrete and by compactness finite.

To prove the second statement, it is clear that

$$\begin{aligned}\iota_M(0, 1) + \Lambda_M &= \left(0, \frac{1}{p}, 0, \frac{2}{p}\right) + \Lambda_A = \left(0, \frac{1}{p} - 1, 0, \frac{2}{p}\right) + \Lambda_A = \iota_N(0, 2) + \Lambda_N, \\ \iota_M(b, pc) + \Lambda_M &= (b, c, 2b, 2c) + \Lambda_A = \left(0, -\left(\frac{p-1}{2}\right)^2 f, \frac{p-1}{2}e, \frac{p-1}{2}f\right) + \Lambda_A \\ &= \iota_N\left(\frac{p-1}{2}e, \frac{p-1}{2}pf\right) + \Lambda_N,\end{aligned}$$

so indeed those elements belong to $\iota_M(M) \cap \iota_N(N)$.

For the converse, let us use once more the real basis for \mathbb{C}^2 . Then

$$(s, t) = (r_1a + r_2pb + r_3, r_1pb + r_2p^2c + pr_4) \text{ for some } r_1, r_2, r_3, r_4 \in \mathbb{R}.$$

Let us assume that $\iota_M(s, t) \in \iota_N(N)$. We want to find conditions on (s, t) by comparing coordinates. Consider general complex numbers a, \dots, f , i.e. $1, a, \dots, f$ are linearly independent over \mathbb{Q} . Then there are no nontrivial rational relations between the i th coordinates of column vectors, $i = 1, \dots, 4$. Denote the projection onto the i th coordinate by π_i . The idea is to show that $r_1, pr_2, r_3, pr_4 \in \mathbb{Z}$ in a general case and by discreteness to extend it to all matrices. This will show that (s, t) is in the subgroup generated by $(0, 1)$ and (b, pc) in M and so $\iota_M(M) \cap \iota_N(N)$ is generated by $(0, \frac{1}{p}, 0, \frac{2}{p})$ and $(b, c, 2b, 2c)$.

We can easily restrict the proof to the case $r_1, r_2, r_3, r_4 \in \mathbb{Q}$ because $\iota_M(M) \cap \iota_N(N)$ is finite and ι_M is an embedding and therefore the order of (s, t) in M has to be finite. Moreover, without loss of generality we can assume that $r_1, r_2, r_3, r_4 \in [0, 1) \cap \mathbb{Q}$.

The first coordinate of $\iota_N(N)$ is 0, so we have $r_1a + r_2pb + r_3 \equiv 0 \pmod{\pi_1(\Lambda_A)}$, i.e. $r_1a + r_2pb + r_3 = ka + lb + m$, for some $k, l, m \in \mathbb{Z}$. Using general a and b , we get $r_1 = r_3 = 0, pr_2 = l \in \mathbb{Z}$. So $(s, t) = (lb, plc + pr_4) = l(b, pc) + pr_4(0, 1)$.

It remains to prove that $pr_4 \in \mathbb{Z}$. As $\iota_M(l(b, pc)) = \iota_N(l(\frac{p-1}{2}e, \frac{p-1}{2}pf))$ and both embeddings are additive maps, we can assume that $\iota_M(pr_4(0, 1)) \in \iota_N(N)$. We will consider the second and the fourth coordinate. We have $r_4 \equiv \frac{(1-p)t}{2p} \pmod{\pi_2(\Lambda_A)}$ and $2r_4 \equiv \frac{t}{p} \pmod{\pi_4(\Lambda_A)}$, for some $t \in \mathbb{C}$. Once more, as $r_4 \in \mathbb{Q}$, for a general Λ_A we have $r_4 \equiv \frac{(1-p)t}{2p} \pmod{\mathbb{Z}}$ and $2r_4 \equiv \frac{t}{p} \pmod{\mathbb{Z}}$, for some $t \in \mathbb{R}$. Therefore, $\frac{t}{p} - t \equiv \frac{t}{p} \pmod{\mathbb{Z}}$, so $t \in \mathbb{Z}$ leading to $pr_4 \in \mathbb{Z}$, as $\frac{1-p}{2}t \in \mathbb{Z}$. \square

Lemma 2.22. *If L is a principal polarisation on A , then ρ^*L is of type $(1, p, 1, p)$ on $M \times N$.*

Proof. We prove this directly, by computing the analytic representations of $\hat{i}_M \phi_L \iota_M$

and $\hat{\iota}_N \phi_L \iota_N$. To shorten notation we write z' for $\text{Im}(z)$. Then

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & \frac{1}{p} & 0 & \frac{2}{p} \end{bmatrix} \begin{bmatrix} a' & b' & 2a' & 2b' \\ b' & (\frac{p-1}{2})^2 f' + c' & 2b' - \frac{p-1}{2} e' & 2c' - \frac{p-1}{2} f' \\ 2a' & 2b' - \frac{p-1}{2} e' & d' + 4a' & 4b' + e' \\ 2b' & 2c' - \frac{p-1}{2} f' & e' + 4b' & 4c' + f' \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{p} \\ 2 & 0 \\ 0 & \frac{2}{p} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1-p}{2p} & 0 & \frac{1}{p} \end{bmatrix} \begin{bmatrix} a' & b' & 2a' & 2b' \\ b' & (\frac{p-1}{2})^2 f' + c' & 2b' - \frac{p-1}{2} e' & 2c' - \frac{p-1}{2} f' \\ 2a' & 2b' - \frac{p-1}{2} e' & d' + 4a' & 4b' + e' \\ 2b' & 2c' - \frac{p-1}{2} f' & e' + 4b' & 4c' + f' \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1-p}{2p} \\ 1 & 0 \\ 0 & \frac{1}{p} \end{bmatrix}$$

are matrices of the analytic representations of hermitian forms

$$\begin{bmatrix} a' & pb' \\ pb' & p^2 c' \end{bmatrix}^{-1} \quad \text{and} \quad \begin{bmatrix} d' & pe' \\ pe' & p^2 f' \end{bmatrix}^{-1}$$

by a direct computation. □

The following argument gives an indirect proof of Lemma 2.22. By lemma 2.21 the kernel of ρ has p^2 points, so ρ is of degree p^2 . From Riemann-Roch [BL, Corollary 3.6.6], we have $\chi(\rho^* L) = p^2 \chi(L) = p^2$. As we can choose M and N general, so the only types of polarisations are multiples of $(1, p)$, it follows that $\rho^* L$ has to be of type $(1, p, 1, p)$.

Remark 2.23. This construction is invertible in the following sense. For any abelian variety

$$\Lambda_{A'} = \left\langle \begin{array}{cccccccc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & 1 & 0 & 0 & 0 \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} & 0 & 1 & 0 & 0 \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,4} & 0 & 0 & 1 & 0 \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} & 0 & 0 & 0 & 1 \end{array} \right\rangle, A' = \mathbb{C}^4 / \Lambda_{A'}$$

with $2a_{1,1} = a_{1,3}$, $2a_{1,2} = a_{1,4}$, $2pa_{1,4} = a_{2,3} + \frac{p-1}{2}a_{3,4}$, $2a_{2,2} = \frac{p-1}{2}a_{4,4} - (p-2)a_{2,4}$, we can find $(1, p)$ abelian surfaces $M' = \mathbb{C}^2 / \Lambda_{M'}$ and $N' = \mathbb{C}^2 / \Lambda_{N'}$, where

$$\Lambda_{M'} = \left\langle \begin{array}{ccc} a_{1,1} & pa_{1,2} & 1 & 0 \\ pa_{1,2} & \frac{p}{2}(a_{2,4} + \frac{p-1}{2}a_{4,4}) & 0 & p \end{array} \right\rangle,$$

$$\Lambda_{N'} = \left\langle \begin{array}{cccc} a_{3,3} - 2a_{1,3} & p(a_{3,4} - 2a_{2,3}) & 1 & 0 \\ p(a_{3,4} - 2a_{2,3}) & p(a_{4,4} - 2a_{2,4}) & 0 & p \end{array} \right\rangle,$$

embedded in A' by $\iota_{M'}$ and $\iota_{N'}$:

$$\iota_{M'}((s, t) + \Lambda_{M'}) = \left(s, \frac{t}{p}, 2s, 2\frac{t}{p}\right) + \Lambda_{A'} \in A'$$

$$\iota_{N'}((s, t) + \Lambda_{N'}) = \left(0, \frac{(1-p)t}{2p}, s, \frac{t}{p}\right) + \Lambda_{A'} \in A'.$$

Then

$$\rho' = \iota_{M'} + \iota_{N'} : M' \times N' \longrightarrow A'$$

is an isogeny of $(1, p, 1, p)$ and principally polarised abelian fourfolds.

The above argument, Theorem 2.18 and Theorem 2.2 lead us to the following.

Theorem 2.24. *Let p be an odd positive integer. Let (A, Θ) be a principally polarised abelian fourfold. The following conditions are equivalent:*

1. *there exists an abelian surface $M \subset A$ such that $\Theta|_M$ is of type $(1, p)$, i.e. $A \in \text{Is}_{(1,p)}^4$;*
2. *there exists a pair (M, N) of complementary abelian surfaces in A of type $(1, p)$;*
3. *$\text{End}(A)$ contains a symmetric endomorphism f with the matrix of the analytic representation given by*

$$\begin{bmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{p-1}{2} \\ 2p & 0 & 0 & 0 \\ 0 & 2 & 0 & p-1 \end{bmatrix};$$

4. *(A, Θ) is isomorphic to an abelian fourfold defined by the period matrix*

$$\begin{bmatrix} a & b & 2a & 2b & 1 & 0 & 0 & 0 \\ b & (\frac{p-1}{2})^2 f + c & 2b - \frac{p-1}{2} e & 2c - \frac{p-1}{2} f & 0 & 1 & 0 & 0 \\ 2a & 2b - \frac{p-1}{2} e & d + 4a & 4b + e & 0 & 0 & 1 & 0 \\ 2b & 2c - \frac{p-1}{2} f & e + 4b & 4c + f & 0 & 0 & 0 & 1 \end{bmatrix}$$

for some $a, b, c, d, e, f \in \mathbb{C}$.

In other words there exists a period matrix of A with coordinates satisfying the following linear equations

$$F_1 : 2a_{1,1} - a_{1,3} = 0,$$

$$F_2 : 2a_{1,2} - a_{1,4} = 0,$$

$$F_3 : -2pa_{1,4} + a_{2,3} + \frac{p-1}{2}a_{3,4} = 0,$$

$$F_4 : 2a_{2,2} + (p-2)a_{2,4} - \frac{p-1}{2}a_{4,4} = 0$$

Proof. $1 \Leftrightarrow 2$ is the content of Proposition 2.1, and $3 \Leftrightarrow 4$ is the content of Lemma 1.64.

We only need to prove that $1 \Leftrightarrow 4$. Remark 2.23 tells us that the set of abelian varieties with a period matrix satisfying (4) is a subset of $\text{Is}_{(1,p)}^4$. Moreover, both of them are closed irreducible subvarieties of \mathcal{A}_4 of dimension 6, which means that they are equal. In other words, the set

$$\{Z = [a_{i,j}] \in \mathfrak{h}_4 : 2a_{1,1} = a_{1,3}, 2a_{1,2} = a_{1,4}, 2pa_{1,4} = a_{2,3} + \frac{p-1}{2}a_{3,4}, \\ 2a_{2,2} = \frac{p-1}{2}a_{4,4} - (p-2)a_{2,4}\}$$

is one of the irreducible components of the preimage of $\text{Is}_{(1,p)}^4$ in \mathfrak{h}_4 . □

Remark 2.25. It is important to note that we need the irreducibility of $\text{Is}_{(1,p)}^4$ to prove Theorem 2.24.

Chapter 3

(1, 3) Theta divisors

The main motivation of this chapter is to understand the construction which for any (1, 3) polarised abelian surface A defines so called (1, 3) Theta divisor C_A . It may be defined as a distinguished curve, which is the zero set of a (-1) -eigensection on a symmetric (1, 3) bundle of characteristic 0. The main result (Theorem 3.14) characterises smooth curves that are the outcome of the construction as hyperelliptic genus 4 curves with Jacobian belonging to $\text{Is}_{(1,3)}^4$.

One of the crucial ingredients in the proof is Andreotti-Mayer theory [AM], which proves that for general (1, 3) polarised abelian surface A , the curve C_A is smooth.

3.1 Construction

Let (A, H) be a (1, 3) polarised abelian surface. Take an isomorphism of A with an abelian surface given by $Z \in \mathfrak{h}_2$. With respect to a standard decomposition $\mathbb{C}^2 = Z\mathbb{R}^2 + \mathbb{R}^2$ there exists a unique line bundle, \mathcal{L} , with $c_1(\mathcal{L}) = H$, of characteristic 0 on A . By Proposition 1.39, the space of its global sections can be identified with a space of classical theta functions and $\theta_{[0]}^0, \theta_{[0]}^{[\omega]}, \theta_{[0]}^{[-\omega]}$, where $\omega = (0, \frac{1}{3})$, form a basis of this vector space. As the line bundle is symmetric, the (-1) action extends to $H^0(\mathcal{L})$. It acts on the basis by

$$(-1)^*\theta_{[0]}^0 = \theta_{[0]}^0, \quad (-1)^*\theta_{[0]}^{[\omega]} = \theta_{[0]}^{[-\omega]}, \quad (-1)^*\theta_{[0]}^{[-\omega]} = \theta_{[0]}^{[\omega]}$$

Hence, we see that, up to a constant, there exists a unique odd theta function on $H^0(\mathcal{L})$, called θ_A , namely $\theta_A = \theta_{[0]}^{[\omega]} - \theta_{[0]}^{[-\omega]}$. The aim is to study the curve of its zeros:

$$C_A = (\theta_A = 0).$$

We made a few choices to define C_A , but actually it can be avoided, due to the proof of proposition [BL, Prop. 4.6.5] adapted to our setting.

Proposition 3.1. *Let (A, H) be a $(1, 3)$ -polarised abelian surface. Let $L \in \text{Pic}^H(X)$ be a symmetric line bundle of characteristic $c = (c_1, c_2) \in \frac{1}{2}\Lambda$ with respect to some decomposition. Then $K(L)_1 = \{0, \omega, -\omega\}$, for some $\omega \in A[3]$. Recall that h_{\pm}^0 are dimensions of the ± 1 -eigenspaces of (-1) action. It follows that*

- if c is even then $h_+^0 = 2, h_-^0 = 1$
- if c is odd then $h_+^0 = 1, h_-^0 = 2$

In both cases

$$\theta_{A,L} = \theta_{\omega}^c - \exp(4\pi i \text{Im}(H)(\omega, c_2))\theta_{-\omega}^c$$

generates the 1-dimensional eigenspace and so for every characteristic c , we have

$$(\theta_A = 0) = t_c^*(\theta_{A,L} = 0)$$

and so C_A is well defined.

Proof. The first useful fact is that, as 3 is an odd number, we can assume that the characteristic c lies in $\frac{1}{2}\Lambda$. Let us recall that $\{\theta_0^c, \theta_{\omega}^c, \theta_{-\omega}^c\}$ form a basis for the space of canonical theta functions.

The Inverse Formula (Theorem 1.49) tells us that for any $k \in K(L)_1$

$$(-1)^* \theta_k^c = \exp(4\pi i \text{Im}(H)(c_1 + k, c_2))\theta_{-k}^c$$

Now, we define

$$\theta_{\omega}^{\pm} = \theta_{\omega}^c \pm \exp(4\pi i \text{Im}(H)(c_1 + \omega, c_2))\theta_{-\omega}^c$$

It is obvious that $\{\theta_0^c, \theta_{\omega}^+, \theta_{\omega}^-\}$ is another basis and θ_{ω}^+ is even, whereas θ_{ω}^- is odd. Again, the inverse formula tells us that θ_0^c is even/odd if c is even/odd, which proves the first part.

For the second part, if c is even, then $\theta_{A,L} = \theta_{\omega}^-$ generates the space of odd sections, whereas for c odd $\theta_{A,L} = \theta_{\omega}^+$ generates the space of even sections, as claimed. To finish the proof, let us note that classical and canonical theta functions have the same zeros. \square

Definition 3.2. We say that C_A is the $(1, 3)$ Theta divisor on A .

3.1.1 Smoothness

We would like to prove that for a general abelian surface A , the curve C_A is smooth. The main ideas are taken from [AM], adapted to our setting.

The singular locus of any theta function $\theta = 0$ is

$$\text{Sing}(\theta) = \{(v, Z) \in \mathbb{C}^2 \times \mathfrak{h}_2 : \theta(v, Z) = 0, \frac{\partial \theta}{\partial v_\alpha}(v, Z) = 0, \text{ for } 1 \leq \alpha \leq 2\}$$

Proposition 3.3. *For a general Z , $\theta(\cdot, Z) = 0$ defines a smooth curve.*

Proof. It suffices to show that $\text{pr}_{\mathfrak{h}_2}(\text{Sing}(\theta))$ is a proper analytic subset of \mathfrak{h}_2 . Surely $\text{Sing}(\theta)$ is analytic. If we define the family over \mathfrak{h}_2 of principally polarised abelian surfaces

$$\mathfrak{X} = (\mathbb{C}^2 \times \mathfrak{h}_2) / \{(\Lambda(Z), Z) : Z \in \mathfrak{h}_2\},$$

then we see that $\text{pr}_{\mathfrak{h}_2}$ factors through \mathfrak{X} .

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathfrak{h}_2 & \xrightarrow{\text{pr}_{\mathfrak{h}_2}} & \mathfrak{h}_2 \\ & \searrow \mu & \nearrow \delta \\ & \mathfrak{X} & \end{array}$$

Certainly, μ is locally an isomorphism, so $S := \mu(\text{Sing}(\theta))$ is analytic. Moreover, δ is proper, since it is closed and preimage of any point is compact. Hence, from the theorem of Remmert, we know that $\delta(S) = \text{pr}_{\mathfrak{h}_2}(\text{Sing}(\theta))$ is analytic.

From the properness of δ we can deduce that if $\delta|_S$ was surjective on an open subset U' , we could construct a holomorphic section of δ on some small connected open subset $U \subset U'$ (i.e. $\sigma : U \rightarrow S$, $\delta \circ \sigma = \text{id}_U$).

As μ is the universal cover, after shrinking U if necessary and choosing a base point, we are able to lift σ to a map

$$U \ni z \mapsto (\tau(z), z) \in V \times U \subset \mathbb{C}^2 \times \mathfrak{h}_2,$$

for some open subset V and some holomorphic map τ .

But, using the following lemma, we will get a contradiction, since θ_A is not identically zero. □

Lemma 3.4. *Let U be an open connected subset of \mathfrak{h}_2 , and $V \times U$ an open connected subset of $\mathbb{C}^2 \times \mathfrak{h}_2$.*

Let $(\tau(z), z): U \rightarrow V \times U$ be a holomorphic section and let $u(w, z): V \times U \rightarrow \mathbb{C}$ be a holomorphic solution of the heat equation (Prop 1.42). If

$$u(\tau(z), z) = 0, \quad \frac{\partial u}{\partial w_i}(\tau(z), z) = 0, \quad 1 \leq i \leq 2$$

then u is identically zero.

Proof. We can change coordinates

$$\begin{cases} w = w - \tau(z) \\ z = z \end{cases}$$

so that τ is reduced to $w = 0$. The Heat Equation (Prop 1.42) is changed into another Heat Equation. As we can act locally, we expand $u(w, z_0)$ in a Taylor series near $(0, z_0)$:

$$u(w, z_0) = \sum_{\underline{k}} a_{\underline{k}}(z_0) w^{\underline{k}}$$

Let \mathfrak{m} be the ideal generated by w_1, w_2 in the ring of formal power series $\mathbb{C}[[w_1, w_2]]$. By the assumption, we have $u(w, z_0) \in \mathfrak{m}^2$. The heat equation (1.42) implies that for any $i > 1$ if $u \in \mathfrak{m}^i$ then $u \in \mathfrak{m}^{i+1}$. Hence $u \in \bigcap \mathfrak{m}^i = 0$ \square

3.1.2 Basic properties of C_A

Lemma 3.5. *Let A be a $(1, 3)$ -polarised surface and C_A be a $(1, 3)$ Theta divisor. Then:*

1. C_A is of (arithmetic) genus 4.
2. C_A passes through at least ten 2-torsion points on A .
3. If C_A is smooth then it is a double cover of \mathbb{P}^1 branched along 10 points, i.e. it is a hyperelliptic curve.
4. For $A = E \times F$ with a product polarisation $\mathcal{O}_E(1) \boxtimes \mathcal{O}_F(3)$, where E, F are elliptic curves, the curve C_A is reducible and consists of one copy of F and three copies of E .

Proof. The adjunction formula says that

$$2p_a(C) - 2 = (C^2) = (L^2).$$

Riemann-Roch implies that

$$3 = h^0(L) = \frac{1}{2}(L^2).$$

Hence

$$p_a(C) = \frac{1}{2}(6 + 2) = 4.$$

Point (2) is a consequence of Proposition 1.56.

For (3), let $A/(-1)$ be the Kummer surface for A . The projection is two to one, so the image of a smooth symmetric curve is a smooth curve. From the Hurwitz formula we can find the genus of the image and the number of branch points:

$$2 \cdot 4 - 2 = 2(2g - 2) + b.$$

As we know that $b \geq 10$, the only possibility is $g = 0$, $b = 10$.

The last part comes from theory of theta functions. In that case, the matrix Z can be chosen to be diagonal, so the theta function is of the form $\theta(v_1, v_2) = f(v_1)g(v_2)$, where $f \in H^0(\mathcal{O}(1))$ on E and $g \in H^0(\mathcal{O}(3))$ on F . Therefore, f has exactly one zero and g has three zeros, which gives the assertion. \square

3.1.3 Product of elliptic curves in detail

This part is devoted to a case in which we can make a few explicit computations. Let

$$E = \mathbb{C}/\tau_1\mathbb{Z} + \mathbb{Z}, \quad F = \mathbb{C}/\tau_2\mathbb{Z} + \mathbb{Z}, \quad \Lambda = \begin{bmatrix} \tau_1 & 0 & 1 & 0 \\ 0 & \tau_2 & 0 & 3 \end{bmatrix}, \quad A = \mathbb{C}^2/\Lambda.$$

Then $A = E \times F$, with the product polarisation. We can take a standard decomposition $\mathbb{C}^2 = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix} \mathbb{R}^2 + \mathbb{R}^2$ and define the line bundle of characteristic 0.

Now we can write theta functions explicitly:

$$\theta_{[0]}^{[\omega]}(v, Z) = \sum_{l_1, l_2 \in \mathbb{Z}} \exp(\pi i l_1^2 \tau_1 + \pi i (l_2 + \frac{1}{3})^2 \tau_2 + 2\pi i v_1 l_1 + 2\pi i v_2 (l_2 + \frac{1}{3})),$$

$$\theta_{[0]}^{[-\omega]}(v, Z) = \sum_{l_1, l_2 \in \mathbb{Z}} \exp(\pi i l_1^2 \tau_1 + \pi i (l_2 - \frac{1}{3})^2 \tau_2 + 2\pi i v_1 l_1 + 2\pi i v_2 (l_2 - \frac{1}{3})).$$

For the computations, let us denote

$$\begin{aligned} a_l &= \exp(\pi i l^2 \tau_1 + 2\pi i v_1 l) \\ b_l &= \exp(\pi i (l + \frac{1}{3})^2 \tau_2 + 2\pi i v_2 (l + \frac{1}{3})) \\ c_l &= \exp(\pi i (l - \frac{1}{3})^2 \tau_2 + 2\pi i v_2 (l - \frac{1}{3})). \end{aligned}$$

Then

$$\theta_A = \theta_{[0]}^{[\omega]} - \theta_{[0]}^{[-\omega]} = \Sigma_{l_1, l_2} a_{l_1} b_{l_2} - \Sigma_{l_1, l_2} a_{l_1} c_{l_2} = \Sigma_{l_1} a_{l_1} (\Sigma_{l_2} b_{l_2} - \Sigma_{l_2} c_{l_2}).$$

For $v_1 = \frac{1}{2} + \frac{1}{2}\tau_1$ we have

$$\begin{aligned} a_l &= \exp(\pi i l^2 \tau_1 + \pi i \tau_1 l + \pi i l) \\ &= \exp(\pi i \tau_1 (l + \frac{1}{2})^2 + \pi i l - \frac{1}{4} \pi i \tau_1) \\ &= (-1)^l \exp(\pi i \tau_1 (l + \frac{1}{2})^2 - \frac{1}{4} \pi i \tau_1). \end{aligned}$$

Now, $a_l = -a_{-l-1}$, so $\Sigma_{l_1} a_{l_1} = 0$ and therefore for any v_2 we find $\theta_A(\frac{1}{2} + \frac{1}{2}\tau_1, v_2) = 0$. The image of this zero set in A is a curve isomorphic to F .

Similar computations work for $v_2 = 0$, $v_2 = \frac{3}{2}$, $v_2 = \frac{1}{2}\tau_2$. For $v_2 = 0$, it is obvious that $b_l = c_{-l}$. For $v_2 = \frac{3}{2}$ it is also true that $b_l = c_{-l}$, as $(-1)^{3l+1} = (-1)^{-3l-1}$. For $v_2 = \frac{1}{2}\tau_2$, we get

$$(l \pm \frac{1}{3})^2 \tau_2 + \tau_2 (l \pm \frac{1}{3}) = \tau_2 (l \pm \frac{1}{3} + \frac{1}{2})^2 - \frac{1}{4} \tau_2,$$

so $b_l = c_{-l-1}$. In all three cases we get $\Sigma_{l_2} b_{l_2} = \Sigma_{l_2} c_{l_2}$ and therefore $\theta_A = 0$. The images in A of those zero sets are isomorphic to E , so by Lemma 3.5, we know we have found all zeros of θ_A .

Remark 3.6. The detailed computation also shows that a general curve C_A is not a finite cover of a curve of smaller positive genus.

Proof. If it was the case, curves $3E \cup F$ would be specialisations and so, they would be covers of possibly singular curves. The only nontrivial case is to show that $3E \cup F$ is not an étale triple cover of a genus 2 curve $E \cup F$. If it was the case, the intersection points of E and three copies of F would generate a cyclic subgroup of 3-torsion points on E . However, the points of intersection are actually 2-torsion points on E , which gives a contradiction. \square

3.1.4 Jacobian of C_A and $\text{Is}_{(1,3)}^4$

We would like to understand the properties of C_A by understanding its Jacobian. We will consider a slightly more general case.

Let C be a smooth genus 4 curve. Fix a base point $O \in C$ and assume that we have an embedding $\iota: C \rightarrow A$, which sends O to 0.

Lemma 3.7. $\iota(C)$ generates A and $\mathcal{O}(\iota(C))$ is a polarising line bundle of type $(1, 3)$.

Proof. The first part follows from the fact that C is of genus 4 and the only abelian subvarieties of A are A , $\{0\}$ and possibly elliptic curves. Hence the only subvariety which contains $\iota(C)$ is A itself.

$\mathcal{O}(\iota(C))$ is a polarising line bundle of some type (d_1, d_2) . Using Riemann-Roch and the adjunction formula, we get $g(C) = 1 + d_1d_2$, so $d_1 = 1$ and $d_2 = 3$. \square

We also have the following diagram, which commutes:

$$\begin{array}{ccc}
 C & \xrightarrow{\iota} & A \\
 \searrow \alpha_O & & \nearrow f \\
 & JC & \\
 \nearrow k & & \\
 K^0 & &
 \end{array} \tag{3.1}$$

where JC is the Jacobian of C , f is the canonical homomorphism defined by the universal property (Thm 1.83) and K^0 is the identity component of kernel of f .

$\iota(C)$ generates A , so f must be surjective, hence K^0 is an abelian surface. The following lemma tells us that in fact $K^0 = \ker(f)$.

Lemma 3.8. K^0 is the kernel of f . We have the exact sequence

$$0 \longrightarrow K^0 \xrightarrow{k} JC \xrightarrow{f} A \longrightarrow 0$$

hence \hat{A} can be embedded in $J(C)$.

Proof. Because of Stein factorisation (Prop 1.5) the kernel of f consists of a finite number of connected components, being copies of the identity component K^0 . We can view $\ker(f)$ as a reduced and effective 2-cycle in JC . Suppose we can write $\ker(f)$ as $K^0 \cup \dots \cup K^t$ for some $t > 0$. Note that every K^i , being translation of K^0 is numerically equivalent to K^0 .

We can also consider the Abel-Jacobi map α_Q defined for any $Q \in C$ and the difference map $\delta: C \times C \longrightarrow JC$.

$$\alpha_Q(P) = \mathcal{O}(P - Q) \in JC,$$

$$\delta(P, Q) = \mathcal{O}(P - Q) \in JC.$$

$\text{Im } \delta$ is also an effective 2-cycle. For this, let us see that for any points $P, Q, R \in C$, we have $\delta(P, R) + \delta(Q, P) = \delta(Q, R)$, so $t_{\delta(P,R)}^* \alpha_P = \alpha_R$. Therefore for any $x \in \text{Im } \delta$ we can define a non-trivial automorphism on $\text{Im } \delta$, namely $t_x^*|_{\text{Im } \delta}$. Now, assuming the image is

of dimension 1, hence isomorphic to C , we could find infinitely many automorphisms of C and hence prove that C is of genus 1, which is a contradiction. Therefore $\text{Im } \delta$ is an effective 2-cycle.

Let us consider the intersection and the intersection number of $\ker(f)$ and $\text{Im } \delta$. Note that for each i , we have $K^i \cdot \text{Im } \delta = K^0 \cdot \text{Im } \delta$. But we will show that

$$\ker(f) \cap \text{Im } \delta = K^0 \cap \text{Im } \delta = \{0\},$$

so $K^0 \cdot \text{Im } \delta > 0$ and for $i > 0$, $K^i \cdot \text{Im } \delta = 0$, which will give a contradiction and hence prove that $\ker(f) = K^0$.

Certainly $\ker(f) \cap \text{Im } \delta \supseteq K^0 \cap \text{Im } \delta \supseteq \{0\}$. To show the remaining inclusion, let us choose a point $\mathcal{O}(P - Q) \in \delta(C \times C) \cap \ker(f)$. It is obviously in the image of α_Q . Now, from the Universal Property of the Jacobians, f makes the following diagram commutative

$$\begin{array}{ccc} C & \xrightarrow{\iota} & A \\ \downarrow \alpha_Q & & \uparrow t_{\iota(Q)} \\ JC & \xrightarrow{f} & A \end{array}$$

But C is embedded in A and in JC so $f|_{\alpha_Q(C)}$ has to be bijective. Hence $\ker(f|_{\alpha_Q(C)}) = \{0\}$ and therefore $\mathcal{O}(P - Q) = 0$.

To embed \hat{A} in JC , we dualise the exact sequence and use the canonical isomorphism between Jacobian and its dual. \square

The next proposition tells us what the induced polarisation from JC to \hat{A} is.

Proposition 3.9. *Let (A, \mathcal{L}) be a $(1, 3)$ polarised surface and let $(J(C), \Theta)$ be the Jacobian of C . Then, using notation from the diagram 3.1, we have $\hat{f}^* \Theta \equiv \widehat{c_1(\mathcal{L})}$ (defined in Prop 1.22).*

Proof. The proposition can be proved using endomorphisms associated to cycles defined in [BL, Sections 5.4 and 11.6]. However, it is a special case of the following.

Proposition 3.10. [BL2, Prop 4.3] *Let C be a smooth curve and (JC, Θ) its Jacobian. Let (A, H) be a polarised abelian surface. Let $\iota: C \rightarrow A$ be a morphism and $f: JC \rightarrow A$ the canonical homomorphisms defined by the universal property. Then the following are equivalent:*

1. $\hat{f}^* \Theta \equiv \hat{H}$;
2. $\iota_*[C] = H$ in $H^2(A, \mathbb{Z})$.

In our case the second statement is obviously true, as $\mathcal{L} = \mathcal{O}(C)$. \square

Proposition 3.10 leads to a nice description of $\mathcal{J} \cap \text{Is}_{(1,3)}^4$, where \mathcal{J} is the locus of Jacobians of curves.

Proposition 3.11. *Let C be a smooth genus 4 curve. Then $JC \in \text{Is}_{(1,3)}^4$ if and only if there exists an abelian surface A and an embedding $\iota: C \rightarrow A$.*

Proof. Assume we have an embedding ι . By Lemma 3.7, $\mathcal{O}(\iota(C))$ is of type $(1, 3)$. Using Proposition 3.10, we get $f: JC \rightarrow A$ and $\hat{f}^*\Theta \equiv \widehat{\mathcal{O}(\iota(C))}$. Moreover, by Lemma 3.8, \hat{f} is an embedding, so $JC \in \text{Is}_{(1,3)}^4$.

If $JC \in \text{Is}_{(1,3)}^4$, then there exists an abelian subvariety M , such that the induced polarisation is of type $(1, 3)$. Let $f: JC \rightarrow \hat{M}$ be the dual map to the embedding of M . Let $\alpha: C \rightarrow JC$ be an Abel-Jacobi map. As $\alpha(C)$ generates JC , $f \circ \alpha(C)$ is a curve which generates \hat{M} . Again, using Proposition 3.10, we get that $\mathcal{O}(f \circ \alpha(C))$ is dual to the induced polarisation on M and hence of type $(1, 3)$. Therefore $f \circ \alpha(C)$ is of genus 4, so isomorphic to C and thus $f \circ \alpha: C \rightarrow \hat{M}$ is an embedding. \square

Remark 3.12. Note, that there is an extra symmetry in Proposition 3.11, because JC has two subvarieties of type $(1, 3)$.

Proposition 3.13. *Using notation from Lemma 3.8, we get that \hat{A} and $\ker(f)$ are complementary abelian subvarieties of (JC, Θ) of type $(1, 3)$.*

Proof. Proposition 3.9 tells us that the induced polarisation $\Theta|_{\hat{A}}$ is the dual polarisation to A and hence of type $(1, 3)$. The exact sequence from Lemma 3.8 tells us that there exists an allowed isogeny $(\hat{A} \times \ker f, \Theta|_{\hat{A}} \boxtimes \Theta|_{\ker(f)}) \rightarrow (JC, \Theta)$. Therefore \hat{A} and $\ker f$ are complementary to each other and both induced polarisations are of type $(1, 3)$. \square

Now we will use the fact that C_A is hyperelliptic to get the main result of this chapter.

Theorem 3.14. *Let C be a smooth hyperelliptic genus 4 curve. Then JC contains a $(1, 3)$ polarised surface M if and only if C can be embedded into \hat{M} , as the $(1, 3)$ Theta divisor.*

Proof. Most of theorem is already proved in Proposition 3.11 and Lemma 3.5. We only need to prove that if C is hyperelliptic with $JC \in \text{Is}_{(1,3)}^4$: then $C = C_{\hat{M}}$ is the $(1, 3)$ Theta divisor.

Let $i: M \rightarrow JC$ be the inclusion. Then $\hat{i}: JC \rightarrow \hat{M}$ is the dual map. Let ι be a hyperelliptic involution, let $o \in C$ be a Weierstrass point and let $\alpha = \alpha_o$ be the Abel-Jacobi map. To shorten notation, identify C with $\alpha(C)$. Then $(-1)^*C = C$ is a symmetric curve because $\alpha(P) = -\alpha(\iota(P))$. Therefore, the image $\hat{i}(C)$ is also symmetric in \hat{M} . Note that by Proposition 3.11, $\hat{i}(C)$ is isomorphic to C .

Let $\pi_{JC}: JC \rightarrow JC/(-1)$ be the Kummer map. Then $\pi_{JC}(C)$ is isomorphic to \mathbb{P}^1 , because $(-1)|_C = \iota$. Let $\pi_{\hat{M}}$ be the Kummer map of \hat{M} . We have the diagram

$$\begin{array}{ccc} JC & \xrightarrow{\hat{i}} & \hat{M} \\ \downarrow \pi_{JC} & & \downarrow \pi_{\hat{M}} \\ JC/(-1) & & \hat{M}/(-1) \end{array} \quad (3.2)$$

and \hat{i} is a homomorphism, so it descends to a map $f: JC/(-1) \rightarrow \hat{M}/(-1)$, given by $f(\pm x) = \pm \hat{i}(x)$, which makes Diagram 3.2 commutative. Now, $f(\pi_{JC}(C))$ has to be a rational curve and, by commutativity of Diagram 3.2, equal to $\pi_{\hat{M}}(\hat{i}(C))$. As $\pi_{\hat{M}}$ is a 2 to 1 map, by the Hurwitz formula it has to be branched in ten points. The only possible branch points are 2-torsion points, so $\hat{i}(C)$ has to go through ten 2-torsion points of \hat{M} . By Proposition 1.56, $\hat{i}(C) \cong C$ is the zero locus of an odd global section, which finishes the proof. \square

Before stating the last corollary of this Chapter, recall that \mathcal{JH} is the locus of hyperelliptic Jacobians.

Corollary 3.15. *The construction of the (1, 3) Theta divisor gives a rational map*

$$\Psi: \mathcal{A}_{(1,3)} \rightarrow \text{Is}_{(1,3)}^4 \cap \mathcal{JH}, \text{ given by } A \mapsto JC_A$$

Theorem 3.14 shows it is a surjective map. It is a 2 to 1 map, because JC_A contains a pair of complementary subvarieties \hat{A} and K^0 , defined in Diagram 3.1. Moreover, we have shown that there is exactly one smooth hyperelliptic curve of genus 4 on a general (1, 3)-polarised abelian surface.

3.2 Unanswered questions

There are still some unanswered questions arising from this construction.

1. We constructed a rational map

$$\Phi_1: \mathcal{A}_{(1,3)} \rightarrow \mathcal{M}_4, \text{ given by } A \mapsto C_A.$$

We characterised the Jacobian of C_A , so theoretically we can use Torelli to understand $C_A \in \mathcal{M}_4$. However, the Torelli theorem is not explicit, so we know nothing about how special the curves are in \mathcal{M}_4 .

2. We also constructed a rational map

$$\Phi_2: \mathcal{A}_{(1,3)} \longrightarrow \mathcal{A}_{(1,3)}, \text{ given by } A \longmapsto K^0$$

where K^0 is defined in Diagram 3.1. It is an involution, which lies over Ψ . What are its other properties?

3. We can dualise the surface A and ask how C_A and $C_{\hat{A}}$ are related, and about the properties of the rational map

$$\Phi_3: \text{im}(\Phi_1) \longrightarrow \text{im}(\Phi_1), \text{ given by } C_A \longmapsto C_{\hat{A}}.$$

Chapter 4

Hyperelliptic Jacobians and $\text{Is}_{(1,p)}^4$

The main result of this chapter proves explicitly that the locus of hyperelliptic genus 4 Jacobians intersects transversely the locus $\text{Is}_{(1,p)}^4$ for $p > 1$ odd. For $p = 3$ the result is weaker than Theorem 3.14, because we already proved that the intersection is not only of expected dimension, but also irreducible. In this chapter, we use theta function techniques developed by Mumford [M3] and expanded by Poor [P]. Then, we use the trick of finding pairs of terms that cancel to prove that many theta series sums to 0. Careful analysis is needed to prove that there are sums that are not zero. As a result, we can find enough information to prove that the matrix of partial derivatives have the expected rank which gives transversality of those loci.

4.1 Locus of hyperelliptic Jacobians in dimension 4

Hyperelliptic Jacobians are characterised in terms of theta constants by Poor [P] elaborating on work of Mumford [M3]. To state the theorem we need to make some definitions, taken from [P]. We restrict our attention to the case $g = 4$, although all definitions are made in general.

Definition 4.1. Denote by $B = \{1, \dots, 9, \infty\}$ and $U = \{1, 3, 5, 7, 9\}$.

Recall from Section 1.3.2 that for vectors $\zeta = (\zeta', \zeta'') \in \mathbb{R}^{4+4}$, $\xi \in \mathbb{R}^8$ and $J = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix}$, we define $e_* = \exp(4\pi i \zeta' \zeta'')$ and $e_2(\zeta, \xi) = \exp(4\pi i {}^t \zeta J \xi)$.

The set B should remind us of the set of Weierstrass points on a hyperelliptic curve. The image of a Weierstrass point under the Abel map is a 2-torsion point. Therefore we are interested in maps between B and $\frac{1}{2}\mathbb{Z}^8$.

Definition 4.2. [P, Def 1.4.11] Let $\eta: B \rightarrow \frac{1}{2}\mathbb{Z}^8$. For $i \in B$ denote by $\eta_i = \eta(i) \in \frac{1}{2}\mathbb{Z}^8$. We can extend η to any $S \subset B$ by defining $\eta_S = \sum_{i \in S} \eta_i$.

Define Ξ_4 to be the set of maps $\eta: B \rightarrow \frac{1}{2}\mathbb{Z}^8$ such that:

1. $\eta_\infty = 0$;
2. $\{S \subset B: |S| \text{ even}\}$, with the symmetric difference, denoted by Δ is a group. Moreover η is a group isomorphism between $\{S \subset B: |S| \text{ even}, \Delta\}$ and $\frac{1}{2}\mathbb{Z}^8/\mathbb{Z}^8$;
3. For all $S_1, S_2 \subset B$, such that $|S_1|$ and $|S_2|$ are even $e_2(\eta_{S_1}, \eta_{S_2}) = (-1)^{|S_1 \cap S_2|}$;
4. U is the unique subset of B such that $e_*(\eta_S) = (-1)^{\lfloor \frac{g+1-|U \Delta S|}{2} \rfloor}$, for all S with $|S|$ even.

Definition 4.3. [P, Def. 1.4.18] Let $\eta \in \Xi_4$. The equations $V_{4,\eta}$, called the *vanishing equations* are given by:

$$\theta[\eta_S](0, Z) = 0,$$

whenever $S \subseteq B$ satisfies $|S|$ even and $|U \Delta S| \neq 5$.

Then the main theorem of Poor's paper [P] in dimension g is

Theorem 4.4. [P, Th. 2.6.1] Let $\eta \in \Xi_g$ and $Z \in \mathfrak{h}_g$. The following two statements are equivalent:

- Z is irreducible (i.e. A_Z is an irreducible variety) and Z satisfies the equations $V_{g,\eta}$.
- There is a marked hyperelliptic Riemann surface C of genus g which has Z as its period matrix and $JC = \mathbb{C}^g/(\mathbb{Z}^g + Z\mathbb{Z}^g)$. Furthermore, there is a model of C , $y^2 = \prod_{i \in B}(x - a_i)$, with a_∞ as the base point of the Abel-Jacobi map $\alpha_{a_\infty}: C \rightarrow JC$ such that $\alpha(a_i) = [[Z \ I]\eta_i]$ in JC .

Mumford [M3, p. 3.99] and Poor [P, p. 17] gave an explicit element $\eta \in \Xi_g$. When $g = 4$ it is defined as function by $\eta_\infty = 0$,

$$\eta_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \eta_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \eta_3 = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\eta_4 = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \eta_5 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \eta_6 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$\eta_7 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \eta_8 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \eta_9 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We will omit the computations that show that indeed $\eta \in \Xi_4$. The easiest way is to use the notion of azygetic basis. [P, Lemma 1.4.13] shows that one must check that $\sum_{i \in B} \eta_i \in \mathbb{Z}^8$, and that $\{\eta_i : i \in B\}$ generates $\frac{1}{2}\mathbb{Z}^8$ and that for all pairs $i, j \in B \setminus \infty, i \neq j$, we have $e_2(\eta_i, \eta_j) = -1$, which are straightforward computations.

Next we find all sets S such that $|S|$ even and $|U \triangle S| \neq 5$, which we put into six classes:

$$\begin{aligned} & \{(i, j) : i \in U, j \in U\}, \{(i, j) : i \notin U, j \notin U\}, \\ & \{(i, j, k, l) : i \in U, j \in U, k \in U, l \notin U\}, \{(i, j, k, l) : i \notin U, j \notin U, k \notin U, l \in U\}, \\ & \{(i, j, k, l) : i \in U, j \in U, k \in U, l \in U\}, \{(i, j, k, l) : i \notin U, j \notin U, k \notin U, l \notin U\}, \end{aligned}$$

There are 120 subsets in the first four classes, and for them η_S are odd 2-torsion points, so by Proposition 1.61 we have $\theta[\eta_S](0, Z) = 0$, for all matrices Z . Therefore we are only interested in the ten functions

$$\begin{aligned} & \eta_{\{1,3,5,7\}}, \eta_{\{1,3,5,9\}}, \eta_{\{1,3,7,9\}}, \eta_{\{1,5,7,9\}}, \eta_{\{3,5,7,9\}}, \\ & \eta_{\{2,4,6,8\}}, \eta_{\{2,4,6,\infty\}}, \eta_{\{2,4,8,\infty\}}, \eta_{\{2,6,8,\infty\}}, \eta_{\{4,6,8,\infty\}}, \end{aligned}$$

given by

$$\begin{aligned} \eta^1 = \eta_{\{3,5,7,9\}} &= \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \eta^2 = \eta_{\{4,6,8,\infty\}} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \\ \eta^3 = \eta_{\{1,5,7,9\}} &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \eta^4 = \eta_{\{2,6,8,\infty\}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \\ \eta^5 = \eta_{\{1,3,7,9\}} &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \eta^6 = \eta_{\{2,4,8,\infty\}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \\ \eta^7 = \eta_{\{1,3,5,9\}} &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \eta^8 = \eta_{\{2,4,6,\infty\}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \\ \eta^9 = \eta_{\{1,3,5,7\}} &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \eta^\infty = \eta_{\{2,4,6,8\}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The notation comes from the fact that η^i comes from adding all η_j of the same parity as i and subtracting η_i .

Definition 4.5. Using the ten functions $\eta^1, \dots, \eta^\infty$, we define

$$\mathcal{V} = \{Z \in \mathfrak{h}_4 : \theta[\eta^1](0, Z) = 0, \dots, \theta[\eta^1](0, Z) = 0\}.$$

For $g = 4$, we can write Theorem 4.4 more simply.

Theorem 4.6. *Let $Z \in \mathfrak{h}_4$. The following two statements are equivalent:*

- A_Z is an irreducible abelian variety and $Z \in \mathcal{V}$;
- There is a marked hyperelliptic Riemann surface C of genus 4 which has Z as its period matrix and $JC = \mathbb{C}^4 / (\mathbb{Z}^4 + ZZ^4)$.

4.2 Transversality

From now on, we will abuse the notation by considering $\text{Is}_{(1,p)}^4$ as a locus in \mathfrak{h}_4 , because we want to prove that $\text{Is}_{(1,p)}^4$ intersects the locus of hyperelliptic Jacobians transversely. We will do it by proving that \mathcal{V} and $\text{Is}_{(1,p)}^4$ intersect transversely in \mathfrak{h}_4 in a component containing hyperelliptic Jacobians. Again, we assume $p > 1$ is an odd number. Denote by $p_1 = \frac{p-1}{2} > 0$. Let us define

$$W = \begin{bmatrix} z_1 & 0 & 2z_1 & 0 \\ 0 & p_1 z_2 & 0 & 0 \\ 2z_1 & 0 & z_3 & 0 \\ 0 & 0 & 0 & 2z_2 \end{bmatrix}, \quad \Lambda_W = W\mathbb{Z}^4 + \mathbb{Z}^4, \quad A_W = \mathbb{C}^4 / \Lambda_W,$$

which depends on z_1, z_2 and z_3 . To make computations easier, we assume the entries are purely imaginary, with imaginary part positive and $\text{Im } z_3$ big enough (to make this precise see Remark 4.10). For convenience and to make notation shorter, we write

$$\theta^i = \theta[\eta^i](0, W)$$

and $\mathbf{e}(\cdot) = \exp(\pi i(\cdot))$. Note that if $l \in \mathbb{Z}$, then $\mathbf{e}(l) = (-1)^l$.

The following lemma is the first step in proving transversality.

Lemma 4.7.

$$W \in \mathcal{V} \cap \text{Is}_{(1,p)}^4.$$

Proof. By condition (4) of Theorem 2.24, $W \in \text{Is}_{(1,p)}^4$. To see that it is in \mathcal{V} we will

look at theta functions, as series.

$$\begin{aligned}
\theta^1 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_2+l_4} \mathbf{e}(z_1(l_1+2l_3+1)^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2(l_4+\frac{1}{2})^2) \\
\theta^2 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_2+l_4} \mathbf{e}(z_1(l_1+2l_3+1)^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2(l_4+\frac{1}{2})^2) \\
\theta^3 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_2+l_4} \mathbf{e}(z_1(l_1+2l_3+\frac{3}{2})^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2l_2^2 + 2z_2(l_4+\frac{1}{2})^2) \\
\theta^4 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_4} \mathbf{e}(z_1(l_1+2l_3+\frac{3}{2})^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2l_2^2 + 2z_2(l_4+\frac{1}{2})^2) \\
\theta^5 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_4} \mathbf{e}(z_1(l_1+2l_3+\frac{1}{2})^2 + (z_3-4z_1)l_3^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2(l_4+\frac{1}{2})^2) \\
\theta^6 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_3+l_4} \mathbf{e}(z_1(l_1+2l_3+\frac{1}{2})^2 + (z_3-4z_1)l_3^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2(l_4+\frac{1}{2})^2) \\
\theta^7 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_3+l_4} \mathbf{e}(z_1(l_1+2l_3+\frac{3}{2})^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2l_4^2) \\
\theta^8 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_3} \mathbf{e}(z_1(l_1+2l_3+\frac{3}{2})^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2l_4^2) \\
\theta^9 &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_1+l_3} \mathbf{e}(z_1(l_1+2l_3+\frac{3}{2})^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2(l_4+\frac{1}{2})^2) \\
\theta^\infty &= \sum_{\ell \in \mathbb{Z}^4} (-1)^{l_2+l_4} \mathbf{e}(z_1(l_1+2l_3+\frac{3}{2})^2 + (z_3-4z_1)(l_3+\frac{1}{2})^2 + p_1z_2(l_2+\frac{1}{2})^2 + 2z_2(l_4+\frac{1}{2})^2)
\end{aligned}$$

To shorten notation, we denote the ℓ th term of the θ^i series by $e^i(\ell)$. To prove that each series sums to 0, it is enough to find pairs of terms that cancel, because of the sign. In fact

$$\begin{aligned}
e^1(l_1, l_2, l_3, l_4) &= -e^1(l_1, -l_2-1, l_3, l_4) \\
e^2(l_1, l_2, l_3, l_4) &= -e^2(l_1, -l_2-1, l_3, l_4) \\
e^3(l_1, l_2, l_3, l_4) &= -e^3(-l_1-1, l_2, -l_3-1, l_4) \\
e^4(l_1, l_2, l_3, l_4) &= -e^4(-l_1-1, l_2, -l_3-1, l_4) \\
e^5(l_1, l_2, l_3, l_4) &= -e^5(-l_1-1, l_2, -l_3, l_4) \\
e^6(l_1, l_2, l_3, l_4) &= -e^6(-l_1-1, l_2, -l_3, l_4) \\
e^7(l_1, l_2, l_3, l_4) &= -e^7(-l_1-2l_3-3, l_2, l_3, l_4) \\
e^8(l_1, l_2, l_3, l_4) &= -e^8(-l_1-2l_3-3, l_2, l_3, l_4) \\
e^9(l_1, l_2, l_3, l_4) &= -e^9(-l_1-2l_3-3, l_2, l_3, l_4) \\
e^\infty(l_1, l_2, l_3, l_4) &= -e^\infty(l_1, -l_2-1, l_3, l_4)
\end{aligned} \tag{4.1}$$

This proves that $W \in \mathcal{V}$. □

We need to use an easy result from analysis.

Lemma 4.8. *For $N < -2$*

$$\sum_{k \in \mathbb{Z}} (-1)^k \left(k + \frac{1}{2}\right) \exp\left(N\left(k + \frac{1}{2}\right)^2\right) \neq 0$$

Proof. As the k th term is equal to the $(-k - 1)$ st term we can write the sum as

$$2 \sum_{k \geq 0} (-1)^k \left(k + \frac{1}{2}\right) \exp\left(N\left(k + \frac{1}{2}\right)^2\right).$$

Now, as the series is alternating, to show that it is not zero we only need to prove that the absolute values of terms decrease. We will use the fact that for $a < \frac{1}{3}, b > 0$ we have

$$a^b > 3a^{b+1} \tag{4.2}$$

Now using inequality (4.2), it is easy to compare the absolute value of the k th and $(k + 1)$ th terms:

$$\left(k + \frac{1}{2}\right) \exp\left(N\left(k + \frac{1}{2}\right)^2\right) > \left(3k + \frac{3}{2}\right) \exp\left(N\left(k + \frac{1}{2}\right)^2 + 1\right) > \left(k + \frac{3}{2}\right) \exp\left(N\left(k + \frac{3}{2}\right)^2\right).$$

□

Theorem 4.9. *W is a smooth point of both $\text{Is}_{(1,p)}^4$ and \mathcal{V} , and $\text{Is}_{(1,p)}^4$ and \mathcal{V} meet transversely at W .*

Proof. Recall from condition (4) of Theorem 2.24 that $\text{Is}_{(1,p)}^4$ is defined by linear equations F_1, \dots, F_4 . We can easily write the matrix of partial derivatives, putting coordinates of \mathfrak{h}_4 in the lexicographical order $a_{1,1}, a_{1,2}, \dots, a_{3,4}, a_{4,4}$:

$$\frac{\partial F_i}{\partial a_{m,n}} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2p & 0 & 1 & 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & (p-2) & 0 & 0 & -p_1 \end{bmatrix}.$$

It is much harder to determine partial derivatives of theta functions, but using the trick of cancelling out terms, we can prove that most of partial derivatives are 0. Recall that

$$\theta_{[c'']}^{[c']}(v, Z) = \sum_{\ell \in \mathbb{Z}^4} \mathbf{e}^{(t(\ell + c')Z(\ell + c') + 2^t(v + c'')(\ell + c'))},$$

so

$$\frac{\partial \theta[c'](0, Z)}{\partial a_{m,n}} = \sum_{\ell \in \mathbb{Z}^4} (l_m + c'_m)(l_n + c'_n) \mathbf{e}^{t(\ell + c')} Z(\ell + c') + 2^t c''(\ell + c').$$

If we have an m, n partial derivative and we do find pairs of terms, with opposite sign and unchanged subscripts m and n , then $\frac{\partial \theta[\eta^i](0, \cdot)}{\partial a_{m,n}} = 0$.

We have put them in the following list:

$$\begin{aligned} e^1(l_1, l_2, l_3, l_4) &= -e^1(l_1, -l_2 - 1, l_3, l_4) = -e^1(l_1, l_2, l_3, -l_4 - 1) \\ e^2(l_1, l_2, l_3, l_4) &= -e^2(l_1, -l_2 - 1, l_3, l_4) = -e^2(l_1, l_2, l_3, -l_4 - 1) \\ e^3(l_1, l_2, l_3, l_4) &= -e^3(-l_1 - 1, l_2, -l_3 - 1, l_4) = -e^3(l_1, l_2, l_3, -l_4 - 1) \\ e^4(l_1, l_2, l_3, l_4) &= -e^4(-l_1 - 1, l_2, -l_3 - 1, l_4) = -e^4(l_1, l_2, l_3, -l_4 - 1) \\ e^5(l_1, l_2, l_3, l_4) &= -e^5(-l_1 - 1, l_2, -l_3, l_4) = -e^5(l_1, l_2, l_3, -l_4 - 1) \\ e^6(l_1, l_2, l_3, l_4) &= -e^6(-l_1 - 1, l_2, -l_3, l_4) = -e^6(l_1, l_2, l_3, -l_4 - 1) \\ e^7(l_1, l_2, l_3, l_4) &= -e^7(-l_1 - 2l_3 - 3, l_2, l_3, l_4) \\ e^8(l_1, l_2, l_3, l_4) &= -e^8(-l_1 - 2l_3 - 3, l_2, l_3, l_4) \\ e^9(l_1, l_2, l_3, l_4) &= -e^9(-l_1 - 2l_3 - 3, l_2, l_3, l_4) \\ e^\infty(l_1, l_2, l_3, l_4) &= -e^\infty(l_1, -l_2 - 1, l_3, l_4) = -e^\infty(l_1, l_2, l_3, -l_4 - 1). \end{aligned} \tag{4.3}$$

As an example, we will show how to check whether $\frac{\partial \theta[\eta^1](0, \cdot)}{\partial a_{m,n}} = 0$. In the first equation we only change l_2 and in the second we change l_4 . Hence all partial derivative apart from $\frac{\partial \theta[\eta^1](0, \cdot)}{\partial a_{2,4}}$ have to be zero.

Using all equations from the list 4.3, we can write the matrix of partial derivatives, where '?' means we do not know the answer.

$$\frac{\partial \theta[\eta^i](0, \cdot)}{\partial a_{m,n}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ ? & ? & ? & ? & 0 & 0 & 0 & 0 & 0 & 0 \\ ? & ? & ? & ? & 0 & 0 & 0 & 0 & 0 & 0 \\ ? & ? & ? & ? & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \end{bmatrix}$$

Now, for η^7, η^8 and η^9 , we can prove that $\frac{\partial \theta}{\partial a_{1,2}} = 0$ and $\frac{\partial \theta}{\partial a_{1,4}} = 0$. This is because, for

them, we have

$$e^n(l_1, l_2, l_3, l_4) = e^n(-l_1 - 1, l_2, -l_3 - 1, l_4), n \in \{7, 8, 9\}. \quad (4.4)$$

When differentiating in the direction $a_{1,2}$, we get an additional factor $(l_1 + \frac{1}{2})(l_2 + \frac{1}{2})$. For $a_{1,4}$ we get $(l_1 + \frac{1}{2})l_4$ or $(l_1 + \frac{1}{2})(l_4 + \frac{1}{2})$. In both cases, changing from l_1 to $-l_1 - 1$ gives us desired factor of (-1) . So those pairs of terms cancel out, and the sum of the series is 0.

Summing up, we get the matrix:

$$\frac{\partial \theta[\eta^i](0, \cdot)}{\partial a_{m,n}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ? \\ ? & 0 & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ? & 0 & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ? & 0 & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \end{bmatrix}.$$

Now we will prove that some of question marks are non-zero. This will allow us to show that the rank of the matrix is 3, so W is a smooth point of \mathcal{V} .

The first non-zero question mark is $\frac{\partial \theta[\eta^1](0, \cdot)}{\partial z_{2,4}}$. To show that, let us define

$$\alpha_{13} = \sum_{l_1, l_3 \in \mathbb{Z}} \mathbf{e}((z_1(l_1 + 2l_3 + 1))^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2) > 0,$$

$$\beta_2 = \sum_{l_2 \in \mathbb{Z}} (-1)^{l_2} (l_2 + \frac{1}{2}) \mathbf{e}(p_1 z_2 (l_2 + \frac{1}{2})^2),$$

$$\beta_4 = \sum_{l_4 \in \mathbb{Z}} (-1)^{l_4} (l_4 + \frac{1}{2}) \mathbf{e}(2z_2 (l_4 + \frac{1}{2})^2).$$

Then

$$\frac{\partial \theta[\eta^1](0, \cdot)}{\partial z_{2,4}} = \alpha_{13} \beta_2 \beta_4.$$

As α_{13} is the sum of positive reals, it is positive. β_2 and β_4 are nonzero by Lemma 4.2.

Let us consider another question mark, namely $\frac{\partial\theta[\eta^5](0,\cdot)}{\partial a_{1,4}}$. Let us define

$$\alpha_2 = \sum_{l_2 \in \mathbb{Z}} \mathbf{e}(p_1 z_2 (l_2 + \frac{1}{2})^2) > 0,$$

$$\beta_{13} = \sum_{l_1, l_3 \in \mathbb{Z}} (-1)^{l_1} (l_1 + \frac{1}{2}) \mathbf{e}((z_1(l_1 + 2l_3 + \frac{1}{2}))^2 + (z_3 - 4z_1)l_3^2).$$

Then

$$\frac{\partial\theta[\eta^5]}{\partial z_{1,4}} = \alpha_2 \beta_{13} \beta_4.$$

Therefore we only need to prove that $\beta_{13} \neq 0$. Note that β_{13} depends on z_1 and z_3 . Putting $l_3 = 0$, we get that the partial sum depends only on z_1 and is non-zero by Lemma 4.2.

If we fix z_1 , then the sum over $l_3 \neq 0$ depends only on z_3 and can be chosen to be arbitrarily small, by choosing sufficiently big $|z_{3,3}| \gg 0$. In other words

Remark 4.10. β_{13} seen as a function of (z_1, z_3) is not identically 0.

Let us consider one more pair of question marks, namely $\frac{\partial\theta[\eta^8](0,\cdot)}{\partial z_{1,1}}$ and $\frac{\partial\theta[\eta^8](0,\cdot)}{\partial z_{1,3}}$.

Let us define

$$\alpha_4 = \sum_{l_4 \in \mathbb{Z}} \mathbf{e}(2z_2 l_4^2) > 0,$$

$$\gamma_{11} = \sum_{l_1, l_3 \in \mathbb{Z}} (-1)^{l_1+l_3} (l_1 + \frac{1}{2})^2 \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2),$$

$$\gamma_{13} = \sum_{l_1, l_3 \in \mathbb{Z}} (-1)^{l_1+l_3} (l_1 + \frac{1}{2})(l_3 + \frac{1}{2}) \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2).$$

Then

$$\frac{\partial\theta[\eta^8]}{\partial z_{1,1}} = \alpha_2 \alpha_4 \gamma_{11},$$

$$\frac{\partial\theta[\eta^8]}{\partial z_{1,3}} = \alpha_2 \alpha_4 \gamma_{13}.$$

We want to prove that

$$\frac{\partial\theta[\eta^8]}{\partial z_{1,1}} \neq -2 \frac{\partial\theta[\eta^8]}{\partial z_{1,3}}.$$

Therefore we consider

$$\gamma_{11} + 2\gamma_{13} = \sum_{l_1, l_3 \in \mathbb{Z}} (-1)^{l_1+l_3} (l_1 + \frac{1}{2})(l_1 + 2l_3 + \frac{3}{2}) \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2).$$

As the (l_1, l_3) term is equal to the $(-l_1 - 1, -l_3 - 1)$ term we can write the sum as

$$\gamma_{11} + 2\gamma_{13} = 2 \sum_{l_3 \geq 0} \sum_{l_1 \in \mathbb{Z}} (-1)^{l_1 + l_3} (l_1 + \frac{1}{2})(l_1 + 2l_3 + \frac{3}{2}) \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2).$$

Now, we want to use a trick of adding and subtracting $2l_3 - 1$ to get

$$\begin{aligned} \gamma_{11} + 2\gamma_{13} &= 2 \sum_{l_3 \geq 0} \sum_{l_1 \in \mathbb{Z}} (-1)^{l_1 + l_3} (l_1 + 2l_3 + \frac{3}{2})^2 \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2) \\ &\quad - 2 \sum_{l_3 \geq 0} \sum_{l_1 \in \mathbb{Z}} (-1)^{l_1 + l_3} (2l_3 + 1)(l_1 + 2l_3 + \frac{3}{2}) \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2). \end{aligned}$$

This helps us, because if we fix $l_3 \geq 0$, the l_1 th and $(-l_1 - 2l_3 - 3)$ th terms of the first double sum cancel, so the first double sum vanishes. Hence $\gamma_{11} + 2\gamma_{13}$ is equal to

$$-2 \sum_{l_3 \geq 0} \sum_{l_1 \in \mathbb{Z}} (-1)^{l_1 + l_3} (2l_3 + 1)(l_1 + 2l_3 + \frac{3}{2}) \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2 + (z_3 - 4z_1)(l_3 + \frac{1}{2})^2)$$

As we can change the index from l_1 to $l_1 + 2l_3$, when summing over \mathbb{Z} , we can separate sums and get

$$\gamma_{11} + 2\gamma_{13} = -2\beta_{123}\beta_3,$$

where

$$\begin{aligned} \beta_{123} &= \sum_{l_1 + 2l_3 \in \mathbb{Z}} (-1)^{l_1 + 2l_3} (l_1 + 2l_3 + \frac{3}{2}) \mathbf{e}(z_1(l_1 + 2l_3 + \frac{3}{2})^2), \\ \beta_3 &= \sum_{l_3 \geq 0} (-1)^{l_3} 2(l_3 + \frac{1}{2}) \mathbf{e}((z_3 - 4z_1)(l_3 + \frac{1}{2})^2). \end{aligned}$$

Both β_{123} and β_3 are non-zero by Lemma 4.2, which means $\gamma_{11} + 2\gamma_{13} \neq 0$.

To sum everything up, we find that the matrix of partial derivatives is

$$\begin{bmatrix} \frac{\partial \theta[\eta^i](0, \cdot)}{\partial a_{m,n}} \\ \frac{\partial F_i(0, \cdot)}{\partial a_{m,n}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{13}\beta_2\beta_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ?0 \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ?0 \\ 0 & 0 & 0 & \alpha_2\beta_{13}\beta_4 & 0 & 0 & 0 & 0 & ? & ?0 \\ 0 & 0 & 0 & ? & 0 & 0 & 0 & 0 & ? & ?0 \\ ? & 0 & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2\beta_4\gamma_{11} & 0 & \alpha_2\beta_4\gamma_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ? & 0 & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2p & 0 & 1 & 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & (p-2) & 0 & 0 & -p_1 \end{bmatrix},$$

where ‘?’ means that, by looking closely at the Table 4.3, it can be proved that there are pairs of terms cancelling out, so the entry is 0. We will skip this, as it is not needed to finish the proof.

Now, $\gamma_{11} + 2\gamma_{13} \neq 0$ means that rows 1, 5, 8, 11, 12, 13, 14 and columns 1, 2, 3, 4, 5, 6, 7, give a non-zero minor, so the rank of the above matrix is at least 7. It cannot be more than 7, as the codimensions of \mathcal{V} and $\text{Is}_{(1,3)}^4$ are 3 and 4 respectively.

Hence the rank of the matrix is equal to the sum of codimensions, so W is a smooth point of both \mathcal{V} and $\text{Is}_{(1,3)}^4$ and they intersect transversely in W . \square

It is obvious that W is reducible, thus not a period matrix of a hyperelliptic Jacobian, although the following proposition shows that it is in the closure of the locus of hyperelliptic Jacobians.

Proposition 4.11. *A_W is a product of elliptic curves. Therefore there exists a family of Jacobians of hyperelliptic curves which specialise to A_W and so $A_W \in \overline{\mathcal{JH}}$.*

Proof. It is obvious that A_W is a product of elliptic curves defined by complex numbers p_1z_2 and $2z_2$ and a principally polarised surface defined by $W' = \begin{bmatrix} z_1 & 2z_1 \\ 2z_1 & z_3 \end{bmatrix}$. By Remark 2.4, the discriminant $\Delta(W') = 1$, so A_W is a product of four elliptic curves. This can also be proved explicitly by finding elliptic curves defined by z_1 and z_3 embedded by inclusions that have analytic representations of the form $s \mapsto (s, 2s)$ and $t \mapsto (0, t)$.

Define a curve C' , which has four elliptic curves E_1, E_2, E_3, E_4 as irreducible components, with three nodes coming from the intersection of E_1 with E_2, E_3 and E_4 . Then $\text{Pic}^0(C') = \prod_{i=1}^4 \text{Pic}^0(E_i) = \prod_{i=1}^4 E_i$. It is an example of a curve of compact type. We can define a so-called generalised Jacobian $JC' = \text{Pic}^0(C') = \prod_{i=1}^4 E_i$. Although it is a very natural definition, it lacks many properties of Jacobians of smooth curves. One of biggest problems is the fact that JC' completely forgets the points of intersection and it does not have a nice Abel map and universal property. However, it is enough to prove our result, because it is obvious that one can write a family of genus 4 smooth hyperelliptic curves that specialise to C' . Then, by functoriality of Pic^0 , their Jacobians specialise to $\prod_{i=1}^4 E_i$. \square

Let us state the main result of this chapter.

Theorem 4.12. *For $p > 1$ odd, the locus $\mathcal{JH} \cap \text{Is}_{(1,p)}^4$ is a nonempty (possibly reducible) subvariety of \mathcal{A}_4 with the smallest dimension of its components being 3.*

Proof. In Theorem 4.9, we proved that \mathcal{V} and the preimage of $\text{Is}_{(1,p)}^4$ in \mathfrak{h}_4 intersect transversely in an explicitly defined point W . Moreover, in Proposition 4.11 we proved that W is in the closure of the preimage of \mathcal{JH} . The transversality is an open condition, so the preimages of \mathcal{JH} and $\text{Is}_{(1,p)}^4$ have to intersect transversely in a neighbourhood of W . In particular the component having W in its closure is non-empty and of dimension 3. \square

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