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Frobenius Categorification of Cluster Algebras

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

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SUMMARY

Cluster categories, introduced by Buan–Marsh–Reineke–Reiten–Todorov [BMR+06] and later generalised by Amiot [Ami09], are certain 2-Calabi–Yau triangulated categories that model the combinatorics of cluster algebras without frozen variables. When frozen variables do occur, it is natural to try to model the cluster combinatorics via a Frobenius category, with the indecomposable projective-injective objects corresponding to these special variables.

Amiot–Iyama–Reiten [AIR15] show how Frobenius categories admitting $(d-1)$-cluster-tilting objects arise naturally from the data of a Noetherian bimodule d-Calabi–Yau algebra A and an idempotent e of A such that $A/\langle e\rangle$ is finite dimensional. In this work, we observe that this phenomenon still occurs under the weaker assumption that A and A^{op} are internally d-Calabi–Yau with respect to e; this new definition allows the d-Calabi–Yau property to fail in a way controlled by e. Under either set of assumptions, the algebra $B = eAe$ is Iwanaga–Gorenstein, and eA is a cluster-tilting object in the Frobenius category $\mathrm{GP}(B)$ of Gorenstein projective B-modules.

Geiß–Leclerc–Schröer [GLS08] define a class of cluster algebras that are, by construction, modelled by certain Frobenius subcategories $\operatorname{Sub} Q_J$ of module categories over preprojective algebras. Buan–Iyama–Reiten–Smith [BIRS11] prove that the endomorphism algebra of a cluster-tilting object in one of these categories is a frozen Jacobian algebra. Following Keller–Reiten [KR07], we observe that such algebras are internally 3-Calabi–Yau with respect to the idempotent corresponding to the frozen vertices, thus obtaining a large class of examples of such algebras.

Geiß–Leclerc–Schröer also attach, via an algebraic homogenization procedure, a second cluster algebra to each category $\operatorname{Sub} Q_J$, by adding more frozen variables. We describe how to compute the quiver of a seed in this cluster algebra via approximation theory in the category $\operatorname{Sub} Q_J$; our alternative construction has the advantage that arrows between the frozen vertices appear naturally. We write down a potential on this enlarged quiver, and conjecture that the resulting frozen Jacobian algebra A and its opposite are internally 3-Calabi–Yau. If true, the algebra may be realised as the endomorphism algebra of a cluster-tilting object in a Frobenius category $\mathrm{GP}(B)$ as above. We further conjecture that $\mathrm{GP}(B)$ is stably 2-Calabi–Yau, in which case it would provide a categorification of this second cluster algebra.

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CONTENTS

CHAPTER 1
INTRODUCTION

1.1 Motivation and Existing Results

Cluster algebras, introduced by Fomin–Zelevinsky [FZ02a], are commutative algebras with a distinguished family of generators, called cluster variables, displaying special combinatorial properties. The variables are organised into overlapping sets of clusters, each having the same cardinality, say n. Each cluster is also equipped with some data that allows it to be transformed into another cluster via a process called mutation; in the level of generality we are interested in, we may assume this data is given by a quiver without loops or 2-cycles, having the elements of the cluster as vertices. Mutation replaces an element of the cluster by a different cluster variable, as dictated by the quiver, and also changes the quiver. Certain cluster variables cannot be mutated – these are called frozen variables, and they occur in every cluster. Any set of $n-1$ cluster variables, containing all the frozen variables, can be completed to a cluster in exactly two ways, if at all.

Since the combinatorics of cluster algebras are often very complicated, it can be useful to model them categorically. Consider a cluster algebra $\mathscr{A}$ with no frozen variables, and such that one of the clusters has an acyclic quiver Q. In this case, Buan–Marsh–Reineke–Reiten–Todorov [BMR$^+$06] introduced the cluster category $\mathcal{C}_Q$, given by the orbit category

$$\mathcal{C}_Q = \mathcal{D}^b(\mathbb{C}Q)/\tau[-1]$$

which is triangulated by a result of Keller [Kel05], and 2-Calabi–Yau by construction. Here τ denotes the Auslander–Reiten translation, and [1] the shift functor on the bounded derived category $\mathcal{D}^b(\mathbb{C}Q)$ of $\mathbb{C}Q$ modules. These categories are equipped with a cluster character $M \mapsto \varphi_M \in \mathscr{A}$ (see [CC06, §3]), mapping objects to elements of the cluster algebra. Under this assignment, the cluster variables are given by φ_M for M an indecomposable rigid object of $\mathcal{C}_Q$, i.e. an indecomposable M such that $\operatorname{Ext}^1_{\mathcal{C}_Q}(M,M)=0$.

The clusters correspond to cluster-tilting objects T, which are objects satisfying

$$\operatorname{add} T = \{X \in \mathcal{C}_Q : \operatorname{Ext}^1_{\mathcal{C}_Q}(X,T) = 0\} = \{X \in \mathcal{C}_Q : \operatorname{Ext}^1_{\mathcal{C}_Q}(T,X) = 0\}.$$

The category $\operatorname{add} T$ is an example of a cluster-tilting subcategory of $\mathcal{C}_Q$; such categories are fundamental in Iyama's higher-dimensional Auslander–Reiten theory [Iya07b]. In the situations we will study, every cluster-tilting subcategory will be of the form $\operatorname{add} T$ for some cluster-tilting object T, so we may focus on the object rather than the category. Moreover, for any T, we always have $\operatorname{add} T = \operatorname{add} T'$ for some object T' which is basic, meaning that in any decomposition of T' into indecomposable summands, no two of these summands are isomorphic. Thus we will usually discuss basic cluster-tilting objects to simplify the exposition. For example, the endomorphism algebra of any basic object $T \in \mathcal{C}_Q$ is a finite dimensional basic algebra [ASS06, §I.6] and so is isomorphic (rather than merely Morita equivalent) to the path algebra of a quiver modulo an admissible ideal of relations [ASS06, §II.3].

Each basic cluster-tilting object of $\mathcal{C}_Q$ has n indecomposable summands $T_1, \dots, T_n$, and the clusters of $\mathscr{A}$ are precisely the sets of the form $\{\varphi_{T_1}, \dots, \varphi_{T_n}\}$. The quiver of each cluster is given by the quiver of the endomorphism algebra $\operatorname{End}_{\mathcal{C}_Q}(T)^{\mathrm{op}}$ of the corresponding cluster-tilting object. Once such cluster-tilting object is $\mathbb{C}Q$ itself, and the quiver of $\operatorname{End}_{\mathcal{C}_Q}(\mathbb{C}Q)^{\mathrm{op}}$ is Q.

The most important feature of cluster-tilting objects in $\mathcal{C}_Q$ is the following *mutation property*. Given a summand T_i of T, there exists a unique (up to isomorphism) indecomposable $T_i' \not\cong T_i$ such that $T/T_i \oplus T_i'$ is cluster-tilting, and the replacement of T_i by T_i' induces a mutation of the corresponding cluster at the variable φ_{T_i}. This mutation property for cluster-tilting objects holds in any 2-Calabi–Yau triangulated category in which the quivers of endomorphism algebras of such objects have no loops, as shown by Iyama–Yoshino [IY08]. In such categories, the mutation T_i' of a summand T_i of T is isomorphic to both the cone of a minimal left $\operatorname{add} T/T_i$-approximation of T_i, and the cocone of a minimal right $\operatorname{add} T/T_i$-approximation of T_i.

These constructions have been generalised to cluster algebras for which no cluster has an acyclic quiver by Amiot [Ami09], via the study of quivers with potential, and associated Ginzburg dg-algebras.

In the case of cluster algebras with frozen variables, a suitable categorical model must have certain objects occurring as summands of every cluster-tilting object. Taking the quotient of this category by the ideal of maps factoring through these objects should correspond to removing the frozen variables from the cluster algebra, and therefore the result should be a triangulated 2-Calabi–Yau category of the kind described above. Thus a natural candidate for such a model is a Frobenius category $\mathcal{E}$, an exact category with enough projective and injective objects, and such that the projective and injective objects coincide. Each projective-injective object Q satisfies

$$\mathrm{Ext}^1_{\mathcal{E}}(Q,-) = 0 = \mathrm{Ext}^1_{\mathcal{E}}(-,Q)$$

by definition, and thus $Q \in \operatorname{add} T$ for any cluster-tilting object $T \in \mathcal{E}$. Moreover, the stable category $\underline{\mathcal{E}}$, formed by taking the quotient by the ideal of morphisms factoring through a projective-injective object, is triangulated by a result of Happel [Hap88]. The stable category $\underline{\mathcal{E}}$ is 2-Calabi–Yau (said differently, $\mathcal{E}$ is stably 2-Calabi–Yau) if there is a functorial duality

$$\mathrm{Ext}^1_{\mathcal{E}}(X,Y) = \mathrm{D}\,\mathrm{Ext}^1_{\mathcal{E}}(Y,X)$$

for all $X, Y \in \mathcal{E}$.

A large family of cluster algebras admitting a Frobenius categorification have been constructed by Geiß–Leclerc–Schröer [GLS08]. To each Dynkin diagram Δ, and each non-empty subset J of the nodes of Δ, they attach a subcategory $\operatorname{Sub} Q_J$ of the category $\operatorname{mod}\Pi$ of modules for the projective algebra $\Pi = \Pi(\Delta)$. Here $Q_J = \bigoplus_{j\in J} Q_j$ denotes the direct sum of the vertex injective Π-modules at vertices $j \in J$, and $\operatorname{Sub} Q_J$ is the full subcategory of $\operatorname{mod}\Pi$ whose objects are submodules of a direct sum of finitely many copies of Q_J. Equivalently, they are the Π-modules with socle supported at J. For each choice of Δ and J, the category $\operatorname{Sub} Q_J$ is a functorially finite Frobenius subcategory of $\operatorname{mod}\Pi$, with projective-injective objects given by the minimal left $\operatorname{Sub} Q_J$-approximations of the projective-injective objects of $\operatorname{mod}\Pi$. The stable category $\underline{\operatorname{Sub}}\, Q_J$ is 2-Calabi–Yau. Moreover, $\operatorname{Sub} Q_J$ contains cluster-tilting objects, and Geiß–Leclerc–Schröer declare a distinguished mutation class $\mathscr{R}_J$ of such objects. In the case that $J = \Delta_0$ consists of all nodes of Δ, we have $\operatorname{Sub} Q_J = \operatorname{mod}\Pi$.

Via some constructions of Lusztig, Geiß–Leclerc–Schröer define [GLS06, §9] a map $M \mapsto \varphi_M$ on the objects of $\operatorname{Sub} Q_J$, taking values in the coordinate ring $\mathbb{C}[N_J]$ of an affine open subset N_J of the partial flag variety F_J determined by the pair (Δ, J) [GLS08, Prop. 9.1]. As a result of a number of remarkable properties of the category $\operatorname{Sub} Q_J$ and of the assignment $M \mapsto \varphi_M$, the functions φ_M, for $M \in \operatorname{Sub} Q_J$ an indecomposable summand of a cluster-tilting object in $\mathscr{R}_J$, are the cluster variables of a cluster algebra $\mathscr{A}_J = \mathbb{C}[N_J]$. If M is an indecomposable projective-injective object, then φ_M is frozen. The clusters of $\mathscr{A}_J$ correspond to the cluster-tilting objects in $\mathscr{R}_J$, and the quiver of a cluster is the quiver of the endomorphism algebra of the corresponding cluster-tilting object. By construction, $\operatorname{Sub} Q_J$ provides a Frobenius categorification of $\mathscr{A}_J$. These methods have been generalised by Fu–Keller [FK10] to associate (abstract) cluster algebras to a wider class of stably 2-Calabi–Yau Frobenius categories; we will recall Fu–Keller's construction in Section 2.7.

The quiver of $\mathrm{End}_\Pi(T)^{\mathrm{op}}$ for a cluster-tilting object $T \in \operatorname{Sub} Q_J$ almost always has arrows between the vertices corresponding to the projective-injective indecomposable summands, resulting in arrows between the frozen variables of the corresponding cluster.

Since these 'frozen' arrows play no role in the combinatorics of the cluster algebra, they are usually ignored. However, they will play an important role in our constructions.

Given a cluster-tilting object $T \in \mathscr{R}_J$, it is shown by Buan–Iyama–Reiten–Smith [BIRS11] that the endomorphism algebra of T is a frozen Jacobian algebra. These algebras are defined as follows. Let Q be a quiver, and W a potential, i.e. a linear combination of cycles of Q. We define the cyclic derivative $\partial_\alpha W$ of W with respect to an arrow α of Q by defining

$$\partial_\alpha \alpha_1 \cdots \alpha_k = \sum_{\alpha_i = \alpha} \alpha_{i+1} \cdots \alpha_k \alpha_1 \cdots \alpha_{i-1}$$

on any cycle $\alpha_1 \cdots \alpha_k$, and extending linearly. Given a subquiver F of Q, called the *frozen subquiver* of Q, we can define the frozen Jacobian algebra

$$\mathcal{J}(Q, F, W) = \mathbb{C}Q / \langle \partial_\alpha W : \alpha \in Q_1 \setminus F_1 \rangle.$$

Taking Q to be the quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ for $T \in \mathscr{R}_J$, let F be the full subquiver on vertices corresponding to projective-injective summands of T. Buan–Iyama–Reiten–Smith then describe a potential W on Q such that

$$\operatorname{End}_\Pi(T)^{\mathrm{op}} \cong \mathcal{J}(Q, F, W).$$

Taking $\underline{Q}$ to be the full subquiver of Q on vertices corresponding to non-projective-injective summands of T, and $\underline{W}$ to be the potential obtained from W by deleting terms given by cycles passing through frozen vertices, they also give an isomorphism

$$\operatorname{End}_{\underline{\operatorname{Sub}}\, Q_J}(T)^{\mathrm{op}} \cong \mathcal{J}(\underline{Q}, \underline{W})$$

of the endomorphism algebra of T in the stable category $\underline{\operatorname{Sub}}\, Q_J$ with the ordinary Jacobian algebra (where the relation $\partial_\alpha \underline{W}$ is imposed for all arrows α).

As well as the cluster algebras $\mathscr{A}_J$, which come with a categorification automatically, Geiß–Leclerc–Schröer also construct more homogeneous cluster algebras inside a multi-graded homogeneous coordinate ring of the flag variety F_J of which N_J is an affine open piece. Each cluster variable of $\mathscr{A}_J$ has a multi-degree in $\mathbb{Z}^{|J|}$, and so $\mathscr{A}_J$ can be homogenised to get a multi-graded cluster algebra $\widetilde{\mathscr{A}_J}$, with $|J|$ extra frozen variables, each having (multi-)degree given by an element ε_j of the standard basis of $\mathbb{Z}^{|J|}$. The condition that $\widetilde{\mathscr{A}_J}$ is multi-graded determines the quiver of each cluster (ignoring, as usual, arrows between frozen variables) via the condition that

$$\sum_{y \to x} \deg y = \sum_{x \to z} \deg z$$

for each unfrozen variable x in the cluster, together with the requirement that this quiver must not have 2-cycles.

We may summarise Geiß–Leclerc–Schröer's two-step construction of the cluster algebra $\widetilde{\mathscr{A}_J}$ in the following schematic.

$$\begin{tikzcd} & \widetilde{\mathscr{A}_J} \\ \operatorname{Sub} Q_J \arrow[r, "\text{decategorify}"'] & \mathscr{A}_J \arrow[u, "\text{homogenise}"'] \end{tikzcd}$$

While the construction of $\widetilde{\mathscr{A}_J}$ from $\mathscr{A}_J$ is purely combinatorial, the fact that the algebras $\mathscr{A}_J$ come with Frobenius categorifications makes it natural to look for such categorifications for $\widetilde{\mathscr{A}_J}$, and this is the main motivation of our work. Informally, we wish to complete the schematic above to a 'commuting square'

$$\begin{tikzcd} \mathcal{C}_J \arrow[r, "\text{decategorify}"] & \widetilde{\mathscr{A}_J} \\ \operatorname{Sub} Q_J \arrow[u, "\text{homogenise}"] \arrow[r, "\text{decategorify}"'] & \mathscr{A}_J \arrow[u, "\text{homogenise}"'] \end{tikzcd}$$

by constructing a 'homogenisation' $\mathcal{C}_J$ of the category $\operatorname{Sub} Q_J$, such that $\mathcal{C}_J$ categorifies $\widetilde{\mathscr{A}_J}$ in the same sense in which $\operatorname{Sub} Q_J$ categorifies $\mathscr{A}_J$. The resulting category $\mathcal{C}_J$ should be Frobenius, stably 2-Calabi–Yau, and enlarge $\operatorname{Sub} Q_J$ by $|J|$ indecomposable projective-injective objects.

Constructing such a category $\mathcal{C}_J$ would continue the programme of additive categorification of cluster algebras begun by Buan–Marsh–Reineke–Reiten–Todorov in [BMR+06], by describing a new family of examples of cluster algebras with frozen variables admitting additive categorifications by stably 2-Calabi–Yau Frobenius categories.

In the case that $\Delta = \mathsf{A}_n$ and J is a singleton, so that F_J is a Grassmannian, the cluster algebra $\widetilde{\mathscr{A}_J}$ has been categorified by Jensen–King–Su [JKS14], using a category of Cohen–Macaulay modules over a certain Gorenstein order. Recent work of Demonet–Iyama constructs Frobenius categorifications of $\widetilde{\mathscr{A}_J}$ for Δ of type A_n or D_n and arbitrary J. Their construction also uses categories of Cohen–Macaulay modules over Gorenstein orders, but it is sometimes necessary to consider certain subcategories of these, depending on (Δ, J). We will discuss these approaches to the construction of $\mathcal{C}_J$ in more detail in Section 3.1.

In order to more readily construct Frobenius categorifications of cluster algebras with frozen variables, we wish to have methods for constructing Frobenius categories admitting cluster-tilting objects. One such method, which we will adapt to our purposes later, is given by Amiot–Iyama–Reiten. Let A be an algebra, and let $A^\varepsilon = A \otimes_{\mathbb{C}} A^{\mathrm{op}}$ be the enveloping algebra of A, so that A^ε-modules are A-bimodules. The algebra A is said to be bimodule d-Calabi–Yau if $A \in \operatorname{per} A^\varepsilon$, i.e. A is isomorphic to a bounded complex of projective A^{ε}-modules when thought of as an object of the derived category $\mathcal{D}A^{\varepsilon}$, and moreover we have a quasi-isomorphism

$$A \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{A^{\varepsilon}}(A, A^{\varepsilon})[d]$$

in $\mathcal{D}A^{\varepsilon}$.

Assume A is Noetherian, and let e be an idempotent of A such that $A/\langle e\rangle$ is finite dimensional. Let $B = eAe$, and let

$$\mathrm{GP}(B) = \{X \in \operatorname{mod} B : \mathrm{Ext}^i_B(X, B) = 0,\ i > 0\}$$

be the category of Gorenstein projective B-modules. Then Amiot–Iyama–Reiten [AIR15, Thm. 2.2] show that B is an Iwanaga–Gorenstein algebra, meaning it is Noetherian with finite injective dimension on both sides. This implies that $\mathrm{GP}(B)$ is Frobenius. Moreover, eA is Gorenstein projective as a B-module, satisfies $\mathrm{End}_B(eA)^{\mathrm{op}} \cong A$, and is $(d-1)$-cluster-tilting in $\mathrm{GP}(B)$. A $(d-1)$-cluster-tilting object T is one satisfying

$$\begin{aligned} \operatorname{add} T &= \{X \in \mathrm{GP}(B) : \mathrm{Ext}^i_B(X, T) = 0,\ 0 < i < d-1\} \\ &= \{X \in \mathrm{GP}(B) : \mathrm{Ext}^i_B(T, X) = 0,\ 0 < i < d-1\}, \end{aligned}$$

so we obtain ordinary (2-)cluster-tilting objects by starting from bimodule 3-Calabi–Yau algebras. Thus one may construct a Frobenius category from a candidate for the endomorphism algebra of a cluster-tilting object in it, and it is this approach that we will take to the construction of our desired categorification $\mathcal{C}_J$. More details on this strategy can by found in Section 3.4.

Cluster-tilting objects are also of interest away from the study of cluster algebras. A 1-cluster-tilting object in a category $\mathcal{C}$ is an additive generator, which exists if and only if $\mathcal{C}$ is representation finite. Beginning with Iyama [Iya07b], many results in the Auslander–Reiten theory of representation finite categories have been adapted to d-representation finite categories, which are those admitting a d-cluster-tilting object. Being able to construct more examples of d-representation finite Frobenius categories is therefore also of interest in this higher Auslander–Reiten theory.

1.2 Overview of Results

We now describe the main results of the thesis. First, we obtain a generalisation of Amiot–Iyama–Reiten's result by weakening the bimodule 3-Calabi–Yau assumption. Given an algebra A and an idempotent $e \in A$, write $\underline{A} = A/\langle e\rangle$. We say that A is *internally d-Calabi–Yau* with respect to e (Definition 5.1) if

(i) $\operatorname{gl.dim} A \le d$, and

(ii) there is a functorial duality

$$\mathrm{D}\,\mathrm{Ext}^i_A(M,N) = \mathrm{Ext}^{d-i}_A(N,M)$$

for all $N \in \operatorname{mod} A$ and all $M \in \operatorname{mod} A/\langle e\rangle$.

We then have the following result.

Theorem 1 (Theorem 5.13). *Let A be a Noetherian algebra and let $e \in A$ be an idempotent such that $A/\langle e\rangle$ is finite dimensional, and both A and A^{op} are internally d-Calabi–Yau with respect to e. Write $B = eAe$ and $\underline{A} = A/\langle e\rangle$. Then*

(i) B is Iwanaga–Gorenstein with Gorenstein dimension at most d, so $\mathrm{GP}(B)$ is a Frobenius category,

(ii) eA is $(d-1)$-cluster-tilting in $\mathrm{GP}(B)$, and

(iii) there are natural isomorphisms $\mathrm{End}_B(eA)^{\mathrm{op}} \xrightarrow{\sim} A$ and $\mathrm{End}_{\underline{\mathrm{GP}}(B)}(eA)^{\mathrm{op}} \xrightarrow{\sim} \underline{A}$.

We say A is *internally bimodule d-Calabi–Yau* with respect to e (Definition 5.7) if

(i) A has projective dimension at most d in the category of A-bimodules, and

(ii) there exists a triangle

$$A[-d] \xrightarrow{\psi} \mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon) \longrightarrow C \longrightarrow A[1-d]$$

in $\mathcal{D}A^\varepsilon$ such that $\underline{A} \overset{\mathbf{L}}{\otimes}_A C = 0 = C \overset{\mathbf{L}}{\otimes}_A \underline{A}$.

This definition, unlike that of being internally d-Calabi–Yau, is left-right symmetric; A is internally bimodule d-Calabi–Yau with respect to e if and only if the same is true of A^{op}. We show (Corollary 5.12) that if A is internally bimodule d-Calabi–Yau with respect to e then it is also internally d-Calabi–Yau with respect to e. By symmetry, it also follows that A^{op} is internally d-Calabi–Yau with respect to e, and so Theorem 1 applies to internally bimodule d-Calabi–Yau algebras.

Theorem 1 works in the opposite direction to results of Keller–Reiten [KR07, §4], which state that if T is a cluster-tilting object of a Hom-finite stably 2-Calabi–Yau Frobenius category $\mathcal{E}$, then the algebra $\mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ is internally 3-Calabi–Yau with respect to the idempotent corresponding to the projective-injective summands.

In the case $d = 3$, we are particularly interested in when a frozen Jacobian algebra $\mathcal{J}(Q,F,W)$ is internally bimodule 3-Calabi–Yau with respect to the idempotent $e = \sum_{v\in F_0} e_v$. For a Jacobian algebra A, Ginzburg [Gin06] (see also Broomhead [Bro12]) defines a complex of projective A-bimodules, based on the combinatorics of the quiver and potential. If this complex is isomorphic to A in the bounded derived category $\mathcal{D}^b A^\varepsilon$ of A-bimodules, then A is bimodule 3-Calabi–Yau. We define an analogous complex $\mathbf{P}(A)$ of A-bimodules for $A = \mathcal{J}(Q,F,W)$ (Definition 5.21), together with a map $\mu_0 \colon \mathbf{P}(A) \to A$ (Lemma 5.22), and prove the following.

Theorem 2 (Theorem 5.24)**.** *Let $A = \mathcal{J}(Q,F,W)$. If $\mu_0 \colon \mathbf{P}(A) \to A$ is a quasi-isomorphism, then A is internally bimodule 3-Calabi–Yau with respect to $e = \sum_{v \in F_0} e_v$.*

Combining this with Theorem 1, we see that such a frozen Jacobian algebra is isomorphic to the endomorphism algebra of a cluster-tilting object in the Frobenius category $\mathrm{GP}(B)$ for $B = eAe$. The requirement that $\mathbf{P}(A)$ is quasi-isomorphic to A is stronger than the requirement that A is internally bimodule 3-Calabi–Yau with respect to e, as it puts extra conditions on modules whose support includes the boundary vertices. Thus we expect that under this stronger assumption we can also deduce some homological properties of B. In particular, we conjecture the following.

Conjecture 1 (Conjecture 5.25)**.** *If A is a frozen Jacobian algebra such that $\mu_0 \colon \mathbf{P}(A) \to A$ is a quasi-isomorphism, and $B = eAe$ for $e = \sum_{v \in F_0} e_v$, then the category $\mathrm{GP}(B)$, which is Frobenius by Theorems 1 and 2, is stably 2-Calabi–Yau.*

By the results of Keller–Reiten [KR07, §4] referred to above, Geiß–Leclerc–Schröer's constructions provide a large supply of finite dimensional internally 3-Calabi–Yau algebras.

Theorem 3 (Proposition 5.5)**.** *Let $T \in \operatorname{Sub} Q_J$ be cluster-tilting. Then $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ is internally 3-Calabi–Yau with respect to the idempotent given by projection onto the projective-injective summands of T.*

In the situation of Theorem 3, since $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ is finite dimensional, $\operatorname{End}_\Pi(T)$ is also internally 3-Calabi–Yau. Thus we can apply Theorem 1 to find that $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ is isomorphic to $\operatorname{End}_B(T')^{\mathrm{op}}$ for some cluster-tilting object $T' \in \mathrm{GP}(B)$, where $B = \operatorname{End}_\Pi(\bigoplus_{i \in \Delta_0} F_i)^{\mathrm{op}}$ is the endomorphism algebra of the projective-injective generator-cogenerator of $\operatorname{Sub} Q_J$. Since $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ has global dimension at most 3 ([GLS10, Thm. 13.6], see also Proposition 4.21), this is already known via a result of Iyama–Kalck–Wemyss–Yang [IKWY15, Thm. 2.7], but Theorem 1 provides a new proof. In fact, [IKWY15, Thm. 2.7] shows that there is even an equivalence $\mathrm{GP}(B) \simeq \operatorname{Sub} Q_J$, and so $\mathrm{GP}(B)$ is stably 2-Calabi–Yau in this case.

Our ultimate aim is to produce Frobenius categorifications of cluster algebras with frozen variables, and we give some partial results in the case of Geiß–Leclerc–Schröer's cluster algebras $\widetilde{\mathscr{A}_J}$. In order to apply Theorem 1 to produce a categorification, we must find an internally 3-Calabi–Yau algebra A such that the quiver of A agrees, up to arrows between the frozen vertices, with the quiver of a cluster of $\widetilde{\mathscr{A}_J}$. Thus our first task is to add arrows between the frozen vertices, which are not provided by Geiß–Leclerc–Schröer's construction, in a sensible way.

Given a cluster-tilting object $T \in \operatorname{Sub} Q_J$, we can produce a quiver $\widetilde{\Gamma}_T$ as follows. Let Γ_T be the quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$, so that vertices of Γ_T correspond to the summands of T via $v \mapsto T_v$. Add a vertex j^* for each $j \in J$. Denoting the simple module at j by S_j, and the Auslander–Reiten translation on $\operatorname{Sub} Q_J$ by τ_J, we add arrows in the following situations.

- Take a minimal left $\operatorname{add} T$-approximation L of S_j, and add an arrow $v \to j^*$ for each summand of L isomorphic to T_v.
- Take a minimal right $\operatorname{add} T$-approximation R of $\tau_J S_j$, and add an arrow $j^* \to v$ for each summand of R isomorphic to T_v.
- Add a number of arrows $k^* \to j^*$ equal to the dimension of the space of maps $S_j \to \tau_J S_k$ modulo those factoring through T.

We call the resulting quiver $\widetilde{\Gamma}_T$, and have the following theorem.

Theorem 4 (Corollary 6.12)**.** *For each cluster-tilting object $T \in \operatorname{Sub} Q_J$ such that $\widetilde{\Gamma}_T$ has no 2-cycles through its mutable vertices, ignoring the arrows in $\widetilde{\Gamma}_T$ between frozen vertices recovers the quiver of the cluster of $\widetilde{\mathscr{A}_J}$ corresponding to the cluster-tilting object T.*

The only way that $\widetilde{\Gamma}_T$ could have 2-cycles is if some summand of T is a summand of both a minimal left $\operatorname{add} T$-approximation of S_j and a minimal right $\operatorname{add} T$-approximation of $\tau_J S_j$. We conjecture (Conjecture 6.9) that this never happens, and so Theorem 4 holds for all T.

Let the frozen subquiver $\widetilde{F}$ of $\widetilde{\Gamma}_T$ be the full subquiver on the new vertices j^* and the vertices of Γ_T corresponding to projective-injective summands of T. Recall that if $T \in \mathscr{R}_J$, then Buan–Iyama–Reiten–Smith show that there is an isomorphism

$$\mathcal{J}(\underline{Q}, \underline{W}) \xrightarrow{\sim} \operatorname{End}_{\underline{\operatorname{Sub}} Q_J}(T)^{\mathrm{op}}$$

of the stable endomorphism algebra of T with a Jacobian algebra. We fix such an isomorphism Φ, and use it, along with the homological algebra of the stable category $\underline{\operatorname{Sub}}\, Q_J$, to define a potential $\widetilde{W}$ on $\widetilde{\Gamma}_T$, and thus a frozen Jacobian algebra

$$\widetilde{A} = \mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W}).$$

We then make the following conjecture.

Conjecture 2 (Conjectures 6.18 and 6.19)**.** *There is an isomorphism*

$$\widetilde{A}/\langle e_{j^*} : j \in J\rangle \xrightarrow{\sim} \operatorname{End}_{\Pi}(T)^{\mathrm{op}}.$$

Moreover, $\mu_0 \colon \mathbf{P}(\widetilde{A}) \to \widetilde{A}$ is a quasi-isomorphism, so $\widetilde{A}$ is internally bimodule 3-Calabi–Yau with respect to the idempotent $e = \sum_{v \in \widetilde{F}_0} e_v$ by Theorem 2.

Write $\widetilde{B} = e\widetilde{A}e$. If Conjectures 1 and 2 hold, then the category $\operatorname{GP}(\widetilde{B})$ has the following properties.

(i) By Theorem 1, $\operatorname{GP}(\widetilde{B})$ is a Frobenius category admitting a cluster-tilting object $e\widetilde{A}$ such that $\operatorname{End}_{\widetilde{B}}(e\widetilde{A})^{\mathrm{op}} \cong \widetilde{A}$. If $\widetilde{\Gamma}_T$ has no 2-cycles, then by Theorem 4 the quiver $\Gamma_{e\widetilde{A}} = \widetilde{\Gamma}_T$ coincides with the quiver of a seed of $\widetilde{\mathscr{A}_J}$, up to arrows between frozen variables.

(ii) The stable category $\underline{\mathrm{GP}}(\widetilde{B})$ is 2-Calabi–Yau, so its cluster-tilting objects have the mutation property [IY08]. It follows that cluster-tilting objects of $\mathrm{GP}(\widetilde{B})$ also have the mutation property at non-projective-injective indecomposable summands.

(iii) The mutation of cluster-tilting objects of $\underline{\mathrm{GP}}(\widetilde{B})$ induces Fomin–Zelevinsky mutations of their Gabriel quivers [BIRS11, §5].

These properties would allow us to view $\mathrm{GP}(\widetilde{B})$ as a *Frobenius categorification* of $\widetilde{\mathscr{A}_J}$. In particular, an isomorphism of quivers as in (i) allows us to associate to each indecomposable summand of $e\widetilde{A}$ a cluster variable of $\widetilde{\mathscr{A}_J}$ (which is frozen if the summand is projective-injective). Then the Fu–Keller cluster character introduced in [FK10, §3] provides a map from objects of $\mathrm{GP}(\widetilde{B})$ to $\widetilde{\mathscr{A}_J}$, inducing a map from cluster-tilting objects of $\mathrm{GP}(\widetilde{B})$ (reachable from $e\widetilde{A}$ via a finite sequence of mutations) to clusters of $\widetilde{\mathscr{A}_J}$. We will recall the definition of Fu–Keller's cluster character in Section 2.7.

1.3 Notation, Terminology and Conventions

We introduce some notation and standard terminology that will be used throughout. All modules are left modules unless otherwise indicated, and compositions of maps and arrows are taken from right to left. All categories and functors are assumed to be $\mathbb{C}$-linear. Given a $\mathbb{C}$-algebra A, write $\operatorname{mod} A$ for the category of finitely generated A-modules. If V is a vector space, we will denote the dual space $\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ by either $\mathrm{D}V$ or $V^{\vee}$, depending on the context. If M is an A-module, then $\mathrm{D}M$ is an A^{op}-module.

Given objects A, B in some category $\mathcal{C}$, we write

$$\hom_{\mathcal{C}}(A,B) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{C}}(A,B),$$
$$\operatorname{ext}^i_{\mathcal{C}}(A,B) = \dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathcal{C}}(A,B).$$

For $M \in \mathcal{C}$, we denote by $\operatorname{add} M$ the full subcategory of $\mathcal{C}$ whose objects are isomorphic to direct sums of summands of M, and by $\operatorname{Sub} M$ the full subcategory of $\mathcal{C}$ whose objects are isomorphic to subobjects of direct sums of copies of M. The categories $\operatorname{proj} \mathcal{C}$ and $\operatorname{inj} \mathcal{C}$ are the full subcategories of $\mathcal{C}$ whose objects are projective and injective respectively. A category $\mathcal{C}$ is said to be *Frobenius* if it is exact (in the sense of Quillen [Qui73], see also [Büh10, Defn. 2.1]), has enough projectives and injectives, and $\operatorname{proj} \mathcal{C} = \operatorname{inj} \mathcal{C}$.

A triangulated category $\mathcal{T}$ with shift functor $[1]$ is said to be *d-Calabi–Yau* if it is Hom-finite and $[d]$ is a Serre functor, meaning that for any objects $X, Y \in \mathcal{T}$, there is an isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(X,Y) \cong \mathrm{D} \operatorname{Hom}_{\mathcal{T}}(Y, X[d]),$$

functorial in X and Y.

The stable category $\underline{\mathcal{E}}$ of a Frobenius category $\mathcal{E}$ has the same objects as $\mathcal{E}$, but has morphisms

$$\mathrm{Hom}_{\underline{\mathcal{E}}}(M,N) := \underline{\mathrm{Hom}}_{\mathcal{E}}(M,N) = \mathrm{Hom}_{\mathcal{E}}(M,N)/\,\mathrm{Hom}^{\mathrm{p}}_{\mathcal{E}}(M,N),$$

where $\mathrm{Hom}^{\mathrm{p}}_{\mathcal{E}}(M,N) \leq \mathrm{Hom}_{\mathcal{E}}(M,N)$ is the subspace of morphisms $f\colon M \to N$ factoring through $\operatorname{proj}\mathcal{E} = \operatorname{inj}\mathcal{E}$. All projective-injective objects of $\mathcal{E}$ are isomorphic to the zero object in $\underline{\mathcal{E}}$. The stable category of a Frobenius category is triangulated [Hap88, I.2], with shift functor given by the inverse syzygy functor Ω^{-1}, taking an object to the cokernel of an injective hull. A Frobenius category $\mathcal{E}$ is said to be *stably d-Calabi–Yau* if the triangulated category $\underline{\mathcal{E}}$ is d-Calabi–Yau.

The derived category of an algebra A is denoted by $\mathcal{D}A$, and the bounded derived category of A by $\mathcal{D}^bA$. The category $\operatorname{per}A$ is the full subcategory of $\mathcal{D}^bA$ consisting of perfect complexes, i.e. those complexes isomorphic in $\mathcal{D}^bA$ to a complex of projective A-modules. These three categories are all triangulated, and we will denote the shift functor by [1] in each case.

A quiver is a quadruple $Q = (Q_0, Q_1, h, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $h, t\colon Q_1 \to Q_0$ determine the heads and tails of arrows respectively. Let $S = \mathbb{C}Q_0$ and view $\mathbb{C}Q_1$ as an S-bimodule by defining

$$e_{h\alpha}\alpha e_{t\alpha} = \alpha.$$

We then define the (complete) *path algebra* of Q to be the direct product

$$\mathbb{C}Q = \prod_{d=0}^{\infty} (\mathbb{C}Q_1)^{\otimes_S d},$$

with multiplication given by tensor product. The radical of $\mathbb{C}Q$ is

$$\mathfrak{m}(\mathbb{C}Q) = \prod_{d=1}^{\infty} (\mathbb{C}Q_1)^{\otimes_S d}.$$

Note that $\mathbb{C}Q/\mathfrak{m}(\mathbb{C}Q) = S$. Moreover, $\mathbb{C}Q$ is a topological algebra with a basic system of open neighbourhoods of zero given by the powers $\mathfrak{m}(\mathbb{C}Q)^k$. The closure of any subset $U \subseteq \mathbb{C}Q$ is

$$\overline{U} = \bigcap_{k=0}^{\infty} U + \mathfrak{m}(\mathbb{C}Q)^k.$$

If $Z \subseteq A$ is a set of elements of A, we denote by $\langle Z\rangle$ the two-sided ideal generated by Z.

We take the direct product rather than the direct sum in the definition of $\mathbb{C}Q$ so that if $A \cong \mathbb{C}Q/I$ for some finite quiver Q and a closed ideal $I \subseteq \mathfrak{m}(\mathbb{C}Q)^2$, then the category $\operatorname{mod}A$ is Krull–Schmidt. We call such ideals are called *admissible*; this is weaker than the definition of admissible in [ASS06, Defn. 2.1], since we often wish to consider infinite dimensional algebras. Moreover, if Z is a minimal set of elements of $\mathbb{C}Q$ such that each is a linear combination of paths with common start and end points and $I = \overline{\langle Z \rangle}$, then the number of elements of Z starting at vertex i and ending at vertex j is given by $\mathrm{ext}^2_A(S_i, S_j)$; see [BIRS11, Prop. 3.4]. If we set $\mathfrak{m}(A) = \mathfrak{m}(\mathbb{C}Q)/I$, then $A/\mathfrak{m}(A) = S$.

For $v \in Q_0$, let e_v be the corresponding basis element in $S = \mathbb{C}Q_0$. The elements e_v for $v \in Q_0$ form a maximal set of pairwise orthogonal primitive idempotents of $\mathbb{C}Q$, and thus of $\mathbb{C}Q/I$ for any admissible ideal I.

The *double quiver* $\overline{Q}$ of Q has $\overline{Q}_0 = Q_0$, and has arrows $\alpha \colon i \to j$ and $\alpha^\vee \colon j \to i$ for each $\alpha \colon i \to j$ in Q_1. The *preprojective algebra* $\Pi(Q)$ of Q is the quotient of $\mathbb{C}\overline{Q}$ by the two-sided ideal generated by

$$\sum_{\alpha \in Q_1} [\alpha, \alpha^\vee].$$

Up to isomorphism, $\Pi(Q)$ depends only on the underlying graph of Q, and so we will sometimes write $\Pi(\Delta) = \Pi(Q)$ when Δ is a graph and Q is any orientation of Δ.

When $\mathcal{C}$ is Krull–Schmidt, we say $M \in \mathcal{C}$ is basic if in any decomposition $M = \bigoplus_{i=1}^N M_i$ of M into indecomposable summands, $M_i \ncong M_j$ for $i \neq j$. In this case, $\mathrm{End}_{\mathcal{C}}(M)^{\mathrm{op}}$ is a basic algebra, meaning it is a basic object of its module category, and we denote its Gabriel quiver [ASS06, §II.3] by Γ_M. By construction, this means there is a (non-canonical) isomorphism

$$\mathbb{C}\Gamma_M/I \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}}(M)^{\mathrm{op}}$$

for some admissible closed ideal I. From Chapter 4 onwards, all complete rigid, maximal rigid or cluster-tilting objects (Definition 4.4) will be assumed without loss of generality to be basic.

CHAPTER 2

CLUSTER ALGEBRAS AND THEIR CATEGORIFICATION

In this chapter, we describe some of the background and motivation for our work in more detail.

2.1 Cluster Algebras

The study of cluster algebras was initiated by Fomin and Zelevinsky in [FZ02a], and further developed in [FZ03, BFZ05, FZ07]. For our purposes it will be sufficient to consider cluster algebras of geometric type associated to skew-symmetric matrices; in this section, our cluster algebras have no frozen variables, but we will introduce these later. As a general reference at this level of generality, we recommend Keller's survey article [Kel10].

Let $\mathbb{F} \cong \mathbb{C}(x_1, \dots, x_n)$ be any purely transcendental field extension of $\mathbb{C}$ of transcendence degree n.

Definition 2.1. A *seed* is a pair (X, B), where

(i) $X = \{v_1, \dots, v_n\}$ is a free generating set for $\mathbb{F}$ over $\mathbb{C}$, indexed by $1, \dots, n$, and

(ii) $B = (b_{ij})$ is a skew-symmetric $(n \times n)$-matrix, with rows and columns indexed by $1, \dots, n$.

The set X is called a *cluster*, and B is its *exchange matrix.* Seeds are considered equivalent if they are related by a permutation of $\{1, \dots, n\}$, the set indexing both the cluster and the rows and columns of the exchange matrix.

Given a seed (X, B) and an index $1 \leq k \leq n$, we define the mutation $\mu_k(X, B) = (X', B')$ as follows. We take $X' = \{v_1', \dots, v_n'\}$, where $v_i' = v_i$ for $i \neq k$, and v_k' is defined via the *exchange relation*

$$v_k v_k' = \prod_{i=1}^{n} v_i^{[b_{ik}]_+} + \prod_{j=1}^{n} v_j^{[b_{kj}]_+},$$

where $[b_{ij}]_+ = \max\{b_{ij}, 0\}$. We define the matrix B' to have entries b'_{ij} satisfying

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(|b_{ij}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise.} \end{cases}$$

We sometimes write $\mu_k(X)$ and $\mu_k(B)$ for the elements X' and B' of the seed $\mu_k(X, B)$. The mutation operation has the following key properties.

(i) The set $\mu_k(X)$ is a free generating set of $\mathbb{F}$, and $\mu_k(B)$ is skew-symmetric, so that $\mu_k(X, B)$ is again a seed.

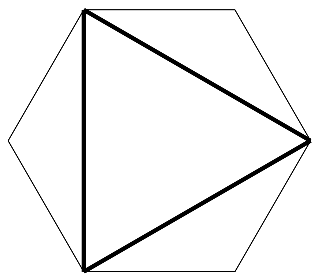
(ii) For any $1 \leq k \leq n$ and any seed (X, B), we have $\mu_k^2(X, B) = (X, B)$.

As B is skew-symmetric, it can be visualised as a quiver Q without loops or 2-cycles, where $Q_0 = \{1, \dots, n\}$ and there are $[b_{ij}]_+$ arrows from $i \to j$. Then we can rewrite the exchange relation as

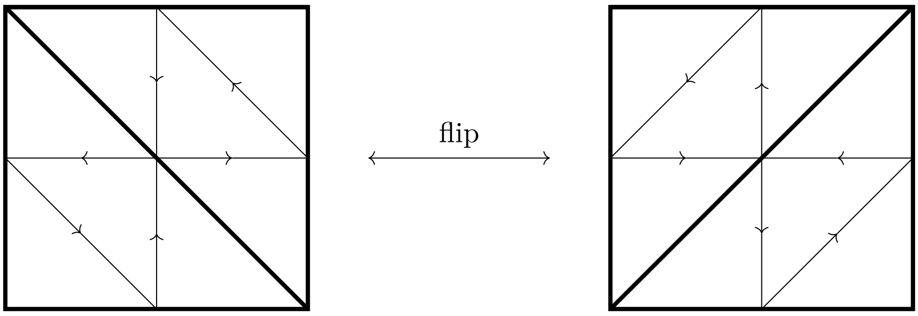
$$v_k v'_k = \prod_{i \to k} v_i + \prod_{k \to j} v_j$$

with the products taken over arrows. The quiver corresponding to $\mu_k(B)$ is obtained from that corresponding to B by adding a composite arrow $i \to j$ for every length 2 path $i \to k \to j$ through k, deleting a maximal collection of 2-cycles, and then reversing all arrows incident to k. This operation on quivers is called *Fomin–Zelevinsky mutation* at the vertex k. Since Q encodes the same data as B, we will sometimes write (X, Q) in place of (X, B).

Example 2.2. Consider a triangulation of the n-gon, by which we mean a maximal collection of pairwise non-crossing arcs, connecting non-adjacent vertices of the n-gon. An example of a triangulation of the hexagon is as follows.



Given such a triangulation, we can obtain a quiver with a vertex for each arc, and an arrow $\alpha \to \beta$ if the arcs α and β are consecutive sides of a triangle (with the sides of each triangle ordered anti-clockwise). Each arc in the triangulation is the diagonal of a quadrilateral, and we can observe the effect on the quiver of 'flipping' this diagonal to get a new triangulation.



We see that the flip causes a Fomin–Zelevinsky mutation of the associated quiver at the vertex corresponding to the flipped diagonal.

Definition 2.3. The *mutation class* of a seed (X, B) is the set of all seeds obtained from (X, B) by arbitrary sequences of mutations. Given a mutation class $\mathcal{S}$ of seeds, let $\mathcal{X}(\mathcal{S})$ be the union of all clusters X of seeds of $\mathcal{S}$, and let $\mathscr{A}(\mathcal{S})$ be the subalgebra of $\mathbb{F}$ generated by $\mathcal{X}(\mathcal{S})$. We call the elements of $\mathcal{X}(\mathcal{S})$ the *cluster variables* of the *cluster algebra* $\mathscr{A}(\mathcal{S})$. Given a seed (X, B), write $\mathscr{A}(X, B) = \mathscr{A}(\mathcal{S})$ for $\mathcal{S}$ the mutation class of (X, B).

Two cluster variables are called *compatible* if they occur in the same cluster, and a product of compatible cluster variables is called a *cluster monomial*. We say that seeds in the same mutation class are *mutation equivalent*, and make an analogous definition for clusters, matrices and quivers.

The cardinality n of any cluster is called the *rank* of $\mathscr{A}(\mathcal{S})$. The *type* of $\mathscr{A}(\mathcal{S})$ is the mutation class of the matrix B.

If $(X, Q) \in S$ is such that Q has a tree (such as a simply laced Dynkin diagram) as its underlying graph, then this tree determines the type of $\mathscr{A}(\mathcal{S})$, as all orientations of a tree are related by Fomin–Zelevinsky mutations at sources and sinks.

Remark 2.4. Under the definitions given above, the mutation operators μ_k define a left action of the group

$$G = \langle \mu_k | \mu_k^2 = 1 \rangle$$

on the set of seeds of rank n (but not on the set of equivalence classes of such under relabelling). This set also carries an action of the symmetric group $\mathfrak{S}_n$ by relabelling, and so of the semi-direct product

$$M_n = \mathfrak{S}_n \ltimes G,$$

where $\sigma\mu_k = \mu_{\sigma(k)}\sigma$. A mutation class is an orbit of the M_n-action on the set of seeds, and so is in particular an M_n-homogeneous space. This point of view can be useful, and it is described in more detail in [KP15].

Example 2.5. Let $\mathbb{F} = \mathbb{C}(x,y)$, and take $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so B corresponds to the quiver $1 \longrightarrow 2$, of type A_2. For the purpose of computing a cluster algebra, we need only consider seeds up to simultaneous relabelling of the cluster variables and the rows and columns of the matrix (in this case interpreted as vertices of the quiver). Thus we can describe a seed more compactly by writing the cluster variables at vertices of the quiver. So taking $X = \{v_1, v_2\}$ with $v_1 = x$ and $v_2 = y$, the seed (X, B) can be drawn as $x \longrightarrow y$. As mutation at a given index is an involution, any seed that can be obtained from (X, B) by a sequence of mutations can be obtained via a sequence that alternates between mutation at 1 and at 2. We compute the following seeds.

$$
\begin{aligned}
(X,B) &= (x \longrightarrow y) \\
\mu_1(X,B) &= \left(\frac{1+y}{x} \longleftarrow y\right) \\
\mu_2\mu_1(X,B) &= \left(\frac{1+y}{x} \longrightarrow \frac{1+x+y}{xy}\right) \\
\mu_1\mu_2\mu_1(X,B) &= \left(\frac{1+x}{y} \longleftarrow \frac{1+x+y}{xy}\right) \\
\mu_2\mu_1\mu_2\mu_1(X,B) &= \left(\frac{1+x}{y} \longrightarrow x\right) \\
\mu_1\mu_2\mu_1\mu_2\mu_1(X,B) &= (y \longleftarrow x)
\end{aligned}
$$

The first and last seeds in the sequence agree, up to swapping the indices 1 and 2, so in this case there are only finitely many seeds mutation equivalent to (X, B), and the cluster algebra $\mathscr{A}(X, B)$ is the subalgebra of $\mathbb{F}$ generated by the five rational functions x, y, $\frac{1+x}{y}$, $\frac{1+y}{x}$ and $\frac{1+x+y}{xy}$. The rank of $\mathscr{A}(X, B)$ is 2, and it has type A_2.

We observe several phenomena in Example 2.5, some of which generalise. Firstly, all of the cluster variables are Laurent polynomials in the initial cluster. This is a special case of the following general result, due to Fomin and Zelevinsky.

Theorem 2.6 (Laurent Phenomenon, [FZ02a, Thm. 3.1])**.** *In any cluster algebra, all cluster variables are expressible as Laurent polynomials in any cluster.*

The phenomenon suggests an alternative definition of the cluster algebra as the intersection of the rings of Laurent polynomials in each cluster. In general, this is called the *upper cluster algebra* and denoted by $\overline{\mathscr{A}}(\mathcal{S})$, and while the Laurent phenomenon implies that $\mathscr{A}(\mathcal{S}) \subseteq \overline{\mathscr{A}}(\mathcal{S})$, this inclusion is strict in general. An example in which the upper cluster algebra is strictly larger than the cluster algebra is given by Berenstein–Fomin–Zelevinsky [BFZ05, Prop. 1.26].

The Laurent phenomenon has many guises and applications, and provides an explanation for the integrality of a number of classical recurrence relations. Further discussion of this topic can be found in Fomin–Zelevinsky [FZ02b]. Cluster algebras are a subclass of the larger family of *Laurent phenomenon algebras*, described by Lam–Pylyavskyy

[LP12]. These algebras have a similar definition to cluster algebras, but allow for more general exchange relations while still exhibiting the Laurent phenomenon.

We also observe in Example 2.5 that all of the coefficients in the numerators of the Laurent polynomials are positive. Fomin–Zelevinsky conjecture [FZ02a, §3] that this is true for the Laurent expression of any cluster variable in terms of any cluster. At our level of generality, in which the matrix B is skew-symmetric, this conjecture has been proved by Lee–Schiffler [LS15].

In Example 2.5, there are only finitely many cluster variables. This is false in general, but the cases in which it is true are classified in [FZ03]. In our context, $\mathscr{A}(\mathcal{S})$ has finitely many cluster variables, or has *finite type*, if and only if its type is a simply laced Dynkin diagram, i.e. one of type A_n, D_n, E_6, E_7 or E_8. In each type, the cluster variables are in bijection with the *almost positive roots* of the root system of the same type, which are the positive roots together with the negative simple roots. Finite type cluster algebras corresponding to the other Dynkin diagrams can be obtained by allowing B to be skew-symmetrisable rather than skew-symmetric.

Finally, we observe that the mutation class of the quiver $1 \longrightarrow 2$ is finite. In general, there will be infinitely many isomorphism classes of quivers in a mutation class. We have the following classification of acyclic quivers with finite mutation classes.

Theorem 2.7 ([Kel10, Thm. 5.5])**.** *If Q is a connected quiver without oriented cycles, then the mutation class of Q is finite if and only if it contains a quiver of simply laced Dynkin type, or a quiver of simply laced affine type, or Q has two vertices.*

Moreover, if $\nu(Q)$ is the supremum of the number of arrows between any two vertices in quivers mutation equivalent to Q, then $\nu(Q) = 1$ if and only if Q is of simply laced Dynkin type, and $\nu(Q) = 2$ if and only if Q is of simply laced affine type. If Q has more than 2 vertices, then $\nu(Q) \geq 3$ if and only if $\nu(Q) = \infty$, if and only if Q is of neither simply laced Dynkin or simply laced affine type.

We may define some combinatorial objects to capture some of the structure of a cluster algebra. Firstly, we define the *exchange graph* of a cluster algebra to have a vertex for each cluster, with two clusters joined by an edge if they are related by a single mutation. This graph is n-valent by Gekhtman–Shapiro–Vainshtein's 'cluster determines seed' result [GSV08, Thm. 3], stating that each cluster occurs in a unique seed (up to relabelling).

In the cluster algebra of type A_2 (Example 2.5), the exchange graph is a pentagon. Because the indices may be permuted to identify seeds, we may not always consistently label each edge in the exchange graph according to the index of the mutation. Indeed, the exchange graph in type A_2 already displays this phenomenon. For cluster algebras of type A_n, the exchange graph coincides with the 1-skeleton of the n-th Stasheff associahedron [FZ03, §3].

Alternatively, we can define a dual object, called the *cluster complex*, with vertices given by cluster variables, and a simplex for each collection of compatible cluster variables. The top dimensional simplices correspond to clusters. Now the statement that clusters determine seeds implies that the cluster complex has a manifold property; any $(n-1)$-dimensional simplex is contained in exactly two n-dimensional simplices.

2.2 Cluster Categories

Since the combinatorics involved in cluster algebras can be complex, it is useful to find categorical models. In the case that $\mathscr{A}$ is a cluster algebra possessing a seed (X,Q) with Q acyclic, such models are given by Buan–Marsh–Reineke–Reiten–Todorov [BMR$^+$06]. For each acyclic quiver Q, they define the *cluster category* $\mathcal{C}_Q$ to be the orbit category

$$\mathcal{C}_Q = \mathcal{D}^b(\mathbb{C}Q)/\tau[-1]$$

where τ is the Auslander–Reiten translation on $\mathcal{D}^b(\mathbb{C}Q)$. This category is triangulated by a result of Keller [Kel05], and is 2-Calabi–Yau by construction. By a result of Caldero–Keller [CK06, Thm. 4], the indecomposable rigid objects (those indecomposable objects M satisfying $\operatorname{Ext}^1_{\mathcal{C}_Q}(M,M)=0$) are in one-to-one correspondence with the cluster variables of $\mathscr{A}$ via the Caldero–Chapoton *cluster character* $M \mapsto \varphi_M$ introduced in [CC06]. Moreover, the cluster-tilting objects of $\mathcal{C}_Q$, which are objects T such that

$$\operatorname{add} T = \{X \in \mathcal{C}_Q : \operatorname{Ext}^1_{\mathcal{C}_Q}(X,T)=0\} = \{X \in \mathcal{C}_Q : \operatorname{Ext}^1_{\mathcal{C}_Q}(T,X)=0\},$$

are in one-to-one correspondence with the clusters of $\mathscr{A}$. For $T \in \mathcal{C}_Q$, the second equality above is implied by the first, since $\mathcal{C}_Q$ is 2-Calabi–Yau.

The results of [BMR$^+$06, §6] show that if $T \in \mathcal{C}_Q$ is cluster-tilting, and M is an indecomposable summand of T, then there exists a unique $M' \ncong M$ such that $T/M \oplus M'$ is cluster-tilting. Moreover, any minimal left $\operatorname{add} T/M$-approximation $f\colon M \to X$ (see Definition 4.2) gives rise to an *exchange triangle*

$$M \xrightarrow{f} X \longrightarrow M' \longrightarrow M[1]$$

in $\mathcal{C}_Q$. Dually, any minimal right $\operatorname{add} T/M$-approximation $g\colon Y \to M$ gives rise to a triangle

$$M[-1] \longrightarrow M' \longrightarrow Y \xrightarrow{g} M.$$

The process of replacing M by M' is called mutation at M. Iyama–Yoshino [IY08] show that this mutation operation is well-defined, with exchange triangles given in the same way, for a cluster-tilting object T in any 2-Calabi–Yau triangulated category $\mathcal{C}$ such that the quiver of $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ has no loops.

Write $T' = T/M \oplus M'$ as above. Then it is shown by Buan–Marsh–Reiten [BMR08, Thm. 6.1] that the quivers of the algebras $\operatorname{End}_{\mathcal{C}_Q}(T)^{\mathrm{op}}$ and $\operatorname{End}_{\mathcal{C}_Q}(T')^{\mathrm{op}}$ are related by a Fomin–Zelevinsky mutation at the vertex corresponding to M. Since $\mathbb{C}Q$ is itself a cluster-tilting object of $\mathcal{C}_Q$, and the quiver of $\mathrm{End}_{\mathcal{C}_Q}(\mathbb{C}Q)^{\mathrm{op}}$ is Q, it follows that the bijection between cluster variables of $\mathscr{A}$ and indecomposable rigid objects of $\mathcal{C}_Q$ induces an isomorphism between the quiver of a cluster and the quiver of $\mathrm{End}_{\mathcal{C}_Q}(T)^{\mathrm{op}}$ for T the corresponding cluster-tilting object.

Many of these results have been extended by Amiot [Ami09] to cluster algebras $\mathscr{A}$ for which no seed has an acyclic quiver. For any quiver with potential (Q, W) such that the Jacobian algebra

$$\mathcal{J}(Q, W) = \mathbb{C}Q/\langle \partial_\alpha W : \alpha \in Q_1 \rangle$$

is finite dimensional, Amiot defines [Ami09, Defn. 3.5] a cluster category

$$\mathcal{C}_{(Q,W)} = \operatorname{per} \Gamma(Q, W)/\mathcal{D}^b\Gamma(Q, W).$$

Here $\Gamma(Q, W)$ denotes the Ginzburg dg-algebra of (Q, W), as defined in [Gin06, §4.2], see also [Ami09, §3.1].

If Q is an acyclic quiver and $T \in \mathcal{C}_Q$ is cluster-tilting, then Buan–Iyama–Reiten–Smith [BIRS11, Cor. 6.8], see also [Kel11, Thm. 6.12], show that

$$\mathrm{End}_{\mathcal{C}_Q}(T)^{\mathrm{op}} \cong \mathcal{J}(Q', W)$$

for some quiver with potential (Q', W). In this case we have

$$\mathcal{C}_Q \simeq \mathcal{C}_{(Q',W)}$$

by [Ami09, Cor. 3.12].

For a more detailed survey of the theory of cluster categories and their cluster-tilting objects, we recommend Reiten [Rei10].

2.3 Frozen Variables

We sometimes wish to have some variables that cannot be mutated, and thus occur in every cluster. To achieve this, we define cluster algebras with frozen variables as subalgebras of a $\mathbb{C}$-algebra $\mathbb{F}_C \cong \mathbb{C}[v_{n+1}, \dots, v_{n+m}](x_1, \dots, x_n)$ isomorphic to the algebra of rational functions in $\{x_1, \dots, x_n\}$ with coefficients given by polynomials in $C = \{v_{n+1}, \dots, v_{n+m}\}$. Elements of C are called *frozen variables*.

In this more general setting, a seed is a triple (X, C, B), where X is a free generating set of $\mathbb{F}$ over $\mathbb{C}[v_{n+1}, \dots, v_{n+m}]$, the set C is the fixed set of frozen variables, and B is an $(n+m) \times n$ matrix with rows indexed by $\{1, \dots, n+m\}$ and columns by $\{1, \dots, n\}$, such that the submatrix B° consisting of the rows labelled by $1, \dots, n$ is skew-symmetric. The submatrix B° is called the *principal part* of B.

We can still visualize such a B as a quiver; the quiver of B is by definition the quiver of the skew-symmetric matrix

$$\begin{pmatrix} B^\circ & B^{\bullet\mathsf{T}} \\ B^\bullet & 0 \end{pmatrix}$$

where $B^\bullet$ is the submatrix of B consisting of the last m rows. There are no arrows between the vertices $n+1,\dots,n+m$, which are called *frozen*.

We can define the mutation $\mu_k(B)$ as before, although only allow mutation at indices $1 \leq k \leq n$. This again corresponds to Fomin–Zelevinsky mutation of the quiver of B, but with the modification that arrows between frozen vertices, which may be produced by the mutation algorithm, are ignored. In the language of matrices, this corresponds to the fact that

$$\mu_k \begin{pmatrix} B^\circ & B^{\bullet\mathsf{T}} \\ B^\bullet & 0 \end{pmatrix} \neq \begin{pmatrix} \mu_k(B)^\circ & \mu_k(B)^{\bullet\mathsf{T}} \\ \mu_k(B)^\bullet & 0 \end{pmatrix}$$

but the only differences occur in the lower-right block, which may be non-zero on the left-hand side.

The exchange relations become

$$v_k v_k' = \prod_{i=1}^{n+m} v_i^{[b_{ik}]_+} + \prod_{j=1}^{n+m} v_j^{[b_{kj}]_+},$$

or in terms of quivers

$$v_k v_k' = \prod_{i\to k} v_i + \prod_{k\to j} v_j$$

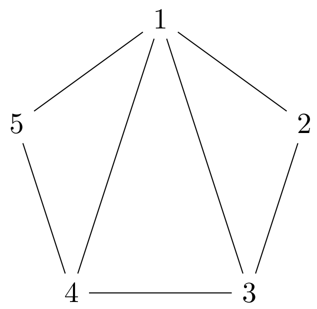
as before. Since we do not allow mutation at frozen vertices, any arrows between such vertices would not contribute to the exchange relations, which is why we omit them at this stage.

Thus for any $1 \leq k \leq n$ and any seed (X, C, B), we have definitions of $\mu_k(X)$ and $\mu_k(B)$, and so we define mutation of a seed at k by $\mu_k(X, C, B) = (\mu_k(X), C, \mu_k(B))$. Then the cluster algebra $\mathscr{A}(X, C, B)$ associated to a seed (X, C, B) is the subalgebra of $\mathbb{F}_C$ generated by C and all elements of clusters occurring in seeds mutation equivalent to (X, C, B).

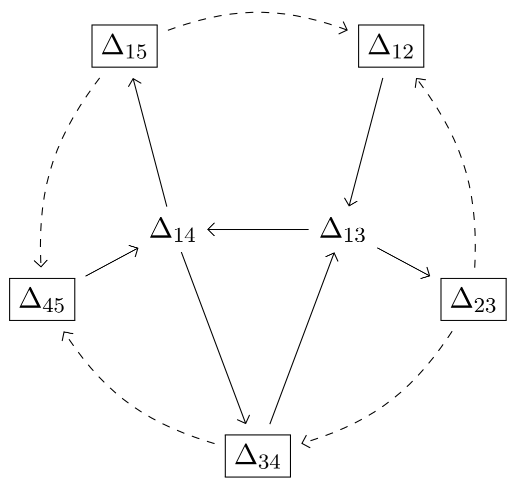
The rank and type of the cluster algebra $\mathscr{A}(X, C, B)$ are taken to be the same as that of the cluster algebra $\mathscr{A}(X, B^\circ)$.

Example 2.8 (Grassmannians, $k = 2$)**.** There is a cluster algebra structure on the homogeneous coordinate ring $\mathbb{C}[G_2^n]$ of the Grassmannian G_2^n of planes in $\mathbb{C}^n$, as described in [FZ02a]. This cluster algebra structure has frozen variables, given by the Plücker coordinates $\Delta_{i,i+1}$ (with indices read cyclically, so Δ_{1n} is also frozen). A triangulation T of an n-gon, with vertices labelled $1,\dots,n$, determines a quiver Q by modifying the construction of Example 2.2. The modification is by also taking (frozen) vertices for the edges of the n-gon, and then drawing arrows as before. We obtain a seed (X, C, Q), by taking X to be the set of Plücker coordinates Δ_{ij} such that there is an internal arc between i and j in the triangulation T, and taking C to be the Plücker coordinates corresponding to sides of the n-gon, i.e. the frozen variables $\Delta_{i,i+1}$ for $1 \leq i \leq n$. Then we have $\mathscr{A}(X, C, Q) = \mathbb{C}[G_2^n]$, the cluster and frozen variables of $\mathcal{A}$ are precisely the Plücker coordinates, and the exchange relations are the short Plücker relations.

For example, we can obtain a cluster of $\mathbb{C}[G_2^5]$ from the following triangulation of the pentagon.



The frozen variables are Δ_{12}, Δ_{23}, Δ_{34}, Δ_{45} and Δ_{15}, as always, corresponding to the sides of the 5-gon. The two non-frozen variables in this cluster are Δ_{13} and Δ_{14}. Constructing the quiver of this triangulation, and labelling vertices by the corresponding cluster or frozen variables, we obtain



where the boxed vertices/variables are frozen. The dashed arrows, between the frozen vertices, are suggested by the combinatorics of the triangulation, but play no immediate role in the cluster algebra structure (although cf. [BKM14, §3]). If we, for example, mutate at the vertex labelled by Δ_{13}, then we produce a new variable x satisfying

$$\Delta_{13}x = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23},$$

so $x = \Delta_{24}$ and the above exchange relation is a short Plücker relation. This can be seen via the combinatorics of the triangulation; the diagonal between 2 and 4 is obtained by flipping that between 1 and 3, as in Example 2.2.

This construction has been generalised by Scott [Sco06] to define a cluster algebra structure on the homogeneous coordinate ring $\mathbb{C}[G^n_k]$ of the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. The frozen variables are always given by the Plücker coordinates indexed by k cyclically consecutive indices from $\{1,\dots,n\}$, and the other Plücker coordinates are always cluster variables. However, for $k > 3$ there are also other cluster variables. This cluster algebra has finite type only when $k = 1$ (in which case the clusters are empty, and the cluster algebra is just the polynomial ring $\mathbb{C}[\Delta_1,\dots,\Delta_n]$ in the frozen variables), when $k = 2$, and when $k = 3$ and $n \in \{6,7,8\}$.

2.4 Graded Cluster Algebras

Cluster algebras can also be equipped with gradings. For a fuller discussion of graded cluster algebras, see Grabowski [Gra15]. In our setting, with seeds given by cluster variables attached to the vertices of a quiver, the definition can be stated as follows.

Definition 2.9. Let $\mathscr{A}$ be a cluster algebra. We say that a seed (X, C, Q) of $\mathscr{A}$ is *balanced* with respect to a function $\deg\colon X \cup C \to \mathbb{Z}^d$ if

$$\sum_{i\to k} \deg(v_i) = \sum_{k\to j} \deg(v_j)$$

for all $1 \le k \le n$ (i.e. for all k such that v_k is not frozen).

Any function $\deg\colon X \cup C \to \mathbb{Z}^d$ extends to a grading of the ring of Laurent polynomials in $X \cup C$, which contains the cluster algebra $\mathscr{A}(X, C, Q)$. Thus deg determines a grading of $\mathscr{A}(X, C, Q)$ as an algebra, and the balancing of a seed provides a compatibility between the graded structure and the cluster structure.

Definition 2.10. If (X, C, Q) is balanced by the function $\deg\colon X \cup C \to \mathbb{Z}^d$, then $\mathscr{A}(X, C, Q)$ is said to be a *graded cluster algebra* with respect to the grading induced by deg.

Definition 2.10 is justified by the following observation of Jensen–King–Su.

Proposition 2.11 ([JKS14, Lem. 2.2])**.** *Let (X, C, Q) be balanced by $\deg\colon X \cup C \to \mathbb{Z}^d$. Then every cluster variable of $\mathscr{A}(X, C, Q)$ is homogeneous, and every seed of $\mathscr{A}(X, C, Q)$ balanced, with respect to the induced grading.*

We also observe that is possible to balance, or homogenise, an unbalanced seed.

Definition 2.12. Let (X, C, Q) be a seed and $\deg\colon X \cup C \to \mathbb{Z}^d$ a function. Define the *homogenisation* $(X, \widetilde{C}, \widetilde{Q})$ of (X, C, Q) as follows. The set $\widetilde{C}$ is obtained by adjoining d formal frozen variables $c^*_1, \dots, c^*_d$ to C. For each $1 \le k \le n$, let

$$\theta_k = \sum_{i\to k} \deg(v_i) - \sum_{k\to j} \deg(v_j) \in \mathbb{Z}^d.$$

Then define $\widetilde{Q}$ to be the quiver with vertex set $Q_0 \cup \{1^*, \ldots, d^*\}$ (where the vertex i^* corresponds to the frozen variable c_i^*) such that the full subquiver on Q_0 is given by Q, and each vertex i^* has $(\theta_k)_i$ arrows from k for each k such that $(\theta_k)_i > 0$, and $(\theta_k)_i$ arrows to k for each k such that $(\theta_k)_i < 0$.

In the situation of Definition 2.12, we may extend the definition of deg to the new frozen variables, by defining $\deg(c_i^*) = \varepsilon_i$ to be the i-th standard basis vector of $\mathbb{Z}^d$. Then, by construction, the homogenisation of a seed (X, C, Q) is balanced, and thus determines a graded cluster algebra in $\mathbb{F}_{\widetilde{C}}$. By [JKS14, Lem. 2.2], this graded cluster algebra depends only on the mutation class of (X, C, Q), and thus the procedure above can be interpreted as a homogenisation of the cluster algebra $\mathscr{A}(X, C, Q)$. The original cluster algebra can be recovered from such a homogenisation by setting the frozen variables c_i^* to 1.

2.5 Partial Flag Varieties

By work of Geiß–Leclerc–Schröer [GLS08], the definition of a cluster algebra structure on the Grassmannians of planes, as described in Example 2.8, can be generalised to obtain cluster algebras in multi-homogeneous coordinate rings of more general partial flag varieties, as we now explain. Let G be a simple connected algebraic group over $\mathbb{C}$ and let Δ be the Dynkin diagram of G. We assume that G is such that Δ is simply laced, and identify the vertex set I of Δ with the set of conjugacy classes of maximal parabolic subgroups of G. Thus any non-empty subset $J \subseteq I$ determines a conjugacy class of parabolic subgroups of G. Parabolic subgroups in this class are said to have *type* J. We write $K = I \setminus J$ for the complementary subset of I. We sometimes refer to the pair (Δ, J) as an *icon*.

Any homogeneous space X for G such that the stabiliser of some (and hence every) point in X is a parabolic subgroup of type J is canonically isomorphic as a G-space to the set F_J of parabolic subgroups of type J, by identifying each point with its stabiliser. We call F_J the *partial flag variety* of type J. The parabolic subgroups of type I are Borel subgroups, and F_I is called the *full flag variety*.

For consistency, we fix a vertex set I for each Dynkin diagram Δ as follows.

$$\mathsf{A}_n\colon \quad 1 - 2 - 3 - \cdots - n$$

$$\mathsf{D}_n\colon \quad \begin{array}{ccccccccc} 1 & - & 2 & - & \cdots & - & n-2 & - & n \\ & & & & & & | & & \\ & & & & & & n-1 & & \end{array}$$

$$\mathsf{E}_6\colon \quad \begin{array}{ccccccccc} 1 & - & 3 & - & 4 & - & 5 & - & 6 \\ & & & & | & & & & \\ & & & & 2 & & & & \end{array}$$

$$\mathsf{E}_7\colon \quad \begin{array}{ccccccccccc} 1 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 \\ & & & & | & & & & & & \\ & & & & 2 & & & & & & \end{array}$$

$$\mathsf{E}_8\colon \quad \begin{array}{ccccccccccccc} 1 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 & - & 8 \\ & & & & | & & & & & & & & \\ & & & & 2 & & & & & & & & \end{array}$$

This numbering agrees with the computer algebra software Sage, and thus with Bourbaki [Bou68]. The reader is warned that this this is not consistent with [GLS08] in type D; the conventions differ by swapping i with $n+1-i$.

Example 2.13. If $G = \mathrm{SL}(V)$ for some n-dimensional vector space V over $\mathbb{C}$, then Δ is of type A_n. Then if $J = \{i_1, \dots, i_r\}$, the space F_J is canonically isomorphic to the variety of flags

$$\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_r \subseteq V_{r+1} = V$$

of V, with $\dim V_j = i_j$, as the stabiliser of such a flag is a parabolic subgroup of $\mathrm{SL}(V)$ of type J. In particular, $F_{\{j\}}$ is the Grassmannian $G_j(V)$ of j-dimensional subspaces of V. We write $G_k^n = G_k(\mathbb{C}^n)$, as in Example 2.8.

Note that if P is any parabolic subgroup of type J, then G/P is canonically isomorphic to F_J, as P is its own stabiliser under the action of G on G/P by left multiplication. Thus we can apply well-known results about the space G/P to F_J; we use Tevelev's book [Tev05] as a general reference.

The canonical isomorphism $F_J \cong G/P$ gives F_J the structure of a projective variety (in the sense that it has the structure of an algebraic variety admitting a very ample line bundle; we will choose a specific projective embedding later). The choice of a maximal torus T and a Borel subgroup B with $T \subseteq B \subseteq G$ determines a cone of dominant weights in the weight lattice $T^\vee = \mathrm{Hom}(T, \mathbb{C}^\times)$, generated by a collection of *fundamental weights* ω_i each naturally associated to a vertex $i \in I$. Let $\mathrm{Pic}(F_J)$ be the lattice of (isomorphism classes of) line bundles on F_J under tensor product. Let $T_J^\vee$ be the sublattice of $T^\vee$ generated by ω_j for $j \in J$. Then, via the Borel–Weil theorem, there is a canonical isomorphism of $\mathrm{Pic}(F_J)$ with $T_J^\vee$ restricting to an isomorphism of the effective cone of $\mathrm{Pic}(F_J)$ with the cone of dominant weights of $T_J^\vee$. Specifically, if $\mathscr{L}$ is a line bundle on F_J, then (the universal cover of) G acts on the projective space $\mathbb{P}\Gamma(\mathscr{L})$, where $\Gamma(\mathscr{L})$ denotes the vector space of global sections of $\mathscr{L}$, as in [Ser95]. This action lifts to an action on $\Gamma(\mathscr{L})$, and so induces a representation of the Lie algebra $\mathfrak{g}$ of G on $\Gamma(\mathscr{L})$. This representation is irreducible and supported on $T_J^\vee$. The Borel–Weil isomorphism associates $\mathscr{L}$ to the highest weight of the representation of $\mathfrak{g}$ on $\Gamma(\mathscr{L})$. Let $\varepsilon_j \in \mathrm{Pic}(X)$ be a line bundle such that $\Gamma(\varepsilon_j) \cong L(\omega_j)$ is the line bundle associated to the fundamental weight ω_j of $T^\vee$.

We conclude, independent of the additional choices of T and B, that $\mathrm{Pic}(F_J) = \mathbb{Z}^J$ is a rank $|J|$ lattice, generated by *fundamental line bundles* ε_j naturally associated to $j \in J$. We will write the operation on Pic additively, so

$$\sum_{j \in J} n_j \varepsilon_j = \bigotimes_{j \in J} \varepsilon_j^{\otimes n_j}.$$

Any line bundle $\mathscr{L}$ on F_J determines a map from F_J to the projective space $\mathbb{P}\Gamma(\mathscr{L})^\vee$, by taking a point x to the hyperplane of $\Gamma(\mathscr{L})$ comprising sections vanishing at x. The line bundle $\mathscr{L}$ is said to be *very ample* if this map is an embedding. The line bundle $\sum_{j\in J} \varepsilon_j$, given by the tensor product of the fundamental line bundles, is very ample, so F_J embeds into the projective space $\mathbb{P}\Gamma(\sum_{j\in J} \varepsilon_j)^\vee$. This embedding is the composition of the map $F_J \to \prod_{j\in J} \mathbb{P}\Gamma(\varepsilon_j)^\vee$ with the Segre embedding $\prod_{j\in J} \mathbb{P}\Gamma(\varepsilon_j)^\vee \to \mathbb{P}\Gamma(\sum_{j\in J} \varepsilon_j)^\vee$. Thus

$$F_J \to \prod_{j \in J} \mathbb{P}\Gamma(\varepsilon_j)^\vee$$

is itself an embedding. The homogeneous coordinate ring $\mathbb{C}[F_J]$ of F_J in this embedding is given by the product of homogeneous coordinate rings arising from the embeddings in each of the factors, and so is naturally (multi-)graded by $\mathrm{Pic}(F_J) = \mathbb{Z}^J$. Under the additional choice of a Borel subgroup B, which identifies $\mathrm{Pic}(F_J)$ with $T_J^\vee$, the homogeneous component of $\mathbb{C}[F_J]$ with degree $\sum_{j\in J} n_j\varepsilon_j$ is an irreducible representation of $\mathfrak{g}$ with highest weight $\sum_{j\in J} n_j \omega_j$.

2.6 Cluster Algebra Structures on Partial Flag Varieties

Now, following Geiß–Leclerc–Schröer, we define a cluster algebra $\widetilde{\mathscr{A}_J}$ inside $\mathbb{C}[F_J]$. In order to describe particular elements of the homogeneous coordinate ring, we will realise F_J explicitly as a quotient of G by a parabolic subgroup of the appropriate type. We may then identify F_J with the set of left cosets of the chosen parabolic subgroup.

From now on, we fix a maximal torus T of G, and opposite Borel subgroups B^+ and B^- of G such that $B^+ \cap B^- = T$. As above, let $T^\vee = \mathrm{Hom}(T, \mathbb{C}^\times)$ be the weight lattice of T. The choice of B^+ determines a choice of simple roots $\alpha_i \in T^\vee$, and thus a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Let e_i, f_i, h_i be a set of Chevalley generators for $\mathfrak{g}$ under this Cartan decomposition, and write

$$x_i(t) = \exp(te_i),$$
$$y_i(t) = \exp(tf_i)$$

for the corresponding one-parameter subgroups of G. Let P_J^- be the parabolic subgroup of G generated by B^- and the one parameter subgroups $x_k(t)$ for $k \in K = I \setminus J$. This is called the *standard parabolic subgroup* associated to T, B^+ and K. We henceforth identify F_J with G/P_J^-. (The reader is warned that in [GLS08], this parabolic subgroup is indexed by $K = I \setminus J$, and F_J is taken to be the quotient $P_J^- \backslash G$ by the left action of P_J^- on G.) As in Section 2.5, the homogeneous coordinate ring $\mathbb{C}[G/P_J^-]$ is defined via the embedding

$$F_J \to \prod_{j \in J} \mathbb{P}\Gamma(\varepsilon_j)^\vee,$$

and so is graded by $\mathrm{Pic}(G/P_J^-)$. We identify $\mathrm{Pic}(G/P_J^-)$ with $\mathbb{Z}^J$ by associating ε_j to the j-th standard basis element.

Let N^+ and N^- be the unipotent radicals of B^+ and B^-. As in [GLS08, §2.3], we observe that G/N^- is the affine multi-cone on $G/B^- = G/P_I^-$ in the chosen embedding, and so identify $\mathbb{C}[G/B^-]$ with $\mathbb{C}[G/N^-]$; both are graded by $\mathbb{Z}^I$. The coordinate ring of $\mathbb{C}[G/P_J^-]$ is identified with the subalgebra of $\mathbb{C}[G/N^-]$ generated by homogeneous elements of degree ε_j for $j \in J$, and so is graded by $\mathbb{Z}^J$.

Let $W = N_G(T)/T$ be the Weyl group of G, and let $u \in W$ and $j \in J$. Fomin–Zelevinsky define [FZ99, §1.4] a regular function $\Delta_{\omega_j, u\omega_j}$ on G, called a *generalised minor*. This function in fact depends only on the weight $u\omega_j$, rather than on the particular choice of u [FZ99, §2.3]. It is immediate from the definition that $\Delta_{\omega_j,u\omega_j}(xn) = \Delta_{\omega_j,u\omega_j}(x)$ for all $x \in G$ and $n \in N^-$, and so $\Delta_{\omega_j,u\omega_j} \in \mathbb{C}[G/N^-]$. Moreover, $\Delta_{\omega_j,u\omega_j}$ has degree ε_j, and so $\Delta_{\omega_j,u\omega_j} \in \mathbb{C}[G/P_J^-]$. In the case that $\Delta = \mathsf{A}_n$ and J is a singleton, so G/P_J^- is a Grassmannian, the generalised minors coincide with the Plücker coordinates. The cluster algebra $\widetilde{\mathscr{A}_J}$ constructed by Geiß–Leclerc–Schröer always contains these generalised minors.

Now let P_J^+ be the parabolic subgroup generated by B^+ and the one parameter subgroups $y_k(t)$ for $k \in K$. Its unipotent radical N_J^+ embeds into G/P_J^- as a dense open subset [Bor91, Prop. 14.21]. By [GLS08, §2.4], this open subset is defined by the non-vanishing of the generalised minors $\Delta_{\omega_j,\omega_j}$ for $j \in J$, and so we write

$$\mathbb{C}[N_J^+] = \mathbb{C}[G/P_J^-]/(\Delta_{\omega_j,\omega_j} - 1 : j \in J).$$

In [GLS08], Geiß–Leclerc–Schröer define a cluster algebra $\mathscr{A}_J \subseteq \mathbb{C}[N_J^+]$, and homogenise it as in Section 2.4 it to obtain a cluster algebra $\widetilde{\mathscr{A}_J} \subseteq \mathbb{C}[G/P_J^-]$ of the same type, but with $|J|$ more frozen variables.

We first outline the construction of $\mathscr{A}_J$. Let $\Pi = \Pi(\Delta)$ be the preprojective algebra of type Δ, and denote the vertex projective at vertex i by P_i, the vertex injective at i by Q_i, and the vertex simple at i by S_i. Let $Q_J = \bigoplus_{j\in J} Q_j$. Then $\operatorname{Sub} Q_J$ is the full subcategory of $\operatorname{mod}\Pi$ whose objects are isomorphic to a submodule of a direct sum of copies of Q_J, or equivalently those modules with socle isomorphic to $\bigoplus_{j\in J} S_j^{n_j}$ for some natural numbers n_j. The category $\operatorname{Sub} Q_J$ is closed under submodules so it has kernels which agree with those in $\operatorname{mod}\Pi$, but it does not have cokernels in general. However, it is extension-closed (Proposition 2.15), so it inherits the structure of an exact category in which a sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

in $\operatorname{Sub} Q_J$ is exact if and only if the sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact in $\operatorname{mod}\Pi$. For background on homological algebra in exact categories, we refer to the survey of Bühler [Büh10].

For $M \in \operatorname{mod}\Pi$, let $\theta_J(M)$ be the minimal submodule of M such that $M/\theta_J(M) \in \operatorname{Sub} Q_J$, or equivalently the maximal submodule of M supported on K. Then [AS80, Thm. 4.8(b)] the projection $M \to M/\theta_J(M)$ is a minimal left $\operatorname{Sub} Q_J$-approximation of M (see Definition 4.2). We write $F_i = Q_i/\theta_J(Q_i)$; by [GLS08, Prop. 3.2], these are the indecomposable projective-injective objects in $\operatorname{Sub} Q_J$, and their direct sum is a minimal finite cover and minimal finite cocover of $\operatorname{Sub} Q_J$ (cf. [AS81, Prop. 3.1]). Hence $\operatorname{Sub} Q_J$ is a Frobenius category.

Another description of the F_i is as follows (cf. [GLS08, Prop. 4.2]). Let $\Pi_K = \Pi(\Delta_K)$ be the preprojective algebra of type Δ_K, the full subgraph of Δ on K. For $k \in K$, let q_k be the injective Π_K-module with socle at k, and let $q_j = 0$ for $j \in J$; these are also Π-modules as Π_K is a quotient of Π. Then as Π_K is also a subalgebra of Π, we have that q_i is a submodule of Q_i for all $i \in I$. Since q_i is supported on K, we have $q_i \subseteq \theta_J(Q_i)$, and since $\theta_J(Q_i)$ is a Π_K-module with simple socle at i, we have $\theta_J(Q_i) \subseteq q_i$. So $q_i = \theta_J(Q_i)$, and $F_i = \theta_J(Q_i)/q_i$.

Remark 2.14. As $\operatorname{Sub} Q_J$ is merely exact, and not abelian in general, the F_i are only projective-injective in the sense that

$$\operatorname{Ext}^1_{\operatorname{Sub} Q_J}(F_i, M) = 0 = \operatorname{Ext}^1_{\operatorname{Sub} Q_J}(M, F_i)$$

for all $M \in \operatorname{Sub} Q_J$. As $\operatorname{Sub} Q_J$ is closed under submodules, it contains $\ker f$ for any epimorphism f (where the kernel is taken in $\operatorname{mod}\Pi$), meaning that every epimorphism is admissible [Büh10, Defn. 2.1], and the usual projective lifting property

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r] & 0 \\ & F_i \arrow[u] \arrow[ul, dashed, "\exists"] & \end{tikzcd}$$

holds [Büh10, Prop. 11.3]. Not every monomorphism is admissible, but if f is a monomorphism such that $\operatorname{coker} f \in \operatorname{Sub} Q_J$, then f is admissible and the injective lifting property

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d] & N \arrow[r] \arrow[dl, dashed, "\exists"] & \operatorname{coker} f \arrow[r] & 0 \\ & F_i & & & \end{tikzcd}$$

holds. This happens, for example, if f is an injective hull.

Using some constructions of Lusztig [Lus91, Lus00], Geiß–Leclerc–Schröer describe [GLS06, §9] how each module $M \in \operatorname{mod} \Pi$ determines a function $\varphi_M \in \mathbb{C}[N^+]$. By [GLS08, Prop. 9.1], the coordinate ring $\mathbb{C}[N_J^+]$ is isomorphic to the subspace of $\mathbb{C}[N^+]$ spanned by φ_M for $M \in \operatorname{Sub} Q_J$.

We write $\underline{\operatorname{mod}}\, \Pi = \underline{\operatorname{mod} \Pi}$ and $\underline{\operatorname{Sub}}\, Q_J = \underline{\operatorname{Sub} Q_J}$ for the stable categories of $\operatorname{mod} \Pi$ and $\operatorname{Sub} Q_J$, and write

$$\begin{aligned} \underline{\operatorname{Hom}}_\Pi(M, N) &= \underline{\operatorname{Hom}}_{\operatorname{mod} \Pi}(M, N), \\ \underline{\operatorname{Hom}}_J(M, N) &= \underline{\operatorname{Hom}}_{\operatorname{Sub} Q_J}(M, N) \end{aligned}$$

for their morphism spaces. Note that while $\operatorname{Hom}_{\operatorname{Sub} Q_J}(M, N) = \operatorname{Hom}_\Pi(M, N)$, in general $\underline{\operatorname{Hom}}_J(M, N) \neq \underline{\operatorname{Hom}}_\Pi(M, N)$ because the projective-injective objects in $\operatorname{Sub} Q_J$ are different from those in $\operatorname{mod} \Pi$.

We summarise some important properties of $\operatorname{Sub} Q_J$ which we will use frequently.

Proposition 2.15 ([BIRS09, Cor. II.2.7a]). *The category $\operatorname{Sub} Q_J$ is a functorially-finite, extension closed, stably 2-Calabi–Yau Frobenius subcategory of $\operatorname{mod} \Pi$.*

As $\operatorname{Sub} Q_J$ is full and extension closed, we have $\operatorname{Ext}^1_{\operatorname{Sub} Q_J}(M, N) = \operatorname{Ext}^1_\Pi(M, N)$ for all $M, N \in \operatorname{Sub} Q_J$, and we will prefer to write the latter.

There are several important autoequivalences of the stable category $\underline{\operatorname{Sub}}\, Q_J$. Firstly, as $\operatorname{Sub} Q_J$ is Frobenius, $\underline{\operatorname{Sub}}\, Q_J$ is triangulated with shift given by the inverse syzygy functor Ω_J^{-1}, where $\Omega_J^{-1} M$ is the cokernel of an injective hull $M \to Q$, by [Hap88, I.2]. The inverse shift is Ω_J, with $\Omega_J M$ given by the kernel of a projective cover $P \to M$.

We now recall a little Auslander–Reiten theory; for an overview of this subject, we recommend the notes of Angeleri Hügel [AH06]. For more a more detailed description of this rich subject, we recommend the books of Assem–Simson–Skowroński [ASS06] and Auslander–Reiten–Smalø [ARS97].

Definition 2.16. Let $\mathcal{C}$ be an exact category. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is called a *split monomorphism* if there exists $f'\colon Y \to X$ such that $f'f = 1_X$. Dually, f is called a *split epimorphism* if there exists $f'\colon Y \to X$ such that $ff' = 1_Y$.

A morphism $f\colon X \to Y$ in $\mathcal{C}$ is *left almost split* if it is not a split monomorphism, and for any $g\colon X \to Z$ that is not a split monomorphism, there exists $h\colon Y \to Z$ such that $g = hf$. Dually, f is *right almost split* if it is not a split epimorphism, and for any $g\colon Z \to Y$ that is not a split epimorphism, there exists $h\colon Z \to X$ such that $g = fh$.

An exact sequence

$$0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

is called an *almost split sequence*, or sometimes an *Auslander–Reiten sequence*, if f is left almost split and g is right almost split. The category $\mathcal{C}$ is said to *have almost split sequences* if every non-projective object $M \in \mathcal{C}$ appears as the rightmost term in an almost split sequence.

We may make analogous definitions in a triangulated category; a triangle

$$N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow N[1]$$

is almost split if f is left almost split and g is right almost split.

By [AS81, Prop. 6.1], $\operatorname{Sub} Q_J$ has almost split sequences, inducing an Auslander–Reiten translation τ_J on $\operatorname{Sub} Q_J$. Here $\tau_J M$ and $\tau_J^{-1} M$ are defined up to isomorphism on non-projective-injective objects by the property that there are almost split sequences

$$0 \longrightarrow \tau_J M \longrightarrow E_1 \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow E_2 \longrightarrow \tau_J^{-1} M \longrightarrow 0$$

in $\operatorname{Sub} Q_J$. If M is projective-injective, then $\tau_J M = 0 = \tau_J^{-1} M$. While this means that τ_J and τ_J^{-1} are not inverses to each other on $\operatorname{Sub} Q_J$, we do have $\tau_J \tau_J^{-1} M \cong M \cong \tau_J^{-1} \tau_J M$ in the stable category $\underline{\operatorname{Sub}}\, Q_J$.

Proposition 2.17. *The category* $\operatorname{Sub} Q_J$ *has the Auslander–Reiten formula*

$$\underline{\operatorname{Hom}}_J(N, \tau_J M) = \operatorname{D} \operatorname{Ext}^1_{\Pi}(M, N).$$

In other words, $\tau_J \Omega_J^{-1}$ *is a Serre functor for* $\underline{\operatorname{Sub}}\, Q_J$.

Proof. This is equivalent to the existence of almost split sequences, by [LNP13, Thm. 2.7] (see also [GR92, Thm. 9.3, Cor. 9.4]). □

We will make frequent use of the following result of Geiß–Leclerc–Schröer.

Proposition 2.18 ([GLS08, Prop. 3.4])**.** *For any indecomposable non-projective module $M \in \operatorname{Sub} Q_J$, we have*

$$\tau_J M = \Omega_J^{-1} M.$$

It follows that $\tau_J = \Omega_J^{-1}$ and $\tau_J^{-1} = \Omega_J$ as autoequivalences of $\underline{\operatorname{Sub}}\, Q_J$. In the presence of Proposition 2.17 (which is equivalent to the existence of almost split sequences), Proposition 2.18 is equivalent to the fact that $\underline{\operatorname{Sub}}\, Q_J$ is 2-Calabi–Yau, as follows. The Auslander–Reiten formula states that $\tau_J \Omega_J^{-1}$ is a Serre functor on $\underline{\operatorname{Sub}}\, Q_J$. Thus if $\tau_J = \Omega_J^{-1}$, then Ω_J^{-2} is a Serre functor, which is precisely the statement that $\underline{\operatorname{Sub}}\, Q_J$ is 2-Calabi–Yau. Conversely if both $\tau_J \Omega_J^{-1}$ and Ω_J^{-2} are Serre functors on $\underline{\operatorname{Sub}}\, Q_J$, then they are isomorphic, and hence $\tau_J^{-1} = \Omega_J$.

A different construction of $\operatorname{Sub} Q_J$ is given by Buan–Iyama–Reiten–Scott [BIRS09, §III.3]. Write $W(\Delta)$ for the Coxeter group

$$W(\Delta) = \langle s_i : i \in I \mid (s_i s_j)^{n_{ij}} \rangle,$$

where

$$n_{ij} = \begin{cases} 1, & i = j, \\ 2, & i \text{ and } j \text{ are different and non-adjacent in } \Delta, \\ 3, & \text{otherwise.} \end{cases}$$

The s_i are called *simple reflections*. Given $w \in W(\Delta)$, a *word* for w is an expression $w = s_{i_1} \cdots s_{i_k}$ for w as a product of simple reflections. We say the *length* of the word $s_{i_1} \cdots s_{i_k}$ is k, and the *length* $\ell(w)$ of w is the minimal length of a word for w. A *reduced expression* for w is a sequence $(i_1, \ldots, i_k)$ of vertices of Δ such that $w = s_{i_1} \cdots s_{i_k}$ and $k = \ell(w)$. In this case we call the word $s_{i_1} \cdots s_{i_k}$ *reduced.*

Buan–Iyama–Reiten–Scott construct, for each $w \in W(\Delta)$, a subcategory $\mathcal{C}_w$ of $\operatorname{mod} \Pi$. They do not in fact require that Δ is a Dynkin quiver, but we will restrict to this case. For i a vertex of Δ, let e_i be the corresponding idempotent in Π, and let I_i be the two-sided ideal generated by $1 - e_i$.

Proposition 2.19 ([BIRS09, Prop. III.1.8])**.** *The following identities hold.*

(i) $I_i^2 = I_i$.

(ii) $I_i I_j = I_j I_i$ *if i and j are not adjacent in Δ.*

(iii) $I_i I_j I_i = I_j I_i I_j$ *if i and j are adjacent in Δ.*

It follows that for any reduced expression $(i_1, i_2, \ldots, i_k)$, the ideal $I_{i_1} I_{i_2} \cdots I_{i_k}$ is determined by $s_{i_1} \cdots s_{i_k} \in W(\Delta)$.

Definition 2.20. Let $w \in W(\Delta)$, and define $I_w = I_{i_1} I_{i_2} \cdots I_{i_k}$, where $(i_1, i_2, \ldots, i_k)$ is any reduced expression for w; this is well-defined by Proposition 2.19. Define $\Pi_w = \Pi / I_w$ and $\mathcal{C}_w = \operatorname{Sub} \Pi_w$.

Since Δ is a Dynkin diagram, the group $W(\Delta)$ is finite, and has a unique element w_0 of maximal length. Similarly, $W(\Delta_K)$ has a unique longest element w_0^K. In view of the remark following [GLS10, Lem. 22.15], the following result is dual to [GLS10, Lem. 22.19].

Proposition 2.21 (cf. [GLS10, Lem. 22.19])**.** *Let w_0 be the longest element of $W(\Delta)$, let w_0^K be the longest element of $W(\Delta_K)$, and let $w_J = w_0^K w_0$. Then*

$$\mathcal{C}_{w_J} = \operatorname{Sub} Q_J.$$

Proposition 2.21 allows us to apply results on the categories $\mathcal{C}_w$ to $\operatorname{Sub} Q_J$, which we will typically do without further comment.

The cluster algebra structure on $\mathbb{C}[N_J^+]$ is induced from a mutation property of maximal rigid objects (see Definition 4.4) in $\operatorname{Sub} Q_J$. Geiß–Leclerc–Schröer show in [GLS08, Prop. 8.1], building on results from [GLS06, §5, §6], that if $M \oplus T$ is a maximal rigid module in $\operatorname{Sub} Q_J$, and M is indecomposable and not projective-injective, then there is a unique indecomposable $M' \ncong M$ in $\operatorname{Sub} Q_J$ such that $M' \oplus T$ is a maximal rigid module. Moreover $\operatorname{ext}^1(M, M') = 1 = \operatorname{ext}^1(M', M)$, and if

$$0 \longrightarrow M \xrightarrow{f} X \xrightarrow{g} M' \longrightarrow 0$$
$$0 \longrightarrow M' \xrightarrow{h} Y \xrightarrow{i} M \longrightarrow 0$$

are non-split extensions between M and M', then f and h are minimal left $\operatorname{add} T$-approximations, and g and i are minimal right $\operatorname{add} T$-approximations (see Definition 4.2). The module $M' \oplus T$ is called the *mutation* of $M \oplus T$ at M, and it follows that $M \oplus T$ is the mutation of $M' \oplus T$ at M'. Moreover, by [GLS08, Prop. 8.2], the quiver $\Gamma_{M' \oplus T}$ of $\operatorname{End}_\Pi(M' \oplus T)^{\mathrm{op}}$ is given by the Fomin–Zelevinsky mutation of $\Gamma_{M \oplus T}$ at the vertex corresponding to M. The pair $\{M, M'\}$ is called the *exchange pair* associated to T.

Since $\operatorname{Sub} Q_J$ is stably 2-Calabi–Yau, its cluster-tilting objects have the above mutation property by [IY08]. We will see later (Theorem 4.22) that the maximal rigid objects in $\operatorname{Sub} Q_J$ coincide with the cluster-tilting objects, giving another explanation for these observations.

Let $\mathbf{i} = (i_1, \dots, i_r)$ be a reduced expression for $w_J = w_0^K w_0$. This determines a cluster-tilting object $T_{\mathbf{i},J}$ of $\operatorname{Sub} Q_J$ by [BIRS09, Thm. III.2.6]. We recall the construction of $T_{\mathbf{i},J}$ in Section 6.1.

By [GLS08, Prop. 6.1], if M is an indecomposable summand of $T_{\mathbf{i},J}$, then φ_M is the restriction of a generalised minor to N_J^+. Let $\mathscr{R}_J$ be the set of maximal rigid objects of $\operatorname{Sub} Q_J$ that can be obtained from $T_{\mathbf{i},J}$ by sequences of mutations; this is independent of the choice of word $\mathbf{i}$, and we call such objects *reachable*. Then each cluster-tilting object $T = T_1 \oplus \cdots \oplus T_r \in \mathscr{R}_J$, where the T_i are indecomposable summands, projects to a cluster $\{\varphi_{T_1}, \dots, \varphi_{T_r}\}$ of a cluster algebra $\mathscr{A}_J \subseteq \mathbb{C}[N_J^+]$; mutation in this cluster algebra is induced from the mutation of basic maximal rigid modules in $\operatorname{Sub} Q_J$. Geiß–Leclerc–Schröer have shown [GLS11, Thm. 17.4] that there is in fact an equality $\mathscr{A}_J = \mathbb{C}[N_J^+]$.

Moreover, in the case $J = I$, we have $\mathscr{A}_I = \mathbb{C}[N^+]$, and by [GLS06, Thm. 2.8(3)] the cluster monomials of $\mathscr{A}_I$ belong to the dual of Lusztig's semicanonical basis, introduced in [Lus00]. The cluster monomials coincide exactly with this basis if and only if $\mathscr{A}_I$ has finite type, if and only if $\Delta = \mathsf{A}_n$ and $n \leq 4$.

To obtain the desired cluster algebra $\widetilde{\mathscr{A}_J} \subseteq \mathbb{C}[G/P_J^-]$, the cluster algebra $\mathscr{A}_J$ can be homogenised as follows. First, each cluster variable x in $\mathbb{C}[N_J^+]$ can be canonically lifted to a homogeneous element $\widetilde{x} \in \mathbb{C}[G/P_J^-]$ by [GLS08, Lem. 2.4]; for example, if x is the restriction of a generalised minor $\Delta_{\omega_j,u\omega_j}$ to N_J^+, then $\widetilde{x} = \Delta_{\omega_j,u\omega_j}$. Thus each $x \in \mathbb{C}[N_J^+]$ has an associated degree $\deg \widetilde{x} \in \mathbb{Z}^J$. For $M \in \operatorname{Sub} Q_J$ and $i \in I$, let $d_i = \hom_\Pi(S_i, M)$, where S_i is the simple module at vertex i. Note that M has socle concentrated at J, so $d_k = 0$ for all $k \in K$. Then as in [GLS08, §10.1], we have

$$\deg \widetilde{\varphi_M} = \sum_{j \in J} d_j \varepsilon_j \in \mathbb{Z}^J.$$

To obtain a cluster algebra $\widetilde{\mathscr{A}_J} \subseteq \mathbb{C}[G/P_J^-]$, we homogenise $\mathscr{A}_J$ as in Definition 2.12. Pick any seed of $\mathscr{A}_J$, replace each cluster variable x_i by its lift $\widetilde{x_i}$, and add $|J|$ additional frozen variables given by the generalised minors $\Delta_{\omega_j,\omega_j}$ for $j \in J$, which satisfy $\deg \Delta_{\omega_j,\omega_j} = \varepsilon_j$. In the quiver of this seed, add arrows adjacent to the new frozen variables until the exchange relations are homogeneous; equivalently, add arrows so that the balancing condition

$$\sum_{u \to v} \deg \widetilde{x_u} = \sum_{v \to w} \deg \widetilde{x_w}$$

is satisfied at each mutable vertex v. This condition, along with the requirement that there are no 2-cycles, uniquely determines the additional arrows. Define $\widetilde{\mathscr{A}_J} \subseteq \mathbb{C}[G/P_J^-]$ to be the (graded) cluster algebra generated by this homogenised seed; this is strongly isomorphic to the abstract homogenisation of $\mathscr{A}_J$ described in Definition 2.12, but has explicit cluster variables chosen in $\mathbb{C}[G/P_J^-]$.

It is conjectured [GLS08, Conj. 10.4] that $\widetilde{\mathscr{A}_J}$ and $\mathbb{C}[G/P_J^-]$ coincide after localization at the multiplicative submonoid of $\widetilde{\mathscr{A}_J}$ generated by the minors $\Delta_{\omega_j,\omega_j}$ for $j \in J$ such that the highest weight representation $L(\omega_j)$ is not minuscule. This conjecture is proved in [GLS08, §10] for types A_n and D_4.

2.7 Cluster Characters

The map $M \mapsto \varphi_M$ from objects of $\operatorname{Sub} Q_J$ to the coordinate ring $\mathbb{C}[N_J^+]$ is an example of a cluster character. These functions, first described by Caldero–Chapoton [CC06], allow one to recover a cluster algebra from its categorification. In this section, we will describe some of Fu–Keller's work on cluster characters, in the context of stably 2-Calabi–Yau Frobenius categories such as $\operatorname{Sub} Q_J$.

Definition 2.22 ([FK10, Defn. 3.1]). Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category, and let R be a commutative ring. A cluster character on $\mathcal{E}$ is a map ζ on the set of objects of $\mathcal{E}$, taking values in R, such that

(i) if $M \cong M'$ then $\zeta_M = \zeta_{M'}$,

(ii) $\zeta_{M\oplus N} = \zeta_M \zeta_N$, and

(iii) if $\operatorname{ext}^1_{\mathcal{E}}(M,N) = 1$ (equivalently, $\operatorname{ext}^1_{\mathcal{E}}(N,M) = 1$), and

$$\begin{aligned} 0 \longrightarrow M \longrightarrow E_1 \longrightarrow N \longrightarrow 0,\\ 0 \longrightarrow N \longrightarrow E_2 \longrightarrow M \longrightarrow 0 \end{aligned}$$

are non-split sequences, then

$$\zeta_M \zeta_N = \zeta_{E_1} + \zeta_{E_2}.$$

Let $\mathcal{E}$ be a Frobenius category such that $\underline{\mathcal{E}}$ is Hom-finite and stably 2-Calabi–Yau, and assume there exists a cluster-tilting object $T \in \mathcal{E}$. Then $\mathcal{E}$ admits a cluster character, as we now explain. Assume without loss of generality that T is basic, and let $T = \bigoplus_{i=1}^n T_i$ be a decomposition of T into pairwise non-isomorphic indecomposable summands. Let $A = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$, and $\underline{A} = \operatorname{End}_{\underline{\mathcal{E}}}(T)^{\mathrm{op}}$. We write

$$\begin{aligned} F &= \operatorname{Hom}_{\mathcal{E}}(T,-)\colon \mathcal{E} \to \operatorname{mod} A,\\ G &= \operatorname{Ext}^1_{\mathcal{E}}(T,-)\colon \mathcal{E} \to \operatorname{mod} \underline{A}. \end{aligned}$$

If $M,N \in \operatorname{mod} A$ are such that $\operatorname{Ext}^i_A(M,N)$ is finite dimensional for all i, we write

$$\begin{aligned} \langle M,N\rangle_1 &= \hom_A(M,N) - \operatorname{ext}^1_A(M,N),\\ \langle M,N\rangle_3 &= \hom_A(M,N) - \operatorname{ext}^1_A(M,N) + \operatorname{ext}^2_A(M,N) - \operatorname{ext}^3_A(M,N). \end{aligned}$$

This is the case, for example, if one of M or N is finite dimensional. In particular, $\underline{A}$ is finite dimensional since $\underline{\mathcal{E}}$ is Hom-finite, so any $M \in \operatorname{mod} \underline{A}$ is finite dimensional. Fu–Keller show [FK10, Prop. 3.2] that if $M \in \operatorname{mod} \underline{A} \subseteq \operatorname{mod} A$, then $\langle M,N\rangle_3$ depends only on the dimension vector $(\hom_A(P_i,M))_{i=1}^n$, where the $P_i = FT_i$ are a complete set of indecomposable projective A-modules. Thus if $e \in \mathbb{Z}^d$, we define $\langle e,N\rangle_3 = \langle M,N\rangle$ for any $M \in \operatorname{mod} \underline{A}$ with dimension vector e, or $\langle e,N\rangle_3 = 0$ if no such M exists.

Let $R = \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be the ring of Laurent polynomials in $x_1,\dots,x_n$. We define a map $M \to \zeta_M$ on objects of $\mathcal{E}$, taking values in R, via the formula

$$\zeta_M = \prod_{i=1}^n x_i^{\langle FM,S_i\rangle_1} \sum_{e\in\mathbb{Z}^r} \chi(\operatorname{Gr}_e(GM)) \prod_{i=1}^n x_i^{-\langle e,S_i\rangle_3}.$$

Here $\operatorname{Gr}_e(GM)$ denotes the projective variety of submodules of GM with dimension vector e, and $\chi(\operatorname{Gr}_e(GM))$ denotes its Euler characteristic. The modules S_i are the simple tops of the projective modules P_i.

By [FK10, Thm. 3.3], the map $M \mapsto \zeta_M$ is a cluster character, with the property that $\zeta_{T_i} = x_i$. In the case of $\mathcal{E} = \operatorname{Sub} Q_J$, we can recover Geiß–Leclerc–Schröer's map $M \mapsto \varphi_M$ by composing $M \mapsto \zeta_M$ with the map $\operatorname{im} \zeta \to \mathbb{C}[N_J^+]$ determined by $x_i \mapsto \varphi_{T_i}$. However, we note that the definition of φ_M, unlike that of ζ_M, does not depend on the choice of a cluster-tilting object $T \in \operatorname{Sub} Q_J$.

CHAPTER 3

GORENSTEIN PROJECTIVE MODULES AND ENDOMORPHISM ALGEBRAS

3.1 Categorical Models for $\widetilde{\mathscr{A}_J}$

We have now discussed enough background to describe the motivation for our work more precisely. Recall that our ultimate aim is to find a Frobenius category $\mathcal{C}_J$ categorifying the cluster algebra $\widetilde{\mathscr{A}_J}$. In particular, $\mathcal{C}_J$ should be stably 2-Calabi–Yau, and have $|J|$ more indecomposable projective-injective objects than $\operatorname{Sub} Q_J$. More precisely, there should be an indecomposable projective-injective $F_j^* \in \mathcal{C}_J$ for each $j \in J$, and there should be an equivalence $\mathcal{C}_J/\langle F_j^* : j \in J\rangle \simeq \operatorname{Sub} Q_J$ between $\operatorname{Sub} Q_J$ and the quotient of $\mathcal{C}_J$ by the ideal of maps factoring through the objects F_j^*.

When $\Delta = \mathsf{A}_n$ and $J = \{j\}$ is a singleton (so G/P_J^- is a Grassmannian), the category $\mathcal{C}_{\{j\}}$ can be taken to be that defined by Jensen–King–Su in [JKS14]. The construction is as follows. Recall that an algebra is called *Iwanaga–Gorenstein* if it is Noetherian and has finite injective dimension on both sides. Given an Iwanaga–Gorenstein algebra B, define

$$\operatorname{GP}(B) = \{X \in \operatorname{mod} B : \operatorname{Ext}^i_A(X, B) = 0 \ \forall \ i > 0\}.$$

The category $\operatorname{GP}(B)$ is Frobenius, and objects of $\operatorname{GP}(B)$ are called *Gorenstein projective B-modules*, as in [IKWY15]. For j a vertex of A_n, let $\widetilde{B}_j$ be the algebra defined in [JKS14, §3], isomorphic to a quotient of the (completed) preprojective algebra of type $\widetilde{\mathsf{A}}_{n+1}$. The main result of Jensen–King–Su's paper is that $\operatorname{GP}(\widetilde{B}_j)$ is a Frobenius categorification of the cluster algebra structure $\widetilde{\mathscr{A}_j}$ on the homogeneous coordinate ring of the Grassmannian G_j^n. Denoting the vertex projective at the Euclidean node by F_j^*, we have $\operatorname{GP}(\widetilde{B}_j)/\langle F_j^*\rangle \simeq \operatorname{Sub} Q_j$ [JKS14, Thm. 4.5].

Remark 3.1. The category $\operatorname{GP}(B)$ is sometimes, for example in [AIR15], [Buc87] and [JKS14], denoted by $\operatorname{CM}(B)$, and the objects called (maximal) Cohen–Macaulay B-modules. The relevant algebra $\widetilde{B}_j$ in [JKS14] is a Gorenstein order over a commutative ring $Z \cong \mathbb{C}[[t]]$, and so the notion of 'Gorenstein projective over $\widetilde{B}_j$' coincides with that of 'Cohen–Macaulay over Z'. However, Iwanaga–Gorenstein rings are not always Gorenstein orders, so these two notions do not always coincide (see [IKWY15, Rem. 3.3]), and we use the term Gorenstein projective to minimise any possible confusion.

In view of Jensen–King–Su's work, we aim to construct, for any icon (Δ, J), an algebra $\widetilde{B}_J$ such that $\mathcal{C}_J = \mathrm{GP}(\widetilde{B}_J)$ provides the desired categorification of $\widetilde{\mathscr{A}_J}$. Further motivation for this approach comes from work of Iyama–Kalck–Wemyss–Yang, which we recall in Section 3.2.

While this objective has not yet been achieved, this work provides some progress towards it. In particular, in Chapter 5 we provide some theoretical tools for constructing Frobenius categories admitting cluster-tilting objects with certain endomorphism algebras, subject to certain symmetry conditions on these algebras. These symmetry conditions are best understood in the case of frozen Jacobian algebras, discussed in Sections 3.3 and 5.4. In Chapter 6, we describe some frozen Jacobian algebras that are good candidates for being endomorphism algebras of cluster-tilting objects in a categorification of $\widetilde{\mathscr{A}_J}$; in particular, their quivers have the correct shape. However, more work needs to be done to be able to check that these algebras have the necessary symmetry, and to show that the Frobenius categories constructed via our methods are stably 2-Calabi–Yau. Nevertheless, as a proof of concept, we will be able to recreate the categorification of $\mathscr{A}_J$ by $\operatorname{Sub} Q_J$ in our more general framework.

Recent independent work of Demonet–Iyama [DI15] constructs a homogenised category for all flag varieties in types A and D, using a very different approach. In their construction, each icon (Δ, J), with Δ of type A or D, determines a Gorenstein order. This order is constructed for Δ of type A in [DI15, §6.3] (cf. [DL14a]) and for Δ of type D in [DI15, §6.4] (cf. [DL14b]). It is always a quotient of the preprojective algebra of affine type $\widetilde{\mathsf{A}}$ or $\widetilde{\mathsf{D}}$, matching the type of Δ, but with the number of vertices depending additionally on J. The set J also determines an idempotent of the order, which in turn determines a subcategory of the category of Cohen–Macaulay modules over the order, and this subcategory categorifies $\widetilde{\mathscr{A}_J}$.

Our proposed method for constructing the categorification $\mathcal{C}_J$ has the advantage of also producing descriptions of the endomorphism algebras of various cluster-tilting objects in them, as well as the endomorphism algebra $\widetilde{B}_J$ of the basic projective-injective generator-cogenerator. Our construction is also uniform with respect to the type of Δ. The trade-off is that the algebras $\widetilde{B}_J$ we construct are less well-behaved than in Demonet–Iyama's work; in particular, they are very rarely orders. However, our categories are constructed explicitly as categories of Gorenstein projective modules for $\widetilde{B}_J$, which are better understood; see for example [Buc87]. Such categories may also be considered as 'normal forms' for Frobenius categories admitting cluster-tilting objects, via work of Iyama–Kalck–Wemyss–Yang; see Theorem 3.4 below.

We also expect that the methods we develop in Chapter 5 for constructing Frobenius categories with cluster-tilting objects will have applications to more algebras than those arising in the context of categorifying $\widetilde{\mathscr{A}_J}$. Indeed, we expect that they have implications for a wide class of frozen Jacobian algebras (see Sections 3.3 and 5.4).

Remark 3.2. In the case that $\Delta = \mathsf{A}_n$ and either $J = \{1\}$ or $J = \{n\}$, the simple module S_j is projective-injective in $\operatorname{Sub} Q_J$, which will cause some of our arguments to fail. However, in this case the cluster algebras $\mathscr{A}_J$ and $\widetilde{\mathscr{A}_J}$ are degenerate, each having a single seed consisting entirely of frozen variables. Thus we can safely exclude this possibility without sacrificing any interesting examples. In all other cases, no simple module is projective-injective, and neither is the radical of any projective-injective, and we restrict to this case from now on.

Even in this degenerate case, the category $\mathrm{GP}(\mathbb{C}Q) = \mathrm{proj}(\mathbb{C}Q)$, where Q is of type A_{n+1}, provides an appropriate categorification of $\widetilde{\mathscr{A}_J}$, as it has $n+1$ indecomposable objects up to isomorphism, all of which are projective-injective.

3.2 Non-Commutative Resolutions

Many of our results and methods are related to the theory of non-commutative resolutions of singularities; we recommend Wemyss [Wem14] for a survey. We will restrict our attention to non-commutative resolutions of Frobenius categories.

Definition 3.3 ([IKWY15, Defn. 2.4])**.** Let $\mathcal{E}$ be a Frobenius category. Assume that $\operatorname{proj} \mathcal{E} = \operatorname{add} P$ for some $P \in \operatorname{proj} \mathcal{E}$. We call $M \in \mathcal{E}$ a *non-commutative resolution* of $\mathcal{E}$ if

(i) $P \in \operatorname{add} M$, and

(ii) $\operatorname{End}_{\mathcal{E}}(M)^{\mathrm{op}}$ is Noetherian and has finite global dimension.

Not every Frobenius category admits a non-commutative resolution in this sense [IKWY15, Rem. 2.5], but we will see later (using Lemma 4.17) that the categories $\operatorname{Sub} Q_J$ have non-commutative resolutions given by the cluster-tilting objects. The following result of Iyama–Kalck–Wemyss–Yang gives a 'normal form' for a Frobenius categories admitting a non-commutative resolution.

Theorem 3.4 ([IKWY15, Thm. 2.7])**.** *Let $\mathcal{E}$ be a Frobenius and idempotent complete category, and assume* $\operatorname{proj} \mathcal{E} = \operatorname{add} P$ *for some* $P \in \operatorname{proj} \mathcal{E}$*. If $\mathcal{E}$ has a non-commutative resolution, the algebra* $\operatorname{End}_{\mathcal{E}}(P)^{\mathrm{op}}$ *is Iwanaga–Gorenstein and* $\mathcal{E} \simeq \mathrm{GP}(\operatorname{End}_{\mathcal{E}}(P)^{\mathrm{op}})$*.*

Note that functions are composed from left to right in [IKWY15], so our $\operatorname{End}_{\mathcal{E}}(P)^{\mathrm{op}}$ is written $\operatorname{End}_{\mathcal{E}}(P)$.

Recall that the indecomposable projective-injective objects in $\operatorname{Sub} Q_J$ are given by $F_i = Q_i/\theta_J(Q_i)$. Write $F = \bigoplus_{i\in I} F_i$ and $B_J = \operatorname{End}_{\Pi}(F)^{\mathrm{op}}$. Let T_0 be a maximal rigid object in $\operatorname{Sub} Q_J$. By [GLS10, Prop. 13.5] (reproduced here as Lemma 4.17) if the quiver Γ_{T_0} of $\operatorname{End}_\Pi(T_0)^{\mathrm{op}}$ has no loops, then $\operatorname{gl.dim}\operatorname{End}_\Pi(T_0)^{\mathrm{op}} \leq 3$. It then follows from [GLS10, Thm. 13.6] (see Proposition 4.21) that $\operatorname{gl.dim}\operatorname{End}_\Pi(T)^{\mathrm{op}} \leq 3$ for any maximal rigid object T of $\operatorname{Sub} Q_J$, and Γ_T has no loops for any such T.

We have $\operatorname{proj}\operatorname{Sub} Q_J = \operatorname{add} F$, and F is a summand of every maximal rigid object of $\operatorname{Sub} Q_J$. If $T_{\mathbf{i},J}$ is one of the maximal rigid objects associated to a reduced expression for w_J, a complete description of $\Gamma_{T_{\mathbf{i},J}}$ is given in [BIRS11, Thm. III.4.1(a)], see also Section 6.1, and this quiver has no loops. Then by the remarks of the previous paragraph, every maximal rigid object is a non-commutative resolution of $\operatorname{Sub} Q_J$, and therefore $\operatorname{Sub} Q_J \simeq \operatorname{GP}(B_J)$ by Theorem 3.4.

In light of the above discussion, it is pertinent to study the algebras $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ for maximal rigid objects $T \in \operatorname{Sub} Q_J$. These algebras have a very special form, which we now describe.

3.3 Frozen Jacobian Algebras

In this section we introduce ice quivers with potential, and their associated frozen Jacobian algebras. We also describe how to mutate ice quivers with potential, and give some properties of this operation. Many of the results of this section are mild generalisations of work of Derksen–Weyman–Zelevinsky [DWZ08] on quivers with potential.

Definition 3.5 (cf. [BIRS11, Defn. 1.1], [DL14a, §2.1], [Fra12, §6.1])**.** An *ice quiver* (Q, F) consists of a quiver Q and a (not necessarily full) subquiver F of Q. Two elements p and p' of $\mathbb{C}Q$ are said to be *cyclically equivalent* if $p - p'$ lies in the closure of the ideal of $\mathbb{C}Q$ generated by elements of the form

$$a_1a_2\cdots a_d - a_2\cdots a_d a_1$$

for cyclic paths $a_1\cdots a_d$. A *potential* on Q is a linear combination W of cyclic paths of Q. An *ice quiver with potential* is a triple (Q, F, W), where (Q, F) is an ice quiver without loops, and W is a potential on Q. A vertex or arrow of Q is called *frozen* if it is a vertex or arrow of F, and *mutable* or *unfrozen* otherwise. For brevity, we write $Q_0^m = Q_0 \setminus F_0$ and $Q_1^m = Q_1 \setminus F_1$ for the sets of mutable vertices and unfrozen arrows respectively. For $\alpha \in Q_1$ and $\alpha_1 \dots \alpha_n$ a cycle in Q, write

$$\partial_\alpha \alpha_1 \cdots \alpha_n = \sum_{\alpha_i = \alpha} \alpha_{i+1}\cdots\alpha_n\alpha_1\cdots\alpha_{i-1}$$

and extend linearly. The *frozen Jacobian algebra* of (Q, F, W) is then defined to be

$$\mathcal{J}(Q,F,W) = \mathbb{C}Q/\overline{\langle \partial_\alpha W : \alpha \in Q_1^m\rangle}.$$

The ideal $\langle \partial_\alpha W : \alpha \in Q_1^m\rangle$ is called the *Jacobian ideal.*

Write $A = \mathcal{J}(Q,F,W)$. The idempotent $e = \sum_{v \in F_0} e_v$ of A is called the *frozen* *idempotent*. We will call $B = eAe$ the *boundary algebra* of A.

Remark 3.6. If $F = \varnothing$, then $\mathcal{J}(Q, \varnothing, W) = \mathcal{J}(Q, W)$ is the usual Jacobian algebra.

If W and W' are cyclically equivalent potentials, then

$$\overline{\langle \partial_\alpha W : \alpha \in Q_1^m \rangle} = \overline{\langle \partial_\alpha W' : \alpha \in Q_1^m \rangle},$$

by [DWZ08, Lem. 3.3], and so $\mathcal{J}(Q, F, W) = \mathcal{J}(Q, F, W')$. Thus we may always replace W by any cyclically equivalent potential without affecting the isomorphism class of $\mathcal{J}(Q, F, W)$.

Definition 3.7 (cf. [DWZ08, Defn. 4.2])**.** Let (Q, F) and (Q', F') be ice quivers such that $Q_0 = Q_0'$ and $F_0 = F_0'$. Write $e_F = \sum_{v \in F_0} e_v$. Then an isomorphism $\varphi \colon \mathbb{C}Q \to \mathbb{C}Q'$ is said to be a *right equivalence* of the ice quivers with potential (Q, F, W) and (Q', F', W') if φ restricts to the identity on $S = \mathbb{C}Q_0$, restricts to an isomorphism $\mathbb{C}F \xrightarrow{\sim} \mathbb{C}F'$, and is such that $\varphi(W)$ is cyclically equivalent to W'.

Let

$$\mathbb{C}Q \mathbin{\widehat{\otimes}_{\mathbb{C}}} \mathbb{C}Q = \prod_{m,n \geq 0} (\mathbb{C}Q_1)^{\otimes_S m} \otimes_{\mathbb{C}} (\mathbb{C}Q_1)^{\otimes_S n}.$$

For any path $p = \alpha_k \cdots \alpha_1$ of $\mathbb{C}Q$, and any $\alpha \in Q_1$, we may define

$$\Delta_\alpha(p) = \sum_{\alpha_i = \alpha} \alpha_k \cdots \alpha_{i+1} \otimes \alpha_{i-1} \cdots \alpha_1 \in \mathbb{C}Q \mathbin{\widehat{\otimes}_{\mathbb{C}}} \mathbb{C}Q$$

and extend by linearity to a map $\Delta_\alpha \colon \mathbb{C}Q \to \mathbb{C}Q \mathbin{\widehat{\otimes}_{\mathbb{C}}} \mathbb{C}Q$. For $f \in \mathbb{C}Q \mathbin{\widehat{\otimes}_{\mathbb{C}}} \mathbb{C}Q$ and $g \in \mathbb{C}Q$, we define $f \bullet g$ in $\mathbb{C}Q$ by setting

$$(u \otimes v) \bullet g = vgu$$

and extending linearly. These definitions allow us to state a chain rule for cyclic derivatives.

Lemma 3.8 ([DWZ08, Lem. 3.9])**.** *If Q and Q' share a vertex set Q_0, and $\varphi \colon \mathbb{C}Q \to \mathbb{C}Q'$ is an algebra homomorphism restricting to the identity on $S = \mathbb{C}Q_0$, then for any potential W on Q and any $\alpha \in Q_1'$, we have*

$$\partial_\alpha \varphi(W) = \sum_{\beta \in Q_1} \Delta_\alpha(\varphi(\beta)) \bullet \varphi(\partial_\beta W).$$

Proposition 3.9 (cf. [DWZ08, Prop. 3.7])**.** *If φ is a right equivalence of (Q, F, W) and (Q', F', W'), then φ induces an isomorphism*

$$\mathcal{J}(Q, F, W) \xrightarrow{\sim} \mathcal{J}(Q', F', W').$$

Proof. By Lemma 3.8, for any unfrozen arrow α of Q', we have

$$\partial_\alpha\varphi(W) = \sum_{\beta\in Q_1} \Delta_\alpha(\varphi(\beta)) \bullet \varphi(\partial_\beta W).$$

Since φ restricts to an isomorphism $\mathbb{C}F \xrightarrow{\sim} \mathbb{C}F'$, if $\beta \in F_1$ then no term of $\varphi(\beta)$ can include the unfrozen arrow α, and so we have $\Delta_\alpha(\varphi(\beta)) = 0$. Thus we may instead write

$$\partial_\alpha\varphi(W) = \sum_{\beta\in Q_1^m} \Delta_\alpha(\varphi(\beta)) \bullet \varphi(\partial_\beta W),$$

and see that

$$\langle \partial_\alpha W' : \alpha \in {Q'_1}^m \rangle = \langle \partial_\alpha\varphi(W) : \alpha \in {Q'_1}^m \rangle \subseteq \langle \varphi(\partial_\beta W) : \beta \in Q_1^m \rangle,$$

with the equality coming from the cyclic equivalence of W' and $\varphi(W)$. Applying the same argument to φ^{-1}, which is also a right equivalence, we obtain the reverse inclusion, and the result follows. $\square$

If Q and Q' share the same vertex set Q_0, we can define $Q \oplus Q'$ to be the quiver with vertex set Q_0, and arrows $Q_1 \cup Q'_1$. Thus if (Q, F, W) and (Q', F', W') are ice quivers with potential such that $Q_0 = Q'_0$ and $F_0 = F'_0$, we can define

$$(Q, F, W) \oplus (Q', F', W') = (Q \oplus Q', F \oplus F', W + W').$$

Definition 3.10. An ice quiver with potential (Q, F, W) is *trivial* if $\mathcal{J}(Q, F, W) = S$, as in [DWZ08, Defn. 4.3]. Exactly as in [DWZ08, Prop. 4.4], we may assume in this case that Q_1 has exactly $2N$ arrows $\alpha_1, \beta_1, \ldots, \alpha_N, \beta_N$, all unfrozen, such that $\alpha_i\beta_i$ is a 2-cycle for all i, and $W = \sum_{i=1}^N \alpha_i\beta_i$.

We call (Q, F, W) *reduced* if

$$W = \sum_{i=1}^N \alpha_i\beta_i + W',$$

where each α_i is unfrozen, each β_i is frozen and does not appear in W', and any 2-cycle appearing in W' consists of two frozen arrows. We say (Q, F, W) is *strongly reduced* if the sum in the above expression is empty, so the only 2-cycles in W consist of a pair of frozen arrows.

Our definition of reduced (and even of strongly reduced) is slightly weaker than that given by Amiot–Reiten–Todorov [ART11, §1.3].

Theorem 3.11 (cf. [DWZ08, Thm. 4.6])**.** *Any ice quiver with potential (Q, F, W) is right equivalent to*

$$(Q_{\mathrm{red}}, F_{\mathrm{red}}, W_{\mathrm{red}}) \oplus (Q_{\mathrm{triv}}, F_{\mathrm{triv}}, W_{\mathrm{triv}}),$$

where $(Q_{\text{red}}, F_{\text{red}}, W_{\text{red}})$ *is reduced and* $(Q_{\text{triv}}, F_{\text{triv}}, W_{\text{triv}})$ *is trivial.*

Proof. Up to cyclic equivalence and rescaling arrows, we have

$$W = \sum_{i=1}^{N} (\alpha_i\beta_i + \alpha_i p_i + q_i\beta_i) + W'$$

for some $p_i, q_i \in \mathfrak{m}(\mathbb{C}Q)^2$, such that the $\alpha_i\beta_i$ are the only 2-cycles in W, and no term of W' contains α_i or β_i. The proof of [DWZ08, Lem. 4.7] applies in our situation to show that, up to right equivalence, we may assume that $p_i = 0$ whenever β_i is unfrozen, and $q_i = 0$ whenever α_i is unfrozen. Indeed, this lemma can be used to construct a right equivalence $\varphi\colon \mathbb{C}Q \xrightarrow{\sim} \mathbb{C}Q$ which is the identity on vertices, all frozen arrows and all unfrozen arrows different from the α_i and β_i, and takes W to a potential of the required form.

Thus we may assume

$$W = \sum_{i=1}^{K} \alpha_i\beta_i + \sum_{i=K+1}^{M} (\alpha_i\beta_i + \alpha_i p_i) + \sum_{i=M+1}^{N} (\alpha_i\beta_i + \alpha_i p_i + q_i\beta_i) + W'$$

for some $p_i, q_i \in \mathfrak{m}(\mathbb{C}Q)^2$, where the arrow α_i is frozen if and only if $i > M$, the arrow β_i is frozen if and only if $i > K$, and the potential W' contains no 2-cycles or any of the arrows α_i or β_i.

Now we take $(Q_{\text{triv}}, F_{\text{triv}})$ to be the subquiver of (Q, F) consisting of all vertices and the arrows α_i, β_i for $i \leq K$, and W_{triv} to be the trivial potential $\sum_{i=1}^{K} \alpha_i\beta_i$. Take $(Q_{\text{red}}, F_{\text{red}})$ to be the subquiver of (Q, F) consisting of all vertices and all arrows different from α_i, β_i, and W_{red} to be the reduced potential $W - W_{\text{triv}}$. We then have that (Q, F, W) is right equivalent to $(Q_{\text{red}}, F_{\text{red}}, W_{\text{red}}) \oplus (Q_{\text{triv}}, F_{\text{triv}}, W_{\text{triv}})$ as required. □

Remark 3.12. As in Derksen–Weyman–Zelevinsky's result [DWZ08, Thm. 4.6], the ice quivers with potential $(Q_{\text{red}}, F_{\text{red}}, W_{\text{red}})$ and $(Q_{\text{triv}}, F_{\text{triv}}, W_{\text{triv}})$ appearing in Theorem 3.11 are determined up to right equivalence by (Q, F, W). Indeed, every right equivalence of quivers with potential constructed in the proof of [DWZ08, Thm. 4.6] is also a right equivalence of ice quivers with potential. Specifically, every automorphism $\varphi\colon \mathbb{C}Q \to \mathbb{C}Q$ occurring in the proof has $\varphi(\alpha) = \alpha + p_\alpha$, where p_α is the coefficient of $\partial_\alpha W$ in an expression for an element of the Jacobian ideal. If α is frozen, $\partial_\alpha W$ is not an element of the Jacobian ideal, so we may treat this coefficient as zero, meaning that φ restricts to the identity on $\mathbb{C}F$.

Proposition 3.13. *Let* (Q, F, W) *be reduced, so that*

$$W = \sum_{i=1}^{N} \alpha_i\beta_i + W',$$

where each α_i *is unfrozen, each* β_i *is frozen and does not appear in* W'*, and any 2-cycle* *appearing in W' consists of two frozen arrows. Let (Q', F') be the ice quiver obtained from (Q, F) by deleting each β_i and freezing each α_i. Then (Q', F', W') is strongly reduced, and we have an isomorphism $\varphi\colon \mathcal{J}(Q, F, W) \xrightarrow{\sim} \mathcal{J}(Q', F', W')$ given by the identity on vertices and all arrows different from the β_i, and by $\varphi(\beta_i) = -\partial_{\alpha_i} W'$.*

Proof. By definition, there are no 2-cycles in W' containing unfrozen arrows, and so (Q', F', W') is strongly reduced.

The map φ is well-defined as follows. If γ is unfrozen and not equal to α_i for any i, then

$$\varphi(\partial_\gamma W) = \varphi(\partial_\gamma W') = \partial_\gamma W' = 0,$$

since β_i does not appear in W', and γ is unfrozen in Q'. On the other hand

$$\varphi(\partial_{\alpha_i} W) = \varphi(\beta_i + \partial_{\alpha_i} W') = -\partial_{\alpha_i} W' + \partial_{\alpha_i} W' = 0.$$

Let $\psi\colon \mathcal{J}(Q', F', W') \to \mathcal{J}(Q, F, W)$ be given by the identity on vertices and arrows. This is also well-defined, as for each mutable γ in Q' we have

$$\psi(\partial_\gamma W') = \partial_\gamma W' = \partial_\gamma W = 0,$$

as γ is not one of the α_i. Moreover

$$\psi(-\partial_{\alpha_i} W') = -\partial_{\alpha_i} W' = -\partial_{\alpha_i} W + \beta_i = \beta_i$$

in $\mathcal{J}(Q, F, W)$, so ψ and φ are inverses. □

Definition 3.14. In the notation of Proposition 3.13, we call the ice quiver with potential (Q', F', W') the *strong reduction* of (Q, F, W).

Definition 3.15 (cf. [BIRS11, §1.2], [DWZ08, §5])**.** Let (Q, F, W) be an ice quiver with potential, and let $k \in Q_0^m$ be a mutable vertex such that Q has no 2-cycles through k. Then we define the *mutation* $\mu_k(Q, F, W) = (\mu_k Q, \mu_k F, \mu_k W)$ of (Q, F, W) at k to be the result of the following procedure.

(i) Replace W by a cyclically equivalent potential such that no terms of W start at k.

(ii) Add new unfrozen arrows $[\beta\alpha]\colon i \to j$ for all pairs $\alpha\colon i \to k$, $\beta\colon k \to j$ of arrows in Q_1, and replace $\beta\alpha$ by $[\beta\alpha]$ each time it occurs as a factor of a term of W.

(iii) Replace each (necessarily unfrozen) arrow $\alpha\colon i \to k$ in Q_1 by $\alpha^*\colon k \to i$, and each arrow $\beta\colon k \to j$ in Q_1 by $\beta^*\colon j \to k$.

(iv) For every new arrow $[\beta\alpha]$ added in step (ii), add the term $\beta^*[\beta\alpha]\alpha^*$ to W.

(v) The resulting ice quiver with potential (Q', F', W') is right equivalent to

$$(Q'_{\mathrm{red}}, F'_{\mathrm{red}}, W'_{\mathrm{red}}) \oplus (Q'_{\mathrm{triv}}, F'_{\mathrm{triv}}, W'_{\mathrm{triv}})$$

by Theorem 3.11, so we define $\mu_k(Q,F,W)$ to be the strong reduction of the reduced part $(Q'_{\mathrm{red}}, F'_{\mathrm{red}}, W'_{\mathrm{red}})$.

In the notation of Definition 3.15, we have an isomorphism

$$\mathcal{J}(Q'_{\mathrm{red}}, F'_{\mathrm{red}}, W'_{\mathrm{red}}) \xrightarrow{\sim} \mathcal{J}(\mu_k Q, \mu_k F, \mu_k W)$$

by Proposition 3.13.

Remark 3.16. We allow unfrozen arrows between frozen vertices, unlike [BIRS11, Defn. 1.1], as this is more compatible with our description of mutation. This point of view is implicit in [BKM14, Rem. 12.3(d)].

Definition 3.17. The *trace space* of $A = \mathcal{J}(Q,F,W)$ is

$$\mathrm{Tr}(A) = A/[A,A],$$

where $[A,A]$ is the closure of the ideal generated by commutators. The *deformation space* of W is $\mathrm{Tr}(A)/S$, recalling that $S = A/\mathfrak{m}(A)$, and we call W *rigid* if $\mathrm{Tr}(A)/S = 0$.

Proposition 3.18 (cf. [DWZ08, Prop. 8.1, Cor. 6.11])**.** *If (Q,F,W) is rigid and reduced, then it has no 2-cycles through its mutable vertices. Moreover, all of its mutations are also rigid and strongly reduced.*

Proof. The proof of [DWZ08, Prop. 8.1] applies in this context to show that (Q,F,W) has no 2-cycles through its mutable vertices, after noting that the derivatives of W lie in $\mathfrak{m}(\mathbb{C}Q)^3 + \mathbb{C}F_1$. Since we only allow mutation at mutable vertices, and strong reduction does not affect the isomorphism class of the Jacobian algebra by Proposition 3.13, the proof of [DWZ08, Cor. 6.11] also applies to show that all mutations of (Q,F,W) are rigid. They are strongly reduced by definition. □

Proposition 3.19 ([DWZ08, Prop. 7.1])**.** *Let (Q,F,W) be rigid and reduced. Then for any mutable vertex k of Q, the quiver $\mu_k Q$ of $\mu_k(Q,F,W)$ agrees with the Fomin–Zelevinsky mutation of Q at k, up to arrows between the frozen vertices.*

Proof. As the only difference between a mutation of (Q,F,W) at a mutable vertex and the corresponding mutation of (Q,W) is the strong reduction step in Definition 3.15(v), and this step only affects arrows between frozen vertices, the desired result follows immediately from the corresponding statement [DWZ08, Prop. 7.1] for ordinary quivers with potential. □

Example 3.20. Consider the ice quiver with potential (Q,F,W) given by

$$Q = \begin{tikzcd} \boxed{1} \arrow[dr, "\alpha_1"'] & & \boxed{3} \arrow[ll, dashed, "\alpha_3"'] \\ & 2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

where F is the full subquiver on $\{1,3\} \subseteq Q_0$ and $W = \alpha_3\alpha_2\alpha_1$. Mutating at vertex 2 produces

$$(Q', F') = \begin{tikzcd} \boxed{1} \arrow[rr, bend right, "{[\alpha_2\alpha_1]}"'] & & \boxed{3} \arrow[ll, dashed, bend right, "\alpha_3"'] \arrow[dl, "\alpha_2^*"] \\ & 2 \arrow[ul, "\alpha_1^*"] & \end{tikzcd}$$

with potential $\alpha_2^*[\alpha_2\alpha_1]\alpha_1^* + \alpha_3[\alpha_2\alpha_1]$; the only frozen arrow is α_3. This ice quiver with potential is reduced, so $\mu_2(Q, F, W)$ is given by the strong reduction, which is the ice quiver

$$(\mu_2 Q, \mu_2 F) = \begin{tikzcd} \boxed{1} \arrow[rr, dashed, "{[\alpha_2\alpha_1]}"] & & \boxed{3} \arrow[dl, "\alpha_2^*"] \\ & 2 \arrow[ul, "\alpha_1^*"] & \end{tikzcd}$$

with potential $\alpha_2^*[\alpha_2\alpha_1]\alpha_1^*$. The quiver agrees with the Fomin–Zelevinsky mutation of Q at vertex 2 up to frozen arrows. Mutating at 2 again gives

$$((\mu_2 Q)', (\mu_2 F)') = \begin{tikzcd} \boxed{1} \arrow[rr, dashed, bend right, "{[\alpha_2\alpha_1]}"'] \arrow[dr, "\alpha_1"'] & & \boxed{3} \arrow[ll, bend right, "{[\alpha_1^*\alpha_2^*]}"'] \\ & 2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

where we write $\alpha_i = (\alpha_i^*)^*$. The potential is $\mu_2^2 W = [\alpha_2\alpha_1][\alpha_1^*\alpha_2^*] + [\alpha_1^*\alpha_2^*]\alpha_2\alpha_1$, so strong reduction gives

$$(\mu_2^2 Q, \mu_2^2 F) = \begin{tikzcd} \boxed{1} \arrow[dr, "\alpha_1"'] & & \boxed{3} \arrow[ll, dashed, "{[\alpha_1^*\alpha_2^*]}"'] \\ & 2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

with potential $[\alpha_1^*\alpha_2^*]\alpha_2\alpha_1$. Thus we have recovered the original ice quiver with potential, up to right equivalence.

Recall that for a maximal rigid object $T \in \operatorname{Sub} Q_J$, we denote the quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ by Γ_T. We make Γ_T into an ice quiver by letting F be the full subquiver on the vertices of Γ_T corresponding to the projective-injective summands of T. The following theorem is due to Buan–Iyama–Reiten–Smith.

Theorem 3.21 ([BIRS11, Thm. 6.6])**.** *Let $T \in \mathscr{R}_J$ be a reachable maximal rigid object in $\operatorname{Sub} Q_J$. Then there exists a rigid potential W on Γ_T such that $\operatorname{End}_\Pi(T)^{\mathrm{op}} \cong \mathcal{J}(\Gamma_T, F, W)$.*

The potential W has an explicit description when $T = T_{\mathbf{i},J}$, which we recall in Section 6.1. A similar theorem [BIRS11, Thm. 6.4] holds for $T \in \underline{\operatorname{Sub}}\, Q_J$, which is still maximal rigid. These theorems in fact apply to any cluster-tilting object (within a particular mutation class of such objects) in any category of the form $\mathcal{C}_w$ from Definition 2.20. By Proposition 2.21, this class of categories includes $\operatorname{Sub} Q_J$.

If we fix an isomorphism $A = \operatorname{End}_\Pi(T)^{\mathrm{op}} \cong \mathcal{J}(\Gamma_T, F, W)$, which we treat as an identification, then we can define the boundary algebra eAe of A. This is equal to $\operatorname{End}_\Pi(F_I)^{\mathrm{op}} = B_J$, since e is the sum of idempotents of A corresponding to the indecomposable projective-injective summands of T. In particular, the boundary algebra of A is independent of the maximal rigid object T.

3.4 Outline of Strategy

Recall that our ultimate aim is to produce, for any possible icon (Δ, J), an algebra $\widetilde{B}_J$ such that $\mathcal{C}_J = \operatorname{GP}(\widetilde{B}_J)$ categorifies $\widetilde{\mathscr{A}_J}$. Rather than attempting to describe $\widetilde{B}_J$ directly, as in the work of Jensen–King–Su [JKS14], and to some extent Demonet–Iyama [DI15], our approach is to construct a candidate for the endomorphism algebra of a cluster-tilting object in a categorification, and use this algebra to determine $\widetilde{B}_J$, and hence the category $\operatorname{GP}(\widetilde{B}_J)$. More precisely, our strategy is the following.

We start by taking a reachable maximal rigid object $T \in \operatorname{Sub} Q_J$, so that $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ is a frozen Jacobian algebra with quiver Γ_T. We will then define an enlarged quiver $\widetilde{\Gamma}_T$ containing Γ_T as a full subquiver, and with $|J|$ more vertices. The frozen subquiver $\widetilde{F}$ of $\widetilde{\Gamma}_T$ will be the full subquiver on the frozen vertices of Γ_T and the $|J|$ new vertices. We then define an algebra $\widetilde{A}$ by putting a potential $\widetilde{W}$ on $\widetilde{\Gamma}_T$, and taking $\widetilde{A} = \mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W})$. Denote the boundary algebra of $\widetilde{A}$ by $\widetilde{B}_J$. We aim to construct $\widetilde{A}$ in such a way that $\operatorname{GP}(\widetilde{B}_J)$ is a Frobenius category admitting a cluster-tilting object $\widetilde{T}$ with an isomorphism

$$\operatorname{End}_{\widetilde{B}_J}(\widetilde{T})^{\mathrm{op}} \xrightarrow{\sim} \widetilde{A}.$$

Our main theoretical result, Theorem 5.13, gives a sufficient condition on $\widetilde{A}$ for this to be the case, namely that $\widetilde{A}$ is internally 3-Calabi–Yau (Definition 5.1) with respect to its boundary idempotent.

Our inspiration, and the term boundary algebra, comes from work of Baur–King–Marsh on dimer models on disks, presented in [BKM14]. In brief, their construction is the following. Given a disk with n marked points on the boundary, a Postnikov diagram is a collection of n strands in the disk satisfying various combinatorial conditions; see [BKM14, Defn. 2.1] for the formal definition in this context, and [Sco06, Defn. 2] for a more general definition. Each such diagram yields a cluster of Scott's cluster structure on the homogeneous coordinate ring $\mathbb{C}[G_k^n]$ of the Grassmannian G_k^n of k-planes in $\mathbb{C}^n$, and the clusters occurring in this way have the property that all of their cluster variables are Plücker coordinates. Postnikov diagrams also determine bipartite tilings of the disk, i.e. dimer models, in such a way that the strands in the Postnikov diagram become the zig-zag paths in the dimer model. We refer to Broomhead [Bro12] for background on dimer models on closed surfaces.

We point out one important feature of dimer models on the disk that does not occur in the case of closed surfaces. In the dimer arising from a Postnikov diagram D, all of the vertices lie in the interior of the disk, but there are n 'half-edges' which join a vertex to the boundary rather than to another vertex. This gives rise to n tiles adjacent to the boundary. There is an ice quiver with potential (Q, F, W) associated to the dimer model; Q and W are defined in the usual way (see [Bro12, §2.1.2] for example), F_0 is the set of vertices dual to boundary tiles, and F_1 is the set of arrows dual to half edges. We replace the usual Jacobian algebra of the dimer by the frozen Jacobian algebra $\widetilde{A}_D = \mathcal{J}(Q, F, W)$. The 'missing' relations are the derivatives of W with respect to the arrows of Q corresponding to the half-edges of the dimer model. The boundary algebra is given by multiplying $\widetilde{A}_D$ on each side by the sum of idempotents coming from the boundary tiles, hence the name. It is shown [BKM14, Cor. 10.4] that this boundary algebra is isomorphic to $\widetilde{B}_j$, the algebra from [JKS14] such that $\mathrm{GP}(\widetilde{B}_j)$ categorifies the cluster algebra $\mathbb{C}[G_k^n]$. (In fact, in [BKM14], the isomorphism is with $\widetilde{B}_j^{\mathrm{op}}$, but the difference can be absorbed into various choices of convention.) Moreover, the entire frozen Jacobian algebra $\widetilde{A}_D$ is isomorphic to $\mathrm{End}_{\widetilde{B}_j}(T_D)^{\mathrm{op}}$ for some maximal rigid object $T_D \in \mathrm{GP}(B_J)$ determined by D. Thus we are attempting to replicate this situation in wider generality, although the frozen Jacobian algebras we will construct do not in general arise from dimer models on surfaces with boundary.

CHAPTER 4

MINIMAL APPROXIMATIONS AND RIGID OBJECTS

4.1 Minimal Approximations

We begin by recalling some of the definitions and basic results of approximation theory; as general references we suggest Auslander–Smalø [AS80, AS81] and Kleiner [Kle97], in particular the results on subcategories closed under submodules. We assume $\mathcal{C}$ is an exact Krull–Schmidt category, with full subcategory $\mathcal{D}$.

Definition 4.1. Let $f\colon M \to N$ be a morphism in $\mathcal{C}$. Then f is said to be *left minimal* if every $g\colon N \to N$ with $gf = f$ is an isomorphism, and *right minimal* if every $h\colon M \to M$ with $fh = f$ is an isomorphism.

Definition 4.2. Let $M \in \mathcal{C}$. Then a morphism $f\colon M \to L$ is a left $\mathcal{D}$-approximation of M if $L \in \mathcal{D}$ and the induced morphism $\operatorname{Hom}_{\mathcal{C}}(f, N)\colon \operatorname{Hom}_{\mathcal{C}}(L, N) \to \operatorname{Hom}_{\mathcal{C}}(M, N)$ is surjective for all $N \in \mathcal{D}$. Dually, a morphism $g\colon R \to M$ is a right $\mathcal{D}$-approximation of M if $R \in \mathcal{D}$ and the induced morphism $\operatorname{Hom}_{\mathcal{C}}(N, g)\colon \operatorname{Hom}_{\mathcal{C}}(N, R) \to \operatorname{Hom}_{\mathcal{C}}(N, M)$ is surjective for all $N \in \mathcal{D}$. If, for every $M \in \mathcal{C}$, there exists both a left $\mathcal{D}$-approximation and a right $\mathcal{D}$-approximation of M, then $\mathcal{D}$ is called *functorially finite*.

A left minimal left $\mathcal{D}$-approximation will be referred to merely as a minimal left $\mathcal{D}$-approximation, and similarly for right minimal right $\mathcal{D}$-approximations. If some left $\mathcal{D}$-approximation of M is a monomorphism, then all such approximations are. Similarly if some right $\mathcal{D}$-approximation of M is an epimorphism, then all such approximations are. A minimal right $\operatorname{proj}\mathcal{C}$-approximation is a projective cover, and a minimal left $\operatorname{inj}\mathcal{C}$-approximation is an injective hull.

If $f_1\colon M \to T_1$ and $f_2\colon M \to T_2$ are minimal left $\mathcal{D}$-approximations, then $T_1 \cong T_2$. Moreover, if both f_1 and f_2 are admissible in $\mathcal{C}$, then $\operatorname{coker} f_1 \cong \operatorname{coker} f_2$. The dual results hold for minimal right $\operatorname{add} T$-approximations and their kernels.

Given $T \in \mathcal{C}$, a map $f\colon M \to L$ is a left $\operatorname{add} T$-approximation of M if $L \in \operatorname{add} T$ and $\operatorname{Hom}_{\mathcal{C}}(f, T)\colon \operatorname{Hom}_{\mathcal{C}}(L, T) \to \operatorname{Hom}_{\mathcal{C}}(M, T)$ is surjective. Similarly, $g\colon R \to M$ is a right $\operatorname{add} T$-approximation if $R \in \operatorname{add} T$ and $\operatorname{Hom}_{\mathcal{C}}(T,g)\colon \operatorname{Hom}_{\mathcal{C}}(T,R) \to \operatorname{Hom}_{\mathcal{C}}(T,M)$ is surjective.

If $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ for some Artin algebra Λ (for example, if $\mathcal{C} = \operatorname{Sub} Q_J \subseteq \operatorname{mod} \Pi$), then $\operatorname{add} T$ is functorially finite for all $T \in \mathcal{C}$ by [AS80, Prop. 4.2]. By [AS80, Prop. 1.2(d)] and dually [AS80, Prop. 1.4(d)], it follows that every object in $\mathcal{C}$ admits a minimal left $\operatorname{add} T$-approximation and a minimal right $\operatorname{add} T$-approximation.

Proposition 4.3. *Let $\mathcal{E}$ be a Frobenius category. For any $M, T \in \mathcal{E}$ with* $\operatorname{proj} \mathcal{E} \subseteq \operatorname{add} T$, *let $f\colon M \to L$ be a minimal left* $\operatorname{add} T$*-approximation of M. Then the class $\underline{f}$ of f in the stable category $\underline{\mathcal{E}}$ is a minimal left* $\operatorname{add} T$*-approximation of M in $\underline{\mathcal{E}}$. The dual result for minimal right* $\operatorname{add} T$*-approximations also holds.*

Proof. We only prove the statement for minimal left $\operatorname{add} T$-approximations, the proof of the dual statement being essentially the same. First we show that $\underline{\operatorname{Hom}}_{\mathcal{E}}(f,T)$ is surjective; this follows from the commutativity of the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{E}}(L,T) \arrow[r, "\operatorname{Hom}_{\mathcal{E}}(f,T)"] \arrow[d, "\pi_1"'] & \operatorname{Hom}_{\mathcal{E}}(M,T) \arrow[d, "\pi_2"] \\
\underline{\operatorname{Hom}}_{\mathcal{E}}(L,T) \arrow[r, "\underline{\operatorname{Hom}}_{\mathcal{E}}(\underline{f},T)"'] & \underline{\operatorname{Hom}}_{\mathcal{E}}(M,T)
\end{tikzcd}$$

in which π_1 and π_2 are the canonical projections, and $\operatorname{Hom}_{\mathcal{E}}(f,T)$ is surjective because f is a left $\operatorname{add} T$-approximation. Thus $\underline{f}$ is a left $\operatorname{add} T$-approximation in $\underline{\mathcal{E}}$.

Recall that the set of morphisms $X \to Y$ in $\mathcal{E}$ factoring through $\operatorname{proj} \mathcal{E}$ is denoted by $\operatorname{Hom}^{\mathrm{p}}_{\mathcal{E}}(X,Y)$. Let $g \in \operatorname{End}_{\mathcal{E}}(L)$ be such that $\underline{gf} = \underline{f}$ in $\underline{\mathcal{E}}$, so there exists $h \in \operatorname{Hom}^{\mathrm{p}}_{\mathcal{E}}(M,L)$ such that $gf = f + h$. As h factors through $\operatorname{proj} \mathcal{E} \subseteq \operatorname{add} T$, it must factor through the left $\operatorname{add} T$-approximation f, so there exists $h' \in \operatorname{Hom}^{\mathrm{p}}_{\mathcal{E}}(L,L)$ with $h = h'f$. Now $(g-h')f = f$, so $g - h'$ is an isomorphism in $\mathcal{E}$ by minimality of f. It follows that $\underline{g} = \underline{g-h'}$ is an isomorphism in $\underline{\mathcal{E}}$, so $\underline{f}$ is minimal. ☐

4.2 Rigid Objects

We now describe various rigidity conditions on objects of $\mathcal{E}$ and some of their consequences. We will be particularly interested in results concerning $\operatorname{add} T$-approximations for an object T satisfying one of these conditions. Some of these ideas are explored in a higher level of generality in recent work of Chen–Koenig [CK15].

Definition 4.4. Let $\mathcal{C}$ be an exact category, and let $T \in \mathcal{C}$. We say that T is *rigid* if $\operatorname{Ext}^1_{\mathcal{C}}(T,T) = 0$. A rigid object is said to be

(i) *complete rigid* if the number $\Sigma(T)$ of non-isomorphic indecomposable summands of T is maximal among rigid objects of $\mathcal{C}$,

(ii) *maximal rigid* if $M \in \operatorname{add} T$ whenever $M \oplus T$ is rigid, or

(iii) *d-cluster-tilting* (sometimes *maximal* $(d-1)$*-orthogonal*, see [Iya07b]) for $d \geq 2$ if

$$\begin{aligned}\operatorname{add} T &= \{M \in \mathcal{C} : \operatorname{Ext}^i_{\mathcal{C}}(M, T) = 0 \,\forall\, 0 < i < d\} \\ &= \{M \in \mathcal{C} : \operatorname{Ext}^i_{\mathcal{C}}(T, M) = 0 \,\forall\, 0 < i < d\}.\end{aligned}$$

A 2-cluster-tilting object will be called simply *cluster-tilting.*

We make the same definitions for an object T of a triangulated category $\mathcal{T}$, for which $\operatorname{Ext}^i_{\mathcal{T}}(M, N) = \operatorname{Hom}_{\mathcal{T}}(M, N[i])$.

If T is an object of a Frobenius category $\mathcal{E}$, then

$$\operatorname{Ext}^i_{\mathcal{E}}(M, N) = \underline{\operatorname{Hom}}_{\mathcal{E}}(M, \Omega^{-i}N) = \operatorname{Ext}^i_{\underline{\mathcal{E}}}(M, N)$$

for $i > 0$. It follows that T is rigid, maximal rigid or d-cluster-tilting in $\mathcal{E}$ if and only if it has the same property in $\underline{\mathcal{E}}$. Moreover, if $\mathcal{E}$ is stably d-Calabi–Yau, then the second equality in the definition of d-cluster-tilting holds for any T, and so it is only necessary to check the first. In order to simplify the exposition, whenever we refer to a complete rigid, maximal rigid or cluster-tilting object, it will always be assumed to be basic.

Remark 4.5. The definition of d-cluster-tilting makes sense for $d = 1$; a 1-cluster-tilting object is an additive generator (and need not be rigid). Thus an exact category $\mathcal{C}$ admits a 1-cluster-tilting object if and only if $\mathcal{C}$ has finitely many indecomposable objects up to isomorphism. If $\mathcal{C} = \operatorname{mod} \Lambda$ for a non-semisimple Artin algebra Λ, then the endomorphism algebra of a 1-cluster-tilting object in $\mathcal{C}$ is an Auslander algebra [ARS97, VI.5], which in particular has global dimension 2 (cf. Lemma 4.17).

If $\mathcal{E}$ is Frobenius, and T is either complete rigid, maximal rigid or cluster-tilting, then $\operatorname{add} T$ contains $\operatorname{proj} \mathcal{E} = \operatorname{inj} \mathcal{E}$, and thus T is a generator and a cogenerator of $\mathcal{E}$. So for every $M \in \mathcal{E}$, any left $\operatorname{add} T$-approximation of M is a monomorphism, and any right $\operatorname{add} T$-approximation of M is an epimorphism.

Given any exact category $\mathcal{C}$, it is immediate that every cluster-tilting object in $\mathcal{C}$ is maximal rigid, and that every complete rigid object is maximal rigid. While the converse statements do not hold in general, all three classes of objects coincide in the categories $\operatorname{Sub} Q_J$; see Section 4.4 and the following proposition, which collects several results of Geiß–Leclerc–Schröer [GLS08].

Proposition 4.6 ([GLS08, Prop. 7.1, Prop. 7.3, §7.4])**.** *The category* $\operatorname{Sub} Q_J$ *has a complete rigid object* T*, with* $\ell(w_J) = \ell(w_0) - \ell(w_0^K)$ *indecomposable summands up to isomorphism. Furthermore, all maximal rigid objects of* $\operatorname{Sub} Q_J$ *are complete rigid.*

The rest of the chapter is devoted to properties of maximal rigid and cluster-tilting objects in various categories. We are particularly interested in the case of stably 2-Calabi–Yau Frobenius categories, such as $\operatorname{Sub} Q_J$. Most of the arguments in this section and the next are essentially the same as those by Geiß–Leclerc–Schröer in [GLS10], but for convenience, and to emphasise the significance of the mutation property in the arguments, we reproduce them in our context and with our conventions. The main exceptions are Theorems 4.12 and 4.14, which will be important for our constructions in Section 6.2. While we believe that Theorem 4.12 is well-known, we were unable to find a suitable reference in the literature.

Any small exact category $\mathcal{C}$ embeds into an abelian category $\mathcal{A}$ as a full extension-closed subcategory, in such a way that the exact sequences of $\mathcal{C}$ are precisely those mapping to exact sequences of $\mathcal{A}$, by the Gabriel–Quillen embedding theorem ([TT90], see also [Büh10, Thm. A.1]). From now on, any time we refer to an exact category $\mathcal{C}$ we will assume implicitly that it is a full extension closed subcategory of an abelian category. Since we will only work with finite diagrams, there is no loss of generality in this assumption. In any case, we will be most interested in applying our results to $\operatorname{Sub} Q_J$, which is already an explicit full extension-closed subcategory of the abelian category $\operatorname{mod} \Pi$.

Proposition 4.7. *Let $\mathcal{C}$ be an exact category, and let $M, T \in \mathcal{C}$.*

(i) If there is an admissible monomorphism $M \to T'$ for $T' \in \operatorname{add} T$, then any left $\operatorname{add} T$-approximation $f\colon M \to L$ is an admissible monomorphism.

(ii) If there is an admissible epimorphism $T'' \to M$ with $T'' \in \operatorname{add} T$, then any right $\operatorname{add} T$-approximation $g\colon R \to M$ is an admissible epimorphism.

Proof. We prove only (i), as (ii) is dual. Pick an admissible monomorphism $i\colon M \to T'$ with $T' \in \operatorname{add} T$; by definition this fits into an exact sequence

$$0 \longrightarrow M \xrightarrow{i} T' \longrightarrow N \longrightarrow 0$$

of $\mathcal{C}$. Now take a left $\operatorname{add} T$-approximation $f\colon M \to L$. As $T' \in \operatorname{add} T$, the monomorphism i factors through f, and so f is a monomorphism. Let C be the cokernel of f in the ambient abelian category. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "1_M"] & L \arrow[r, "x"] \arrow[d, "z"] & C \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & M \arrow[r, "i"'] & T' \arrow[r, "w"'] & N \arrow[r] & 0
\end{tikzcd}$$

with exact rows. A diagram chase yields the exact sequence

$$0 \longrightarrow L \xrightarrow{\left(\begin{smallmatrix} x \\ -z \end{smallmatrix}\right)} T' \oplus C \xrightarrow{(y\ w)} N \longrightarrow 0$$

with the outer terms in $\mathcal{C}$. It follows that $C \in \mathcal{C}$, since $\mathcal{C}$ is extension closed. $\square$

Corollary 4.8. *Let $\mathcal{E}$ be a Frobenius category, and let $T \in \mathcal{E}$ be maximal rigid. Then for any $M \in \mathcal{E}$, any left* $\operatorname{add} T$*-approximation of M is an admissible monomorphism, and any right* $\operatorname{add} T$*-approximation of M is an admissible epimorphism.*

Proof. As $\mathcal{E}$ is Frobenius, there exists an admissible epimorphism $P \to M$ with P projective-injective, and admissible monomorphism $M \to Q$ with Q projective-injective. Since T is maximal rigid, both P and Q lie in $\operatorname{add} T$, so the result follows from Proposition 4.7. □

Corollary 4.8 allows us to refer to cokernels and kernels of left and right $\operatorname{add} T$-approximations whenever T is a maximal rigid object in a Frobenius category. In the case of $\operatorname{Sub} Q_J$, it implies that these cokernels and kernels, taken in the ambient abelian category $\operatorname{mod} \Pi$, in fact lie in $\operatorname{Sub} Q_J$.

Lemma 4.9 (cf. [GLS06, Lem 5.1])**.** *Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category, and let $M, T \in \mathcal{E}$ be rigid.*

(i) If C is the cokernel of a monomorphic left $\operatorname{add} T$*-approximation of M, then $T \oplus C$ is rigid.*

(ii) If K is the kernel of an epimorphic right $\operatorname{add} T$*-approximation of M, then $T \oplus K$ is rigid.*

Proof. We only prove (i), as (ii) is dual. By assumption, we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} L \xrightarrow{g} C \longrightarrow 0,$$

in which $f\colon M \to L$ is a left $\operatorname{add} T$-approximation. Applying $\operatorname{Hom}_{\mathcal{E}}(-, T)$ yields the exact sequence

$$\operatorname{Hom}_{\mathcal{E}}(L, T) \xrightarrow{f_L^*} \operatorname{Hom}_{\mathcal{E}}(M, T) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(C, T) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(L, T) = 0.$$

Since f is a left $\operatorname{add} T$-approximation, f_L^* is surjective, and so $\operatorname{Ext}^1_{\mathcal{E}}(C, T) = 0$. As $\mathcal{E}$ is stably 2-Calabi–Yau, we also have $\operatorname{Ext}^1_{\mathcal{E}}(T, C) = 0$.

It remains to show that C is itself rigid. Applying $\operatorname{Hom}_{\mathcal{E}}(M, -)$ to our original exact sequence yields

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{E}}(M, M) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(M, L) \xrightarrow{g_*^M} \operatorname{Hom}_{\mathcal{E}}(M, C) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(M, M) = 0,$$

so g_*^M is surjective. Instead applying $\operatorname{Hom}_{\mathcal{E}}(-, C)$, we get

$$\operatorname{Hom}_{\mathcal{E}}(L, C) \xrightarrow{f_C^*} \operatorname{Hom}_{\mathcal{E}}(M, C) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(C, C) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(L, C) = 0.$$

So to obtain our desired conclusion, it suffices to show that f_C^* is surjective. Let $h\colon M \to C$. By the surjectivity of g_*^M, there exists $t\colon M \to L$ with $gt = h$. Moreover, since f is a left $\operatorname{add} T$-approximation, there exists $s \in \operatorname{End}_{\mathcal{E}}(L)$ with $sf = t$, and so $h = gsf \in \operatorname{im} f_C^*$. Thus f_C^* is surjective, and $\operatorname{Ext}^1_{\mathcal{E}}(C, C) = 0$. $\square$

Proposition 4.10. *Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category, and let $M, T \in \mathcal{E}$ with T maximal rigid. Pick left and right $\operatorname{add} T$-approximations $f\colon M \to L$ and $g\colon R \to M$, and write $C = \operatorname{coker} f$ and $K = \ker g$. Then $C, K \in \operatorname{add} T$ if either M is rigid or T is cluster-tilting.*

Proof. Since T is maximal rigid, f is a monomorphism and g is an epimorphism. Thus if M is rigid, $T \oplus C$ and $T \oplus K$ are rigid by Lemma 4.9, so $C, K \in \operatorname{add} T$ by maximality. On the other hand, if M is arbitrary, but T is cluster-tilting, then the sequence

$$\operatorname{Hom}_{\mathcal{E}}(L, T) \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{E}}(M, T) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(C, T) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(L, T) = 0$$

is exact, and f^* is surjective. Thus $\operatorname{Ext}^1_{\mathcal{E}}(C, T) = 0$, and so $C \in \operatorname{add} T$. The proof for K is dual. $\square$

Proposition 4.11. *Let $\mathcal{E}$ be a Frobenius category, and let $M, T \in \mathcal{C}$ with T maximal rigid. Pick a minimal left $\operatorname{add} T$-approximation $f\colon M \to L$ with cokernel C, and a minimal right $\operatorname{add} T$-approximation $g\colon R \to M$ with kernel K. Then no indecomposable summand of C is projective, and no indecomposable summand of K is injective.*

Proof. Again we only give the proof for C, as the result for K is dual. Let P be a projective summand of C, so there exist maps $i\colon P \to C$ and $\pi\colon C \to P$ with $\pi i = 1_P$. Let $h\colon L \to C$ be the cokernel of f. Then h is an admissible epimorphism, so $i\colon P \to C$ lifts to $i'\colon P \to L$ such that $i = hi'$. In summary, we have the commutative diagram

$$\begin{tikzcd}
 & & & P \arrow[dl, "i'"'] \arrow[d, bend right, "i"'] & \\
0 \arrow[r] & M \arrow[r, "f"'] & L \arrow[r, "h"'] & C \arrow[r] \arrow[u, bend right, "\pi"'] & 0
\end{tikzcd}$$

in which the bottom row is exact. Now $\pi h i' = \pi i = 1_P$, so P is a summand of L. The projection $1_L - i'\pi h$ away from P satisfies $(1_L - i'\pi h)f = f$ because $hf = 0$. As f is minimal, this means that $1_L - i'\pi h$ is an isomorphism, and so $P = 0$. $\square$

The following result will be crucial in our later constructions; see Theorem 6.11.

Theorem 4.12. *Let $\mathcal{C}$ be a triangulated category with shift τ, and let $T \in \mathcal{C}$ be rigid. Let $M \in \mathcal{C}$, and let $f\colon M \to L$ be a left $\operatorname{add} T$-approximation of M. Let*

$$M \xrightarrow{f} L \longrightarrow R \xrightarrow{g} \tau M$$

be a triangle starting with f. If $R \in \operatorname{add} T$, then g is a right $\operatorname{add} T$-approximation of τM. Moreover, if f is minimal then g is minimal.

Proof. Given any map $h\colon T \to \tau M$, the diagram

$$\begin{tikzcd}
\tau^{-1}T \arrow[r] \arrow[d, "\tau^{-1}h"'] & 0 \arrow[r] \arrow[d] & T \arrow[r, "1_T"] & T \arrow[d, "h"] \\
M \arrow[r, "f"] & L \arrow[r] & R \arrow[r, "g"] & \tau M
\end{tikzcd}$$

commutes, since $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T, L) = 0$ by rigidity of T. Thus there exists a morphism $T \to R$ completing the diagram above to a morphism of triangles in $\mathcal{C}$, and hence h factors through g.

Now assume f is minimal. Let $h\colon R \to R$ be such that $gh = g$, so we have the commutative diagram

$$\begin{tikzcd}
\tau^{-1}R \arrow[r, "\tau^{-1}g"] \arrow[d, "\tau^{-1}h"'] & M \arrow[r, "f"] \arrow[d, "1_M"] & L \arrow[r] & R \arrow[d, "h"] \\
\tau^{-1}R \arrow[r, "\tau^{-1}g"] & M \arrow[r, "f"] & L \arrow[r] & R
\end{tikzcd}$$

which can be completed to a morphism of triangles by $h'\colon L \to L$. Since f is minimal, h' is an isomorphism. Thus by [Hap88, Prop. 1.2(c)], the diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "1_M"] & L \arrow[r] \arrow[d, "h'"] & R \arrow[r, "g"] \arrow[d, "h"] & \tau M \arrow[d, "1_{\tau M}"] \\
M \arrow[r, "f"] & L \arrow[r] & R \arrow[r, "g"] & \tau M
\end{tikzcd}$$

is an isomorphism of triangles, so in particular h is an isomorphism and g is minimal. □

Corollary 4.13. *Let $\mathcal{E}$ be a Frobenius category, let $M \in \mathcal{E}$, and let $T \in \mathcal{E}$ be rigid. Pick a minimal left* $\mathrm{add}\,T$*-approximation $f\colon M \to L$ and a minimal right* $\mathrm{add}\,T$*-approximation $g\colon R \to \Omega^{-1}M$. Write $C = \mathrm{coker}\,f$ and $K = \ker g$.*

(i) If f is a monomorphism, and $C \in \mathrm{add}\,T$, then $C \cong R$ in $\underline{\mathcal{E}}$.

(ii) If g is an epimorphism, and $K \in \mathrm{add}\,T$, then $K \cong L$ in $\underline{\mathcal{E}}$.

Proof. We prove only (i), the proof of (ii) being dual. Since f is a monomorphism, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f} L \longrightarrow C \longrightarrow 0,$$

inducing an exact triangle

$$M \xrightarrow{f} L \longrightarrow C \xrightarrow{h} \Omega^{-1}M$$

in $\underline{\mathcal{E}}$. By Proposition 4.3, $f\colon M \to L$ is a minimal left $\mathrm{add}\,T$-approximation in $\underline{\mathcal{E}}$. Since $C \in \mathrm{add}\,T$ by assumption, Theorem 4.12 implies that h is a minimal right $\mathrm{add}\,T$-approximation of $\Omega^{-1}M$ in $\underline{\mathcal{E}}$, recalling that Ω^{-1} is the shift functor on $\underline{\mathcal{E}}$. By Proposition 4.3 again, g is also such an approximation, so $C \cong R$ in $\underline{\mathcal{E}}$, as minimal approximations are unique up to isomorphism. □

If, in the setting of Corollary 4.13, we also assume that T is maximal rigid, then f and g are always monomorphisms, and C and K lie in $\operatorname{add} T$ if either M is rigid or T is cluster-tilting, by Proposition 4.10. In either situation, Proposition 4.11 implies that the indecomposable summands of C are precisely the non-projective-injective indecomposable summands of R, and similarly for K and L.

Theorem 4.14. *Let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category with shift τ, and let $T \in \mathcal{C}$ be cluster-tilting. For each $M, N \in \mathcal{C}$, let $\operatorname{Hom}_{\mathcal{C}}^{T}(M,N)$ denote the subspace of $\operatorname{Hom}_{\mathcal{C}}(M,N)$ consisting of maps factoring through T, and write*

$$\underline{\operatorname{Hom}}_T(M,N) = \operatorname{Hom}_{\mathcal{C}}(M,N)/\operatorname{Hom}_{\mathcal{C}}^{T}(M,N).$$

Then for any $M, N \in \mathcal{T}$, there is a duality

$$\operatorname{D}\operatorname{Hom}_{\mathcal{C}}^{T}(M,\tau N) \cong \underline{\operatorname{Hom}}_T(N,\tau M).$$

The same conclusion holds for T maximal rigid, if one assumes additionally that M and N are rigid.

Proof. Under either set of assumptions, any minimal left $\operatorname{add} T$-approximation $f \colon M \to L$ gives rise to a triangle

$$M \xrightarrow{f} L \longrightarrow R \xrightarrow{g} \tau M,$$

in which g is a minimal right $\operatorname{add} T$-approximation, by Theorem 4.12. Applying the functor $\operatorname{Hom}_{\mathcal{C}}(-,\tau N)$ to this triangle produces the exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(R,\tau N) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(L,\tau N) \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{C}}(M,\tau N).$$

Since f is a minimal left $\operatorname{add} T$-approximation, the image of f^* is $\operatorname{Hom}_{\mathcal{C}}^{T}(M,\tau N)$, so after taking duals we get the exact sequence

$$0 \longrightarrow \operatorname{D}\operatorname{Hom}_{\mathcal{C}}^{T}(M,\tau N) \longrightarrow \operatorname{D}\operatorname{Hom}_{\mathcal{C}}(L,\tau N) \longrightarrow \operatorname{D}\operatorname{Hom}_{\mathcal{C}}(R,\tau N).$$

Applying $\operatorname{Hom}_{\mathcal{C}}(N,-)$ to our original triangle gives an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(N,R) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{C}}(N,\tau M) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(N,\tau L) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(N,\tau R).$$

Since g is a minimal right $\operatorname{add} T$-approximation, the image of g_* is $\operatorname{Hom}_{\mathcal{C}}^{T}(N,\tau M)$, and so we can truncate to the exact sequence

$$0 \longrightarrow \underline{\operatorname{Hom}}_T(N,\tau M) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(N,\tau L) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(N,\tau R).$$

We have a natural isomorphism of functors $\operatorname{D}\operatorname{Hom}_{\mathcal{C}}(-,\tau N) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(N,\tau(-))$ since $\mathcal{C}$

is 2-Calabi–Yau. From this we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{D}\operatorname{Hom}^T_{\mathcal{C}}(M,\tau N) \arrow[r] & \mathrm{D}\operatorname{Hom}_{\mathcal{C}}(L,\tau N) \arrow[r] \arrow[d] & \mathrm{D}\operatorname{Hom}_{\mathcal{C}}(R,\tau N) \arrow[d] \\
0 \arrow[r] & \underline{\operatorname{Hom}}_T(N,\tau M) \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(N,\tau L) \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(N,\tau R)
\end{tikzcd}$$

in which the vertical maps are isomorphisms, yielding the required isomorphism

$$\mathrm{D}\operatorname{Hom}^T_{\mathcal{C}}(M,\tau N) \xrightarrow{\sim} \underline{\operatorname{Hom}}_T(N,\tau M). \qquad \square$$

Corollary 4.15. *Under the assumptions of Theorem 4.14,* $\operatorname{Hom}^T_{\mathcal{C}}(M,\tau N)$ *is the annihilator of* $\operatorname{Hom}^T_{\mathcal{C}}(N,\tau M)$ *under the duality* $\operatorname{Hom}_{\mathcal{C}}(N,\tau M) = \mathrm{D}\operatorname{Hom}_{\mathcal{C}}(M,\tau N)$*.*

Another interpretation of Theorem 4.14 is that a cluster-tilting object T determines a Lagrangian subspace, or polarisation

$$\operatorname{Hom}^T_{\mathcal{C}}(M,\tau N) \oplus \operatorname{Hom}^T_{\mathcal{C}}(N,\tau M)$$

of the symplectic vector space

$$\operatorname{Hom}_{\mathcal{C}}(M,\tau N) \oplus \operatorname{Hom}_{\mathcal{C}}(N,\tau M),$$

where the symplectic form is induced from the Calabi–Yau duality of the two summands.

4.3 Mutation, Global Dimension and Derived Equivalence

We now focus our attention more closely on the category $\operatorname{Sub} Q_J$. We begin with a precise statement of the mutation property for maximal rigid objects in $\operatorname{Sub} Q_J$, discussed earlier in Section 2.6.

Proposition 4.16 ([GLS08, Prop. 8.1])**.** *Let $T \in \operatorname{Sub} Q_J$ be maximal rigid and let M be an indecomposable non-projective-injective summand of T. Then there exists an indecomposable object $M' \in \operatorname{Sub} Q_J$, unique up to isomorphism, such that $(T/M) \oplus M'$ is maximal rigid. Moreover,* $\operatorname{ext}^1_\Pi(M,M') = \operatorname{ext}^1_\Pi(M',M)$*, and for any non-split extensions*

$$0 \longrightarrow M \xrightarrow{f} X \xrightarrow{g} M' \longrightarrow 0$$
$$0 \longrightarrow M' \xrightarrow{h} Y \xrightarrow{i} M \longrightarrow 0$$

the maps f,g,h,i are all $\operatorname{add}(T/M)$*-approximations of M or M'.*

Note that while [GLS08, Prop. 8.1] is stated for complete rigid modules, these coincide with maximal rigid modules in $\operatorname{Sub} Q_J$ by Proposition 4.6. We call the pair $\{M, M'\}$ appearing in Proposition 4.16 the *exchange pair* associated to T/M. The mutation property allows us to deduce various homological properties of the endomorphism algebras $\mathrm{End}_\Pi(T)^{\mathrm{op}}$ for $T \in \mathrm{Sub}\, Q_J$ maximal rigid.

Lemma 4.17 (cf. [GLS10, Prop. 13.5])**.** *Let T be a maximal rigid object in $\mathrm{Sub}\, Q_J$. If the quiver Γ_T of $\mathrm{End}_\Pi(T)^{\mathrm{op}}$ has no loops, then $\mathrm{gl.dim}\,\mathrm{End}_\Pi(T)^{\mathrm{op}} \leq 3$.*

Proof. Write $A = \mathrm{End}_\Pi(T)^{\mathrm{op}}$ for brevity. If M is an indecomposable summand of T, write S_M for the simple top of the projective A-module $\mathrm{Hom}_\Pi(T, M)$. As A is finite dimensional, to show that $\mathrm{gl.dim}\, A \leq 3$ it suffices to show that $\mathrm{p.dim}_A\, S_M \leq 3$ for each indecomposable summand M of T.

Assume that M is an indecomposable summand of T that is not projective-injective. Then M is part of an exchange pair $\{M, M'\}$ associated to T/M, and $\mathrm{ext}^1_\Pi(M, M') = \mathrm{ext}^1_\Pi(M', M) = 1$. Let

$$0 \longrightarrow M \longrightarrow X \longrightarrow M' \longrightarrow 0,$$
$$0 \longrightarrow M' \longrightarrow Y \xrightarrow{f} M \longrightarrow 0$$

be non-split extensions and recall from Proposition 4.16 that $X, Y \in \mathrm{add}(T/M) \subseteq \mathrm{add}\, T$. Applying the functor $\mathrm{Hom}_\Pi(T, -)\colon \mathrm{Sub}\, Q_J \to \mathrm{mod}\, A$, we obtain

$$0 \longrightarrow \mathrm{Hom}_\Pi(T, M) \longrightarrow \mathrm{Hom}_\Pi(T, X) \longrightarrow \mathrm{Hom}_\Pi(T, M') \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Hom}_\Pi(T, M') \longrightarrow \mathrm{Hom}_\Pi(T, Y) \xrightarrow{f_*} \mathrm{Hom}_\Pi(T, M) \longrightarrow \mathrm{Ext}^1_\Pi(T, M') \longrightarrow 0.$$

As M is the only summand of T not appearing as a summand of the maximal rigid object $T/M \oplus M'$, we have $\mathrm{Ext}^1_\Pi(T, M') = \mathrm{Ext}^1_\Pi(M, M')$. Thus

$$\dim \mathrm{coker}\, f_* = \mathrm{ext}^1_\Pi(M, M') = 1.$$

Moreover, as $\mathrm{Hom}_\Pi(T, M)$ is the indecomposable projective A-module with top S_M, we have $\mathrm{coker}\, f_* \cong S_M$. Combining the above sequences yields the projective resolution

$$0 \longrightarrow \mathrm{Hom}_\Pi(T, M) \longrightarrow \mathrm{Hom}_\Pi(T, X) \longrightarrow \mathrm{Hom}_\Pi(T, Y) \longrightarrow \mathrm{Hom}_\Pi(T, M) \longrightarrow S_M \longrightarrow 0$$

of S_M, meaning $\mathrm{p.dim}_A\, S_M \leq 3$.

Now assume F is an indecomposable projective-injective summand of T, so $S = \mathrm{top}\, F$ is a simple Π-module. Let $U = \ker(F \to \mathrm{top}\, F)$, so $F/U \cong S$. Applying $\mathrm{Hom}_\Pi(U, -)$ to the exact sequence

$$0 \longrightarrow U \longrightarrow F \longrightarrow S \longrightarrow 0$$

we obtain

$$0 \longrightarrow \mathrm{Hom}_\Pi(U, U) \longrightarrow \mathrm{Hom}_\Pi(U, F) \longrightarrow \mathrm{Hom}_\Pi(U, S) \longrightarrow \mathrm{Ext}^1_\Pi(U, U) \longrightarrow 0.$$

As there are no loops in the quiver of Π, there are no summands of $\mathrm{top}(U)$ isomorphic to S. It follows that $\operatorname{Hom}_\Pi(U,S)=0$, and so $\operatorname{Ext}^1_\Pi(U,U)=0$ and U is rigid. Pick a projective cover P of U. As S is not a summand of $\operatorname{top}(U)$, the projective module F is not a summand of P. Thus $P \in \operatorname{add}(T/F)$, so any minimal right $\operatorname{add}(T/F)$-approximation $f\colon T' \to U$ is an epimorphism. From such an approximation we obtain an exact sequence

$$0 \longrightarrow V \longrightarrow T' \xrightarrow{f} U \longrightarrow 0.$$

By Lemma 4.9, the module $T/F \oplus V$ is rigid, so by maximality of T, every summand of V is either in $\operatorname{add}(T/F)$ or is isomorphic to F, and so $V \in \operatorname{add} T$. Let $i\colon U \to F$ be the inclusion, and let $h = if$. Thus we obtain the exact sequence

$$0 \longrightarrow V \longrightarrow T' \xrightarrow{h} F.$$

Applying $\operatorname{Hom}_\Pi(T,-)$ yields

$$0 \longrightarrow \operatorname{Hom}_\Pi(T,V) \longrightarrow \operatorname{Hom}_\Pi(T,T') \xrightarrow{h_*} \operatorname{Hom}_\Pi(T,F) \longrightarrow \operatorname{coker} h_* \longrightarrow 0,$$

which is a projective resolution of $\operatorname{coker} h_*$. Note that

$$\operatorname{Hom}_\Pi(T,F) = \operatorname{Hom}_\Pi(T/F,F) \oplus \operatorname{Hom}_\Pi(F,F).$$

As F is projective, no map $g\colon T/F \to F$ can be an epimorphism, so for any such g there exists $g'\colon T/F \to U$ such that $g = ig'$. Thus there exists $g''\colon T/F \to T'$ such that $g' = fg''$, as f is a right $\operatorname{add}(T/F)$-approximation. We obtain the commutative diagram

$$\begin{tikzcd} T/F \arrow[r, "g"] \arrow[d, "g''"'] \arrow[dr, "g'"] & F \\ T' \arrow[r, "f"'] & U \arrow[u, "i"'] \end{tikzcd}$$

from which it follows that $g = ig' = ifg'' = hg''$, so g factors through h. As there is no loop at F in Γ_T, every non-isomorphism $F \to F$ factors through T/F, and so factors through h by the previous argument. Thus $\dim \operatorname{coker} h_* = 1$, and so $\operatorname{coker} h_* \cong S_F$. It follows that $\operatorname{p.dim}_A S_F \le 2$, completing the proof. □

Lemma 4.18. *Let $M, T \in \operatorname{mod}\Pi$, let $T' \in \operatorname{add} T$, and let $A = \operatorname{End}_\Pi(T)^{\mathrm{op}}$. Then the map*

$$i_{T',M,T}\colon \operatorname{Hom}_\Pi(T',M) \to \operatorname{Hom}_A(\operatorname{Hom}_\Pi(T,T'), \operatorname{Hom}_\Pi(T,M))$$

taking $\varphi\colon T' \to M$ to $\operatorname{Hom}(T,\varphi)\colon \operatorname{Hom}_\Pi(T,T') \to \operatorname{Hom}_\Pi(T,M)$ is an isomorphism.

Proof. Let $T' = \bigoplus_{i=1}^r T'_i$, where T'_i is indecomposable for each i. Then

$$\operatorname{Hom}_\Pi(T',N) \cong \bigoplus_{i=1}^r \operatorname{Hom}_\Pi(T'_i,N)$$

and

$$\mathrm{Hom}_A(\mathrm{Hom}_\Pi(T,T'),\mathrm{Hom}_\Pi(T,N)) \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_A(\mathrm{Hom}_\Pi(T,T'_i),\mathrm{Hom}_\Pi(T,N)),$$

so without loss of generality we may assume T' is indecomposable. In particular, T' is a summand of T; denote by $i\colon T' \to T$ and $\pi\colon T \to T'$ the inclusion and projection. Now let $\varphi,\psi\colon T' \to N$ and assume $\varphi f = \psi f$ for all $f\colon T \to T'$. In particular, $\varphi\pi = \psi\pi$, so $\varphi = \psi$, and $i_{T',N,T}$ is injective. Now let $\alpha\colon \mathrm{Hom}(T,T') \to \mathrm{Hom}(T,N)$ be a homomorphism of A-modules, and let $f\colon T \to T'$. We have $\pi i = 1_{T'}$, so $\pi i f = f$. Thus

$$\alpha(f) = \alpha(\pi i f) = \alpha(\pi) i f,$$

and so $\alpha = \mathrm{Hom}_\Pi(T, \alpha(\pi)i)$. Therefore $i_{T',N,T}$ is an isomorphism. $\square$

Definition 4.19. Let A be an algebra and let $T \in \operatorname{mod} A$. Then T is called a *tilting module* if there exists $d \geq 1$ such that

(i) $\operatorname{p.dim}_A T \leq d$,

(ii) $\mathrm{Ext}^i_A(T,T) = 0$ for all $i \geq 1$, and

(iii) there exists an exact sequence

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_d \longrightarrow 0$$

of A-modules with $T_i \in \operatorname{add} T$ for all i.

We call T a *classical tilting module* if conditions (i)–(iii) hold for $d = 1$.

Proposition 4.20 (cf. [GLS10, Thm. 10.2])**.** *Let T_1 and T_2 be maximal rigid modules in $\operatorname{Sub} Q_J$, and write $A_i = \mathrm{End}_\Pi(T_i)^{\mathrm{op}}$. Then $\mathrm{Hom}_\Pi(T_1,T_2)$ is a classical tilting module over A_1, and*

$$\mathrm{End}_{A_1}(\mathrm{Hom}_\Pi(T_1,T_2))^{\mathrm{op}} \cong A_2.$$

It follows that A_1 and A_2 are derived equivalent.

Proof. Let $\mathbf{T} = \mathrm{Hom}_\Pi(T_1,T_2)$. Let $f\colon T'_1 \to T_2$ be a right $\operatorname{add} T_1$-approximation, which is an epimorphism as T_1 is a generator for $\operatorname{Sub} Q_J$. So we have the exact sequence

$$0 \longrightarrow T''_1 \xrightarrow{g} T'_1 \xrightarrow{f} T_2 \longrightarrow 0.$$

Since T_2 is rigid, $T''_1 \in \operatorname{add} T_1$ by Proposition 4.10. Applying $\mathrm{Hom}_\Pi(T_1,-)$ gives a projective resolution

$$0 \longrightarrow \mathrm{Hom}_\Pi(T_1,T''_1) \xrightarrow{g_*} \mathrm{Hom}_\Pi(T_1,T'_1) \xrightarrow{f_*} \mathbf{T} \longrightarrow 0$$

of $\mathbf{T}$, so $\operatorname{p.dim}_{A_1} \mathbf{T} \leq 1$, as required. This also shows that $\operatorname{Ext}^i_{A_1}(\mathbf{T}, \mathbf{T}) = 0$ for all $i \geq 2$. Now applying $\operatorname{Hom}_{A_1}(-, \mathbf{T})$ and truncating, we get the exact sequence

$$\operatorname{Hom}_{A_1}(\operatorname{Hom}_\Pi(T_1, T_1'), \mathbf{T}) \xrightarrow{G} \operatorname{Hom}_{A_1}(\operatorname{Hom}_\Pi(T_1, T_1''), \mathbf{T}) \longrightarrow \operatorname{Ext}^1_{A_1}(\mathbf{T}, \mathbf{T}) \longrightarrow 0 \tag{4.1}$$

because $\operatorname{Hom}_\Pi(T_1, T_1')$ is a projective A_1-module. The diagram

$$\begin{tikzcd}
\operatorname{Hom}_\Pi(T_1', T_2) \arrow[r, "\operatorname{Hom}_\Pi(g,T_2)"] \arrow[d, "i_{T_1',T_2,T_1}"'] & \operatorname{Hom}_\Pi(T_1'', T_2) \arrow[d, "i_{T_1'',T_2,T_1}"] \\
\operatorname{Hom}_{A_1}(\operatorname{Hom}_\Pi(T_1, T_1'), \mathbf{T}) \arrow[r, "G"'] & \operatorname{Hom}_{A_1}(\operatorname{Hom}_\Pi(T_1, T_1''), \mathbf{T})
\end{tikzcd}$$

is commutative, and the vertical maps are isomorphisms by Lemma 4.18. Moreover, $\operatorname{Hom}_\Pi(g, T_2)$ is surjective because T_2 is rigid. Hence G is surjective, and so we obtain from Sequence 4.1 that $\operatorname{Ext}^1_{A_1}(\mathbf{T}, \mathbf{T}) = 0$.

Similarly, since T_1 is rigid, Proposition 4.10 gives a short exact sequence

$$0 \longrightarrow T_1 \longrightarrow T_2' \longrightarrow T_2'' \longrightarrow 0$$

with $T_2', T_2'' \in \operatorname{add} T_2$. Apply $\operatorname{Hom}_\Pi(T_1, -)$, to obtain the exact sequence

$$0 \longrightarrow A_1 \longrightarrow \operatorname{Hom}(T_1, T_2') \longrightarrow \operatorname{Hom}(T_1, T_2'') \longrightarrow 0$$

of A_1-modules, which is of the form required by Definition 4.19(iii).

Now write $\xi\colon \operatorname{End}_\Pi(T_2) \to \operatorname{End}_{A_1}(\mathbf{T})$ for the map defined by $\xi(h)(h'') = hh''$ for all $h \in \operatorname{End}_\Pi(T_2)$ and $h''\colon T_1 \to T_2$. Write $F = \operatorname{Hom}_{A_1}(f_*, \mathbf{T})$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{End}_{A_1}(\mathbf{T}) \arrow[r, "F"] & \operatorname{Hom}_{A_1}(\operatorname{Hom}_\Pi(T_1, T_1'), \mathbf{T}) \arrow[r, "G"] & \operatorname{Hom}_{A_1}(\operatorname{Hom}_\Pi(T_1, T_1''), \mathbf{T}) \\
0 \arrow[r] & \operatorname{End}_\Pi(T_2) \arrow[r, "\operatorname{Hom}(f,T_2)"'] \arrow[u, "\xi"] & \operatorname{Hom}_\Pi(T_1', T_2) \arrow[r, "\operatorname{Hom}(g,T_2)"'] \arrow[u, "i_{T_1',T_2,T_1}"'] & \operatorname{Hom}_\Pi(T_1'', T_2) \arrow[u, "i_{T_1'',T_2,T_1}"']
\end{tikzcd}$$

commutes and has exact rows. The maps i_{T_1',T_2,T_1} and i_{T_1'',T_2,T_1} are isomorphisms by Lemma 4.18. Thus ξ is an isomorphism, and so $\operatorname{End}_{A_1}(\mathbf{T})^{\mathrm{op}} \cong \operatorname{End}_\Pi(T_2)^{\mathrm{op}} = A_2$. The final conclusion is then [Hap87, §1.7]. $\square$

Proposition 4.21 (cf. [GLS10, Thm. 13.6])**.** *Assume that* $\operatorname{Sub} Q_J$ *has a maximal rigid object* T_0 *such that* Γ_{T_0} *has no loops. Then* $\operatorname{gl.dim} \operatorname{End}_\Pi(T)^{\mathrm{op}} \leq 3$ *for any maximal rigid object* T *of* $\operatorname{Sub} Q_J$*, so all maximal rigid objects of* $\operatorname{Sub} Q_J$ *are non-commutative resolutions, as in Definition 3.3.*

Proof. Write $A_0 = \operatorname{End}_\Pi(T_0)^{\mathrm{op}}$, and $A = \operatorname{End}_\Pi(T)^{\mathrm{op}}$. By Lemma 4.17, $\operatorname{gl.dim} A_0 \leq 3$. The algebras A_0 and A are derived equivalent by Proposition 4.20, so $\operatorname{gl.dim} A < \infty$ by [Hap87, §1.4]. By a result of Igusa [Igu90, Thm. 4.5], the quiver Γ_T of A has no loops, and so $\operatorname{gl.dim} A \leq 3$ by applying Lemma 4.17 again. $\square$

4.4 Maximal Rigid Versus Cluster-Tilting

It is immediate from the definition that any cluster-tilting object is maximal rigid. In this section, we recall a result of Zhou–Zhu [ZZ11, Thm. 2.6], see also [BIRS09, Thm. II.1.8], showing that the converse holds in stably 2-Calabi–Yau Frobenius categories with cluster-tilting objects. The result in [ZZ11] is stated for 2-Calabi–Yau triangulated categories, but here we lift the statement, and its proof, to any stably 2-Calabi–Yau Frobenius category.

Theorem 4.22 (cf. [ZZ11, Thm. 2.6], [BIRS09, Thm. II.1.8]). *Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category, and assume that $\mathcal{E}$ admits a cluster-tilting object T. Then every maximal rigid object of $\mathcal{E}$ is cluster-tilting.*

Proof. Let $R \in \mathcal{E}$ be maximal rigid, and assume $X \in \mathcal{E}$ satisfies $\operatorname{Ext}^1_{\mathcal{E}}(X,R) = 0$; as $\mathcal{E}$ is stably 2-Calabi–Yau this is equivalent to $\operatorname{Ext}^1_{\mathcal{E}}(R,X) = 0$. Let $g\colon T_0 \to X$ be a minimal right $\operatorname{add} T$-approximation of X, so we have a short exact sequence

$$0 \longrightarrow T_1 \xrightarrow{f} T_0 \xrightarrow{g} X \longrightarrow 0$$

with $T_1 \in \operatorname{add} T$, by Proposition 4.10. Let $\alpha_0\colon T_0 \to R'$ be a minimal left $\operatorname{add} R$-approximation of T_0. Since T_0 is rigid, we have a short exact sequence

$$0 \longrightarrow T_0 \xrightarrow{\alpha_0} R' \longrightarrow R_0 \longrightarrow 0$$

with $R_0 \in \operatorname{add} R$, again by Proposition 4.10. Since $\operatorname{Ext}^1_{\mathcal{E}}(X,R) = 0$, there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{E}}(X,R) \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{E}}(T_0,R) \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{E}}(T_1,R) \longrightarrow 0.$$

In particular, f^* is surjective. So for any $\varphi_1\colon T_1 \to R$, there exists $\varphi_0\colon T_0 \to R$ with $\varphi_1 = \varphi_0 f$. Moreover, φ_0 must factor through the approximation $\alpha_0\colon T_0 \to R'$, so there is $\psi\colon R' \to R$ with $\varphi_1 = \psi\alpha_0 f$. This means that $\alpha_1 := \alpha_0 f$ is a left R-approximation of T_1. Since T_1 is rigid, by Proposition 4.10 we have an exact sequence

$$0 \longrightarrow T_1 \xrightarrow{\alpha_1} R' \longrightarrow R_1 \longrightarrow 0$$

with $R_1 \in \operatorname{add} R$. This sequence fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_1 \arrow[r, "\alpha_1"] \arrow[d, "f"] & R' \arrow[r] \arrow[d, "1_{R'}"] & R_1 \arrow[r] \arrow[d, "f'"] & 0 \\
0 \arrow[r] & T_0 \arrow[r, "\alpha_0"'] & R' \arrow[r] & R_0 \arrow[r] & 0
\end{tikzcd}$$

with exact rows, to which we may apply the snake lemma to see that

$$\ker f' \cong \operatorname{coker} f \cong X.$$

Thus we have an exact sequence

$$0 \longrightarrow X \longrightarrow R_1 \xrightarrow{f'} R_0 \longrightarrow 0,$$

which splits as $\operatorname{Ext}^1_{\mathcal{E}}(R_0, X) = 0$. It follows that $X \in \operatorname{add} R$, and so R is cluster-tilting. □

In [BIRS09, §III.2], Buan–Iyama–Reiten–Scott construct cluster-tilting objects for the categories $\mathcal{C}_w$ described in Definition 2.20. By Proposition 2.21, $\mathcal{C}_{w_J} = \operatorname{Sub} Q_J$, so each $\operatorname{Sub} Q_J$ admits a cluster-tilting object. Since $\operatorname{Sub} Q_J$ is stably 2-Calabi–Yau by Proposition 2.15, we may apply Theorem 4.22 to see that every maximal rigid object of $\operatorname{Sub} Q_J$ is cluster-tilting.

Alternatively, in [GLS08, §9], building on results of [GLS06, §7] and [BFZ05, §2], Geiß–Leclerc–Schröer give a description of most of the quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ for a particular maximal rigid object T of $\operatorname{mod} \Pi$ (cf. Section 6.1). It is immediate from the construction that this part of the quiver has no loops. The only missing arrows are those between vertices corresponding to the indecomposable projective-injective summands of T, which are the indecomposable projective Π-modules. Since the quiver of Π has no loops, the quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ has no loops at these vertices either. Thus by Lemma 4.17, $\operatorname{gl.dim} \operatorname{End}_\Pi(T)^{\mathrm{op}} \leq 3$. Now a result of Iyama [Iya07a, Thm. 5.1(c)] implies that T is cluster-tilting in $\operatorname{mod} \Pi$. By [GLS08, Prop. 7.4], the minimal left $\operatorname{Sub} Q_J$-approximation of T is cluster-tilting in $\operatorname{Sub} Q_J$, and so we can again apply Theorem 4.22 to see that every maximal rigid object of $\operatorname{Sub} Q_J$ is cluster-tilting.

From now on, we will usually refer to cluster-tilting objects in $\operatorname{Sub} Q_J$, rather than using the equivalent terminology of complete rigid or maximal rigid, but we will apply results referring to complete rigid or maximal rigid objects without comment.

CHAPTER 5

INTERNAL CALABI–YAU CONDITIONS

If we are to produce Frobenius categories modelling the combinatorics of cluster algebras with frozen variables, we require tools for readily producing Frobenius categories admitting cluster-tilting objects. In this chapter, we show how such categories arise from algebras satisfying certain Calabi–Yau conditions with respect to an idempotent. Some variations on this theme already appear in papers by Keller–Reiten [KR07] and Amiot–Reiten–Todorov [ART11], in which it is observed that some of these conditions are satisfied by endomorphism algebras of cluster-tilting objects in stably 2-Calabi–Yau Frobenius categories.

5.1 Internally d-Calabi–Yau Algebras

Definition 5.1. Let A be a $\mathbb{C}$-algebra, and let e be an idempotent of A. We say A is *internally d-Calabi–Yau* with respect to e if

(i) $\operatorname{gl.dim} A \leq d$, and

(ii) there is a functorial duality

$$\mathrm{D}\operatorname{Ext}^i_A(M,N) = \operatorname{Ext}^{d-i}_A(N,M)$$

for all $N \in \operatorname{mod} A$ and all finite dimensional $M \in \operatorname{mod} A/\langle e\rangle$, thought of as A-modules via the quotient map.

Remark 5.2. An algebra A is d-Calabi–Yau if and only if it is internally d-Calabi–Yau with respect to 0. It is internally d-Calabi–Yau with respect to 1 if and only if $\operatorname{gl.dim} A \leq d$. We also note that if A is internally d-Calabi–Yau with respect to e then it is internally d-Calabi–Yau with respect to $e+e'$ for any idempotent $e' \in A$.

Remark 5.3. A finite dimensional algebra A is internally d-Calabi–Yau with respect to e if and only if the same is true of A^{op}; since A is finite dimensional, it is (left and right)

Noetherian, so $\operatorname{gl.dim} A = \operatorname{gl.dim} A^{\mathrm{op}}$ [Wei94, Ex. 4.1.1], and D induces an equivalence $\operatorname{mod} A^{\mathrm{op}} \xrightarrow{\sim} (\operatorname{mod} A)^{\mathrm{op}}$ yielding the required functorial duality for A^{op}. Definition 5.1 is not necessarily left-right symmetric in this way if A is infinite dimensional, so in Section 5.2 we describe a way of correcting this.

Example 5.4. Consider the ice quiver with potential (Q, F, W), where

$$Q = \begin{tikzcd} \boxed{1} \arrow[dr, "\alpha_1"'] & & \boxed{3} \arrow[ll, dashed, "\alpha_3"'] \\ & 2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

the frozen subquiver is the full subquiver on vertices 1 and 3, indicated by boxed vertices and a dashed arrow, and $W = \alpha_3\alpha_2\alpha_1$; cf. Example 3.20. It is straightforward to check (for example, by Theorem 5.24 below) that the frozen Jacobian algebra $\mathcal{J}(Q, F, W)$ is internally 3-Calabi–Yau with respect to $e_1 + e_3$ (although the ordinary Jacobian algebra $\mathcal{J}(Q, W)$ is not 3-Calabi–Yau).

We now explain a result of Keller–Reiten [KR07, §4] which implies that if T is a cluster-tilting object in $\operatorname{Sub} Q_J$, then $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ is internally 3-Calabi–Yau with respect to the projection onto the projective-injective summands of T, thus providing us with a large class of examples of such algebras.

Proposition 5.5 (cf. [KR07, §4])**.** *Let T be a cluster-tilting object of $\operatorname{Sub} Q_J$, and let e be the idempotent of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ given by projection onto the projective-injective summands of T. Then $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ is internally 3-Calabi–Yau with respect to e.*

Proof. Write $A = \operatorname{End}_\Pi(T)^{\mathrm{op}}$. We have $\operatorname{gl.dim} A \leq 3$ by Proposition 4.21, so it only remains to check the necessary duality. Let $M \in \operatorname{mod} A/\langle e\rangle$; since A is finite dimensional, so is M. Choose a projective presentation

$$\operatorname{Hom}_\Pi(T, T_1) \xrightarrow{f_*} \operatorname{Hom}_\Pi(T, T_0) \longrightarrow M \longrightarrow 0$$

of M, and let $p\colon F_0 \to T_0$ be a projective cover. As $M \in \operatorname{mod} A/\langle e\rangle$, and e is the idempotent given by projection onto the projective-injective summands of T, we have

$$\operatorname{Hom}_A(\operatorname{Hom}_\Pi(T, F_0), M) = 0.$$

It follows that the diagram

$$\begin{tikzcd} \operatorname{Hom}_A(\operatorname{Hom}_\Pi(T, F_0), \operatorname{Hom}_\Pi(T, T_1)) \arrow[r] \arrow[d] & \operatorname{Hom}_A(\operatorname{Hom}_\Pi(T, F_0), \operatorname{Hom}_\Pi(T, T_0)) \arrow[r] \arrow[d] & 0 \\ \operatorname{Hom}_\Pi(F_0, T_1) \arrow[r] & \operatorname{Hom}_\Pi(F_0, T_0) \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows; the horizontal maps are induced from f, and the vertical maps are isomorphisms from Yoneda's lemma. This diagram shows that $p = fp'$ for some

$p' \colon P_0 \to T_1$. It then follows from the commutative diagram

$$\begin{tikzcd} T_1 \oplus P_0 \arrow[r, "(f\ p)"] \arrow[d, "(1\ p')"'] & T_0 \\ T_1 \arrow[ur, "f"'] & \end{tikzcd}$$

that f is surjective, since p is. Since $\operatorname{Sub} Q_J$ is closed under submodules, the kernel K of f is in $\operatorname{Sub} Q_J$, and we have an exact sequence

$$0 \longrightarrow K \longrightarrow T_1 \xrightarrow{f} T_0 \longrightarrow 0.$$

As $\operatorname{Ext}^1_\Pi(T, T_1) = 0$, applying $\operatorname{Hom}_\Pi(T, -)$ to this sequence shows that $M \cong \operatorname{Ext}^1_\Pi(T, K)$ as an A-module.

Pick a right $\operatorname{add} T$-approximation $T_2 \to K$ fitting into the exact sequence

$$0 \longrightarrow T_3 \longrightarrow T_2 \longrightarrow K \longrightarrow 0$$

with $T_3 \in \operatorname{add} T$ by Proposition 4.10. As $\operatorname{Ext}^1_\Pi(T, T_3) = \operatorname{Ext}^1_\Pi(T, T_1) = 0$, applying $\operatorname{Hom}_\Pi(T, -)$ and combining the previous two sequences gives a projective resolution

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_\Pi(T, T_3) \arrow[r] & \operatorname{Hom}_\Pi(T, T_2) \arrow[r] & \operatorname{Hom}_\Pi(T, T_1) \arrow[r] & \operatorname{Hom}_\Pi(T, T_0) \arrow[d] & \\ & & & & \operatorname{Ext}^1_\Pi(T, K) \arrow[r] & 0 \end{tikzcd}$$

of $M \cong \operatorname{Ext}^1_\Pi(T, K)$. Now we have

$$\begin{aligned} \operatorname{Hom}_{\operatorname{per} A}(M, M[3]) &= \operatorname{coker}(\operatorname{Hom}_A(\operatorname{Hom}_\Pi(T, T_2), M) \to \operatorname{Hom}_A(\operatorname{Hom}_\Pi(T, T_3), M)) \\ &= \operatorname{coker}(\operatorname{Ext}^1_\Pi(T_2, K) \to \operatorname{Ext}^1_\Pi(T_3, K)) =: C \end{aligned}$$

by Yoneda's lemma applied to the functor $\operatorname{Ext}^1_\Pi(-, K)$ on $\operatorname{add} T$. Applying $\operatorname{Hom}_\Pi(-, K)$ to the exact sequence

$$0 \longrightarrow T_3 \longrightarrow T_2 \longrightarrow K \longrightarrow 0$$

yields

$$\operatorname{Ext}^1_\Pi(T_2, K) \longrightarrow \operatorname{Ext}^1_\Pi(T_3, K) \longrightarrow \operatorname{Ext}^2_\Pi(K, K),$$

and so there is an inclusion $C \hookrightarrow \operatorname{Ext}^2_\Pi(K, K)$. Now since $\operatorname{Sub} Q_J$ is stably 2-Calabi–Yau, there is a functorial trace map $\phi \colon \operatorname{Ext}^2_\Pi(K, K) = \underline{\operatorname{Hom}}_J(K, \tau_J^2 K) \to \mathbb{C}$, given by pairing with the identity map under the functorial duality $\mathrm{D}\,\underline{\operatorname{Hom}}_J(K, \tau_J^2 K) = \underline{\operatorname{Hom}}_J(K, K)$. We define $\psi \colon C \to \mathbb{C}$ to be the restriction of ϕ to C. From this we get a map

$$\operatorname{Hom}_{\operatorname{per} A}(-, M[3]) \to \mathrm{D} \operatorname{Hom}_{\operatorname{per} A}(M, -),$$

sending f to the map $g \mapsto \psi(fg)$. We claim this is an isomorphism of cohomological functors; to see this it suffices to evaluate it at all shifts of projective A-modules $\mathrm{Hom}_\Pi(T,T')$ for $T' \in \operatorname{add} T$. We have

$$\mathrm{Hom}_{\mathrm{per}\,A}(\mathrm{Hom}_\Pi(T,T')[i], M[3]) = \mathrm{Ext}_A^{3-i}(\mathrm{Hom}_\Pi(T,T'), M),$$

which vanishes if $i \neq 3$ and is canonically isomorphic to $\mathrm{Ext}^1_\Pi(T', K)$ if $i = 3$, by Yoneda's lemma again. On the other hand, to compute $\mathrm{Hom}_{\mathrm{per}\,A}(M, \mathrm{Hom}_\Pi(T,T'))$, we compute the homology of

$$0 \longrightarrow \mathrm{Hom}_\Pi(T_0,T') \longrightarrow \mathrm{Hom}_\Pi(T_1,T') \longrightarrow \mathrm{Hom}_\Pi(T_2,T') \longrightarrow \mathrm{Hom}_\Pi(T_3,T') \longrightarrow 0.$$

Using the exact sequences

$$0 \longrightarrow \mathrm{Hom}_\Pi(T_0,T') \longrightarrow \mathrm{Hom}_\Pi(T_1,T') \longrightarrow \mathrm{Hom}_\Pi(K,T') \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}_\Pi(K,T') \longrightarrow \mathrm{Hom}_\Pi(T_2,T') \longrightarrow \mathrm{Hom}_\Pi(T_3,T') \longrightarrow \mathrm{Ext}^1_\Pi(K,T') \longrightarrow 0,$$

we see that the homology is zero except at $\mathrm{Hom}_\Pi(T_3,T')$, where it is $\mathrm{Ext}^1_\Pi(K,T')$. Now the functorial form ϕ induces an isomorphism

$$\mathrm{Ext}^1_\Pi(T',K) \xrightarrow{\sim} \mathrm{D}\,\mathrm{Ext}^1_\Pi(K,T')$$

agreeing with the map given by ψ. Thus we indeed have a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{per}\,A}(-, M[3]) \xrightarrow{\sim} \mathrm{D}\,\mathrm{Hom}_{\mathrm{per}\,A}(M,-).$$

Since A is finite dimensional, $\operatorname{mod} A$ is abelian. Moreover A has finite global dimension, so $\mathrm{per}\,A = \mathcal{D}^b A$ and the functorial isomorphism above yields the required isomorphism

$$\mathrm{Ext}_A^{3-i}(N,M) \xrightarrow{\sim} \mathrm{D}\,\mathrm{Ext}^i_A(M,N)$$

for all $N \in \operatorname{mod} A$. □

Remark 5.6. In [KR07, §5.4], Keller–Reiten give analogous arguments for the case of d-cluster-tilting objects. For example, if T is a d-cluster-tilting object in a Hom-finite Frobenius category $\mathcal{E}$ with kernels, then A is internally $(d+1)$-Calabi–Yau with respect to projection onto the projective-injective summands.

5.2 Internally Bimodule d-Calabi–Yau Algebras

Denote by $A^\varepsilon = A \otimes_{\mathbb{C}} A^{\mathrm{op}}$ the enveloping algebra of A, so that an A-bimodule is the same as an A^ε-module. Recall [AIR15, Defn. 2.1] that an algebra A is said to be *bimodule* d-Calabi–Yau if $A \in \operatorname{per} A^\varepsilon$ (i.e. A is quasi-isomorphic to a bounded complex of projective A-bimodules) and there is an isomorphism $A \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon)[d]$ in $\mathcal{D}A^\varepsilon$. This definition is slightly weaker than that of Ginzburg [Gin06, 3.2.5], as we will not need to impose any 'self-duality' condition on the isomorphism $A \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon)[d]$.

Definition 5.7. An algebra A is *internally bimodule d-Calabi–Yau* with respect to an idempotent $e \in A$ if

(i) $\operatorname{p.dim}_{A^\varepsilon} A \leq d$, and

(ii) there exists a triangle

$$A[-d] \xrightarrow{\psi} \mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon) \longrightarrow C \longrightarrow A[1-d]$$

in $\mathcal{D}A^\varepsilon$ such that $\underline{A} \overset{\mathbf{L}}{\otimes}_A C = 0 = C \overset{\mathbf{L}}{\otimes}_A \underline{A}$, where $\underline{A} = A/\langle e \rangle$.

Remark 5.8. An algebra A is internally bimodule d-Calabi–Yau with respect to 0 if and only if ψ can be chosen to be a quasi-isomorphism, or equivalently if A is bimodule d-Calabi–Yau. In this case, we need only assume that A has finite projective dimension as an A-bimodule, or equivalently that $A \in \operatorname{per} A^\varepsilon$, and then it follows from (ii) that this dimension is at most d [AIR15, Prop. 2.4(b)]. When $e \neq 0$, this implication does not hold, and so we must make the stronger condition part of the definition.

An algebra A is internally bimodule d-Calabi–Yau with respect to 1 if and only if $\operatorname{p.dim}_{A^\varepsilon} A \leq d$; in this case $\underline{A} = 0$, so condition (ii) is satisfied for any ψ.

Remark 5.9. There is an isomorphism $A^\varepsilon \xrightarrow{\sim} (A^{\mathrm{op}})^\varepsilon$ given by reversing the order of the tensor product. The resulting equivalence $\operatorname{mod} A^\varepsilon \xrightarrow{\sim} \operatorname{mod} (A^{\mathrm{op}})^\varepsilon$ takes A to A^{op}. As a result, Definition 5.7 is left-right symmetric, meaning that A is internally bimodule d-Calabi–Yau with respect to e if and only if the same is true of A^{op}.

Write $\Omega_A = \mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon)$. We view Ω_A as a complex in $\mathcal{D}A^\varepsilon$ via the 'inner' multiplication on A^ε; for any homomorphism $f \colon M \to A^\varepsilon$ of A-bimodules such that $f(m) = u \otimes v$ and any $x \otimes y \in A^\varepsilon$, let $xfy(m) = ux \otimes yv$. We then have the following lemma.

Lemma 5.10 ([Kel08, Lem. 4.1])**.** *Assume $A \in \operatorname{per} A^\varepsilon$. For all objects $M, N \in \mathcal{D}A$ such that M has finite dimensional total cohomology, there is a functorial isomorphism*

$$\mathrm{D}\,\mathrm{Hom}_{\mathcal{D}A}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(\Omega_A \overset{\mathbf{L}}{\otimes}_A N, M).$$

If A is bimodule d-Calabi–Yau, then $\Omega_A \cong A[-d]$ in $\mathcal{D}A^\varepsilon$. It then follows from Lemma 5.10 that for any $M, N \in \operatorname{mod} A$, with M finite dimensional, we have

$$\begin{aligned}
\mathrm{D}\operatorname{Ext}^i_A(M,N) &= \mathrm{D}\operatorname{Hom}_{\mathcal{D}A}(M, N[i]) \\
&\cong \operatorname{Hom}_{\mathcal{D}A}(\Omega_A \overset{\mathbf{L}}{\otimes}_A N[i], M) \\
&\cong \operatorname{Hom}_{\mathcal{D}A}(N[i-d], M) \\
&= \operatorname{Ext}^{d-i}_A(N, M).
\end{aligned}$$

We now use Lemma 5.10 to prove a similar result for internally bimodule d-Calabi–Yau algebras.

Theorem 5.11. *Let A be internally bimodule d-Calabi–Yau with respect to e. Then for any $N \in \mathcal{D}A$ and any $M \in \mathcal{D}(A/\langle e\rangle)$ with finite dimensional total cohomology, we have a functorial isomorphism*

$$\mathrm{D}\operatorname{Hom}_{\mathcal{D}A}(M,N) = \operatorname{Hom}_{\mathcal{D}A}(N[-d], M).$$

Proof. Pick a triangle

$$A[-d] \xrightarrow{\psi} \Omega_A \longrightarrow C \longrightarrow A[1-d]$$

as in Definition 5.7. Applying $- \overset{\mathbf{L}}{\otimes}_A N$ yields a triangle

$$N[-d] \longrightarrow \Omega_A \overset{\mathbf{L}}{\otimes}_A N \longrightarrow C \overset{\mathbf{L}}{\otimes}_A N \longrightarrow N[1-d]$$

in $\mathcal{D}A$. Now apply $\mathbf{R}\operatorname{Hom}_A(-, M)$, to get a triangle

$$\begin{tikzcd}
\mathbf{R}\operatorname{Hom}_A(\Omega_A \overset{\mathbf{L}}{\otimes}_A N, M) \arrow[rr] & & \mathbf{R}\operatorname{Hom}_A(N[-d], M) \arrow[dl] \\
& \mathbf{R}\operatorname{Hom}_A(C \overset{\mathbf{L}}{\otimes}_A N, M) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

Write $\underline{A} = A/\langle e\rangle$. Since $M \in \mathcal{D}\underline{A}$, we have $M = \mathbf{R}\operatorname{Hom}_{\underline{A}}(\underline{A}, M)$, and so

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_A(C \overset{\mathbf{L}}{\otimes}_A N, M) &= \mathbf{R}\operatorname{Hom}_A(M, \mathbf{R}\operatorname{Hom}_A(C, M)) \\
&= \mathbf{R}\operatorname{Hom}_A(M, \mathbf{R}\operatorname{Hom}_A(C, \mathbf{R}\operatorname{Hom}_{\underline{A}}(\underline{A}, M))) \\
&= \mathbf{R}\operatorname{Hom}_A(M, \mathbf{R}\operatorname{Hom}_{\underline{A}}(\underline{A} \overset{\mathbf{L}}{\otimes}_A C, M)) = 0.
\end{aligned}$$

Thus $\mathbf{R}\operatorname{Hom}_A(\Omega_A \overset{\mathbf{L}}{\otimes}_A N, M) \cong \mathbf{R}\operatorname{Hom}_A(N[-d], M)$. We obtain the result by taking 0-th cohomology and applying Lemma 5.10. $\square$

Corollary 5.12. *If A is internally bimodule d-Calabi–Yau with respect to e, then it is internally d-Calabi–Yau with respect to e in the sense of Definition 5.1.*

Proof. Since $\operatorname{p.dim}_{A^\varepsilon} A \leq d$, there is an exact sequence

$$0 \longrightarrow P_d \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

of A-bimodules, in which each P_i is a projective bimodule [Wei94, Lem. 4.1.6]. If X is any A-module, then $P_i \otimes_A X$ is a projective A-module, and so applying $- \otimes_A X$ to the above sequence gives a projective resolution

$$0 \longrightarrow P_d \otimes_A X \longrightarrow \cdots \longrightarrow P_1 \otimes_A X \longrightarrow P_0 \otimes_A X \longrightarrow X \longrightarrow 0$$

of X. It follows that $\operatorname{gl.dim} A \leq d$.

Now by Theorem 5.11, if $N \in \operatorname{mod} A$ and $M \in \operatorname{mod} A/\langle e\rangle$ is finite dimensional, we have

$$\begin{aligned} \mathrm{D}\operatorname{Ext}^i_A(M,N) &= \mathrm{D}\operatorname{Hom}_{\mathcal{D}A}(M,N[i]) \\ &= \operatorname{Hom}_{\mathcal{D}A}(N[i-d],M) \\ &= \operatorname{Ext}^{d-i}_A(N,M). \end{aligned}$$ □

5.3 From Internally d-Calabi–Yau Algebras to d-Cluster-Tilting Objects

Proposition 5.5 shows how internally 3-Calabi–Yau algebras arise as endomorphism algebras of cluster-tilting objects in $\operatorname{Sub} Q_J$; see Keller–Reiten [KR07, §4] for the same conclusion in a slightly wider context. In this section we work in the opposite direction, and prove the following generalisation of a result of Amiot–Iyama–Reiten [AIR15, Thm. 2.2] on bimodule d-Calabi–Yau algebras.

Theorem 5.13 (cf. [AIR15, Thm. 2.2])**.** *Let A be a Noetherian algebra and let $e \in A$ be an idempotent such that $A/\langle e\rangle$ is finite dimensional, and both A and A^{op} are internally d-Calabi–Yau with respect to e. Write $B = eAe$ and $\underline{A} = A/\langle e\rangle$. Then*

(i) B is Iwanaga–Gorenstein with Gorenstein dimension at most d, so $\operatorname{GP}(B)$ is a Frobenius category,

(ii) eA is $(d-1)$-cluster-tilting in $\operatorname{GP}(B)$, and

(iii) there are natural isomorphisms $\operatorname{End}_B(eA)^{\mathrm{op}} \xrightarrow{\sim} A$ and $\operatorname{End}_{\underline{\operatorname{GP}}(B)}(eA)^{\mathrm{op}} \xrightarrow{\sim} \underline{A}$.

Remark 5.14. While all of the conclusions of Theorem 5.13, except for B being Iwanaga–Gorenstein, refer only to left B-modules, the proof we will give uses the assumption that A^{op} is internally d-Calabi–Yau to draw conclusions about right A-modules.

This applies in particular to showing that the right A-module eA is cluster-tilting in the category of Gorenstein projective B-modules; see Lemmas 5.19 and 5.20.

Since the assumptions of Theorem 5.13 are symmetric in A and A^{op}, we may also conclude that $\mathrm{GP}(B^{\mathrm{op}})$ is a Frobenius category in which Ae is a $(d-1)$-cluster-tilting object, and there are natural isomorphisms $\mathrm{End}_{B^{\mathrm{op}}}(Ae)^{\mathrm{op}} \xrightarrow{\sim} A^{\mathrm{op}}$ and $\mathrm{End}_{\underline{\mathrm{GP}}(B^{\mathrm{op}})}(Ae)^{\mathrm{op}} \xrightarrow{\sim} \underline{A}^{\mathrm{op}}$.

We emphasize two cases in which the assumptions of Theorem 5.13 may be made to appear one-sided. Firstly, as in Remark 5.3, if A is a finite dimensional algebra then it is internally d-Calabi–Yau with respect to e if and only if the same is true of A^{op}. Secondly, if A is internally bimodule d-Calabi–Yau with respect to e, then (Remark 5.9) so is A^{op}, and therefore both A and A^{op} are internally d-Calabi–Yau with respect to e by Corollary 5.12.

The rest of the section is devoted to proving Theorem 5.13, so we let A, e, $\underline{A}$ and B be as in the assumptions of this theorem. We begin with the following observation.

Proposition 5.15 (cf. [AIR15, Lem. 2.5])**.** *For any $X \in \operatorname{mod} \underline{A}$, we have*

(i) $\mathrm{Ext}^i_A(X, A) = 0$ *for $i \neq d$, and*

(ii) $\mathrm{Ext}^i_A(X, Ae) = 0$ *for any $i \in \mathbb{Z}$.*

Proof. Since A is internally d-Calabi–Yau and $X \in \operatorname{mod} \underline{A}$, we have

$$\mathrm{Ext}^i_A(X, A) = \mathrm{D}\mathrm{Ext}^{d-i}_A(A, X) = 0$$

and

$$\mathrm{Ext}^i_A(X, Ae) = \mathrm{D}\,\mathrm{Ext}^{d-1}_A(Ae, X) = 0$$

for $i \neq d$, since both A and Ae are projective A-modules. We also have

$$\mathrm{Ext}^d_A(X, Ae) = \mathrm{D}\mathrm{Hom}_A(Ae, X) = 0,$$

again using that $X \in \operatorname{mod} \underline{A}$. ☐

The assumption of part (i) of Proposition 5.15 is slightly more restrictive than that of [AIR15, Lem. 2.5(a)]. This is necessary for the result to hold in our setting, since our A is only internally 3-Calabi–Yau. However, this stronger assumption is satisfied whenever [AIR15, Lem. 2.5(a)] is used in the proof of [AIR15, Thm. 2.2].

The following results (Proposition 5.16 and Lemmas 5.17, 5.19 and 5.20) are now close analogues of [AIR15, Prop. 2.6, Lem. 2.8–2.10], with very similar proofs. For the convenience of the reader, we reproduce the arguments from [AIR15] using our notation and conventions.

Proposition 5.16 (cf. [AIR15, Prop. 2.6]). *We have*

$$\operatorname{Ext}^i_B(eA, B) \cong \begin{cases} Ae, & i = 0, \\ 0, & i \neq 0, \end{cases}$$

$$\operatorname{Ext}^i_B(eA, eA) \cong \begin{cases} A^{\mathrm{op}}, & i = 0, \\ 0, & 0 < i < d-1. \end{cases}$$

Proof. We can compute $\operatorname{Ext}^i_B(eA, B)$ as the cohomology of

$$\mathbf{R}\mathrm{Hom}_B(eA, B) \cong \mathbf{R}\mathrm{Hom}_B(eA, \mathbf{R}\mathrm{Hom}_A(Ae, Ae)) \cong \mathbf{R}\mathrm{Hom}_A(Ae \overset{\mathbf{L}}{\otimes}_B eA, Ae),$$

and wish to show that this is isomorphic to the cohomology of

$$Ae \cong \mathbf{R}\mathrm{Hom}_A(A, Ae).$$

Thus it is sufficient to show that

$$\mathbf{R}\mathrm{Hom}_A(Ae \overset{\mathbf{L}}{\otimes}_B eA, Ae) \cong \mathbf{R}\mathrm{Hom}_A(A, Ae).$$

Let f be the composition of the natural map

$$Ae \overset{\mathbf{L}}{\otimes}_B eA \to \mathrm{H}^0(Ae \overset{\mathbf{L}}{\otimes}_B eA) = Ae \otimes_B eA$$

with the multiplication map $Ae \otimes_B eA \to A$, and let X be the mapping cone of f, so we have a triangle

$$Ae \overset{\mathbf{L}}{\otimes}_B eA \xrightarrow{\ f\ } A \longrightarrow X \longrightarrow Ae \overset{\mathbf{L}}{\otimes}_B eA[1] \tag{5.1}$$

in the derived category $\mathcal{D}A^{\varepsilon}$ of A-bimodules. The map $eA \otimes_A f$ is the natural isomorphism $B \overset{\mathbf{L}}{\otimes}_B eA \xrightarrow{\sim} A$, so $eA \otimes_A X = 0$. It follows that $e\mathrm{H}^i(X) = 0$, and hence, forgetting the right module structure, $\mathrm{H}^i(X) \in \operatorname{mod} \underline{A}$ for all $i \in \mathbb{Z}$. Thus by Proposition 5.15, $\operatorname{Ext}^j_A(\mathrm{H}^i(X), Ae) = 0$ for all $i, j \in \mathbb{Z}$.

We can compute $\mathrm{H}^k(\mathbf{R}\mathrm{Hom}_A(X, Ae))$ via a hypercohomology spectral sequence ${}^{II}E_r^{ij}$ [Wei94, §5.7.9, see also Defn. 5.6.2], in which

$${}^{II}E_2^{ij} = \operatorname{Ext}^j_A(\mathrm{H}^i(X), Ae) = 0$$

as above. It follows that $\mathrm{H}^k(\mathbf{R}\mathrm{Hom}_A(X, Ae)) = 0$ for all k, and so $\mathbf{R}\mathrm{Hom}_A(X, Ae) = 0$. Now applying $\mathbf{R}\mathrm{Hom}_A(-, Ae)$ to the triangle (5.1) yields the required isomorphism

$$\mathbf{R}\mathrm{Hom}_A(Ae \overset{\mathbf{L}}{\otimes}_B eA, Ae) \cong \mathbf{R}\mathrm{Hom}_A(A, Ae)$$

in $\mathcal{D}(A \otimes_{\mathbb{C}} B^{\mathrm{op}})$, from which the first assertion follows by our initial calculations.

Similarly, we have isomorphisms

$$\mathbf{R}\mathrm{Hom}_B(eA, eA) \cong \mathbf{R}\mathrm{Hom}_B(eA, \mathbf{R}\mathrm{Hom}_A(Ae, A)) \cong \mathbf{R}\mathrm{Hom}_A(Ae \overset{\mathbf{L}}{\otimes}_B eA, A)$$

in $\mathcal{D}(A \otimes_{\mathbb{C}} B^{\mathrm{op}})$, and so to obtain the second assertion we wish to show that

$$\mathbf{R}\mathrm{Hom}_A(Ae \overset{\mathbf{L}}{\otimes}_B eA, A) \cong \mathbf{R}\mathrm{Hom}_A(A, A).$$

Again we use the triangle (5.1). As Ae and eA are concentrated in degree 0, we have $\mathrm{H}^i(Ae \overset{\mathbf{L}}{\otimes}_B eA) = 0$ for $i > 0$, and so $\mathrm{H}^i(X) = 0$ for $i > 0$. Recalling that $\mathrm{H}^i(X) \in \mathrm{mod}\,\underline{A}$, it follows from Proposition 5.15 that $\mathrm{Ext}^j_A(\mathrm{H}^i(X), A) = 0$ for $j \neq d$. So by an analogous spectral sequence argument to above, $\mathrm{H}^i(\mathbf{R}\mathrm{Hom}_A(X, A)) = 0$ for $i < d$.

From (5.1), we obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{D}A}(X, A[i]) \longrightarrow \mathrm{Hom}_{\mathcal{D}A}(A, A[i]) \longrightarrow \mathrm{Hom}_{\mathcal{D}A}(Ae \overset{\mathbf{L}}{\otimes}_B eA, A[i]) \longrightarrow \cdots$$

As $\mathrm{Hom}_{\mathcal{D}A}(X, A[i]) = 0$ for $i < d$ as above, it follows from our initial calculations that

$$\mathrm{Ext}^i_B(eA, eA) \cong \mathrm{Hom}_{\mathcal{D}A}(Ae \overset{\mathbf{L}}{\otimes}_B eA, A[i]) \cong \mathrm{Hom}_{\mathcal{D}A}(A, A[i]) \cong \begin{cases} A^{\mathrm{op}}, & i = 0, \\ 0, & 0 < i < d-1 \end{cases}$$

as required. ☐

Lemma 5.17 (cf. [AIR15, Lem. 2.8]). *For any $X \in \mathrm{mod}\,B$, we have*

$$\mathrm{p.\,dim}_{A^{\mathrm{op}}} \mathrm{Hom}_B(X, eA) \leq d - 2.$$

Proof. Let $P_1 \to P_0 \to X \to 0$ be a projective presentation of X in $\mathrm{mod}\,B$, and apply $\mathrm{Hom}_B(-, eA)$ to obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_B(X, eA) \longrightarrow \mathrm{Hom}_B(P_0, eA) \longrightarrow \mathrm{Hom}_B(P_1, eA)$$

of A^{op}-modules. Since $\mathrm{Hom}_B(P_i, eA)$ is a projective A^{op}-module, the above sequence shows that $\mathrm{Hom}_B(X, eA)$ is a second syzygy module. Then as A^{op} is internally d-Calabi–Yau, we have $\mathrm{gl.\,dim}\,A^{\mathrm{op}} \leq d$, and so $\mathrm{p.\,dim}_{A^{\mathrm{op}}} \mathrm{Hom}_B(X, eA) \leq d - 2$. ☐

Remark 5.18. We can obtain the statement that $\mathrm{gl.\,dim}\,A^{\mathrm{op}} \leq d$ needed in the proof of Lemma 5.17 without assuming that A^{op} is internally d-Calabi–Yau. Since A is Noetherian, we have $\mathrm{gl.\,dim}\,A^{\mathrm{op}} = \mathrm{gl.\,dim}\,A$ [Wei94, Ex. 4.1.1], and $\mathrm{gl.\,dim}\,A \leq d$ by the assumption that A is internally d-Calabi–Yau. However, the next two results, Lemmas 5.19 and 5.20, will use the assumption that A^{op} is internally d-Calabi–Yau in a more fundamental way.

Lemma 5.19 (cf. [AIR15, Lem. 2.9])**.** *If* $X \in \mathrm{GP}(B)$ *and* $\mathrm{Ext}^i_B(X, eA) = 0$ *for all* $0 < i < d-1$*, then* $X \in \operatorname{add}_B(eA)$*.*

Proof. Pick an exact sequence

$$0 \longrightarrow Y \longrightarrow P_{d-3} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

in which each P_i is projective. By the assumption on the vanishing of $\mathrm{Ext}^i_B(X, eA)$, we can apply $\mathrm{Hom}_B(-, eA)$ to obtain an exact sequence

$$0 \to \mathrm{Hom}_B(X, eA) \to \mathrm{Hom}_B(P_0, eA) \to \cdots \to \mathrm{Hom}_B(P_{d-3}, eA) \to \mathrm{Hom}_B(Y, eA) \to 0$$

of A^{op}-modules. Each $\mathrm{Hom}_B(P_i, eA)$ is a projective A^{op}-module, and by Lemma 5.17 we have $\operatorname{p.dim}_{A^{\mathrm{op}}} \mathrm{Hom}_B(Y, eA) \leq d-2$, so $\mathrm{Hom}_B(X, eA)$ is also a projective A^{op}-module. Therefore $\mathrm{Hom}_B(X, B) = \mathrm{Hom}_B(X, eA)e \in \operatorname{add}_{B^{\mathrm{op}}}(Ae)$. By [AIR15, Prop. 1.3(b)] there are quasi-inverse dualities

$$\begin{aligned} \mathrm{Hom}_B(-, B)\colon\ & \mathrm{GP}(B) \to \mathrm{GP}(B^{\mathrm{op}}), \\ \mathrm{Hom}_{B^{\mathrm{op}}}(-, B)\colon\ & \mathrm{GP}(B^{\mathrm{op}}) \to \mathrm{GP}(B). \end{aligned}$$

Since we are assuming A^{op} is also internally d-Calabi–Yau with respect to e, we can apply Proposition 5.16 to A^{op} to obtain an isomorphism $\mathrm{Hom}_{B^{\mathrm{op}}}(Ae, B) \xrightarrow{\sim} eA$ of B-modules. Therefore

$$X \cong \mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_B(X, B), B) \in \operatorname{add}_B(\mathrm{Hom}_{B^{\mathrm{op}}}(Ae, B)) = \operatorname{add}_B(eA)$$

as required. ☐

Lemma 5.20 (cf. [AIR15, Lem. 2.10])**.** *If* $X \in \mathrm{GP}(B)$ *and* $\mathrm{Ext}^i_B(eA, X) = 0$ *for all* $0 < i < d-1$*, then* $X \in \operatorname{add}_B(eA)$*.*

Proof. The quasi-inverse dualities

$$\begin{aligned} \mathrm{Hom}_B(-, B)\colon\ & \mathrm{GP}(B) \to \mathrm{GP}(B^{\mathrm{op}}), \\ \mathrm{Hom}_{B^{\mathrm{op}}}(-, B)\colon\ & \mathrm{GP}(B^{\mathrm{op}}) \to \mathrm{GP}(B) \end{aligned}$$

from [AIR15, Prop. 1.3(b)] preserve extension groups. Since $\mathrm{Hom}_B(eA, B) \cong Ae$ by Proposition 5.16, we have $\mathrm{Ext}^i_{B^{\mathrm{op}}}(Ae, \mathrm{Hom}_B(X, B)) = 0$ for all $0 < i < d-1$. Thus by applying Lemma 5.19 to A^{op} and $\mathrm{Hom}_B(X, B) \in \mathrm{GP}(A^{\mathrm{op}})$, we find that $\mathrm{Hom}_B(X, B) \in \operatorname{add}_{B^{\mathrm{op}}}(Ae)$. Then, as in Lemma 5.19, applying $\mathrm{Hom}_{B^{\mathrm{op}}}(-, B)$ gives $X \in \operatorname{add}_B(eA)$. ☐

We are now ready to prove Theorem 5.13.

Proof of Theorem 5.13. (i) First we show B is Noetherian. Any left ideal I of B is of the form $e\widetilde{I}$ for an ideal $\widetilde{I} = AI$ of A. So any strictly ascending chain of left ideals of B determines such a chain of ideals of A, which stabilizes as A is Noetherian. A similar argument shows that B is right Noetherian.

Now we show that $\operatorname{Ext}_B^{d+1}(X, B) = 0$ for all $X \in \operatorname{Mod} B$. Given such an X, let $Y = Ae \otimes_B X$, and let $\mathbf{P}$ be a projective resolution of Y. Then $e\mathbf{P}$ is a bounded complex in the full subcategory $\operatorname{add}(eA)$ of $\operatorname{Mod} B$, quasi-isomorphic to $eY = X$. By Proposition 5.16, $\operatorname{Ext}_B^i(eA, B) = 0$ for $i > 0$, so another spectral sequence argument (now using ${}^{I}E^r_{pq}$ from [Wei94, Defn. 5.6.1]) shows that

$$\operatorname{Ext}_B^{d+1}(X, B) \cong \mathrm{H}^{d+1}(\operatorname{Hom}_B(e\mathbf{P}, B)),$$

where $\operatorname{Hom}_B(e\mathbf{P}, B)$ denotes the complex obtained by applying $\operatorname{Hom}_B(-, B)$ to $e\mathbf{P}$. Since

$$\begin{aligned} \operatorname{Hom}_B(e\mathbf{P}, B) &= \operatorname{Hom}_B(eA \otimes_A \mathbf{P}, B) \\ &= \operatorname{Hom}_A(\mathbf{P}, \operatorname{Hom}_B(eA, B)) \cong \operatorname{Hom}_A(\mathbf{P}, Ae), \end{aligned}$$

with the final isomorphism coming from Proposition 5.16, it follows that

$$\operatorname{Ext}_B^{d+1}(X, B) \cong \mathrm{H}^{d+1}(\operatorname{Hom}_A(\mathbf{P}, Ae)) \cong \operatorname{Ext}_A^{d+1}(Y, Ae) = 0,$$

since $\operatorname{gl.dim} A \leq d$ by assumption. A dual argument, using that A^{op} is internally d-Calabi–Yau with respect to e, shows that $\operatorname{Ext}_{B^{\mathrm{op}}}^{d+1}(X, B^{\mathrm{op}}) = 0$ for all $X \in \operatorname{Mod} B^{\mathrm{op}}$. It follows that B is Iwanaga–Gorenstein of dimension at most d, and so $\operatorname{GP}(B)$ is Frobenius [AIR15, Prop. 1.3(a)].

(ii) Since A is Noetherian, the left ideal $\langle e \rangle = AeA$ is finitely-generated. Thus there is a finite generating set of the A-module AeA contained in eA, which must generate $eA \subseteq AeA$ as a B-module, so $eA \in \operatorname{mod} B$. Now $eA \in \operatorname{GP}(B)$ and $\operatorname{Ext}_B^i(eA, eA) = 0$ for $0 < i < d-1$ by Proposition 5.16. This, together with Lemmas 5.19 and 5.20, shows that eA is $(d-1)$-cluster-tilting in $\operatorname{GP}(B)$.

(iii) We have $\operatorname{End}_B(eA)^{\mathrm{op}} \cong A$ by Proposition 5.16, and thus we have an equivalence

$$\operatorname{Hom}_B(eA, -)\colon \operatorname{add}{}_B(eA) \xrightarrow{\sim} \operatorname{add}{}_A A.$$

By Proposition 5.16 again, $\operatorname{Hom}_B(eA, B) \cong Ae$. It follows that

$$\begin{aligned} \operatorname{End}_{\underline{\operatorname{GP}}(B)}(eA)^{\mathrm{op}} &= \operatorname{End}_B(eA)^{\mathrm{op}} / \langle \operatorname{add}{}_B B \rangle \\ &\cong \operatorname{End}_A(A)^{\mathrm{op}} / \langle \operatorname{add}{}_A(Ae) \rangle \cong A/\langle e \rangle = \underline{A}, \end{aligned}$$

where $\langle \mathcal{C} \rangle$ denotes the ideal of maps factoring through the subcategory $\mathcal{C}$. $\square$

Let $T \in \operatorname{Sub} Q_J$ be cluster-tilting, let $A = \operatorname{End}_\Pi(T)^{\mathrm{op}}$, and let e be the idempotent of A corresponding to projection onto the projective-injective summand $F = \bigoplus_{i \in \Delta_0} F_i$.

By Proposition 5.5, the finite dimensional algebra A is internally 3-Calabi–Yau with respect to e. As in Remark 5.3, A^{op} is also internally 3-Calabi–Yau with respect to e, so we may apply Theorem 5.13 to to conclude that A is isomorphic to the endomorphism algebra of a cluster-tilting object in $\mathrm{GP}(B)$, where $B = eAe \cong \mathrm{End}_{\Pi}(F)^{\mathrm{op}}$. Since A has finite global dimension, this also follows from [IKWY15, Thm. 2.7] (stated here as Theorem 3.4), which implies that $\mathrm{Sub}\,Q_J \simeq \mathrm{GP}(B)$. In particular, $\underline{\mathrm{GP}}(B) \simeq \underline{\mathrm{Sub}}\,Q_J$ is 2-Calabi–Yau.

5.4 A Bimodule Complex for Frozen Jacobian Algebras

Given a quiver with potential (Q, W), Ginzburg [Gin06, 5.1.5] (see also [Bro12, §7]) defines a complex of projective bimodules over the associated Jacobian algebra. For (Q, W) a quiver with potential determined by a dimer model on a torus, Broomhead shows in [Bro12, Thm. 7.7] that if the dimer model is consistent (in one of several possible senses), then this complex is isomorphic to $A = \mathcal{J}(Q, W)$ in $\mathcal{D}^b A^{\varepsilon}$, and thus provides a projective bimodule resolution of A. It follows in this case that A is 3-Calabi–Yau, with this property arising from a natural symmetry in the bimodule resolution.

We will now define an analogous complex $\mathbf{P}(A)$ for a frozen Jacobian algebra $A = \mathcal{J}(Q, F, W)$. The presentation of A suggests a preferred *boundary idempotent* $\sum_{v \in F_0} e_v$, which we will denote by e for the remainder of the section. Our main result (Theorem 5.24) will be that if $\mathbf{P}(A)$ is isomorphic to A in $\mathcal{D}^b A^{\varepsilon}$, then A is internally bimodule 3-Calabi–Yau with respect to e, in the sense of Definition 5.7. While we will write $\mathbf{P}(A)$ for this complex in order to save space, the definition depends not only on A but on the ice quiver with potential (Q, F, W) giving the presentation of A as $\mathcal{J}(Q, F, W)$.

Recall that we write $Q_0^m = Q_0 \setminus F_0$ for the set of mutable vertices and $Q_1^m = Q_1 \setminus F_1$ for the set of unfrozen arrows. We also write v^+ for the set of arrows with tail at v, and v^- for the set of arrows with head at v. As usual, let $\mathfrak{m}(A)$ denote the arrow ideal of A, and let $S = A/\mathfrak{m}(A)$. For the remainder of this section, we write $\otimes = \otimes_S$.

Introduce formal symbols ρ_α for each $\alpha \in Q_1$ and ω_v for each $v \in Q_0$, and define S-bimodule structures on the vector spaces

$$\begin{aligned}
\mathbb{C}Q_0 &= \bigoplus_{v \in Q_0} \mathbb{C}e_v, & \mathbb{C}Q_0^m &= \bigoplus_{v \in Q_0^m} \mathbb{C}e_v, & \mathbb{C}F_0 &= \bigoplus_{v \in F_0} \mathbb{C}e_v,\\
\mathbb{C}Q_1 &= \bigoplus_{\alpha \in Q_1} \mathbb{C}\alpha, & \mathbb{C}Q_1^m &= \bigoplus_{\alpha \in Q_1^m} \mathbb{C}\alpha, & \mathbb{C}F_1 &= \bigoplus_{\alpha \in F_1} \mathbb{C}\alpha,\\
\mathbb{C}Q_2 &= \bigoplus_{\alpha \in Q_1} \mathbb{C}\rho_\alpha, & \mathbb{C}Q_2^m &= \bigoplus_{\alpha \in Q_1^m} \mathbb{C}\rho_\alpha, & \mathbb{C}F_2 &= \bigoplus_{\alpha \in F_1} \mathbb{C}\rho_\alpha,\\
\mathbb{C}Q_3 &= \bigoplus_{v \in Q_0} \mathbb{C}\omega_v, & \mathbb{C}Q_3^m &= \bigoplus_{v \in Q_0^m} \mathbb{C}\omega_v, & \mathbb{C}F_3 &= \bigoplus_{v \in F_0} \mathbb{C}\omega_v,
\end{aligned}$$

via the formulae

$$\begin{aligned} e_v \cdot e_v \cdot e_v &= e_v, \\ e_{h\alpha} \cdot \alpha \cdot e_{t\alpha} &= \alpha, \\ e_{t\alpha} \cdot \rho_\alpha \cdot e_{h\alpha} &= \rho_\alpha, \\ e_v \cdot \omega_v \cdot e_v &= e_v. \end{aligned}$$

For each i, the S-bimodule $\mathbb{C}Q_i$ splits as the direct sum

$$\mathbb{C}Q_i = \mathbb{C}Q_i^m \oplus \mathbb{C}F_i.$$

Since $\mathbb{C}Q_0 \cong S$, the A-bimodule $A \otimes \mathbb{C}Q_0 \otimes A$ is canonically isomorphic to $A \otimes A$, and we will use the two descriptions interchangeably.

We define maps $\bar{\mu}_i \colon A \otimes \mathbb{C}Q_i \otimes A \to A \otimes \mathbb{C}Q_{i-1} \otimes A$ for $1 \leq i \leq 3$. The map $\bar{\mu}_1$ is defined by

$$\bar{\mu}_1(x \otimes \alpha \otimes y) = x \otimes e_{h\alpha} \otimes \alpha y - x\alpha \otimes e_{t\alpha} \otimes y,$$

or, composing with the natural isomorphism $A \otimes \mathbb{C}Q_0 \otimes A \xrightarrow{\sim} A \otimes A$, by

$$\bar{\mu}_1(x \otimes \alpha \otimes y) = x \otimes \alpha y - x\alpha \otimes y.$$

For any path $p = \alpha_m \cdots \alpha_1$ of $\mathbb{C}Q$, we may define

$$\Delta_\alpha(p) = \sum_{\alpha_i = \alpha} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \otimes \alpha_{i-1} \cdots \alpha_1,$$

and extend by linearity to obtain a map $\Delta_\alpha \colon \mathbb{C}Q \to A \otimes \mathbb{C}Q_1 \otimes A$ similar to the map of the same name introduced in Section 3.3. We then define

$$\bar{\mu}_2(x \otimes \rho_\alpha \otimes y) = \sum_{\beta \in Q_1} x \Delta_\beta(\partial_\alpha W) y.$$

Finally, let

$$\bar{\mu}_3(x \otimes \omega_v \otimes y) = \sum_{\alpha \in v^+} x \otimes \rho_\alpha \otimes \alpha y - \sum_{\beta \in v^-} x\beta \otimes \rho_\beta \otimes y.$$

Definition 5.21. For $A = \mathcal{J}(Q, F, W)$, let $\mathbf{P}(A)$ be the sequence

$$A \otimes \mathbb{C}Q_3^m \otimes A \xrightarrow{\mu_3} A \otimes \mathbb{C}Q_2^m \otimes A \xrightarrow{\mu_2} A \otimes \mathbb{C}Q_1 \otimes A \xrightarrow{\mu_1} A \otimes \mathbb{C}Q_0 \otimes A$$

of A-bimodules, where $\mu_1 = \bar{\mu}_1$, μ_2 is the restriction of $\bar{\mu}_2$ to $A \otimes \mathbb{C}Q_2^m \otimes A$ and μ_3 is the restriction of $\bar{\mu}_3$ to $A \otimes \mathbb{C}Q_3^m \otimes A$. As $v^+ \cup v^- \subseteq Q_1^m$ for any $v \in Q_0^m$, the map μ_3 takes values in $A \otimes \mathbb{C}Q_2^m \otimes A$ as claimed.

If $F = \varnothing$, then $\mathbf{P}(A)$ is the complex associated to (Q, W) by Ginzburg [Gin06, 5.1.5] and Broomhead [Bro12, §7]. In the general case, $\mathbf{P}(A)$ has already appeared in work of

Amiot–Reiten–Todorov [ART11, Proof of Prop. 2.2].

Lemma 5.22. *For a frozen Jacobian algebra $A = \mathcal{J}(Q,F,W)$, the sequence $\mathbf{P}(A)$ in Definition 5.21 is a complex of projective A-bimodules, and there is a morphism*

$$\begin{tikzcd}
A \otimes \mathbb{C}Q_3^m \otimes A \arrow[r, "\mu_3"] \arrow[d] & A \otimes \mathbb{C}Q_2^m \otimes A \arrow[r, "\mu_2"] \arrow[d] & A \otimes \mathbb{C}Q_1 \otimes A \arrow[r, "\mu_1"] \arrow[d] & A \otimes A \arrow[d, "\mu_0"] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A
\end{tikzcd}$$

from $\mathbf{P}(A)$ to A, where $\mu_0 \colon A \otimes A \to A$ is the multiplication in A.

Proof. Each term of $\mathbf{P}(A)$ is a projective A-bimodule since A is a projective A-module on each side, so we only need to check that $\mu_i \circ \mu_{i+1} = 0$ for each i (including $i = 0$, to obtain the required morphism). We check these identities by evaluating on the generators of the relevant projective A^ε-modules. First, for any $\alpha \in Q_1$ we have

$$\mu_0(\mu_1(1 \otimes \alpha \otimes 1)) = \mu_0(\alpha \otimes 1 - 1 \otimes \alpha) = \alpha - \alpha = 0.$$

Let $p = \alpha_m \cdots \alpha_1$ be any path. We have

$$\sum_{\gamma \in Q_1} \Delta_\gamma(p) = \sum_{i=1}^{m} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \otimes \alpha_{i-1} \cdots \alpha_1 \in A \otimes \mathbb{C}Q_1 \otimes A.$$

Applying μ_1, we see that

$$\begin{aligned}
\mu_1\left(\sum_{\gamma \in Q_1} \Delta_\gamma(p)\right) &= \left(\sum_{i=1}^{m} \alpha_m \cdots \alpha_i \otimes \alpha_{i-1} \cdots \alpha_1\right) - \left(\sum_{i=1}^{m} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1\right) \\
&= p \otimes 1 - 1 \otimes p.
\end{aligned}$$

It follows by linearity that

$$\begin{aligned}
\mu_1(\mu_2(1 \otimes \rho_\alpha \otimes 1)) &= \mu_1\left(\sum_{\gamma \in Q_1} \Delta_\gamma(\partial_\alpha W)\right) \\
&= \partial_\alpha W \otimes 1 - 1 \otimes \partial_\alpha W = 0
\end{aligned}$$

for any $\alpha \in Q_1^m$, since $\partial_\alpha W = 0$ in A for such α.

Now let $v \in Q_0$, and write

$$W_v = \sum_{\alpha \in v^+} (\partial_\alpha W)\alpha = \sum_{\beta \in v^-} \beta(\partial_\beta W).$$

We can calculate $\sum_{\gamma\in Q_1}\Delta_\gamma(W_v)$ using each of the two expressions, to get

$$\sum_{\gamma\in Q_1}\Delta_\gamma(W_v)=\sum_{\alpha\in v^+}\sum_{\gamma\in Q_1}\Delta_\gamma(\partial_\alpha W)\alpha+\sum_{\alpha\in v^+}\partial_\alpha W\otimes\alpha\otimes 1,$$
$$\sum_{\gamma\in Q_1}\Delta_\gamma(W_v)=\sum_{\beta\in v^-}\sum_{\gamma\in Q_1}\beta\Delta_\gamma(\partial_\beta W)+\sum_{\beta\in v^-}1\otimes\beta\otimes\partial_\beta W.$$

If $v\in Q_0^m$, then all arrows incident with v are unfrozen, and so $\partial_\alpha W=0=\partial_\beta W$ in A for any $\alpha\in v^+$ and $\beta\in v^-$. Thus in this case we have

$$\sum_{\alpha\in v^+}\sum_{\gamma\in Q_1}\Delta_\gamma(\partial_\alpha W)\alpha=\sum_{\gamma\in Q_1}\Delta_\gamma(W_v)=\sum_{\beta\in v^-}\sum_{\gamma\in Q_1}\beta\Delta_\gamma(\partial_\beta W).$$

It follows that

$$\begin{aligned}\mu_2(\mu_3(1\otimes\omega_v\otimes 1))&=\mu_2\left(\sum_{\alpha\in v^+}1\otimes\rho_\alpha\otimes\alpha-\sum_{\beta\in v^-}\beta\otimes\rho_\beta\otimes 1\right)\\&=\sum_{\alpha\in v^+}\sum_{\gamma\in Q_1}\Delta_\gamma(\partial_\alpha W)\alpha-\sum_{\beta\in v^-}\sum_{\gamma\in Q_1}\beta\Delta_\gamma(\partial_\beta W)\\&=0.\end{aligned}$$

This completes the proof. □

If the map

$$\begin{tikzcd}
A\otimes\mathbb{C}Q_3^m\otimes A \arrow[r, "\mu_3"] \arrow[d] & A\otimes\mathbb{C}Q_2^m\otimes A \arrow[r, "\mu_2"] \arrow[d] & A\otimes\mathbb{C}Q_1\otimes A \arrow[r, "\mu_1"] \arrow[d] & A\otimes A \arrow[d, "\mu_0"] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A
\end{tikzcd}$$

from Lemma 5.22 is a quasi-isomorphism, then $\mathbf{P}(A)$ is a projective bimodule resolution of A. This means that, for the presentation of A as a frozen Jacobian algebra, with relations given by certain deriviatives of the superpotential, the first syzygies are dual to the mutable vertices, and there are no higher syzygies. In particular, $\operatorname{gl.dim} A\leq 3$. By standard results on presentations of algebras, see for example Butler–King [BK99, 1.2], the vertical maps starting at $A\otimes A$ and $A\otimes\mathbb{C}Q_1\otimes A$ induce isomorphisms on cohomology; in particular, μ_0 is the cokernel of μ_1. It follows that the above map of complexes is a quasi-isomorphism if and only if the cohomology of $\mathbf{P}(A)$ vanishes at $A\otimes\mathbb{C}Q_2\otimes A$ and $A\otimes\mathbb{C}Q_3\otimes A$ (cf. [Bro12, Rem. 7.4]). We will usually abuse notation and denote the map $\mathbf{P}(A)\to A$ from Lemma 5.22 by μ_0.

Example 5.23. Let Q be the quiver with vertex set $\mathbb{Z}_n$ and arrows $\alpha_i\colon i\to i+1$. Let the frozen subquiver be the arrow α_n and its two end-points, and let $W=\alpha_n\cdots\alpha_1$. The case $n=3$ is the ice quiver with potential from Example 5.4. Let $A=\mathcal{J}(Q,F,W)$ be the corresponding frozen Jacobian algebra. It is straightforward to compute that the alternating sum of dimensions of terms of the complex $\mathbf{P}(A) \xrightarrow{\mu_0} A$ is $3-n$, so this complex can only be a bimodule resolution when $n=3$. One can also readily check that $\mu_0\colon \mathbf{P}(A) \to A$ is a quasi-isomorphism when $n=3$; cf. Example 5.4.

If $F=\varnothing$, the map $\mu_0\colon \mathbf{P}(A) \to A$ being a quasi-isomorphism implies that A is 3-Calabi–Yau [Gin06, Cor. 5.3.3], [Bro12, Thm. 7.7]. We now show that, in the general case, μ_0 being a quasi-isomorphism implies that A is internally bimodule 3-Calabi–Yau with respect to e.

Theorem 5.24. *If A is a frozen Jacobian algebra such that $\mu_0\colon \mathbf{P}(A) \to A$ is a quasi-isomorphism, then A is internally bimodule 3-Calabi–Yau with respect to the frozen idempotent $e=\sum_{v\in F_0} e_v$.*

Proof. Since $\mathbf{P}(A) \in \operatorname{per} A^\varepsilon$, the quasi-isomorphism $\mu_0\colon \mathbf{P}(A) \xrightarrow{\sim} A$ makes $\mathbf{P}(A)$ into a projective resolution of A, implying immediately that $\operatorname{p.dim}_{A^\varepsilon} \le 3$. It remains to check condition (ii) from Definition 5.7.

We begin by describing $\Omega_A = \mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon) \in \mathcal{D}^b A^\varepsilon$. Denoting $\mathrm{Hom}_{A^\varepsilon}(-,-)$ by $(-,-)$, the complex Ω_A is given by

$$(A\otimes A, A^\varepsilon) \xrightarrow{-\mu_1^*} (A\otimes \mathbb{C}Q_1 \otimes A, A^\varepsilon) \xrightarrow{\mu_2^*} (A\otimes \mathbb{C}Q_2^m \otimes A, A^\varepsilon) \xrightarrow{-\mu_3^*} (A\otimes \mathbb{C}Q_3^m \otimes A, A^\varepsilon)$$

with $\mu_i^*\colon f \mapsto f\circ\mu_i$; see Keller [Kel08, §2.7] for the signs on the differentials.

There are A-bimodule isomorphisms $A\otimes A \cong \bigoplus_{v\in Q_0} Ae_v \otimes_{\mathbb{C}} e_vA$ and $A^\varepsilon \cong A\otimes_{\mathbb{C}} A$. Introducing the shorthand notation

$$\mathbf{x}\otimes\mathbf{y} = \sum_{i=1}^{k} x^i \otimes y^i$$

for elements of $A\otimes_{\mathbb{C}} A$, a homomorphism $f_0\colon A\otimes A \to A^\varepsilon$ is uniquely determined by the values

$$f_0(1\otimes e_v \otimes 1) = \mathbf{x}_v \otimes \mathbf{y}_v$$

for each $v\in Q_0$. Since $1\otimes e_v\otimes 1 = e_v\otimes e_v \otimes e_v$, we must have

$$\mathbf{x}_v\otimes\mathbf{y}_v = e_v\mathbf{x}_v \otimes \mathbf{y}_v e_v \in e_vA\otimes_{\mathbb{C}} Ae_v,$$

but $\mathbf{x}_v$ and $\mathbf{y}_v$ may otherwise be chosen freely. If follows that we have an isomorphism

$$(A\otimes A, A^\varepsilon) \xrightarrow{\sim} A\otimes \mathbb{C}Q_3 \otimes A, \qquad f_0 \mapsto \sum_{v\in Q_0} \mathbf{y}_v \otimes \omega_v \otimes \mathbf{x}_v$$

of A-bimodules. Similar arguments yield explicit isomorphisms

$$\begin{aligned}
(A \otimes \mathbb{C}Q_1 \otimes A, A^\varepsilon) &\xrightarrow{\sim} A \otimes \mathbb{C}Q_2 \otimes A, & f_1 &\mapsto \sum_{\alpha \in Q_1} \mathbf{y}_\alpha \otimes \rho_\alpha \otimes \mathbf{x}_\alpha, \\
(A \otimes \mathbb{C}Q_2^m \otimes A, A^\varepsilon) &\xrightarrow{\sim} A \otimes \mathbb{C}Q_1^m \otimes A, & f_2 &\mapsto \sum_{\alpha \in Q_1^m} \mathbf{y}'_\alpha \otimes \alpha \otimes \mathbf{x}'_\alpha, \\
(A \otimes \mathbb{C}Q_3^m \otimes A, A^\varepsilon) &\xrightarrow{\sim} A \otimes \mathbb{C}Q_0^m \otimes A, & f_3 &\mapsto \sum_{v \in Q_0^m} \mathbf{y}'_v \otimes e_v \otimes \mathbf{x}'_v,
\end{aligned}$$

where the functions f_1, f_2 and f_3 are uniquely determined by the values

$$\begin{aligned}
f_1(1 \otimes \alpha \otimes 1) &= \mathbf{x}_\alpha \otimes \mathbf{y}_\alpha \in e_{h\alpha} A \otimes_{\mathbb{C}} A e_{t\alpha}, \\
f_2(1 \otimes \rho_\alpha \otimes 1) &= \mathbf{x}'_\alpha \otimes \mathbf{y}'_\alpha \in e_{t\alpha} A \otimes_{\mathbb{C}} A e_{h\alpha}, \\
f_3(1 \otimes \omega_v \otimes 1) &= \mathbf{x}'_v \otimes \mathbf{y}'_v \in e_v A \otimes_{\mathbb{C}} A e_v.
\end{aligned}$$

Since $\alpha \in F_1$ implies that $h\alpha, t\alpha \in F_0$, the map $\bar{\mu}_1 \colon A \otimes \mathbb{C}Q_1 \otimes A \to A \otimes \mathbb{C}Q_0 \otimes A$ restricts to a map $A \otimes \mathbb{C}F_1 \otimes A \to A \otimes \mathbb{C}F_0 \otimes A$, and thus taking quotients yields a map $\mu_1^\vee \colon A \otimes \mathbb{C}Q_1^m \otimes A \to A \otimes \mathbb{C}Q_0^m \otimes A$. Explicitly, $\mu_1^\vee$ is given by

$$\mu_1^\vee(1 \otimes \alpha \otimes 1) = 1 \otimes (1-e)\alpha - \alpha(1-e) \otimes 1.$$

Define $\mu_2^\vee$ to be the composition of $\bar{\mu}_2$ with the projection $A \otimes \mathbb{C}Q_1 \otimes A \to A \otimes \mathbb{C}Q_1^m \otimes A$; explicitly

$$\mu_2^\vee(1 \otimes \rho_\alpha \otimes 1) = \sum_{\beta \in Q_1^m} \Delta_\beta(\partial_\alpha W).$$

Finally, let $\mu_3^\vee = \bar{\mu}_3$. Then one can check that the isomorphisms of A-bimodules defined above induce an isomorphism of Ω_A with the complex

$$A \otimes \mathbb{C}Q_3 \otimes A \xrightarrow{\mu_3^\vee} A \otimes \mathbb{C}Q_2 \otimes A \xrightarrow{\mu_2^\vee} A \otimes \mathbb{C}Q_1^m \otimes A \xrightarrow{\mu_1^\vee} A \otimes \mathbb{C}Q_0^m \otimes A.$$

As an example, we show that our isomorphisms relate the map $\mu_2^* \colon (A \otimes \mathbb{C}Q_1 \otimes A, A^\varepsilon) \to (A \otimes \mathbb{C}Q_2^m \otimes A, A^\varepsilon)$ to the map $\mu_2^\vee \colon A \otimes \mathbb{C}Q_2 \otimes A \to A \otimes \mathbb{C}Q_1^m \otimes A$. It suffices to check this on the generators $1 \otimes \rho_\alpha \otimes 1$ of $A \otimes \mathbb{C}Q_2 \otimes A$. First observe that under the isomorphism $(A \otimes \mathbb{C}Q_1 \otimes A, A^\varepsilon) \xrightarrow{\sim} A \otimes \mathbb{C}Q_2 \otimes A$, the preimage of $1 \otimes \rho_\alpha \otimes 1 = e_{t\alpha} \otimes \rho_\alpha \otimes e_{h\alpha}$ is the A-bimodule homomorphism f_α determined by

$$f_\alpha(1 \otimes \beta \otimes 1) = \begin{cases} e_{h\alpha} \otimes e_{t\alpha}, & \beta = \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

We then calculate for each $\beta \in Q_1^m$ that

$$\begin{aligned}\mu_2^*(f_\alpha)(1 \otimes \rho_\beta \otimes 1) &= f_\alpha \mu_2(1 \otimes \rho_\beta \otimes 1) \\ &= f_\alpha\Big(\sum_{\gamma \in Q_1} \Delta_\gamma(\partial_\beta W)\Big) \\ &= \mathbf{x}_\beta \otimes \mathbf{y}_\beta,\end{aligned}$$

where

$$\Delta_\alpha(\partial_\beta W) = \mathbf{x}_\beta \otimes \alpha \otimes \mathbf{y}_\beta.$$

It follows from the above formula that

$$\Delta_\beta(\partial_\alpha W) = \mathbf{y}_\beta \otimes \beta \otimes \mathbf{x}_\beta.$$

Thus the isomorphism $(A \otimes \mathbb{C}Q_2^m \otimes A, A^\varepsilon) \xrightarrow{\sim} A \otimes \mathbb{C}Q_1^m \otimes A$ takes $\mu_2^*(f_\alpha)$ to

$$\sum_{\beta \in Q_1^m} \mathbf{y}_\beta \otimes \beta \otimes \mathbf{x}_\beta = \sum_{\beta \in Q_1^m} \Delta_\beta(\partial_\alpha W) = \mu_2^\vee(1 \otimes \rho_\alpha \otimes 1)$$

as required.

We can express all of this data in the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \otimes \mathbb{C}F_1 \otimes A \arrow[r] \arrow[d] & A \otimes \mathbb{C}F_0 \otimes A \arrow[d] \\
A \otimes \mathbb{C}Q_3^m \otimes A \arrow[r, "-\mu_3"] \arrow[d, "+"] & A \otimes \mathbb{C}Q_2^m \otimes A \arrow[r, "-\mu_2"] \arrow[d, "-"] & A \otimes \mathbb{C}Q_1 \otimes A \arrow[r, "-\mu_1"] \arrow[d, "+"] & A \otimes \mathbb{C}Q_0 \otimes A \arrow[d, "-"] \\
A \otimes \mathbb{C}Q_3 \otimes A \arrow[r, "\mu_3^\vee"] \arrow[d] & A \otimes \mathbb{C}Q_2 \otimes A \arrow[r, "\mu_2^\vee"] \arrow[d] & A \otimes \mathbb{C}Q_1^m \otimes A \arrow[r, "\mu_1^\vee"] \arrow[d] & A \otimes \mathbb{C}Q_0^m \otimes A \arrow[d] \\
A \otimes \mathbb{C}F_3 \otimes A \arrow[r] \arrow[d] & A \otimes \mathbb{C}F_2 \otimes A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd} \tag{5.2}$$

in which the columns are split exact, the second row is $\mathbf{P}(A)[-3]$, the third row is Ω_A, and the signs on the vertical arrows indicate whether the corresponding map is the inclusion or its negative.

The diagram (5.2) provides us with a map of complexes $A[-3] \cong \mathbf{P}(A)[-3] \to \Omega_A$ in $\mathcal{D}^b A^\varepsilon$, and shows that the cone of this map has the form

$$C = A \otimes \mathbb{C}F_3 \otimes A \longrightarrow A \otimes (\mathbb{C}F_2 \oplus \mathbb{C}F_1) \otimes A \longrightarrow A \otimes \mathbb{C}F_0 \otimes A.$$

Since $e(\mathbb{C}F_i)e = \mathbb{C}F_i$ for all i, we have $\underline{A} \overset{\mathbf{L}}{\otimes}_A C = 0 = C \overset{\mathbf{L}}{\otimes}_A \underline{A}$ as required. $\square$

The statement that there exists a quasi-isomorphism $\mathbf{P}(A) \xrightarrow{\sim} A$ is stronger than the statement that A is internally 3-Calabi–Yau with respect to e. For example, if $i \in F_0$ then we have $A \otimes \mathbb{C}Q_3^m \otimes S_i = 0$, and so $\operatorname{p.dim} S_i \leq 2$. Recall from Proposition 4.21 and Lemma 4.17 that if $T \in \operatorname{Sub} Q_J$ is cluster-tilting, then the simple $\operatorname{End}_\Pi(T)^{\mathrm{op}}$-modules corresponding to the projective-injective summands of T have projective dimension at most 2.

Buan–Iyama–Reiten–Smith [BIRS11, Thm. 6.6] show that for reachable cluster-tilting objects $T \in \mathscr{R}_J \subseteq \operatorname{Sub} Q_J$, there exists an isomorphism

$$\Phi\colon A = \mathcal{J}(Q, F, W) \xrightarrow{\sim} \operatorname{End}_\Pi(T)^{\mathrm{op}}$$

for some ice quiver with potential (Q, F, W), such that $\Phi(e)$ is the map given by projection onto the projective-injective summands of T. The construction of (Q, F, W) will be recalled in Section 6.1. The existence of a quasi-isomorphism $\mathbf{P}(A) \xrightarrow{\sim} A$ in this case would explain the above observation on the projective dimensions of simple $\operatorname{End}_\Pi(T)^{\mathrm{op}}$-modules. Even more strikingly, applying Φ to the projective resolutions of simple $\operatorname{End}_\Pi(T)^{\mathrm{op}}$-modules computed in Lemma 4.17 shows that, for any $i \in Q_0$, the complex $\mathbf{P}(A) \otimes_A S_i$ is a projective resolution of the simple A-module S_i.

For $A \cong \operatorname{End}_\Pi(T)^{\mathrm{op}}$ as above, we also have $\operatorname{GP}(eAe) \simeq \operatorname{Sub} Q_J$ by Theorem 3.4, and so $\underline{\operatorname{GP}}(eAe) \simeq \underline{\operatorname{Sub}}\, Q_J$ is 2-Calabi–Yau. We conjecture that this property would also follow directly from the existence of a quasi-isomorphism $\mathbf{P}(A) \xrightarrow{\sim} A$.

Conjecture 5.25. *Let $A = \mathcal{J}(Q, F, W)$ be a frozen Jacobian algebra, let $e = \sum_{v \in F_0} e_v$ be its frozen idempotent and write $B = eAe$. Assume that A is Noetherian, $A/\langle e \rangle$ is finite-dimensional, and $\mu_0\colon \mathbf{P}(A) \to A$ is a quasi-isomorphism (meaning that A and A^{op} are internally 3-Calabi–Yau with respect to e, by Theorem 5.24 and Corollary 5.12). Then $\operatorname{GP}(B)$, which is a Frobenius category by Theorem 5.13, is stably 2-Calabi–Yau.*

We close with some observations about the homological algebra of B under the assumptions of Conjecture 5.25, that we hope may lead to a proof of this conjecture; see the discussion of Question 2 in Section 7.1 below.

Proposition 5.26 (cf. [AIR15, Rem. 2.7])**.** *Let A be a frozen Jacobian algebra such that $\mu_0\colon \mathbf{P}(A) \to A$ is a quasi-isomorphism, and let $B = eAe$. Let $\Omega_B = \mathbf{R}\mathrm{Hom}_{B^\varepsilon}(B, B^\varepsilon)$. Then $\Omega_B \cong e\Omega_A e$ in $\mathcal{D}^b B^\varepsilon$.*

Proof. Write $P_i = A \otimes \mathbb{C}Q_i \otimes A$ for $i = 0, 1$ and $P_i = A \otimes \mathbb{C}Q_i^m \otimes A$ for $i = 2, 3$. By Theorem 5.24, A and A^{op} are internally 3-Calabi–Yau with respect to e, so we have $\operatorname{Ext}^i_B(eA, B) = 0 = \operatorname{Ext}^i_{B^{\mathrm{op}}}(Ae, B)$ for all $i > 0$ by Proposition 5.16. Thus we may calculate

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{B^\varepsilon}(eA\otimes_A Ae, B^\varepsilon) &= \mathbf{R}\mathrm{Hom}_B(eA,B)\otimes_A \mathbf{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Ae,B)\\ &= \mathrm{Hom}_B(eA,B)\otimes_A \mathrm{Hom}_{B^{\mathrm{op}}}(Ae,B)\\ &= \mathrm{Hom}_{B^\varepsilon}(eA\otimes_A Ae, B^\varepsilon).\end{aligned}$$

Thus we see that the terms eP_ie of the sequence $e\mathbf{P}(A)e \cong B$ satisfy $\mathrm{Ext}^i_{B^\varepsilon}(eP_ie, B^\varepsilon) = 0$ for $i > 0$, and so

$$\mathbf{R}\mathrm{Hom}_{B^\varepsilon}(B, B^\varepsilon) \cong \mathrm{Hom}_{B^\varepsilon}(e\mathbf{P}(A)e, B^\varepsilon).$$

By Theorem 5.13(iii), the functor $eA\otimes_A - \otimes_A Ae\colon \operatorname{proj} A^\varepsilon \to \operatorname{mod} B^\varepsilon$ is fully faithful, and so

$$\mathrm{Hom}_{B^\varepsilon}(eP_ie, B^\varepsilon) \cong \mathrm{Hom}_{A^\varepsilon}(P_i, Ae\otimes_{\mathbb{C}} eA) = e\,\mathrm{Hom}_{A^\varepsilon}(P_i, A^\varepsilon)e.$$

It follows that

$$\begin{aligned}\Omega_B &= \mathbf{R}\mathrm{Hom}_{B^\varepsilon}(B, B^\varepsilon)\\ &\cong \mathrm{Hom}_{B^\varepsilon}(e\mathbf{P}(A)e, B^\varepsilon)\\ &\cong e\,\mathrm{Hom}_{A^\varepsilon}(\mathbf{P}(A), B^\varepsilon)e\\ &\cong e\,\mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon)e = e\Omega_A e.\end{aligned}$$

□

Proposition 5.27. *Let A be a frozen Jacobian algebra such that $\mu_0\colon \mathbf{P}(A) \to A$ is a quasi-isomorphism, and let $B = eAe$. Then for any $X \in \mathcal{D}^bB$, we have*

$$\Omega_B \overset{\mathbf{L}}{\otimes}_B X \cong X[-3]$$

in the quotient category $\mathcal{D}^bB/\operatorname{per} B \simeq \underline{\mathrm{GP}}(B)$.

Proof. The proof of Theorem 5.24 constructs a map $A[-3] \to \Omega_A$ with mapping cone

$$C = A\otimes \mathbb{C}F_3 \otimes A \longrightarrow A\otimes(\mathbb{C}F_2\oplus\mathbb{C}F_1)\otimes A \longrightarrow A\otimes\mathbb{C}F_0\otimes A.$$

Since each S-bimodule $\mathbb{C}F_i$ has the property that $e(\mathbb{C}F_i)e = \mathbb{C}F_i$, we can instead write C as

$$Ae\otimes \mathbb{C}F_3 \otimes eA \longrightarrow Ae\otimes(\mathbb{C}F_2\oplus\mathbb{C}F_1)\otimes eA \longrightarrow Ae\otimes\mathbb{C}F_0\otimes eA.$$

Now applying the functor $eA\otimes_A - \otimes_A Ae$ to the triangle $A[-3] \to \Omega_A \to C \to A[-2]$ in $\operatorname{per} A^\varepsilon$ yields the triangle

$$B[-3] \longrightarrow e\Omega_A e \longrightarrow eCe \longrightarrow B[-2]$$

in $\mathcal{D}^b B^\varepsilon$. We have

$$eCe = B \otimes \mathbb{C}F_3 \otimes B \longrightarrow B \otimes (\mathbb{C}F_2 \oplus \mathbb{C}F_1) \otimes B \longrightarrow B \otimes \mathbb{C}F_0 \otimes B \in \operatorname{per} B^\varepsilon$$

and $e\Omega_A e \cong \Omega_B$ by Proposition 5.26. So applying $- \overset{\mathbf{L}}{\otimes}_B X$ to the above triangle yields the triangle

$$X[-3] \longrightarrow \Omega_B \overset{\mathbf{L}}{\otimes}_B X \longrightarrow eCe \overset{\mathbf{L}}{\otimes}_B X \longrightarrow X[-2]$$

in $\mathcal{D}^b B$. Since $eCe \in \operatorname{per} B^\varepsilon$, we have $eCe \overset{\mathbf{L}}{\otimes}_B X \in \operatorname{per} B$, and so the above triangle shows that $\Omega_B \overset{\mathbf{L}}{\otimes}_B X \cong X[-3]$ in the quotient $\mathcal{D}^b B / \operatorname{per} B$. This quotient is equivalent to $\underline{\mathrm{GP}}(B)$ by a result of Buchweitz [Buc87, Thm. 4.4.1], see also Keller–Vossieck [KV87]. $\square$

CHAPTER 6

SEEDS, ENDOMORPHISM ALGEBRAS AND CATEGORIFICATION

Theorem 5.13 suggests a method for obtaining a Frobenius categorification of a cluster algebra with frozen variables. Given the quiver of a seed of such a cluster algebra, one can attempt to add arrows between the frozen vertices to get an ice quiver (Q, F). If there is a potential W on Q such that the frozen Jacobian algebra $A = \mathcal{J}(Q, F, W)$ is internally bimodule 3-Calabi–Yau with respect to the boundary idempotent $e = \sum_{v \in F_0} e_v$, then it follows from Theorem 5.13 that $A \cong \mathrm{End}_B(eA)^{\mathrm{op}}$, where $B = eAe$, and eA is cluster-tilting in the Frobenius category $\mathrm{GP}(B)$.

In this chapter, we will attempt to apply this procedure to seeds of Geiß–Leclerc–Schröer's cluster algebra $\widetilde{\mathscr{A}_J} \subseteq \mathbb{C}[F_J]$. Unfortunately, we do not yet have sufficient technology to easily verify that a given algebra, particularly an infinite dimensional one, is internally bimodule 3-Calabi–Yau. Thus it is currently only conjectural that our constructions provide examples of such algebras. However, as well as providing a target for future techniques to aim at, our constructions have some interesting combinatorial properties and suggest methods for constructing Frobenius categories with cluster-tilting objects in wider generality. The main result of this section is Theorem 6.11, which shows that our construction categorifies Geiß–Leclerc–Schröer's homogenisation procedure for obtaining a seed of $\widetilde{\mathscr{A}_J}$ from one of $\mathscr{A}_J$. We will also show by example that our constructions can recover the endomorphism algebras of cluster-tilting objects of categorifications of $\widetilde{\mathscr{A}_J}$ where they exist.

To begin, we recall descriptions of some of the seeds of $\mathscr{A}_J$ and $\widetilde{\mathscr{A}_J}$, together with descriptions of the algebras $\mathrm{End}_\Pi(T)^{\mathrm{op}}$ for the corresponding cluster-tilting objects T in $\operatorname{Sub} Q_J$.

6.1 The Inhomogeneous Case

In this section, we describe how to construct seeds for the cluster algebra $\mathscr{A}_J$, each of which corresponds to a cluster-tilting object $T \in \operatorname{Sub} Q_J$. For convenience, we reproduce the constructions from [GLS08], [BIRS09] and [BIRS11]. In fact we will end up describing much more than just a seed in each case; we will describe a method for computing a presentation of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ as a frozen Jacobian algebra, and thus of the stable endomorphism algebra $\underline{\operatorname{End}}_J(T)^{\mathrm{op}} = \operatorname{End}_{\underline{\operatorname{Sub}} Q_J}(T)^{\mathrm{op}}$ as a Jacobian algebra. All of the results in this section are due to either Buan–Iyama–Reiten–Scott [BIRS09] or Buan–Iyama–Reiten–Smith [BIRS11], as indicated.

We first recall how to construct a cluster-tilting object $T_{\mathbf{i},J}$ of $\operatorname{Sub} Q_J$ from the data of a reduced expression $\mathbf{i}$ for the Weyl group element $w_J = w_0^K w_0 \in W(\Delta)$. These special cluster-tilting objects are sometimes called *standard* [ART11]. If $T_{\mathbf{i},J}$ is a standard cluster-tilting object, then a presentation of $\operatorname{End}_\Pi(T_{\mathbf{i},J})^{\mathrm{op}}$ as a frozen Jacobian algebra can be computed purely combinatorially from $\mathbf{i}$, as in [BIRS11, §6].

Recall from Section 2.6 that all of the standard cluster-tilting objects lie in the same mutation class $\mathscr{R}_J$, and we call cluster-tilting objects in this class *reachable*. If $T \in \mathscr{R}_J$, then we can present $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ as a frozen Jacobian algebra by finding a sequence of mutations from a standard cluster-tilting object $T_{\mathbf{i},J}$ to T, and applying the same sequence of mutations to the frozen Jacobian algebra $\operatorname{End}_\Pi(T_{\mathbf{i},J})^{\mathrm{op}}$; see Definition 3.15 and Theorem 6.6.

Let $\mathbf{i} = (i_1, \dots, i_r)$ be a reduced expression for w_J. Recall from Proposition 2.19 that each $w \in W(\Delta)$ gives rise to an ideal I_w of Π. For each $1 \le k \le r$, let $(w_J)_{\le k} = s_{i_1} \cdots s_{i_k}$, and let $T_k = P_{i_k}/I_{(w_J)_{\le k}} P_{i_k}$, where $P_{i_k} = \Pi e_{i_k}$ is the indecomposable projective Π-module at vertex i_k. Define $T_{\mathbf{i},J} = \bigoplus_{k=1}^r T_k$.

Theorem 6.1 ([BIRS09, Thm. III.2.8(b)])**.** *The object $T_{\mathbf{i},J}$ is cluster-tilting in* $\operatorname{Sub} Q_J$.

As usual, given any cluster-tilting object T of $\operatorname{Sub} Q_J$, we denote the Gabriel quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ by Γ_T, and equip it with the structure of an ice quiver by taking the frozen subquiver $F(T)$ to be the full subquiver on the vertices corresponding to the projective-injective summands of T. If $T = T_{\mathbf{i},J}$, we can compute this quiver combinatorially from the expression $\mathbf{i}$. Given $1 \le k \le r$, define

$$k^+ = \begin{cases} \min\{l > k : i_l = i_k\}, & \text{if defined,} \\ r+1, & \text{otherwise.} \end{cases}$$

We use this data to determine an ice quiver $(\Gamma_{\mathbf{i},J}, F_{\mathbf{i},J})$ with frozen vertices. The vertex set is $\{i : 1 \le i \le r\}$. We have two types of arrows. Firstly, there are *horizontal* arrows $k \to k^+$ for each k such that $k^+ \ne r+1$. We also have a *slanted* arrow $l \to k$ when

(i) i_k and i_l are adjacent in Δ, and

(ii) $k < l < k^+ \le l^+$.

The frozen subquiver $F_{\mathbf{i},J}$ is the full subquiver of $\Gamma_{\mathbf{i},J}$ on vertices k such that $k^+ = r+1$; these are precisely the k for which T_k is projective-injective in $\operatorname{Sub} Q_J$. Note that if $k < l$, then $k^+ = l^+$ if and only if k and l are both frozen. It follows that $F_{\mathbf{i},J}$ is some orientation of Δ, determined by the order in which the vertices occur for the final time in $\mathbf{i}$.

Definition 6.2. Define $\Phi\colon \mathbb{C}\Gamma_{\mathbf{i},J} \to \operatorname{End}_\Pi(T_{\mathbf{i},J})^{\mathrm{op}}$ as follows. We take $\Phi(e_k)$ to be the composition $T \to T_k \to T$ of the natural projection and inclusion maps. Since $I_{(w_J)_{\leq k}} \supseteq I_{(w_J)_{\leq k^+}}$ for each k, there is a projection $T_{k^+} \to T_k$; the image of Φ on a horizontal arrow is taken to be this projection, appropriately composed with the natural projections and inclusions between T and its summands. If there is a slanted arrow $k \to l$, then i_k and i_l are adjacent vertices in Δ and so there is a unique arrow $\alpha\colon i_l \to i_k$ in Π. The image of Φ on the slanted arrow $k \to l$ is taken to be the map $T_l \to T_k$ given by right multiplication by α, again composed appropriately with projections and inclusions. Recall that T_k is realised explicitly as a quotient of Πe_{i_k}, so this multiplication is well-defined.

Proposition 6.3 ([BIRS09, Thm. III.4.1], see also [BIRS11, Lem. 6.1])**.** *The map Φ from Definition 6.2 is surjective, and has admissible kernel. It follows that $\Gamma_{\mathbf{i},J} = \Gamma_{T_{\mathbf{i},J}}$.*

When drawing Γ_T for a cluster-tilting object T, we will usually represent the vertex k by a picture of the radical filtration of T_k. However, it is important to remember that Γ_T is the quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$, and so an arrow $k \to l$ represents a map $T_l \to T_k$.

Example 6.4. As an example, we take Δ of type A_4, with nodes labelled

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4$$

and take $J = \{1,3\}$. The word $\mathbf{i} = (1,2,3,4,2,3,1,2)$ is a reduced expression for w_J, giving rise to a cluster-tilting object $T_{\mathbf{i},J} = \bigoplus_{k=1}^{8} T_k$, with

$$T_1 = \begin{smallmatrix} 1 \end{smallmatrix} \qquad\qquad T_5 = \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix}$$

$$T_2 = \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix} \qquad\qquad T_6 = \begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \end{smallmatrix} = F_2$$

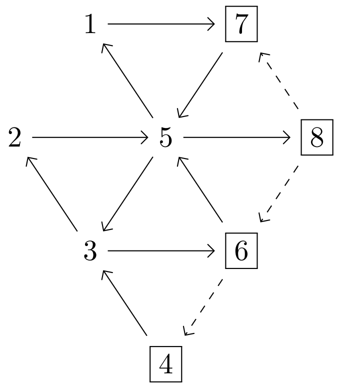
$$T_3 = \begin{smallmatrix} & & 3 \\ & 2 & \\ 1 & & \end{smallmatrix} \qquad\qquad T_7 = \begin{smallmatrix} 1 & & \\ & 2 & \\ & & 3 \end{smallmatrix} = F_4$$

$$T_4 = \begin{smallmatrix} & & & 4 \\ & & 3 & \\ & 2 & & \\ 1 & & & \end{smallmatrix} = F_1 \qquad\qquad T_8 = \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix} = F_3$$

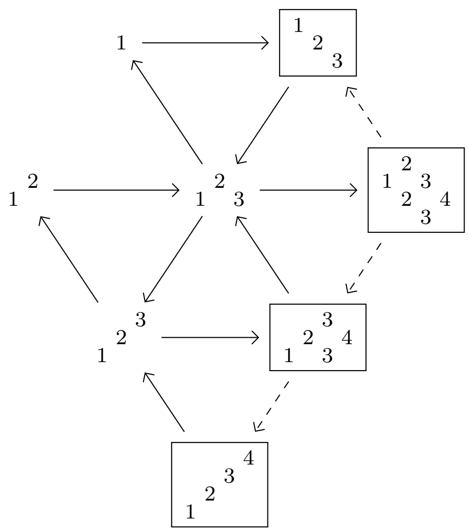
Our combinatorial data is

i_k	1	2	3	4	2	3	1	2
k	1	2	3	4	5	6	7	8
k^+	7	5	6	9	8	9	9	9

from which we compute that $(\Gamma_{\mathbf{i},J}, F_{\mathbf{i},J})$ is the quiver



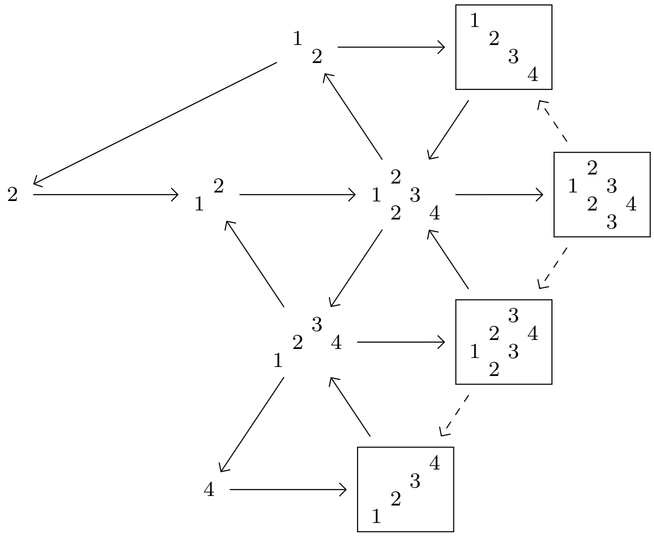
where the frozen vertices are those whose labels appear in boxes, and the frozen arrows are dashed. We can also compute directly that $\Gamma_{T_{\mathbf{i},J}}$ is the isomorphic quiver



where the full subquiver on vertices corresponding to projective-injective summands is frozen.

Recall that we denote the vertex set of Δ by I. From the reduced expression $\mathbf{i}' = (2, 4, 1, 2, 3, 4, 2, 3, 1, 2)$ for $w_I = w_0$, we may also compute $(\Gamma_{\mathbf{i}',I}, F_{\mathbf{i}',I})$, drawn below as

$\Gamma_{T_{\mathbf{i}',I}}$.



Fix an orientation of Δ, and use it to give a sign $\varepsilon(\alpha)$ to each slanted arrow $\alpha\colon l \to k$; we let $\varepsilon(\alpha) = 1$ if the edge between i_k and i_l is oriented towards i_k, and $\varepsilon(\alpha) = -1$ otherwise. Now define a potential $W_{\mathbf{i},J}$ on $\Gamma_{\mathbf{i},J}$ as follows. For every slanted arrow $\alpha\colon l \to k$, we define a cycle W_α. If there is an arrow $\alpha^*\colon k \to l'$ such that $i_{l'} = i_l$, then by construction this arrow is unique, and there is a unique path $p\colon l' \to l$ consisting solely of horizontal arrows, so we let $W_\alpha = \varepsilon(\alpha)p\alpha^*\alpha$. Otherwise, $W_\alpha = 0$. Let

$$W_{\mathbf{i},J} = \sum_\alpha W_\alpha,$$

with the sum taken over slanted arrows.

Theorem 6.5 ([BIRS11, Thm. 6.5]). *The potential $W_{\mathbf{i},J}$ is rigid (see Definition 3.17).*

Let $\underline{\Gamma}_{\mathbf{i},J}$ be the full subquiver of $\Gamma_{\mathbf{i},J}$ on the mutable vertices, and let $\underline{W}_{\mathbf{i},J}$ be the potential on $\underline{\Gamma}_{\mathbf{i},J}$ obtained by deleting terms of $W_{\mathbf{i},J}$ given by cycles passing through frozen vertices, or equivalently by summing the W_α for α unfrozen. Then we have the following.

Theorem 6.6 ([BIRS11, Thm. 6.6]). *The map Φ from Definition 6.2 induces isomorphisms*

$$\begin{aligned} \mathcal{J}(\Gamma_{\mathbf{i},J}, F_{\mathbf{i},J}, W_{\mathbf{i},J}) &\xrightarrow{\sim} \operatorname{End}_\Pi(T_{\mathbf{i},J})^{\mathrm{op}}, \\ \mathcal{J}(\underline{\Gamma}_{\mathbf{i},J}, \underline{W}_{\mathbf{i},J}) &\xrightarrow{\sim} \underline{\operatorname{End}}_J(T_{\mathbf{i},J})^{\mathrm{op}}. \end{aligned}$$

Thus by [BIRS11, Thm. 5.3], *if $T \in \mathscr{R}_J$, meaning that T is related to $T_{\mathbf{i},J}$ by a sequence of mutations, then applying the same sequence of mutations to $(\Gamma_{\mathbf{i},J}, F_{\mathbf{i},J}, W_{\mathbf{i},J})$ and*

$(\underline{\Gamma}_{\mathbf{i},J}, \underline{W}_{\mathbf{i},J})$ *gives* (Γ, F, W) *and* $(\underline{\Gamma}, \underline{W})$ *such that*

$$\mathcal{J}(\Gamma, F, W) \cong \operatorname{End}_J(T)^{\mathrm{op}},$$
$$\mathcal{J}(\underline{\Gamma}, \underline{W}) \cong \underline{\operatorname{End}}_J(T)^{\mathrm{op}}.$$

6.2 The Homogeneous Case

We fix a cluster-tilting object $T \in \operatorname{Sub} Q_J$ for the entire section. Recall from Section 2.6 that if $T \in \mathscr{R}_J$, then T determines a seed of the cluster algebra $\mathscr{A}_J \subseteq \mathbb{C}[N]$, which can be homogenised to obtain a seed of the cluster algebra $\widetilde{\mathscr{A}_J} \subseteq \mathbb{C}[F_J]$. In this section, we lift this homogenisation at the level of seeds to a construction of a frozen Jacobian algebra $\widetilde{A} = \mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W})$. Let $e = \sum_{v \in \widetilde{F}_0} e_v$ be the frozen idempotent of $\widetilde{A}$, and write $\widetilde{B}_J = e\widetilde{A}e$ for the boundary algebra. We will attempt to construct $\widetilde{A}$ in such a way that the Frobenius category $\operatorname{GP}(\widetilde{B}_J)$ categorifies the cluster algebra $\widetilde{\mathscr{A}_J}$, and that $\widetilde{A} \cong \operatorname{End}_{\widetilde{B}_J}(\widetilde{T})^{\mathrm{op}}$ for some $\widetilde{T} \in \operatorname{GP}(\widetilde{B}_J)$. Unfortunately, since $\widetilde{A}$ is an infinite dimensional algebra with a complicated description, we will be unable to verify these properties except in very specific cases. However, our construction has some interesting combinatorial properties, and categorifies Geiß–Leclerc–Schröer's construction of the enlarged quiver $\widetilde{\Gamma}_T$; see Theorem 6.11.

For now, we do not need to assume that $T \in \mathscr{R}_J$, but we will add this assumption later. Let $\underline{\Gamma}_T$ be the quiver of the stable endomorphism algebra $\underline{\operatorname{End}}_J(T)^{\mathrm{op}}$. Pick a decomposition

$$T = \left(\bigoplus_{v \in (\underline{\Gamma}_T)_0} T_v \right) \oplus \left(\bigoplus_{i \in \Delta_0} F_i \right)$$

of T into indecomposable summands, which are pairwise non-isomorphic since we always assume that T is basic. It will be convenient to write $T_i = F_i$ for $i \in \Delta_0$, so that we can take the set of vertices of the quiver Γ_T of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$ to be $(\underline{\Gamma}_T)_0 \cup \Delta_0$, and write

$$T = \bigoplus_{v \in (\Gamma_T)_0} T_v.$$

Let $\widetilde{\Delta}_0^J = \Delta_0 \cup J^*$, where J^* is the set of symbols $\{j^* : j \in J\}$. For each $i \in \widetilde{\Delta}_0^J$, let

$$U_i = \begin{cases} \mathfrak{m}(F_i) = \ker(F_i \to \operatorname{top} F_i), & i \in \Delta_0, \\ \tau_J S_j, & i = j^*, \end{cases}$$

and write $V_i = \tau_J^{-1} U_i$. To improve legibility, we will usually write $U_{j^*} = U_j^*$, $V_{j^*} = V_j^*$, and so on.

Recall that U_i is never zero in $\underline{\operatorname{Sub}}\, Q_J$ since we are excluding degenerate cases (Remark 3.2), but it may not be indecomposable. It follows that V_i is also non-zero in $\underline{\operatorname{Sub}}\, Q_J$. For each $i \in \widetilde{\Delta}_0^J$, make an explicit choice of minimal left $\operatorname{add} T$-approximation

$l_i\colon V_i \to L_i$, with L_i realised as a direct sum of the summands T_v, for $v \in (\underline{\Gamma}_T)_0$, of T in $\underline{\mathrm{Sub}}\, Q_J$. Pick a triangle in $\underline{\mathrm{Sub}}\, Q_J$ with l_i as its first map; by Theorem 4.12, such a triangle has the form

$$V_i \xrightarrow{l_i} L_i \longrightarrow R_i \xrightarrow{r_i} U_i = \tau_J V_i$$

for $r_i\colon R_i \to U_i$ a minimal right $\operatorname{add} T$-approximation of U_i in $\underline{\mathrm{Sub}}\, Q_J$. By composing with an isomorphism if necessary, we may also assume that R_i is realised as a direct sum of the summands T_v of T. Fixing l_i and r_i in this way uniquely determines the above triangle.

For each $X, Y \in \underline{\mathrm{Sub}}\, Q_J$, write $\underline{\mathrm{Hom}}_J^T(X,Y)$ for the subspace of $\underline{\mathrm{Hom}}_J(X,Y)$ consisting of maps factoring through T, and pick a basis $\{d_{ij}^k\}_k$ of $\underline{\mathrm{Hom}}_J^T(V_i,U_j)$ for each $(i,j) \in (\widetilde{\Delta}_0^J)^2$. Writing

$$\underline{\mathrm{Hom}}_T(X,Y) = \underline{\mathrm{Hom}}_J(X,Y)/\,\underline{\mathrm{Hom}}_J^T(X,Y)$$

for the space of maps $X \to Y$ modulo those factoring through T, we have a duality

$$\mathrm{D}\,\underline{\mathrm{Hom}}_J^T(V_i,U_j) \cong \underline{\mathrm{Hom}}_T(V_j,U_i)$$

by Theorem 4.14. Thus our choice of basis $\{d_{ij}^k\}_k$ of $\underline{\mathrm{Hom}}_J^T(V_i,U_j)$ determines a dual basis of $\underline{\mathrm{Hom}}_T(V_j,U_i)$.

Proposition 6.7. *For any $i,j \in \widetilde{\Delta}_0^J$, we have*

$$\underline{\mathrm{hom}}_J(V_i,U_j) = \mathrm{ext}^1_\Pi(V_j,V_i) = \mathrm{ext}^1_\Pi(V_i,V_j) = \underline{\mathrm{hom}}_J(V_j,U_i).$$

Moreover, if $(i,j) \in \Delta_0^2 \cup (J^*)^2 \subseteq (\widetilde{\Delta}_0^J)^2$, then $\underline{\mathrm{hom}}_J(V_j,U_i) \le 1$.

Proof. The equalities all follow from the Auslander–Reiten formula (Proposition 2.17), recalling that $U_i = \tau_J V_i$, and the fact that $\operatorname{mod}\Pi$ is stably 2-Calabi–Yau. For the inequality, there are two cases.

(i) If $i,j \in \Delta_0$, then $U_i = \mathfrak{m}(F_i)$ and $U_j = \mathfrak{m}(F_j)$. Since F_i has simple top $S_{\nu(i)}$, where ν is the Nakayama involution, we have a short exact sequence

$$0 \longrightarrow U_i \longrightarrow F_i \longrightarrow S_{\nu(i)} \longrightarrow 0.$$

Applying $\mathrm{Hom}_\Pi(-,S_{\nu(j)})$ yields

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_\Pi(S_{\nu(i)},S_{\nu(j)}) & \xrightarrow{\sim} & \mathrm{Hom}_\Pi(F_i,S_{\nu(j)}) & \longrightarrow & \mathrm{Hom}_\Pi(U_i,S_{\nu(j)}) \\ & & & & & & \downarrow \\ & & & & & & \mathrm{Ext}^1_\Pi(S_{\nu(i)},S_{\nu(j)}), \end{array}$$

and so

$$\hom_\Pi(U_i, S_{\nu(j)}) \leq \operatorname{ext}^1_\Pi(S_{\nu(i)}, S_{\nu(j)}) \leq 1.$$

Similarly, we have a short exact sequence

$$0 \longrightarrow U_j \longrightarrow F_j \longrightarrow S_{\nu(j)} \longrightarrow 0,$$

and applying $\operatorname{Hom}_\Pi(U_i, -)$ yields

$$\begin{array}{c} 0 \longrightarrow \operatorname{Hom}_\Pi(U_i, U_j) \longrightarrow \operatorname{Hom}_\Pi(U_i, F_j) \longrightarrow \operatorname{Hom}_\Pi(U_i, S_{\nu(j)}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Ext}^1_\Pi(U_i, U_j) \longrightarrow 0 \end{array}$$

from which it follows that $\operatorname{ext}^1_\Pi(U_i, U_j) \leq \hom_\Pi(U_i, S_{\nu(j)}) \leq 1$. Since $U_i = \tau_J V_i$ and $U_j = \tau_J V_j$, we also have $\operatorname{ext}^1_\Pi(V_i, V_j) \leq 1$, as required.

(ii) If $i = i^*$ and $j = j^*$ are in J^*, then we have $V_i = S_i$ and $V_j = S_j$, so

$$\operatorname{ext}^1_\Pi(V_i, V_j) = \operatorname{ext}^1_\Pi(S_i, S_j) \leq 1. \qquad \square$$

It follows from Proposition 6.7 that if $(i, j) \in \Delta_0^2 \cup (J^*)^2$ then $\underline{\operatorname{Hom}}_J^T(V_i, U_j)$ is either 0 or equal to $\underline{\operatorname{Hom}}_J(V_i, U_j)$, and in the second case our chosen basis consists of a single non-zero element d_{ij}. We believe that $\underline{\hom}_J(V_i, U_j) \leq 1$ for any pair $(i, j) \in (\widetilde{\Delta}_0^J)^2$.

We start by describing the ice quiver $(\widetilde{\Gamma}_T, \widetilde{F})$. The mutable vertices of $\widetilde{\Gamma}_T$ are the vertices of $\underline{\Gamma}_T$, one for each indecomposable summand of T in $\underline{\operatorname{Sub}}\, Q_J$, and the frozen vertices are given by the set $\widetilde{\Delta}_0^J$. The full subquiver on the mutable vertices is taken to be $\underline{\Gamma}_T$, and the arrows incident to the frozen vertices are defined as follows.

(i) For each frozen vertex i and mutable vertex v, there is an arrow $\alpha\colon v \to i$ for each component $l_i^\alpha\colon V_i \to T_v$ of $l_i\colon V_i \to L_i$.

(ii) For each frozen vertex i and mutable vertex v, there is an arrow $\beta\colon i \to v$ for each component $r_i^\beta\colon T_v \to U_i$ of $r_i\colon R_i \to U_i$.

(iii) For each pair i, j of frozen vertices, there is an arrow $\delta_{ij}^k\colon j \to i$ for each element of our chosen basis $\{d_{ij}^k\}_k$ of $\underline{\operatorname{Hom}}_J^T(V_j, U_i)$.

In part (iii), we think of the arrows δ_{ij}^k as corresponding to the basis of $\underline{\operatorname{Hom}}_T(V_i, U_j) = \mathrm{D}\,\underline{\operatorname{Hom}}_J^T(V_j, U_i)$ dual to $\{d_{ij}^k\}_k$. The frozen subquiver $\widetilde{F}$ is taken to be the full subquiver on the set $\widetilde{\Delta}_0^J$ of frozen vertices.

Remark 6.8. For $i \in \Delta_0$, the objects U_i and V_i have been chosen so that any non-split map $T \to F_i$ in $\operatorname{Sub} Q_J$ factors through U_i, and any non-split map $F_i \to T$ in $\operatorname{Sub} Q_J$ factors through V_i. Thus the full subquiver of $\widetilde{\Gamma}_T$ on the vertices $(\underline{\Gamma}_T)_0 \cup \Delta_0$ is, by construction, the quiver of $\operatorname{End}_\Pi(T)^{\mathrm{op}}$.

The following conjecture is based on computations of $\widetilde{\Gamma}_T$ in a large number of examples.

Conjecture 6.9. *The quiver $\widetilde{\Gamma}_T$ has no 2-cycles through its mutable vertices.*

Remark 6.10. By Remark 6.8, Conjecture 6.9 reduces to the statement that L_j^* and R_j^* have no isomorphic summands for any $j \in J$, and for certain T it is clear that this is the case. For example, if there are no edges of Δ between any two vertices of J, then the semi-simple module $S_J = \bigoplus_{j \in J} S_j$ is rigid, and thus can be extended (non-uniquely) to a maximal rigid object T_0 of $\operatorname{Sub} Q_J$, which is cluster-tilting by Theorem 4.22. Then a left $\operatorname{add} T_0$-approximation of S_j is an isomorphism $S_j \xrightarrow{\sim} S_j$, and so $L_j^* \cong S_j$. As

$$\underline{\operatorname{Hom}}_J(S_j, \tau_J S_j) = \operatorname{Ext}^1_\Pi(S_j, S_j) = 0,$$

any map $S_j \to \tau_J S_j$ factors through a projective-injective object, which is in particular an object of $\operatorname{add} T$ not containing S_j as a summand by Remark 3.2. Thus S_j cannot be isomorphic to a summand of R_j^*, and so $\widetilde{\Gamma}_{T_0}$ has no 2-cycles through its mutable vertices.

We now show that if $T \in \mathscr{R}_J$ is such that $\widetilde{\Gamma}_T$ has no 2-cycles through its mutable vertices, then the quiver of the seed of $\widetilde{\mathscr{A}_J}$ determined by T is obtained from $\widetilde{\Gamma}_T$ by removing the frozen arrows. Most importantly, we have to verify that $\widetilde{\Gamma}_T$ satisfies the balancing condition

$$\sum_{u \to v} \deg(u) = \sum_{v \to w} \deg(w)$$

at each mutable vertex v, where if $u \in (\Gamma_T)_0$ we set

$$\deg(u) = \deg(\varphi_{T_u}) = \sum_{j \in J} \hom(S_j, T_u)\varepsilon_j,$$

and for the other vertices we set

$$\deg(j^*) = \deg \Delta_{\omega_j, \omega_j} = \varepsilon_j,$$

recalling from Section 2.6 that $\Delta_{\omega_j,\omega_j}$ is one of the frozen variables of $\widetilde{\mathscr{A}_J}$ that is specialised to 1 to recover $\mathscr{A}_J$. Let v be a mutable vertex, and let

$$0 \longrightarrow T_v \longrightarrow X \longrightarrow T_v' \longrightarrow 0,$$
$$0 \longrightarrow T_v' \longrightarrow Y \longrightarrow T_v \longrightarrow 0$$

be the associated exchange sequences. From the definition of Γ_T, we have

$$X \cong \bigoplus_{u \to v} T_u,$$
$$Y \cong \bigoplus_{v \to w} T_w,$$

with the sums taken over arrows in Γ_T. Thus, writing $\mathrm{mult}_M(N)$ for the number of indecomposable summands of N isomorphic to M, we have

$$\begin{aligned}\sum_{u\to v}\deg(u)&=\sum_{j\in J}(\hom_\Pi(S_j,X)+\mathrm{mult}_{T_v}(R_j^*))\varepsilon_j,\\ \sum_{v\to w}\deg(w)&=\sum_{j\in J}(\hom_\Pi(S_j,Y)+\mathrm{mult}_{T_v}(L_j^*))\varepsilon_j,\end{aligned}$$

so the balancing condition at v is equivalent to the property that

$$\hom_\Pi(S_j,X)-\hom_\Pi(S_j,Y)=\mathrm{mult}_{T_v}(L_j^*)-\mathrm{mult}_{T_v}(R_j^*)$$

for all j.

Theorem 6.11. *Let T be a cluster-tilting object in $\mathrm{Sub}\,Q_J$, let v be a mutable vertex of $\widetilde{\Gamma}_T$, and let*

$$0\longrightarrow T_v\longrightarrow X\longrightarrow T_v'\longrightarrow 0,$$
$$0\longrightarrow T_v'\longrightarrow Y\longrightarrow T_v\longrightarrow 0$$

be the corresponding exchange sequences. Then for each $j\in J$ we have

$$\hom_\Pi(S_j,X)-\hom_\Pi(S_j,Y)=\mathrm{mult}_{T_v}(L_j^*)-\mathrm{mult}_{T_v}(R_j^*),$$

so $\widetilde{\Gamma}_T$ satisfies the balancing condition at each mutable vertex.

Proof. Fix $j\in J$, and write $L=L_j^*$ and $R=R_j^*$. Apply $\mathrm{Hom}_\Pi(L,-)$ and $\mathrm{Hom}_\Pi(R,-)$ to the exchange sequences, noting that

$$\mathrm{Ext}^1_\Pi(L,T_v)=\mathrm{Ext}^1_\Pi(L,Y)=\mathrm{Ext}^1_\Pi(R,T_v)=\mathrm{Ext}^1_\Pi(R,Y)=0,$$

to obtain exact sequences

$$\begin{array}{l}0\longrightarrow\mathrm{Hom}_\Pi(L,T_v)\longrightarrow\mathrm{Hom}_\Pi(L,X)\longrightarrow\mathrm{Hom}_\Pi(L,T_v')\longrightarrow 0,\\ 0\longrightarrow\mathrm{Hom}_\Pi(L,T_v')\longrightarrow\mathrm{Hom}_\Pi(L,Y)\longrightarrow\mathrm{Hom}_\Pi(L,T_v)\longrightarrow\mathrm{Ext}^1_\Pi(L,T_v')\longrightarrow 0,\\ 0\longrightarrow\mathrm{Hom}_\Pi(R,T_v)\longrightarrow\mathrm{Hom}_\Pi(R,X)\longrightarrow\mathrm{Hom}_\Pi(R,T_v')\longrightarrow 0,\\ 0\longrightarrow\mathrm{Hom}_\Pi(R,T_v')\longrightarrow\mathrm{Hom}_\Pi(R,Y)\longrightarrow\mathrm{Hom}_\Pi(R,T_v)\longrightarrow\mathrm{Ext}^1_\Pi(R,T_v')\longrightarrow 0.\end{array}$$

It follows that from the exactness of these sequences that

$$\mathrm{ext}^1_\Pi(L,T_v')-\mathrm{ext}^1_\Pi(R,T_v')=\hom_\Pi(L,X)-\hom_\Pi(L,Y)+\hom_\Pi(R,Y)-\hom_\Pi(R,X).\tag{6.1}$$

All of the summands of L are either summands of the maximal rigid object $T/T_v\oplus T_v'$, or are isomorphic to T_v, which satisfies $\mathrm{ext}^1_\Pi(T_v,T_v')=1$. Thus $\mathrm{ext}^1_\Pi(L,T_v')=\mathrm{mult}_{T_v}(L)$, and similarly $\mathrm{ext}^1_\Pi(R,T_v')=\mathrm{mult}_{T_v}(R)$. Now let C be the cokernel of the minimal left

$\operatorname{add} T$-approximation l_j^*, so we have a short exact sequence

$$0 \longrightarrow S_j \xrightarrow{l_j^*} L \longrightarrow C \longrightarrow 0.$$

By Proposition 4.10, C is in $\operatorname{add} T$, so applying $\operatorname{Hom}_\Pi(-, X)$ yields

$$0 \longrightarrow \operatorname{Hom}_\Pi(C, X) \longrightarrow \operatorname{Hom}_\Pi(L, X) \longrightarrow \operatorname{Hom}_\Pi(S_j, X) \longrightarrow 0.$$

Therefore

$$\hom_\Pi(L, X) = \hom_\Pi(S_j, X) + \hom_\Pi(C, X),$$

and a similar argument shows that

$$\hom_\Pi(L, Y) = \hom_\Pi(S_j, Y) + \hom_\Pi(C, Y).$$

Now C and R are isomorphic in the stable category $\underline{\operatorname{Sub}}\, Q_J$ by Corollary 4.13, and C has no projective-injective summands by Proposition 4.11, so $R \cong C \oplus P$ for some projective-injective object P. Thus $\hom_\Pi(R, X) = \hom_\Pi(C, X) + \hom_\Pi(P, X)$ and $\hom_\Pi(R, Y) = \hom_\Pi(C, Y) + \hom_\Pi(P, Y)$. By applying $\operatorname{Hom}_\Pi(P, -)$ to the exchange sequences, we see that

$$\hom_\Pi(P, X) = \hom_\Pi(P, T_v) + \hom_\Pi(P, T_v') = \hom_\Pi(P, Y),$$

and so

$$\hom_\Pi(R, Y) - \hom_\Pi(R, X) = \hom_\Pi(C, Y) - \hom_\Pi(C, X).$$

Taking into account these observations, the equation (6.1) becomes

$$\operatorname{mult}_{T_v}(L) - \operatorname{mult}_{T_v}(R) = \hom_\Pi(S_j, X) - \hom_\Pi(S_j, Y),$$

as required. □

Corollary 6.12. *If $T \in \mathscr{R}_J$ and $\widetilde{\Gamma}_T$ has no 2-cycles through its mutable vertices, then the ice quiver $(\widetilde{\Gamma}_T, \widetilde{F})$ agrees with the ice quiver $(\widetilde{\Gamma}', \widetilde{F}')$ of the seed of $\widetilde{\mathscr{A}_J}$ attached to T in* [GLS08, §10]*, see also Section 2.6, up to frozen arrows.*

Proof. The ice quiver $(\widetilde{\Gamma}', \widetilde{F}')$ is, by definition, the unique ice quiver with $\widetilde{\Gamma}'_0 = (\widetilde{\Gamma}_T)_0$ and $\widetilde{F}'_0 = \widetilde{F}_0$ such that

(i) the full subquiver of $\widetilde{\Gamma}'$ on $(\Gamma_T)_0$ is Γ_T,

(ii) $\widetilde{\Gamma}'$ satisfies the balancing condition at every mutable vertex,

(iii) there are no 2-cycles in $\widetilde{\Gamma}'$, and

(iv) $F_1' = \varnothing$.

The quiver obtained from $\widetilde{\Gamma}_T$ by removing all frozen arrows satisfies (i) by Remark 6.8, (ii) by Theorem 6.11, (iii) by assumption and (iv) since $\widetilde{F}$ is full. Thus $(\widetilde{\Gamma}_T, \widetilde{F})$ and $(\widetilde{\Gamma}', \widetilde{F}')$ agree up to frozen arrows. □

If T and T' are two cluster-tilting objects of $\operatorname{Sub} Q_J$ related by mutating the summand T_v of T, and neither $\widetilde{\Gamma}_T$ nor $\widetilde{\Gamma}_{T'}$ have 2-cycles, it follows from Corollary 6.12 together with [GLS08, Thm. 10.2] that $\widetilde{\Gamma}_{T'}$ is the Fomin–Zelevinsky mutation of $\widetilde{\Gamma}_T$ at v, up to arrows between frozen vertices.

Before continuing with the construction, we point out that the frozen subquiver in $\widetilde{\Gamma}_T$ is always some orientation of a graph determined by (Δ, J). Let $\widetilde{\Delta}^J$ be the graph with vertex set $\widetilde{\Delta}_0^J$, and $\underline{\hom}_J(V_i, U_j)$ edges between i and j; this condition is symmetric in i and j by Proposition 6.7. Note that this graph depends only on the category $\underline{\operatorname{Sub}}\, Q_J$, and not on a choice of cluster-tilting object. If $J = I$, then it is straightforward to check that $\widetilde{\Delta}^I$ is given by two copies of Δ, one with vertex set Δ_0 and the other with vertex set $I^* = \Delta_0^*$, joined by additional edges between i and i^* for all $i \in \Delta_0$.

We see in the next proposition, using Theorem 4.14, that the data of a cluster-tilting object of $\operatorname{Sub} Q_J$ (or, equivalently, a cluster-tilting object of $\underline{\operatorname{Sub}}\, Q_J$) determines an orientation of $\widetilde{\Delta}^J$, which appears as the frozen subquiver of $\widetilde{\Gamma}_T$.

Proposition 6.13. *For any cluster-tilting object T of $\operatorname{Sub} Q_J$, the frozen subquiver $\widetilde{F}$ of $\widetilde{\Gamma}_T$ is an orientation of $\widetilde{\Delta}^J$.*

Proof. For any i, j in $\widetilde{\Delta}_0^J$, we have the short exact sequences

$$0 \longrightarrow \underline{\operatorname{Hom}}_J^T(V_j, U_i) \longrightarrow \underline{\operatorname{Hom}}_J(V_j, U_i) \longrightarrow \underline{\operatorname{Hom}}_T(V_j, U_i) \longrightarrow 0,$$
$$0 \longrightarrow \underline{\operatorname{Hom}}_J^T(V_i, U_j) \longrightarrow \underline{\operatorname{Hom}}_J(V_i, U_j) \longrightarrow \underline{\operatorname{Hom}}_T(V_i, U_j) \longrightarrow 0,$$

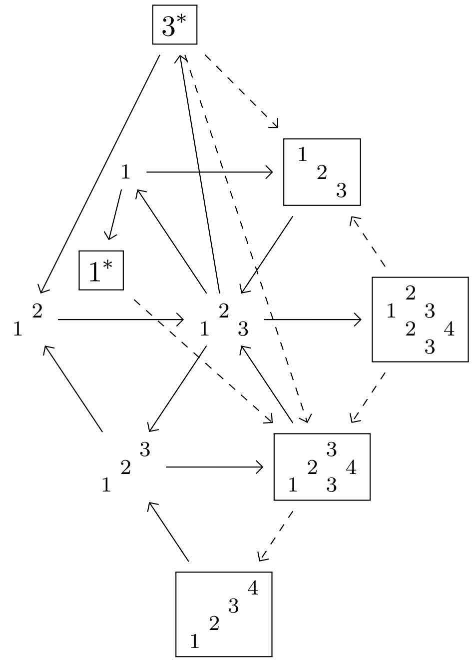
which are dual by Theorem 4.14. By definition, $\underline{\hom}_T(V_i, U_j)$ is the number of arrows $j \to i$ in $\widetilde{\Gamma}_T$, and $\underline{\hom}_J(V_i, U_j)$ is the number of edges of $\widetilde{\Delta}^J$ between i and j. From the above dual sequences, we see that

$$\underline{\hom}_T(V_i, U_j) + \underline{\hom}_T(V_j, U_i) = \underline{\hom}_J(V_i, U_j)$$

as required. □

Example 6.14. We continue Example 6.4, and compute that the ice quiver $(\widetilde{\Gamma}_{T_{\mathrm{i},J}}, \widetilde{F})$

is



where, as usual, the frozen vertices are boxed and the frozen arrows are dashed. To illustrate the necessary calculations, we explain the presence of the arrows adjacent to the vertex 3^*. We have $V_3^* = S_3$, which has a minimal left $\operatorname{add} T$-approximation

$$l_3^* \colon 3 \to \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix}$$

in $\underline{\mathrm{Sub}}\, Q_J$, so we draw an arrow from $\begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix}$ to 3^* in $\widetilde{\Gamma}_{T_{\mathbf{i},J}}$. This approximation fits into the triangle

$$V_3^* = 3 \xrightarrow{l_3^*} \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix} \xrightarrow{r_3^*} \begin{smallmatrix} 1 & & 3 & \\ & 22 & & 4 \\ 1 & & 3 & \end{smallmatrix} = U_3^*$$

in $\underline{\mathrm{Sub}}\, Q_J$, so r_3^* is a minimal right $\operatorname{add} T$-approximation of U_3^*, and we draw an arrow from 3^* to $\begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}$. We also have 1-dimensional spaces of maps

$$\begin{aligned} V_3^* &= 3 \to \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \to \begin{smallmatrix} & 2 & & 4 \\ 1 & & 3 & \end{smallmatrix} = U_2, \\ V_3^* &= 3 \to \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \to \begin{smallmatrix} 2 & \\ & 3 \end{smallmatrix} = U_4 \end{aligned}$$

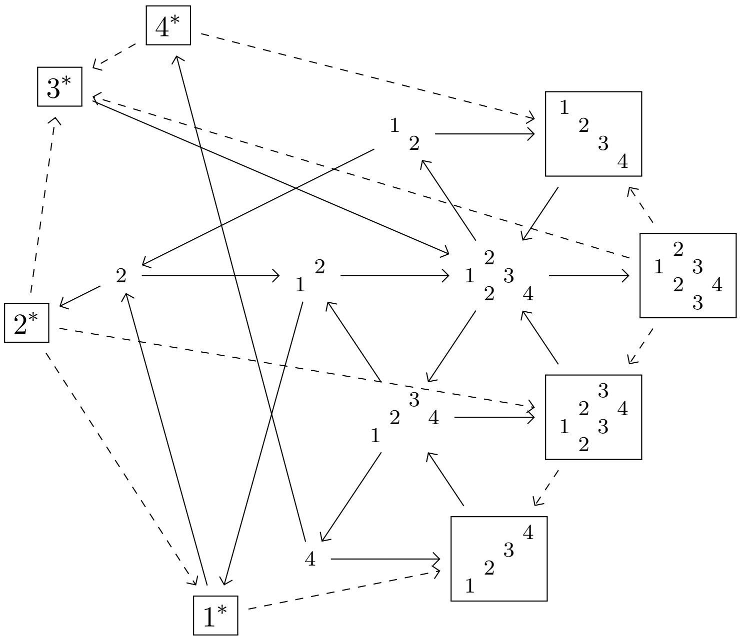
factoring through $\operatorname{add} T$, leading to the two frozen arrows leaving vertex 3^*. We have $\underline{\mathrm{Hom}}_J^T(V_3^*, U_i) = 0$ for i different from 2 and 4, and $\underline{\mathrm{Hom}}_J^T(V_i, U_3^*) = 0$ for all $i \in \widetilde{\Delta}_0^J$, so there are no more frozen arrows adjacent to 3^*.

The graph $\widetilde{\Delta}^J$ is

$$\begin{array}{ccccccc} 1 & & & & 3 & & \\ & \diagdown & & \diagup & & \diagdown & \\ & & 2 & & & & 4 \\ & \diagup & & \diagdown & & \diagup & \\ 1^* & & & & 3^* & & \end{array}$$

and $\widetilde{F}(T_{\mathbf{i},J})$ is an orientation of it, as predicted by Proposition 6.13.

Applying the same procedure to $T_{\mathbf{i}',I}$, we compute that $(\widetilde{\Gamma}_{T_{\mathbf{i}',I}}, \widetilde{F})$ is



The graph $\widetilde{\Delta}^I$ is

$$\begin{array}{ccccccc} 1 & — & 2 & — & 3 & — & 4 \\ | & & | & & | & & | \\ 1^* & — & 2^* & — & 3^* & — & 4^* \end{array}$$

and again $\widetilde{F}(T_{\mathbf{i},I})$ is an orientation of it.

From now on, we make the additional assumption that $T \in \mathscr{R}_J$, so that, by Theorem 6.6, there exists a potential $\underline{W}$ on $\underline{\Gamma}_T$ such that there is an isomorphism

$$\Phi \colon \mathcal{J}(\underline{\Gamma}_T, \underline{W}) \xrightarrow{\sim} \underline{\mathrm{End}}_J(T)^{\mathrm{op}}$$

with the property that $\Phi(e_v)(T) = T_v$ for all $v \in (\underline{\Gamma}_T)_0$. We fix such an isomorphism, and note that it induces a surjection $\mathbb{C}\underline{\Gamma}_T \to \underline{\mathrm{End}}_J(T)^{\mathrm{op}}$ with admissible kernel, which we also call Φ. Using Φ and our earlier choices of maps l_i, r_i and d_{ij}^k, we may extend $\underline{W}$ to a potential $\widetilde{W}$ on the ice quiver $(\widetilde{\Gamma}_T, \widetilde{F})$ as follows.

First, let $\delta_{ij}^k \colon j \to i$ be a frozen arrow of $\widetilde{\Gamma}_T$, corresponding to an element $d_{ij}^k \colon V_j \to U_i$ of our chosen basis of the space of maps $\underline{\mathrm{Hom}}_J^T(V_j, U_i)$ factoring through T in $\underline{\mathrm{Sub}}\, Q_J$. Since each map d_{ij}^k factors through T, it factors through the minimal left and right $\operatorname{add} T$-approximations of V_j and U_i as

$$d_{ij}^k = V_j \xrightarrow{l_j} L_j \xrightarrow{h_{ij}^k} R_i \xrightarrow{r_i} U_i.$$

Pick $p_{ij}^k \in \mathbb{C}\underline{\Gamma}_T$ such that $\Phi(p_{ij}^k) = h_{ij}^k$, and write

$$W_{ij}^k = \sum_{\substack{\alpha\colon v\to j\\ \beta\colon i\to w}} \alpha p_{ij}^k \beta \delta_{ij}^k.$$

Now let $i \in \widetilde{\Delta}_0^J$ be a frozen vertex, and consider the chosen minimal left and right $\operatorname{add} T$-approximations $l_i\colon V_i \to L_i$ and $r_i\colon R_i \to U_i$. These approximations have been chosen so that they fit into a triangle

$$V_i \xrightarrow{l_i} L_i \xrightarrow{h_i} R_i \xrightarrow{r_i} U_i$$

in $\underline{\operatorname{Sub}}\, Q_J$, uniquely determined by l_i and r_i. For each pair of components $l_i^\alpha\colon V_i \to T_v$ and $r_i^\beta\colon T_w \to U_i$, or equivalently for each pair of unfrozen arrows $\alpha\colon v \to i$ and $\beta\colon i \to w$ in $\widetilde{\Gamma}_T$, let $h_i^{\beta\alpha} \in \underline{\operatorname{Hom}}_J(T_v, T_w)$ be the component of h_i having non-zero composition with both l_i^α and r_i^β, and pick $p_i^{\beta\alpha} \in \mathbb{C}\underline{\Gamma}_T$ such that $\Phi(p_i^{\beta\alpha}) = h_i^{\beta\alpha}$. Then define

$$W_i^{\beta\alpha} = \alpha p_i^{\beta\alpha} \beta.$$

We can now define a potential $\widetilde{W}$ on $\widetilde{\Gamma}_T$ by

$$\widetilde{W} = \underline{W} + \sum_{\delta_{ij}^k \in \widetilde{F}_1} W_{ij}^k + \sum_{i \in \widetilde{\Delta}_0^J} \sum_{\substack{\alpha \in i^- \cap Q_1^m\\ \beta \in i^+ \cap Q_1^m}} W_i^{\beta\alpha},$$

and thus obtain a frozen Jacobian algebra

$$\widetilde{A} = \mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W}).$$

We now illustrate the construction of $\widetilde{A}$ via a pair of small but detailed examples.

Example 6.15. Let $\Delta = \mathsf{A}_3$, and $J = \{1, 2\}$. Then a cluster-tilting object in $\operatorname{Sub} Q_J$ is given by $T_u \oplus T_v \oplus F_1 \oplus F_2 \oplus F_3$, where

$$T_u = {}_2{}^3$$
$$T_v = {}^1{}_2{}^3$$

We then have that $\underline{\Gamma}_T$ is the quiver

$$u \xleftarrow{\alpha_1} v$$

In order to compute the potential on $\widetilde{\Gamma}_T$, we will need to be completely precise about the realisation of T as an explicit Π-module. Labelling the vertices and arrows of Π by

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\alpha^\vee"] \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\beta^\vee"] \end{tikzcd}$$

we let F_1 be the projective Π-module generated by e_3, and F_2 be the projective Π-module generated by e_2, so as vector spaces

$$\begin{aligned} F_1 &= \langle e_3, \beta^\vee, \alpha^\vee\beta^\vee\rangle, \\ F_2 &= \langle e_2, \alpha^\vee, \beta, \alpha\alpha^\vee\rangle. \end{aligned}$$

Via the preprojective relations, we have $\alpha\alpha^\vee = \beta^\vee\beta$, but for consistency we will always write the former. Having made the above choices, we may realise every other module we will need to consider as an explicit submodule of F_1 or F_2. For instance, we can take F_3 to be the submodule of F_2 spanned by $\alpha^\vee$ and $\alpha\alpha^\vee$. Similarly, we may take

$$\begin{aligned} T_u &= \langle \beta, \alpha\alpha^\vee\rangle \le F_2, \\ T_v &= \langle \alpha^\vee, \beta, \alpha\alpha^\vee\rangle \le F_2, \end{aligned}$$

so there is an isomorphism

$$\Phi\colon \mathcal{J}(\underline{\Gamma}_T, 0) = \mathbb{C}\underline{\Gamma}_T \xrightarrow{\sim} \underline{\mathrm{End}}_J(T)^{\mathrm{op}}$$

sending α_1 to the inclusion $T_u \to T_v$.

First we compute U_i and $V_i = \tau_J^{-1}U_i$ for all $i \in \widetilde{\Delta}_0^J$. Depicting modules via their radical filtrations and then choosing particular elements of each isomorphism class, we take

$$\begin{aligned}
U_1 &= \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix} = \langle \beta^\vee, \alpha^\vee\beta^\vee\rangle \le F_1, & V_1 &= \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix} = \langle \beta, \alpha\alpha^\vee\rangle \le F_2, \\
U_2 &= \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix} = \langle \alpha^\vee, \beta, \alpha\alpha^\vee\rangle \le F_2, & V_2 &= \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix} = \langle \beta^\vee, \alpha^\vee\beta^\vee\rangle \le F_1, \\
U_3 &= \begin{smallmatrix} 2 \end{smallmatrix} = \langle \alpha\alpha^\vee\rangle \le F_2, & V_3 &= \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix} = \langle \alpha^\vee, \beta, \alpha\alpha^\vee\rangle \le F_2, \\
U_1^* &= \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix} = \langle \beta, \alpha\alpha^\vee\rangle \le F_2, & V_1^* &= \begin{smallmatrix} 1 \end{smallmatrix} = \langle \alpha^\vee\beta^\vee\rangle \le F_1, \\
U_2^* &= \begin{smallmatrix} 1 \end{smallmatrix} = \langle \alpha^\vee\beta^\vee\rangle \le F_1, & V_2^* &= \begin{smallmatrix} 2 \end{smallmatrix} = \langle \alpha\alpha^\vee\rangle \le F_2.
\end{aligned}$$

Next we pick minimal left and right $\operatorname{add} T$-approximations

$$
\begin{aligned}
l_1 &\colon V_1 \xrightarrow{\equiv} T_u, & r_1 &\colon 0 \to U_1,\\
l_2 &\colon V_2 \to 0, & r_2 &\colon T_v \xrightarrow{\equiv} U_2,\\
l_3 &\colon V_3 \xrightarrow{\equiv} T_v, & r_3 &\colon 0 \to U_3,\\
l_1^* &\colon V_1^* \to 0, & r_1^* &\colon T_u \xrightarrow{\equiv} U_1^*,\\
l_2^* &\colon V_2^* \to T_u, & r_2^* &\colon T_v \to U_2^*
\end{aligned}
$$

in $\underline{\operatorname{Sub}}\, Q_J$; under our explicit realisations, every map above between non-zero modules is the identity, except for l_2^*, which is the inclusion, and r_2^*, which is determined by $\alpha^\vee \mapsto \alpha^\vee\beta^\vee$. We also compute that the graph $\widetilde{\Delta}^J$ is

$$
\begin{tikzcd}
 & 1 \arrow[rr, no head, bend left] \arrow[dl, no head, bend right] & & 2 \arrow[dr, no head, bend left] & \\
1^* \arrow[drr, no head, bend right] & & & & 3 \arrow[dll, no head, bend left] \\
 & & 2^* & &
\end{tikzcd}
$$

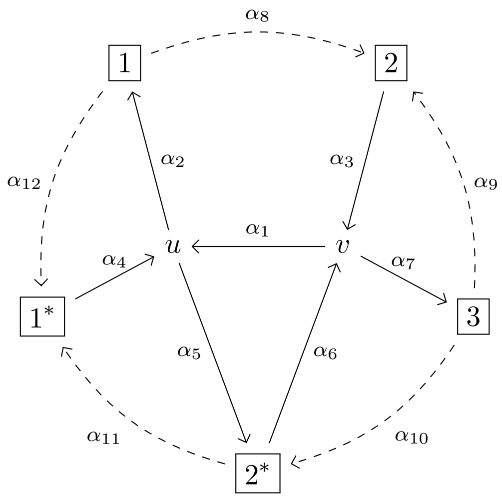
For each edge (i, j) of $\widetilde{\Delta}^J$, we choose bases of the spaces $\underline{\operatorname{Hom}}_J^T(V_j, U_i)$ and $\underline{\operatorname{Hom}}_J^T(V_i, U_j)$ of maps factoring through T. For a given (i, j), we know from Proposition 6.13 that one of these two spaces will be zero and the other will have dimension 1. For this example we find that there are non-zero maps

$$
\begin{aligned}
d_{21} &\colon V_1 \to U_2,\\
d_{23} &\colon V_3 \xrightarrow{\equiv} U_2,\\
d_{2^*3} &\colon V_3 \to U_2^*,\\
d_{1^*2^*} &\colon V_2^* \to U_1^*,\\
d_{1^*1} &\colon V_1 \xrightarrow{\equiv} U_1^*
\end{aligned}
$$

factoring through T, where d_{23} and d_{1^*1} are the identity, and the other maps are defined by

$$
\begin{aligned}
d_{21} &\colon \beta \mapsto -\beta,\\
d_{2^*3} &\colon \begin{array}{l} \alpha^\vee \mapsto -\alpha^\vee\beta^\vee,\\ \beta \mapsto 0, \end{array}\\
d_{1^*2^*} &\colon \alpha\alpha^\vee \mapsto -\alpha\alpha^\vee.
\end{aligned}
$$

Looking at the data of the maps l_i, r_i and d_{ij}, we see that $\widetilde{\Gamma}_T$ is given by



cf. Example 2.8.

Finally, we compute the potential $\widetilde{W}$. First we get a term W_{ij} for each arrow $j \to i$ of $\widetilde{F}$; we omit the superscript since it is always equal to 1. The maps d_{ij} factor through our chosen approximations as

$$\begin{aligned}
d_{21}\colon V_1 &\xrightarrow{l_1} T_u \xrightarrow{\Phi(-\alpha_1)} T_v \xrightarrow{r_2} U_2,\\
d_{23}\colon V_3 &\xrightarrow{l_3} T_v \xrightarrow{\Phi(e_v)} T_v \xrightarrow{r_2} U_2,\\
d_{2^*3}\colon V_3 &\xrightarrow{l_3} T_v \xrightarrow{\Phi(-e_v)} T_v \xrightarrow{r_2^*} U_2^*,\\
d_{1^*2^*}\colon V_2^* &\xrightarrow{l_2^*} T_u \xrightarrow{\Phi(-e_u)} T_u \xrightarrow{r_1^*} U_1^*,\\
d_{1^*1}\colon V_1 &\xrightarrow{l_1} T_u \xrightarrow{\Phi(e_u)} T_u \xrightarrow{r_1^*} U_1^*,
\end{aligned}$$

and so we have

$$\begin{aligned}
W_{21} &= -\alpha_2\alpha_1\alpha_3\alpha_8,\\
W_{23} &= \alpha_7\alpha_3\alpha_9,\\
W_{2^*3} &= -\alpha_7\alpha_6\alpha_{10},\\
W_{1^*2^*} &= -\alpha_5\alpha_4\alpha_{11},\\
W_{1^*1} &= \alpha_2\alpha_4\alpha_{12}.
\end{aligned}$$

The only triangle of the form

$$V_i \xrightarrow{l_i} L_i \xrightarrow{h_i} R_i \xrightarrow{r_i} U_i$$

in which h_i is non-zero is

$$V_2^* \xrightarrow{l_2^*} T_u \xrightarrow{\Phi(\alpha_1)} T_v \xrightarrow{r_2^*} U_2^*,$$

so we have

$$W_{2^*}^{65} = \alpha_5\alpha_1\alpha_6$$

and $W_i^{\beta\alpha} = 0$ whenever $i \neq 2^*$. Thus we define the potential

$$\widetilde{W} = -\alpha_2\alpha_1\alpha_3\alpha_8 + \alpha_7\alpha_3\alpha_9 - \alpha_7\alpha_6\alpha_{10} - \alpha_5\alpha_4\alpha_{11} + \alpha_2\alpha_4\alpha_{12} + \alpha_5\alpha_1\alpha_6$$

on $\widetilde{\Gamma}_T$, and obtain the frozen Jacobian algebra

$$\widetilde{A} = \mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W}).$$

Embedding $\widetilde{\Gamma}_T$ into the disk as indicated by the above picture and taking its dual gives a dimer model, with the bipartite structure coming from the two different orientations of the cycles. Then the frozen Jacobian algebra attached to this dimer model by Baur–King–Marsh [BKM14] coincides exactly with $\widetilde{A}$.

Let B be the algebra from [BKM14, §7], see also [JKS14], in the case $k = 2$ and $n = 5$. By [BKM14, Thm. 10.3, Cor. 10.4], we have that $B \cong e\widetilde{A}e$, where e is the idempotent corresponding to the frozen vertices, and $\widetilde{A} \cong \operatorname{End}_B(\widetilde{T})^{\mathrm{op}}$ for some cluster-tilting object $\widetilde{T}$ in $\operatorname{GP}(B)$. These are some of the conclusions we would obtain from Theorem 5.13 if we knew $\widetilde{A}$ was internally bimodule 3-Calabi–Yau with respect to e. Moreover, $\operatorname{GP}(B)$ is stably 2-Calabi–Yau by [JKS14, Cor. 4.6] and Proposition 2.15, so the conclusion of Conjecture 5.25 also holds in this example.

By Jensen–King–Su's arguments [JKS14], $\operatorname{GP}(B)$ provides a Frobenius categorification of Geiß–Leclerc–Schröer's cluster algebra structure on the Grassmannian G_2^5. This cluster algebra is strongly isomorphic to the cluster algebra $\widetilde{\mathscr{A}_J}$ in this example, since the two cluster algebras have seeds with matching quivers. We conclude that $\operatorname{GP}(B) \simeq \operatorname{GP}(e\widetilde{A}e)$ is a Frobenius categorification of $\widetilde{\mathscr{A}_J}$.

Example 6.16. We now give an example in which the potential $\underline{W}$ is non-zero; for brevity, we will be less explicit about the realisations of the modules and maps that we use than in Example 6.15, and leave the reader to check that it is possible to make choices leading to our conclusions.

Let $\Delta = \mathsf{A}_4$ and $J = \{1, 2\}$. A cluster-tilting object of $\operatorname{Sub} Q_J$ is given by $T_u \oplus T_v \oplus$

$T_w \oplus F_1 \oplus F_2 \oplus F_3 \oplus F_4$, where

$$
\begin{aligned}
T_u &= 2 & F_1 &= \begin{smallmatrix} &&&4\\ &&3&\\ &2&&\\ 1&&& \end{smallmatrix} \\
T_v &= \begin{smallmatrix} &2&&4\\ 1&&3&\\ &2&& \end{smallmatrix} & F_2 &= \begin{smallmatrix} &&3&\\ &2&&4\\ 1&&3&\\ &2&& \end{smallmatrix} \\
T_w &= \begin{smallmatrix} &&4\\ &3&\\ 2&& \end{smallmatrix} & F_3 &= \begin{smallmatrix} &2&\\ 1&&3\\ &2& \end{smallmatrix} \\
& & F_4 &= \begin{smallmatrix} 1&\\ &2 \end{smallmatrix}
\end{aligned}
$$

Then $\underline{\Gamma}_T$ is the quiver

$$
\begin{tikzcd}
& u \arrow[dr, "\alpha_3"] & \\
w \arrow[ur, "\alpha_1"] & & v \arrow[ll, "\alpha_2"]
\end{tikzcd}
$$

If we define $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ to be the inclusions, and $\Phi(\alpha_3)$ to be the projection of T_v onto the summand $S_2 \cong T_u$ of its top, we get an isomorphism

$$\Phi\colon \mathcal{J}(\underline{\Gamma}_T, \underline{W}) \xrightarrow{\sim} \underline{\mathrm{End}}_{\Pi}(T)^{\mathrm{op}},$$

where $\underline{W} = \alpha_1\alpha_2\alpha_3$. The modules U_i and V_i are

$$
\begin{aligned}
U_1 &= \begin{smallmatrix} &&3\\ &2&\\ 1&& \end{smallmatrix} & V_1 &= \begin{smallmatrix} &&4\\ &3&\\ 2&& \end{smallmatrix} \\
U_2 &= \begin{smallmatrix} &2&&4\\ 1&&3&\\ &2&& \end{smallmatrix} & V_2 &= \begin{smallmatrix} &&3\\ &2&\\ 1&& \end{smallmatrix} \\
U_3 &= \begin{smallmatrix} 1&&3\\ &2& \end{smallmatrix} & V_3 &= \begin{smallmatrix} &2&&4\\ 1&&3&\\ &2&& \end{smallmatrix} \\
U_4 &= 2 & V_4 &= \begin{smallmatrix} 1&&3\\ &2& \end{smallmatrix} \\
U_1^* &= \begin{smallmatrix} &&4\\ &3&\\ 2&& \end{smallmatrix} & V_1^* &= 1 \\
U_2^* &= 1 & V_2^* &= 2
\end{aligned}
$$

We obtain minimal approximations

$$
\begin{aligned}
l_1 &\colon V_1 \to T_w, & r_1 &\colon 0 \to U_1, \\
l_2 &\colon V_2 \to 0, & r_2 &\colon T_v \to U_2, \\
l_3 &\colon V_3 \to T_v, & r_3 &\colon 0 \to U_3, \\
l_4 &\colon V_4 \to 0, & r_4 &\colon T_u \to U_4, \\
l_1^* &\colon V_1^* \to 0, & r_1^* &\colon T_w \to U_1^*, \\
l_2^* &\colon V_2^* \to T_u, & r_2^* &\colon 0 \to U_2^*
\end{aligned}
$$

by taking each non-zero map to be the identity. For every i, the triangle

$$V_i \xrightarrow{l_i} L_i \xrightarrow{h_i} R_i \xrightarrow{r_i} U_i$$

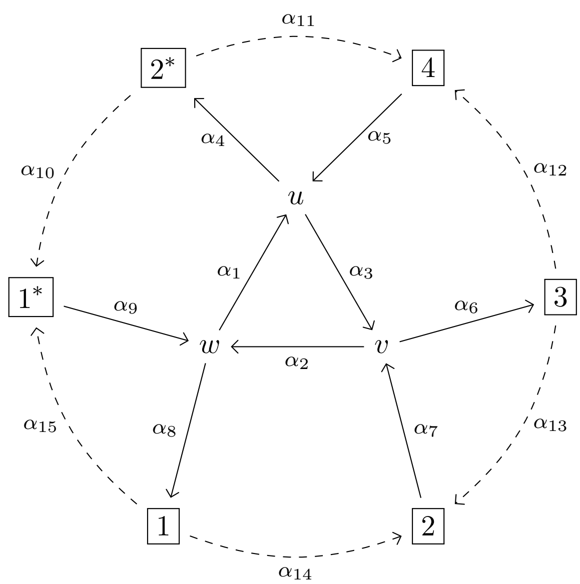
has $h_i = 0$, so we will not have any terms of the form $W_i^{\beta\alpha}$ in our potential. Up to scale, the only maps $V_i \to U_j$ factoring through T are

$$\begin{aligned}
d_{21} &\colon V_1 \to U_2,\\
d_{23} &\colon V_3 \to U_2,\\
d_{43} &\colon V_3 \to U_4,\\
d_{42^*} &\colon V_2^* \to U_4,\\
d_{1^*2^*} &\colon V_2^* \to U_1^*,\\
d_{1^*1} &\colon V_1 \to U_1^*,
\end{aligned}$$

where

$$\begin{aligned}
d_{21} &= \Phi(-\alpha_2),\\
d_{43} &= \Phi(-\alpha_3),\\
d_{1^*2^*} &= \Phi(-\alpha_1),
\end{aligned}$$

and the other three maps are the identity. Thus we compute that the quiver $\widetilde{\Gamma}_T$ is



and our potential is

$$\widetilde{W} = \alpha_1\alpha_2\alpha_3 - \alpha_8\alpha_2\alpha_7\alpha_{14} + \alpha_6\alpha_7\alpha_{13} - \alpha_6\alpha_3\alpha_5\alpha_{12} \\ + \alpha_4\alpha_5\alpha_{11} - \alpha_4\alpha_1\alpha_9\alpha_{10} + \alpha_8\alpha_5\alpha_{15}.$$

As in Example 6.15, this is the potential obtained from $\widetilde{\Gamma}_T$ by thinking of it as the quiver of a dimer model in the disk. The seed of $\widetilde{\mathscr{A}_J}$ coming from T agrees with a seed of the cluster algebra structure on the Grassmannian G_2^6 coming from this dimer model, so again by results of Jensen–King–Su [JKS14] and Baur–King–Marsh [BKM14] we see that $\mathrm{GP}(\widetilde{B}_J)$ is stably 2-Calabi–Yau and categorifies the cluster algebra $\widetilde{\mathscr{A}_J}$.

We use this example to point out some of the ambiguity in the construction. Since $\Phi(\alpha_1\alpha_2) = 0$, we have

$$d_{1^*2^*} = \Phi(-\alpha_1) = \Phi(-\alpha_1 + \alpha_1\alpha_2\alpha_3\alpha_1).$$

Choosing this second pre-image of $d_{1^*2^*}$ in the construction gives an alternative potential

$$\begin{aligned}\widetilde{W}' = \alpha_1\alpha_2\alpha_3 - \alpha_8\alpha_2\alpha_7\alpha_{14} + \alpha_6\alpha_7\alpha_{13} - \alpha_6\alpha_3\alpha_5\alpha_{12} + \alpha_4\alpha_5\alpha_{11}&\\ - \alpha_4\alpha_1\alpha_9\alpha_{10} + \alpha_4\alpha_1\alpha_2\alpha_3\alpha_1\alpha_9\alpha_{10} + \alpha_8\alpha_5\alpha_{15},&\end{aligned}$$

and it is not immediately clear whether or not there is an isomorphism $\mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W}) \cong \mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W}')$.

If $\widetilde{A}$ is internally bimodule 3-Calabi–Yau with respect to the boundary idempotent $e = \sum_{i\in\widetilde{\Delta}_0^J} e_i$, then Theorem 5.13 implies that $\widetilde{A} \cong \mathrm{End}_{\widetilde{B}_J}(\widetilde{T})^{\mathrm{op}}$ for the cluster-tilting object $\widetilde{T} = e\widetilde{A}$ in the category $\mathrm{GP}(\widetilde{B}_J)$, where $\widetilde{B}_J = e\widetilde{A}e$ is the boundary algebra of $\widetilde{A}$. Note that the quiver of $\widetilde{B}_J$ necessarily contains $\widetilde{F}$ as a full subquiver. The following result gives some of the structure of $\widetilde{B}_J$.

Proposition 6.17. *For any reachable cluster-tilting object $T \in \mathscr{R}_J$, write $\overline{F}$ for the double quiver of the frozen subquiver $\widetilde{F}$ of $\widetilde{\Gamma}_T$. Then the map $\varphi\colon \mathbb{C}\overline{F} \to \widetilde{B}_J$ defined by $\varphi(e_v) = e_v$, $\varphi(\alpha) = \alpha$ for each arrow α of $\widetilde{F}$, and $\varphi(\overline{\alpha}) = \partial_\alpha\widetilde{W}$, induces a map $\Pi(\widetilde{F}) \to \widetilde{B}_J$.*

Proof. It suffices to check that $\ker(\varphi)$ contains the preprojective relation $\sum_{\alpha\in\widetilde{F}_1}[\alpha,\overline{\alpha}]$. We have that

$$\sum_{\alpha\in(\widetilde{\Gamma}_T)_1}[\alpha, \partial_\alpha\widetilde{W}] = 0,$$

so we deduce that

$$\sum_{\alpha\in\widetilde{F}_1}\varphi([\alpha,\overline{\alpha}]) = \sum_{\alpha\in\widetilde{F}_1}[\alpha,\partial_\alpha\widetilde{W}] = \sum_{\alpha\in(\widetilde{\Gamma}_T)_1\setminus\widetilde{F}_1}[\partial_\alpha\widetilde{W},\alpha] = 0,$$

with the final equality coming from the fact that $\partial_\alpha\widetilde{W}$ is a relation of $\widetilde{A}$ whenever $\alpha \notin \widetilde{F}_1$. Thus φ induces the required map $\Pi(\widetilde{F}) \to \widetilde{B}_J$. □

It is immediate from the construction that we have an isomorphism

$$\widetilde{A}/\langle e\rangle \xrightarrow{\sim} \underline{\mathrm{End}}_\Pi(T)^{\mathrm{op}}.$$

By Theorem 5.13, the algebra on the left is isomorphic to $\underline{\mathrm{End}}_{\widetilde{B}_J}(\widetilde{T})^{\mathrm{op}}$ if $\widetilde{A}$ is internally bimodule 3-Calabi–Yau with respect to e. We conjecture that we can recover $\mathrm{End}_{\Pi}(T)^{\mathrm{op}}$ via a 'partial stabilisation'.

Conjecture 6.18. *There is an isomorphism*

$$\widetilde{\Phi}\colon \widetilde{A}/\langle e_j^* : j \in J\rangle \xrightarrow{\sim} \mathrm{End}_{\Pi}(T)^{\mathrm{op}}.$$

6.3 The Variety of Isotropic Lines

As explained in [GLS08, §12], when $\Delta = \mathsf{D}_n$ and $J = \{1\}$ is the fundamental node, the flag variety F_J is a quadric $\mathcal{Q} \subseteq \mathbb{P}^{2n-1}(\mathbb{C})$, which can be realised as the variety of isotropic lines of the vector space $\mathbb{C}^{2n}$ equipped with a quadratic form. Recall that our labelling of the Dynkin diagram D_n is different from that in [GLS08]; we swap the labels i and $n+1-i$. We give a brief account of our construction as applied to these examples.

As in [GLS08, §12.5], it is possible to describe all of the objects of $\mathrm{Sub}\,Q_1$. The projective-injective objects with simple socle are $F_1 = Q_1$, F_{n-1} and F_n, the latter two being the submodules of F_1 with dimension vectors $(1,\dots,1,0,1)$ and $(1,\dots,1,1,0)$ respectively. There are also $n-3$ indecomposable submodules of Q_1^2, which all have dimension vector $(1,1,2,\dots,2)$ and socle S_1^2. These are the other projective-injective objects $F_2,\dots,F_{n-2}$ of $\mathrm{Sub}\,Q_1$.

For $2 \leq i \leq n-2$, the radical U_i of F_i splits as $U_i^+ \oplus U_i^-$. Here U_i^+ denotes the summand with complete support; it usually has simple top at $i+1$, except when $i = n-2$, in which case its top is $S_{n-1} \oplus S_n$. The summand U_i^- has simple top at $i-1$, and is supported on $1 \leq j \leq i-1$. The radical U_1 of F_1 is the indecomposable module with dimension vector $(1,2,\dots,2,1,1)$, and we have that $U_{n-1} = U_n$ is the indecomposable module with dimension vector $(1,\dots,1,0,0)$. The n projective-injectives, plus the $2n-4$ indecomposable modules $U_i^{(\pm)}$, are all of the indecomposable objects of $\mathrm{Sub}\,Q_1$. It will be notationally convenient to write $U_1 = U_1^+$, and $U_{n-1} = U_{n-1}^- = U_n^- = U_n$. Then we may observe that $\tau_J U_i^- = U_{i-1}^+$ for $2 \leq i \leq n$ and $\tau_J U_i^+ = U_{i+1}^-$ for $1 \leq i \leq n-2$. A cluster-tilting object of $\mathrm{Sub}\,Q_J$ is then specified by choosing, for each $1 \leq i \leq n-2$, a summand T_i isomorphic to one of the modules U_i^+ or U_{i+1}^-. Mutation at T_i replaces T_i by $\tau_J T_i$, which was the other possible choice. Thus we get a cluster structure of type $(\mathsf{A}_1)^{n-2}$, as in [GLS08, Table 2].

If we take T_0 to be the cluster-tilting object given by choosing $T_i = U_{i+1}^-$ for each i, then the quiver of $\mathrm{End}_\Pi(T)^{\mathrm{op}}$ is

$$\begin{tikzcd}
& U_2^- \arrow[dl] & & U_3^- \arrow[dl] & & U_{n-3}^- \arrow[dl] & & U_{n-2}^- \arrow[dl] & & F_{n-1} \arrow[d] \\
F_1 \arrow[rr, dashed] & & F_2 \arrow[ul] \arrow[rr, dashed] & & \cdots \arrow[ul] \arrow[rr, dashed] & & F_{n-3} \arrow[ul] \arrow[rr, dashed] & & F_{n-2} \arrow[ul] \arrow[ur, dashed] \arrow[dr, dashed] & U_{n-1}^- \arrow[l] \\
& & & & & & & & & F_n \arrow[u]
\end{tikzcd}$$

where, for $2 \leq i \leq n-1$, the unfrozen arrows come from injective hulls and projective covers of $U_i^- = \tau_J U_{i-1}^-$, and the frozen arrows come from compositions

$$F_i \to \tau_J U_i^- = U_{i-1}^+ \to F_{i-1}.$$

If $i = n$, replace each instance of $i-1$ in the above by $i-2$.

If T is some cluster-tilting object of $\mathrm{Sub}\, Q_1$, and T' is the mutation of T at T_i, we obtain the quiver $\Gamma_{T'}$ from Γ_T by reversing the direction of the 3-cycles on which T_i lies. In this way we can compute the quivers of all cluster-tilting objects of $\mathrm{Sub}\, Q_1$, starting from T_0.

Since $U_2^- \cong S_1$, we see that the quiver $\widetilde{\Gamma}_{T_0}$ is given by

$$\begin{tikzcd}
1 \arrow[drr, dashed] & & & U_3^- \arrow[dl] & & U_{n-3}^- \arrow[dl] & & U_{n-2}^- \arrow[dl] & & n-1 \arrow[d] \\
U_2^- \arrow[u] \arrow[d] & & 2 \arrow[ll] \arrow[rr, dashed] & & \cdots \arrow[ul] \arrow[rr, dashed] & & n-3 \arrow[ul] \arrow[rr, dashed] & & n-2 \arrow[ul] \arrow[ur, dashed] \arrow[dr, dashed] & U_{n-1}^- \arrow[l] \\
1^* \arrow[urr, dashed] & & & & & & & & & n \arrow[u]
\end{tikzcd}$$

We equip this quiver with the potential given by the sum of all 3-cycles. The quiver of the boundary algebra $\widetilde{B}_J$ is the double quiver of the frozen subquiver $\widetilde{F}$, and we can see that the preprojective relations are satisfied, either by Proposition 6.17 or by direct observation. It follows that $\widetilde{B}_J$ is a quotient of the preprojective algebra $\Pi(\widetilde{\mathsf{D}}_n)$ of affine type D_n. We may also observe some additional relations; the 2-cycles $\alpha\alpha^\vee$ and $\alpha^\vee\alpha$ are zero for all arrows $\alpha\colon i \to i+1$ with $i \leq 2 \leq n-3$, and the length 2 paths

$$\begin{gathered}
1 \longrightarrow 2 \longrightarrow 1^*, \\
1^* \longrightarrow 2 \longrightarrow 1, \\
(n-1) \longrightarrow (n-2) \longrightarrow n, \\
n \longrightarrow (n-2) \longrightarrow (n-1)
\end{gathered}$$

are all zero.

Under the Morita equivalence of $\Pi(\widetilde{\mathsf{D}}_n)$ with the twisted group ring $\mathbb{C}[[x,y]]{*}\mathrm{BiDih}_{n-2}$

coming from Auslander's version of the McKay correspondence [Aus86], these relations correspond to restricting the action of BiDih_{n-2} to the coordinate axes, cut out by $xy = 0$. Here BiDih_{n-2} acts as the finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ generated by

$$\alpha = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$
$$\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where ω is a $(n-2)$-th root of unity, so the action is defined by $\alpha \cdot x = \omega x$, $\alpha \cdot y = \omega^{-1} y$, $\beta \cdot x = y$ and $\beta \cdot y = -x$. Thus xy is a semi-invariant, satisfying $\alpha \cdot xy = xy$ and $\beta \cdot xy = -xy$. Moreover, the weight of xy is given by the character of BiDih_{n-2} corresponding to the node 1 of D_n under its canonical embedding into the McKay graph $\widetilde{\mathsf{D}}_n$ of BiDih_{n-2}.

This is highly reminiscent of the constructions of Jensen–King–Su [JKS14], where the algebra B such that $\mathrm{GP}(B)$ categorifies the cluster algebra $\widetilde{\mathscr{A}_j} = \mathbb{C}[G^n_j]$ is given by the twisted group ring $\mathbb{C}[\![x, y]\!]/\langle x^j - y^{n-j}\rangle * \mu_n$, where $\mu_n = \{\omega \in \mathbb{C} : \omega^n = 1\}$ is a cyclic group acting by $\omega \cdot x = \omega x$ and $\omega \cdot y = \omega^{-1} y$. In this case $x^j - y^{n-j}$ is a semi-invariant of the μ_n-action whose weight is given by the character of μ_n corresponding to the node j under the canonical embedding of A_n into $\widetilde{\mathsf{A}}_n$.

6.4 Categorification

Fix a reachable cluster-tilting object $T \in \mathscr{R}_J$, and let $\widetilde{B}_J = e\widetilde{A}e$ be the boundary algebra of the frozen Jacobian algebra $\widetilde{A}$ constructed from T in Section 6.2. We claim that the Frobenius category $\mathrm{GP}(\widetilde{B}_J)$ provides the desired additive categorification of the cluster algebra $\widetilde{\mathscr{A}_J}$. Precisely, we conjecture the following.

Conjecture 6.19. *Let T be a maximal rigid object of $\operatorname{Sub} Q_J$. Then $\mu_0 \colon \mathbf{P}(\widetilde{A}) \to \widetilde{A}$ is a quasi-isomorphism, so $\widetilde{A}$ is internally bimodule 3-Calabi–Yau with respect to its boundary idempotent by Theorem 5.24. Moreover, we have an equivalence*

$$\underline{\mathrm{GP}}(\widetilde{B}_J) \simeq \underline{\operatorname{Sub}}\, Q_J.$$

Recall that $A = \operatorname{End}_\Pi(T)^{\mathrm{op}}$ is internally 3-Calabi–Yau with respect to its boundary idempotent e, and has the property that $\underline{\mathrm{GP}}(eAe) \simeq \underline{\operatorname{Sub}}\, Q_J$, by Proposition 5.5 and Theorem 3.4. In the case of Example 6.14, we saw that $\mathrm{GP}(\widetilde{B}_J)$ is equivalent to Jensen–King–Su's categorification of the Grassmannian G^5_2, and so $\underline{\mathrm{GP}}(\widetilde{B}_J) \simeq \underline{\operatorname{Sub}}\, Q_J$ by [JKS14, Cor. 4.6].

If we could find a functor $F \colon \underline{\mathrm{GP}}(\widetilde{B}_J) \to \underline{\operatorname{Sub}}\, Q_J$ with $F(e\widetilde{A}) \cong T$ such that F induces the isomorphism $\underline{\operatorname{End}}_B(e\widetilde{A})^{\mathrm{op}} \xrightarrow{\sim} \underline{\operatorname{End}}_\Pi(T)^{\mathrm{op}}$, then F would be an equivalence by [AIR15, Prop. 1.7] (see also [KR08, §4.5], noting that the 2-Calabi–Yau assumption is not used in the proof).

For the rest of the section, we assume that Conjecture 6.19 holds, and also that the potential $\widetilde{W}$ constructed in Section 6.2 is rigid. In this case we can show that $\mathrm{GP}(\widetilde{B}_J)$ provides the required categorification of $\widetilde{\mathscr{A}_J}$. This follows from a result of Buan–Iyama–Reiten–Smith [BIRS11, Thm. 5.7], which we now recall. Since $\widetilde{A}$ is internally bimodule 3-Calabi–Yau, $\widetilde{T} = e\widetilde{A}$ is a cluster-tilting object in $\mathrm{GP}(\widetilde{B}_J)$ by Theorem 5.13. Moreover, the quiver of $\mathrm{End}_B(\widetilde{T})^{\mathrm{op}}$ is $\widetilde{\Gamma}_T$, which is the quiver of a seed of $\widetilde{\mathscr{A}_J}$, so all that remains for us to check is that cluster-tilting objects in $\mathrm{GP}(\widetilde{B}_J)$ have the mutation property, and that a mutation of a cluster-tilting object induces a mutation of the quiver (with potential) of its endomorphism algebra.

Any cluster-tilting object $\widetilde{T}$ in $\mathrm{GP}(\widetilde{B}_J)$ is also cluster-tilting in $\underline{\mathrm{GP}}(\widetilde{B}_J)$. As we are assuming Conjecture 6.19, we have an equivalence $\underline{\mathrm{GP}}(\widetilde{B}_J) \simeq \underline{\mathrm{Sub}}\, Q_J$, and so $\underline{\mathrm{GP}}(\widetilde{B}_J)$ is 2-Calabi–Yau by Proposition 2.15. This means that we can mutate the cluster tilting object $\widetilde{T}$ in this category (see Iyama–Yoshino [IY08, §5]). For any indecomposable summand M of $\widetilde{T}$ in $\underline{\mathrm{GP}}(\widetilde{B}_J)$, there is a unique (up to isomorphism) basic cluster-tilting object $\mu_M\widetilde{T}$ in $\mathrm{GP}(\widetilde{B}_J)$ that, when thought of as an object of the stable category, agrees with the mutation of $\widetilde{T}$ in $\underline{\mathrm{GP}}(\widetilde{B}_J)$; the non-projective-injective summands of $\mu_M\widetilde{T}$ are determined by its stable isomorphism class, so $\mu_M\widetilde{T}$ is the direct sum of these with the projective-injective generator-cogenerator $\widetilde{B}_J$.

Remark 6.20. If Conjecture 5.25 holds, then $\mathbf{P}(\widetilde{A})$ being quasi-isomorphic to $\widetilde{A}$ would be enough to conclude that $\underline{\mathrm{GP}}(\widetilde{B}_J)$ is 2-Calabi–Yau. This is the only consequence of the statement '$\underline{\mathrm{GP}}(\widetilde{B}_J) \simeq \underline{\mathrm{Sub}}\, Q_J$' from Conjecture 6.19 that we will use in this section.

Under the assumption that Conjecture 6.19 holds, Theorem 5.13 provides an isomorphism $\widetilde{A} \cong \mathrm{End}_{\widetilde{B}_J}(\widetilde{T})$, inducing a functor Φ from the complete path category $\mathbb{C}\widetilde{\Gamma}_T$ to $\mathrm{GP}(\widetilde{B}_J)$, with $\Phi(v) = e\widetilde{A}e_v$. It follows that $\bigoplus_{v\in(\widetilde{\Gamma}_T)_0} \Phi(v) = \widetilde{T}$, and $\bigoplus_{v\in\widetilde{F}_0} \Phi(v) = \widetilde{B}_J$.

Given $X, Y \in \mathrm{GP}(\widetilde{B}_J)$, write $\mathrm{radHom}_{\widetilde{B}_J}(X, Y)$ for the set of non-split maps $X \to Y$ in $\mathrm{GP}(\widetilde{B}_J)$. We recall the following definition from Buan–Iyama–Reiten–Smith.

Definition 6.21 ([BIRS11, Defn. 4.4])**.** A sequence

$$X \longrightarrow U_1 \longrightarrow U_0 \longrightarrow X$$

in $\mathrm{add}\,\widetilde{T}$ is called *weak* 2*-almost split* if there are induced exact sequences

$$\mathrm{Hom}_{\widetilde{B}_J}(\widetilde{T}, U_1) \longrightarrow \mathrm{Hom}_{\widetilde{B}_J}(\widetilde{T}, U_0) \longrightarrow \mathrm{radHom}_{\widetilde{B}_J}(\widetilde{T}, X) \longrightarrow 0,$$
$$\mathrm{Hom}_{\widetilde{B}_J}(U_0, \widetilde{T}) \longrightarrow \mathrm{Hom}_{\widetilde{B}_J}(U_1, \widetilde{T}) \longrightarrow \mathrm{radHom}_{\widetilde{B}_J}(X, \widetilde{T}) \longrightarrow 0.$$

For any cycle $\alpha_m \cdots \alpha_1$ in $\mathbb{C}\widetilde{\Gamma}_T$, write

$$\partial_{(\alpha,\beta)}\alpha_m \cdots \alpha_1 = \sum_{\substack{\alpha_i=\alpha\\ \alpha_{i+1}=\beta}} \alpha_{i+2} \cdots \alpha_m\alpha_1 \cdots \alpha_{i-1},$$

reading indices cyclically, so $\alpha_{m+1} = \alpha_1$, and extend linearly. Note that $\Phi(\partial_{(\alpha,\beta)}\widetilde{W})$ is a map $\Phi(h\beta)$ to $\Phi(t\alpha)$.

Proposition 6.22. *For each mutable vertex v of $\widetilde{\Gamma}_T$, the sequence*

$$\Phi(v) \xrightarrow{\Phi(\beta)_\beta} \bigoplus_{\beta\colon v\to w} \Phi(w) \xrightarrow{\Phi(\partial_{(\alpha,\beta)}\widetilde{W})_{\alpha,\beta}} \bigoplus_{\alpha\colon u\to v} \Phi(w) \xrightarrow{\Phi(\alpha)_\alpha} \Phi(v)$$

is weak 2-almost split.

Proof. Since Φ induces an isomorphism $\widetilde{A} \xrightarrow{\sim} \operatorname{End}_{\widetilde{B}_J}(\widetilde{T})^{\mathrm{op}}$, and $\widetilde{A} = \mathcal{J}(\widetilde{\Gamma}_T, \widetilde{F}, \widetilde{W})$, this is [BIRS11, Thm. 4.6]. □

It follows from Proposition 6.22 and the preceding discussion that the isomorphism of the Jacobian algebra $\widetilde{A}/\langle e\rangle$ with $\underline{\operatorname{End}}_{\operatorname{GP}(\widetilde{B}_J)}(\widetilde{T})^{\mathrm{op}}$ from Theorem 5.13 is liftable to the Frobenius category $\operatorname{GP}(\widetilde{B}_J)$, in the terminology of [BIRS11, §5.1].

Theorem 6.23 ([BIRS11, Thm. 5.3])**.** *Let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category with a basic cluster-tilting object T. If we have a liftable isomorphism $\operatorname{End}_{\mathcal{C}} \cong \mathcal{J}(Q, W)$ for a quiver with potential (Q, W), and no 2-cycles start in the vertex k of Q, then we have a liftable isomorphism $\operatorname{End}_{\mathcal{C}}(\mu_{T_k}T) \cong \mathcal{J}(\mu_k Q, \mu_k W)$.*

As we are assuming that $\widetilde{W}$ is rigid, it follows from Proposition 3.18 that there are no 2-cycles through any mutable vertex of $\widetilde{\Gamma}_T$. Thus we may apply Theorem 6.23 to our situation, by taking $\mathcal{C} = \underline{\operatorname{GP}}(\widetilde{B}_J)$, containing the basic cluster-tilting object $\widetilde{T} \in \underline{\operatorname{GP}}\,\widetilde{B}_J$, letting Q be the full subquiver of $\widetilde{\Gamma}_T$ on the mutable vertices, and letting W be the restriction of the potential $\widetilde{W}$ to this subquiver. By Proposition 3.18 again, the quiver of $\mu_{T_k}\widetilde{T}$ also has no 2-cycles through its mutable vertices for any k, and so we can continue mutating indefinitely.

As a consequence, we observe that the isomorphism class of the boundary algebra $\widetilde{B}_J$ produced by our algorithm is independent of the choice of cluster-tilting object to input.

Corollary 6.24. *Assume that Conjecture 6.19 holds and that the potential $\widetilde{W}$ is rigid. Let $T_1, T_2 \in \mathscr{R}_J$, and write $\widetilde{A}_1$ and $\widetilde{A}_2$ for the associated frozen Jacobian algebras. Then the boundary algebras $\widetilde{B}_1 = e_1\widetilde{A}_1e_1$ and $\widetilde{B}_2 = e_2\widetilde{A}_2e_2$ are isomorphic.*

Proof. Let $\widetilde{T}_1 = e_1\widetilde{A}_1 \in \operatorname{GP}(\widetilde{B}_1)$ and $\widetilde{T}_2 = e_2\widetilde{A}_2 \in \operatorname{GP}(\widetilde{B}_2)$. Since $\mathscr{R}_J$ is a mutation class, there is a sequence of mutations from T_1 to T_2 in $\underline{\operatorname{Sub}}\,Q_J$. Let T' be the result of applying this sequence of mutations to $\widetilde{T}_1$ in $\operatorname{GP}(\widetilde{B}_1)$, and let e be the idempotent of $\operatorname{End}_{\widetilde{B}_1}(\widetilde{T}')^{\mathrm{op}}$ given by projection onto the projective-injective summands of $\widetilde{T}'$. By Theorems 6.23 and 5.13 there are isomorphisms

$$\operatorname{End}_{\widetilde{B}_1}(\widetilde{T}')^{\mathrm{op}} \xrightarrow{\sim} \operatorname{End}_{\widetilde{B}_2}(\widetilde{T}_2)^{\mathrm{op}} \xrightarrow{\sim} \widetilde{A}_2,$$

with the composition taking e to the frozen idempotent $e_2 \in \widetilde{A}_2$. In particular, there is an isomorphism

$$\widetilde{B}_1 \cong e \operatorname{End}_{\widetilde{B}_1}(\widetilde{T}')^{\mathrm{op}} e \cong e_2 \widetilde{A}_2 e_2 = \widetilde{B}_2. \qquad \square$$

CHAPTER 7

FUTURE WORK

7.1 Remaining Questions

In this section, we give a summary of the remaining open problems that most directly affect the construction of Frobenius categorifications of the cluster algebras $\widetilde{\mathscr{A}_J}$ via our methods.

Question 1 (cf. Conjecture 5.25)**.** *For the frozen Jacobian algebra $\widetilde{A}$ constructed in Section 6.2, is the bimodule complex $\mathbf{P}(\widetilde{A})$ constructed in Section 5.4 quasi-isomorphic to $\widetilde{A}$? In particular, is $\widetilde{A}$ internally bimodule 3-Calabi–Yau with respect to its boundary idempotent?*

A positive answer to Question 1 would allow us to apply Theorem 5.13 to our constructions to produce a Frobenius category $\mathrm{GP}(\widetilde{B}_J)$ admitting a cluster-tilting object with endomorphism algebra $\widetilde{A}$. We strongly believe that the answer to Question 1 is yes if $\widetilde{A}$ is replaced by $A = \mathrm{End}_\Pi(T)^{\mathrm{op}}$ for $T \in \mathrm{Sub}\, Q_J$ cluster-tilting, since these algebras are internally 3-Calabi–Yau by Proposition 5.5, and the projective resolutions of simple A-modules appearing in Lemma 4.17 are of a form consistent with the existence of a quasi-isomorphism $\mathbf{P}(A) \xrightarrow{\sim} A$.

In the case of a Jacobian algebra A arising from a dimer model on a torus, Broomhead [Bro12, Thm. 7.7] shows that $\mathbf{P}(A)$ is quasi-isomorphic to A under certain consistency conditions on the dimer model. In some cases, such as in Example 6.15, our algebras $\widetilde{A}$ are the frozen Jacobian algebras attached to dimer models on closed surfaces in [BKM14]. In these cases, we could look for analogues of Broomhead's consistency conditions (such as the existence of an anomaly-free R-charge, see [Bro12, §3.1], [BKM14, §5]) that imply the existence of a quasi-isomorphism $\mathbf{P}(\widetilde{A}) \xrightarrow{\sim} \widetilde{A}$. However, many of our examples, such as those in Example 6.14, do not arise from dimer models on closed surfaces, so we will also need more general methods.

Inspired by Broomhead's approach to the proof of [Bro12, Thm. 7.7], one way to make Question 1 potentially more tractable would be to find a grading of $\widetilde{A}$ such that every arrow has strictly positive degree. This would, for example, allow us to reduce the problem of finding a quasi-isomorphism $\mathbf{P}(\widetilde{A}) \xrightarrow{\sim} \widetilde{A}$ to that of finding a quasi-isomorphism

$$\mathbf{P}(\widetilde{A}) \otimes_A S \xrightarrow{\sim} S := \widetilde{A}/\mathfrak{m}(\widetilde{A}),$$

or equivalently to the problem of showing that $\mathbf{P}(\widetilde{A}) \otimes_A S_i$ is a projective resolution of the simple A-module S_i, as in [Bro12, Prop. 7.5].

It is also necessary to better understand when the Frobenius categories obtained from internally 3-Calabi–Yau algebras via Theorem 5.13 are stably 2-Calabi–Yau, and thus have some chance of categorifying a cluster algebra. The definition of internally 3-Calabi–Yau given in Definition 5.1 is probably too weak to expect this in full generality, but we may ask the following.

Question 2 (cf. Conjecture 5.25)**.** *Let $A = \mathcal{J}(Q, F, W)$ be a frozen Jacobian algebra, with frozen idempotent e and boundary algebra $B = eAe$. Assume that A is Noetherian, $A/\langle e\rangle$ is finite dimensional, and $\mathbf{P}(A)$ is isomorphic to A in $\mathcal{D}^b A^\varepsilon$. Then A is internally bimodule* 3*-Calabi–Yau with respect to e by Theorem 5.24, and* $\mathrm{GP}(B)$ *is a Frobenius category by Theorem 5.13. Is* $\mathrm{GP}(B)$ *stably* 2*-Calabi–Yau?*

We observe that if $A = \mathrm{End}_\Pi(T)^{\mathrm{op}}$ for a cluster-tilting object $T \in \mathrm{Sub}\, Q_J$, and $B = \mathrm{End}_\Pi(F)^{\mathrm{op}}$ is its boundary algebra, then A is an internally 3-Calabi–Yau frozen Jacobian algebra by Proposition 5.5 and Theorem 6.6, $\mathrm{GP}(B) \simeq \mathrm{Sub}\, Q_J$ by Proposition 4.21 and Theorem 3.4, and so $\mathrm{GP}(B)$ is stably 2-Calabi–Yau by Proposition 2.15.

One possible approach to showing that Question 2 has a positive answer is the following. By a result of Buchweitz [Buc87, Thm. 4.4.1], see also Keller–Vossieck [KV87], there is an equivalence

$$\underline{\mathrm{GP}}(B) \simeq \mathcal{D}^b B/\operatorname{per} B.$$

We could thus attempt to show that there is an autoequivalence $\nu\colon \mathcal{D}^b B \to \mathcal{D}^b B$ restricting to an autoequivalence of $\operatorname{per} B$, and having the property that $\nu X \cong X[-3]$ in the quotient $\mathcal{D}^b B/\operatorname{per} B$. Moreover, we want there to be a bifunctorial bilinear form

$$\beta_{X,P}\colon \mathrm{Hom}_{\mathcal{D}^b B}(X, P) \times \mathrm{Hom}_{\mathcal{D}^b B}(\nu P, X) \to \mathbb{C}$$

for each $X \in \mathcal{D}^b B$ and $P \in \operatorname{per} B$. Via a slight modification of Amiot's methods in [Ami09, §1], see also [Ami08, §4], this induces a bifunctorial bilinear form

$$\beta'_{X,Y}\colon \underline{\mathrm{Hom}}_B(X, Y) \times \underline{\mathrm{Hom}}_B(Y[-2], X) \to \mathbb{C}$$

on the quotient $\mathcal{D}^b B/\operatorname{per} B \simeq \underline{\mathrm{GP}}(B)$. If this form is non-degenerate [Ami09, §1.2], then $\underline{\mathrm{GP}}(B)$ is 2-Calabi–Yau. A good candidate for ν is the functor $\Omega_B \overset{\mathbf{L}}{\otimes}_B -$, where $\Omega_B = \mathbf{R}\mathrm{Hom}_{B^\varepsilon}(B, B^\varepsilon)$ (cf. Lemma 5.10 and Proposition 5.27), but we do not yet know how to construct a suitable form $\beta_{X,P}$.

For the algebra $\widetilde{B}_J$ constructed in Section 6.2, while we expect that $\underline{\mathrm{GP}}(\widetilde{B}_J) \simeq \underline{\mathrm{Sub}}\, Q_J$ is Hom-finite, the category $\mathcal{D}^b(\widetilde{B}_J)$ will be Hom-infinite, since $\widetilde{B}_J$ is infinite dimensional. Thus it is unlikely that a form $\beta_{X,P}$ as above is non-degenerate, as is the case in Amiot's applications [Ami09]. However, we hope that we may be able to find a form such that the induced form $\beta'_{X,Y}$ is non-degenerate.

Question 3 (cf. Conjecture 6.9)**.** *Let $T \in \mathscr{R}_J$. Is the potential $\widetilde{W}$ on $\widetilde{\Gamma}_T$, as constructed in Section 6.2, rigid? In particular, can $\widetilde{\Gamma}_T$ have 2-cycles through its mutable vertices?*

A positive answer to Question 3 (as well as to Question 2) would mean that $\mathrm{GP}(\widetilde{B}_J)$ provides a Frobenius categorification of $\widetilde{\mathscr{A}_J}$, by ensuring that the mutation of cluster-tilting objects of $\mathrm{GP}(\widetilde{B}_J)$ is compatible with mutation of seeds of $\widetilde{\mathscr{A}_J}$. The description of $\widetilde{W}$ for a general cluster-tilting object $T \in \mathscr{R}_J$ is fairly involved, and depends on many choices. It seems likely at this stage that to answer Question 3 we will need to have a better description of $\widetilde{W}$, possibly for certain special cluster-tilting objects such as the standard objects $T_{\mathbf{i},J}$ discussed in Section 6.1. Indeed, the isomorphisms of Theorem 6.23 are proved by constructing rigid potentials for these standard cluster-tilting objects, and thus obtaining potentials on the quivers of endomorphism algebras of other reachable cluster-tilting objects via mutation. Since $\widetilde{\Gamma}_T$ has no 2-cycles in any example we have computed (hence Conjecture 6.9), it seems plausible that there always exists some rigid potential on $\widetilde{\Gamma}_T$, even if it is not $\widetilde{W}$.

7.2 Related Problems

In addition to the remaining open problems described in Section 7.1, our results suggest a number of possibilities for future research, and we close with a somewhat speculative discussion of some of these related problems.

One possible direction is to study our approach to the construction of Frobenius categories from the perspective of Riedtmann's work on the classification of representation-finite finite dimensional selfinjective algebras. This classification uses general methods introduced in [Rie80a, Rie80b] and is completed in [Rie83, BLR81]. Riedtmann starts from the data of a representation-finite triangulated category $\mathcal{T}$, and gives a combinatorial recipe for constructing a Frobenius category $\mathcal{E}$ with stable category $\mathcal{T}$. The construction is constrained in such a way that $\mathcal{E} = \operatorname{mod}\Lambda$ for some finite dimensional selfinjective algebra Λ. This work has been extended by Luo [Luo14] and Keller–Scherotzke [KS13] to include Frobenius categories of different flavours.

The discussion in Section 6.2 suggests an extension of this work to the case of triangulated categories which are (weakly) d-representation-finite, meaning they admit a d-cluster-tilting object (cf. Iyama–Oppermann [IO11, Defn. 2.2]). Our aim in Section 6.2 is to start with a 2-representation finite 2-Calabi–Yau triangulated category $\mathcal{T}$, and try to find a Frobenius category $\mathcal{E}$ with $\underline{\mathcal{E}} = \mathcal{T}$, via Theorem 5.13. To do this, we pick a cluster-tilting object $T \in \mathcal{T}$, and specify which objects of $\mathcal{T}$ should be the radicals of the indecomposable projective injective objects in $\mathcal{E}$; these are the U_i in the notation of Section 6.2. We then attempt to use this information to compute what the endomorphism algebra of the pre-image of T in $\mathcal{E}$ must be, if such a category exists, and use it to determine $\mathcal{E}$. This is in many ways similar to Riedtmann's work – in this language, she works with a 1-representation-finite triangulated category $\mathcal{T}$ in which the unique basic 1-cluster-tilting object is the unique basic additive generator, and computes an algebra A which turns out to be the endomorphism algebra of a 1-cluster-tilting object, or equivalently an additive generator, of a Frobenius category $\mathcal{E}$ with $\underline{\mathcal{E}} = \mathcal{T}$.

Thus it would be interesting to try to make our methods for constructing the algebra A more general, and extend them to the case of d-representation-finite triangulated categories, thus developing a higher-dimensional version of Riedtmann's theory. Possible applications include constructions, or even a classification, of d-representation-finite selfinjective algebras. Such algebras have been previously studied by, for example, Herschend–Iyama [HI11] in the case $d = 2$.

As already mentioned in the discussion of Question 1 above, we need to develop methods for verifying that a given ice quiver with potential determines an internally bimodule 3-Calabi–Yau frozen Jacobian algebra. Various consistency conditions on dimer models on closed surfaces, as discussed by Broomhead in [Bro12], imply that the resulting Jacobian algebra is bimodule 3-Calabi–Yau. In view of Theorem 5.13, it would be interesting to discover whether there are natural consistency conditions for dimer models on surfaces with boundary that would imply that the corresponding frozen Jacobian algebra is internally bimodule 3-Calabi–Yau. A candidate for such a condition is the existence of a consistent boundary R-charge, as defined by Baur–King–Marsh [BKM14, §5]. Indeed, the analogous condition for closed surfaces is one of the consistency conditions leading to bimodule 3-Calabi–Yau Jacobian algebras [Bro12, Thm. 6.1, Thm. 7.7]. Results in this direction may have implications in mathematical physics, where dimer models on surfaces with boundary are also studied; see, for example, work of Franco [Fra12].

It would also be interesting to better understand the role of the graph $\widetilde{\Delta}^J$. By Proposition 6.17 we have a map from the preprojective algebra $\Pi(\widetilde{\Delta}^J)$ to the boundary algebra $\widetilde{B}_J = e\widetilde{A}e$. We expect that this map is surjective, and so realises $\widetilde{B}_J$ as a quotient of $\Pi(\widetilde{\Delta}^J)$, just as $\mathrm{End}_\Pi(F)^{\mathrm{op}}$ is a quotient of the preprojective algebra $\Pi(\Delta)$. In any case in which $\widetilde{A}$ is internally 3-Calabi–Yau, it would then follow that our candidate categorification $\mathrm{GP}(\widetilde{B}_J)$ is a subcategory of $\operatorname{mod}\Pi(\widetilde{\Delta}^J)$. Thus it is tempting to speculate that there may be a construction of $\widetilde{\mathscr{A}_J}$ analogous to Geiß–Leclerc–Schröer's construction of $\mathscr{A}_J$ via $\operatorname{Sub} Q_J \subseteq \operatorname{mod}\Pi(\Delta)$. While cluster-tilting objects for certain subcategories of $\operatorname{mod}\Pi$ for Π a preprojective algebra of arbitrary type have been well-studied, beginning with Buan–Iyama–Reiten–Scott [BIRS09, §III], these subcategories consist of finite-length Π-modules, whereas we will need to consider infinite-length modules as in [JKS14], and so require new methods.

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