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Twistor constructions of quaternionic manifolds and asymptotically hyperbolic Einstein–Weyl spaces

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**Twistor constructions of
quaternionic manifolds and
asymptotically hyperbolic
Einstein–Weyl spaces**

submitted by

Aleksandra Borówka

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

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Signature of Author

Aleksandra Borówka

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Summary

Let S be a $2n$ -dimensional complex manifold equipped with a line bundle with a real-analytic complex connection such that its curvature is of type $(1, 1)$, and with a real analytic h-projective structure such that its h-projective curvature is of type $(1, 1)$. For $n = 1$ we assume that S is equipped with a real-analytic Möbius structure.

Using the structure on S , we construct a twistor space of a quaternionic $4n$ -manifold M . We show that M can be identified locally with a neighbourhood of the zero section of the twisted (by a unitary line bundle) tangent bundle of S and that M admits a quaternionic S^1 action given by unit scalar multiplication in the fibres. We show that S is a totally complex submanifold of M and that a choice of a connection D in the h-projective class on S gives extensions of a complex structure from S to M . For any such extension, using D , we construct a hyperplane distribution on Z which corresponds to the unique quaternionic connection on M preserving the extended complex structure. We show that, in a special case, the construction gives the Feix–Kaledin construction of hypercomplex manifolds, which includes the construction of hyperkähler metrics on cotangent bundles. We also give an example in which the construction gives the quaternion-Kähler manifold $\mathbb{H}\mathbb{P}^n$ which is not hyperkähler.

We show that the same construction and results can be obtained for $n = 1$. By convention, in this case, M is a self-dual conformal 4-manifold and from Jones–Tod correspondence we know that the quotient B of M by an S^1 action is an asymptotically hyperbolic Einstein–Weyl manifold. Using a result of LeBrun [49], we prove that B is an asymptotically hyperbolic Einstein–Weyl manifold. We also give a natural construction of a minitwistor space T of an asymptotically hyperbolic Einstein–Weyl manifold directly from S , such that T is the Jones–Tod quotient of Z . As a consequence, we deduce that the Einstein–Weyl manifold constructed using T is equipped with a distinguished Gauduchon gauge.

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Introduction

In this thesis we show that a $2n$ -dimensional h-projective Cartan geometry can be used to construct quaternionic $4n$ -manifolds and, for $n = 1$, Einstein–Weyl 3-manifolds. The techniques that we use include techniques from twistor theory, differential geometry, representation theory and algebraic geometry.

In 1976 R. Penrose [57] found that there is a correspondence between self-dual conformal 4-manifolds and complex 3-manifolds containing a family of projective lines with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^2$. Such complex 3-manifolds are called twistor spaces and the projective lines are called twistor lines. Twistor theory has proven to be very useful as it enables studying differential manifolds using methods from complex analytic geometry. A special class of self-dual conformal 4-manifolds are self-dual Einstein 4-manifolds. As Einstein manifolds are solutions of vacuum Einstein field equations, this made twistor theory an object of study not only for mathematicians but also for physicists.

A quaternionic manifold M is a smooth $4n$ -dimensional manifold together with a rank 3 bundle $Q \subseteq \text{End}(TM)$ spanned point-wise by anti-commuting almost complex structures, and with a torsion-free connection preserving Q . S. Salamon [58] in 1986 developed the theory of quaternionic manifolds and found that there exists a twistor correspondence between quaternionic $4n$ -manifolds and complex $(2n + 1)$ -dimensional manifolds containing projective lines with normal bundles $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$, which generalises the Penrose correspondence (note that quaternionic manifold in dimension 4 are, by convention, self-dual conformal 4-manifolds).

When Q is trivial and admits three anti-commuting integrable almost complex structures (I, J, K) , then the manifold is said to be hypercomplex. A special class of such manifolds are hyperkähler manifolds i.e., the manifolds which admit a metric which is Kähler with respect to all (I, J, K) . They are currently

a subject of interest not only in differential geometry (R. Bielawski, A. Dancer [13], A.S. Dancer, A. Swann [31]) but also in algebraic geometry (global Torelli theorem, see for example M. Verbitsky [61]). The first explicit non-trivial example of a hyperkähler metric was the Calabi metric on $T^*\mathbb{C}P^n$ [17]. This motivated the question of existence of hyperkähler metrics on the cotangent bundles of Kähler manifolds. B. Feix [32], [33] and D. Kaledin [43], [44] independently showed that, at least locally, such metrics exist. More precisely, they proved that there exists a hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle of a Kähler manifold. They also generalised this construction, i.e., they showed that there exists a hypercomplex structure on a neighbourhood of the zero section of the tangent bundle of a manifold S , where S is a complex manifold equipped with a real analytic complex connection with curvature of type $(1, 1)$. In both cases, manifolds obtained in this way admit circle actions given by unit scalar multiplication in the fibres.

One of the main results of the thesis is to generalise the Feix construction in such a way that we obtain quaternionic manifolds which are not-necessarily hypercomplex. The Feix approach is to complexify a complex manifold S equipped with a connection D which has curvature of type $(1, 1)$ and then, using the fact that D is flat along leaves of $(1, 0)$ and $(0, 1)$ foliations, to construct bundles $V^{(1,0)}$ and $V^{(0,1)}$ over the leaf spaces of the foliations given fibrewise by the spaces of affine functions along the leaves of the corresponding foliations. Then we glue open subsets of $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$ and, as a result, obtain a twistor space Z of a hypercomplex manifold. Our key observation is that, to proceed with this construction, we only need to have flat projective structures along leaves of the foliations; in this case we replace bundles of affine functions by bundles of affine sections of $\mathcal{O}(1)$ along the leaves and obtain a manifold Z which is a twistor space of a quaternionic manifold M which is not necessarily hypercomplex. This places the construction into a natural context of h-projective geometry: if we assume that S is a $2n$ -manifold equipped with a real analytic h-projective structure (or Möbius structure in dimension 2) such that the h-projective curvature is of type $(1, 1)$, we obtain flat projective structures along leaves of the foliations on S^c . To maximise generality, we also suppose that S is equipped with a holomorphic line bundle \mathcal{L} with a compatible real-analytic complex connection with curvature of type $(1, 1)$.

We show that M can be locally identified with a neighbourhood of the zero section of the twisted (by some unitary line bundle) tangent bundle of S and admits an S^1 action given by unit multiplication in the fibres. We also prove, that S is a totally complex submanifold of M , and that a choice of a real-analytic connection D in the h-projective class, gives local extensions of the complex structure from S to M . Using D , we will also construct a hyperplane distribution on the twistor space Z of M transversal to twistor lines; such distributions correspond to quaternionic connections on M . We also show that, when the h-projective structure admits a real-analytic complex connection D with curvature of type $(1, 1)$ and the line bundle is $\mathcal{O}(-1)$ equipped with the connection induced from D , the unitary twist is trivial and our construction is the same as the B. Feix construction. We call our construction the generalised Feix–Kaledin construction of quaternionic manifolds.

A type of twistor correspondence was found in 1982 by N. Hitchin [38] for Einstein–Weyl 3-manifolds [30]. It shows that there exists a correspondence between Einstein–Weyl 3-manifolds and complex surfaces containing families of projective lines called minitwistor lines. Such complex surfaces are called minitwistor spaces. The Penrose and Hitchin correspondences were connected by P.E. Jones and K.P. Tod [42] in 1985. They showed that quotients by a conformal vector field of self-dual conformal 4-manifolds admit Weyl structures which are Einstein–Weyl, and that the self-dual conformal 4-manifolds can be reconstructed from Einstein–Weyl 3-manifolds with an abelian monopole. Moreover, they showed that these operations commute with the corresponding operations on twistor spaces, where the conformal action on a self-dual conformal 4-manifold corresponds to a local holomorphic action on its twistor space. An explicit calculation of the Einstein–Weyl structure obtained by the Jones–Tod construction can be found in [22].

The case when the conformal vector field integrates to an S^1 action has been studied by C. LeBrun [49]. He has shown that quotients by semi-free conformal S^1 actions of compact self-dual conformal 4-manifolds that have non-negative scalar curvature and are not conformally flat, are hyperbolic Einstein–Weyl manifolds. He has also introduced the notion of an asymptotically hyperbolic Einstein–Weyl manifold and has proved that the quotient of any self-dual conformal 4-manifold by a semi-free, or, more generally, so called docile circle

action is an asymptotically hyperbolic Einstein–Weyl manifold.

The S^1 action obtained by the generalised Feix–Kaledin construction for quaternionic manifold is semi-free, hence, in the case when S is a Möbius 2-manifold, we can apply the result of LeBrun. Therefore, our second main result is that the quotient by the S^1 action of a quaternionic manifold, obtained by the generalised Feix–Kaledin construction for quaternionic manifolds in the case when S is a Möbius 2-manifold, is an asymptotically hyperbolic Einstein–Weyl manifold with S as an asymptotic end.

Our third main result is to construct a minitwistor space T of an asymptotically hyperbolic Einstein–Weyl 3-manifold B directly from a surface S equipped with a real-analytic conformal Cartan connection. Hence, B solves the Einstein–Weyl equation for a boundary value problem, where the boundary condition is the conformal Cartan geometry on S . We also show a natural construction of a twistor space Z from S , such that T is a quotient of Z ; in fact the minitwistor lines in T are constructed as quotients of some twistor lines from Z . In other words, the Einstein–Weyl manifolds constructed from the conformal Cartan 2-dimensional geometries are not only asymptotically hyperbolic, but also admit a distinguished Gauduchon gauge.

Because we use twistor methods for the construction, the results that we obtain are only local.

Now we present the structure of the thesis. Chapter 1 introduces basics and known results which we need as a background for our constructions. We start from a brief introduction to complex manifolds and holomorphic bundles (see Section 1.1). Then we discuss theory of quaternionic manifolds (see Section 1.2) and Einstein–Weyl 3-manifolds (see Section 1.3). The theory of parabolic geometries is discussed in Section 1.4 and twistor theory is introduced in Section 1.5.

Chapter 2 contains the generalised Feix–Kaledin construction for quaternionic manifold. We start from discussing basic and known tools, namely complexifications and projective bundles, that we need for the construction (see Section 2.3). In Section 2.4 we discuss the construction of a complex $(2n + 1)$ -dimensional manifold Z and in Sections 2.5 and 2.6.1 we prove that Z is a twistor space of a quaternionic manifold. In the remaining subsections of Section 2.6, we discuss properties of the obtained manifold. We prove that M

can locally be identified with a neighbourhood of the zero section of $TS \otimes \mathcal{Q}$, where \mathcal{Q} is some unitary line bundle, and show that it admits quaternionic S^1 action given by unit scalar multiplication in the fibres (see Sections 2.6.2 and 2.6.4). In Section 2.6.5 we show that S is a totally complex submanifold of M . The extensions of the complex structure from S to M are discussed in Section 2.6.3. Finally, in Section 2.7 we construct quaternionic connections on Z and discuss difficulties that lie behind showing that the quaternionic structure on M gives on S the h-projective structure we started from.

In Chapter 3 we construct asymptotically hyperbolic Einstein–Weyl 3-spaces. In Section 3.1 we apply a result of LeBrun [49] to show that the quotient by the S^1 action of a quaternionic manifold obtained by the generalised Feix–Kaledin construction of quaternionic manifolds is an asymptotically hyperbolic Einstein–Weyl 3-manifold. In Section 3.2.2, starting from a surface with a real-analytic conformal Cartan connection, we construct a complex surface T , while in Section 3.2.3 we construct a 3-dimensional complex manifold Z such that T is a quotient of Z . In Section 3.2.4 we show that Z is a twistor space of a self-dual conformal 4-manifold and that images, by quotients, of some of real twistor lines on Z are real minitwistor lines on T . As a result, we conclude that T is a minitwistor space of an asymptotically hyperbolic Einstein–Weyl 3-manifold. Finally in Section 3.3 we compare the constructions of twistor spaces obtained in Chapters 2 and 3.

Chapter 1

Background material

1.1 Complex manifolds and vector bundles

1.1.1 Complex manifolds

In this section we will discuss some basic facts about smooth and complex manifolds. Smooth, real-analytic or holomorphic manifolds are topological manifolds equipped with an atlas such that the transition functions are smooth, real-analytic or holomorphic respectively. An almost complex structure I on a smooth manifold S is an endomorphism of the tangent bundle TS such that $I^2 = -\text{id}$, and a manifold equipped with an almost complex structure is called an almost complex manifold. In this case the complexified tangent bundle to S satisfies the following decomposition into $\pm i$ eigenspaces of I

$$TS \otimes \mathbb{C} = T^{1,0}S \oplus T^{0,1}S.$$

An almost complex structure is called an integrable almost complex structure or a complex structure if the $T^{1,0}S$ distribution is invariant under Lie bracket. A smooth manifold S equipped with a complex structure is called a complex manifold. By the Newlander–Nirenberg theorem [53], complex manifolds are holomorphic and the converse is also true. Hence we can treat holomorphic manifolds as smooth manifolds equipped with a complex structure.

A Riemannian manifold is a smooth manifold equipped with a Riemannian metric g , that is, a positive-definite section of S^2T^*S . A Riemannian smooth

manifold (S, g, I) is called a Kähler manifold if I is a complex structure, the metric g is hermitian with respect to I (that is $g(IX, IY) = g(X, Y)$), and the 2-form $\omega(X, Y) := g(X, IY)$ is closed.

Examples of Kähler manifolds include:

- Flat spaces \mathbb{C}^n ,
- Complex tori \mathbb{C}^n/Λ with the metric induced from \mathbb{C}^n ,
- The Grassmannian $Gr_r(\mathbb{C}^n)$ of r -dimensional subspaces of \mathbb{C}^n ,
- The unit complex ball B^n with the Bergman metric,
- Calabi–Yau manifolds (not necessarily simply-connected), in particular K3 surfaces.

1.1.2 Complex and holomorphic vector bundles

In this section we discuss very briefly the properties of holomorphic vector bundles (see [4], [62]).

Definition 1. *A holomorphic vector bundle \mathcal{A} is a complex vector bundle such that the transition functions are holomorphic.*

On the complex manifold S the differential operator d decomposes into holomorphic and anti-holomorphic parts:

$$d = \partial + \bar{\partial}.$$

Moreover, a function f on S is holomorphic if $\bar{\partial}f = 0$.

Now for holomorphic vector bundle \mathcal{A} , using holomorphic trivialisations of \mathcal{A} and holomorphic transition functions, we get that $\bar{\partial}$ operator on functions induces a $\bar{\partial}_{\mathcal{A}}$ operator on \mathcal{A} . Suppose now that D is a connection on a complex vector bundle \mathcal{A} . Then D decomposes into sum of connections

$$D = D^{(1,0)} + D^{(0,1)}.$$

This induces a decomposition of the curvature F of D into

$$F = F^{(2,0)} + F^{(1,1)} + F^{(0,2)}.$$

It is a well known fact (see for example [52]) that if the $F^{(0,2)}$ part of curvature of the connection D vanishes, then there exists a holomorphic structure on \mathcal{A} such that $D^{(0,1)} = \bar{\partial}_{\mathcal{A}}$.

Definition 2. A covariant derivative D on \mathcal{A} is called compatible with the holomorphic structure on \mathcal{A} if $D^{(0,1)} = \bar{\partial}_{\mathcal{A}}$.

In this thesis we take a holomorphic connection to mean a covariant derivative on a holomorphic vector bundle which is compatible with the holomorphic structure.

1.2 Quaternionic Manifolds

Quaternions are a well known generalisation of complex numbers. They are a 4-dimensional real vector space \mathbb{H} with a bilinear operation, called the multiplication, which in the standard basis $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfies

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Quaternions have all the properties of a field except commutativity of multiplication.

Recall that a complex manifold is a $2n$ -real manifold with an integrable complex structure I . These ideas motivate generalisations of complex manifolds to ‘quaternion-like’ manifolds called hypercomplex manifolds and quaternionic manifolds.

1.2.1 Hypercomplex and hyperkähler manifolds

The first generalisation of complex manifolds from the quaternionic point of view are hypercomplex manifolds (see [14]). These are smooth manifolds such that their tangent spaces are equipped with triples of anti-commuting integrable complex structures (I, J, K) which satisfy $I^2 = J^2 = K^2 = IJK = -1$.

Remark 1. It is enough to have two anti-commuting complex structures I, J as then $K := IJ$ is the third one.

Moreover, from these complex structures we obtain a sphere of complex structures given by $aI + bJ + cK$ for $(a, b, c) \in S^2 \subset \mathbb{R}^3$. This means that we have a

natural action of the quaternions \mathbb{H} on the tangent bundle of the hypercomplex manifold (compare [12]).

Proposition 1. *On any hypercomplex manifold M there exists a unique torsion free connection D with $DI = DJ = DK = 0$, which is called the Obata connection [54].* \square

A hypercomplex 4-manifold M is naturally equipped with a conformal structure such that X, IX, JX, KX is a conformal frame for any non-zero vector field X . We will see later that from this point of view, dimension 4 is special for other ‘quaternion-like’ geometries.

An important subclass of hypercomplex manifolds are the hyperkähler manifolds introduced by Berger [8] and further investigated by Calabi [17].

Definition 3. *Hyperkähler manifolds (M, g, I, J, K) are Riemannian hypercomplex manifolds such that the metric g is hermitian with respect to each of the complex structures I, J, K , and the 2-forms $\omega_I(X, Y) := g(X, IY)$, $\omega_J(X, Y) := g(X, JY)$, $\omega_K(X, Y) := g(X, KY)$ are closed.*

In other words a hyperkähler manifold is a hypercomplex manifold which is Kähler with respect to all complex structures from the hypercomplex family.

Proposition 2. *The curvature of the Levi-Civita connection of a Kähler metric has type $(1, 1)$. As a consequence the Levi-Civita connection of a hyperkähler metric has type $(1, 1)$ with respect to all complex structures from the hypercomplex family.*

Proof. This is a standard fact in Kähler geometry, for proof see for example [32]. \square

Examples of hyperkähler manifolds include the following:

- \mathbb{H}^n is a hyperkähler manifold.
- The only compact hyperkähler 4-manifolds are K3 surfaces and tori.
- An explicit non-trivial example of a hyperkähler manifold is $T^*\mathbb{C}P^n$ equipped with the Calabi metric, see [17].

Any hyperkähler manifold has trivial canonical bundle hence it is a (non-necessarily simply-connected) Calabi-Yau manifold.

1.2.2 Quaternionic manifolds

Quaternionic manifolds (see [58]) are a generalisation of hypercomplex manifolds. Now we state the definition of quaternionic manifolds for dimension $4n \geq 8$. In Section 1.2.4 we will discuss the 4-dimensional case.

Definition 4. *Let $n \geq 2$. A $4n$ -manifold M is called a quaternionic manifold if M is a smooth manifold equipped with a rank 3 subbundle $Q \subset \text{End}(TM)$ such that Q is locally generated by three almost complex structures I, J, K satisfying*

$$I^2 = J^2 = K^2 = IJK = -1$$

and such that there exists a torsion-free connection D which preserves Q (but only as a whole i.e., for an vector field X we have that $D_X I, D_X J, D_X K \in Q$). Such a connection will be called a quaternionic connection.

Note that a frame I, J, K exists only locally, which means that there is still an S^2 -bundle of almost complex structures but, unlike in the hypercomplex case, this bundle may not be trivial and the almost complex structures may be not integrable. Moreover, note that although quaternionic manifolds locally admit integrable almost complex structures there may not exist a global integrable almost complex structure; thus quaternionic manifolds do not need to be complex (see Example 1). The example that justifies this definition of quaternionic manifolds is the quaternionic projective space.

Example 1. Let $\mathbb{H}P^n$ be the quaternionic projective space, i.e., the space of all 1-dimensional quaternionic subspaces of \mathbb{H}^{n+1} . This space looks locally like a quaternionic affine space (and thus is locally hypercomplex) so one would expect it to be a ‘quaternionic’ manifold. Indeed it is clear that this space is a quaternionic manifold according to our definition. But the topological constraints (see for example [10]) show that $\mathbb{H}P^n$ does not have a global almost complex structure so it is not a complex manifold. However, $\mathbb{H}P^n$ is an integrable quaternionic manifold in the sense that there exist local quaternionic coordinates, i.e., maps from neighbourhoods of points in $\mathbb{H}P^n$ to \mathbb{H} giving an isomorphism between neighbourhoods of points in $\mathbb{H}P^n$ and subsets of \mathbb{H}^{n+1} .

Kulkarni [47] showed that any compact and simply connected quaternionic manifold on which there exist quaternionic coordinates is diffeomorphic to

$\mathbb{H}\mathbb{P}^n$. Examples of compact and simply connected quaternionic manifolds that are not diffeomorphic to $\mathbb{H}\mathbb{P}^n$ are the symmetric Wolf spaces other than $\mathbb{H}\mathbb{P}^n$ [63] which will be discussed in the next section.

Note that we can always find Riemannian metrics which are hermitian with respect to every almost complex structure from Q . Indeed, given any Riemannian metric g on a quaternionic manifold M , if (I, J, K) span Q_x then we can define a Riemannian metric g' compatible with Q by:

$$g'(X, Y) = \frac{1}{4}(g(X, Y) + g(IX, IY) + g(JX, JY) + g(KX, KY)), \quad X, Y \in T_x M.$$

However, the Levi-Civita connection of g' is usually not a quaternionic connection.

A quaternionic connection is not unique and, unlike in the hypercomplex case, there is no preferred choice of such. In Section 1.4 we will say more about the class of compatible quaternionic connections.

Proposition 3 ([1], [58]). *The curvature of any quaternionic connection decomposes into two parts $R^D = W_Q + R_B^D$, where W_Q , called the quaternionic Weyl tensor, is in the kernel of the Ricci contraction and R_B^D is determined by the normalised quaternionic Ricci tensor. Moreover, the quaternionic Weyl tensor W_Q is independent of the choice of the quaternionic connection D .*

Proof. The proof involves the decomposition of quaternionic curvature tensors into irreducible components. For $n > 1$, the kernel of the Ricci contraction and thus the Weyl part is the irreducible submodule of $\mathcal{R}(\mathfrak{sl}_n(\mathbb{H}))$. In [1], [58] it is also shown that the curvature of a quaternionic connection satisfies $[R_{X,Y}^D, Q] \subseteq Q$ and that $[(W_Q)_{X,Y}, Q] = 0$. \square

Remark 2. The tensor R_B^D is in fact equal to $-[[\text{Id} \wedge r_Q^D]]_Q$. In Section 1.4 we give a precise formula for r_Q^D and $[[\cdot, \cdot]]_Q$.

The fact that Weyl curvature lie in an irreducible submodule implies the following proposition.

Proposition 4. *The quaternionic Weyl tensor is of type $(1, 1)$ with respect to any almost complex structure from Q .*

Proof. This comes from the fact that quaternionic Weyl curvatures of type $(1, 1)$ form a submodule of irreducible module of all quaternionic Weyl curvatures. Moreover this submodule is non-zero as the Weyl curvature of any hyperkähler manifold is the whole curvature which is type of $(1, 1)$. \square

1.2.3 Quaternion-Kähler manifolds

For a quaternionic manifold (M, Q, D) , there may not exist a quaternionic connection which is the Levi-Civita connection of a metric compatible with Q . If such a connection (and thus such a metric) exists then the manifold will be called a quaternion-Kähler manifold.

Remark 3. Recall that the Levi-Civita connection of a compatible metric is ‘only’ a quaternionic connection so it preserves Q only as a whole. Thus a quaternion-Kähler manifold may not be Kähler. An example of this case is again $\mathbb{H}P^n$ which is naturally a quaternion-Kähler manifold.

S. Salamon [59] showed that if M is a quaternion-Kähler manifold then TM is itself a quaternionic manifold. Moreover, unless M is flat, TM is not quaternion-Kähler. This provides examples of quaternionic manifolds which are not quaternion-Kähler. An important property of quaternion-Kähler manifolds is that they are Einstein (see [9]).

Definition 5. Let S be a submanifold of a quaternionic manifold (M, Q) . Then S is called a totally complex submanifold if in a neighbourhood of $S \subset M$ there exists an integrable complex structure $J \in Q$ such that

1. $J(TS) \subseteq TS$
2. for all $I \in Q$ anti-commuting with J we have $I(TS) \cap TS = 0$

A special case of totally complex submanifolds are those of maximal dimension i.e., of dimension $2n$ where $4n$ is the dimension of the quaternionic manifold M .

Proposition 5 ([2]). *Let (S, J) be a $2n$ -dimensional totally complex submanifold of a quaternionic $4n$ -manifold M . Then*

1. *The space $NS := I(TS)$ does not depend upon the choice of I anticommuting with J and this gives a direct sum decomposition $TM = TS \oplus NS$,*
2. *Every quaternionic connection \mathfrak{D} on TM induces by projection a complex connection D on TS ,*
3. *\mathfrak{D} torsion free $\Rightarrow D$ torsion free,*
4. *\mathfrak{D} preserves a metric $\Rightarrow D$ preserves the induced metric,*
5. *M is quaternion-Kähler $\Rightarrow S$ is Kähler.*

Moreover, the condition 1 is an equivalent condition for a complex submanifold S to be a totally complex submanifold of M .

Proof. To prove 1, choose $I \in Q$ anti-commuting with J and identify $NS = I(TS)$. This definition does not depend on the choice of I as all almost complex structures anti-commuting with J in Q are spanned by I and IJ and $IJ(TS) = I(JTS) = I(TS)$. As the dimension of M is $4n$ and we have found two subbundles of TM dimension $2n$ which are transversal to each other, the whole TM is a direct sum of them.

For 2, let $X, Y \in TS$ and $D_X Y := \pi(\mathfrak{D}_X Y)$. Then

$$\begin{aligned} (D_X J)Y &= D_X(JY) - JD_X Y = \pi(\mathfrak{D}_X(JY)) - J\pi(\mathfrak{D}_X Y) = \\ &= \pi(\mathfrak{D}_X J)Y + \pi(J\mathfrak{D}_X Y) - J\pi(\mathfrak{D}_X Y) \subseteq \pi(NS) = 0, \end{aligned}$$

as $\mathfrak{D}_X J$ is an element of Q spanned by the structures anti-commuting with J .

For 3, observe that

$$D_X Y - D_Y X - [X, Y] = \pi(\mathfrak{D}_X Y) - \pi(\mathfrak{D}_Y X) - \pi([X, Y]) = \pi(\mathfrak{D}_X Y - \mathfrak{D}_Y X - [X, Y]).$$

For 4, using 1 we have that

$$(D_X g)(X, Y) = \partial_X(g(Y, Z)) - g(D_X Y, Z) - g(Y, D_X Z) =$$

$$= \partial_X(g(Y, Z)) - g(\mathcal{D}_X Y, Z) - g(Y, \mathcal{D}_X Z) = 0.$$

As S is a complex manifold, 5 is a direct consequence of 4. □

The complete classification of symmetric quaternion-Kähler manifolds with a non-zero Ricci curvature was given by Wolf [63] and these spaces are called the Wolf spaces. It turns out that all compact examples have positive Ricci curvature while all non-compact examples have negative Ricci curvature and that for each type of compact simple Lie group (except $SU(2)$) we have exactly one example of a compact symmetric Wolf space. More precisely

- $M = SU(n + 2)/S(U(n) \times U(2)) = Gr_2(\mathbb{C}^{n+2}),$
- $M = SO(n + 4)/[SO(n) \times SO(4)] = Gr_4(\mathbb{R}^{n+4}),$
- $M = Sp(n + 1)/[Sp(n) \times Sp(1)] = \mathbb{H}P^n$

are Wolf spaces with positive Ricci curvature and

- $M = SU(n, 2)/S(U(n) \times U(2)),$
- $M = SO(n, 4)/[SO(n) \times SO(4)],$
- $M = Sp(n, 1)/[Sp(n) \times Sp(1)]$

are Wolf spaces with negative Ricci curvature. Additionally we have an example with positive Ricci curvature and an example with negative Ricci curvature for each of the exceptional groups G_2, F_4, E_6, E_7 and E_8 .

The Wolf spaces are the only known examples of compact simply connected quaternion-Kähler manifolds which are not hyperkähler and there is a conjecture due to C. LeBrun and S. Salamon that all compact quaternion-Kähler manifolds with positive Ricci curvature are Wolf spaces.

1.2.4 Quaternionic manifolds in dimension 4

The Weyl part of a quaternionic-type curvature for dimension $4n > 4$ is an element of the irreducible submodule $\mathcal{R}(\mathfrak{sl}_n(\mathbb{H}))$. For dimension $4n = 4$ it takes values in $\mathcal{R}(\mathfrak{sp}_1) \oplus \mathcal{R}(\mathfrak{sp}'_1)$ which is clearly reducible. This is one of the reasons why 4-manifolds equipped with a quaternionic structure and a quaternionic

connection have different properties than quaternionic manifolds in dimension $4n > 4$. However, the manifolds with self-dual quaternionic Weyl tensor, i.e., such that the $\mathcal{R}(\mathfrak{sp}'_1)$ part of the Weyl tensor which vanishes, have a similar theory to quaternionic manifolds of higher dimension. This motivates the definition of quaternionic 4-manifolds as those smooth manifolds which admit a quaternionic subbundle Q of $\text{End}(TM)$ and a compatible quaternionic connection with a self-dual quaternionic Weyl tensor.

As we mentioned before, the quaternionic 4-manifolds are conformal in the natural way and, in fact, the quaternionic structure for dimension 4 is exactly the same data as the conformal structure. Moreover, the conformal Weyl tensor coincides with the quaternionic one and the notions of self-duality coincide.

The usual convention is to define quaternion-Kähler 4-manifolds as self-dual Einstein manifolds (see [10]).

1.2.5 Holonomy

A very common (see [9], [10]) and equivalent way to define hyperkähler and quaternion-Kähler manifolds in dimension $4n > 4$ is via holonomy groups (for hyperkähler manifolds this can be done also in dimension 4).

From this point of view we can define hyperkähler $4n$ -manifolds as Riemannian $4n$ -manifolds which have holonomy group contained in $\text{Sp}(n)$. To see that both definitions coincide observe that $\text{Sp}(n)$ is isomorphic to the group of orthogonal transformations of quaternionic n -plane \mathbb{H}^n which are linear with respect to complex structures from the quaternionic structure. Thus, as the hyperkähler connection preserves those complex structures, the holonomy group must be contained in $\text{Sp}(n)$. Conversely, suppose that the holonomy group of the Riemannian manifold is contained in $\text{Sp}(n)$. If we define a quaternionic structure over some point in such a way that the tangent space at this point becomes a quaternionic vector space, then we can parallel transport this structure to obtain the quaternionic structure over the manifold.

In the similar way one can treat quaternion-Kähler $4n$ -manifolds for $n > 1$ as those Riemannian manifolds which have holonomy group contained in $\text{Sp}(n) \times_{\mathbb{Z}/2} \text{Sp}(1)$ (see for example [10]).

1.3 Conformal and Einstein–Weyl geometry

Now we will introduce some basic and well known facts about Einstein–Weyl manifolds (see [22], [21]). As they come naturally from the conformal geometry point of view rather than Riemannian geometry, we will first introduce conformal manifolds.

1.3.1 Conformal manifolds

Classically a conformal structure is defined as an equivalence class of Riemannian metrics, where two Riemannian metrics g, g' are equivalent if there exists a positive function f such that $g = fg'$. This approach brings technical problems, because when studying conformal manifolds (i.e., smooth manifolds equipped with the conformal structure), one has to check how objects change under the change of a representative metric.

Fortunately there is another, equivalent approach (see [22], [21]) based on the fact that a conformal structure on M is a metric \langle, \rangle on the weightless tangent bundle – the tangent bundle tensored with the -1 -density line bundle L^{-1} (the 1 -density on a manifold S is the line bundle L such that $L^{\otimes n} \cong \wedge TS$; see [21]).

A Weyl connection is a connection which is torsion free and preserves the conformal structure. Such a connection is not unique. In fact they form an affine space modelled on 1 -forms (see [22], [21] for more on this approach) and other Weyl connections are determined by the formula

$$D'_X Y = (D + \omega)_X Y := D_X Y + \omega(X)Y + \omega \Delta X(Y),$$

where $\omega \Delta X(Y) = \omega(Y)X - \langle X, Y \rangle \omega^\sharp$ and the \sharp operator is the inverse of $Z \mapsto \langle Z, \cdot \rangle$ for vector fields tensored with some density line bundle.

The notation $D + \omega$ is motivated by the fact that the connections on the density line bundle differ by a 1 -form and there is a bijection between Weyl derivatives (i.e. curvatures of induced connections on the 1 -density line bundle) and Weyl connections.

For this reason we will often write a Weyl connection D in the form (g, ω) where D^g is the Levi-Civita of some compatible metric g , ω is a 1 -form and $D = D^g + \omega$. The choice of the Levi-Civita connection in this formula will

be called a choice of gauge and it corresponds to a choice of a non-vanishing positive section of a density bundle. The change of a gauge corresponds to a change of a metric in the conformal class which can be obtained by a conformal rescaling of the metric by e^f for some function f . The Weyl connection in the new gauge is given by $(e^f g, \omega')$, where $\omega' = \omega + df$.

A conformal manifold together with a Weyl connection will be called a Weyl manifold.

1.3.2 Einstein–Weyl manifolds

Definition 6. *A Weyl manifold (M, c, D) will be called an Einstein–Weyl manifold if the symmetric trace-free part of the Ricci tensor of the Weyl connection vanishes.*

In this thesis we will be interested in Einstein–Weyl 3-manifolds, although the definition makes sense also for higher dimensional manifolds (see for example [18],[56]).

If the Weyl connection is the Levi-Civita connection of a compatible metric then the Einstein–Weyl manifold is Einstein. This means that for Riemannian manifolds the Einstein–Weyl equation is the same as the Einstein equation so it brings nothing new. The situation in general is much different – there are many examples of Einstein–Weyl manifolds which are not Einstein. In particular for dimension 3 the only examples of Einstein manifolds are those of constant curvature whereas there exist many non-trivial examples of Einstein–Weyl 3-manifolds.

One of the non-trivial examples (see [42]) is the Berger sphere with the Weyl connection given by

$$g = d\theta^2 + \sin^2 \theta d\phi^2 + a^2(d\psi + \cos \theta d\phi)^2$$

$$\omega = b(d\psi + \cos \theta d\phi),$$

where $b^2 = a^2(1 - a^2)$.

There is a special gauge on Weyl manifolds, called the Gauduchon gauge, which is especially important for Einstein–Weyl manifolds because of the Jones–Tod correspondence [42] (see Section 1.5.4). This is a gauge which satisfies $d *_g \omega = 0$. Such a gauge always exists but, unless the manifold is

compact, it may be not unique. In 3-dimensional case, it can be interpreted in the language of abelian monopoles.

Definition 7. *An abelian monopole on a Weyl manifold (B, D, c) is a section μ of the L^{-1} -density bundle together with a connection on a rank 1 trivial bundle of the form $d + A$ such that the following equation is satisfied*

$$*D\mu = dA.$$

This equation is called an abelian monopole equation.

Note that the formula $d + A$ is non-canonical in the sense, that it depends on gauge transformations. More precisely one can notice that

$$(d + A)(e^f s) = e^f (d + A + df)(s).$$

But as the possible changes are of the form $A \rightarrow A + df$, they do not change the monopole equation.

If μ is non-vanishing then it is a gauge and the Weyl derivative is $D = D^\mu + \omega$, where $D^\mu \mu = 0$ and $\omega = \mu^{-1} D\mu$, and the metric associated to the gauge μ is $g = \mu^2 c$. Hence $*_g \omega = *D\mu = dA$, and finally $d*_g \omega = 0$. Thus the gauge μ is a Gauduchon gauge.

1.3.3 Asymptotically hyperbolic Einstein–Weyl manifolds

Hyperbolic 3-manifolds are Riemannian 3-manifolds of constant sectional curvature equal to -1 . They are basic examples of Einstein and thus Einstein–Weyl geometries. LeBrun [49] stated the definition which generalises this notion.

Definition 8 ([49]). *An Einstein–Weyl 3-manifold (B, c, \mathcal{D}) is said to be asymptotically hyperbolic if there exist:*

- *a connected Riemannian 3-manifold (Y, \tilde{h}) with a boundary ∂Y ,*
- *a defining function of ∂Y , i.e., a function $f : Y \rightarrow \mathbb{R}^+ \cup \{0\}$ such that $f|_{\partial Y} = 0$ and $(df)|_{\partial Y} \neq 0$,*
- *a smooth 1-form ω on Y vanishing along ∂Y ,*

such that (B, c, \mathcal{D}) is isomorphic to $(Y - \partial Y)$ equipped with the Weyl structure given by the conformal class of $[\tilde{h}]$ and the Weyl connection given by $(f^{-2}\tilde{h}, \omega)$ (see Section 1.3.1).

In the same paper LeBrun also asked if all asymptotically hyperbolic Einstein–Weyl manifolds (in the global sense, i.e., when Y from the definition is compact) are in fact hyperbolic.

One of the aims of this thesis (Chapter 3.1) is to construct examples of Einstein–Weyl manifolds which are locally (i.e., in the neighbourhood of $y \in \partial Y$) asymptotically hyperbolic.

1.4 Parabolic geometries

As we have already discussed, a well known approach to conformal geometry is to consider the class of connections which are torsion free and preserve the conformal structure (see [22], [21]). Such connections are called conformal Weyl connections and form an affine space modelled on 1-forms. Moreover, one can define an algebraic bracket $\llbracket \cdot, \cdot \rrbracket$ which can be used to write down explicitly the formula for $D + \gamma$ for any Weyl connection D and any 1-form γ . This leads to an approach to study a conformal structure from an equivalence class of connections point of view, though it is important to remember that in order to be able to define $\llbracket \cdot, \cdot \rrbracket$ we need the conformal structure. The same situation appears also in other geometries like projective geometries, h -projective geometries and quaternionic geometries. There is a Lie-algebra-theoretic explanation for those facts. All these geometries can be approached from the Cartan connections point of view and are examples of parabolic Cartan geometries with an abelian nilradical. In this section we discuss some of the theory of parabolic geometries (see [20], [30], [27], [28], [29]).

1.4.1 General theory

Let M be a manifold such that $\dim M = \dim G/P$, where G, P are Lie groups.

Definition 9. A *principal G -connection* on a principal G bundle \tilde{G} on M is a 1-form $\tilde{\eta} : T\tilde{G} \rightarrow \mathfrak{g}$ which is G -equivariant and satisfies $\tilde{\eta}(X_\xi) = \xi$, where X_ξ is the vector

field on \tilde{G} associated with the derivative of the 1-parameter subgroup of G defined by the map $t \mapsto \exp(t\xi)$.

Let G/P be a homogeneous space and let \tilde{G} be a principal G -bundle with a principal G -connection $\tilde{\eta}$. Suppose also that we have a reduction of a structure group to $P \leq G$. Then $\tilde{\eta}$ induces a 1-form η_M on M with values in the bundle $\mathfrak{g}_M/\mathfrak{p}_M$ associated to $\mathfrak{g}/\mathfrak{p}$ in the following way. The reduction of the structure group to P means that we have a new principal bundle \tilde{P} over M which is contained in \tilde{G} . The pull-back of $\tilde{\eta}$ defines a 1-form $\eta : T\tilde{P} \rightarrow \mathfrak{g}$ such that $\eta \bmod \mathfrak{p} : T\tilde{P} \rightarrow \mathfrak{g}/\mathfrak{p}$ is P -equivariant and horizontal hence defines $\eta_M : TM \rightarrow \tilde{P} \times_P \mathfrak{g}/\mathfrak{p}$.

Definition 10. A *Cartan geometry* on a manifold M modelled on the homogeneous space G/P is

- A principal G bundle \tilde{G} ,
- a principal G -connection $\tilde{\eta}$ on \tilde{G} ,
- a reduction of a structure group to $P \leq G$ such that the 1-form η_M is a bundle isomorphism (Cartan condition).

Definition 11. *Parabolic geometries* are Cartan geometries modelled on G/P , where \mathfrak{g} is semisimple and \mathfrak{p} is parabolic, that is, \mathfrak{p}^\perp (with respect to the Killing form on \mathfrak{g}) is the nilradical of \mathfrak{p} .

Parabolic geometries with abelian nilradical are parabolic geometries such that \mathfrak{p}^\perp is abelian.

Note that, using the Killing form, $\mathfrak{p}^\perp \cong (\mathfrak{g}/\mathfrak{p})^*$. Using this, as $TM \cong \tilde{P} \times_P \mathfrak{g}/\mathfrak{p}$, we have that $T^*M \cong \tilde{P} \times_P \mathfrak{p}^\perp$. As a consequence $T^*M \subseteq \mathfrak{g}_M$ is a bundle of Lie algebras. As the curvature of a Cartan connection \mathcal{D} satisfies $R^\mathcal{D} \in \wedge^2 T^*M \otimes \mathfrak{g}_M$, we can use a bracket induced by $T^*M \subseteq \mathfrak{g}_M$ to define

$$\partial R^\mathcal{D} := \Sigma_i[\varepsilon^i, R_{e_i}^\mathcal{D}].$$

Definition 12. A Cartan connection \mathcal{D} is normal if $\partial R^\mathcal{D} = 0$.

For parabolic geometries with abelian nilradical the Lie bracket on the graded bundle $TM \oplus \mathfrak{p}_M^0 \oplus T^*M$ associated to \mathfrak{g}_M induces an algebraic bracket

$$\llbracket \cdot, \cdot \rrbracket : TM \times T^*M \rightarrow \mathfrak{p}_M^0 \subseteq \mathfrak{gl}(TM).$$

The first order structure for parabolic Cartan geometries is a reduction of the structure group of the frame bundle of M to $P_0 := P/\exp(\mathfrak{p}^\perp)$. It is important to understand that generally the first structures for parabolic structures may or may not encode the whole Cartan geometry. For example the conformal structure is a first order structure for the normal conformal Cartan connection. For dimension $n > 2$ it encodes the conformal Cartan geometry but for dimensions 1 and 2 it does not.

For the corresponding parabolic structure a second order structure can be defined as an equivalence class Π of torsion-free P_0 -connections with the equivalence relation defined by:

$$D \sim D' \Leftrightarrow \exists \gamma \in \Omega^1(M) : D' = D + \gamma := D + \llbracket \cdot, \gamma \rrbracket.$$

A. Cap and H. Schichl in [26] showed that, for parabolic geometries in almost all cases, a first order structure together with a second order structure (as above) determine normal Cartan connection.

There is a useful method for finding out which functions on Π are invariant, i.e., do not depend on the choice of the connection in the equivalence class. For any function F on Π we can define $\partial_\gamma F(D) := \frac{d}{dt} F(D + t\gamma)|_{t=0}$. Then F is invariant if for any $D \in \Pi$ and $\gamma \in \Omega^1(M)$ we have that $\partial_\gamma F(D) = 0$.

Straightforward calculations show that $\partial_\gamma D_X s = \llbracket X, \gamma \rrbracket \cdot s$, where s is a section of a vector bundle E associated to the frame bundle and \cdot is the natural action of \mathfrak{p}_M^0 on E . This gives:

$$\partial_\gamma D_{X,Y}^2 s = \llbracket X, \gamma \rrbracket \cdot D_Y s + \llbracket Y, \gamma \rrbracket \cdot D_X s - D_{\llbracket X, \gamma \rrbracket \cdot Y} s + \llbracket Y, D_X \gamma \rrbracket \cdot s$$

and the curvature $R^D \in \Omega^2(M, \mathfrak{p}_M^0)$ of D satisfies

$$\partial_\gamma R_{X,Y}^D = -\llbracket \text{Id} \wedge D\gamma \rrbracket_{X,Y},$$

where $\llbracket \text{Id} \wedge \alpha \rrbracket_{YZ} = \llbracket Y, \alpha(Z, \cdot) \rrbracket - \llbracket Z, \alpha(Y, \cdot) \rrbracket$ for a T^*M -valued 1-form α .

For the parabolic structure one can also define the normalised Ricci tensor $r^D \in \Omega^1(M, T^*M)$ which satisfies $\partial_\gamma r^D = -D\gamma$. The normalised Ricci tensor arises from the decomposition of the Ricci curvature tensor with respect to the parabolic structure by normalising the coefficients in the decomposition. Then $R^D - \llbracket \text{Id} \wedge r^D \rrbracket$ is invariant and we call $W := R^D - \llbracket \text{Id} \wedge r^D \rrbracket$ the Weyl curvature tensor of the corresponding parabolic structure (of course this definition depends on the considered structure). We also define the Cotton-York tensor by $C^D = d^D r^D$ and it can be shown that $\partial_\gamma C^D_{X,Y} = \llbracket W_{X,Y}, \gamma \rrbracket$. As a consequence, when W vanishes, the Cotton-York tensor is invariant.

1.4.2 Projective, h-projective and quaternionic structures

Projective structure

Definition 13. An n -manifold $(M, [D]_p)$, $n > 1$ is called a *projective manifold* if $[D]_p$ is a **projective structure**, i.e., an equivalence class of torsion-free connections under the equivalence

$$D \sim_p D' \Leftrightarrow \exists \gamma : \forall Y, Z \in TM \quad D'_Y Z = D_Y Z + \llbracket Y, \gamma \rrbracket_p Z,$$

where γ is a 1-form and $\llbracket Y, \gamma \rrbracket_p Z = \gamma(Y)Z + \gamma(Z)Y$.

A **holomorphic projective structure** on a holomorphic manifold is an equivalence class of local holomorphic connections under the relation defined above, where γ is a holomorphic 1-form and Y, Z are holomorphic vector fields.

The projective structure is a parabolic Cartan geometry with $G = PGL(n+1, \mathbb{R})$ and $P_0 = GL_n \mathbb{R}$, hence there is no reduction of the structure group of the frame bundle.

This definition coincides with a classical definition of projective equivalence of connections (see for example [28]). More precisely, it can be shown that two connections D and D' are projectively equivalent in the classical sense (i.e., have the same unparametrised geodesics) if there exists $\gamma \in \Omega^1(M)$ such that $D'_Y Z = D_Y Z + \llbracket Y, \gamma \rrbracket_p Z$ for any $Y, Z \in TM$.

The normalised Ricci tensor in this case is:

$$r_p^D(Y, Z) = \frac{1}{n-1} \text{Ric}^D(Y, Z) + \frac{1}{(n-1)(n+1)} [\text{Ric}^D(Y, Z) - \text{Ric}^D(Z, Y)],$$

and the projective Weyl curvature $W_p := R^D - \llbracket \text{Id} \wedge r_p^D \rrbracket_p$ is invariant.

It can be shown that a projective structure is flat for $n > 2$ if the projective Weyl tensor vanishes. A projective structure is flat for $n = 2$, if the projective Cotton-York tensor vanishes (the projective Weyl tensor for $n = 2$ always vanishes).

H-projective structure An h-projective structure (see [40], [41], [64]) is a complex but not-necessary holomorphic version of a real projective structure. Let (M, J) be a complex manifold of dimension $2n$. The curve c is called h-planar with respect to a connection D if for all T tangent to c we have that $D_T T \in \text{span}\{T, JT\}$. Classically, two connections are h-projectively equivalent if they have the same h-planar curves. This motivates the following definition:

Definition 14. A $2n$ -manifold $(M, J, [D]_h)$, $n > 1$, is called *an h-projective manifold* if J is a complex structure and $[D]_h$ is an h-projective structure, i.e., an equivalence class of complex ($DJ = 0$) torsion-free connections under the equivalence

$$D \sim_h D' \Leftrightarrow \exists \gamma : \forall Y, Z \in TM \quad D'_Y Z = D_Y Z + \llbracket Y, \gamma \rrbracket_h Z,$$

where $\gamma \in T^*M$ and $\llbracket Y, \gamma \rrbracket_h(Z) = \frac{1}{2}(\gamma(Y)Z + \gamma(Z)Y - \gamma(JY)JZ - \gamma(JZ)JY)$.

A *holomorphic h-projective structure* is an h-projective structure for which locally there exist holomorphic connections in the class.

H-projective structure is a parabolic Cartan geometry with $G = PGL(n+1, \mathbb{C})$ and the reduction of the structure group of the frame bundle is $GL(m, \mathbb{C}) \subset GL(2m, \mathbb{R})$. Again it can be shown that the notion of an h-projective structure coincides with the classical notion of the h-projective equivalence.

The h-projective normalised Ricci tensor satisfies

$$\begin{aligned} r_h^D(Y, Z) &= \\ &= \frac{1}{n+1} [\text{Ric}^D(Y, Z) + \frac{1}{2(n-1)} (\text{Ric}^D(Y, Z) + \text{Ric}^D(Z, Y) - \text{Ric}^D(JY, JZ) - \text{Ric}^D(JZ, JY))], \end{aligned}$$

and the h-projective Weyl curvature tensor

$$W_h = R^D - \llbracket \text{Id} \wedge r_h^D \rrbracket_h$$

is invariant.

Definition 15. *In this thesis we will say that **the h-projective curvature on a 2n-dimensional manifold has type (1, 1)** when the h-projective Weyl curvature tensor has type (1, 1) and, additionally for $n = 2$, the h-projective Cotton-York tensor has type (1, 1).*

Note, that on a complex manifold M , the h-projective bracket on vector fields of the form $Y - iJY$ or of the form $Y + iJY$ gives a projective bracket:

$$\llbracket Y - iJY, \gamma \rrbracket_h(Z - iJZ) = \llbracket Y - iJY, \gamma \rrbracket_p(Z - iJZ),$$

$$\llbracket Y + iJY, \gamma \rrbracket_h(Z + iJZ) = \llbracket Y + iJY, \gamma \rrbracket_p(Z + iJZ).$$

We will use this fact in Chapter 2 to show that a complexified h-projective structure on a complexification of h-projective manifold gives holomorphic projective structures along the leaves of (1, 0) and (0, 1) foliations (we will discuss complexifications in Section 2.3).

Quaternionic structure As we have already discussed in Section 1.2, any quaternionic manifold (M, Q) admits by definition a quaternionic connection, i.e., a connection which preserves Q . Quaternionic connections are not unique. We say that connections on M are quaternionically equivalent if they preserve the same quaternionic structure Q (see [1]).

Let (M, Q) be a $4n$ - quaternionic manifold. Then the class of all quaternionic connections $[D]_q$ is given by an equivalence relation

$$D \sim_q D' \Leftrightarrow \exists \gamma : \forall Y, Z \in TM \quad D'_Y Z = D_Y Z + \llbracket Y, \gamma \rrbracket_q Z, \quad (1.1)$$

where $\gamma \in T^*M$ and

$$\llbracket Y, \gamma \rrbracket_q(Z) = \frac{1}{2}(\gamma(Y)Z + \gamma(Z)Y - \sum_{i=1}^3 (\gamma(I_i Y)I_i Z + \gamma(I_i Z)I_i Y))$$

where I_1, I_2, I_3 is a pointwise frame (see [1]).

As a consequence if we are given one quaternionic connection we can construct all others using (1.1).

Recall that the projective structure can be defined as the class of connections which have the same unparametrised geodesics and the h-projective structure can be defined as the class of complex connections which have the same h-projective geodesics. Following [1] we can describe a class of quaternionic connections in the similar way, namely as those which have the same so called quaternionic planar curves. More precisely, if (Q, D) is a quaternionic structure on a manifold M , then quaternionic planar curves for the connection D are such curves c that for all T tangent to c we have that $D_T T \in \text{span}\{T, QT\}$. It can be shown (see [1]) that all quaternionic connections on (M, Q) have the same quaternionic planar curves.

Quaternionic structures are examples of parabolic Cartan geometries with abelian nilradical with $G = PGL(n + 1, \mathbb{H})$ and the reduction of the structure group of the frame bundle is $GL(n, \mathbb{H}) \times_{\mathbb{Z}/2} Sp(1) \subseteq GL(4n, \mathbb{R})$. As for all parabolic geometries, the curvature of any quaternionic connection decomposes into the Weyl part and the part which comes from the normalised Ricci tensor. The normalised Ricci tensor in the quaternionic case (see [1]) is given by

$$r_Q^D = \frac{1}{4(n+1)}(\text{Ric}^D)^a + \frac{1}{4n}(\text{Ric}^D)^s - \frac{1}{2n(n+2)}\mathcal{F}((\text{Ric}^D)^s),$$

where $(\text{Ric}^D)^a$ is the antisymmetric part of the Ricci tensor, $(\text{Ric}^D)^s$ is the symmetric part of the Ricci tensor and

$$\mathcal{F}((\text{Ric}^D)^s) = \frac{1}{4}((\text{Ric}^D)^s + \Sigma_{i=1}^3(\text{Ric}^D)^s(I_i \cdot, I_i \cdot))$$

Proposition 6. *The quaternionic structure of a $4n$ -manifold M restricted to a totally complex $2n$ -submanifold (S, I) (see Definition 5) gives an h-projective structure on S .*

Proof. Applying Proposition 5, it is enough to show that the quaternionic bracket gives the h-projective bracket along the submanifold. If (S, I) is a maximal totally complex submanifold of M and $X \in TS$ we have $\pi(JX) = \pi(KX) = 0$ and $IX \in TS$, where π is the projection from TM to TS . Hence $\llbracket \cdot, \cdot \rrbracket_q$ for $Y, Z \in TS$ gives $\llbracket \cdot, \cdot \rrbracket_h$:

$$\begin{aligned} \llbracket Y, \gamma \rrbracket_q Z &= \\ \frac{1}{2}(\gamma(Y)Z + \gamma(Z)Y - (\gamma(IY)IZ + \gamma(IZ)IY + \gamma(JY)JZ + \gamma(JZ)JY + \gamma(KY)KZ + \gamma(KZ)KY)) &= \\ = \frac{1}{2}(\gamma(Y)Z + \gamma(Z)Y - (\gamma(IY)IZ + \gamma(IZ)IY)) &= \llbracket Y, \gamma \rrbracket_h Z, \end{aligned}$$

for γ denoting both a 1-form on M and a pull-back of γ to S . □

1.4.3 Conformal structure

Recall (see Section 1.3.1) that the formula for the bracket in the conformal case is

$$\llbracket Y, \gamma \rrbracket_c Z = \gamma(Y)Z + \gamma \Delta Y(Z),$$

where $\gamma \Delta X(Z) = \gamma(Z)Y - \langle Y, Z \rangle \gamma^\sharp$.

For $n > 2$ the normalised conformal Ricci tensor (see [34]) is given by:

$$r_c^D = \frac{1}{n-2} \text{sym}_0 \text{Ric}^D + \frac{1}{2n(n-1)} \text{scal}^D \text{id} - \frac{1}{2} F^D,$$

where F^D is the curvature on the connection induced on the density line bundle. Again we can define the conformal Weyl tensor by

$$W_c = R^D - \llbracket \text{Id} \wedge r_c^D \rrbracket_c.$$

For $n > 3$ the vanishing of the conformal Weyl tensor is equivalent to the condition that the conformal structure is flat. For $n = 3$, W_c is always zero. That is why for low dimensional conformal structures one considers another tensor, the Cotton–York tensor $C^D = d^D r^D$. For $n > 3$ if the Weyl tensor vanishes then the Cotton–York tensor vanishes.

1.4.4 Linear representations of Cartan connections

Given a Cartan connection modelled on G/P and a faithful representation V of G , we can study the Cartan connection using the induced linear connection on

the bundle $\tilde{G} \times_G V$ associated to V . Here we will discuss some examples of this approach that we will use in this thesis.

Following [15], one can define a normal conformal Cartan connection as follows:

Definition 16. *A conformal Cartan geometry on an n -manifold S is a quadruple $(V, \langle, \rangle, \Lambda, \mathcal{D})$ where:*

- V is a rank $n + 2$ vector bundle with inner product \langle, \rangle over S ,
- $\Lambda \subset V$ is a null line subbundle over S ,
- \mathcal{D} is a linear metric connection satisfying the Cartan condition, that is $\delta := \mathcal{D} \lfloor_{\Lambda} \text{ mod } \Lambda$ is an isomorphism from $TS \otimes \Lambda$ to $\Lambda^{\perp} / \Lambda$.

In the real Riemannian case we require that $\langle \cdot, \cdot \rangle$ has signature $(n + 1, 1)$. A complex conformal Cartan geometry on a complex n -manifold is a quadruple $(V, \langle, \rangle, \Lambda, \mathcal{D}^S)$ as above, where all objects are holomorphic.

Remark 4. The line bundle Λ in the Definition 16 is the -1 -density line bundle L^{-1} (see Section 1.3.1).

Similarly, following [15], we can define a projective Cartan connection:

Definition 17. *A projective Cartan geometry on an n -manifold S is a triple $(W, \tilde{\Lambda}, \mathcal{D})$ where:*

- W is a rank $n + 1$ vector bundle with a volume form ,
- $\tilde{\Lambda} \subset W$ is a line subbundle over S
- \mathcal{D}^M is a unimodular connection satisfying the Cartan condition, that is $\mathcal{D}^S \lfloor_{\tilde{\Lambda}} \text{ mod } \tilde{\Lambda}$ is an isomorphism from $TS \otimes \tilde{\Lambda}$ to $W / \tilde{\Lambda}$.

Definition 18. *Let S be a smooth (complex) n -manifold. Then we define a line bundle $\mathcal{O}(1)$ over S as a line bundle that satisfies $\mathcal{O}(n + 1) \cong \Lambda^n TS$, where $\mathcal{O}(n + 1) = \mathcal{O}(1)^{\otimes n+1}$. We denote $\mathcal{O}(-1) := (\mathcal{O}(1))^*$.*

Remark 5. The line bundle $\tilde{\Lambda}$ from Definition 17 is $\mathcal{O}(-1)$.

Another interpretation of a normal projective Cartan connection can be obtained from the above using a decomposition $W^* = (T^*S \otimes \tilde{\Lambda})^* \oplus \tilde{\Lambda}^*$ given by a connection in the projective class. It can be described as follows:

Suppose now that $[D]$ is a (complex) projective structure on a (complex) manifold S . Then for any $D \in [D]$ we can define the following connection on $\mathcal{O}(1) \oplus (T^*S \otimes \mathcal{O}(1))$:

$$\mathcal{D}_Y^D \begin{pmatrix} l \\ \alpha \end{pmatrix} = \begin{pmatrix} D_Y l - \alpha(Y) \\ D_Y \alpha + (\mathfrak{r}_p^D)_Y l \end{pmatrix}.$$

Proposition 7. \mathcal{D}^D is flat iff the projective structure $[D]$ is flat.

Proof. By direct calculation of the curvature we have that

$$\mathbf{R}^{\mathcal{D}^D} \begin{pmatrix} l \\ \alpha \end{pmatrix} = \begin{pmatrix} W_p l \\ W_p \alpha + C_p \alpha \end{pmatrix},$$

where $C_p = d^D \mathfrak{r}_p^D$ is the projective Cotton–York tensor. As for $n > 2$ if $W_p = 0$, then $C_p = 0$, this finishes the proof. \square

Observe that a choice of D gives a splitting of the 1-jet sequence

$$0 \rightarrow T^*S \otimes \mathcal{O}(1) \rightarrow J^1\mathcal{O}(1) \rightarrow \mathcal{O}(1) \rightarrow 0$$

and this gives an isomorphism $J^1\mathcal{O}(1) \cong (\mathcal{O}(1) \oplus T^*S \otimes \mathcal{O}(1))$.

Definition 19. Denote by $\begin{pmatrix} l \\ \alpha \end{pmatrix}_D$ an element of $J^1\mathcal{O}(1)$ such that $l \in \mathcal{O}(1)$ and $\alpha \in T^*S \otimes \mathcal{O}(1)$ are given by projections from $J^1\mathcal{O}(1)$ defined by the splitting by D .

Proposition 8. The term $\mathcal{D}^D \begin{pmatrix} l \\ \alpha \end{pmatrix}_D$ does not depend on the choice of $D \in [D]$.

Proof. We use techniques introduced in Section 1.4.1. Observe that if $D' = D + \gamma$ then $D'l = Dl + \gamma l$ and thus

$$\partial_\gamma \begin{pmatrix} l \\ \alpha \end{pmatrix}_D = - \begin{pmatrix} 0 \\ \gamma l \end{pmatrix}.$$

Then by the Leibniz rule

$$\partial_\gamma \begin{pmatrix} D_Y l - \alpha(Y) \\ D_Y \alpha + (r_p^D)_Y l \end{pmatrix}_D = - \begin{pmatrix} 0 \\ \gamma(D_Y l - \alpha(Y)) \end{pmatrix}_D + \begin{pmatrix} \gamma(Y)l \\ \llbracket Y, \gamma \rrbracket_p \cdot \alpha - \llbracket Y, D\gamma \rrbracket_p \cdot l \end{pmatrix}_D.$$

We have that $\llbracket D\gamma, Y \rrbracket_p \cdot l = (D_Y \gamma)l$ and, as α is $\mathcal{O}(1)$ -valued 1-form,

$$\llbracket Y, \gamma \rrbracket_p \cdot \alpha = -\alpha(Y)\gamma.$$

Hence

$$\partial_\gamma \begin{pmatrix} D_Y l - \alpha(Y) \\ D_Y \alpha + (r_p^D)_Y l \end{pmatrix}_D = \begin{pmatrix} \gamma(Y)l \\ -\gamma D_Y l - (D_Y \gamma)l \end{pmatrix}_D = \mathcal{D}_Y^D \partial_\gamma \begin{pmatrix} l \\ \alpha \end{pmatrix}_D = \mathcal{D}_Y^D \begin{pmatrix} 0 \\ -\gamma l \end{pmatrix}_D$$

Thus we obtain that

$$\mathcal{D}^D \partial_\gamma \begin{pmatrix} l \\ \alpha \end{pmatrix}_D - \partial_\gamma \begin{pmatrix} D_Y l - \alpha(Y) \\ D_Y \alpha + (r_p^D)_Y l \end{pmatrix}_D = 0,$$

which by the Leibniz rule completes the proof. \square

Proposition 9. *If the projective structure on the projective n -manifold $(M, [D])$, $n > 1$ is flat (i.e., the projective curvature tensor vanishes) we can construct a family of sections of the line bundle $\mathcal{O}(1)$ over M which give rise to parallel sections of \mathcal{D}^D , and this condition does not depend on the choice of $D \in [D]$.*

Proof. Note that if (l, α) is parallel for \mathcal{D}^D then $\alpha = Dl$. Moreover, by Proposition 8, if (l, Dl) is parallel for \mathcal{D}^D then for any $D' \in [D]$ we have that $(l, D'l)$ is parallel for $\mathcal{D}^{D'}$. Thus there is a one to one correspondence between sections $l \in \mathcal{O}(1)$ such that (l, Dl) is parallel for \mathcal{D}^D and the parallel sections of \mathcal{D}^D . Moreover, if $l \in \mathcal{O}(1)$ gives a parallel section for \mathcal{D}^D , then it also gives a parallel section for $\mathcal{D}^{D'}$, for any $D' \in [D]$. \square

Definition 20. *Sections of the line bundle $\mathcal{O}(1)$ over M which give rise to parallel sections of \mathcal{D}^D will be called **affine sections** of $\mathcal{O}(1)$. By Proposition 9 being an affine section of $\mathcal{O}(1)$ does not depend on the choice of a connection in the projective class.*

There is a similar interpretation of the conformal Cartan connections [16]. A choice of D from a conformal class gives the decomposition $V = \Lambda^* \oplus (T^*M \otimes$

$\Lambda) \oplus \Lambda$ and we can define a connection on $\Lambda^* \oplus (T^*M \otimes \Lambda) \oplus \Lambda$, where Λ is a density line bundle, by:

$$\mathcal{D}_Y^D \begin{pmatrix} \sigma \\ \theta \\ l \end{pmatrix}_D = \begin{pmatrix} D_Y \sigma + r_Y^D(\theta) \\ D_Y \theta + r_X^D l - \sigma X \\ D_X l - \theta(X) \end{pmatrix}_D.$$

Dimension 2 In this paragraph we will discuss some properties of complex conformal Cartan geometry in dimension $n = 2$ from the point of view of Definition 16. More precisely, we will prove the following proposition.

Proposition 10. *The complex Cartan geometry (see Definition 16) defines on S two foliations by null curves C^+ and C^- . The null line bundle Λ decomposes into a tensor product of two line bundles*

$$\Lambda = \Lambda^+ \otimes \Lambda^-,$$

such that Λ^+ is trivial along curves from the family C^+ and Λ^- is trivial along curves from the family C^- .

Proof. Using the fact that $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ is a double cover of $SO(4, \mathbb{C})$ we can find bundles W and \tilde{W} such that $\wedge^2 W$ and $\wedge^2 \tilde{W}$ are trivial and

$$V \cong W \otimes \tilde{W}.$$

We can choose also non-vanishing sections ϵ of $\wedge^2 W$ and $\hat{\epsilon}$ of $\wedge^2 \tilde{W}$ such that $\epsilon \otimes \hat{\epsilon} = \langle \cdot, \cdot \rangle$. Then we have that for $w_1, w_2 \in W$ and $\hat{w}_1, \hat{w}_2 \in \tilde{W}$

$$\langle w_1 \otimes \hat{w}_1, w_2 \otimes \hat{w}_2 \rangle = \epsilon(w_1, w_2) \hat{\epsilon}(\hat{w}_1, \hat{w}_2).$$

Thus

$$\langle w_1 \otimes \hat{w}_1, w_1 \otimes \hat{w}_2 \rangle = 0 \quad \text{and} \quad \langle w_1 \otimes \hat{w}_1, w_2 \otimes \hat{w}_1 \rangle = 0,$$

and hence bundles of the form $w_1 \otimes \tilde{W}$ and $W \otimes \hat{w}$ are null subbundles. Let w_1, w_2 be a basis for W and \hat{w}_1, \hat{w}_2 be a basis for \tilde{W} . Then $w_1 \otimes \hat{w}_1, w_1 \otimes \hat{w}_2, w_2 \otimes \hat{w}_1, w_2 \otimes \hat{w}_2$ is a basis of $W \otimes \tilde{W}$ and being a null vector implies that the determinant vanishes. This implies that null vectors are of the form $w \otimes \hat{w}$, where $w \in W$ and $\hat{w} \in \tilde{W}$.

As a result, the null subbundle $\Lambda \subseteq V$ must be of the form

$$\Lambda = \text{span}\{w \otimes \hat{w}\},$$

for some $w \in W$ and $\hat{w} \in \hat{W}$.

Then

$$\Lambda^\perp = U + \hat{U} = w \otimes \hat{W} + W \otimes \hat{w}.$$

Using the Cartan condition

$$TS \cong \Lambda^\perp / \Lambda = w \otimes (\hat{W} / \langle \hat{w} \rangle) \oplus (W / \langle w \rangle) \otimes \hat{w}.$$

As a consequence, TS decomposes into a direct sum of rank 2 subbundles which integrate to two families of integral curves. Call them the C^+ family and the C^- family. We have that $\Lambda^+ := \langle w \rangle \subset W$ is constant along C^+ curves and $\Lambda^- := \langle \hat{w} \rangle \subset \hat{W}$ is constant along C^- curves, which finishes the proof. \square

1.4.5 Projective space

In this section we will discuss properties of the complex projective n -space which is a motivation and a flat model for h-projective geometry (see [51], [35]). In this thesis for any vector space A we denote by $A^\times := A \setminus \{0\}$.

Definition 21. *The complex projective n -space $\mathbb{P}(A)$ is the space of all 1-dimensional vector subspaces of an $n + 1$ -dimensional vector space A . As all $n + 1$ -dimensional vector spaces are isomorphic to \mathbb{C}^{n+1} for any $n + 1$ -dimensional vector space A we have $\mathbb{C}\mathbb{P}^n := \mathbb{P}(\mathbb{C}^{n+1}) \cong \mathbb{P}(A)$.*

Remark 6. We will use the notation $\mathbb{P}(A)$ when we consider structures which depend on an isomorphism between A and \mathbb{C}^{n+1} e.g. when considering projective bundles. We use the notation $\mathbb{C}\mathbb{P}^n$ for projective n -space when the vector space (which we projectivise) is unimportant.

Equivalently, complex projective space may be defined as a quotient of A^\times by the \mathbb{C}^\times action given by complex multiplication. It is a standard fact from algebraic geometry that complex projective spaces are compact holomorphic manifolds (see for example [35])

Definition 22. *The tautological bundle $\mathcal{O}(-1)$ over the projective n -space $\mathbb{P}(A)$ is a holomorphic line bundle defined by $\mathcal{O}(-1)_l := l \subset A$, where l is a 1-dimensional subspace of A . The total space of $\mathcal{O}(-1)$ is a submanifold of $\mathbb{P}(A) \times A$.*

The above definition of the tautological bundle depends on the choice of a vector space A . That is why when we want to avoid ambiguity we will denote the tautological line bundle corresponding to A by $\mathcal{O}(-1)^A$.

Remark 7. Observe that for any 1-dimensional vector space L we have a canonical isomorphism between $\mathbb{P}(A)$ and $\mathbb{P}(A \otimes L)$. But as $\mathcal{O}(-1)^{A \otimes L} = \mathcal{O}(-1)^A \otimes L$, isomorphisms between $\mathcal{O}(-1)^A$ are not canonical: they depend on the choice of a trivialisation of L .

Definition 23. *Denote $\mathcal{O}(1) := \mathcal{O}(-1)^*$ and $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$.*

Remark 8. It can be shown (see for example [35]) that $\mathcal{O}(n+1) \cong \wedge^{n+1} T\mathbb{P}(A)$, hence the above definition of the line bundle $\mathcal{O}(1)$ agrees with the definition from Section 1.4.4.

Consider the space $H^0(\mathbb{P}(A), \mathcal{O}(1))$ of global holomorphic sections of $\mathcal{O}(1)$.

Proposition 11. *The space $H^0(\mathbb{P}(A), \mathcal{O}(1))$ is canonically isomorphic to the vector space A^* .*

Proof. Pick $f \in \text{Hom}(A, \mathbb{C}) = A^*$. It induces a holomorphic section s of $\mathcal{O}(1)$ by restriction: for $u \in \mathbb{P}(A)$ define $s(u) = f|_u \in \text{Hom}(u, \mathbb{C}) = u^*$. It can be shown (see [35]) that this defines an isomorphism between A^* and $H^0(\mathbb{P}(A), \mathcal{O}(1))$. \square

Corollary 1. *The above proof implies that for $\dim A = 2$ any non-zero holomorphic section of $\mathcal{O}(1)$ vanishes at exactly one point.*

Observation 1. We can use the space $H^0(\mathbb{P}(A), \mathcal{O}(1))$ to define affine sections of $\mathcal{O}(1)$. Suppose that $U \subset \mathbb{P}(A)$. Then we can define the space $\text{Aff}(\mathcal{O}(1))$ over U as the space of sections given by restrictions of elements of $H^0(\mathbb{P}(A), \mathcal{O}(1))$. Then the space of affine sections of $\mathcal{O}(1)$ over U is isomorphic to A^* . It can be shown that the space $H^0(\mathbb{P}(A), \mathcal{O}(1))$ coincides with the space of affine sections $\mathcal{O}(1)$ for the flat projective structure on $\mathbb{P}(A)$.

The idea of projective structures is to generalise this situation by turning it around. We use a differential operator to define affine sections of $\mathcal{O}(1)$ and then use them to define the space A^* as the space of affine sections.

Now we will introduce some basic properties of blow-ups and blow-downs.

Definition 24. Let B be a non-singular holomorphic submanifold of a holomorphic manifold A . A map $p : \hat{A} \rightarrow A$ is a **blow-up** of a manifold A along B if

- (i) $p^{-1}(B) = E \cong \mathbb{P}(NB)$, where $NB = TA|_B/TB$,
- (ii) $p|_{\hat{A} \setminus E} : \hat{A} \setminus E \rightarrow A \setminus B$ is a biholomorphism.

$$\begin{array}{ccc} E & \subseteq & \hat{A} \\ \downarrow & & \downarrow p \\ B & \subseteq & A \end{array}$$

In this situation we say that A (and p) is a blow-down of \hat{A} along E .

From the definition of the projective space and the tautological line bundle we deduce the following proposition.

Proposition 12. Let A be an $n + 1$ -dimensional vector space. The blow-down at the zero section of the bundle $\mathcal{O}(-1)$ over $\mathbb{P}(A)$ can be naturally identified with the space A . The (restriction of the) blow-down at the zero section of the bundle $\mathcal{O}(-1)$ over a connected open $U \subseteq \mathbb{P}(A)$ is an $n + 1$ -dimensional cone in A with the vertex at 0. \square

The above proposition can be summarised in the following diagram:

$$\begin{array}{ccccccc} \mathbb{P}(A) & \leftarrow & \mathcal{O}(-1) & \xrightarrow{\text{blow-down}} & A & \xrightarrow{\pi} & \mathbb{P}(A) \\ \uparrow i & & \cup & & \cup & & \cup \\ U & \leftarrow & i^*\mathcal{O}(-1) & \xrightarrow{\text{blow-down}} & \pi^{-1}(U) & \xrightarrow{\pi} & U, \end{array}$$

where $\pi^{-1}(U)$ is an $n + 1$ -dimensional cone in A with singularity at 0.

1.4.6 Möbius structure

In dimension $n \leq 2$ a conformal structure is not enough to recover a normal conformal Cartan connection. This is the reason why in many situations that are, even non-explicitly, connected with the conformal geometry, the low dimensional cases are exceptional. That is why it is convenient to introduce

a structure which extends the notion of the conformal structure such that a normal conformal Cartan connection can be recovered in all dimensions (see [16]).

As we do not deal in this thesis with the 1-dimensional case we will discuss here only those aspects of the theory which are relevant for dimension $n = 2$.

Firstly for l a section of the density line bundle L and a Weyl connection D , we define the operator

$$\mathcal{H}^D l = \text{sym}_0 D^2 l \in C^\infty(M, S_0^2 T^* M \otimes L).$$

A Möbius structure (see [16]) on a conformal manifold (M, c) is a linear differential operator $\mathcal{H} : C^\infty(M, L) \rightarrow C^\infty(M, S_0^2 T^* M \otimes L)$ such that $\mathcal{H}l - \mathcal{H}^D l$ is zero order in l for some (hence any) Weyl connection D .

Now observe that for dimension $n \geq 3$ we can define:

$$\mathcal{H}^c l = \mathcal{H}^D l + r_0^D l,$$

which is independent of the choice of D . As a consequence we have that on the conformal manifold (S, c) of dimension $n \geq 3$ there is a canonical Möbius structure \mathcal{H}^c , i.e., the conformal structure is enough data to define the Möbius structure. In dimension 2 the situation is different and the conformal structure is not enough to define the Möbius structure. We will need these extra data encoded in the Möbius structure to be able to extend the generalised Feix–Kaledin construction of quaternionic manifolds (see Chapter 2) for dimension 2.

Möbius structures form an affine space modelled on $C^\infty(M, S_0^2 T^* M)$ (see [16]) and similarly to other parabolic geometries, it is possible to define the Möbius normalised conformal Ricci tensor and thus the Möbius Weyl curvature for \mathcal{H} . These notions will extend the notions from conformal geometry.

Let (c, \mathcal{H}) be a Möbius structure on a 2-manifold M . Then the Möbius Ricci curvature of a Weyl derivative D with respect to \mathcal{H} (see [16]) is defined by

$$r^{D, \mathcal{H}} := \mathcal{H} - \mathcal{H}^D - \frac{1}{2} F^D,$$

and the Weyl curvature $W^{\mathcal{H}}$ is given by

$$W^{\mathcal{H}} := R^D - \llbracket \text{Id} \wedge r^{D;\mathcal{H}} \rrbracket_c.$$

Note that $W^{\mathcal{H}} = 0$ in dimension 2. The Cotton–York curvature of \mathcal{H} with respect to D is given by

$$C^{\mathcal{H},D} := d^D r^{D;\mathcal{H}}.$$

The normalised Möbius Ricci curvature, similarly to other parabolic geometries, has the property that

$$\partial_\gamma r^{D;\mathcal{H}} = -D\gamma$$

and thus the Weyl curvature $W^{\mathcal{H}}$ is an invariant of the Möbius structure. However, in dimensions bigger than 2 it does not have to be equal to W^c . Therefore it is natural to distinguish a class of Möbius structures with the property $W^c = W^{\mathcal{H}}$. We call such Möbius structures conformal Möbius structures (see [16]).

To summarise, a conformal Möbius structure on a 2-manifold M is a conformal metric together with a second order linear differential operator \mathcal{H}^c from the density line bundle L to $\mathcal{S}_0^2 T^*M \otimes L$ such that $\mathcal{H}^c - \mathcal{H}^D$ is zero order. A conformal Möbius structure on a n -manifold M with $n \geq 3$ is a conformal metric and \mathcal{H}^c is the canonical Möbius operator of c defined earlier. A Möbius manifold is a manifold equipped with a conformal Möbius structure (c, \mathcal{H}^c) .

The important fact about a conformal Möbius structure is that it is essentially the same structure as a normal conformal Cartan connection. More precisely the conformal Cartan connection on the manifold defines a canonical conformal Möbius structure and vice-versa. However, the point of view of Möbius structures can be more convenient in some cases. Further information and proofs about Möbius structures can be found in [16].

1.5 Twistor theory

In this section we will discuss twistor theory for self-dual conformal 4-manifolds, Einstein–Weyl 3-manifolds and quaternionic manifolds.

As an important ingredient of twistor theory is Kodaira deformation theory, we will start by introducing the Kodaira theorem.

1.5.1 Kodaira deformation theory

The Kodaira theorem [46] allows us to study the properties of the parameter spaces of families of compact submanifolds of a complex manifold. More precisely let W be a complex manifold and V a compact complex smooth submanifold of W . If $H^1(V, N) = 0$, then locally near V the parameter space M of all smooth submanifolds isomorphic with V is a manifold and the tangent space at $\tilde{V} \in M$ is isomorphic to $H^0(\tilde{V}, N)$. Now we will state precise definitions and theorem given by K. Kodaira in [46]. We assume that all manifolds considered are paracompact and connected.

Definition 25. *An analytic family of compact submanifolds of dimension d of a complex manifold W of dimension $r + d$ is a pair (\mathcal{V}, M) of a complex manifold M (the parameter space) and a complex analytic submanifold \mathcal{V} of $W \times M$ of codimension $r + d$ such that:*

- *For each point $t \in M$, the intersection $\mathcal{V} \cap (W \times t) := V_t$ is a connected, compact d -dimensional submanifold of $W \times t$,*
- *The manifold \mathcal{V} is defined locally as the zero set of r holomorphic functions on $W \times M$ such that their Jacobian is of the maximal rank (equal r).*

The analytic (\mathcal{V}, M) family of compact submanifolds of W is maximal at a point $t_0 \in M$ if any other analytic family $(\tilde{\mathcal{V}}, \tilde{M})$ of compact submanifolds of W with $\tilde{V}_{s_0} = V_{t_0}$ for some $s_0 \in \tilde{M}$ is, locally near s_0 , a sub-family of (\mathcal{V}, M) near t_0 . Let Ψ_t denote the sheaf over V_t of germs of holomorphic sections of the normal bundle of V_t in W .

Theorem 1 ([46]). *If $H^1(V, N) = 0$ then there exists an analytic family of (\mathcal{V}, M) of compact submanifolds V_t of W maximal for each $t \in M$ such that $V_0 = V$ and the tangent space $T_t M$ is isomorphic to $H^0(V_t, \Psi_t)$.*

Note that in particular the Kodaira theorem implies that if we have one submanifold V of W with normal bundle such that $H^1(V, N) = 0$, then we have

a whole family of submanifolds of W isomorphic to V , and the parameter space of those submanifolds form a complex manifold of dimension equal to the dimension of $H^0(V, N)$.

Note that the maximality of the Kodaira moduli space is local to a point in M . Therefore, twistor constructions are local. We abuse notation and talk about twistor spaces as moduli spaces of twistor lines.

K. Kodaira in [45] showed also a similar result for families of surfaces with ordinary singularities.

1.5.2 Twistor theory for self-dual conformal 4-manifolds

For a complex conformal manifold M of dimension 4 the conformal Hodge $*$ operator defines an automorphism on the space $\wedge^2 TM$ of 2-forms. This automorphism is an involution i.e., $*^2|_{\wedge^2 T^*M} = \text{Id}$ and has two eigenvalues: 1 and -1 . The subbundles corresponding to these eigenvalues have rank 3 and are denoted $\wedge^+ T^*M$ and $\wedge^- T^*M$ respectively. Thus we get the splitting

$$\wedge^2 T^*M = \wedge^+ T^*M \oplus \wedge^- T^*M$$

with $*|_{\wedge^+ T^*M} = \text{Id}$ and $*|_{\wedge^- T^*M} = -\text{Id}$. This decomposition induces the decomposition of conformal Weyl tensor into self-dual and anti-self-dual part. This decomposition coincides with the decomposition of the quaternionic Weyl tensor in dimension 4 (see Section 1.4.2). The conformal 4-manifold is called self-dual if its anti-self-dual conformal Weyl tensor vanishes. Such manifolds are sometimes called half-conformally flat.

For complex self-dual conformal 4-manifolds we have the Penrose correspondence [57] which we describe here in more detail.

In Penrose original approach **the twistor space** Z is the space of α -surfaces. Z is a manifold of real dimension 6 with an almost complex structure which is integrable if M is self-dual. On Z we have families of curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ given by points in M . Such curves are called **twistor lines**.

Conversely, let Z be a complex 3-manifold containing a non-singular rational curve c ($c \cong \mathbb{C}\mathbb{P}^1$) with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Suppose that Z is equipped the real structure which preserves some of the twistor lines. Moreover, suppose that the real structure induces on some of the twistor lines the

antipodal map. Then, as $H^1(\mathcal{O}(1)) = 0$, we can apply Theorem 1 and the moduli space of non-singular curves c obtained as the analytic family is a complex 4-manifold M^c with the tangent space $T_c M^c = H^0(c, N|_c)$, where $N|_c$ denotes the normal bundle to the curve c . The conformal 4-manifold is obtained by taking the real submanifold M of M^c (the real structure is induced from the real structure on Z). The conformal structure is obtained by requiring that points of M corresponding to intersecting curves are null separated. It can be shown that M is automatically self-dual and that these two constructions commute.

This gives a correspondence between a self-dual conformal 4-manifolds M and its twistor space Z ; the twistor lines correspond to points in M , and α -planes in M correspond to points in Z and we have the following theorem.

Theorem 2 (Penrose, [57]). *Let Z be a complex 3-manifold such that:*

- (i) *There is a family of non-singular holomorphic projective lines $\mathbb{C}P^1$ each with normal bundle isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$,*
- (ii) *Z has a real structure which on lines from the family which are invariant under this real structure induces the antipodal map of $\mathbb{C}P^1$.*

Then the parameter space of projective lines invariant under the real structure is a self-dual conformal 4-manifold for which Z is the twistor space.

Another, equivalent, approach for constructing the twistor space Z was given by M. Atiyah, N. Hitchin and I. Singer in [5]., where they construct Z as a 2-sphere bundle over M .

The self-dual conformal manifold M is Ricci-flat if it admits a holomorphic projection from Z to $\mathbb{C}P^1$ parametrising twistor lines.

The classical and motivating example for this construction is $Z = \mathbb{C}P^3$. It turns out that this is a twistor space of $\mathbb{H}P^1 \cong S^4$. The other example is the flag manifold $F_{1,2}(\mathbb{C}^3)$ which is the twistor space of $\mathbb{C}P^2$.

1.5.3 Twistor theory for Einstein–Weyl manifolds

As we have already mentioned a similar construction holds for Einstein–Weyl 3-manifolds. Here we will discuss how the twistor construction looks in this case in more detail (see [38]).

Let B be a complex Einstein–Weyl 3-manifold. Then **the minitwistor space** T is defined as the space of totally geodesic null hypersurfaces in B . Again the Einstein–Weyl equation is a condition under which T admits a complex structure. Thus T is a complex 2-manifold. It admits a family of rational curves $c \cong \mathbb{C}P^1$ given by points in B . The normal bundle to those curves is $O(2)$ and they are called **minitwistor lines**.

Conversely let T be a complex surface equipped with a real structure, containing a projective line with the normal bundle $O(2)$ which is invariant under the real structure and the real structure restricted to the line is antipodal. As $H^1(O(2)) = 0$, from Theorem 1 we get that the moduli space of deformations of the curve is a complex 3-manifold B^c with tangent bundle $T_c B = H^0(c, N|_c)$. The real structure on T induces a real structure on B^c and its fixed points set is Einstein–Weyl. To construct the conformal structure and the Weyl connection we firstly define which vectors of the tangent space are going to be null with respect to the conformal structure on B . These are the vectors which are represented by sections of $N|_c$ that have double zeros. To define a Weyl connection on B let $c' \in B$ (which can be thought of as minitwistor line in T) and let $x, y \in T$ such that $x, y \in c'$ (including the possibility that $x = y$). Then the geodesic in the direction of the tangent vector given by a section of $N|_c$ which has zeros in x and y , is the curve which consists of twistor lines that intersect c' in x and y . It can be shown that B is an Einstein–Weyl manifold. Hence we have the following theorem.

Theorem 3 (Hitchin, [38]). *Let T be a surface such that:*

- (i) *There is a family of non-singular holomorphic projective lines $\mathbb{C}P^1$ each with normal bundle isomorphic to $O(2)$,*
- (ii) *T has a real structure which on lines from the family which are invariant under this real structure induces the antipodal map of $\mathbb{C}P^1$.*

Then the parameter space of projective lines invariant under the real structure is an Einstein–Weyl manifold.

The classical examples here are $\mathbb{C}P^1 \times \mathbb{C}P^1$ which after removing the diagonal is the twistor space of the hyperbolic 3-space and $T\mathbb{C}P^1$ as the twistor space of the flat space.

1.5.4 Jones–Tod correspondence

The Penrose and the Hitchin correspondences have been related by P.E. Jones and K.P. Tod in [42]. Here we will briefly present their results.

Firstly let Z be a twistor space of the self-dual conformal 4-manifold M . Furthermore suppose that we have a holomorphic vector field V on Z . They show that such a vector field induces a conformal vector field \tilde{V} on M . Moreover, any conformal vector field on M gives rise to a holomorphic vector field on Z .

If the vector field V is such that there exists a twistor line transversal to V then the quotient space of Z by V is a complex surface. Choose such a twistor line c . Then V defines a non-vanishing section of the normal bundle of c and thus the trivial line subbundle. The image c' of the twistor line c is a rational curve in Z/V , as the action is transversal, and the normal bundle is the quotient bundle. These calculations show that the normal bundle of c' is $\mathcal{O}(2)$. Thus locally the quotient space of the twistor space of a self-dual manifold by a holomorphic action which is transversal to at least one twistor line is a minitwistor space of an Einstein–Weyl manifold. P.E. Jones and K.P. Tod [42] actually showed that the minitwistor space obtained in the way described above is the minitwistor space of the manifold obtained by the quotient of M by the conformal action of \tilde{V} . More precisely they showed that M/\tilde{V} is Einstein–Weyl and that the minitwistor space of M/\tilde{V} coincide with Z/V . Moreover they gave some formulae for the metric and connection on M/\tilde{V} .

In the paper they also showed how to invert this construction. They showed that any minitwistor space arises in this way and that this construction induces on the Einstein–Weyl manifold B an abelian monopole. Moreover, they have shown that given an Einstein–Weyl manifold B with an abelian monopole we can construct from these data a conformal self-dual 4-manifold and that the quotient of this self-dual 4-manifold by an induced S^1 action gives back the manifold B .

An example of such a correspondence is $Z = \mathbb{C}\mathbb{P}^3$ which is a twistor space of the Klein quadric $M = \mathbb{H}\mathbb{P}^2$. On Z we have a \mathbb{C}^\times action discussed in more detail in Section 2.8.1 and this action gives an S^1 action on M . The quotient of Z by the \mathbb{C}^\times action is $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, while the quotient of M by the S^1 action is the hyperbolic 3-space. These operations commute as $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ contains the

minitwistor space of B . By choosing on B a particular Gauduchon gauge we can also reconstruct M from B .

1.5.5 Twistor theory for quaternionic manifolds

S. Salamon [58] showed that we also have a twistor-type construction of quaternionic manifolds. This is in fact a generalisation of the Penrose construction of self-dual conformal 4-manifolds.

Let M be a quaternionic $4n$ -manifold. As we discussed in Section 1.2 over each point of M we have a sphere (i.e., $\mathbb{C}\mathbb{P}^1$) of anti-commuting complex structures. The **twistor space** Z of M is defined to be the total space of this sphere bundle. S. Salamon showed that Z is integrable and the rational curves given by fibres over M have normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$.

Moreover the following theorem is true:

Theorem 4 (Salamon, [58]). *Let Z be a complex manifold of dimension $2n + 1$ such that:*

- (i) *There is a family of holomorphic projective lines $\mathbb{C}\mathbb{P}^1$ each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$,*
- (ii) *Z has a real structure which on lines from the family which are invariant under this real structure induces the antipodal map of $\mathbb{C}\mathbb{P}^1$.*

Then the parameter space of projective lines invariant under the real structure is a $4n$ -manifold with natural quaternionic structure for which Z is the twistor space.

If, additionally, we have a holomorphic projection from Z to $\mathbb{C}\mathbb{P}^1$ transversal to real twistor lines such that Z is a trivial $\mathbb{C}\mathbb{P}^1$ bundle over M then M is hypercomplex.

If moreover Z admits a twisted (by $\mathcal{O}(2)$) symplectic form along the fibres then M is hyperkähler.

A $2n + 1$ -manifold Z is called a contact manifold if there exists a $2n$ -dimensional distribution $\mathcal{H} \subseteq TZ$ called a contact structure such that there exists a 1-form $\alpha \in T^*Z$ called the contact form with the property that $\mathcal{H} = \ker \alpha$ and $\alpha \wedge (d\alpha)^n \neq 0$. The complex contact structure is the contact structure where all objects are defined in the sense of complex geometry.

If a manifold Z is a twistor space of a quaternionic manifold M such that Z is a complex contact manifold, then M is a quaternion-Kähler manifold.

As we have already mentioned for dimension $4n = 4$ this construction coincides with the Penrose twistor construction.

For example, the twistor space of $\mathbb{H}\mathbb{P}^n$ is $\mathbb{C}\mathbb{P}^{2n+1}$ and the twistor space of $Gr_2(\mathbb{C}^n)$ is the flag manifold $F_{1,2}(\mathbb{C}^n)$. Note that $\mathbb{H}\mathbb{P}^n$ and $Gr_2(\mathbb{C}^n)$ are examples of symmetric Wolf spaces (see 1.2.3).

Chapter 2

The generalised Feix–Kaledin construction.

B. Feix [32], [33] and D. Kaledin [43], [44] independently showed that there exists a hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle of a real analytic Kähler manifold. Moreover, they generalised this construction: there exists a hyperkähler structure on a neighbourhood of the zero section of the tangent bundle of any complex manifold equipped with a real analytic complex connection with curvature of type $(1, 1)$.

The motivation for the construction in this chapter is mainly the work of Feix [32], where she constructs the twistor space Z of a hypercomplex $4n$ -manifold M from a Kähler $2n$ -manifold S . A sketch of her construction is as follows. Consider the trivial projective line bundle $S^c \times \mathbb{C}P^1$ over a complexification S^c (see Section 2.3) of S . On S^c the $\pm i$ eigendistributions $T^{(1,0)}S^c$ and $T^{(0,1)}S^c$ of J are integrable and hence define two foliations on S^c , called the $(1, 0)$ and $(0, 1)$ foliations. The decomposition $\mathbb{C}P^1 = \{0\} \cup \mathbb{C}^\times \cup \{\infty\}$ defines zero and infinity sections of $S^c \times \mathbb{C}P^1$ and two vector bundles $S^c \times (\{0\} \cup \mathbb{C}^\times)$ and $S^c \times (\mathbb{C}^\times \cup \{\infty\})$, called affine parts, which may be glued over $S^c \times \mathbb{C}^\times$ to give $S^c \times \mathbb{C}P^1$. The idea is to make a blow down of $S^c \times \mathbb{C}P^1$ at the zero section in the direction of the $(0, 1)$ -foliation and at the infinity section in the $(1, 0)$ -foliation. But the resulting space may be singular and to repair this, one has to embed it into a non-singular space which is obtained in the following way. Let $V^{(1,0)}$ and $V^{(0,1)}$ be the bundles over the leaf spaces $S^{(1,0)}$ and $S^{(0,1)}$ of the $(1, 0)$ and the $(0, 1)$ foliations given fibrewise by spaces of affine functions (i.e., functions satisfying

$D(df) = 0$ along the leaves of the corresponding foliations. As $S^c \times \mathbb{C}\mathbb{P}^1$ is a union of two affine parts of $S^c \times \mathbb{C}\mathbb{P}^1$ which are complex trivial line bundles, we can consider evaluations $f \mapsto f(x)l$, where f is a function on S^c and $(x, l) \in S^c \times \mathbb{C}$ is an element of an affine part. This induces maps from affine parts of $S^c \times \mathbb{C}\mathbb{P}^1$ to the vector bundles $(V^{(1,0)})^*$ and $(V^{(1,0)})^*$. Let δ be some annular region in \mathbb{C} , for example $\delta = \{z : |z| \in (\frac{1}{2}, 2)\}$. The twistor space Z is obtained by gluing of open subsets of $(V^{(1,0)})^*$ and $(V^{(1,0)})^*$ on the image by evaluation maps of $S^c \times \delta$ of both affine parts of $S^c \times \mathbb{C}\mathbb{P}^1$. Moreover, we have a smooth map $\phi : S^c \times \mathbb{C}\mathbb{P}^1 \rightarrow Z$ which factors through the blow down.

One can observe that locally along leaves of foliations the evaluation map arises from the following construction (see Section 2.3). Let U be an open subset of a leaf that is biholomorphic to \mathbb{C}^n . Then there exists a trivialisation of the tautological bundle $\mathcal{O}(-1)$ over $U \subseteq \mathbb{C}\mathbb{P}^n$ (here we use definition of $\mathcal{O}(-1)$ from Section 1.4.5) which identifies $\mathcal{O}(-1)|_U$ with the affine part $U \times \mathbb{C}$ of the projective bundle $S^c \times \mathbb{C}\mathbb{P}^1$ restricted to U .

Recall that the tautological bundle $\mathcal{O}(-1)$ of $\mathbb{C}\mathbb{P}^n$ is a blow up of $(\text{Aff}(\mathcal{O}(1)))^*$ at 0, where $\text{Aff}(\mathcal{O}(1))$ is here the space of global holomorphic sections of $\mathcal{O}(1)$ (see Section 1.4.5). Using the trivialisation of $\mathcal{O}(1)$ over U induced from the trivialisation of $\mathcal{O}(-1)$ we obtain an isomorphism between the space $(\text{Aff}(\mathcal{O}(1)))^*$ and $(\text{Aff}(\mathcal{O}))^*$ and the following diagram commutes:

$$\begin{array}{ccc}
 U \times \mathbb{C} & \xrightarrow{\text{evaluation}} & (\text{Aff}(\mathcal{O}))^* \\
 \downarrow & & \downarrow \\
 \mathcal{O}(-1)|_U & \xrightarrow{\text{blow-down}} & (\text{Aff}(\mathcal{O}(1)))^*
 \end{array}$$

In more general situation there is no canonical trivialisation of $\mathcal{O}(1)$ as above. We therefore must introduce explicitly the bundles along leaves corresponding to $\mathcal{O}(1) \rightarrow \mathbb{C}\mathbb{P}^n$. These are the pull back bundles denoted by $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$. To maximise generality we also introduce a holomorphic line bundle \mathcal{L} with a connection ∇ with type $(1, 1)$ curvature and define $\mathcal{L}^{(1,0)}(1) := \mathcal{L}^{(1,0)} \otimes \mathcal{O}(1, 0)$ and $\mathcal{L}^{(0,1)}(1) := \mathcal{L}^{(0,1)} \otimes \mathcal{O}(0, 1)$, where the decomposition $\mathcal{L}^c = \mathcal{L}^{(1,0)} \oplus \mathcal{L}^{(0,1)}$ is induced by the $\pm i$ eigenspaces of the complexified complex structure from \mathcal{L} (see Section 2.3). We generalise the Feix construction by replacing $S^c \times \mathbb{C}\mathbb{P}^1$ by

a projective bundle over S^c of the form $\mathbb{P}(\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1))$.

As in the Feix case, $\mathbb{P}(\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1))$ has zero and infinity section and the manifold obtained by the blow down may have cone singularities. To repair this we will also embed Z into a gluing of vector bundles $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$ which are constructed using affine sections of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ and of $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ respectively, rather than affine functions (see Section 2.4).

We now observe that the only data needed to construct the spaces of affine sections along the leaves is a flat projective structure along the leaves. To obtain this, it is enough to require that S is an h-projective manifold such that the h-projective curvature is of type $(1, 1)$ (see Section 2.4).

Analogously like in Feix construction, we obtain a holomorphic map

$$\phi : \mathbb{P}(\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1)) \rightarrow Z$$

and while the fibres of $\mathbb{P}(\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1))$ have normal bundles $\mathcal{O} \otimes \mathbb{C}^{2n}$, their images in Z by ϕ have normal bundles $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$.

We will prove in Section 2.8.2 that, when the h-projective class admits a real-analytic complex connection D (i.e., $DJ = 0$) such that its curvature is of type $(1, 1)$ and when $\mathcal{L} \cong \mathcal{O}(-1)$ with a connection induced from D , our construction is a generalisation of the Feix construction. In particular, for $S = \mathbb{C}\mathbb{P}^n$ with the Fubini-Study metric the obtained quaternionic manifold is $T^*\mathbb{C}\mathbb{P}^n$ with the Calabi metric. We will also show that if we take S as above together with \mathcal{L} being the trivial bundle with the standard connection then Z is $\mathbb{C}\mathbb{P}^3$ and thus M is $\mathbb{H}\mathbb{P}^1$ with complexification the Klein quadric $Gr_2(\mathbb{C}^4)$. This is a quaternion-Kähler manifold which is not hyperkähler.

In Section 2.6.2 we will show that obtained quaternionic manifolds can locally be identified with a neighbourhood of the zero section of TS twisted by some unitary line bundle depending on \mathcal{L} . In Section 2.6.4 we will show that they admit an S^1 action given by unit scalar multiplication in the fibres. Moreover, the action comes from a holomorphic \mathbb{C}^\times action on Z induced by the scalar multiplication on $\mathbb{P}(\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1))$.

2.1 Motivating example

Consider the space $Z \cong \mathbb{P}(\mathbb{C}^{2n+2})$ which is the twistor space of quaternionic projective n -space. Choose complementary vector $n + 1$ -subspaces A_+ and A_- of \mathbb{C}^{2n+2} . Then $\mathbb{P}(A_+)$ and $\mathbb{P}(A_-)$ are disjoint projective n -subspaces of $\mathbb{P}(\mathbb{C}^{2n+2})$. Observe that the blow-up along $\mathbb{P}(A_+) \sqcup \mathbb{P}(A_-)$ of Z is the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ over $\mathbb{P}(A_+) \times \mathbb{P}(A_-)$, where $\mathcal{O}(1, -1)$ is the pull-back of $\mathcal{O}(1)$ over $\mathbb{P}(A_+)$ tensored with the pull-back of $\mathcal{O}(-1)$ over $\mathbb{P}(A_-)$ (for more about projective bundles see Section 2.3).

Moreover, observe that $Z^{(1,0)} = Z \setminus A_-$ is a vector bundle over A_+ and $Z^{(0,1)} = Z \setminus A_+$ is a vector bundle over A_- . To see this note that if $z \in Z^{(1,0)}$ is a point then there exists exactly one projective $n + 1$ -plane A_z^- containing z and A_- . As A^+ and A^- were disjoint, A_z^- intersects A^+ in exactly one point. This defines a projection of $Z^{(1,0)}$ to A^+ and induces a vector bundle structure on $Z^{(1,0)}$. Analogously we obtain a vector bundle structure on $Z^{(0,1)}$. The blow-down map from $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ induces a gluing of $Z^{(1,0)}$ with $Z^{(0,1)}$ and the resulting space is Z .

The normal bundle to fibres of the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ is the trivial bundle. The images of the fibres by blow-down are projective lines such that the normal bundles in Z are $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$. This is the key fact that we need to check in the general construction.

This example will be discussed in more detail in Section 2.8.1.

2.2 Overview of the construction

We will start from a $2n$ -manifold S with a complex structure J and a real-analytic h-projective structure $[D]$ with h-projective curvature of type $(1, 1)$ and (\mathcal{L}, ∇) a holomorphic line bundle with a real analytic complex connection with type $(1, 1)$ curvature. Using these data we will construct a complex $2n + 1$ -dimensional manifold Z admitting a family of projective lines called canonical twistor lines. To do this, we will consider a complexification S^c of S (see Section 2.3) and observe that the complexified h-projective structure gives flat projective structures along leaves of $(1, 0)$ and $(0, 1)$ foliation. Using the flat projective structures along the leaves and the complexified connection ∇^c flat

along the leaves, we define bundles $V^{(0,1)}$ and $V^{(1,0)}$ of affine sections over leaf spaces of the foliations (for definition of affine sections see Section 2.4).

We will also construct isomorphisms (given by blow-down maps) between the multiplicative part (i.e., points not on the zero section) of the line bundle $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ over S^c and an open subset of $(V^{(0,1)})^*$ and between the multiplicative part of the line bundle $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ over S^c and an open subset of $(V^{(1,0)})^*$ (for definitions of the line bundles see Section 2.4). The gluing of the bundles $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ and $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ to the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$ will induce gluing of $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ on some open subset. Unfortunately the manifold obtained in this way would not be Hausdorff. We will repair this by gluing open subsets of $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ instead of the whole bundles and shrinking the open set on which we glue. We denote the manifold obtained by the above construction by Z . We will prove the following main theorem

Theorem. *Z is a twistor space of a quaternionic structure on a neighbourhood of the zero section of $TS \otimes \mathcal{Q}$, where \mathcal{Q} is a unitary line bundle on S .*

We will refer to projective lines which arise as the images of the fibres of $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$ by the blow-down map as canonical twistor lines. To prove that Z is indeed a twistor space of a quaternionic manifold and that the canonical twistor lines are twistor lines in Z , the key point is the observation that the blow down changes the normal bundle of the canonical twistor lines to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$. This will be proved in Section 2.5. Then, we investigate how the normal bundle to canonical twistor lines (which are parametrised by points in S^c) changes when we change a point in S^c . We start by observing that $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$ is a subspace of the incidence space of the twistor space Z (see Section 2.5). Then we decompose the normal bundles into a direct product of rank n subbundles and 'untwist' them using some tautological bundles along the fibres on $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$ (see Section 2.3). As a result we obtain two rank n bundles $W^{(1,0)}$ and $W^{(0,1)}$ trivial along canonical twistor lines (represented by fibres of $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$). Hence the bundles $W^{(1,0)}$ and $W^{(0,1)}$ are pull-backs from S^c and encode the information about how the normal bundle to canonical twistor line q changes with a change of $q \in S^c$. Using this we show in Section 2.6.2 that the quaternionic manifold M

obtained from the twistor space Z can be locally (near $S \subset M$) embedded into a neighbourhood of the zero section of $TS \otimes \mathcal{Q}$.

2.3 Tools

Complexifications Here we discuss complexifications which will be an important tool in our construction (see [11],[51, p.66])

Definition 26. A *real structure* θ on a complex manifold S is an anti-holomorphic involution, i.e., an anti-holomorphic map $\theta : S \rightarrow S$ such that $\theta^2 = \text{id}$. The set of fixed points $S^\theta = \{x \in S : \theta x = x\}$ is a real analytic submanifold.

Definition 27. A *complexification* (S^c, θ) of a real m -dimensional manifold S is a complex manifold S^c of complex dimension m together with a real structure θ on S^c with fixed point set $(S^c)^\theta \simeq S$.

Let S be a real-analytic manifold of real dimension m . We can construct a complexification S^c of S using a holomorphic extension of real-analytic coordinates on S . Indeed, as transition functions are given by real-analytic functions, we can locally holomorphically extend them to obtain a holomorphic atlas and the real structure θ is given by complex conjugation. Clearly $S \equiv (S^c)^\theta$. Conversely, if S^c is a complexification then near the submanifold S , the transition functions of S^c must coincide with holomorphic extensions of transition functions from S .

Definition 28. Let (S_1^c, θ_1) and (S_2^c, θ_2) be complexifications of real-analytic manifolds S_1 and S_2 . Then the holomorphic map f between S_1^c and S_2^c will be called *real holomorphic* if $f \circ \theta_1 = \theta_2 \circ f$.

By local uniqueness of the holomorphic extensions of real analytic functions we obtain the following proposition.

Proposition 13. Let S_1^c and S_2^c be complexifications of a real analytic manifold S ; then exists a real holomorphic isomorphism from a neighbourhood of S in S_1^c to a neighbourhood of S in S_2^c . \square

We have a real structure $\overline{\theta}^*$ on the sheaf of holomorphic functions defined by $\overline{\theta}^*(f) \mapsto \overline{f \circ \theta}$ which induces the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_{S^c} & \xrightarrow{\bar{\theta}^*} & \mathcal{O}_{S^c} \\
\downarrow & & \downarrow \\
S^c & \xrightarrow{\theta} & S^c
\end{array}$$

Henceforth $\bar{\theta}^*$ will be denoted by θ .

Now assume that $m = 2n$ and S is a $2n$ -manifold equipped with a real analytic almost complex structure J and let $S_{\mathbb{R}}$ be the underlying real-analytic manifold to S . Then $S = (S_{\mathbb{R}}, J)$. The tangent bundle to any complexification $S_{\mathbb{R}}^c$ of $S_{\mathbb{R}}$ decomposes into two distributions

$$TS^c = T^{(1,0)}S^c \oplus T^{(0,1)}S^c,$$

given by the $\pm i$ eigenspaces of complexification of J . We define $\bar{S} := (S_{\mathbb{R}}, -J)$.

Suppose now that J is integrable. Then $T^{(1,0)}S^c$ and $T^{(0,1)}S^c$ are integrable and induce a direct product structure on S^c . To emphasise that the identity map $S \rightarrow \bar{S}$ is anti-holomorphic we denote it by $x \mapsto \bar{x}$.

Proposition 14. $S \times \bar{S}$ is a complexification of $S_{\mathbb{R}}$ with the real structure θ given by $\theta(x, \bar{x}) = (\tilde{x}, \bar{x})$. \square

Corollary 2. By Propositions 13 and 14, if S is a complex manifold then any complexification of $S_{\mathbb{R}}$ can be identified with a neighbourhood of the diagonal in $S \times \bar{S}$.

By abuse of notation we will denote the manifold $S_{\mathbb{R}}^c$ by S^c .

Integrable distributions $T^{(1,0)}S^c$ and $T^{(0,1)}S^c$ define two transverse foliations which will be called $(1,0)$ foliation and $(0,1)$ foliation respectively. In such a case, $S_{\mathbb{R}}^c$ can be described in the local coordinates as follows. There exist holomorphic coordinates $z = (z_1, \dots, z_n)$ on S such that

$$v = (v_1, \dots, v_n) := \operatorname{Re} z, \quad w = (w_1, \dots, w_n) := \operatorname{Im} z$$

are real-analytic. Then S^c is a manifold with holomorphic coordinates z, \tilde{z} , where $\tilde{z} := v^c - iw^c$, and where v^c, w^c denote complexifications of v and w respectively. The real structure θ is given by $\theta(z, \tilde{z}) = (\tilde{z}, \bar{z})$ and S is given by $S = \{x^c \in S^c : \tilde{z}(x^c) = \overline{z(x^c)}\}$. In this setting, the leaves of the $(1,0)$ foliation

are given by $\tilde{z} = \text{const}$, the leaves of $(0, 1)$ foliation by $z = \text{const}$, and the real structure interchanges the foliations.

Suppose now that D is a real-analytic connection on $S_{\mathbb{R}}$. This means that the connection forms of D are given by real analytic functions, thus we can holomorphically extend them near $S_{\mathbb{R}}$ to obtain a holomorphic affine connection D^c (in the sense that the connection forms are holomorphic) on some neighbourhood S^c of $S_{\mathbb{R}}$ in $S \times \bar{S}$. Now if we assume that $[D]$ is a real analytic h-projective structure on a complex manifold S (in the sense of existence of a real-analytic connection in the class) we can complexify the real analytic connections from the class. As $(D + \llbracket \cdot, \gamma \rrbracket)^c = D^c + \llbracket \cdot, \gamma^c \rrbracket$ we can define the complexified h-projective structure on S^c as the holomorphic h-projective class of connections containing the complexifications of real-analytic connections from $[D]$.

Remark 9. The h-projective Weyl curvature tensor W_h^c of the holomorphic h-projective structure $[D^c]$ arising as a complexification of an h-projective structure $[D]$ is just a complexification of the h-projective curvature Weyl curvature tensor W_h of $[D]$.

For most of the time in the construction we will use another description of the complexification. Note that the leaves of the $(1, 0)$ foliation are locally near the real submanifold biholomorphic to S and the leaves of the $(0, 1)$ are locally biholomorphic to \bar{S} . We define $S^{(1,0)}$ to be the local leaf space of $(0, 1)$ foliation and $S^{(0,1)}$ the local leaf space of $(1, 0)$ foliation (if necessary we shrink spaces $S^{(1,0)}$ and $S^{(0,1)}$ to ensure that their product is contained in the neighbourhood of the diagonal in $S \times \bar{S}$). As the foliations are transverse, locally we can identify the space $S^{(1,0)}$ with some leaf of the $(1, 0)$ foliation and $S^{(0,1)}$ with a leaf of the $(0, 1)$ foliation. Note that this gives a local biholomorphism of spaces $S^{(1,0)}$ and $S^{(0,1)}$ with S and \bar{S} respectively. There are canonical projections from S^c to the leaf spaces:

$$\begin{array}{ccc} & S^c & \\ \pi_{(1,0)} \swarrow & & \searrow \pi_{(0,1)} \\ S^{(1,0)} & & S^{(0,1)} \end{array}$$

The projections $\pi_{(1,0)}$ and $\pi_{(0,1)}$ are locally jointly injective and thus locally near

the real submanifold define an embedding

$$(\pi_{(1,0)}, \pi_{(0,1)}) : S^c \hookrightarrow S^{(1,0)} \times S^{(0,1)},$$

which allows us to view a complexification of S as a subset of $S^{(1,0)} \times S^{(0,1)}$.

Let \mathcal{L} be a real-analytic vector bundle of rank k over S . Then after possible shrinking of S^c , we can complexify transition functions to obtain a holomorphic vector bundle \mathcal{L}^c of complex rank k on S^c . Analogous to the complexification of real analytic manifolds, \mathcal{L}^c is not unique, but any two complexifications of a vector bundle \mathcal{L} are locally isomorphic near \mathcal{L} .

The real structure gives an isomorphism between $\mathcal{L}_{x,\bar{x}}^c$ and $\overline{\mathcal{L}_{\bar{x},x}^c}$:

$$\mathcal{L}^c \cong \overline{\theta^* \mathcal{L}^c}.$$

If \mathcal{L} is a complex vector bundle then the bundle \mathcal{L}^c has a decomposition into $\pm i$ eigenspaces of the complex structure:

$$\mathcal{L}^c = \mathcal{L}^{(1,0)} \oplus \mathcal{L}^{(0,1)}.$$

Remark 10. Similarly as for complexification of a complex manifold S we abuse the notation here. When \mathcal{L} is a rank k complex vector bundle, then we should consider the underlying real rank $2k$ bundle $\mathcal{L}_{\mathbb{R}}$ and denote its complexification by $\mathcal{L}_{\mathbb{R}}^c$. To simplify the notation we denote $\mathcal{L}_{\mathbb{R}}^c$ by \mathcal{L}^c .

If \mathcal{L} is a holomorphic line bundle then the $\bar{\partial}_{\mathcal{L}}$ operator on \mathcal{L} extends to \mathcal{L}^c . It trivialises $\mathcal{L}^{(1,0)}$ along the leaves of $(0,1)$ foliation, and $\mathcal{L}^{(0,1)}$ along the leaves of $(1,0)$ foliation. Moreover, we have that $\mathcal{L}^c|_S \cong \mathcal{L} \otimes \mathbb{C}$, $\mathcal{L}^{(1,0)}|_S \cong \mathcal{L}$ and $\mathcal{L}^{(0,1)}|_S \cong \overline{\mathcal{L}}$. Hence $\mathcal{L}^{(1,0)}$ and $\mathcal{L}^{(0,1)}$ are the pull-backs of bundles \mathcal{L} on $S^{(1,0)}$ by $\pi_{(1,0)}$ and $\overline{\mathcal{L}}$ on $S^{(0,1)}$ by $\pi_{(0,1)}$ respectively. The above observations can be summarised in the following proposition.

Proposition 15. *Let S be a complex manifold and \mathcal{L} a holomorphic vector bundle on S . Then the bundle $\mathcal{L}^{(1,0)} \oplus \mathcal{L}^{(0,1)}$ on $S^{(1,0)} \times S^{(0,1)}$ is a complexification of the bundle \mathcal{L} on S , where $\mathcal{L}^{(1,0)}$ and $\mathcal{L}^{(0,1)}$ are pull-backs of bundles \mathcal{L} on $S^{1,0}$ by $\pi_{(1,0)}$ and $\overline{\mathcal{L}}$ on $S^{(0,1)}$ by $\pi_{(0,1)}$ respectively. Moreover the real structure $\overline{\theta}_{\mathcal{L}}$ induces isomorphisms $\mathcal{L}^{(1,0)} \cong \overline{\theta^* \mathcal{L}^{(0,1)}}$ and $\mathcal{L}^{(0,1)} \cong \overline{\theta^* \mathcal{L}^{(1,0)}}$. \square*

Suppose now that \mathcal{L} admits a real analytic complex connection ∇ compatible with the holomorphic structure i.e., a complex connection such that $\nabla^{0,1} = \bar{\partial}_{\mathcal{L}}$. Then locally we can complexify the connection (by holomorphic extensions of the connection forms) to obtain a complexified connection ∇^c on \mathcal{L}^c .

Projective bundles In this paragraph we discuss some properties of projective bundles - the bundles such that their fibres are isomorphic to projective spaces (see Section 1.4.5).

Definition 29. *Suppose that \mathcal{A} is a rank $k + 1$ holomorphic vector bundle over a complex manifold S . Then \mathcal{A} induces a projective bundle $\mathbb{P}(\mathcal{A})$ over S by requiring that $\mathbb{P}(\mathcal{A})_x = \mathbb{P}(\mathcal{A}_x)$ for any $x \in S$.*

Remark 11. Let \mathcal{L} be a holomorphic line bundle and \mathcal{A} be a rank $k + 1$ holomorphic vector bundle over a complex manifold S . Then as a consequence of Remark 7, $\mathbb{P}(\mathcal{A})$ is canonically isomorphic to $\mathbb{P}(\mathcal{A} \otimes \mathcal{L})$.

Suppose that \mathcal{P} is a projective bundle which arise as a projectivisation of some vector bundle. As we observed, such a vector bundle is not unique. The choice of a vector bundle \mathcal{A} such that $\mathcal{P} = \mathbb{P}(\mathcal{A})$ canonically induces the following line bundle over the total space of \mathcal{P} .

Definition 30. *A **tautological bundle along the fibres** of the projective bundle \mathcal{P} corresponding to a vector bundle \mathcal{A} , where $\mathcal{P} = \mathbb{P}(\mathcal{A})$ is the bundle $\Delta_{\mathcal{A}}$ defined by $(\Delta_{\mathcal{A}})_x = l_x$, where l_x is a 1-dimensional subspace in \mathcal{A}_x .*

The bundle $\Delta_{\mathcal{A}}$ after restriction to any fibre of \mathcal{P} is isomorphic to the tautological bundle $\mathcal{O}(-1)$.

Summary of the notation

- For a complex manifold S we denote by S^c the complexification of S which is a neighbourhood of the diagonal in $S^{(1,0)} \times S^{(0,1)}$, where $S^{(1,0)}$ is the local leaf space of the $(0, 1)$ foliation, $S^{(0,1)}$ is the local leaf space of the $(1, 0)$ foliation. Manifolds $S^{(1,0)}$ and $S^{(0,1)}$ can be identified locally with some leaf of the $(1, 0)$ and $(0, 1)$ foliation respectively. Observe that $S^{(1,0)} \times S^{(0,1)}$ is contained in the neighbourhood of the diagonal in $S \times \bar{S}$, where \bar{S} is real-analytically isomorphic to S but admits the opposite complex structure.

- θ is the real structure on S^c corresponding to a complexification and also denotes induced real structures.
- $S_{\mathbb{R}}$ is the real submanifold of S^c defined as the set of fixed points for θ . It is the underlying real-analytic manifold to S and S^c is in fact a complexification of $S_{\mathbb{R}}$.
- We denote points on the diagonal of $S^c = S \times \bar{S}$ by (x, \bar{x}) to emphasise that holomorphic functions of $x \in S$ correspond to anti-holomorphic functions of $\bar{x} \in \bar{S}$.
- $\pi_{(1,0)}$ and $\pi_{(0,1)}$ denote the corresponding projections from S^c to $S^{(1,0)}$ and $S^{(0,1)}$ respectively.
- \mathcal{L}^c denotes a complexification of a bundle \mathcal{L} over S . If the real rank of \mathcal{L} is k , then \mathcal{L}^c is a complex vector bundle over S^c of the complex rank k . $\mathcal{L}^c = \mathcal{L}^{(1,0)} \oplus \mathcal{L}^{(0,1)}$ and $\mathcal{L}^{(1,0)}$ is trivial along leaves of the $(0, 1)$ foliation and $\mathcal{L}^{(0,1)}$ is trivial along leaves of the $(1, 0)$ foliation,
- If D is a connection on S then D^c denotes a complexification of D and is a connection on S^c . If ∇ is a connection on \mathcal{L} then ∇^c denotes a complexification of ∇ and is a connection on \mathcal{L}^c .
- We take local leaf spaces $S^{(1,0)}$ and $S^{(0,1)}$ such that all above objects are defined on the whole $S^c = S^{(1,0)} \times S^{(0,1)}$,
- $\Delta_{\mathcal{A}}$ is the tautological bundle along the fibres of the projective bundle $\mathbb{P}(\mathcal{A})$ corresponding to the vector bundle \mathcal{A} .
- For any vector space A , we denote $A^\times := A \setminus \{0\}$ and for any vector bundle \mathcal{A} we denote $\mathcal{A}^\times := \mathcal{A} \setminus \underline{0}$, where $\underline{0}$ is the zero section of \mathcal{A} .

2.4 The construction

Preliminaries Let S be a $2n$ -manifold, $n > 1$ with a complex structure J and a real-analytic h-projective structure $[D]$ with h-projective curvature of type $(1, 1)$ (see Definition 15). Let \mathcal{L} be a holomorphic line bundle on S and ∇ be a real-analytic complex connection on \mathcal{L} compatible with the holomorphic structure

such that the curvature is of type $(1, 1)$. Let us fix complexifications S^c , $[D^c]$, \mathcal{L}^c and ∇^c of S , $[D]$, \mathcal{L} and ∇ respectively. We restrict S^c such that leaves of the $(1, 0)$ and $(0, 1)$ foliation are simply connected.

Proposition 16. *The complexified h-projective structure $[D^c]$ on S^c induces holomorphic projective structures on leaves of the $(1, 0)$ and $(0, 1)$ foliations.*

Proof. Recall that in Section 1.4.2 we showed that the h-projective bracket on the vector fields of the form $Y - iJY$ or $Z - iJZ$ is the same as projective bracket. Hence the holomorphic h-projective bracket on S^c gives holomorphic projective brackets along leaves of the $(1, 0)$ and $(0, 1)$ foliations, which finishes the proof. \square

Let us now observe that using the identification of S^c (locally near $S_{\mathbb{R}}$) with $S^{(1,0)} \times S^{(0,1)}$ we obtain the following splitting:

$$\wedge^2 T^* S^c = T^* S^{(1,0)} \oplus (T^* S^{(1,0)} \otimes T^* S^{(0,1)}) \oplus \wedge^2 T^* S^{(0,1)}.$$

Moreover, complexifications of holomorphic 1-forms on S are sections of $T^* S^{(1,0)}$ and complexifications of anti-holomorphic 1-forms on S are sections of $T^* S^{(0,1)}$.

Proposition 17. *The curvature of the connection ∇^c vanishes along leaves of the $(1, 0)$ and the $(0, 1)$ foliation. The projective structures induced from $[D^c]$ on the leaves of $(1, 0)$ and the $(0, 1)$ foliation are flat.*

Proof. As the curvature of the connection ∇ is of type $(1, 1)$, the curvature of ∇^c is a section of $(T^* S^{(1,0)} \otimes T^* S^{(0,1)}) \otimes \text{End}(\mathcal{L}^c)$ and thus vanishes along leaves of the foliations.

As we observed in Section 1.4.2, the h-projective Weyl curvature tensor W_h^c induces Weyl projective curvature tensors on leaves of the foliations. As W_h^c on S is of type $(1, 1)$, we have that W_h^c is a section $(T^* S^{(1,0)} \otimes T^* S^{(0,1)}) \otimes \text{End}(TS^c)$ and hence the induced Weyl projective curvature tensors on leaves vanish. Similarly, for $n = 2$ we show that the h-projective Cotton-York tensor C_h (which we assume is of type $(1, 1)$) induce vanishing projective Cotton-York tensors on leaves. \square

Definition 31. *We define:*

$$\mathcal{L}^{(1,0)}(1) := (\pi_{(1,0)}^* \mathcal{O}(1)) \otimes \mathcal{L}^{(1,0)},$$

$$\mathcal{L}^{(0,1)}(1) := (\pi_{(0,1)}^* \mathcal{O}(1)) \otimes \mathcal{L}^{(0,1)},$$

where the bundle $\mathcal{O}(1)$ is the bundle from Definition 18.

The properties of the pull back and of the bundles $\mathcal{L}^{(1,0)}$ and $\mathcal{L}^{(0,1)}$ imply the following fact:

Remark 12. The bundles $\mathcal{L}^{(1,0)}(1)$ and $\mathcal{L}^{(1,0)}(1)^*$ are trivial along the leaves of the $(0, 1)$ foliation and the bundles $\mathcal{L}^{(0,1)}(1)$ and $\mathcal{L}^{(0,1)}(1)^*$ are trivial along the leaves of the $(1, 0)$ foliation.

Observe that if J is a complex structure on \mathcal{L} , then it extends to \mathcal{L}^c and as $\nabla^c J = 0$, the connection ∇^c preserves J -eigenspaces. As a consequence we have the following proposition.

Proposition 18. *The connection ∇^c on \mathcal{L}^c induces a connection on $\mathcal{L}^{(1,0)}$ which is flat along leaves of the $(1, 0)$ foliation and a connection on $\mathcal{L}^{(0,1)}$ which is flat along leaves of the $(0, 1)$ foliation. For simplification the connections on $\mathcal{L}^{(1,0)}$ and $\mathcal{L}^{(0,1)}$ are also denoted by ∇^c . \square*

Affine sections along the leaves. Let D^c be a connection in the complexified h-projective class.

Define connections on $(\mathcal{O}(1) \oplus (T^*S \otimes \mathcal{O}(1))) \otimes \mathcal{L}^{(1,0)}$ and on $(\mathcal{O}(1) \oplus (T^*S \otimes \mathcal{O}(1))) \otimes \mathcal{L}^{(0,1)}$ by

$$(\mathcal{D}^{D^c} \otimes \nabla^c)_Y \begin{pmatrix} l \otimes u \\ \alpha \otimes u \end{pmatrix} = \begin{pmatrix} (D_Y^c l - \alpha(Y)) \otimes u + l \otimes \nabla_Y^c u \\ (D_Y^c \alpha + (r_p^{D^c})_Y l) \otimes u + \alpha \otimes \nabla_Y^c u \end{pmatrix}.$$

Note that to simplify the notation we denote both connections by $\mathcal{D}^{D^c} \otimes \nabla^c$.

Proposition 19. *For each leaf of the $(1, 0)$ and the $(0, 1)$ foliation, the connection $\mathcal{D}^{D^c} \otimes \nabla^c$ admits the $n + 1$ -dimensional vector space of parallel sections. Moreover, being a parallel section along the leaf does not depend on the choice of D^c in the complexified projective class.*

Proof. The connection $\mathcal{D}^{D^c} \otimes \nabla^c$ is a tensor product connection of \mathcal{D}^{D^c} (which is a linear representation of the Cartan connection associated with the h-projective structure) and ∇^c . As both connections are flat along the leaves of foliations

(see Proposition 7 and Proposition 17), we have that $\mathcal{D}^{D^c} \otimes \nabla^c$ is flat along leaves of these foliations. The rank of $(\mathcal{O}(1) \oplus (T^*S \otimes \mathcal{O}(1))) \otimes \mathcal{L}^c$ is $n + 1$ thus along each leaf we have the $n + 1$ -dimensional vector space of parallel sections.

Now we need to show that being a parallel section of this connection along some leaf does not depend on the choice of D . Recall that in Proposition 8 we have shown that the term

$$\mathcal{D}^D \begin{pmatrix} l \\ \alpha \end{pmatrix}_D$$

does not depend on the choice of $D \in [D]$. Using the same techniques, observe that

$$[\partial_\gamma(\mathcal{D}^c \otimes \nabla^c)] \left[\begin{pmatrix} l \\ \alpha \end{pmatrix}_{D^c} \otimes u \right] = \partial_\gamma(\mathcal{D}^c \otimes \nabla^c) \left[\begin{pmatrix} l \\ \alpha \end{pmatrix}_{D^c} \otimes u \right] - [\mathcal{D}^c \otimes \nabla^c] [\partial_\gamma \left(\begin{pmatrix} l \\ \alpha \end{pmatrix}_{D^c} \otimes u \right)].$$

Using Leibniz rule we get that the right hand side of the above equation consists of two terms that sum up to $\partial_\gamma \mathcal{D}^{D^c} \begin{pmatrix} l \\ \alpha \end{pmatrix}_{D^c}$ and two extra terms $\partial_\gamma \left[\begin{pmatrix} l \\ \alpha \end{pmatrix}_{D^c} \otimes \nabla^c u \right]$ and $-\partial_\gamma \left[\begin{pmatrix} l \\ \alpha \end{pmatrix}_{D^c} \otimes \nabla^c u \right]$ which cancel each other out.

□

Observe that if (t, β) is a parallel section for $\mathcal{D}^{D^c} \otimes \nabla^c$ then $\beta = (D^c \otimes \nabla^c)t$. This means that all parallel sections along the leaves for $\mathcal{D}^{D^c} \otimes \nabla^c$ are of the form $(t, D^c \otimes \nabla^c t)$. As a consequence, being a parallel section for $\mathcal{D}^{D^c} \otimes \nabla^c$ on $(\mathcal{O}(1) \oplus (T^*S \otimes \mathcal{O}(1))) \otimes \mathcal{L}^{(1,0)}$ along the leaves of the $(1, 0)$ foliation is in fact a condition on sections of $\mathcal{O}(1) \otimes \mathcal{L}^{(1,0)}$. Thus for any leaf of the $(1, 0)$ foliation we have a distinguished $n + 1$ -dimensional family of those sections of $\mathcal{O}(1) \otimes \mathcal{L}^{(1,0)}$ which gives the parallel sections for $\mathcal{D}^{D^c} \otimes \nabla^c$. Analogously for any leaf of the $(0, 1)$ foliation we have a distinguished $n + 1$ -dimensional family of those sections of $\mathcal{O}(1) \otimes \mathcal{L}^{(0,1)}$ which gives rise to the parallel sections for $\mathcal{D}^{D^c} \otimes \nabla^c$.

Definition 32. Any section s of $\mathcal{L}^{(1,0)}(1)$ such that

$$(s, \mathcal{D}^{D^c} \otimes \nabla^c s) \in (\mathcal{O}(1) \oplus (T^*S \otimes \mathcal{O}(1))) \otimes \mathcal{L}^{(1,0)}$$

is parallel for \mathcal{D}^{D^c} along the leaves of $(1, 0)$ foliation is called an **affine section** of $\mathcal{L}^{(1,0)}(1)$.

Any section \tilde{s} of $\mathcal{L}^{(0,1)}(1)$ such that

$$(\tilde{s}, \mathcal{D}^{D^c} \otimes \nabla^c \tilde{s}) \in (\mathcal{O}(1) \oplus (T^*S \otimes \mathcal{O}(1))) \otimes \mathcal{L}^{(0,1)}$$

is parallel for \mathcal{D}^{D^c} along the leaves of $(0,1)$ foliation is called an **affine section** of $\mathcal{L}^{(0,1)}(1)$.

By $\text{Aff}(\mathcal{A})$ we denote the space of the affine sections of a bundle \mathcal{A} .

Suppose \mathcal{A} and \mathcal{B} are vector bundles over some simply-connected manifold such that \mathcal{B} is trivial line bundle with the fibre B and \mathcal{A} has a connection which defines the space $\text{Aff}(\mathcal{A})$. Then using the tensor product connection of the connection on \mathcal{A} and the trivial connection on \mathcal{B} we can define the space $\text{Aff}(\mathcal{A} \otimes \mathcal{B})$ and we have

$$\text{Aff}(\mathcal{A} \otimes \mathcal{B}) := \text{Aff}(\mathcal{A}) \otimes B.$$

This implies the following observation.

Observation 2. Let $\mathcal{B}^{(1,0)}$ be a vector bundle on S^c which is trivial along the leaves of the $(0,1)$ foliation and $\mathcal{B}^{(0,1)}$ a vector bundle on S^c which is trivial along the leaves of the $(1,0)$ foliation. Then for any leaf of the $(1,0)$ foliation $\tilde{a} \in S^{(0,1)}$, we can define the space of affine sections of $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{B}^{(0,1)}$ along the leaf \tilde{a} . Similarly, for any leaf of the $(0,1)$ foliation $a \in S^{(1,0)}$, we can define the space of affine sections of $\mathcal{L}^{(0,1)}(1) \otimes \mathcal{B}^{(1,0)}$ along the leaf a .

Construction of bundles of affine sections Recall that $\mathcal{L}^{(0,1)}(1)^*$ is trivial along the leaves of the $(1,0)$ foliation and $\mathcal{L}^{(1,0)}(1)^*$ is trivial along the leaves of the $(0,1)$ foliation. By Observation 2 we can define the following vector bundles.

Definition 33. Let $V^{(1,0)}$ be a bundle on the leaf space of the $(1,0)$ -foliation defined fibrewise by the requirement that for a leaf $\tilde{a} \in S^{(0,1)}$ the space $V_{\tilde{a}}^{(1,0)}$ is the space of affine sections of $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ on the leaf \tilde{a} .

Let $V^{(0,1)}$ be a bundle on the leaf space of the $(0,1)$ foliation defined fibrewise by the requirement that for a leaf $a \in S^{(1,0)}$ the space $V_a^{(0,1)}$ is the space of affine sections of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ on the leaf a .

Note that $V^{(0,1)}$ and $V^{(1,0)}$ are complex rank $n + 1$ bundles over manifolds of complex dimension n .

Dimension 4 In this paragraph we will discuss how to obtain affine sections of $\mathcal{L}^{(1,0)}(1)$ and $\mathcal{L}^{(0,1)}(1)$ along the leaves and thus the bundles $V^{(1,0)}$ and $V^{(0,1)}$ in the 4-dimensional case.

Firstly observe that a conformal structure on M is the same data as the complex structure on M and that in the dimension 2

$$\llbracket X, \gamma \rrbracket^c = \llbracket X, \gamma \rrbracket^h.$$

Hence, conformal connections are the same as complex connections and both form the same affine space modelled on 1-forms. Moreover, note that in dimension 2 we have that the corresponding density bundles are the same, i.e., $\mathcal{O}(1) = L$.

Let $\mathcal{H} : \Gamma L \rightarrow \Gamma \mathcal{S}_0^2 T^* S$ be a real analytic Möbius operator (see Section 1.4.6). Then for any Weyl connection D there exists $r_0^D \in \Gamma \mathcal{S}_0^2 T^* S$ such that

$$\mathcal{H} = \mathcal{H}^D + r_0^D = \text{sym}_0 D^2 + r_0^D$$

and we have that $\partial_\gamma r_0^D = -\text{sym}_0 D\gamma$. For D real-analytic, r_0^D is real-analytic. Hence we can complexify it to obtain a section of $\mathcal{S}_0^2 T^* S^c$. Using the decomposition $T^* S^c = TS^{(1,0)} \oplus TS^{(0,1)}$, we get that

$$\mathcal{S}_0^2 T^* S^c = TS^{(1,0)} \otimes_{\mathbb{C}} TS^{(1,0)} \oplus TS^{(0,1)} \otimes_{\mathbb{C}} TS^{(0,1)}$$

and this defines a decomposition

$$(r_0^D)^c = (r_0^D)^{(2,0)} \oplus (r_0^D)^{(0,2)}.$$

Using this we can define a connection along the leaves of the $(1,0)$ foliation by

$$\mathcal{D}_Y^{D^{(1,0)}} \begin{pmatrix} l \\ \alpha \end{pmatrix} = \begin{pmatrix} D_Y^{(1,0)} l - \alpha(Y) \\ D_Y^{(1,0)} \alpha + (r_p^D)_Y^{(2,0)} l \end{pmatrix},$$

where Y is a $(1,0)$ -vector field, l is a section of $\mathcal{O}(1)$ and α is a $(1,0)$ -form with values in $\mathcal{O}(1)$.

As for any $(1,0)$ -vector field X we have $D_X = D_X^{(1,0)}$, we get that

$$\partial_\gamma D_X = \llbracket X, \gamma \rrbracket^c = \llbracket X, \gamma \rrbracket^h.$$

But the same calculation as in Section 1.4.2 shows that the h -projective bracket on the complexified manifold gives a projective bracket along the leaves of the $(1,0)$ foliation:

$$\llbracket X, \gamma \rrbracket^h = \llbracket X, \gamma^{(1,0)} \rrbracket^p,$$

for any $(1,0)$ -vector field X . Now we can use the same proof as the proof of Proposition 8 to conclude that the connection $\mathcal{D}^{D^{(1,0)}}$ defines a family of affine sections of $\mathcal{O}(1)$ along the leaves of the $(1,0)$ foliation and that this definition does not depend on the choice of a Weyl connection D . We can also construct affine sections of $\mathcal{O}(1)$ along the leaves of the $(0,1)$ foliation. Now proceeding in the same way as we did to define bundles $V^{(1,0)}$ and $V^{(0,1)}$ for S of dimension greater than 2 (see Definition 33), we can define bundles $V^{(1,0)}$ and $V^{(0,1)}$ when S is a surface equipped with a conformal Möbius operator.

The blow down maps Let $(x, \tilde{x}) \in S^c$ and $l \in \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*|_{x, \tilde{x}}$. Then, using the evaluation, (x, \tilde{x}) and l define $t \in (V^{(0,1)})_x^*$ by $t(s) = \langle s(\tilde{x}), l \rangle$. This allows us to define the following maps.

Definition 34. *Define:*

$$\begin{aligned} \phi_{(0,1)}: \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^* &\longrightarrow (V^{(0,1)})^* \\ (x, \tilde{x}, l) &\mapsto (x, s \mapsto \langle s(\tilde{x}), l \rangle) \end{aligned}$$

$$\begin{aligned} \phi_{(1,0)}: \mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1) &\longrightarrow (V^{(1,0)})^* \\ (x, \tilde{x}, l) &\mapsto (\tilde{x}, s \mapsto \langle s(x), l \rangle) \end{aligned}$$

Lemma 1. *If U is sufficiently small, the maps $\phi_{(1,0)}|_{(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)^\times}$ and $\phi_{(0,1)}|_{(\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))^\times}$ are injective and their images are open subsets of $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$. Fibrewise the images of $\phi_{(1,0)}$ and $\phi_{(0,1)}$ are cones with vertex at 0.*

Proof. We will prove the lemma for $\phi_{(0,1)}$. The proof for $\phi_{(1,0)}$ is analogous. Re-

strict U so that for any x there exists a non-vanishing section l_0 of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ over the leaf over x and that affine sections provide coordinates on leaves. To see that this is possible observe that if f_0 does not vanish on U , there exists a unique connection D on $\mathcal{O}(1)|_U$ such that $Dl_0 = 0$. Then as l_0 is affine, $D^2l_0 + r^Dl_0 = 0$ hence $r^D = 0$. As a consequence, for any affine section l_i , we have that $D^2l_i = 0$ and hence $D(d(\frac{l_i}{l_0})) = 0$. By definition, for any point \tilde{x} on the leaf, affine sections are determined by the initial condition $(l(\tilde{x}), Dl_{\tilde{x}})$, hence we can find affine sections l_1, \dots, l_n such that $(d\frac{l_1}{l_0}, \dots, d\frac{l_n}{l_0})|_{\tilde{x}}$ is a basis for $T_{\tilde{x}}U$. Hence $\frac{l_1}{l_0}, \dots, \frac{l_n}{l_0}$ provide coordinates on a neighbourhood of \tilde{x} in U . As $d\frac{l_i}{l_0}$ are covariantly constant we have that $\frac{l_i}{l_0}$ are coordinates on the whole U . Note that (l_0, l_1, \dots, l_n) is a basis of affine sections.

• **Injectivity:** Let $(x, \tilde{x}, l_x), (y, \tilde{y}, l_y) \in (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)^\times$ be such that

$$\phi_{(0,1)}(x, \tilde{x}, l_x) = \phi_{(0,1)}(y, \tilde{y}, l_y).$$

Then obviously $x = y$. Let l_0 be a non-vanishing section of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ on the leaf over x and choose l_0, l_1, \dots, l_n a basis of affine sections. Then $z_i := \frac{l_i}{l_0}$ for $i \in \{1, \dots, n\}$ are local coordinates on this leaf in a neighbourhood of \tilde{x} and \tilde{y} . For any $j \in \{1, \dots, n\}$ define

$$s_j = l_j - z_j(\tilde{x}l_0).$$

Then

$$0 = \phi_{(0,1)}(x, \tilde{x}, l_x)(s_j) = \phi_{(0,1)}(y, \tilde{y}, l_y)(s_j)$$

and

$$\phi_{(0,1)}(y, \tilde{y}, l_y)(s_j) = \langle s_j(\tilde{y}), l_y \rangle = (z_j(\tilde{y}) - z_j(\tilde{x})) \langle l_0(\tilde{y}), l_y \rangle.$$

Hence $z_j(\tilde{y}) = z_j(\tilde{x})$ for any $j \in \{1, \dots, n\}$ thus $\tilde{x} = \tilde{y}$ and hence $l_x = l_y$. • **Image:** Fix $x \in S^{0,1}$. Then we can choose affine sections $l_1, \dots, l_n \in V_x^{(0,1)}$ so that $z_i := \frac{l_i}{l_0}$ are a coordinate system centred at 0. Thus, there exists $\Omega \subset \mathbb{C}^n$ such that $(k_1, \dots, k_n) \in \Omega$ if and only if there exists $\tilde{x} \in S^{1,0}$ so that $(x, \tilde{x}) \in U$ and $z_i(\tilde{x}) = k_i$. We will show that the image consists of all sections s^* of $(V^{(0,1)})^*$ that satisfy $s^*(l_0) = k_0, s^*(l_i) = k_i k_0$ for $(k_1, \dots, k_n) \in \Omega$. Pick such an s^* . We claim that s^* is an image of (x, \tilde{x}, l) such that $z_i(\tilde{x}) = k_i, \langle l_0(\tilde{x}), l \rangle = k_0$. This is true because $\langle l_i(\tilde{x}), l \rangle = \frac{l_i}{l_0}(\tilde{x}) \langle l_0(\tilde{x}), l \rangle = k_0 k_i$. Conversely any element of the image must be of

this form. □

We restrict S^c to U from the above lemma. We can also require that U is invariant under the real structure.

Remark 13. The maps $\phi_{(1,0)}$ and $\phi_{(0,1)}$ behave like blow down maps: they contract the $2n$ -dimensional zero sections of the bundles $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ and $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ over S^c to the n -dimensional zero sections of the bundles $V^{(0,1)}$ and $V^{(1,0)}$.

Real structure In this section we will construct an anti-holomorphic isomorphism between $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$.

Recall that θ is the real structure on S^c defined by complexification. As described in Section 2.3, it induces a real structure on any bundle on S^c which arise as a complexification of a bundle on S , in particular on \mathcal{L}^c . As the real structure on \mathcal{L}^c interchanges the $\mathcal{L}^{(1,0)}$ and $\mathcal{L}^{(0,1)}$ subbundles, we also have a natural anti-holomorphic isomorphism between $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ and $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$. Note that this anti-holomorphic isomorphism is a bundle isomorphism over θ (not over identity). As the h-projective structure $[D^c]$ on S^c and the connection ∇^c on \mathcal{L}^c are real in the sense that they are compatible with the real structure θ , we also have an induced anti-holomorphic isomorphism between the bundles of affine sections $V^{(0,1)}$ and $V^{(1,0)}$ and thus between $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$. Note that this anti-holomorphic isomorphism maps a fibre a of $(V^{(0,1)})^*$ into a fibre \bar{a} of $(V^{(1,0)})^*$.

Definition 35. Denote by θ the anti-holomorphic isomorphism between $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ defined as a composition of the natural anti-holomorphic isomorphism between $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ (which has been introduced above) with the $-\text{id}$.

Proposition 20. The anti-holomorphic isomorphism θ between $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$ for points in the image of $\phi_{(1,0)}$ has the property that

$$\theta(\phi_{(1,0)}(x, \tilde{x}, l)) = \phi_{(0,1)}(\theta(x, \tilde{x}, -l)).$$

The anti-holomorphic isomorphism θ between $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ defined above for

points in the image of $\phi_{(0,1)}$ has the property that

$$\theta(\phi_{(0,1)}(x, \tilde{x}, l)) = \phi_{(1,0)}(\theta(x, \tilde{x}, -l)).$$

Proof.

$$\begin{aligned} \theta(\phi_{(0,1)}(x, \tilde{x}, l)) &= \theta(x, (s \mapsto \langle s(\tilde{x}), l \rangle)) = \\ &= (\bar{x}, (s \mapsto -\langle s(\tilde{x}), \theta l \rangle)) = \phi_{(1,0)}(\theta(x, \tilde{x}, -l)) \end{aligned}$$

The proof of second equality is analogous. □

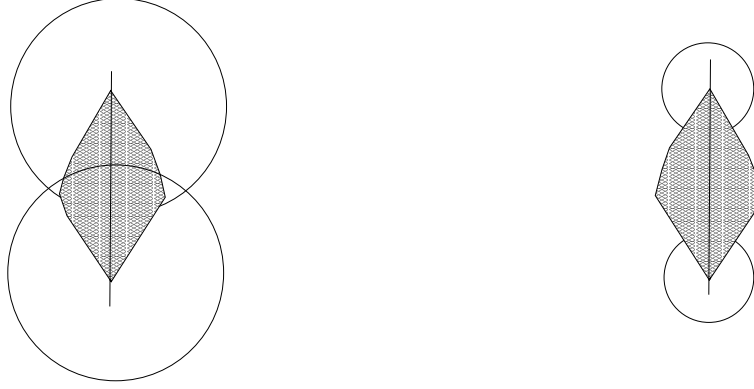
Construction of two halves of the twistor space In this section we will choose open subsets of the bundles $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$ which later will be glued to a twistor space Z of a quaternionic manifold.

Definition 36. Denote by η the isomorphism between bundles $(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)^\times$ and $(\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))^\times$ given fibrewise by $l \mapsto \frac{1}{l} := l^*$.

Using $\phi_{(0,1)}$, $\phi_{(1,0)}$ and η we can glue $V^{(0,1)}$ and $V^{(1,0)}$ on

$$\phi_{(0,1)}((\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)^\times) \cong \phi_{(1,0)}((\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))^\times).$$

However, the space we obtain may be not be Hausdorff. To repair this, we will construct open subsets of $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ such that their gluing will be a Hausdorff manifold. They will be unions of some small neighbourhoods of the zero sections of the corresponding bundles and the images of $\phi_{(0,1)}$ and $\phi_{(1,0)}$ respectively. By taking the tubular neighbourhoods small enough we will ensure that the manifold obtained by gluing will be Hausdorff. The open subsets constructed in this way will also have the property, that they contain (and thus are neighbourhoods of) images by $\phi_{(1,0)}$ and $\phi_{(0,1)}$ of fibres of the corresponding bundles. The situation that we want to obtain is illustrated in the following picture:



Picture 1. In the above picture the white balls correspond to tubular neighbourhoods of the zero and infinity section, the shaded region corresponds to the image of ϕ (which fibrewise is a cone) and the line is a real twistor line. In the first picture there are points on the boundary of the gluing region which are contained in both white balls - those point are double points and obstruct Hausdorffness. In the second picture the balls do not intersect which means that we have no double points.

Now we will present a formal description of the construction. Choose an open neighbourhood \tilde{W} of the zero section of $(V^{(0,1)})^*$ such that we can choose a locally finite cover $\{W_i\}$ of \tilde{W} so that kW_i , for $k > 0$ are a basis of the filter of neighbourhoods of the zero section restricted to W_i , where the multiplication is the scalar multiplication in the fibres. This means that using the scalar multiplications in the fibres we can shrink \tilde{W} locally as much as we want.

Now for each $i \in I$ consider open sets $\theta(kW_i) \in (V^{(1,0)})^*$. From the definition of θ , they are a filter basis of neighbourhoods of the zero section of $(V^{(1,0)})^*$ restricted to $\theta(W_i)$ (and thus we can shrink them by scalar multiplication as much as we want). \tilde{W} and $\theta(\tilde{W})$ are tubular neighbourhoods of the zero sections. Now for each $i \in I$ choose $k_i \in (0, 1)$ such that

$$\phi_{(0,1)} \circ \eta \circ \phi_{(1,0)}^{-1}(\theta(k_i W_i)|_{\text{Im } \phi_{(1,0)}}) \cap \tilde{W} = \emptyset$$

i.e., such that the images of $\theta(k_i W_i)$ and $k_i W_i$ in the gluing of $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ do not intersect (note that if we make \tilde{W} smaller this condition still holds). Such a choice is possible because $\phi_{(0,1)} \circ \eta \circ \phi_{(1,0)}^{-1}$ is a map which takes the zero

section to the infinity (as η is the dual map). Define

$$W = \bigcup_{i \in I} k_i W_i.$$

Then W is a neighbourhood of the zero section in $(V^{(0,1)})^*$ such that

$$\phi_{(0,1)} \circ \eta \circ \phi_{(1,0)}^{-1}(\theta(W)|_{\text{Im } \phi_{(1,0)}}) \cap W = \emptyset.$$

This condition means that we obtained tubular neighbourhoods W and $\theta(W)$ of zero sections of $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ respectively such that their images after gluing by $\phi_{(0,1)} \circ \eta \circ \phi_{(1,0)}^{-1}$ do not intersect.

Define open subsets

$$Z^{(0,1)} = \text{Im } \phi_{(0,1)} \cup W$$

of $(V^{(0,1)})^*$ and

$$Z^{(1,0)} = \text{Im } \phi_{(1,0)} \cup \theta(W)$$

of $(V^{(1,0)})^*$ (they are open as $\text{Im } \phi_{(0,1)} \setminus \underline{0}$ and $\text{Im } \phi_{(1,0)} \setminus \underline{0}$ are open, where $\underline{0}$ denotes the zero sections).

The twistor space

Definition 37. *Set:*

$$Z = Z^{(0,1)} \bigsqcup_{\sim} Z^{(1,0)}$$

where for $(x, s^*) \in \text{Im } \phi_{(0,1)} \setminus \underline{0} \subset Z^{(0,1)}$, $(\tilde{x}, \tilde{s}^*) \in \text{Im } \phi_{(1,0)} \setminus \underline{0} \subset Z^{(1,0)}$ we define the equivalence relation by

$$((x, s^*) \sim (\tilde{x}, \tilde{s}^*)) \Leftrightarrow (\tilde{x}, \tilde{s}^*) = \phi_{(1,0)} \circ \eta \circ \phi_{(0,1)}^{-1}(x, s^*)$$

Proposition 21. *Z defined above is a complex manifold.*

Proof. As $Z^{(1,0)}$ and $Z^{(0,1)}$ are open and are glued on an open set, it is enough to show that Z is Hausdorff. Choose $a \neq b \in Z$. If one of the conditions $a, b \in (V^{(0,1)})^*$ or $a, b \in (V^{(1,0)})^*$ is satisfied, then as $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ are Hausdorff we can find separating neighbourhoods. If this is not true, then, as W is the only part of Z not included in $(V^{(1,0)})^*$ and $\theta(W)$ is the only part of Z not included in $(V^{(0,1)})^*$,

we must have that $a \in W$ and $b \in \theta(W)$. Then W and $\theta(W)$ are separating neighbourhoods as $W \cap \theta(W) = \emptyset$. \square

Note that the gluing defines a map

$$\phi : \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) \longrightarrow Z. \quad (2.1)$$

This is because gluing $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ with $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ by η gives $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$. To see this note that

$$\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) = \mathbb{P}((\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)) \oplus \mathcal{O}).$$

Then fibrewise $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^* \subset \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ corresponds to affine part $[\mathbb{C} : 1]$ of $\mathbb{C}\mathbb{P}^1$ near $0 = [0 : 1]$ and $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ corresponds to the affine part $[1 : \mathbb{C}]$ near $\infty = [1 : 0]$ and the gluing map between those affine parts is given by $z \rightarrow \frac{1}{z}$.

Denote also

$$\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))^\times := \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) \setminus (\underline{0} \cup \underline{\infty})$$

where $\underline{0}$ denotes the zero section of $\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*$ and $\underline{\infty}$ denotes the zero section of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$.

Definition 38. *The images by ϕ of fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ in Z will be called **canonical twistor lines**. Canonical twistor lines invariant under θ are called **canonical real twistor lines**. Note that canonical real twistor lines are exactly those which are images of fibres over the real submanifold $S_{\mathbb{R}} \subset S^c$.*

Remark 14. Fibrewise $Z^{(0,1)}$ and $Z^{(1,0)}$ look like cones with added small balls around 0 and they are glued on the cone part away from balls.

The idea of the construction is that the map induced by ϕ from fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ (which have trivial normal bundles) to canonical twistor lines changes the normal bundle to the desired one (i.e., $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$). To be able to use Kodaira deformation theory we need to see a small neighbourhood of the canonical twistor line in the space in which it has a desired normal bundle. The image of ϕ is such a neighbourhood everywhere except points $\underline{0}$ and $\underline{\infty}$ where such neighbourhoods are W and $\theta(W)$.

2.5 Computing the normal bundle to canonical twistor lines

Now we will prove that the normal bundle to canonical twistor lines is isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$. One way to do this is by direct computation using local coordinates in a similar way to Feix [32]. However we present here a different approach, which will be needed also in Section 2.6.2.

Incidence space Firstly note that, in fact, the construction of Z using $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ and the map ϕ can be viewed as an incidence relation. Observe that the diagram:

$$\begin{array}{ccc}
 & \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) & \\
 \phi \swarrow & & \searrow \pi_{S^c} \\
 Z & & S^c
 \end{array} \tag{2.2}$$

is isomorphic to the incidence relation for canonical twistor lines:

$$\begin{array}{ccc}
 & \tilde{F} = \{(z, u) \in Z \times S^c : z \in u\} & \\
 \pi_Z \swarrow & & \searrow \pi_{S^c} \\
 Z & & S^c
 \end{array} \tag{2.3}$$

In this setting, using the correspondence between canonical twistor lines and fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$, we parametrise canonical twistor lines by points in S^c . We write $z \in u$ to mean that, given $u \in S^c$, z is on the twistor line parametrised by u . Note that $\tilde{F} \cong \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$, ϕ corresponds to π_Z and the fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ are given by $\pi_{S^c}^{-1}(u)$.

Normal bundle A fundamental property of the incidence space is that it separates twistor lines; in our case $\tilde{F} \cong \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ separates canonical twistor lines. Therefore, \tilde{F} (and thus $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$) is a convenient space to investigate how bundles defined along canonical twistor lines change with a change of a twistor line. More precisely, we can pull-back bundles along canonical twistor lines by the map ϕ and form a vector bundle on

the whole of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ and study its properties. In particular, we can consider bundles on $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ after restriction to the zero (or infinity) section, which is isomorphic to the parameter space S^c of the canonical twistor lines.

In this and next paragraph we will discuss the properties of bundles formed from normal bundles to canonical twistor lines viewed in Z and in $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$.

Definition 39. Let N be a bundle on $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ defined fibrewise by

$$N_{(z,u)} = \phi^*(T_z Z / T_z u),$$

where (z, u) denotes the point (x, \tilde{x}, l) in $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ such that the image of the fibre (x, \tilde{x}) under ϕ is the canonical twistor line u and $\phi(x, \tilde{x}, l) = z$ (compare (2.2) and (2.3)).

Observe that after restriction to any fibre of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$, the bundle N is the normal bundle to this line in Z .

Proposition 22. The bundle N admits a natural direct sum decomposition into two rank n subbundles $N^{(1,0)} \oplus N^{(0,1)}$.

Proof. For any canonical twistor line (x, \tilde{x}) we will define submanifolds $\mathcal{E}_x^{(1,0)}$ and $\mathcal{E}_{\tilde{x}}^{(0,1)}$ containing (x, \tilde{x}) . Then we will show that the normal bundles to (x, \tilde{x}) in $\mathcal{E}_x^{(1,0)}$ and $\mathcal{E}_{\tilde{x}}^{(0,1)}$ give the required decomposition. First we fix notation. Recall that Z is a union of open subsets $Z^{(1,0)}$ and $Z^{(0,1)}$ of two vector bundles $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$; by $\underline{0}$ we denote the image of the zero section of $(V^{(0,1)})^*$ in Z , and by $\underline{\infty}$ the image of the zero section of $(V^{(1,0)})^*$ in Z . Any canonical twistor line u has the property that sets $u \cap \underline{0} = \{u_0\}$ and $u \cap \underline{\infty} = \{u_\infty\}$ are singletons and $u \setminus \{u_0\} \subset (V^{(1,0)})^*$, $u \setminus \{u_\infty\} \subset (V^{(0,1)})^*$.

Let (x_0, \tilde{x}_0) be a canonical twistor line. We define $n + 1$ -dimensional submanifolds of Z by

$$\begin{aligned} \mathcal{E}_{\tilde{x}_0}^{(1,0)} &= Z_{\tilde{x}_0}^{(1,0)} \cup \text{im } \phi_{(0,1)}|_{\pi_{S^c}^{-1} \circ \pi_{(0,1)}^{-1}(\tilde{x}_0)}, \\ \mathcal{E}_{x_0}^{(0,1)} &= Z_{x_0}^{(0,1)} \cup \text{im } \phi_{(1,0)}|_{\pi_{S^c}^{-1} \circ \pi_{(1,0)}^{-1}(x_0)}. \end{aligned}$$

To see that they are well defined smooth submanifolds recall that the maps $\phi_{(1,0)}$ and $\phi_{(0,1)}$ are smooth and, after restriction to any leaf of the $(0, 1)$ foliation,

$\phi_{(0,1)}$ is an isomorphism on image and that after restriction to any leaf of the $(1,0)$ foliation $\phi_{(1,0)}$ is an isomorphism on image. Hence $Z_{\tilde{x}_0}^{(1,0)}$, $\text{im } \phi_{(0,1)}|_{\pi_{sc}^{-1} \circ \pi_{(0,1)}^{-1}(\tilde{x}_0)}$, $Z_{x_0}^{(0,1)}$ and $\text{im } \phi_{(1,0)}|_{\pi_{sc}^{-1} \circ \pi_{(1,0)}^{-1}(x_0)}$ are smooth. Moreover $Z_{\tilde{x}_0}^{(1,0)} \cap \text{im } \phi_{(0,1)}|_{\pi_{sc}^{-1} \circ \pi_{(0,1)}^{-1}(\tilde{x}_0)}$ and $Z_{x_0}^{(0,1)} \cap \text{im } \phi_{(1,0)}|_{\pi_{sc}^{-1} \circ \pi_{(1,0)}^{-1}(x_0)}$ are open which proves that $\mathcal{E}_{\tilde{x}_0}^{(1,0)}$ and $\mathcal{E}_{x_0}^{(0,1)}$ are smooth submanifolds of Z . We define rank n subbundles of N by

$$N^{(1,0)}|_{(x,\tilde{x})} = T\mathcal{E}_{\tilde{x}}^{(1,0)} / Tu_{(x,\tilde{x})}$$

$$N^{(0,1)}|_{(x,\tilde{x})} = T\mathcal{E}_x^{(0,1)} / Tu_{(x,\tilde{x})},$$

where $u_{(x,\tilde{x})} = \text{im } \phi|_{\pi_{sc}^{-1}(x,\tilde{x})}$ is the twistor line in Z corresponding to (x, \tilde{x}) .

As for any (x, \tilde{x}) we have

$$TZ|_{u_{(x,\tilde{x})}} = T\mathcal{E}_{\tilde{x}}^{(1,0)} + T\mathcal{E}_x^{(0,1)},$$

we conclude that $N = N^{(1,0)} \oplus N^{(0,1)}$. □

Remark 15. The bundles $N^{(0,1)}$ and $N^{(1,0)}$ can be described in the following way.

- $N^{(0,1)}$:

For points on canonical twistor lines contained in $Z^{(0,1)}$, the bundle $N^{(0,1)}$ is given by the equivalence classes of vectors tangent to fibres of $(V^{(0,1)})^*$ (i.e., by the vertical direction in $(V^{(0,1)})^*$). For points of the form u_∞ the bundle $N^{(0,1)}$ is given by the equivalence classes of vectors tangent to $\underline{\infty}$ (i.e., it is the tangent space to the zero section in $(V^{(1,0)})^*$).

- $N^{(1,0)}$:

For points on canonical twistor lines contained in $Z^{(1,0)}$, the bundle $N^{(1,0)}$ is given by equivalence classes of vectors tangent to fibres of $(V^{(1,0)})^*$ (i.e., by the vertical direction in $(V^{(1,0)})^*$), and for points of the form u_0 by the equivalence classes of vectors tangent to $\underline{0} \subseteq (V^{(0,1)})^*$.

Normal bundle to fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ Now observe that the smooth map ϕ defines a map

$$d\phi : T\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) \longrightarrow \phi^*TZ, \quad (2.4)$$

which is an isomorphism everywhere except on $\underline{0}$ and $\underline{\infty}$. This motivates us to consider a normal bundle to fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ and to use the isomorphism $d\phi$ (for all points not in $\underline{0}$ and $\underline{\infty}$) to study the properties of the bundle N .

Consider the following exact sequence

$$0 \longrightarrow V_{\pi_{S^c}} \longrightarrow T\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) \longrightarrow \pi_{S^c}^* TS^c \longrightarrow 0,$$

where $V_{\pi_{S^c}}$ denotes the vertical bundle of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$. Observe that the restriction of the above exact sequence to a fibre of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ gives the standard exact sequence for the normal bundle to the fibre in $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$.

Definition 40. We define χ to be the bundle $\pi_{S^c}^* TS^c$ on $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$.

Note that after restriction to fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$, χ is the normal bundle to fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$.

Remark 16. The decomposition $TS^c = TS^{(1,0)} \oplus TS^{(0,1)}$ induces a decomposition of the bundle χ into two rank n subbundles $\chi^{(1,0)} \oplus \chi^{(0,1)}$.

For points not in the zero or infinity sections of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ we have that $d\phi(\chi^{(1,0)}) = N^{(1,0)}$ and $d\phi(\chi^{(0,1)}) = N^{(0,1)}$.

Moreover, the map ϕ is an isomorphism at the zero section after restriction to any leaf of the $(1, 0)$ foliation and at the infinity section after restriction to any leaf of the $(0, 1)$ foliation. Hence $d\phi$ gives an isomorphism between the bundle $\chi^{(1,0)}$ and $N^{(1,0)}$ at the complement of the infinity section and an isomorphism between the bundle $\chi^{(0,1)}$ and $N^{(0,1)}$ at the complement of the zero section.

Definition 41. For $l \in \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))_{(x,\bar{x})} \setminus \{\infty\}$ denote by $\phi_l^*(n)$ the element of $N_l^{(1,0)}$ corresponding to $n \in TS_{(x,\bar{x})}^{(1,0)}$ via the pull-back $\pi_{S^c}^*$ to $\chi_l^{(1,0)}$ and the isomorphism $d\phi$. For $l \in \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))_{(x,\bar{x})} \setminus \{0\}$ denote by $\phi_l^*(\tilde{n})$ the element of $N_l^{(0,1)}$ corresponding to $\tilde{n} \in TS_{(x,\bar{x})}^{(0,1)}$ via the pull-back $\pi_{S^c}^*$ to $\chi_l^{(0,1)}$ and the isomorphism $d\phi$.

Tautological line bundles along the twistor lines In this paragraph we will discuss properties of the following bundles which we will use to describe properties of the bundle N .

Definition 42. Using Definition 30, define

$$\Delta_{(1,0)} := \Delta_{(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^* \oplus \mathcal{O}}$$

$$\Delta_{(0,1)} := \Delta_{\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))}.$$

Along canonical twistor lines (i.e., the fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$), the bundles $\Delta_{(1,0)}$ and $\Delta_{(0,1)}$ are isomorphic to the tautological bundle $\mathcal{O}(-1)$ of $\mathbb{C}\mathbb{P}^1$. This isomorphism depends on the base point $x \in S^c$. Later we will use the bundles $\Delta_{(1,0)}$ and $\Delta_{(0,1)}$ to untwist the normal bundle such that the resulting bundle will be trivial along the canonical twistor lines (viewed as fibres in $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$).

Firstly recall that

$$\Delta_{(1,0)} \subset \pi_{S^c}^*((\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^* \oplus \mathcal{O})$$

$$\Delta_{(0,1)} \subset \pi_{S^c}^*(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)))$$

and that the bundles can be described in coordinates as follows:

- $\Delta_{(1,0)}$:

Using the isomorphism

$$\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) \setminus \underline{\infty} \cong \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*_{(x,\tilde{x})},$$

denote the points of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)) \setminus \underline{\infty}$ by $(x, \tilde{x}, \tilde{l})$, where $\tilde{l} \in \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*_{(x,\tilde{x})}$.

The remaining points of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$, i.e., lying on $\underline{\infty}$, will be denoted here by (x, \tilde{x}, ∞) .

Set

$$(\Delta_{(1,0)})_{(x,\tilde{x},\tilde{l})} = \{(a\tilde{l}, a), a \in \mathbb{C}\} \in ((\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^* \oplus \mathcal{O})_{(x,\tilde{x})},$$

$$(\Delta_{(1,0)})_{(x,\tilde{x},\infty)} = (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^* \oplus 0.$$

- $\Delta_{(0,1)}$:

Similarly as for $\Delta_{(1,0)}$, denote points in the affine part $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ by (x, \tilde{x}, l) and the remaining points (i.e., lying on $\underline{0} \in \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ by (x, \tilde{x}, ∞) .

Set

$$\begin{aligned} (\Delta_{(0,1)})_{(x,\tilde{x},l)} &= \{(a, al), a \in \mathbb{C}\} \in (\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)))_{(x,\tilde{x})} \\ (\Delta_{(0,1)})_{(x,\tilde{x},\infty)} &= 0 \oplus (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)). \end{aligned}$$

Twisted normal bundles trivial along canonical twistor lines Our aim is to show that

$$\begin{aligned} N^{(1,0)} &= \pi_{S^c}^* [TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))] \otimes \Delta_{(0,1)}^*, \\ N^{(0,1)} &= \pi_{S^c}^* [TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)] \otimes \Delta_{(1,0)}^*. \end{aligned}$$

To do this, we consider the bundles $N^{(1,0)} \otimes \Delta_{(0,1)}$ and $N^{(0,1)} \otimes \Delta_{(1,0)}$ and show that along fibres of $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ they admit families of trivialising sections given by elements of $TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))$ and $TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$ respectively. To simplify the notation we define

$$W^{(1,0)} = N^{(1,0)} \otimes \Delta_{(0,1)},$$

$$W^{(0,1)} = N^{(0,1)} \otimes \Delta_{(1,0)}.$$

The bundles $W^{(1,0)}$ and $W^{(0,1)}$ should be viewed as bundles which encode how the normal bundle to the canonical twistor lines changes when we change the canonical twistor line. This will be used to identify the quaternionic manifold arising by our construction with the neighbourhood of the zero section of the twisted tangent bundle of S .

Using the notation from Definition 41 we define the maps

$$\begin{aligned} \pi_{W^{(1,0)}} : W^{(1,0)}|_{\mathbb{P}(\mathcal{O}_{\oplus}(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))^\times} &\longrightarrow TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)) \\ (\phi_l^*(b) \otimes (a, al)) &\mapsto (b \otimes l) \end{aligned}$$

$$\begin{aligned} \pi_{W^{(0,1)}} : W^{(0,1)}|_{\mathbb{P}(\mathcal{O}_{\oplus}(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))^\times} &\longrightarrow TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*) \\ (\phi_l^*(\tilde{b}) \otimes (a\tilde{l}, a)) &\mapsto (\tilde{b} \otimes \tilde{l}). \end{aligned}$$

Lemma 2. *The inverse image of any non-zero point in $(TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)))_{(x, \tilde{x})}$ by $\pi_{W^{(1,0)}}$ extends to a family of non-vanishing sections of $W^{(1,0)}|_{\mathbb{P}(\mathcal{O}_{\oplus}(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))_{(x, \tilde{x})}}$ parametrised by $a \in \mathbb{C}^\times$.*

The inverse image of any non-zero point in $(TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^))_{(x, \tilde{x})}$ by $\pi_{W^{(0,1)}}$ extends to a family of non-vanishing sections of $W^{(0,1)}|_{\mathbb{P}(\mathcal{O}_{\oplus}(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))_{(x, \tilde{x})}}$ parametrised by $a \in \mathbb{C}^\times$.*

Proof. We will prove the lemma for $\pi_{W^{(1,0)}}$. The proof for $\pi_{W^{(0,1)}}$ is analogous.

Let $(b \otimes l) \in (TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)))_{(x, \tilde{x})} \setminus \{0\}$. Then $\pi_{W^{(1,0)}}^{-1}(b \otimes l)$ is a subset of $W^{(1,0)}|_{\mathbb{P}(\mathcal{O}_{\oplus}(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))^\times_{(x, \tilde{x})}$ such that over a point $(x, \tilde{x}, \frac{l}{\alpha}) \in (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))^\times_{(x, \tilde{x})}$ we have

$$\pi_{W^{(1,0)}}^{-1}(b \otimes l)|_{\frac{l}{\alpha}} = \{\phi_{\frac{l}{\alpha}}^*(ab) \otimes (a, a\frac{l}{\alpha}) ; a \in \mathbb{C}\}.$$

Observe that for any $a \in \mathbb{C}^\times$ the set

$$w := \{\phi_{\frac{l}{\alpha}}^*(ab) \otimes (a, a\frac{l}{\alpha})\}_{\frac{l}{\alpha}}$$

is a section of $W^{(1,0)}|_{\mathbb{P}(\mathcal{O}_{\oplus}(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))^\times_{(x, \tilde{x})}}$. We will show that w extends to a section of $W^{(1,0)}|_{\mathbb{P}(\mathcal{O}_{\oplus}(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))_{(x, \tilde{x})}}$.

Fix $a \in \mathbb{C}^\times$. Clearly, w is given by

$$w(\frac{l}{\alpha}) = \phi_{\frac{l}{\alpha}}^*(ab) \otimes (a, a\frac{l}{\alpha}).$$

By properties of the tensor product,

$$\phi_{\frac{l}{\alpha}}^*(ab) \otimes (a, a\frac{l}{\alpha}) = \phi_{\frac{l}{\alpha}}^*(b) \otimes (a\alpha, al),$$

so that w extends through $\alpha = 0$. Hence it remains to show that w extends through $\alpha = \infty$.

Fix a twistor line (x, \tilde{x}) , and $b \in TS^{(1,0)}$ and $l \in (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))_{(x, \tilde{x})}$. Observe that near $\alpha = \infty$, the manifold $\mathcal{E}_{\tilde{x}}^{(1,0)}$ is contained in a single fibre $(V^{(1,0)})_{\tilde{x}}^*$. In Lemma 1 we have chosen affine sections $f_0, \dots, f_n \in V_{\tilde{x}}^{(1,0)}$ such that locally f_0 is non-vanishing and $z_i := \frac{f_i}{f_0}$ for $i \neq 0$ are affine coordinates on the leaf \tilde{x} . Observe that we can require that $f_0^*(x) = l$, where f_0^* is a section of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ along the leaf \tilde{x} defined by $f_0^*(p) = \frac{1}{f_0(p)}$. We can also require that $f_i(x) = 0$ for all $i \neq 0$ i.e., that the coordinates z_i are centred at x . For any $y \in S^{(1,0)}$ and $t \in \mathbb{C}$ sufficiently small, define points $y_t \in S^{(1,0)}$ such that $z_i(y_t) = z_i(y) \frac{t}{t+1}$. Then there exists $y \in S^{(1,0)}$ such that b is a vector tangent to a curve $t \mapsto y_t$. Define

$$y_t^\alpha \in S^{(1,0)} : z_i(y_t^\alpha) = z_i(y) \frac{t}{t + \alpha^{-1}}, \quad i \in \{1, \dots, n\}$$

and consider curves in $\mathbb{P}(\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*))$ defined by

$$\gamma^\alpha : t \mapsto (y_t^\alpha, \tilde{x}, f_0^*(y_t^\alpha)(t + \alpha^{-1})).$$

We have that $\gamma^\alpha(0) = \frac{1}{\alpha}$ and, by direct computations, we get that the derivative of γ^α at 0 is αb . Now, the application of ϕ to γ^α gives

$$\phi(\gamma^\alpha(t))(f_i) = f_i(y_t^\alpha) f_0^*(y_t^\alpha)(t + \alpha^{-1}) = z_i(y_t^\alpha)(t + \alpha^{-1}) = z_i(y)t$$

and

$$\phi(\gamma^\alpha(t))(f_0) = f_0(y_t^\alpha) f_0^*(y_t^\alpha)(t + \alpha^{-1}) = t + \alpha^{-1}$$

If we use the base of the vector space $(V^{(1,0)})_{\tilde{x}}^*$ given by evaluations on f_i , we obtain that curves $\phi(\gamma^\alpha)$ are

$$t \mapsto (t + \alpha^{-1}, y_1 t, \dots, y_n t)$$

and their derivatives

$$(1, y_1, \dots, y_n)$$

do not depend on α . In this basis, the tangent space to the canonical twistor line (x, \tilde{x}) is spanned by $(1, 0, \dots, 0)$; hence the section w extends over $\alpha = \infty$

and $w(\frac{1}{\infty}) \neq 0$.

□

Remark 17. In the proof of Lemma 2 we have shown that the value of the section w depends on a point $y \in S^{(1,0)}$ defined using $b \in TS^{(1,0)}$. Observe that by varying b , we can obtain any $y \in S^{(1,0)}$. Hence the possible values at ∞ of sections constructed in Lemma 2 span the fibre at ∞ of the normal bundle to the canonical twistor line (x, \tilde{x}) .

Proposition 23. *The bundles $W^{(1,0)}$ and $W^{(0,1)}$ are trivial along the canonical twistor lines and*

$$W^{(1,0)} = \pi_{S^c}^*[TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))],$$

$$W^{(0,1)} = \pi_{S^c}^*[TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)].$$

Proof. To prove that $W^{(1,0)}$ and $W^{(0,1)}$ are trivial along canonical twistor lines we have to show that we can choose n non-vanishing and (fibrewise) linearly independent sections of $W^{(1,0)}$ and $W^{(0,1)}$ respectively. We will prove this for the bundle $W^{(1,0)}$; the proof for $W^{(0,1)}$ is analogous.

Firstly, note that values at ∞ of sections constructed in Lemma 2 span the fibre at ∞ of the normal bundle to the canonical twistor line (x, \tilde{x}) . Hence we can choose sections w_1, \dots, w_n which come from $b_1, \dots, b_n \in T_{(x, \tilde{x})}S^{(1,0)}$ in the way described in Lemma 2 and such that $w_1(\infty), \dots, w_n(\infty)$ form a basis of the fibre at ∞ of the normal bundle to the canonical twistor line (x, \tilde{x}) .

Being a frame is an open condition; therefore we have that also for neighbouring points of ∞ on the twistor line (x, \tilde{x}) , the values of sections w_1, \dots, w_n are linearly independent. By construction and using the isomorphism ϕ^* , this means that b_1, \dots, b_n have to be independent in $TS^{(1,0)}$. Using again the isomorphism ϕ^* , this implies that the values of sections w_1, \dots, w_n are independent for remaining points, which completes the proof. □

Proposition 24. *The normal bundles to canonical twistor lines are isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.*

Proof. The bundles $\Delta_{(1,0)}$ and $\Delta_{(0,1)}$ restricted to any twistor line from S^c are isomorphic to the tautological bundle, and $W^{(1,0)} = N^{(1,0)} \otimes \Delta_{(0,1)}$ and $W^{(0,1)} = N^{(0,1)} \otimes \Delta_{(1,0)}$ are trivial along canonical twistor lines. Thus for any canonical

twistor line (x, \bar{x})

$$N^{(0,1)}|_{\pi_{Sc}^{-1}(x,\bar{x})} \cong \mathbb{C}^n \otimes \mathcal{O}(1)$$

and

$$N^{(1,0)}|_{\pi_{Sc}^{-1}(x,\bar{x})} \cong \mathbb{C}^n \otimes \mathcal{O}(1)$$

□

2.6 Main theorem

2.6.1 Construction of the quaternionic manifold

Theorem 5. *The complex manifold Z constructed in the previous section is a twistor space of a quaternionic manifold.*

Proof. • By Proposition 21, Z is a complex $2n + 1$ -manifold.

- By Proposition 24, there exists a family of projective lines in Z , called canonical twistor lines, such that their normal bundle are $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.
- θ is a well defined anti-holomorphic involution of Z and thus it is a well defined real structure on Z .
- The real structure induced on canonical real twistor lines is antipodal i.e., does not have fixed points; Let $u = (x, \bar{x})$ be a canonical twistor line. Firstly note that the point $u_0 \in u$ lying on $\underline{0}$ is mapped to $u_\infty \in u$ (lying on $\underline{\infty}$) and vice-versa so u_0 and u_∞ are not fixed points. Let now $\phi_{(1,0)}(x, \bar{x}, l)$ be a point on u with $l \neq 0$. Then, by Proposition 20, we have

$$\theta(\phi_{(1,0)}(x, \bar{x}, l)) = \phi_{(0,1)}(\theta(x, \bar{x}, -l)) = \phi_{(0,1)}(x, \bar{x}, \theta(-l)) = \phi_{(1,0)}(x, \bar{x}, \theta(-l)^*).$$

Thus $\phi_{(1,0)}(x, \bar{x}, l)$ is a fixed point iff $l = -\theta(l)^*$. As θ for points of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)$ over $S_{\mathbb{R}}$ acts as a complex conjugation this is equivalent to the equation

$$\bar{l} = -1,$$

which does not have solutions.

Thus by Theorem 4, Z is indeed a twistor space of quaternionic manifold. □

Definition 43. Denote by M the quaternionic manifold obtained from the twistor space Z from Theorem 5. The above construction of Z and M will be called **the generalised Feix–Kaledin construction for quaternionic manifolds**.

Observe that (locally) the moduli space of curves arising as deformations of canonical twistor lines (in the sense of Kodaira [46]) is a complexification M^c of M . The real submanifold M of M^c arises as moduli space of real twistor lines i.e., those twistor lines which are invariant under the real structure on Z . Because the moduli space of canonical real twistor lines is just $S_{\mathbb{R}} \subset S^c$ and $S_{\mathbb{R}}$ can be naturally identified with S , we get that S is a submanifold of M . Later we will show that S is a totally complex submanifold of M .

Remark 18. The incidence space of the complexified quaternionic manifold M^c can be described by the following diagram.

$$\begin{array}{ccc}
 & F = \{(z, u) \in Z \times M^c : z \in u\} & \\
 \pi_Z \swarrow & & \searrow \pi_{M^c} \\
 Z & & M^c
 \end{array}$$

We observe that (2.3) arises from the above diagram by restriction: $S^c \subseteq M^c$ (represented by canonical twistor lines), $\tilde{F} = (\pi_{M^c})^{-1}(S^c)$ and $\pi_{S^c} = \pi_{M^c}|_{\tilde{F}}$.

2.6.2 Local identification of M with $TS \otimes \mathcal{Q}$

In this section we will show that after restriction to some neighbourhood of the submanifold S , the quaternionic manifold M can be identified with a neighbourhood of the zero section of $TS \otimes \mathcal{Q}$, where \mathcal{Q} is a unitary line bundle. By Kodaira deformation theory ([46]) we have that

$$T_q M^c \cong H^0(\pi_{M^c}^{-1}(q), N|_q).$$

We want to compute $H^0(\pi_{M^c}^{-1}(q), N|_q)$ for canonical twistor lines $q \in S^c \subseteq M^c$. We already know that $N|_q \cong \mathbb{C}^{2n} \otimes \mathcal{O}(1)$ but this isomorphism is not canonical. Our aim is to compute how $H^0(\pi_{M^c}^{-1}(q), N|_q)$ depends on $q \in S^c$. To do this, we define the following bundle over S^c .

Definition 44. By abuse of notation, let H^0 to be the bundle over S^c defined by $H_q^0 = H^0(\pi_{M^c}^{-1}(q), N|_q)$, for $q \in S^c$.

Theorem 6. The quaternionic manifold M obtained by the generalised Feix–Kaledin construction for quaternionic manifolds is, locally near S , isomorphic with a neighbourhood of the zero section of the tangent bundle of S twisted by the unitary line bundle \mathfrak{Q} with fibre $\mathfrak{Q}_{(x,\bar{x})} = (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)_{(x,\bar{x})}$.

Proof. Let q be the twistor line corresponding to $(x, \bar{x}) \in S^c$. Using the bundles $W^{(1,0)}$, $W^{(0,1)}$, $\Delta_{(0,1)}$ and $\Delta_{(1,0)}$ and the fact that

$$W^{(1,0)} = N^{(1,0)} \otimes \Delta_{(0,1)} \quad W^{(0,1)} = N^{(0,1)} \otimes \Delta_{(1,0)}$$

we get that

$$H_{(x,\bar{x})}^0 \cong [H^0(q, W^{(1,0)}|_q) \otimes H^0(q, (\Delta_{(0,1)})^*|_q)] \oplus [H^0(q, W^{(0,1)}|_q) \otimes H^0(q, (\Delta_{(1,0)})^*|_q)]$$

and this isomorphism is canonical. Recall that the bundles $W^{(1,0)}$ and $W^{(0,1)}$ are trivial along the twistor lines and

$$W^{(1,0)} = \pi_{S^c}^*[TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))],$$

$$W^{(0,1)} = \pi_{S^c}^*[TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)].$$

As a consequence,

$$H^0(q, W^{(1,0)}|_q) = [TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))]_{(x,\bar{x})}$$

$$H^0(q, W^{(0,1)}|_q) \cong [TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)]_{(x,\bar{x})}.$$

By Proposition 11,

$$H^0(q, (\Delta_{(0,1)})^*|_q) \cong [\mathcal{O} \oplus (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))]_{(x,\bar{x})}^*$$

$$H^0(q, (\Delta_{(1,0)})^*|_q) \cong [(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*) \oplus \mathcal{O}]_{(x,\bar{x})}^*.$$

As the isomorphisms are canonical, we get

$$T_{(x,\bar{x})}M^c \cong$$

$$\begin{aligned} &\cong (T_{(x,\bar{x})}S^{(1,0)} \oplus (T_{(x,\bar{x})}S^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)_{(x,\bar{x})}) \oplus \\ &\oplus (T_{(x,\bar{x})}S^{(0,1)} \oplus (T_{(x,\bar{x})}S^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))_{(x,\bar{x})}). \end{aligned}$$

Note that in this decomposition term $TS^{(1,0)} \oplus TS^{(0,1)}$ corresponds to the tangent space to the submanifold S^c of M . Also for $q = (x, \bar{x})$, the real structure interchanges $TS^{(1,0)}$ with $TS^{(0,1)}$ and $TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$ with $TS^{(1,0)} \otimes (\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))$.

Hence we can identify

$$T_x M \cong [TS^{(1,0)} \oplus TS^{(0,1)} \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)]_{(x,\bar{x})}.$$

Thus

$$(N_M S)_x \cong T_x S \otimes (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)_{(x,\bar{x})}$$

which by the Tubular Neighbourhood Theorem completes the proof that the quaternionic manifold constructed from the twistor space Z is locally (smoothly) isomorphic in the neighbourhood of the submanifold S with a neighbourhood of the zero section of $TS \otimes \mathcal{Q}$, where $\mathcal{Q} := (\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)|_{S_{\mathbb{R}}}$.

Now it remains to prove that \mathcal{Q} is a unitary line bundle. This follows from the fact, that the map θ gives an isomorphism

$$(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)_{(x,\bar{x})} \cong \overline{(\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))_{(x,\bar{x})}}.$$

□

Remark 19. We have shown that M can be smoothly identified with a neighbourhood of the zero section in $TS \otimes \mathcal{Q}$. Note that this identification depends on the choice of a fibration.

Note that, in fact, we obtained a local identification of M with a neighbourhood of the zero section in $\overline{TS} \otimes \mathcal{Q}$.

2.6.3 Extension of the complex structure

In this section we will show that a choice of a real-analytic connection in the h-projective class together with a choice of a holomorphic n -dimensional submanifold of S^c transverse to the $(1, 0)$ (or $(0, 1)$ respectively) foliation gives a

local extension of the complex structure J ($-J$ respectively) from S to M .

Let $D \in [D]$ be a real-analytic connection in the h-projective class on S and choose $(x_0, \bar{x}_0) \in S_{\mathbb{R}}$. Choose a holomorphic n -dimensional submanifold Ψ of S^c such that Ψ is transversal to the $(1, 0)$ -foliation and $(x_0, \bar{x}_0) \in \Psi$. Note that the transversality condition means that Ψ is biholomorphic to $S^{(0,1)}$ (recall that $S^{(0,1)}$ is the leaf space of the $(1, 0)$ -foliation). For any $\tilde{a} \in S^{(0,1)}$, we denote by $\Psi(\tilde{a})$ the intersection point of Ψ and the leaf \tilde{a} . Note that $\Psi(\bar{x}_0) = (x_0, \bar{x}_0)$. Observe that the manifold $\theta(\Psi)$ is a holomorphic n -dimensional submanifold of S^c which is transversal to the $(0, 1)$ -foliation and $(x_0, \bar{x}_0) \in \theta(\Psi)$.

Remark 20. Note that, in particular, we can take Ψ to be a leaf of the $(0, 1)$ -foliation.

Proposition 25. D and Ψ define a rank n holomorphic subbundle $K^{(\Psi, D)}$ of $(V^{(1,0)})^*$ and a rank n holomorphic subbundle $\tilde{K}^{(\Psi, D)}$ of $(V^{(0,1)})^*$.

Proof. Recall that $V^{(1,0)}$ and $V^{(0,1)}$ are bundles of affine sections along the leaves of the $(1, 0)$ and $(0, 1)$ foliations and that affine sections were constructed using parallel sections of

$$(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*) \oplus (T^*S \otimes \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)$$

and

$$(\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)) \oplus (T^*S \otimes \mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))$$

along the leaves of corresponding foliations (see Definition 32 and Definition 33). By construction, when we fix a point a on a leaf, the space of affine sections on the leaf is determined by initial conditions $(f(a), Df|_a)$. Define a line subbundle $K_0^{(\Psi, D)}$ of $V^{(1,0)}$ by

$$(K_0^{(\Psi, D)})_{\tilde{x}} = \{f \in V_{\tilde{x}}^{(1,0)} : Df|_{\Psi(\tilde{x})} = 0\}.$$

Then the bundle $K^{(\Psi, D)}$ defined by

$$(K^{(\Psi, D)})_{\tilde{x}} := \{s^* \in (V^{(1,0)})_{\tilde{x}}^* : s^*(f) = 0, \forall f \in (K_0^{(\Psi, D)})_{\tilde{x}}\}$$

is a rank n holomorphic subbundle of $(V^{(1,0)})^*$. Analogously, using $\theta(\Psi)$, we can define a holomorphic rank n subbundle $\tilde{K}^{(\Psi, D)}$ of a vector bundle $(V^{(0,1)})^*$. \square

Remark 21. From the construction and properties of complexification (see Section 2.3), we have that

$$(\tilde{K}^{(\Psi, D)})_{\bar{x}} \cong \overline{K_{\tilde{x}}^{(\Psi, D)}}.$$

Note that $K \cap Z$ and $\tilde{K} \cap Z$ are neighbourhoods of the zero sections of K and \tilde{K} and are n -dimensional submanifolds of Z .

Lemma 3. *The bundles $K^{(\Psi, D)}$ and $\tilde{K}^{(\Psi, D)}$ are transversal to twistor lines which are close enough to the canonical real twistor line given by $\Psi(\bar{x}) \in S^c$.*

Proof. We will prove the lemma for $K^{(\Psi, D)}$. The proof for $\tilde{K}^{(\Psi, D)}$ is analogous. Firstly observe that, if $f_0 \in (K_0^{(\Psi, D)})_{\tilde{x}}$ and $f_0(\Psi(\tilde{x})) = 0$, then $f_0 = 0$. Let u be the canonical twistor line corresponding to $\Psi(\tilde{x})$, for $\tilde{x} \in S^{(0,1)}$, and set

$$t := u \cap (V^{(1,0)})^*.$$

Then we have

$$t = \{s^* \in (V^{(1,0)})_{\tilde{x}}^* : s^*(f) = f(\Psi(\tilde{x}))l, \forall f \in V_{\tilde{x}}^{(1,0)}; l \in \mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)_{\Psi(\tilde{x})}\}.$$

Hence, for any non-zero $f_0 \in (K_0^{(\Psi, D)})_{\tilde{x}}$ and non-zero $s^* \in t$, we have that $s^*(f_0) \neq 0$ and therefore,

$$t \cap K_{\tilde{x}}^{(\Psi, D)} = \{0\}.$$

Since t is a vector subspace of the corresponding fibre of $(V^{(1,0)})^*$, we have that $K^{(\Psi, D)}$ is transversal to canonical twistor lines corresponding to $\Psi(\tilde{x})$, and in particular to $\Psi(\bar{x}_0)$. As transversality is an open condition, $K^{(\Psi, D)}$ is also transversal to neighbouring twistor lines. In particular $K^{(\Psi, D)}$ is transverse for real twistor lines close enough to the canonical real twistor line corresponding to $\Psi(\bar{x}_0)$, which completes the proof. □

Using the transversality we obtain that real twistor lines which are close enough (in the moduli space) to canonical twistor lines intersect each K and \tilde{K} in exactly one point. We can summarise this in the following proposition.

Proposition 26. *After restriction to some neighbourhood of the point corresponding to canonical twistor line $\Psi(\bar{x}_0)$, the quaternionic manifold M can be identified with a*

neighbourhood of the zero section of $K^{(\Psi, D)}$ or equivalently $\tilde{K}^{(\Psi, D)}$. As neighbourhoods of the zero sections of K and \tilde{K} are holomorphic submanifolds of Z , this induces an extension of a complex structure from S to a neighbourhood of $\Psi(\bar{x}_0)$ in M .

Proof. Real twistor lines do not intersect each other, thus there is locally a one-to-one correspondence between points in $K^{(\Psi, D)}$ (or $\tilde{K}^{(\Psi, D)}$ respectively) and real twistor lines which are transversal to $K^{(\Psi, D)}$ (or $\tilde{K}^{(\Psi, D)}$ respectively). \square

Remark 22. • Note that $K^{(\Psi, D)}$ and $\tilde{K}^{(\Psi, D)}$ admit opposite complex structures and induce extensions of complex structures J and $-J$ from S and \bar{S} respectively.

- Recall that all twistor lines that are close enough to real twistor lines intersect $K^{(\Psi, D)}$ and $\tilde{K}^{(\Psi, D)}$ in exactly one point. As nearby twistor lines intersect each other in at most one point, the complexified quaternionic manifold M^c can be identified near S with a neighbourhood of the zero section of $K^{(\Psi, D)} \oplus \tilde{K}^{(\Psi, D)}$ and the real structure on M^c interchanges $K^{(\Psi, D)}$ and $\tilde{K}^{(\Psi, D)}$.

Definition 45. Denote by $J^{(\Psi, D)}$ the complex structure on (an open subset of) M given by $K^{(\Psi, D)}$.

2.6.4 The S^1 action

Definition 46. Let \circ be the local \mathbb{C}^\times action on Z defined by scalar multiplication in the fibres of $(V^{(0,1)})^*$ and by the inverse of scalar multiplication in the fibres of $(V^{(1,0)})^*$.

Remark 23. The above definition is compatible with the gluing of open subsets of $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ thus the action is well defined.

Proposition 27. The \mathbb{C}^\times action \circ , maps twistor lines to twistor lines. The S^1 action induced by \circ maps real twistor lines to real twistor lines and thus, it descends to an S^1 action on the quaternionic manifold M .

Proof. Firstly, note that as the \mathbb{C}^\times action is invertible, it maps submanifolds isomorphic to $\mathbb{C}P^1$ into submanifolds isomorphic to $\mathbb{C}P^1$. It is also immediate to write explicit isomorphism between the normal bundle to a twistor line and the normal bundle to its image by the action of any element of \mathbb{C}^\times : Suppose

that $\gamma(t)$ is a curve in Z such that $\gamma(0) = u_z$, where u_z is a point on a twistor line u and choose $\lambda \in \mathbb{C}^\times$. Then $\lambda \circ \gamma(t)$ is a curve in Z with the property $\gamma(0) = \lambda \circ u_z \in \lambda \circ u$ and thus this induces an invertible map between $T_{u_z}Z$ and $T_{\lambda \circ u_z}Z$. Moreover, if the curve γ is contained in the twistor line u then the curve $\lambda \circ \gamma$ is contained in the twistor line $\lambda \circ u$, which induces an isomorphism between tangent spaces to the twistor lines, and thus between normal spaces to the twistor lines. Thus the action maps twistor lines to twistor lines.

To see that the S^1 action maps real twistor lines to real twistor lines let u be a real twistor line, u_z a point on it and $\lambda \in S^1$. We want to check whether $\theta(\lambda \circ u_z)$ belongs to the line $\lambda \circ u$. We claim that

$$\theta(\lambda \circ u_z) = \frac{1}{\lambda} \circ \theta(u_z).$$

This is because θ is an anti-holomorphic isomorphism between $(V^{(0,1)})^*$ and $(V^{(1,0)})^*$ such that it maps $\lambda s^* \in (V^{(0,1)})^*$ to

$$\theta(\lambda s^*) = \bar{\lambda} \theta(s^*) \in (V^{(1,0)})^*.$$

Because the action \circ is given by scalar multiplication in the fibres of $(V^{(0,1)})^*$ and the inverse of scalar multiplication in the fibres of $(V^{(1,0)})^*$, we have that

$$\bar{\lambda} \theta(s^*) = (\bar{\lambda})^{-1} \circ \theta(s^*).$$

But as $\lambda \in S^1$ we have that $\lambda \bar{\lambda} = 1$; thus

$$\frac{1}{\lambda} \circ \theta(u_z) = \lambda \circ \theta(u_z) \in \lambda \circ u.$$

□

Recall that the identification of M with a neighbourhood of the zero section of the tangent bundle is not unique, in the sense that there is no canonical choice of fibration of M .

Proposition 28. *We can choose a fibration of M over S such that the action \circ coincide with the action given by unit scalar multiplication in the fibres of $TS \otimes \mathcal{Q}$.*

Proof. In Section 2.6.3 we constructed holomorphic rank n subbundles $K^{(\Psi, D)}$ of $V^{(1,0)}$. Fix a real-analytic connection D in the h-projective class on S and let

$$K := K^{(S_{\mathbb{R}}, D)}$$

be a smooth complex subbundle of $V^{(1,0)}$ constructed in the same way as $K^{(\Psi, D)}$ (see Section 2.6.3) by taking Ψ to be $S_{\mathbb{R}}$. Note that, as $S_{\mathbb{R}}$ is not a holomorphic submanifold of S^c , the bundle K is not a holomorphic subbundle of $V^{(1,0)}$ and, as a consequence, $K \cap Z$ is a smooth (and not holomorphic) submanifold of Z . As K is transversal to canonical real twistor lines, we can identify smoothly a neighbourhood of the zero section of K with a neighbourhood of S in M .

The S^1 action on K given by unit scalar multiplication in the fibres coincide with the action \circ . To see this, note that if a twistor line u intersects K in the point $u_z \in K_x$, then the twistor line $t \circ u$ intersects K in the point $tu_z \in K_x$. By construction,

$$K \cong (TS^{(0,1)} \otimes \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*)|_{S_{\mathbb{R}}}$$

which means that K is isomorphic with $TS \otimes \mathcal{Q}$, which finishes the proof. \square

2.6.5 Totally complex submanifold

It is a standard technique in quaternion geometry (see for example [55]) to decompose the complexified tangent bundle of a quaternionic manifold M into a tensor product of bundles

$$TM \otimes \mathbb{C} = E \otimes H,$$

where E and H are holomorphic vector bundles of rank $2n$ and 2 respectively, defined in the following way. Recall that in Section 2.5 we constructed a bundle N on the (restricted) incidence space \tilde{F} given by the normal bundles to canonical twistor lines. Analogously, we can define such a bundle on the whole F (see Remark 18) by

$$N_{(z,u)} := T_z Z / T_z u.$$

Now we decompose

$$N = E \otimes L,$$

where E is a rank $2n$ bundle trivial along twistor lines and L is a line bundle. Note that, by properties of the normal bundle to twistor lines, L along twistor lines is isomorphic to $\mathcal{O}(1)$. Now by abuse of notation we set

$$E_u := H^0(u, E|_u)$$

and

$$H_u = H^0(u, L|_u).$$

Note that the bundles E and L are unique only up to tensoring by line bundle.

Recall from Section 2.5 that $\Delta_{(1,0)}$ is a tautological bundle along the fibres on $\mathbb{P}(\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1)) \cong \tilde{F}$ corresponding to a vector bundle $(\mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*) \oplus \mathcal{O}$ and $\Delta_{(0,1)}$ is a tautological bundle along the fibres corresponding to a vector bundle $(\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)) \oplus \mathcal{O}$. We have also shown that

$$N^{(1,0)} = W^{(1,0)} \otimes (\Delta_{(0,1)})^* \quad \text{and} \quad N^{(0,1)} = W^{(0,1)} \otimes (\Delta_{(1,0)})^*,$$

where

$$W^{(1,0)} = \pi_{S^c}^*(TS^{(1,0)} \otimes \mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1))$$

$$W^{(0,1)} = \pi_{S^c}^*(TS^{(0,1)} \otimes \mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)).$$

Recall that

$$\mathbb{P}(\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1)) \cong \tilde{F} = \pi_{M^c}^{-1}(S^c).$$

Observe that a tensor product of a bundle which is trivial along canonical twistor lines with a pull-back of a bundle on S^c is trivial along canonical twistor lines. Also a tensor product of a line bundle which along canonical twistor lines is isomorphic with the tautological line bundle with a pull-back of a line bundle on S^c is along canonical twistor lines isomorphic to the tautological line bundle. Using this define

$$\Gamma := (\Delta_{(1,0)} \otimes \pi_{S^c}^* \mathcal{L}^{(1,0)}(1)^*)^* = (\Delta_{(0,1)} \otimes \pi_{S^c}^* \mathcal{L}^{(0,1)}(1)^*)^*,$$

$$W := (W^{(1,0)} \otimes \pi_{S^c}^* \mathcal{L}^{(0,1)}(1)^*) \oplus (W^{(0,1)} \otimes \pi_{S^c}^* \mathcal{L}^{(1,0)}(1)^*).$$

The above considerations imply the following proposition.

Proposition 29. $N = \Gamma \otimes W$ which means that W is the restriction of E to \tilde{F} and Γ is the restriction of L to \tilde{F} for some choice of E and L . Using the analogous argument as in Section 2.6.2, we obtain that, for canonical twistor lines, $H_q = H^0(q, \Gamma|_q) = (\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1))_q$. \square

Proposition 30. S is a totally complex submanifold of M of maximal dimension.

Proof. From [55], we know that complex structures from the quaternionic structure on M are given by endomorphisms of the bundle H , which in our case is isomorphic to $\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1)$ for points $q \in S^c$. The complex structure J from S^c is given by the matrix

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and applying it to

$$TM^c|_{S^c} = [(TS^{(1,0)} \otimes \mathcal{L}^{(1,0)}(1)^*) \oplus (TS^{(0,1)} \otimes \mathcal{L}^{(0,1)}(1)^*)] \otimes [\mathcal{L}^{(1,0)}(1) \oplus \mathcal{L}^{(0,1)}(1)]$$

we get that

$$JTS^c \subseteq TS^c.$$

Let s be a unitary section of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)|_{S_{\mathbb{R}}}$. Then the complex structure I_s corresponding to s is an endomorphism of the bundle H given by the matrix

$$\begin{pmatrix} 0 & -s^{-1} \\ s & 0 \end{pmatrix}.$$

Applying this to TM^c we get that on S^c

$$ITS^c \subseteq [TS^{(1,0)} \otimes \mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)] \oplus [TS^{(0,1)} \otimes \mathcal{L}^{(1,0)}(1) \otimes \mathcal{L}^{(0,1)}(1)^*].$$

\square

2.7 Quaternionic connections

Let (x_0, \bar{x}_0) be a point of $S_{\mathbb{R}}$. Recall (see Section 2.6.3) that for any real-analytic connection D from the h-projective class on S and any holomorphic submani-

fold Ψ transversal to $(1, 0)$ -foliation and passing through (x_0, \bar{x}_0) , we defined a subbundle $K^{(\Psi, D)}$ of $V^{(1, 0)}$ which induces a complex structure $J^{(\Psi, D)}$ on a neighbourhood of (x_0, \bar{x}_0) in M . For the considerations in this section, we take M to mean some neighbourhood of (x_0, \bar{x}_0) in M and Z to mean some neighbourhood of the real twistor line (x_0, \bar{x}_0) .

Proposition 31. *A choice of a real-analytic connection D from the h -projective class on S together with a choice of holomorphic submanifold Ψ transversal to $(1, 0)$ -foliation and passing through (x_0, \bar{x}_0) induces a distinguished connection on a neighbourhood of (x_0, \bar{x}_0) in M .*

Proof. By Proposition 26, the choice of D and Ψ gives an extension $J^{(\Psi, D)}$ of the complex structure J from S to M . By [3], there exists a unique quaternionic connection on M which preserves $J^{(\Psi, D)}$. \square

Remark 24. As Z is the total space of the S^2 -bundle of the complex structures from the quaternionic structure, the quaternionic connections on M induce hyperplane distributions on Z transversal to real twistor lines.

Observe that any complex structure (from the quaternionic structure) on M is a holomorphic hypersurface in Z . In this setting, the complex structure $J^{(\Psi, D)}$ ($-J^{(\Psi, D)}$ respectively) is the hypersurface $K^{(\Psi, D)} \cap Z$ ($\bar{K}^{(\Psi, D)} \cap Z$ respectively). The quaternionic connection preserving $J^{(\Psi, D)}$ on M gives a hyperplane distribution on Z which at points of the hypersurface $K^{(\Psi, D)}$ is tangent to $K^{(\Psi, D)}$.

Corollary 3. *Let $D^{(\Psi, D)}$ be the quaternionic connection preserving $J^{(\Psi, D)}$ (see Proposition 31). Then the hyperplane distribution on Z given by $D^{(\Psi, D)}$, after restriction to $K^{(\Psi, D)} \cap Z$, is the tangent bundle to $K^{(\Psi, D)} \cap Z$. Moreover, the hyperplane distribution on Z given by $D^{(\Psi, D)}$ after restriction to $\bar{K}^{(\Psi, D)} \cap Z$ is the tangent bundle to $\bar{K}^{(\Psi, D)} \cap Z$.*

Lemma 4. *Using the connection D and Ψ , we can construct $2n$ -dimensional hyperplane distributions on $(V^{(1, 0)})^* \cap Z$ and $(V^{(0, 1)})^* \cap Z$ which are transversal to real twistor lines that are close enough to canonical twistor lines.*

Proof. Our aim is to construct two transversal rank n distributions on $(V^{(1, 0)})^* \cap Z$. Note that D defines a horizontal distribution on the bundle $(V^{(1, 0)})^*$ and that horizontal distributions on the bundle $(V^{(1, 0)})^*$ are n -dimensional distributions on $Z \cap (V^{(1, 0)})^*$ transversal to canonical twistor lines.

To define a complementary rank n distribution on $Z \cap (V^{(1,0)})^*$, we consider the following affine subspaces of the fibres of $(V^{(1,0)})^*$.

Let $\tilde{x} \in S^{(0,1)}$ and $s_0^* \in (V^{(1,0)})_{\tilde{x}}^*$. We define an n -dimensional affine subspace of $(V^{(1,0)})_{\tilde{x}}^*$ passing through s_0^* in the following way. Recall that $(K_0^{(\Psi,D)})_{\tilde{x}} = \{f \in V_{\tilde{x}}^{(1,0)} : Df|_{\Psi(\tilde{x})} = 0\}$. The affine hyperplane is defined by

$$K_{s_0^*}^{(\Psi,D)} := \{s^* \in (V^{(1,0)})_{\tilde{x}}^* : s^*|_{(K_0^{(\Psi,D)})_{\tilde{x}}} = s_0^*|_{(K_0^{(\Psi,D)})_{\tilde{x}}}\}.$$

We claim that for s_0^* close enough to the canonical real twistor line $u := (\bar{x}, \tilde{x})$, $K_{s_0^*}^{(\Psi,D)}$ is transversal to the real twistor line intersecting s_0^* . To see that, firstly note that it is true for s_0^* belonging to the canonical twistor line u . This follows from the fact that u is transversal to $K_{\tilde{x}}^{(\Psi,D)}$ (recall that in this section we take M to be an open subset of M such that canonical real twistor lines are transversal to $K^{(\Psi,D)}$), and hence $f_0(\bar{x}) \neq 0$ for any non-zero $f_0 \in (K_0^{(\Psi,D)})_{\tilde{x}}$. But this implies that $K_{s_0^*}^{(\Psi,D)}$ is transversal to u . As transversality is an open condition, we get that $K_{s_0^*}^{(\Psi,D)}$ are transversal to the real twistor lines through s_0^* for s_0^* close enough to canonical real twistor lines.

As by definition, for any s_0^* we have that $K_{s_0^*}^{(\Psi,D)}$ is transversal to the distribution of the bundle $(V^{(1,0)})^*$ at s_0^* , we can define a rank $2n$ distribution $H^{(\Psi,D)}$ on $Z' \cap (V^{(1,0)})^*$ by setting $H_{s_0^*}^{(\Psi,D)}$ to be the sum of $K_{s_0^*}^{(\Psi,D)}$ and of the horizontal distribution of $(V^{(1,0)})^*$ at s_0^* . Using again the fact that being a horizontal is an open condition, we get that $H^{(\Psi,D)}$ is horizontal to real twistor lines close enough to the canonical real twistor lines in Z' .

Analogously using $\theta(\Psi)$, we can define affine hyperplanes $\tilde{K}_{s_0^*}^{(\Psi,D)} \subset (V^{(0,1)})_{\tilde{x}}^*$ and hence we obtain a horizontal distribution $\tilde{H}^{(\Psi,D)}$ on $Z' \cap (V^{(0,1)})^*$.

□

Proposition 32. *The distributions $H^{(\Psi,D)}$ and $\tilde{H}^{(\Psi,D)}$ coincide on the common part of Z and hence define a distribution $\mathcal{H}^{(\Psi,D)}$ on the whole Z which is transversal to real twistor lines close enough to canonical real twistor lines. Moreover, $\mathcal{H}^{(\Psi,D)}$ preserves $K^{(\Psi,D)} \cap Z$ and $\tilde{K}^{(\Psi,D)} \cap Z$*

Proof. The fact that $H^{(\Psi,D)}$ preserves $K^{(\Psi,D)}$ is implied by the observation that for $s_0^* \in K_{\tilde{x}}^{(\Psi,D)}$, we have $K_{s_0^*}^{(\Psi,D)} = K_{\tilde{x}}^{(\Psi,D)}$. Analogously, $\tilde{H}^{(\Psi,D)}$ preserves $\tilde{K}^{(\Psi,D)}$.

To check that these definitions coincide on the gluing region of $Z^{(1,0)}$ and $Z^{(0,1)}$ we will show that the rank n distribution on $Z \cap (V^{(1,0)})^*$ given by the affine

subspaces of the fibres corresponds by the gluing to the rank n distribution on $Z \cap (V^{(0,1)})^*$ defined using D as a horizontal distribution of $(V^{(0,1)})^*$.

Consider the space $\phi_{(1,0)}^{-1}(K_{s_0^*}^{(\Psi,D)})$ for $s_0^* = \phi_{(1,0)}(x_0, \tilde{x}_0, l_0)$. Choosing $f_0 \in (K_0^{(\Psi,D)})_{x_0}$ we have

$$\phi_{(1,0)}^{-1}(K_{s_0^*}^{(\Psi,D)}) = \{(x, \tilde{x}_0, [(f_0(x))^* f_0(x_0)l_0]) : x \in S^{(1,0)}\},$$

where by $(f_0(x))^*$ or $f_0^*(x)$ we understand an element dual to $f_0(x)$. As f_0 is defined by the property $Df_0|_{\Psi(\tilde{x}_0)} = 0$, the section f_0^* defines a trivialisation of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)|_{\{\tilde{x}=\tilde{x}_0\}}$ compatible with D and, in other words, f_0^* is a section in the direction of horizontal distribution of $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)|_{\{\tilde{x}=\tilde{x}_0\}}$ given by D . Hence, $\phi_{(1,0)}^{-1}(K_{s_0^*}^{(\Psi,D)})$ is in the direction of the horizontal distribution on $\mathcal{L}^{(1,0)}(1)^* \otimes \mathcal{L}^{(0,1)}(1)|_{\{\tilde{x}=\tilde{x}_0\}}$ defined by D . Now ,

$$\phi_{(0,1)} \circ \phi_{(1,0)}^{-1}(K_{s_0^*}) = \{[x, g \mapsto (g(\tilde{x}_0)(f_0(\tilde{x}))^* f_0(\tilde{x}_0)l_0)] : x \in S^{(1,0)}\}$$

which agrees with the horizontal distribution on $V^{(0,1)}$ induced by D (for this we use again the fact that f_0 is in the direction of the horizontal distribution given by D). \square

Corollary 4. *As horizontal distributions on Z correspond to quaternionic connections on M (see [1], [55], [58]), $\mathcal{H}^{(\Psi,D)}$ is the horizontal distribution on the (restricted) twistor space corresponding to $D^{(\Psi,D)}$.*

Corollary 5. *Choosing a not-necessarily holomorphic submanifold Ψ transversal to the $(1,0)$ foliation and passing through (x_0, \bar{x}_0) , we also can construct a distribution $\mathcal{H}^{(\Psi,D)}$ (in the analogous way) and hence, implicitly, a connection on M which does not come from an extension of the complex structure from S to M .*

Remark 25. The hypothesis is that the connections $D^{(\Psi,D)}$ on the quaternionic manifold M give back D on the submanifold $S \subseteq M$. However, the only description of $D^{(\Psi,D)}$ that we obtain was a description of $D^{(\Psi,D)}$ as a bundle on Z . In the literature the proofs that such connections give connections on M are implicit (see for example [55]), therefore it is hard to obtain a formula for $D^{(\Psi,D)}$ on the tangent bundle to M and, hence, to deduce how it behaves on S .

The other problem is that the construction of Z involves using the connection

induced by D on $\mathcal{O}(1)$, but from a connection on $\mathcal{O}(1)$ we can not recover back D without knowing its h-projective structure.

Finally note that we cannot use the methods that Feix uses to get back the connection on S . This is because she applies the theory of hyperholomorphic bundles and uses the correspondence between them and connections of type $(1, 1)$. But in our case the h-projective structure may not admit a connection with curvature of type $(1, 1)$.

2.8 Examples

In this section we will provide explicit examples and describe the results of the construction in special cases.

Firstly consider the case when \mathcal{L} is a trivial bundle. We will show that in this case for $S = \mathbb{C}\mathbb{P}^n$ with the Fubini-Study metric we will obtain the flat example of quaternionic geometry, namely $\mathbb{H}\mathbb{P}^n$ which is not hypercomplex.

To see when we obtain hypercomplex or hyperkähler manifolds we will discuss briefly the Feix construction and show that in the case when S is a manifold with a connection D with curvature of type $(1, 1)$ and $\mathcal{L} \cong \mathcal{O}(1)$ with the connection induced from D , both constructions coincide.

2.8.1 $Z = \mathbb{C}\mathbb{P}^{2n+1}$

In this section we will show that the motivating example discussed in Section 2.1 is the result of our construction when $S = \mathbb{C}\mathbb{P}^n$ with the h-projective structure given by Fubini-Study metric and \mathcal{L} the trivial line bundle on S with the trivial connection.

Firstly, note that affine sections of $\mathcal{O}(1)$ over $\mathbb{C}\mathbb{P}^n$ are the global holomorphic sections of $\mathcal{O}(1)$ i.e., they form $H^0(\mathbb{C}\mathbb{P}^n, \mathcal{O}(1))$. By Proposition 11, the space of the global holomorphic sections of $\mathcal{O}(1)$ of $\mathbb{P}(\mathbb{C}^{n+1})$ is \mathbb{C}^{n+1} , hence the spaces $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$ can be fibrewise naturally identified with \mathbb{C}^{n+1} . Observe also that in this case, the projective bundle is $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ (see Definition 34). As by Proposition 12 the blow down of the tautological bundle on $\mathbb{P}(\mathbb{C}^{n+1})$ along the zero section is \mathbb{C}^{n+1} , we get that the maps $\phi_{(0,1)}$ and $\phi_{(1,0)}$ are surjective. Hence the gluing of $(V^{(1,0)})^*$ and $(V^{(0,1)})^*$ is defined globally and is Hausdorff.

Now, using the notation from Section 2.1, it is enough to prove that the image of the map $\phi_{(1,0)}$ is $\mathbb{P}(\mathbb{C}^{n+1}) \setminus \mathbb{P}(A_-)$. Take A_+ and A_- to be subspaces of $\mathbb{C}^{n+2} = \{(z_1, \dots, z_{n+1}, w_1, \dots, w_{n+1})\}$ given by $(w_1, \dots, w_{n+1}) = 0$ for A_+ and $(z_1, \dots, z_{n+1}) = 0$ for A_- . Let $([z_1 : \dots : z_{n+1}], [w_1 : \dots : w_{n+1}], l(w_1, \dots, w_{n+1}))$ be an element of $\mathcal{O}(1, -1)$ over $\mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1})$ written in such a way that $l \in \mathcal{O}(1, 0)$ and $(w_1, \dots, w_{n+1}) \in \mathcal{O}(0, -1)$. Then the blow-down map is given by:

$$([z_1 : \dots : z_{n+1}], [w_1 : \dots : w_{n+1}], l(w_1, \dots, w_{n+1})) \mapsto [z_1 : \dots : z_{n+1} : lw_1 : \dots : lw_{n+1}].$$

This map is well defined. Indeed, if we change the representative of element $[z_1 : \dots : z_{n+1}]$ to $[az_1 : \dots : az_{n+1}]$ then l changes to al and we get the same element of $\mathbb{P}(\mathbb{C}^{n+1})$. This proves that $Z^{(1,0)} = \mathbb{P}(\mathbb{C}^{n+1}) \setminus \mathbb{P}(A_-)$ and $Z^{(1,0)}$ is a vector space over A_+ . Analogously we get that $Z^{(0,1)} = \mathbb{P}(\mathbb{C}^{n+1}) \setminus \mathbb{P}(A_+)$ and $Z^{(0,1)}$ is a vector space over A_- . Moreover, the gluing induced by the blow down maps glue $Z^{(1,0)}$ and $Z^{(0,1)}$ to $\mathbb{P}(\mathbb{C}^{n+1})$ which proves that we indeed obtain $Z = \mathbb{C}\mathbb{P}^n$. We can summarise the result of this section in the following way.

Example 2. When $S = \mathbb{C}\mathbb{P}^n$ with the h-projective structure given by Fubini-Study metric and \mathcal{L} the trivial line bundle on S with the trivial connection the twistor space obtained by the generalised Feix–Kaledin construction for quaternionic manifolds (see Theorem 5) is $Z = \mathbb{C}\mathbb{P}^n$.

2.8.2 The Feix–Kaledin construction

In this section we prove that the special case our construction is the Feix–Kaledin construction. More precisely we will prove the following theorem.

Theorem 7. *In the case when the h-projective structure on S is given by a real-analytic connection with curvature of type $(1, 1)$, the generalised Feix–Kaledin construction for quaternionic manifolds is exactly the same construction as the one given by Feix ([33]) and thus it gives the hypercomplex manifold. Furthermore, when D is the Levi-Civita connection of a Kähler metric, then the generalised Feix–Kaledin construction for quaternionic manifolds gives the hyperkähler manifolds obtained by Feix.*

Firstly we briefly summarise the Feix construction. Using the fact that the curvature of the connection D on S is of type $(1, 1)$ she deduces that a

complexification D^c of D is flat along the leaves of the $(1, 0)$ and $(0, 1)$ foliations on S^c . She defines affine functions along leaves as those functions f that df is covariantly constant along leaves of the foliation. Using this she constructs vector bundles V^+ and V^- over $S^{(0,1)}$ and $S^{(1,0)}$ as bundles of affine functions along the leaves. She also constructs maps Φ^\pm from $S^c \times \mathbb{C}$ to $(V^+)^*$ and $(V^-)^*$ respectively defined by

$$\Phi^\pm : (x, \tilde{x}, l) \mapsto [x, (f \mapsto f(\tilde{x})l)] \in (V^\pm)^*,$$

which gives the gluing of $(V^+)^*$ and $(V^-)^*$ on some open set by $l \mapsto \frac{1}{l}$. Then she proves that the space Z obtained by a gluing of open subsets of $(V^+)^*$ and $(V^-)^*$ is a twistor space of a hypercomplex manifold. The holomorphic projection from Z to $\mathbb{C}P^1$ is given by the evaluation of elements of $(V^+)^*$ and $(V^-)^*$ on the constant function '1'.

Proof of Theorem 7. Firstly observe that in the case when $\mathcal{L} \cong \mathcal{O}(-1)$, the bundles $V^{(1,0)}$ and $V^{(0,1)}$ from our construction are defined fibrewise as spaces of 'affine' functions. We firstly have to check, that the spaces of 'affine' functions from our construction coincide with the spaces of affine function from the construction of Feix.

The 'affine' functions in our construction are defined by the tensor product connection of the connection induced from D on $\mathcal{O}(-1)$ with the Cartan connection. Note that the Cartan connection was given by the prolongation of $l \mapsto D^2l + r^Dl$ for $l \in \mathcal{O}(1)$. As a consequence the notions of affine functions coincide when the h-projective structure on S is given by a connection with r^D of type $(1, 1)$ as then r^D vanishes along the leaves of the foliations. But when the h-projective curvature is of type $(1, 1)$ and r^D is of type $(1, 1)$, the whole curvature of D is of type $(1, 1)$ which is the case considered by Feix. Hence we have shown that $V^+ = V^{(1,0)}$ and $V^- = V^{(0,1)}$. By construction it is straightforward that $\Phi^+ = \phi_{(1,0)}$ and $\Phi^- = \phi_{(0,1)}$ and hence that the gluings coincide.

Now using this and the Feix construction for the hyperkähler manifolds we conclude that when S is a Kähler manifold then M is hyperkähler. \square

Chapter 3

Construction of Einstein–Weyl 3-manifolds

The aim of this chapter is to show that using a real analytic surface S with a conformal Cartan geometry we can construct a twistor space of an asymptotically hyperbolic Einstein–Weyl manifold, such that its asymptotic end is the surface S . The motivation is the example of the hyperbolic 3-space: the boundary of the hyperbolic 3-space is the 2-sphere S^2 with the trivial conformal Cartan connection. Thus the hyperbolic 3-space is the solution of the boundary value problem for the Einstein–Weyl equation where S^2 with trivial Cartan geometry is the boundary condition. Similarly we can consider quotients of open subsets of S^2 which are examples of flat conformal Cartan geometries. Taking them as a boundary, we get that the solutions to these boundary value problems are hyperbolic 3-manifolds. Our aim is to show that, taking as a boundary any 2-manifold with conformal Cartan geometry, the solutions of this boundary value problem will be asymptotically hyperbolic Einstein–Weyl manifolds.

We will firstly use the results of the C. LeBrun [49] to show that the quotients by the S^1 action of the manifolds obtained in Chapter 2 are, in the case $4n = 4$, asymptotically hyperbolic Einstein–Weyl manifolds. After that, we will show how to obtain the minitwistor space T of an asymptotically hyperbolic Einstein–Weyl manifold B directly from the conformal Cartan connection. We will also construct a twistor space Z such that T is a quotient of Z by an S^1 action. Thus we will show that B admits a distinguished Gauduchon gauge.

3.1 Asymptotically hyperbolic Einstein–Weyl manifolds as quotients of self-dual conformal 4-manifolds with an S^1 action

In the previous chapter we constructed a quaternionic manifold M and we identified it with a neighbourhood of the zero section of the tangent bundle of an h-projective manifold S . We also showed that M admits an S^1 action given by unit scalar multiplication in the fibres. In this section, we will apply a result of LeBrun [49] to M in the 4-dimensional case, i.e., when M is a self-dual conformal 4-manifold with an S^1 action. We will show that the quotient of M by an S^1 action is an asymptotically hyperbolic Einstein–Weyl space (see Definition 8) with S as the asymptotic end. Therefore, we will show that a manifold S with a real analytic Möbius structure, is a boundary condition of a well-posed Einstein–Weyl equation.

We will apply the following result of LeBrun [49]. Note that in [49] LeBrun stated and proved this result for a more general class of circle actions, which he called docile actions. Recall that a semi-free action of a group G is such that the stabiliser is either trivial or the whole G .

Lemma 5 ([49]). *Suppose that (M, g) is a self-dual manifold with an S^1 action which has a non-empty surface S of fixed points, is semi-free and does not have isolated fixed points. Let B be a maximal smooth manifold (without boundary) contained in $Y = M/S^1$ i.e., the smooth point subset of Y . Then the Einstein–Weyl structure defined by the Jones–Tod correspondence (see Section 1.5.4) on B is an asymptotically hyperbolic Einstein–Weyl structure.*

Proof. We will give an idea of the proof. The neighbourhood of the fixed point surface S can be identified with a neighbourhood of the zero section of the normal bundle of S in M , and, by [42], the metric there takes the form

$$g = dr^2 + r^2(1 + f)^2(dt + \mu)^2 + g_S,$$

where r is a radius coordinate meaning the distance from S , t is an angular coordinate on the fibres of the normal bundle, μ is a 1-form and g_S is a metric on S depending on r . Moreover, f, μ, g_S depend on r and on local coordinates

of S and are even with respect to r . We also have that $f = 0$ when $r = 0$.

The gauge from the Jones–Tod correspondence (see [42]) is given by the metric

$$h = \frac{dr^2 + g_S}{r^2(f+1)^2}$$

and the smooth one form

$$v = \frac{*(dt + \mu) \wedge d\mu}{(1+f)^2} = *_h d\mu.$$

We have that $*_h d\mu = r(1+f) *_h d\mu$ thus

$$v = r(1+f) *_h d\mu,$$

which is a smooth 1-form vanishing along S . To sum up,

$$\tilde{h} = dr^2 + g_S$$

is a Riemannian metric extending to the boundary $r = 0$ (see Definition 8), $r(1+f)$ is a defining function of $r = 0$, v is a smooth 1-form vanishing along $r = 0$ and the Einstein–Weyl structure on M is given by the Weyl connection defined by (h, v) . \square

In Section 2.6.4 we showed, that the S^1 action comes from a local holomorphic \mathbb{C}^\times action on the twistor space, which means that it must be conformal (see for example [42]). It is also clearly semi-free and the zero section is the fixed point set. Hence we obtain the following theorem.

Theorem 8. *The quotient by the S^1 action of the self-dual conformal 4-manifold obtained by the generalised Feix–Kaledin construction for quaternionic manifold (see Chapter 2) is an asymptotically hyperbolic Einstein–Weyl manifold with S as an asymptotic end.* \square

3.2 Construction of the minitwistor space of Einstein–Weyl 3-manifold

In this section, from a complex surface Σ with a Cartan geometry, we will construct a complex surface T containing family of line pairs parametrised by points in Σ . We will also assume that the complex Cartan geometry on Σ arises as a complexification of the Cartan geometry on the real manifold S such that $S^c = \Sigma$ and thus we will obtain a real structure on T induced by the complexification. Note that the real submanifold $\Sigma_{\mathbb{R}}$ of Σ can be identified with S . This is because Σ looks locally like $S \times \bar{S}$ and the real submanifold is the diagonal (see Section 2.3). We will show that the real structure on T preserves the line-pairs corresponding to the real submanifold $\Sigma_{\mathbb{R}}$ of Σ . To show that T is a minitwistor space of an asymptotically hyperbolic Einstein–Weyl manifold we would need to show that there exists a minitwistor line in T , i.e., a projective line with normal bundle $\mathcal{O}(2)$. To do this, first we construct a twistor space Z of a self-dual conformal 4-manifold such that T is a quotient of Z by a local \mathbb{C}^\times action. Then we deduce, that images by the projection from Z to T of some real twistor lines are real minitwistor lines. As we will show that S is the fixed point set of the \mathbb{C}^\times action on Z , we conclude, that T is a minitwistor space of an asymptotically hyperbolic Einstein–Weyl manifold with the asymptotic boundary S .

Note that the fact that the space T arises in a natural way as a quotient of some twistor space of a self-dual conformal 4-manifold means, that the Einstein–Weyl manifolds arising from our construction admit a distinguished abelian monopole (compare [42]).

3.2.1 Motivating Example

In this section, by $\mathbb{C}\mathbb{P}^n$ we denote the space $\mathbb{P}(\mathbb{C}^{n+1})$ and by $(\mathbb{C}\mathbb{P}^n)^*$ we denote the space $\mathbb{P}((\mathbb{C}^{n+1})^*)$. We will discuss now a motivating example for the construction of the twistor space of asymptotically hyperbolic Einstein–Weyl manifold from a 2-manifold equipped with a conformal Cartan connection.

Firstly, we will discuss some geometric properties of the space $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$.

Example 3. Let $T \cong (\mathbb{C}\mathbb{P}^1)^* \times (\mathbb{C}\mathbb{P}^1)^*$ be a quadric in $(\mathbb{C}\mathbb{P}^3)^*$. On T we have two

families of lines with normal bundle O called α -lines and β -lines. α -lines do not intersect each other, β -lines do not intersect each other and any α -line intersects any β -line in exactly one point. Moreover, any point in T belongs to exactly one α -line and one β -line. Thus we can consider a family of line pairs parametrised by points in T . Now, note that the projective tangent space to T at a point t (i.e., the projective plane in $(\mathbb{C}\mathbb{P}^3)^*$ through t such that its tangent space at t is the tangent space to T at t) intersects T in the line pair intersecting at the point t . Thus any line pair is determined by the intersection of the tangent space through their intersecting point with T . Thus the moduli space of line pairs of T coincides with the space $T^* \subseteq (\mathbb{C}\mathbb{P}^3)$ of tangent spaces to T . This is a quadric in $(\mathbb{C}\mathbb{P}^3)$.

The space of deformations of hyperplanes tangent to T is the whole space $\mathbb{C}\mathbb{P}^3$ viewed as the space of all hyperplanes in $(\mathbb{C}\mathbb{P}^3)^*$. By the adjunction formula (see [35]) the hyperplanes which are non-tangent to T intersect T in curves which have normal bundles isomorphic to $O(2)$.

Remark 26. In the above example we constructed $O(2)$ -curves as ‘deformations’ of the line pairs. Note that as the line pairs are singular, to be able to obtain non-singular deformations we had to consider non-singular manifolds uniquely determining the line pair. Then we deformed the non-singular manifolds to obtain a non-singular manifold determining a non-singular twistor line. In the general construction we will have a similar situation; to be able to construct non-singular twistor lines we will have to construct some non-singular surfaces determining line pairs.

Now we will interpret the above correspondence using a conformal Cartan geometry on a quadric Σ .

Let $\Sigma = \{[z] \in \mathbb{C}\mathbb{P}^3 \mid z_0z_1 + z_2z_3 = 0\}$ with the following complex Cartan geometry $(V, \langle, \rangle, \Lambda, \mathcal{D})$ (see definition 16):

- $V = \Sigma \times \mathbb{C}^4$ is a rank 4 trivial vector bundle,
- $\langle x, y \rangle = x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2$ is an inner product on \mathbb{C}^4 ,
- $\Lambda := \{([z], w) \in V \mid w = \alpha z \text{ for some } \alpha \in \mathbb{C}\}$ is a rank 1 null subbundle of V ,
- \mathcal{D} is the standard flat connection on the trivial bundle V .

Proposition 33. Σ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the above complex Cartan connection arise as a complexification of (real) Cartan connection on the real 2-sphere S^2 .

Proof. The map

$$\begin{aligned} \Upsilon: \mathbb{CP}^1 \times \mathbb{CP}^1 &\longrightarrow \Sigma \subseteq \mathbb{CP}^3 \\ ([u_0, u_1], [v_0, v_1]) &\mapsto [z] = [-u_0v_0, u_1v_1, u_0v_1, u_1v_0] \end{aligned}$$

is the required isomorphism. □

Observe that for any $[z] = \Upsilon([u_0, u_1], [v_0, v_1])$ the map Υ defines a pair of lines by

$$\alpha_{[z]} := \Upsilon([u_0, u_1], [x, y]) = \{[-u_0x, u_1y, u_0y, u_1x] \in \Sigma, : [x, y] \in \mathbb{CP}^1\},$$

$$\beta_{[z]} := \Upsilon([x, y], [v_0, v_1]) = \{[-xv_0, yv_1, xv_1, yv_0] \in \Sigma, : [x, y] \in \mathbb{CP}^1\}.$$

Now note that $\langle \cdot, \cdot \rangle$ gives an isomorphism between \mathbb{C}^4 and $(\mathbb{C}^4)^*$ by $z \mapsto \langle z, \cdot \rangle$ and hence we have an isomorphism between $\Lambda^0 \subset V^*$ - the annihilator of Λ and $\Lambda^\perp \subset V$. Using this, we abuse the notation and define

$$\Lambda_{([z])}^\perp = \{\langle b, \cdot \rangle \in (\mathbb{C}^4)^* : \langle b, z \rangle = 0\}.$$

We have that

$$\Lambda_{[z]}^\perp = \{a \in (\mathbb{C}^4)^* : a_1(z_0) + a_0(z_1) + a_3(z_2) + a_2(z_3) = 0\}.$$

Furthermore, we have that $\langle \alpha_{[z]}, \cdot \rangle \subseteq \mathbb{P}(\Lambda_{[z]}^\perp)$ and $\langle \beta_{[z]}, \cdot \rangle \subseteq \mathbb{P}(\Lambda_{[z]}^\perp)$. The induced degenerated inner product on $\Lambda_{[z]}^\perp$ defines two null planes $U_{[z]}^+ \subseteq \Lambda^\perp$ and $U_{[z]}^- \subseteq \Lambda^\perp$. They are given by the equation

$$0 = \langle a, a \rangle = -a_0a_1 + a_2a_3,$$

for $a \in \Lambda^\perp$.

Remark 27. The straightforward calculations show, that the null planes have the property that $\mathbb{P}(U_{[z]}^+) = \langle \alpha_{[z]}, \cdot \rangle$ and $\mathbb{P}(U_{[z]}^-) = \langle \beta_{[z]}, \cdot \rangle$.

Definition 47. We define the following fibre bundles over Σ :

- F^+ to be a bundle of projective lines defined fibrewise by $F_{[z]}^+ := \mathbb{P}(U_{[z]}^+)$,
- F^- to be a bundle of projective lines defined fibrewise by $F_{[z]}^- := \mathbb{P}(U_{[z]}^-)$
- F to be a bundle of pairs of projective lines defined fibrewise by $F_{[z]} := F_{[z]}^+ \cup F_{[z]}^-$.

Corollary 6. As for any $[z] \in \Sigma$ we have that $F_{[z]} \subseteq (\mathbb{C}\mathbb{P}^3)^*$, we can consider the space $T = \bigcup_{[z]} F_{[z]}$. Using Remark 27, we get that T is a quadric in $(\mathbb{C}\mathbb{P}^3)^*$ dual to Σ .

As a result we have the following diagram

$$\begin{array}{ccc} & F \subset \Sigma \times \Sigma^* & \\ \pi \swarrow & & \searrow \tilde{\pi} \\ \Sigma & & T \end{array} .$$

We already know that the fibre of F over $[z] \in \Sigma$ is dual to the line pair through $[z]$, hence

$$F = \{([z], \langle [a], \cdot \rangle) \in \Sigma \times \Sigma^* : [a] \in (\alpha_{[z]} \cup \beta_{[z]})\}.$$

We would like to understand what are fibres of F over T .

Observe that on Σ we have two families of null curves, namely the α lines and the β lines. Denote by t^+ the distribution of the tangent bundle defining the foliation by α -lines and by t^- the distribution defining the foliation by β -lines.

Proposition 34. We can horizontally lift α -lines to F^+ and β lines to F^- .

Proof. Let $u \in \Gamma U^+$ and $X \in \Gamma t^+$. As $U^+ \subseteq \Lambda^+$ is totally null, for any $\sigma \in \Gamma \Lambda$ we have that

$$\langle \mathcal{D}_X u, \sigma \rangle = d_X \langle u, \sigma \rangle - \langle u, \mathcal{D}_X \sigma \rangle \in \langle U^+, U^+ \rangle = 0.$$

Moreover, differentiating the equation $\langle u, u \rangle = 0$ we get that

$$\langle \mathcal{D}_X u, u \rangle = 0,$$

thus

$$\mathcal{D}_X u \in (U^+)^{\perp} = U^+.$$

□

Observe that $F_{[z]}^+ \cap F_{[z]}^- = \mathbb{P}(\Lambda^*)$ is a point for any $[z] \in \Sigma$. This gives a foliation on F by a family of intersecting line pairs.

Proposition 35. *T is isomorphic with the leaf space of the foliation of F by horizontal lifts of null curves. The fibres of F over T are pairs of null lines.*

Proof. Any horizontal lift of a line pair is uniquely given by the point in which both lines intersect $\mathbb{P}(\Lambda^*) \cong \Sigma^*$. Now choose $b \in T$. It is given by horizontal lift b_F to F of a line pair $(l_1, l_2) \subset \Sigma$ such that b intersect $\mathbb{P}(\Lambda^*)$ over $[z] \in \Sigma$ and $l_1 \cap l_2 = [z]$. By definition of the horizontal lift, for any $[a] \in l_1 \cup l_2$ there exists a point $b_{[a]} \in b_F \in F$. Hence the fibre of F over b is $l_1 \cup l_2$. □

Remark 28. Note that as $T = \Sigma^*$ is a quadric in $(\mathbb{C}\mathbb{P}^3)^*$, it also admits two families of lines: they are duals to α and β -lines from Σ . Now observe that $\tilde{\pi}(b_F)$ from the above proof is the pair of lines in T which intersect in b . Those lines are the dual lines to l_1 and l_2 and hence the fibre of F over $[b] \in T$ is the pair of lines which are dual to the pair of projective lines intersecting in $[b]$.

Observation 3. We have shown that in the above example the situation is symmetrical: T is dual to Σ and the fibres of F over T are duals to fibres of F over Σ . This symmetry does not exist any more if we proceed with the construction for $U \subsetneq \Sigma$. Then the null curves in Σ are pieces of lines but the fibres of F over Σ are still pairs of projective lines. As a consequence, we obtain that T is a subspace of quadric in $(\mathbb{C}\mathbb{P}^3)^*$ consisting of families of projective lines and $U^* \subsetneq T$. This changes the situation from local (a manifold containing pieces of lines) to global (a manifold containing whole projective lines).

Remark 29. In this example we constructed F as a projective subbundle of $\mathbb{P}(V^*)$. This approach was useful to see the geometry of the situation. However, we observed that $\langle \cdot, \cdot \rangle$ induces an isomorphism between V and V^* . Using this, we will simplify the notation in the general construction and will construct F as a fibre subbundle of $\mathbb{P}(V)$.

3.2.2 Construction of a complex surface containing family of line pairs

Let Σ be a complex surface with a complex Cartan connection $(V, \langle \cdot, \cdot \rangle, \Lambda, \mathcal{D})$ (see Section 1.4.4) which is a complexification of a real surface S with a Cartan connection.

The general idea is as follows. We start by forming a fibre bundle $F = F^+ \cup F^-$ from V such that the fibres of F over Σ are pairs of projective lines (defined by the projectivisation of the pair of null planes in Λ^\perp). We also have two families of null curves in Σ (through each point pass exactly 2 null curves). The Cartan condition enables us to lift them horizontally to F such that curves from one family are lifted to F^+ and curves from second family to F^- . After a possible restriction of Σ , these horizontal lifts will define foliations of F^+ and F^- . In some cases it may also be possible to define a foliation of F by pairs of lifted curves and then we may define T as a leaf space of the foliation. However, in general we will have to consider two manifolds T^+ and T^- defined as leaf spaces of foliations of F^+ and F^- and glue them locally. In both cases, we finally obtain a complex surface T with two families of projective lines (defined by fibres of F^+ and F^- over Σ). On the subset on which we glue (isomorphic with Σ) through each point go exactly two lines - one from each family (through other points of the manifold goes only one line). The real structure from the complexification will induce the real structure on T preserving some of the line pairs.

This construction changes the situation from local (complex manifold Σ with pieces of null curves in it) to global (manifold T with a family of pairs of projective lines).

Null curves and their horizontal lifts Now define the following objects:

- U^+ and U^- - the null plane subbundles of $\Lambda^\perp \subset V$ defined by the degenerate inner product on Λ^\perp . Note that $\Lambda \subset \Lambda^\perp$,
- t^+ and t^- - the line subbundles of the tangent bundle $T\Sigma$ such that

$$\delta(\Lambda \otimes t^+) = U^+/\Lambda \quad \text{and} \quad \delta(\Lambda \otimes t^-) = U^-/\Lambda,$$

where δ denotes the isomorphism between $T\Sigma \otimes \Lambda$ and Λ^\perp/Λ in the Cartan

condition,

- C^+ and C^- - families of curves in Σ defined by t^+ and t^- (by requiring that t^\pm is a tangent space to curves from C^\pm families),
- Σ^+ denotes the leaf space of C^+ curves and Σ^- the leaf space of C^- curves,
- $F^+ = \mathbb{P}(U^+)$, $F^- = \mathbb{P}(U^-)$ - fibre bundles over Σ .

Proposition 36. *The connection \mathcal{D} satisfies $\mathcal{D}_X(U^+) \subseteq U^+$ for $X \in t^+$ and $\mathcal{D}_Y(U^-) \subseteq U^-$ for $Y \in t^-$.*

Proof. The same as the proof of Proposition 34. □

Corollary 7. *The above proposition implies that \mathcal{D} induces a connection on F^+ along curves from C^+ and on F^- along curves from C^- . Hence we can horizontally lift curves from C^+ to F^+ and from C^- to F^- .*

We restrict Σ to an open subset on which curves from families C^+ and C^- are simply connected .

Leaf spaces of foliations of horizontal lifts of null curves

Proposition 37. *Locally the horizontal lifts of curves from C^+ and C^- to F^+ and F^- define foliations of the spaces F^+ and F^- respectively. Locally, the leaf spaces of the foliations are manifolds.*

Proof. We will prove the proposition for C^+ curves. The proof for C^- curves is analogous. Observe that because C^+ is a family of integral curves for t^+ distribution, we can restrict Σ such that the curves from C^+ do not intersect each other. Horizontal lifts of any curve $c^+ \in C^+$ are sections of F^+ over c^+ . Hence horizontal lifts of two different curves do not intersect. Moreover, from properties of horizontal lifts, locally two different horizontal lifts of a c^+ do not intersect each other.

Hence the horizontal lifts of C^+ curves define a foliation of F^+ and thus the leaf space of the foliation is (locally in Σ) a Hausdorff manifold. □

Definition 48. *Denote by T^+ the leaf space of the foliation of F^+ by horizontal lifts of curves from C^+ and by T^- the leaf space of the foliation of F^- by horizontal lifts from C^- .*

The lifted curves are transversal to the fibres of F^+ and F^- respectively. Thus we can further restrict Σ such that any curve from T^+ (T^- respectively) intersects any fibre of F^+ (F^- respectively) at most once. Moreover, through each point of a particular fibre of F^+ (F^- respectively) goes exactly one curve from T^+ (T^- respectively). Recall also that all curves intersecting a particular fibre of F^+ or F^- arise as horizontal lifts of one curve from the C^+ or C^- family respectively. As a consequence we obtain the following proposition.

Proposition 38. *Any curve $c^+ \in C^+$ defines a projective line in T^+ given by horizontal lifts of c^+ . Any curve $c^- \in C^-$ defines a projective line in T^- given by horizontal lifts of c^- . \square*

As an immediate consequence of the above proposition we obtain that T^+ and T^- are foliated by projective lines.

Definition 49. *For $c^+ \in C^+$ we denote by $l_{c^+}^+ \subset T^+$ the projective line given by horizontal lifts of c^+ . For $c^- \in C^-$ we denote by $l_{c^-}^- \subset T^-$ the projective line given by horizontal lifts of c^- .*

As any $z \in \Sigma$ defines exactly one curve c^+ from C^+ and c^- from C^- such that $z \in c^+$ and $z \in c^-$ we set $l_z^+ := l_{c^+}^+$ and $l_z^- := l_{c^-}^-$.

Note that $\mathbb{P}(\Lambda)$ is a section of both F^+ and F^- .

Proposition 39. *The elements of T^+ and T^- , i.e., the horizontal lifts of null curves which intersect $\mathbb{P}(\Lambda)$ are transversal to $\mathbb{P}(\Lambda)$.*

Proof. From the Cartan condition we have that $\mathcal{D}|_\Lambda$ is an isomorphism between $T\Sigma \otimes \Lambda$ and Λ^\perp/Λ . Hence $\mathcal{D}|_\Lambda$ is 0 in Λ^\perp/Λ only for the zero section and consequently the induced connections on F^+ and F^- have the property that non-zero elements of $T\Sigma \otimes \mathbb{P}(\Lambda)$ leave $\mathbb{P}(\Lambda)$. In particular the horizontal lifts of curves $c^+ \in C^+$ and $c^- \in C^-$ must be transversal to $\mathbb{P}(\Lambda)$ \square

Definition 50. *For each $z \in \Sigma$ we have that $F_z^+ \cap F_z^- = \mathbb{P}(\Lambda)_z$ thus we can define a gluing of T^+ with T^- by identifying those lifted curves in F^+ and F^- which intersect at $\mathbb{P}(\Lambda)_z = F_z^+ \cap F_z^-$ for some $z \in \Sigma$:*

$$T := T^+ \bigsqcup_{\sim} T^-,$$

where for $t^+ \in T^+$ and $t^- \in T^-$ we have $t^+ \sim t^-$ if $t^+ \cap t^- = \{\mathbb{P}(\Lambda)_z\}$ for some $z \in \Sigma$.

Note that as lifted curves are transversal to $\mathbb{P}(\Lambda)$, by possible further restriction of Σ we can ensure that any element of T^+ is glued with at most one element of T^- and vice versa. Hence T arises as a gluing of manifolds T^+ and T^- on some set. Without loss of generality we can assume that this set is open and connected.

Remark 30. Recall that the Cartan connection on Σ is a complexification of a Cartan connection on S . Let θ be a real structure on Σ coming from complexification (compare Section 2.3). As the Cartan connection $(V, \langle \cdot, \cdot \rangle, \Lambda, \mathcal{D})$ comes from complexification, θ interchanges t^+ and t^- and thus C^+ and C^- curves. Moreover, the real structure θ on V defines also an anti-holomorphic isomorphism between F^+ and F^- which induces an anti-holomorphic isomorphism between T^+ and T^- . For simplification the anti-holomorphic isomorphisms defined above will be also denoted by θ .

Recall from Section 2.3 that $\Sigma_{\mathbb{R}}$ denotes the real submanifold of Σ coming from the complexification. Then for any $z \in \Sigma_{\mathbb{R}}$, the curves $s^+ \in T^+$ and $s^- \in T^-$ that intersect $\mathbb{P}(\Lambda)$ over z are interchanged by θ .

Proposition 40. *The open subset on which we glue can be identified with $\mathbb{P}(\Lambda)$ and thus with Σ .*

After possible restriction of Σ we get that every projective line from T^+ (defined as in Proposition 38) intersects every projective line in T^- in exactly one point.

Proof. For any point of $\mathbb{P}(\Lambda)$ we have exactly one element of T^+ and one element of T^- which intersect in it. As $\mathbb{P}(\Lambda)$ is a section over Σ this proves the first part of the proposition.

To prove the second part, firstly observe that any curve in C^+ intersects any nearby curve from C^- and vice-versa (as t^+ and t^- are transversal to each other). We can restrict Σ to a \pm -convex region i.e., to a region in which every curve from C^+ intersects every curve from C^- and vice-versa. Pick a projective line $l_{c^+}^+ \in T^+$ given by horizontal lifts of some curve $c^+ \in C^+$, and a projective line $l_{c^-}^- \in T^-$ given by horizontal lifts of some curve $c^- \in C^-$. Denote by $z \in \Sigma$ the intersection point of c^+ and c^- . Then the intersection point of $l_{c^+}^+$ with $l_{c^-}^-$ is the element of T given by the lift of c^+ to $\mathbb{P}(\Lambda)_z$ or equivalently by the lift of c^- to $\mathbb{P}(\Lambda)_z$. \square

Proposition 41. *T is a Hausdorff manifold.*

Proof. Let $x, y \in T$ be non-equal points. If both x and y are points of T^+ or respectively T^- , then obviously we can find separating neighbourhoods of x and y . Suppose that $x \in T^+ \setminus \Sigma \subseteq T$ and $y \in T^- \setminus \Sigma \subseteq T$. There exist projective lines l_x and l_y such that $x \in l_x \subset T^+$ and $y \in l_y \subset T^-$. The lines l_x and l_y intersect in a point $z_0 \in \Sigma \subset T$ different from x and y . As $z_0 \in \Sigma$, we can choose an open subset $\tilde{\Sigma} \subset \Sigma$ separating x and y from z_0 , i.e. such that x and y do not belong to $\tilde{\Sigma} \cup \partial\tilde{\Sigma}$. We can also require that $\tilde{\Sigma}$ is \pm convex, i.e. that on $\tilde{\Sigma}$, every C^+ curve intersect every C^- curve. Let $\hat{T}^+ \subset T^+$ be an union of all projective lines from T^+ which intersect $\tilde{\Sigma}$ and let $\hat{T}^- \subset T^-$ be an union of all projective lines from T^- which intersect $\tilde{\Sigma}$. Define open subsets of \hat{T}^+ and \hat{T}^- by

$$\tilde{T}^+ := \hat{T}^+ \setminus (\tilde{\Sigma} \cup \partial\tilde{\Sigma}),$$

$$\tilde{T}^- := \hat{T}^- \setminus (\tilde{\Sigma} \cup \partial\tilde{\Sigma}).$$

As $\hat{T}^+ \cap \hat{T}^- = \tilde{\Sigma}$ we have that $\tilde{T}^+ \cap \tilde{T}^- = \emptyset$. By definition $x \in \tilde{T}^+$ and $x \in \tilde{T}^-$ hence \tilde{T}^+ and \tilde{T}^- are separating neighbourhoods of x and y . \square

Remark 31. As a consequence of Remark 30, we can choose T^+ and T^- such that the manifold T admits a real structure canonically induced by the complexification of the Cartan geometry. The real structure interchanges projective lines from T^+ with projective lines from T^- and on $\Sigma \subset T$ it gives back the real structure θ from complexification. To simplify the notation we denote the real structure on T by θ .

Coordinate description of the gluing Now we will describe the leaf spaces of foliations T^+ and T^- and the gluing procedure in another way. We will also introduce coordinates on T^+ and T^- .

Firstly recall that for any curve $c^\pm \in C^\pm$, horizontal lifts of c^\pm to F^\pm are sections of $F^\pm|_{c^\pm}$. Choose two intersecting curves $c_0^+ \in C^+$ and $c_0^- \in C^-$. As the curves from C^+ and C^- are transversal to each other, by possible restriction of Σ , lifts of curves from C^+ intersect $F^+|_{c_0^-}$ in exactly one point. Similarly, lifts of curves from C^- intersect $F^-|_{c_0^+}$ in exactly one point. As a consequence we obtain the following proposition.

Proposition 42. *With possible restriction of Σ , the manifolds T^+ and T^- can be identified with $F^+|_{c_0^-}$ and $F^-|_{c_0^+}$ respectively. \square*

The gluing of $F^+|_{c_0^-}$ with $F^-|_{c_0^+}$ induced from the gluing of T^+ and T^- can be described using the following coordinates. Let z, \tilde{z} be local coordinates on Σ such that curves from C^+ are given by $z = \text{const}$ and those from C^- by $\tilde{z} = \text{const}$. Suppose moreover, that c_0^+ is given by $z = 0$ and c_0^- by $\tilde{z} = 0$. This induces local coordinates near $\mathbb{P}(\Lambda)$ on $F^+|_{c_0^-}$ and $F^-|_{c_0^+}$ in the following way. The coordinate in the base direction i.e., on c_0^- and c_0^+ respectively is given by restriction of coordinates from Σ to submanifolds c_0^- and c_0^+ . Hence the coordinate in the base direction of $F^+|_{c_0^-}$ is z and of $F^-|_{c_0^+}$ is \tilde{z} . Now we will construct coordinates w and \tilde{w} on the gluing part in the fibre directions of $F^+|_{c_0^-}$ and $F^-|_{c_0^+}$ respectively.

Let s^+ be a lift of some curve from C^+ . If s^+ intersect $F^+|_{c_0^-}$ in the point that is close enough to $\mathbb{P}(\Lambda)$ (i.e., when s^+ is in the gluing part), then there exists a unique point $(z = a, \tilde{z} = b) \in \Sigma$ such that s^+ intersects $\mathbb{P}(\Lambda)$ in this point. We define coordinate w by requiring that $w(s^+) = b$. In other words, the coordinates on a neighbourhood of $\mathbb{P}(\Lambda)$ in $F^+|_{c_0^-}$ are transported by curves from C^+ from Σ by checking where on Σ the curves intersect $\mathbb{P}(\Lambda)$. Analogously, we define coordinate \tilde{w} on $F^-|_{c_0^+}$. The following proposition is an immediate consequence of the way we constructed the coordinates.

Proposition 43. *The gluing $F^+|_{c_0^-}$ and $F^-|_{c_0^+}$ is given by $z = \tilde{w}$, $w = \tilde{z}$. \square*

Recall that projective lines on T^+ or T^- are given by horizontal lifts of a fixed null curve. This corresponds to fibres of $F^+|_{c_0^-}$ and $F^-|_{c_0^+}$ respectively. This implies the following corollary.

Corollary 8. *The normal bundle of any projective lines on T defined by horizontal lifts of a null curve is isomorphic to \mathcal{O} .*

3.2.3 Construction of a twistor space of a self-dual conformal 4-manifold

In this section we will show that there is a canonical construction of a 3-dimensional twistor space Z from the Cartan connection on Σ such that T arises as a canonical quotient of Z . We will use later the twistor lines to construct $O(2)$ - curves on T .

Constriction of halves of a twistor space Recall (see Section 1.4.4) that the null subbundle Λ decomposes into a tensor product of line bundles $\Lambda = \Lambda^+ \otimes \Lambda^-$ where Λ^+ is trivial along the C^+ curves and Λ^- is trivial along C^- curves. Note that Λ^+ is a pull back by π_{Σ^+} of some line bundle on Σ^+ and Λ^- is a pull back by π_{Σ^-} of some line bundle on Σ^- .

Definition 51. *Define*

$$\tilde{U}^+ := U^+ \otimes (\Lambda^+)^* \otimes (\Lambda^+)^* \quad \text{and} \quad \tilde{U}^- := U^- \otimes (\Lambda^-)^* \otimes (\Lambda^-)^*.$$

Observe that $\Lambda^- \otimes (\Lambda^+)^* \subset \tilde{U}^+$ and $\Lambda^+ \otimes (\Lambda^-)^* \subset \tilde{U}^-$.

Using this and Proposition 36 we obtain the following corollary.

Corollary 9. *Using the connection \mathcal{D} we can horizontally lift C^+ curves to \tilde{U}^+ and C^- curves to \tilde{U}^- .*

Similarly like for lifts of curves to F , locally in Σ , horizontal lifts of C^+ curves (and C^- respectively) do not intersect and thus define foliations of \tilde{U}^+ and \tilde{U}^- .

Definition 52. *Denote by V^+ the leaf space of horizontal lifts of C^+ curves to \tilde{U}^+ . Denote by V^- the leaf space of horizontal lifts of C^- curves to \tilde{U}^- .*

Analogously like for the spaces T^+ and T^- , we can prove that the spaces V^+ and V^- are manifolds.

Proposition 44. *With possible restriction of Σ , the manifolds V^+ and V^- can be identified with $\tilde{U}^+|_{c_0^-}$ and $\tilde{U}^-|_{c_0^+}$ respectively.*

Proof. Analogous to the proof of Proposition 42. □

As \tilde{U}^+ arises from U^+ by tensoring by a line bundle we get that the projective bundle $\mathbb{P}(\tilde{U}^+)$ is equal to $\mathbb{P}(U^+)$ (see Remark 11).

Remark 32. We have that $\mathbb{P}(\tilde{U}^+) = F^+$ and $\mathbb{P}(\tilde{U}^-) = F^-$.

Remark 33. The real structure θ induces an anti-holomorphic isomorphism θ between V^+ and V^- (compare Remarks 30 and 31). We define an anti-holomorphic isomorphism between V^+ and V^- by

$$\theta_Z := (-\text{id}) \circ \theta.$$

We will show later that θ_Z induces a real structure on Z which on real twistor lines is antipodal. Observe that for z not in the zero section $\mathbb{P}(\theta_Z(z)) = \theta(\mathbb{P}(z))$.

Gluing In this paragraph we will glue open subsets of the manifolds V^+ and V^- such that the obtained manifold will be Hausdorff. By applying the Proposition 39 we obtain the following proposition.

Proposition 45. *The non-zero curves from V^+ and V^- are transversal to $\Lambda^- \otimes (\Lambda^+)^*$ and $\Lambda^+ \otimes (\Lambda^-)^*$ respectively. \square*

As a consequence, locally in Σ , the non-zero curves from V^+ (V^- respectively) intersect $\Lambda^- \otimes (\Lambda^+)^*$ ($\Lambda^+ \otimes (\Lambda^-)^*$ respectively) in at most one point. Hence through any point of $(\Lambda^- \otimes (\Lambda^+)^*)^\times$ and $(\Lambda^+ \otimes (\Lambda^-)^*)^\times$ goes exactly one curve from V^+ or V^- .

As $\Lambda^+ \otimes (\Lambda^-)^* = (\Lambda^- \otimes (\Lambda^+)^*)^*$ we can define a gluing \hat{Z} of V^+ and V^- as follows:

$$\hat{Z} := V^+ \bigsqcup_{\sim} V^-,$$

where for $v^+ \in V^+$ and $v^- \in V^-$ we have $v^+ \sim v^-$ if $v^+ \cap (\Lambda^- \otimes (\Lambda^+)^*) = (v^- \cap (\Lambda^+ \otimes (\Lambda^-)^*))^*$.

Coordinate description In this paragraph we will discuss what the gluing region of \hat{Z} looks like in more detail. We will also construct local coordinates on the gluing parts of V^+ and V^- .

Firstly recall that in Proposition 44 we have identified manifolds V^+ and V^- with $\tilde{U}^+|_{c_0^-}$ and $\tilde{U}^-|_{c_0^+}$ respectively. Also in Proposition 42 we identified T^+ with $F^+|_{c_0^-}$ and T^- with $F^-|_{c_0^+}$. The gluing region $\mathbb{P}(\Lambda)$ was a union of non-empty open subsets of fibres of $F^+|_{c_0^-}$ and $F^-|_{c_0^+}$ respectively.

Definition 53. *Denote by \tilde{W}^+ the gluing region in $F^+|_{c_0^-}$ and set $\tilde{W}_z^+ := F_z^+ \cap \tilde{W}^+$. Denote by \tilde{W}^- the gluing region in $F^-|_{c_0^+}$ and set $\tilde{W}_z^- := F_z^- \cap \tilde{W}^-$.*

Observe that the bundles \tilde{U}^+ and \tilde{U}^- can be naturally identified with the blow-down of $\Delta_{\tilde{U}^+}$ and $\Delta_{\tilde{U}^-}$ - the corresponding tautological bundles along the fibres over F^+ and F^- respectively (see Section 2.3).

Definition 54. Denote by $W^+ \subset \tilde{U}^+$ the image by blow-down of $\Delta_{\tilde{U}^+|_{\tilde{W}^+}}$. Denote by $W^- \subset \tilde{U}^-$ the image by blow-down of $\Delta_{\tilde{U}^-|_{\tilde{W}^-}}$.

Proposition 46. $W^+ \setminus \underline{0}$ is the gluing region of $V^+ = \tilde{U}^+|_{c_0^-}$. $W^- \setminus \underline{0}$ is the gluing region of $V^- = \tilde{U}^-|_{c_0^+}$.

Proof. Let s^+ be a horizontal lift of a curve from C^+ family to $(\tilde{U}^+)^*$. Then $s^+ \in W^+$ if and only if $\mathbb{P}(s^+) \in \tilde{W}^+$ which, by Definition 54, completes the proof. \square

Using the properties of blow-down (see Section 2.3) we obtain the following corollary.

Corollary 10. For any $z \in c_0^-, \tilde{z} \in c_0^+$ we have that W_z^+ and $W_{\tilde{z}}^-$ are cones in $(\tilde{U}^+)_z$ and $(\tilde{U}^-)_{\tilde{z}}$ with vertex at 0.

Recall that in Section 3.2.2 we constructed coordinates (z, w) and (\tilde{w}, \tilde{z}) on gluing regions \tilde{W}^+ of T^+ and \tilde{W}^- of T^- . Using them, we will now construct local coordinates on W^+ and W^- .

Let (z, \tilde{z}, λ) be coordinates on local trivialisation of $\Lambda^- \otimes (\Lambda^+)^*$ such that (z, \tilde{z}) are coordinates on the base Σ and λ is a coordinate in the direction of fibres. Recall that the coordinates (z, w) on \tilde{W}^+ were induced from coordinates (z, \tilde{z}) on Σ by the intersection point of a curve $s^+ \in \tilde{W}^+$ with Λ . We can induce coordinates (z, w, λ) on $\Delta_{\tilde{U}^+|_{\tilde{W}^+}}$ in the following way. Let s^+ be a non-zero element of $\Delta_{\tilde{U}^+|_{(a,b)}}$, where (a, b) denotes a point of \tilde{W}^+ written using coordinates (z, w) . By definition of the tautological line bundle (see Section 2.3), $\Delta_{\tilde{U}^+|_{(a,b)}}$ can be canonically identified with a vector subspace of $(\tilde{U}^+)_a$. Then the curve in V^+ corresponding via this identification to s^+ intersects $\Lambda^- \otimes (\Lambda^+)^*$ in the point $(z = a, \tilde{z} = b, \lambda = l_0)$ for some $l_0 \neq 0$. Then the coordinates of s^+ are defined to be $(z = a, w = b, \lambda = l_0)$. The coordinates defined above extend through the zero section of $\Delta_{\tilde{U}^+|_{\tilde{W}^+}}$.

Analogously choosing coordinates $(z, \tilde{z}, \tilde{\lambda})$ on local trivialisation of $\Lambda^+ \otimes (\Lambda^-)^*$ we define coordinates $(\tilde{w}, \tilde{z}, \tilde{\lambda})$ on $\Delta_{\tilde{U}^-|_{\tilde{W}^-}}$. We can require that $\tilde{\lambda}$ is a dual coordinate to λ i.e., that if $l \in \Lambda^- \otimes (\Lambda^+)^*$ is given in the coordinates (z, \tilde{z}, λ) by (a, b, l_0) then the point $l^* \in \Lambda^+ \otimes (\Lambda^-)^*$ is given in coordinates $(z, \tilde{z}, \tilde{\lambda})$ by (a, b, l_0^{-1}) .

From the properties of blow-down we obtain the following proposition.

Proposition 47. The local coordinates (z, w, λ) on $\Delta_{\tilde{U}^+|_{\tilde{W}^+}}$ induce coordinates $(z, \lambda w, \lambda)$ on $(W^+ \setminus \underline{0}) \subset \tilde{U}^+|_{c_0^-}$ such that they extend (by $\lambda w = 0$) continuously through $\underline{0} \subset \tilde{U}^+|_{c_0^-}$.

The local coordinates $(\tilde{w}, \tilde{z}, \tilde{\lambda})$ on $\Delta_{\tilde{U}^-|_{\tilde{W}^-}}$ induce coordinates $(\tilde{\lambda}\tilde{w}, \tilde{z}, \tilde{\lambda})$ on $(W^- \setminus \underline{0}) \subset \tilde{U}^-|_{c_0^+}$ such that they extend (by $\tilde{\lambda}\tilde{w} = 0$) continuously through $\underline{0} \subset \tilde{U}^-|_{c_0^+}$.

Proof. Recall that in Corollary 10 we showed that W^+ is a cone subbundle of \tilde{U}^+ . W^+ is the image of a blow-down of $\Delta_{\tilde{U}^+|_{\tilde{W}^+}}$ and the blow-down map after restriction to $\Delta_{\tilde{U}^+|_{\tilde{W}^+}} \setminus \underline{0}$ is an isomorphism on $W^+ \setminus \underline{0}$, thus the coordinates are well defined. Moreover λ tends to 0 as we approach the zero section and w is bounded (at least locally), which completes the proof. \square

By construction we obtain the following coordinate description of the gluing.

Proposition 48. *The gluing of $W^+ \setminus \underline{0}$ with $W^- \setminus \underline{0}$ is given by $z = \tilde{w}$, $w = \tilde{z}$ and $\lambda = \tilde{\lambda}^{-1}$.* \square

Remark 34. The anti-holomorphic isomorphisms θ_Z written in the above coordinates are

$$\theta_Z(z, \lambda w, \lambda) = (-\bar{\lambda}\bar{w}, \bar{z}, -\bar{\lambda})$$

and

$$\theta_Z(\tilde{\lambda}\tilde{w}, \tilde{z}, \tilde{\lambda}) = (\bar{z}, -\bar{\lambda}\bar{w}, -\bar{\lambda}).$$

Hausdorffness In this paragraph we will show how to choose open subsets of V^+ and V^- such that the manifold obtained after gluing is Hausdorff.

The idea is analogous as in Section 2.4. We choose a tubular neighbourhoods B^+ of the zero section in $\tilde{U}^+|_{c_0^-}$ and B^- of the zero section in $\tilde{U}^-|_{c_0^+}$. By analogous argument as in Section 2.4 we can choose B^+ and B^- such that the intersection of the image in $\tilde{U}^-|_{c_0^+}$ by the gluing of $B^+ \cap (W^+ \setminus \underline{0})$ with B^- is empty.

Definition 55. Define $Z^+ := B^+ \cup W^+$ and $Z^- := B^- \cup W^-$ and

$$Z := Z^+ \bigsqcup_{\sim} Z^-,$$

where for $v^+ \in W^+ \setminus \underline{0}$ and $v^- \in W^- \setminus \underline{0}$ we have $v^+ \sim v^-$ iff $v^+ \cap (\Lambda^- \otimes (\Lambda^+)^*) = (v^- \cap (\Lambda^+ \otimes (\Lambda^-)^*))^*$.

Proposition 49. *The space Z defined above is a Hausdorff manifold.*

Proof. Clearly Z is a manifold as it arises as a gluing of two manifolds on an open set. It remains to show that Z is Hausdorff. Note that the condition that the intersection of the image in $\tilde{U}^-|_{c_0^+}$ by the gluing of $B^+ \cap (W^+ \setminus \underline{0})$ with B^- is empty implies that B^+ and B^- are open subsets of Z . Pick $x, y \in Z$. If $x, y \in W^+ \setminus \underline{0}$ are elements of the gluing region then as $W^+ \setminus \underline{0}$ is open and Hausdorff we can choose separating neighbourhoods. Analogously, as B^+ and B^- are open, we can also find separating neighbourhoods if $x, y \in B^+$ or $x, y \in B^-$. In the case when $x \in B^+$ and $y \in B^-$ then B^+ and B^- themselves are separating neighbourhoods. \square

Remark 35. The anti-holomorphic isomorphism θ_Z between V^+ and V^- induces a real structure on Z . We denote this real structure also by θ_Z .

3.2.4 Construction of a family of lines with the normal bundle $O(2)$

Normal bundle to twistor lines

Definition 56. Let $t_{(a,b)}$ be a projective line in Z which is the gluing of a line in $W^+ \subset U^+|_{c_0^-}$ given by $(z, w) = (a, b)$ with a line in $W^- \subset U^-|_{c_0^+}$ given by $(\tilde{w}, \tilde{z}) = (a, b)$.

Proposition 50. The real structure θ_Z induces on $t_{(a,\bar{a})}$ the antipodal map.

Proof. Firstly observe that the real structure interchange the zero in $\tilde{U}^+|_a$ with the zero in $\tilde{U}^-|_{\bar{a}}$. Any other point on $t_{(a,\bar{a})}$ can be written using the coordinates $(z, \lambda w, \lambda)$ as $(a, \bar{\lambda}a, l)$ for some $l \in \mathbb{C}^\times$. By Remark 34, we have

$$\theta_Z(a, \bar{\lambda}a, l) = (-\bar{\lambda}a, \bar{a}, -\bar{l}),$$

where $(-\bar{\lambda}a, \bar{a}, -\bar{l})$ is written using the coordinates $(\tilde{\lambda}\tilde{w}, \tilde{z}, \tilde{\lambda})$. This corresponds (by gluing) in the coordinates $(z, \lambda w, \lambda)$ to a point $(a, -\bar{l}^{-1}\bar{a}, -\bar{l}^{-1}) \in t_{(a,\bar{a})}$ hence the real structure preserves $t_{(a,\bar{a})}$. As the equation

$$l = -\bar{l}^{-1}$$

does not have solutions in the complex numbers, the induced real structure on $t_{(a,\bar{a})}$ is antipodal. \square

Lemma 6. *The normal bundle $N_{(a,b)}$ to projective lines $t_{(a,b)}$ decomposes into two line subbundles $N_{(a,b)} = N_{(a,b)}^+ \oplus N_{(a,b)}^-$.*

Proof. Our aim is to show that the vertical bundle of \tilde{U}^+ induces a line subbundle $N_{(a,b)}^+$ of $N_{(a,b)}$ and the vertical bundle of \tilde{U}^- induces a line subbundle $N_{(a,b)}^-$ of $N_{(a,b)}$.

Firstly, note that at any point $x \in \tilde{U}^+ \cap t_{(a,b)}$ the tangent space to $t_{(a,b)}$ at x is contained in the vertical bundle of $\tilde{U}^+|_{c_0^-}$. As the vertical bundle has rank 2, for $x \in \tilde{U}^+ \cap t_{(a,b)}$ it defines a 1-dimensional subspace $(N_{(a,b)}^+)_x$ of $(N_{(a,b)})_x$. The only point of $t_{(a,b)}$ which is not an element of W^+ is the point lying on the zero section of \tilde{U}^- . Denote it by ∞ . We will show that the bundle $(N_{(a,b)}^+)$ extends over ∞ by

$$(N_{(a,b)}^+)_{\infty} := T_{\infty} \underline{\infty},$$

where $\underline{\infty}$ denotes the zero section of $\tilde{U}^-|_{c_0^+}$.

To prove that this really defines a holomorphic line bundle over $t_{(a,b)}$, we have to construct a local holomorphic description of $N_{(a,b)}^+$ near ∞ . Consider a subspace $R_{(a,b)}$ of V^- of those lifted curves that intersect $\Lambda^+ \otimes (\Lambda^-)^*$ over the curve from C^+ given by $z = a$. This is a holomorphic submanifold of V^- . As $t_{(a,b)} \cap \tilde{U}^-|_{c_0^+}$ consists of those curves from V^- that intersect $\Lambda^+ \otimes (\Lambda^-)^*$ over the point $(z = a, \tilde{z} = b)$ we have that $t_{(a,b)} \cap \tilde{U}^-|_{c_0^+} \subset R_{(a,b)}$. We claim that

$$(N_{(a,b)}^+)_x = T_x R_{(a,b)} / T_x t_{(a,b)},$$

for $x \in t_{(a,b)} \cap \tilde{U}^-|_{c_0^+}$. To prove this for points of $t_{(a,b)} \cap \tilde{U}^-|_{c_0^+} \setminus \infty$, observe that the image by gluing of $R_{(a,b)} \setminus \underline{\infty}$ in V^+ is a surface contained in the fibre $z = a$ of $(\tilde{U}^+)|_{c_0^+}$. Hence for those points, the tangent space to $R_{(a,b)}$ is the vertical bundle of $\tilde{U}^+|_{c_0^-}$. To prove the equality for $z = \infty$ notice that the zero section of $\tilde{U}^-|_{c_0^+}$ (i.e., $\underline{\infty}$) is contained in $R_{(a,b)}$ and is transversal to $t_{(a,b)}$. As $T_{\infty} t_{(a,b)} \subset T_{\infty} R_{(a,b)}$ and $R_{(a,b)}$ is 2-dimensional, this proves the claim.

Analogously we define a subbundle $N_{(a,b)}^-$ as the vertical bundle of $U_p^-|_{c_0^+}$ and the tangent space to the zero section in $\tilde{U}^+|_{c_0^-}$. By definition, the bundles $N_{(a,b)}^+$ and $N_{(a,b)}^-$ are transversal to each other which completes the proof of the lemma. \square

Proposition 51. *The normal bundle to projective lines $t_{(a,b)}$ is $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Proof. We will show that $N_{(a,b)}^+ \cong \mathcal{O}(1)$. The proof that $N_{(a,b)}^- \cong \mathcal{O}(1)$ is analogous.

We will now construct family of curves such that tangent vectors to them at $t = 0$ will form sections of N^+ .

In the U^+ part using the coordinates $(z, \lambda w, \lambda)$ we define

$$\gamma_\lambda(t) = (a, \lambda(b + t), \lambda),$$

Those curves glue to curves in the U^- part given in the coordinates $(\tilde{\lambda}\tilde{w}, \tilde{z}, \tilde{\lambda})$ by

$$\gamma_{\tilde{\lambda}}(t) = (\tilde{\lambda}a, b + t, \tilde{\lambda}).$$

The direct calculation of derivatives at $t = 0$ gives $\frac{d}{dt}\gamma_\lambda|_{t=0} = (0, \lambda, 0)$ and $\frac{d}{dt}\gamma_{\tilde{\lambda}}|_{t=0} = (0, 1, 0)$.

The local coordinates $(z, \lambda w, \lambda)$ and $(\tilde{\lambda}\tilde{w}, \tilde{z}, \tilde{\lambda})$ are defined in a neighbourhood of $t_{(a,b)}$ everywhere except from points $\lambda = 0$ and $\tilde{\lambda} = 0$ hence the family of curves defined above define a holomorphic section, say u , of $N_{(a,b)}^+$ for all points of $t_{(a,b)}$ except from $\lambda = 0$ and $\tilde{\lambda} = 0$.

For points $\tilde{\lambda} = 0$, the coordinates in the direction of N^+ (i.e., \tilde{z}) are also defined in some neighbourhood of $\tilde{\lambda} = 0$. Hence the formula for u in coordinates is also valid there and hence u extends holomorphically over $\tilde{\lambda} = 0$.

Observe that, as coordinates $(z, \lambda w, \lambda)$ extend continuously (by zero) through 0 (see Proposition 47), the section u tends to zero when λ tends to 0. By Riemann's Theorem on Removable Singularities, u extends holomorphically through $\lambda = 0$ and $u(\lambda = 0) = 0$.

Hence we constructed a holomorphic section of $N_{(a,b)}^+$ vanishing at exactly one point. By the Birkhoff–Grothendieck theorem ([36]) $N_{(a,b)}^+ \cong \mathcal{O}(1)$.

□

We conclude with the following theorem:

Theorem 9. *Z is a twistor space of a self-dual conformal 4-manifold.*

Proof. We have shown that Z is a complex 3-manifold with a real structure θ_Z . Moreover, by Proposition 50, $t_{(a,\bar{a})}$ are θ_Z -invariant projective lines on Z such that the induced real structure on $t_{(a,\bar{a})}$ is antipodal. In Proposition 51 we have also shown that normal bundle to $t_{(a,\bar{a})}$ is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ which, by Theorem 2, completes the proof. □

Minitwistor lines

Definition 57. Denote by M the self-dual conformal 4-manifold such that the twistor space of M is Z .

From the theorem of Kodaira [46], there exists a 4-dimensional complex manifold M^c which is a local moduli space of curves (twistor lines) with the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ which comes as deformations of curves $t_{(a,b)}$.

Remark 36. The fact that the normal bundle to twistor lines is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ implies that the nearby twistor lines intersect each other in at most one point. In particular any nearby twistor lines from the moduli space intersect any $t_{(a,b)}$ in at most one point.

Recall that T is a complex surface defined in Section 3.2.2 and that our aim is to prove that T contains a projective line with normal bundle $\mathcal{O}(2)$.

Proposition 52. The projection Π from Z to T is given by a quotient of Z by a local \mathbb{C}^\times action.

Proof. To see this, note that the projection on the gluing part acts by collapsing projective lines $t_{(a,b)}$ (given in each of the gluing parts by a 1-dimensional vector spaces) to a point and on parts where we do not have a gluing, it acts by projectivisations of $U_p^+|_{c_0^-}$ and of $\tilde{U}^-|_{c_0^+}$. Define a local \mathbb{C}^\times action \circ as the action which is the scalar multiplication in the fibres of $\tilde{U}^+|_{c_0^-}$ and inverse of the scalar multiplication of $\tilde{U}^+|_{c_0^-}$. By Proposition 48, the definition of the \circ is compatible with the gluing, thus the local \mathbb{C}^\times action \circ on Z is well-defined. The set of fixed points are the zero sections in Z^+ and Z^- and the projection can be described as a gluing of fibrewise projectivisations of Z^+ and Z^- . \square

Remark 36 implies that there exists a real twistor line t from M^c which is transversal to \circ . Thus the vector field which is a generator of the action \circ induces a non-vanishing section of the normal bundle of t and thus the rank 1 trivial subbundle $N^s(t)$ of the normal bundle $N(t)$ of t . This can be summarised in the following corollary.

Corollary 11. There exists a real twistor line t which is transversal to the action \circ . The normal bundle to such a twistor line admits a trivial line subbundle $N^s(t)$.

Lemma 7. *For any twistor line t which is transversal to the action \circ , we have that $\Pi(t)$ is a projective line in T with the normal bundle $\mathcal{O}(2)$. Hence $\Pi(t)$ is a minitwistor line. Moreover, if t is a real twistor line then $\Pi(t)$ is a real minitwistor line and the induced real structure on $\Pi(t)$ is antipodal.*

Proof. The quotient bundle of N by N^s is the normal bundle $N(\Pi(t))$ to the projective line $\Pi(t) \subset T$. From this, we obtain the following exact sequence:

$$0 \longrightarrow N^s \longrightarrow N(t) \longrightarrow N(\Pi(t)) \longrightarrow 0.$$

As a consequence, the degrees of the bundles satisfy:

$$\deg N^s + \deg N(\Pi(t)) = \deg N(t),$$

thus $\deg N(t) = 2$ and by the Birkhoff–Grothendieck theorem ([36])

$$N(t) \cong \mathcal{O}(2).$$

By direct application of Remark 34 and Proposition 50, we obtain that the real structure on $\Pi(t)$ is antipodal. □

Now we are in position to state the main theorem of this chapter.

Theorem 10. *T is a minitwistor space of an Einstein–Weyl manifold.*

Proof. T is a complex surface with a real structure θ . By Lemma 7, T admits a projective curve t which is θ -invariant and the induced real structure on t is antipodal. Moreover the normal bundle to t is $\mathcal{O}(2)$ thus by Theorem 3, T is a minitwistor space of an Einstein–Weyl 3-manifold. □

Observation 4. The projective lines with normal bundle $\mathcal{O}(1)$ arise as quotients of images in T of (possibly restricted) fibres of bundles U^+ and U^- . The pair of projective lines given by fibres $z = a$ and $\bar{z} = b$ meet in the point given by $\Pi(t_{(a,b)})$. As those the twistor lines which (by quotients) give minitwistor lines arise as deformations of $t_{(a,b)}$ we can view the line pairs as degenerations of the minitwistor lines.

Following the result of LeBrun [49] (see Lemma 5) we obtain the following corollary.

Corollary 12. *The Einstein–Weyl manifold obtained from the twistor space T is asymptotically hyperbolic.*

We sum up the construction in the following remark.

Remark 37. We have shown that there is a natural construction of the minitwistor space of an asymptotically hyperbolic Einstein–Weyl manifold B from a surface Σ equipped with a conformal Cartan connection. The minitwistor space contains a family of pairs of projective lines which are deformations of the minitwistor lines and encode the asymptotic end of the Einstein–Weyl manifold. Moreover, we have shown that there exists a natural construction of a twistor space of the self-dual conformal 4-manifold M from Σ such that S is a submanifold of M and B arises as quotient of M by an S^1 action. As a consequence, the asymptotically hyperbolic Einstein–Weyl manifold B , constructed from Σ , admits a distinguished Gauduchon gauge (see [42]).

3.3 Comparison of the two constructions

Now we would like to study the relation between the two constructions of minitwistor spaces. Recall that on Σ we defined foliations by C^+ curves and C^- curves, where C^+ curves are the integral curves for t^+ distribution and C^- curves are the integral curves for t^- distribution.

Proposition 53. *The bundles V^+ and V^- used for the construction of the twistor space Z from Section 3.2.3 (see Definition 55) are isomorphic to the bundles of affine sections of $\tilde{U}^+ / (\Lambda^- \otimes (\Lambda^+)^*)$ along the leaves of the t^+ -foliation and of $\tilde{U}^- / (\Lambda^+ \otimes (\Lambda^-)^*)$ along the leaves of the t^- -foliation respectively.*

Proof. Firstly note that, as affine sections of U^+ / Λ are determined by initial conditions on the 1-jet bundle, we get that the bundle of affine sections of U^+ / Λ along the leaves of the t^+ -foliation is isomorphic to the space $J^1 U / \Lambda|_{c_0^-}$, where c_0^- is a leaf of the t^- -foliation and is transversal to C^+ curves.

Using the isomorphism from the Cartan condition and the standard exact sequence of the 1-jet bundle along the t^+ foliation (given by Cartan connection), we get

$$\begin{array}{ccccccc}
0 & \rightarrow & t^+ \otimes U^+/\Lambda & \rightarrow & J^1 U^+/\Lambda & \rightarrow & U^+/\Lambda \rightarrow 0 \\
& & \downarrow \cong & & \downarrow & & \parallel \\
0 & \rightarrow & \Lambda & \rightarrow & U^+ & \rightarrow & U^+/\Lambda \rightarrow 0
\end{array}$$

As a consequence, the bundle of affine sections of U^+/Λ along the leaves of the t^+ -foliation is isomorphic to the space $U^+|_{c_0^-}$. As in Proposition 44 we identified $\tilde{U}^+|_{c_0^-}$ with V^+ , tensoring the isomorphism obtained above by $(\Lambda^+)^* \otimes (\Lambda^+)^*$ gives the required identification of the bundle V^+ . Analogously we prove proposition for the bundle V^- . \square

Remark 38. Recall that the twistor space in the generalised Feix–Kaledin construction of quaternionic manifolds was constructed by gluing of bundles dual to bundles of affine sections along the leaves, whereas we have just shown that in the construction in Chapter 3 we construct Z as gluing of bundles of affine sections. This is because, following Remark 29, we were using the isomorphism between the bundle V and V^* given by the inner product.

Proposition 54. *The gluing of open subsets of bundles V^+ and V^- to a twistor space Z (see Definition 55) is given by a blow-down from the projective bundle $\mathbb{P}(\mathcal{O}(1, -1) \oplus \mathcal{O})$ over Σ .*

Proof. Recall that the gluing of V^+ and V^- is induced by a gluing of bundles $\Delta_{\tilde{U}^+|_{\tilde{W}^+}}$ and $\Delta_{\tilde{U}^-|_{\tilde{W}^-}}$. Recall also that bundles $\Delta_{\tilde{U}^+|_{\tilde{W}^+}}$ and $\Delta_{\tilde{U}^-|_{\tilde{W}^-}}$ admit coordinates (z, w, λ) and $(\tilde{w}, \tilde{z}, \tilde{\lambda})$ respectively and that the gluing is given by $z = \tilde{w}$, $w = \tilde{z}$ and $\lambda = \tilde{\lambda}^{-1}$. Note that the coordinates (z, w) and (\tilde{w}, \tilde{z}) are induced from the coordinates (z, \tilde{z}) on Σ .

If we fix $z = z_0$, then the bundle $\Delta_{\tilde{U}^+|_{\{z=z_0\}}}$ over $\mathbb{P}(\tilde{U}^+)|_{\{z=z_0\}}$ is, by definition, the tautological bundle $\mathcal{O}(-1)$. As $\mathbb{P}(\tilde{U}^+)|_{\{z=z_0\}}$ can be identified (after restriction to the gluing part) with the leaf $\{z = z_0\}$ of the t^+ -foliation, we get that the bundle $\Delta_{\tilde{U}^+}$ is isomorphic to $\mathcal{O}(-1)$ along leaves of the t^+ -foliation.

Using the fact that the gluing is given by $\lambda = \tilde{\lambda}^{-1}$, we get that $\Delta_{\tilde{U}^-|_{\{\tilde{w}=z_0\}}}$ is the dual bundle to the tautological bundle i.e., it is isomorphic to $\mathcal{O}(1)$. Hence,

along leaves of the t^+ -foliation, $\Delta_{\bar{U}^-}$ is isomorphic to $\mathcal{O}(1)$. Proceeding with the above argument also for leaves of the t^- foliation we deduce that the gluing of open subsets of the vector bundles Z^+ with Z^- from Section 3.2 is induced by the blow down from the bundle $\mathbb{P}(\mathcal{O}(1, -1) \oplus \mathcal{O})$ over Σ . \square

The above facts imply the following corollary.

Corollary 13. *The construction of the twistor space from Section 3.2 (see Definition 55) is the generalised Feix–Kaledin construction in the case when S is a surface equipped with a real-analytic Möbius structure (which is equivalent structure to conformal Cartan connection) and the line bundle \mathcal{L} on S is trivial.*

Proposition 55. *Any asymptotically hyperbolic Einstein–Weyl manifold constructed in Chapter 3 is equipped with distinguished Gauduchon gauge given by the conformal Cartan connection on S .*

Proof. Using the conformal Cartan connection on S , we canonically constructed a twistor space Z such that the minitwistor space T is a quotient of Z by a holomorphic action. This, by the Jones–Tod correspondence, finishes the proof. \square

The above result motivates the question if the choice of a line bundle \mathcal{L} on S from Chapter 2 comes from the choice of a gauge on the asymptotically hyperbolic Einstein–Weyl space. The following argument shows that this is true. Choose (\mathcal{L}, ∇) a complex line bundle with a complex connection on $\Sigma_{\mathbb{R}}$. Then we can tensor all objects from the linear representation of the Cartan connection on Σ by \mathcal{L} and ∇ respectively. Then the similar construction to those in Section 3.2 gives the twistor space which is the twistor space of a self-dual conformal 4-manifold obtained by the construction in Chapter 2 from the Möbius manifold $\Sigma_{\mathbb{R}}$ and the line bundle \mathcal{L} . The minitwistor spaces obtained by the construction from Section 3.2 does not depend on the choice of the line bundle \mathcal{L} , because $\mathbb{P}(\mathcal{A} \otimes \mathcal{L}) = \mathbb{P}(\mathcal{A})$ for any line bundle \mathcal{L} and a vector bundle \mathcal{A} . Using the Jones–Tod correspondence ([42]), we conclude that this means that the choice of the line bundle on the Cartan geometry on S equips the constructed asymptotically hyperbolic Einstein–Weyl space with a Gauduchon gauge. We can summarise this in the following proposition.

Proposition 56. *Let S be a surface with a real-analytic Möbius structure. The asymptotically hyperbolic Einstein–Weyl spaces obtained as quotients by the S^1 action of self-dual conformal manifolds obtained by the generalised Feix–Kaledin construction for quaternionic manifold, do not depend on the choice of the line bundle \mathcal{L} on S . The choice of the line bundle \mathcal{L} induces a choice of a Gauduchon gauge on the obtained Einstein–Weyl manifold. \square*

Chapter 4

Conclusions and further directions

In this thesis we have shown that a real-analytic h-projective structure with h-projective curvature of type $(1, 1)$ on a $2n$ -manifold S , for $n > 1$, can be used to construct $4n$ -dimensional quaternionic manifold M with an S^1 action, such that M can be identified with a neighbourhood of the tangent bundle of S twisted by some unitary line bundle and S^1 action is the unit scalar multiplication in the fibres. We have also shown that this construction extends to the case $(n = 1)$, provided that we assume that S admits a real-analytic Möbius structure. For any such h-projective manifold S , we have obtained a family of quaternionic manifolds parametrised by line bundles with connection with curvature of type $(1, 1)$ on S , and we have shown that the twist of the tangent bundle used for the identification of M depends upon the unitary part of the chosen line bundle. Moreover, we have shown that a choice of a connection in the h-projective class induces extensions of the complex structure from S to M .

In Chapter 3 we have shown that the quotients of quaternionic manifolds constructed in Chapter 2 by the generalised Feix–Kaledin construction of quaternionic manifolds are asymptotically hyperbolic Einstein–Weyl 3-manifolds. We have also given a natural construction of a minitwistor space from a real-analytic conformal Cartan geometry on a surface. Using this data, we have also constructed a twistor space from which this minitwistor space arises as a quotient. This means that the obtained asymptotically hyperbolic Einstein–Weyl space admits a Gauduchon gauge induced from the conformal Cartan connection at the boundary. We have compared the two constructions of twistor spaces and have deduced that, taking S with a Möbius structure

and varying line bundles, the minitwistor spaces obtained as quotients of the twistor spaces constructed using the generalised Feix–Kaledin construction of quaternionic manifolds are the same. This means that the line bundles in the generalised Feix-Kaledin construction of quaternionic manifolds in the lowest dimensional case correspond to choices of Gauduchon gauges on the quotient Einstein–Weyl 3-manifolds.

The first question that one could ask is if the h-projective structure (or the Cartan connection respectively) is preserved by the generalised Feix–Kaledin construction of quaternionic manifold, i.e., if it is induced on the submanifold S by the quaternionic structure on M . We have observed that, as S is a totally complex submanifold on M of the maximal dimension, the quaternionic structure on M induces an h-projective structure on S . However, the problem if this h-projective structure is the same as the h-projective structure that we started from remains open. The difficulty here lies in the fact that we construct quaternionic connections as fibre bundle connections on the twistor space Z . Although fibre bundle connections on Z induce connections on M , the proof of this fact is rather implicit and hence it is hard to describe how the connection behaves on the tangent bundle to the submanifold S .

The other question is, if the condition that the h-projective curvature is of type $(1, 1)$ is not too restrictive i.e., if there exists an example of an h-projective structure with curvature of type $(1, 1)$ which does not admit any complex connection with curvature of type $(1, 1)$. One of the methods of finding examples of such h-projective structures may involve studying totally complex maximal submanifolds of non quaternion-Kähler quaternionic manifolds, as they admit h-projective structures with h-projective curvature of type $(1, 1)$.

A very interesting question from the point of view of the Haydys–Hitchin construction (see [37], [39]) is in which cases we obtain quaternion-Kähler manifolds. Once one shows that the h-projective structure induced on the submanifold S is the one used for the construction, we would get that the necessary condition for this is that S is a Kähler manifold (as totally complex submanifolds of a quaternion Kähler manifold are Kähler - see [2]). The conjecture is that this is a sufficient condition, possibly with the additional requirement that the line bundle used for construction is unitary. The further direction is to relate our main construction to the Haydys–Hitchin construction. One should

obtain that the Swann bundle in their construction is related to the (unitary) line bundle from the generalised Feix–Kaledin construction of quaternionic manifolds.

There is also a natural question, if, at least for some subclass, all quaternionic manifolds with a totally complex submanifold and quaternionic S^1 action can be obtained in this way. The uniqueness results obtained by N. Hitchin and A. Haydys for quaternion-Kähler manifolds suggest that it may be true at least in some cases.

The most natural case from the generalised Feix–Kaledin construction of quaternionic manifolds point of view, is the case when the line bundle used for the construction is trivial. Note that this is not the case in which the construction coincides with the Feix construction. It would be interesting to know, if there is any deeper reason for that and if the manifolds obtained in this way are in any sense special. The construction from Chapter 3 suggests that the possible answer to this question may be that, at least for the case when S is a surface, the manifolds obtained in this case correspond to some special Gauduchon gauges. It would be also interesting to further investigate the relation between line bundles on S and induced Gauduchon gauges.

The further direction that can help in explanation of the role of a line bundle on S in the generalised Feix–Kaledin construction of quaternionic manifolds, is to investigate the class of manifolds arising as quotients by an S^1 action of the constructed quaternionic $4n$ -manifold for $n > 1$. Those manifolds should admit a type of twistor correspondence and, possibly, will be similarly related to quaternionic manifolds as Einstein–Weyl manifolds are related to conformal self-dual 4-manifolds.

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