



PHD

## Orbifolds, Cohen-Macaulay modules and cluster categories

Collins, Alex

*Award date:*  
2016

*Awarding institution:*  
University of Bath

[Link to publication](#)

### Alternative formats

If you require this document in an alternative format, please contact:  
[openaccess@bath.ac.uk](mailto:openaccess@bath.ac.uk)

Copyright of this thesis rests with the author. Access is subject to the above licence, if given. If no licence is specified above, original content in this thesis is licensed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivs 4.0 International (CC BY-NC-ND 4.0) Licence (<https://creativecommons.org/licenses/by-nc-nd/4.0/>). Any third-party copyright material present remains the property of its respective owner(s) and is licensed under its existing terms.

#### Take down policy

If you consider content within Bath's Research Portal to be in breach of UK law, please contact: [openaccess@bath.ac.uk](mailto:openaccess@bath.ac.uk) with the details. Your claim will be investigated and, where appropriate, the item will be removed from public view as soon as possible.

**Orbifolds,  
Cohen-Macaulay modules  
and cluster categories**

submitted by

Alexander Collins

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

February 2016

**COPYRIGHT**

Attention is drawn to the fact that copyright of this thesis rests with the author. A copy of this thesis has been supplied on condition that anyone who consults it is understood to recognise that its copyright rests with the author and that they must not copy it or use material from it except as permitted by law or with the consent of the author.

This thesis may be made available for consultation within the University Library and may be photocopied or lent to other libraries for the purposes of consultation with effect from . . . .  
.....

Signed on behalf of the Faculty of Science . . . . .

## Summary

We investigate certain Frobenius structures arising on categories  $\text{Vect}(\mathbb{X})$  of vector bundles over one-dimensional orbifolds. These we classify by coloured affine Dynkin diagrams  $\Delta$ .

One can view these Frobenius categories as categories  $\text{MF}_\Gamma(R, f)$  of equivariant matrix factorizations of curve singularities (or more generally, categories of  $p$ -cycles). When the orbifold  $\mathbb{X}$  is Fano, we establish a derived equivalence between the stable category  $\underline{\text{MF}}_\Gamma(R, f) = \underline{\text{CM}}_\Gamma(R/f)$  and the black part  $\Delta^b$  of the diagram  $\Delta$ . We show that  $\underline{\text{CM}}_\Gamma(R/f)$  carries a natural automorphism which is equivalent to the cluster automorphism on  $D^b(\Delta^b)$ . This allows us to construct finite-type cluster categories as stable categories of Cohen-Macaulay modules equivariant with respect to a finite group.

## Acknowledgements

I am enormously grateful to my supervisor, Alastair King, for his patience and perseverance, and for his acuity and skill as a teacher.

My deepest thanks to Beatrice and Ismay, to Rachel and to Freya. For being there, thank you.

Finally, thank you to my parents, Sarah and Chris, without whose love and support this would never have been possible.

# Contents

<b>1</b>	<b>Introduction</b>	<b>5</b>
<b>2</b>	<b>Background</b>	<b>9</b>
2.1	Cohen-Macaulay modules . . . . .	9
2.2	Matrix factorizations . . . . .	13
2.3	Exact and Frobenius categories . . . . .	20
2.4	Orbifolds . . . . .	26
<b>I</b>	<b>Black and white diagrams</b>	<b>35</b>
<b>3</b>	<b>Graded McKay correspondence</b>	<b>36</b>
3.1	Graded McKay correspondence . . . . .	36
<b>4</b>	<b>Frobenius structures for Fano orbifolds</b>	<b>45</b>
4.1	Fano orbifolds . . . . .	45
4.2	Frobenius structures . . . . .	47
<b>5</b>	<b>The black part</b>	<b>52</b>
5.1	Derived equivalence . . . . .	52
5.2	Discussion . . . . .	56
<b>II</b>	<b>Curves and cluster categories</b>	<b>59</b>
<b>6</b>	<b><math>p</math>-cycles</b>	<b>60</b>
6.1	Abelian categories . . . . .	60
6.2	Exact categories . . . . .	67
<b>7</b>	<b>Curves</b>	<b>71</b>
7.1	Frobenius structures for matrix factorizations . . . . .	71

<b>8</b>	<b>Construction of cluster categories</b>	<b>83</b>
8.1	Cluster categories . . . . .	83
8.2	Categorification of the Grassmannian . . . . .	87
<b>A</b>	<b>Tables</b>	<b>91</b>

# Chapter 1

## Introduction

The central theme of this thesis is the construction and investigation of various Frobenius exact structures on categories of vector bundles over one-dimensional orbifolds. The associated stable categories carry a natural automorphism by means of which we construct cluster categories of finite type, relating to Jensen-King-Su's work on the categorification of the Grassmannian [27].

After the background material of Chapter 2 we begin in Chapter 3 by proving (Theorem 3.1.6) a graded McKay correspondence implicit in early work of Geigle and Lenzing [18], [19]. Our proof is essentially the observation that the category  $\mathrm{CM}^{\mathbb{Z}}(S)$  of  $\mathbb{Z}$ -graded Cohen-Macaulay modules on a simple singularity  $\mathbb{C}^2/G = \mathrm{Spec} S$  is equivalent, by a sort of Hartogs' phenomenon, to the category of  $\mathbb{C}^*$ -equivariant bundles on the punctured surface  $\mathrm{Spec} S - 0$ . That is, to the category of vector bundles on the stack

$$\mathbb{X} := \left[ \frac{(\mathbb{C}^2/G) - 0}{\mathbb{C}^*} \right] \cong \left[ \frac{\mathbb{C}^2 - 0}{\Gamma} \right],$$

where  $\Gamma$  is the product  $G \cdot \mathbb{C}^*$  inside  $\mathrm{GL}(2, \mathbb{C})$ , a one-dimensional group with finite determinant-one part. This  $\mathbb{X}$  is a Fano orbifold.

The point here is less the result itself than the three different descriptions of the same additive category

$$\mathrm{CM}^{\mathbb{Z}}(S) \cong \mathrm{Vect}(\mathbb{X}) \cong \mathrm{Vect}_{\Gamma}(\mathbb{C}^2).$$

Each of these provides a different exact structure on the category. Though  $\mathrm{Vect}(\mathbb{X})$  itself has no projective objects, the other two structures are in fact Frobenius; all degree shifts of the trivial module  $R$  are projective in  $\mathrm{CM}^{\mathbb{Z}}(S)$ , while in  $\mathrm{Vect}_{\Gamma}(\mathbb{C}^2)$  everything is projective. Throughout the thesis we use the convention that an equivalence of categories  $\mathcal{A}, \mathcal{B}$  which is only additive is denoted by  $\mathcal{A} \cong \mathcal{B}$ , while an exact equivalence

of exact or triangulated categories is denoted by  $\mathcal{A} = \mathcal{B}$ .

Given a one-dimensional group  $\Gamma$  of the form  $\Gamma = G \cdot \mathbb{C}^* \subset \mathrm{GL}(2, \mathbb{C})$ , where  $G$  is a finite subgroup of  $\mathrm{SL}(2, \mathbb{C})$  (we will refer to such groups as extended McKay groups) we obtain a Fano orbifold  $\mathbb{X}$ . We can now produce a variety of exact structures on  $\mathrm{Vect}(X)$  by taking normal subgroups  $N \triangleleft G$  and considering the categories  $\mathrm{CM}_{\Gamma/N}(R)$  with  $\mathrm{Spec} R = \mathbb{C}^2/N$ . These structures we classify in Chapter 4 by bi-coloured extended Dynkin diagrams, which we call “black and white diagrams”.

The main result of Part I is a derived equivalence between the stable category of equivariant Cohen-Macaulay modules of a surface singularity and the black part of the corresponding black and white diagram.

**Theorem 5.1.1.** Let  $\Gamma$  be a one-dimensional group with finite determinant-one part  $G$ ,  $N \triangleleft G$  a normal subgroup and  $\mathrm{Spec} R = \mathbb{C}^2/N$ . Let  $\Delta$  be the black and white diagram associated to this data and  $\Delta^b$  its black part (a Dynkin diagram). Then

$$D^b(\Delta^b) = \underline{\mathrm{CM}}_{\Gamma/N}(R)$$

as triangulated categories.

This theorem is a direct generalization of Kussin-Lenzing-Meltzer [33] Theorem 5.1; indeed our proof is essentially theirs. It also generalizes Kajiwara-Takahashi-Saito [28] Theorem 3.1. Finally we investigate the not entirely straightforward relationship between this black part and the ADE type of the surface singularity.

In Part II we introduce  $p$ -cycles (as defined by Lenzing [35]), which may be thought of as a generalization of matrix factorizations: categories of matrix factorizations are examples of  $p$ -cycle categories for  $p = 2$ . The  $p$ -cycle construction is an “expansion of abelian categories” in the sense of Chen-Krause [15]; geometrically speaking it takes the data of an orbifold  $\mathbb{X}$  and produces a new orbifold  $\mathbb{X}^\sharp$  whose orbifold weights are multiples of the old, in the sense that

$$\mathrm{Coh}(\mathbb{X}^\sharp) \cong p\text{-cyc}(\mathrm{Coh}(\mathbb{X})).$$

Chapter 6 is largely technical, providing some proofs that appear to be missing from the literature and establishing two important facts for what is to follow.

First, categories of  $p$ -cycles of equivariant modules can be realised as module categories. Let  $R$  be a ring with the action of an algebraic group  $\Gamma$ . Let  $f \in R$  be a semi-invariant for  $\Gamma$  with associated character  $\chi_f$ . Define

$$\mathcal{L}_p(R, f) := R[z]/(f - z^p)$$



and

$$\mathcal{L}_p(\Gamma, \chi_f) := \{(\gamma, u) \in \Gamma \times \mathbb{C}^* \mid \chi_f(\gamma) = u^p\}.$$

**Proposition 6.1.5.**

$$p\text{-cyc}(\text{mod}_\Gamma(R), f) \cong \text{mod}_{\mathcal{L}_p(\Gamma, \chi_f)}(\mathcal{L}_p(R, f)).$$

Second, in Section 6.2 we establish that the  $p$ -cycle construction behaves well with respect to exact (and Frobenius) structures. For example, Proposition 6.1.5 restricts to the subcategory of equivariant Cohen-Macaulay modules (Theorem 6.2.3) and induces an equivalence on the stable categories. So in particular, when  $p = 2$  we have

$$\underline{\text{CM}}_\Gamma(R/f) = \underline{\text{CM}}_{\mathcal{L}(\Gamma)}(\mathcal{L}(R)).$$

In Chapter 7 we are ready to use  $p$ -cycles to construct more Frobenius structures on categories of orbifold vector bundles, induced by the equivalence

$$\text{Vect}(\mathbb{X}) \cong p\text{-cyc}(\text{Vect}(\mathbb{X}^b))$$

for some “lighter” weighted projective line  $\mathbb{X}^b$ . Our first observation is that when  $\mathbb{X}$  is Fano, these new structures agree with those already described in Part I. We then establish under what conditions the orbifold  $\mathbb{X}^b$  exists (Proposition 7.1.3). This allows us to catalogue, Tables 7.1 - 7.3, all of the Frobenius structures that arise in this way. In Appendix A we reorganize this data (Tables A.1-A.4) to exhibit a large variety of Frobenius categories of equivariant matrix factorizations, including various  $\mathbb{Z}$ -graded cases (compare for example those studied by Araya [2]), but also gradings with torsion and noncommutative cases. Examination of these tables leads to the following finite/tame/wild classification for equivariant matrix factorizations. Let  $\Gamma^b \subset \text{GL}(2, \mathbb{C})$  be a one-dimensional group with finite determinant-one part, and  $f$  a semi-invariant (a curve singularity).

**Corollary 7.1.8.** The category of  $\Gamma^b$ -equivariant matrix factorizations

$$\text{MF}_{\Gamma^b}(\mathbb{C}[x, y], f)$$

has finite representation type if and only if  $f$  is a simple singularity, tame if and only if  $f$  is unimodular of type  $X_9$  or  $J_{10}$  and wild otherwise.

The aim of Chapter 8 is to use the various Frobenius categories we have so far constructed to derive finite-type cluster categories. The key example is Jensen-King-Su’s categorification of the Grassmannian [27].

Let  $G_{k,n}$  denote the Grassmannian of  $k$ -planes in  $\mathbb{C}^n$ . The Grassmannian cluster algebra  $\mathbb{C}[G_{k,n}]$  has finite cluster type  $Q$  precisely when  $k = 2$  or  $k = n - 2$  (type  $Q = A_{n-3}$ ); and  $k = 3$  or  $k = n - 3$  and  $n = 6, 7$  or  $8$  (types  $Q = D_4, Q = E_6, Q = E_8$ ).

Let  $f = x^k - y^{n-k}$ ,  $R := \mathbb{C}[[x, y]]/f$  and  $G \subset \mathrm{SL}(2, \mathbb{C})$  be the cyclic group of order  $n$ , acting on  $R$  by  $(x, y) \mapsto (\zeta x, \zeta^{-1}y)$ . Observe that the curve singularity  $f$  has Dynkin type  $Q$ .

**Theorem 8.2.1.** The stable category  $\underline{\mathrm{CM}}_G(R)$  is equivalent to the cluster category of type  $Q$ , the Dynkin type of the singularity  $f$ .

In order to prove this, we first extend  $G$  to a one-dimensional group by multiplying it by a (weighted)  $\mathbb{C}^*$  inside  $\mathrm{GL}(2, \mathbb{C})$ ,  $\Gamma^b := \mathbb{C}^* \cdot G$ . We then consider the category of equivariant matrix factorizations  $\mathrm{MF}_{\Gamma^b}(\mathbb{C}[x, y], f)$ . By the material of Chapter 6, this Frobenius category is additively equivalent to  $\mathrm{Vect}(\mathbb{X})$  for some orbifold  $\mathbb{X}$ , and is classified by a black and white diagram  $\Delta$ . Now stabilize and apply Theorem 5.1.1:

$$\underline{\mathrm{CM}}_{\Gamma^b}(R^b/f) = D^b(\Delta^b).$$

Two further observations are required: first that  $\Delta^b$  is the Dynkin type of  $f$ . Second that the categorical covering

$$\underline{\mathrm{CM}}_{\Gamma^b}(R^b/f) \longrightarrow \underline{\mathrm{CM}}_G(\hat{R}^b/f)$$

given by completion is equivalent to taking the orbit category with respect to the cluster automorphism  $\tau^{-1}[1]$  on  $D^b(\Delta^b)$ . This we prove using a duality theorem of Buchweitz in Theorem 8.1.3.

## Chapter 2

# Background

### 2.1 Cohen-Macaulay modules

Let us recall some basic definitions and properties. In this section rings  $R$  are commutative, local and Noetherian, with maximal ideal  $\mathfrak{m}$  and residue field  $\mathbb{C} = R/\mathfrak{m}$ . Modules  $M$  are finitely generated over  $R$ .

**Definition 2.1.1.** A regular sequence on  $M$  is a sequence of elements  $x_1, \dots, x_n \in \mathfrak{m}$  such that

1. the inclusion  $(x_1, \dots, x_n)M \hookrightarrow M$  is not an isomorphism, and
2. For  $i = 1, \dots, n$ ,  $x_i$  is a nonzerodivisor on  $M/(x_1, \dots, x_{i-1})M$ .

**Definition 2.1.2.** The depth of  $M$  is the maximum length of a regular sequence on  $M$ .

**Definition 2.1.3.** The module  $M$  is maximal Cohen-Macaulay if its depth is equal to the Krull dimension of  $R$ ;  $R$  is Cohen-Macaulay if it is Cohen-Macaulay as a module over itself. A module  $M$  over a non-local Noetherian ring  $R$  is Cohen-Macaulay if  $M_{\mathfrak{m}}$  is Cohen-Macaulay over  $R_{\mathfrak{m}}$  for every maximal ideal  $\mathfrak{m}$  of  $R$ . Such a non-local Noetherian ring is Cohen-Macaulay if it is Cohen-Macaulay as a module over itself. In either case write  $\text{CM}(R)$  for the additive category of Cohen-Macaulay modules over  $R$ .

The following homological characterisation of depth will be useful.

**Lemma 2.1.4.**  $\text{depth } M = \inf\{i \mid \text{Ext}^i(\mathbb{C}, M) \neq 0\}$ .

*Proof.* This is classical, see for example [10] Theorem 1.2.8.

□

### Examples 2.1.5.

- All 0-dimensional Noetherian local rings are Cohen-Macaulay.
- All 1-dimensional reduced local rings are Cohen-Macaulay. The ring  $\mathbb{C}[x, y]/(y^2)$  is non-reduced but Cohen-Macaulay; the ring  $R = \mathbb{C}[x, y]/(x^2, xy)$  is not: every element of its maximal ideal is a zero divisor, so it has depth zero. Geometrically, the Cohen-Macaulay condition fails here because  $\text{Spec } R$  has an embedded component at the double point.
- For local rings of dimension two normal implies Cohen-Macaulay but not conversely: the ring  $\mathbb{C}[x^2, x^3, y]$  is Cohen-Macaulay but not normal. For an example of a 2-dimensional ring that is not Cohen-Macaulay, consider  $R = \mathbb{C}[x, y, z]/(xy, xz)$ . The point  $(0, 0, 0)$  in  $\text{Spec } R$  is the intersection of a plane and a line and therefore lies in two components of different dimensions. This is a failure of equidimensionality; the local scheme of the origin has irreducible components of differing dimensions and hence cannot be Cohen-Macaulay.

**Definition 2.1.6.** A (possibly non-local) Cohen-Macaulay ring  $R$  is called an isolated singularity if the localizations  $R_{\mathfrak{p}}$  are regular local for each prime  $\mathfrak{p}$  distinct from the maximal ideal  $\mathfrak{m}$ .

**Lemma 2.1.7.** Over a regular local ring the Cohen-Macaulay modules are precisely the free modules. Therefore every Cohen-Macaulay module  $M$  over an isolated singularity is locally free on the punctured spectrum, that is,  $M_{\mathfrak{p}}$  is free over  $R_{\mathfrak{p}}$  for all  $\mathfrak{p} \neq \mathfrak{m}$ .

*Proof.* [46] Proposition 1.5 and Lemma 3.3. □

### Examples 2.1.8.

- Let  $k > 1$ . The ring  $R = \mathbb{C}[[x]]/(x^k)$  is an Artinian principal ideal domain. Its finitely generated indecomposable modules are cyclic of finite length and hence isomorphic to one of the  $k$  modules  $M_i := R/(x^i)$ ,  $0 < i \leq k$ . Since  $x$  is a zero divisor all of these modules have depth 0 and are therefore Cohen-Macaulay.
- Rings of formal power series are regular local so Cohen-Macaulay modules for them are free. In particular  $R = \mathbb{C}[[x, y]]$  has one indecomposable Cohen-Macaulay module, namely  $R$  itself.
- The cusp singularity  $R = \mathbb{C}[[x, y]]/(y^2 + x^3)$  has two indecomposable Cohen-Macaulay modules: the free one  $R$ , and the ideal  $(x, y)$  corresponding to the ideal sheaf of the point. See [46] Chapter 9, in particular 9.11.

### 2.1.1 Equivariant Cohen-Macaulay modules

For rings carrying a group action, equivariant Cohen-Macaulay modules will be of central importance to us in what follows.

**Definition 2.1.9.** Let  $R$  be a ring with a left group action

$$\begin{aligned}\Gamma \times R &\longrightarrow R \\ (\gamma, r) &\longmapsto \gamma(r).\end{aligned}$$

A  $\Gamma$ -equivariant  $R$ -module is an  $R$ -module  $M$  with a left  $\Gamma$  action such that

$$\gamma(rm) = \gamma(r)\gamma(m)$$

for all  $\gamma \in \Gamma$ ,  $r \in R$  and  $m \in M$ .

A morphism of  $\Gamma$ -equivariant modules is a morphism  $f : M \rightarrow N$  of modules which is  $\Gamma$ -equivariant, that is, such that

$$f(\gamma m) = \gamma f(m)$$

for all  $\gamma \in \Gamma$  and  $m \in M$ . Write  $\text{mod}_\Gamma(R)$  for the category of  $\Gamma$ -equivariant  $R$ -modules and  $\text{CM}_\Gamma(R)$  for the additive subcategory of  $\Gamma$ -equivariant  $R$ -modules which are Cohen-Macaulay.

**Definition 2.1.10.** Let  $M$  be a  $\Gamma$ -equivariant  $R$ -module and  $\rho : \Gamma \rightarrow \text{End}(V)$  a representation of  $\Gamma$ . We define the tensor product  $M \otimes V$  to be the vector space  $M \otimes V$ , made an  $R$ -module by multiplication on the left factor and with  $\Gamma$ -equivariant structure defined by

$$\gamma(m \otimes v) := \gamma(m) \otimes \rho(\gamma)(v).$$

In particular, when  $\chi$  is a linear character of  $\Gamma$ , define the tensor product  $M \otimes \chi$  to be the  $R$ -module  $M$  with  $\Gamma$ -equivariant structure defined by

$$\gamma(m)_{[M \otimes \chi]} := \chi(\gamma)\gamma(m)_{[M]},$$

where the action on the right hand side is understood to be the old action of  $\Gamma$  on  $M$ . Every linear character of  $\Gamma$  defines in this way an autoequivalence of  $\text{mod}_\Gamma(R)$ .

**Remark 2.1.11.** Suppose the group  $\Gamma$  the direct product of a torus and a finite abelian group. Then  $R$  is naturally graded by the group  $\Gamma^\vee$  of linear characters of  $\Gamma$ ,

$$R = \bigoplus_{\chi \in \Gamma^\vee} R_\chi,$$

where the

$$R_\chi = \{r \in R \mid \gamma(r) = \chi(\gamma)r \quad \forall \gamma \in \Gamma\}$$

are the eigenspaces for the action of  $\Gamma$ . In the same way any  $\Gamma$ -equivariant  $R$ -module gets a  $\Gamma^\vee$ -grading.

Conversely a  $\Gamma^\vee$ -graded module over a  $\Gamma^\vee$ -graded ring becomes a  $\Gamma$ -equivariant module by defining the  $\Gamma$  action (on both  $R$  and  $M$ ) in the obvious way. In this situation then the categories  $\text{mod}^\Gamma(R)$  (respectively  $\text{CM}^\Gamma(R)$ ) of  $\Gamma$ -equivariant (respectively Cohen-Macaulay) modules with equivariant morphisms and  $\text{mod}_{\Gamma^\vee}(R)$  (respectively  $\text{CM}_{\Gamma^\vee}(R)$ ) of  $\Gamma^\vee$ -graded (respectively Cohen-Macaulay) modules with degree 0 morphisms are equivalent.

The autoequivalence of  $\text{mod}_{\Gamma^\vee}(R)$  corresponding to tensoring by a linear character of  $\Gamma$  is of course the functor of degree shift by that character, thought of as an element of  $\Gamma^\vee$ .

In what follows we will be almost exclusively concerned with finite subgroups  $G$  of  $\text{SL}(2, \mathbb{C})$  and the one-dimensional groups arising as their products inside  $\text{GL}(2, \mathbb{C})$  with a  $\mathbb{C}^*$ .

**Definition 2.1.12.** For  $G$  a finite subgroup of  $\text{SL}(2, \mathbb{C})$  we refer to any group  $\Gamma$  of the form  $\Gamma = G \cdot \mathbb{C}^* \subset \text{GL}(2, \mathbb{C})$  as an extended McKay group.

**Definition 2.1.13.** When an extended McKay group  $\Gamma$  acts on a ring  $R$  in such a way that  $0 \in R$  lies in the closure of every  $\Gamma$ -orbit (equivalently:  $R$  has a unique  $\Gamma$ -homogeneous maximal ideal) we say that  $R$  is equivariant-local.

Whenever we are in the situation of an extended McKay group  $\Gamma = G \cdot \mathbb{C}^*$  acting on a polynomial ring or hypersurface singularity  $R$  it is to be understood that the diagonal  $\mathbb{C}^*$  in  $\Gamma$  is acting diagonally on the coordinates  $x_i$ , making the ring  $R$  equivariant-local.

In this situation,  $\Gamma$  induces an isotypic decomposition of  $R$  by the characters of  $G$

$$R = \bigoplus_{\chi \in G^\vee} R_\chi.$$

Each component  $R_\chi$  is graded by the dual of the quotient  $\mathbb{C}^*$

$$R_\chi = \bigoplus_{\lambda \in \mathbb{Z}} R_{\lambda, \chi}.$$

Replacing the direct sum by a direct product here

$$\bigoplus_{\lambda \in \mathbb{Z}} R_{\lambda, \chi} \longmapsto \prod_{\lambda \in \mathbb{Z}} R_{\lambda, \chi}$$

takes us to the complete local ring  $\hat{R}$ , now carrying a  $G$ -action. This operation yields a functor  $\mathrm{CM}_\Gamma(R) \longrightarrow \mathrm{CM}_G(\hat{R})$ .

**Lemma 2.1.14.** Both  $\mathrm{CM}_\Gamma(R)$  and  $\mathrm{CM}_G(\hat{R})$  are Krull-Schmidt categories.

*Proof.* The category  $\mathrm{CM}_\Gamma(R)$  is closed under direct summands (in  $\mathrm{mod}_\Gamma(R)$ ), hence has split idempotents. Moreover the spaces of equivariant morphisms  $\mathrm{Hom}_{\mathrm{CM}_\Gamma(R)}$  are finite dimensional, hence  $\mathrm{CM}_\Gamma(R)$  is Krull-Schmidt ([34] Corollary 4.4).  $\mathrm{CM}_G(\hat{R})$  is module category over a complete local ring, hence Krull-Schmidt classically.  $\square$

At various points in this thesis we will more or less implicitly rely on results that hold true in the complete local case. In particular the existence of almost split sequences in and Frobenius structures (see below, Section 2.3) on  $\mathrm{CM}_\Gamma(R)$ . The  $\mathbb{C}^*$ -equivariant (that is,  $\mathbb{Z}$ -graded) case is dealt with by Yoshino [46] Chapter 15.

Our set-up involves the extra complication of the finite group  $G$ . We appeal to work of Auslander, Reiten and Riedtmann on twisted group rings in the setting of dualizing  $R$ -varieties (see [7] or [8] for a definition).

**Proposition 2.1.15** (Auslander-Reiten). Let  $\hat{R}$  be an isolated singularity, finite over a complete regular local ring. Then  $\mathrm{CM}(\hat{R})$  is a dualizing  $R$ -variety.

*Proof.* [7] Proposition 2.4.  $\square$

**Proposition 2.1.16.** Let  $\hat{R}$  be an isolated singularity, finite over a complete regular local ring, upon which a finite group  $G$  acts. Then  $\mathrm{CM}_G(\hat{R})$  has almost split sequences.

*Proof.* By Proposition 2.1.15 and [40] Theorem 3.8. See also [6] Proposition 5.1.  $\square$

**Proposition 2.1.17.** Let  $\hat{R}$  be an isolated singularity, finite over a complete regular local ring, upon which a finite group  $G$  acts. Then the projective objects and the injective objects coincide in  $\mathrm{CM}_G(\hat{R})$  if and only if they do in  $\mathrm{CM}(\hat{R})$ .

*Proof.* [40] Proposition 3.3 with the induction-restriction adjunction  $\mathrm{CM}(\hat{R}) \rightleftarrows \mathrm{CM}_G(\hat{R})$ .  $\square$

## 2.2 Matrix factorizations

Let  $S$  be a complete local ring and  $0 \neq f \in S$  such that  $R = S/(f)$  is a Cohen-Macaulay hypersurface. Let  $M_0$  and  $M_1$  be finitely generated free  $S$ -modules.

**Definition 2.2.1.** A matrix factorization  $(\phi_0, \phi_1)$  of  $f$  over  $S$  is a pair of morphisms

$$\phi_0 : M_0 \longrightarrow M_1, \quad \phi_1 : M_1 \longrightarrow M_0$$

such that

$$\phi_0 \circ \phi_1 = f \cdot 1_{M_1} \quad \text{and} \quad \phi_1 \circ \phi_0 = f \cdot 1_{M_0}.$$

Since  $S$  is an integral domain multiplication by  $f$  is injective and hence both  $\phi_0$  and  $\phi_1$  are injective. Thus  $M_0$  and  $M_1$  have the same rank; call this number the rank of the matrix factorization.

**Definition 2.2.2.** A morphism between matrix factorizations  $(\phi_0, \phi_1)$  and  $(\phi'_0, \phi'_1)$  is a pair of  $S$ -module morphisms  $(\alpha_0, \alpha_1)$  such that

$$\begin{array}{ccccc} M_0 & \xrightarrow{\phi_0} & M_1 & \xrightarrow{\phi_1} & M_0 \\ \downarrow \alpha_0 & & \downarrow \alpha_1 & & \downarrow \alpha_0 \\ M'_0 & \xrightarrow{\phi'_0} & M'_1 & \xrightarrow{\phi'_1} & M'_0 \end{array}$$

commutes. Write  $\text{MF}(S, f)$  for the additive category of matrix factorizations of  $f$  over  $S$ .

Let  $(\phi_0, \phi_1)$  be a matrix factorization of  $f$  of rank  $n$ , so  $M_0 \cong M_1 \cong S^n$ . Then  $\phi_0$  and  $\phi_1$  induce  $R$ -module morphisms  $\bar{\phi}_0$  and  $\bar{\phi}_1$  on the quotient  $R^n$ . We thus obtain a complex of free  $R$ -modules

$$\dots \xrightarrow{\bar{\phi}_0} R^n \xrightarrow{\bar{\phi}_1} R^n \xrightarrow{\bar{\phi}_0} R^n \xrightarrow{\bar{\phi}_1} \dots$$

The kernel of each map consists of those elements  $r \in R^n$  which are sent to a multiple of  $f = \phi_0 \circ \phi_1 = \phi_1 \circ \phi_0$ . But both  $\phi_0$  and  $\phi_1$  are injective, so the complex is exact. Since  $fS^n \subset \phi_0(S^n)$ , there is a surjection  $R^n \longrightarrow S^n/\phi_0(S^n)$  with kernel  $\bar{\phi}_0(R^n)$ . Thus we have a 2-periodic  $R$ -free resolution

$$\dots \xrightarrow{\bar{\phi}_0} R^n \xrightarrow{\bar{\phi}_1} R^n \xrightarrow{\bar{\phi}_0} R^n \xrightarrow{\bar{\phi}_1} R^n \xrightarrow{\bar{\phi}_0} R^n \longrightarrow S^n/\phi_0(S^n) \longrightarrow 0$$

of the cokernel  $\text{coker}(\phi_0)$ , itself an  $R$ -module since  $f\text{coker}(\phi_0) = 0$ . By the depth lemma it follows that  $\text{coker}(\phi_0)$  is a Cohen-Macaulay module over  $R$ .

Any morphism  $(\phi_0, \phi_1) \longrightarrow (\phi'_0, \phi'_1)$  of matrix factorizations of  $f$  will induce maps on the corresponding resolutions and hence between the cokernels. We thus obtain an



additive functor

$$\begin{aligned} \text{Coker} : \text{MF}(S, f) &\longrightarrow \text{CM}(R) \\ (\phi_0, \phi_1) &\longmapsto \text{coker}(\phi_0). \end{aligned}$$

Observe that  $\text{Coker}(1, f) = 0$  and  $\text{Coker}(f, 1) = R$ .

**Definition 2.2.3.** Let  $\mathcal{C}$  be an additive category and  $\mathcal{P}$  a collection of objects of  $\mathcal{C}$ . We define the stabilization  $\underline{\mathcal{C}}_{\mathcal{P}}$  of  $\mathcal{C}$  with respect to  $\mathcal{P}$  to be the category whose objects are the objects of  $\mathcal{C}$  and whose morphisms are given by

$$\text{Hom}_{\underline{\mathcal{C}}_{\mathcal{P}}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B) / \mathcal{P}(A, B),$$

where  $\mathcal{P}(A, B)$  denotes the subgroup generated by all morphisms which factor through any summand of a finite sum of objects in  $\mathcal{P}$ . Note in particular that all objects in  $\mathcal{P}$  are isomorphic to the zero object in  $\underline{\mathcal{C}}_{\mathcal{P}}$ .

**Theorem 2.2.4** (Eisenbud [16]). Let  $\mathcal{P} = \{(1, f)\} \subset \mathcal{Q} = \{(1, f), (f, 1)\} \subset \text{MF}(S, f)$  and  $\mathcal{R} = \{R\} \subset \text{CM}(R)$ . Then Coker induces equivalences of categories

$$\underline{\text{MF}}_{\mathcal{P}}(f, S) \cong \text{CM}(R)$$

and

$$\underline{\text{MF}}_{\mathcal{Q}}(f, S) \cong \underline{\text{CM}}_{\mathcal{R}}(R).$$

*Proof.* [16] Theorem 6.1 and Corollary 6.3, though see [46] Theorem 7.4. □

**Example 2.2.5.** Let  $S = \mathbb{C}[[x]]$  and  $f = x^k$ ,  $k > 1$ . Then  $R = S/(f)$  is a zero dimensional singularity of type  $A_{k-1}$ ; the  $k$ -fold cover of a point. Since  $S$  is a principal ideal domain, Smith normal form tells us that every object in  $\text{MF}(S, f)$  decomposes as a direct sum of rank one factorizations. There are  $k+1$  ways to factorize the polynomial  $x^k$ , giving us the  $k+1$  indecomposable matrix factorizations

$$M_i := S \xrightarrow{x^i} S \xrightarrow{x^{k-i}} S.$$

For  $0 \leq i < k$  there is precisely one irreducible morphism  $(1, x) : M_i \longrightarrow M_{i+1}$  and for  $k \geq i > 0$  precisely one irreducible morphism  $(x, 1) : M_i \longrightarrow M_{i-1}$ . The Auslander-Reiten quiver of  $\text{MF}(S, f)$  therefore looks like

$$M_0 \rightleftarrows M_1 \rightleftarrows \dots \rightleftarrows M_{k-1} \rightleftarrows M_k.$$

Stabilizing with respect to  $(1, x^k)$  we obtain the category of Cohen-Macaulay modules over  $R = \mathbb{C}[[x]]/(x^k)$  by deleting  $M_0$ .

**Definition 2.2.6** (Simple curve singularities). Let  $S = \mathbb{C}[[x, y]]$ . A polynomial  $f \in S$  has a finite number of isoclasses of indecomposable matrix factorizations if and only if it is a simple singularity. In this case  $f$  is, up to a change of variables, equal to one of the polynomials

$$(A_n) \quad x^2 + y^{n+1} \quad (n \geq 1),$$

$$(D_n) \quad x^2y + y^{n-1} \quad (n \geq 4),$$

$$(E_6) \quad x^3 + y^4,$$

$$(E_7) \quad x^3 + xy^3,$$

$$(E_8) \quad x^3 + y^5$$

([46] Chapters 8 and 9). When we later refer to simple singularities in the non-complete local case it should be understood that we mean polynomials  $f$  isomorphic to one of these five types.

The calculation of  $\text{MF}(S, f)$  for the simple singularities is elementary and well-known (see for example [46]); for illustration we reproduce some of the facts here.

( $A_n$ ) Let  $f = x^2 - y^{n+1}$ . Then  $R = S/(f)$  is a simple singularity of type  $A_n$ . When  $n$  is even this variety is irreducible; when  $n$  is odd it has two isomorphic components, degree  $\frac{n+1}{2}$  curves meeting at a point.

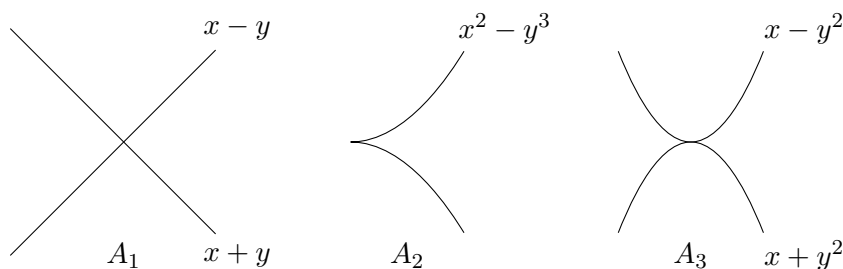


Figure 2-1:  $A_n$  singularities.

There are always the two trivial matrix factorizations of  $f$ ,

$$R := (f, 1), \quad \bar{R} := (1, f);$$

and the rank two factorizations

$$I_j := \left( \begin{pmatrix} x & -y^j \\ y^{n+1-j} & -x \end{pmatrix}, \begin{pmatrix} x & -y^j \\ y^{n+1-j} & -x \end{pmatrix} \right)$$

whose cokernels are isomorphic to the ideals  $(x, y^j)$  for  $1 \leq j \leq \frac{n-1}{2}$ . In addition, when  $n$  is odd, there are two factorizations

$$N_+ := ((x + y^{\frac{n+1}{2}}), (x - y^{\frac{n+1}{2}})), \quad N_- := ((x - y^{\frac{n+1}{2}}), (x + y^{\frac{n+1}{2}}))$$

corresponding to the structure sheaves of the two components.

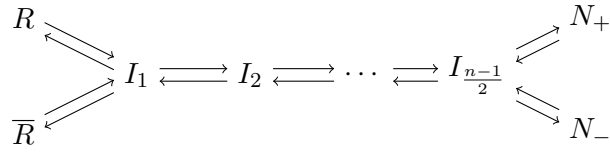


Figure 2-2: Matrix factorizations of  $A_n$  ( $n$  odd).

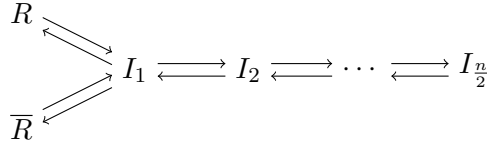


Figure 2-3: Matrix factorizations of  $A_n$  ( $n$  even).

It is worth observing that the  $A_1$  case is faintly degenerate, in that no matrix factorizations of type  $I_j$  appear ( $I_0 \cong R$ ). Thus the category of matrix factorizations over  $R$  in this case looks like figure 2-4.

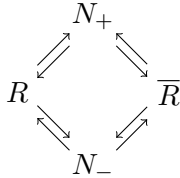


Figure 2-4: Matrix factorizations of  $A_1$ .

( $D_4$ ) Let  $f = x^2y - y^3 = y(x + y)(x - y)$  then  $R = S/(f)$  is a simple curve singularity of type  $D_4$ . It consists of three lines intersecting in a point.

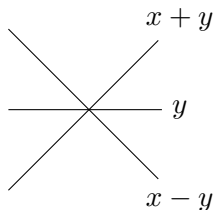


Figure 2-5:  $D_4$ .

There are ten indecomposable matrix factorizations of  $f$  over  $S$ , as follows. First we have the two trivial factorizations

$$R := (f, 1), \quad \bar{R} := (1, f)$$

whose cokernels are respectively the  $R$ -modules  $R$  and  $0$  corresponding to the structure sheaf of the singularity and the zero sheaf. Next there are the three pairs

$$B := ((x + y)(x - y), y), \quad A := (y, (x + y)(x - y));$$

$$C_+ := ((x + y)y, (x - y)), \quad D_+ := ((x - y), y(x + y))$$

and

$$C_- := ((x - y)y, (x + y)), \quad D_- := ((x + y), y(x - y)),$$

corresponding to the sheaves locally free away from  $(0, 0)$  supported respectively on each of the three components and its complement. Finally there are two rank-two factorizations

$$X := \left( \begin{pmatrix} x & -y \\ -y^2 & xy \end{pmatrix}, \begin{pmatrix} xy & y \\ y^2 & x \end{pmatrix} \right), \quad Y := \left( \begin{pmatrix} xy & y \\ y^2 & x \end{pmatrix}, \begin{pmatrix} x & -y \\ -y^2 & xy \end{pmatrix} \right)$$

whose cokernels are isomorphic to the ideals  $(x, y^2)$  and  $(x, y)$  of  $R$ , corresponding to the ideal sheaf of the intersection point and of the space of 1-jets in the  $x$  direction, respectively.

### 2.2.1 Equivariant matrix factorizations

Observe that all of the simple curve singularities have a natural scaling, with a  $\mathbb{C}^*$  acting diagonally on the coordinates  $x, y$ . In many cases there are also symmetries of

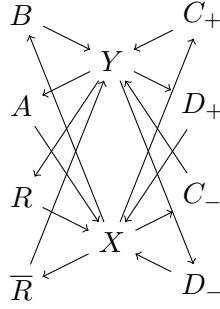


Figure 2-6: Matrix factorizations of  $D_4$ .

finite order. For instance, on the  $D_4$  singularity an  $S_3$  acts by permuting the three isomorphic components. We are led to consider matrix factorizations equivariant with respect to the action of these groups.

Suppose  $S$  is a ring upon which an extended McKay group  $\Gamma$  acts and that  $f \in S$  is a semi-invariant for this action. Tensoring by the associated character  $\chi_f$  defines a shift functor

$$\begin{aligned} [2] : \text{mod}_\Gamma(S) &\longrightarrow \text{mod}_\Gamma(S) \\ M &\longmapsto M[2] := M \otimes \chi_f. \end{aligned}$$

Since the natural map  $f : M \longrightarrow M[2]$  is equivariant, multiplication by  $f$  defines a natural transformation from the identity functor to  $[2]$ .

**Definition 2.2.7.** A  $\Gamma$ -equivariant matrix factorization of  $f$  over  $S$  is a diagram

$$\dots \xrightarrow{\phi_{-1}} M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} \dots,$$

of  $\Gamma$ -equivariant free  $S$  modules, 2-periodic in the sense that  $M_{i+2} = M_i[2]$ ,  $\phi_{i+2} = \phi_i[2]$  and  $\phi_{i+1} \circ \phi_i = f$  for all  $i \in \mathbb{Z}$ .

A morphism  $\alpha : M \longrightarrow M'$  of  $\Gamma$ -equivariant matrix factorizations of  $f$  is a sequence of morphisms  $\alpha_i : M_i \longrightarrow M'_i$ , 2-periodic in the sense that  $\alpha_{i+2} = \alpha_i$  for all  $i$ , such that every square

$$\begin{array}{ccc} M_i & \xrightarrow{\phi_i} & M_{i+1} \\ \downarrow \alpha_i & & \downarrow \alpha_{i+1} \\ M'_i & \xrightarrow{\phi_i} & M'_{i+1} \end{array}$$

commutes. Denote the additive category of  $\Gamma$ -equivariant matrix factorizations of  $f$  by  $\text{MF}_\Gamma(S, f)$ .

The category  $\mathbf{MF}_\Gamma(S, f)$  comes with an embedding

$$\begin{aligned} j : \mathbf{Free}_\Gamma(S) &\longrightarrow \mathbf{MF}_\Gamma(S, f) \\ M &\longmapsto \dots \rightarrow M \xrightarrow{\text{id}} M \xrightarrow{f} M[2] \rightarrow \dots \end{aligned}$$

and an autoequivalence

$$\sigma : \mathbf{MF}_\Gamma(S, f) \longrightarrow \mathbf{MF}_\Gamma(S, f),$$

shifting all diagrams one place to the left and negating differentials. Abusing notation, let  $[2]$  denote the shift functor that  $\mathbf{MF}_\Gamma(S, f)$  inherits from  $\mathbf{mod}_\Gamma(S)$ .

**Remark 2.2.8.** If  $\Gamma$  is diagonalizable then  $R$  may be thought of as a  $\Gamma^\vee$ -graded ring and  $\mathbf{Free}_\Gamma(S)$  is equivalent to the category of free  $\Gamma^\vee$ -graded  $R$ -modules. Definition 2.2.7 gives in this situation the definition of  $\Gamma^\vee$ -graded matrix factorizations of the homogeneous function  $f$ .

We restate Theorem 2.2.4 for the equivariant situation.

**Theorem 2.2.9.** Let  $\mathcal{P} = \{(1, f)\} \subset \mathcal{Q} = \{(1, f), (f, 1)\} \subset \mathbf{MF}_\Gamma(S, f)$  and  $\mathcal{R} = \{R\} \subset \mathbf{CM}_\Gamma(S/f)$ . Then Coker induces equivalences of categories

$$\underline{\mathbf{MF}}_{\Gamma, \mathcal{P}}(f, S) \cong \mathbf{CM}_\Gamma(S/f)$$

and

$$\underline{\mathbf{MF}}_{\Gamma, \mathcal{Q}}(f, S) \cong \underline{\mathbf{CM}}_{\Gamma, \mathcal{R}}(R).$$

*Proof.* This is a minor generalization of [29] Theorem 2.2, which deals with the case of  $\Gamma$  abelian.  $\square$

## 2.3 Exact and Frobenius categories

For a detailed discussion of (Quillen) exact categories, see [11]. We will make do here with the following definition.

**Definition 2.3.1.** Let  $\mathcal{E}$  be a chosen class of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

in some additive category  $\mathcal{A}$ , for which each  $i$  (the *inflation*) is a kernel of its  $d$  (the *deflation*), and each  $d$  a cokernel of its  $i$ . The pairs  $(i, d)$  are sometimes called *conflations*, though we will simply refer to them as exact sequences. The class  $\mathcal{E}$  constitutes a (Quillen) exact structure on  $\mathcal{A}$  if it satisfies the following axioms.

E0 The identity morphism  $0 \rightarrow 0$  on the zero object is a deflation.

E1 The composition of two deflations is a deflation.

E1\* The composition of two inflations is an inflation.

E2 For any deflation  $d : Y \rightarrow Z$  and morphism  $f : Z' \rightarrow Z$  there exists a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ \downarrow f' & & \downarrow f \\ Y & \xrightarrow{d} & Z \end{array}$$

with  $d'$  a deflation.

E2\* For any inflation  $i : X \rightarrow Y$  and morphism  $g : X \rightarrow X'$  there exists a pushforward diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow g & & \downarrow g' \\ X' & \xrightarrow{i'} & Y' \end{array}$$

with  $i'$  an inflation.

**Lemma 2.3.2.** Let  $\mathcal{A}$  be an abelian category,  $\mathcal{C} \subset \mathcal{A}$  a full extension-closed subcategory. Then  $\mathcal{C}$  is exact with respect to the class of conflations consisting of the sequences which are short exact in  $\mathcal{C}$ .

*Proof.* Check the axioms. □

**Example 2.3.3.** The category of vector bundles on  $\mathbb{P}^1$  is not an abelian category. For example, the map  $x : \mathcal{O}(-1) \rightarrow \mathcal{O}$  in  $\text{Vect}(\mathbb{P}^1)$  has kernel and cokernel both equal to zero, but it is not an isomorphism. However, being a full extension-closed subcategory of  $\text{Coh } \mathbb{P}^1$ ,  $\text{Vect}(\mathbb{P}^1)$  inherits an exact structure: the conflations are the short exact sequences of vector bundles.

**Example 2.3.4.** The category of maximal Cohen-Macaulay modules over a Cohen-Macaulay ring is not abelian, but it is exact (again by the lemma).

An object  $P$  in an exact category  $\mathcal{C}$  is projective if the functor  $\text{Hom}_{\mathcal{C}}(P, -)$  takes conflations to short exact sequences. Dually it is injective if  $\text{Hom}_{\mathcal{C}}(-, P)$  takes inflations to short exact sequences. We say that  $\mathcal{A}$  has *enough projectives* (dually, injectives) if every object fits at the end (dually, beginning) of a conflation whose middle term is projective (dually, injective).

**Definition 2.3.5.** If the category  $\mathcal{A}$  has enough projectives and injectives with respect to some exact structure  $\mathcal{E}$  and its classes of injective and projective objects coincide then it is said to be Frobenius with respect to this structure.

Let  $R$  be a Gorenstein ring with canonical module  $K_R \cong R$ . Then it is classical that  $\text{CM}(R)$  is a Frobenius category (see for example [20] 1.3, or 2.3.8 immediately below for an argument).

Now suppose  $R$  carries an action of an extended McKay group  $\Gamma$ . The extension making  $\text{Ext}^i(k, K_R)$  nontrivial in the dimension of  $R$  is most probably not equivariant. That is to say that in general, the regular module with its natural equivariant structure is not canonical in  $\text{CM}_\Gamma(R)$ : it needs to be twisted by a character of  $\Gamma$ . We are almost exclusively concerned with two classes of rings: polynomial rings  $R = \mathbb{C}[x_1, \dots, x_n]$  and isolated hypersurface singularities  $S = R/f$ . In this case we can be quite explicit about the form of  $K_R$  and  $K_S$ .

**Proposition 2.3.6** (Equivariant canonical module). Let  $R = \mathbb{C}[x_1, \dots, x_n]$  be a polynomial ring acted on by an extended McKay group  $\Gamma$ ,  $f$  a semiinvariant and  $S = R/f$ . Then the equivariant canonical modules look like

$$K_R \cong R \otimes \omega_\Gamma^{-1}$$

and

$$K_S \cong S \otimes \omega_\Gamma^{-1} \otimes \chi_f$$

where  $\omega_\Gamma$  is the determinant representation and  $\chi_f$  the character of  $\Gamma$  associated to  $f$ .

*Proof.* The  $\Gamma$ -equivariant Koszul complex of the homogeneous maximal ideal  $(x_1, \dots, x_n)$  of  $R$  is a free resolution of  $\mathbb{C}$  in  $\text{CM}_\Gamma(R)$  with last term  $R \otimes \omega_\Gamma^{-1}$ . So  $\text{Ext}^n(\mathbb{C}, R \otimes \omega_\Gamma^{-1}) \cong \mathbb{C}$  and all other Exts vanish. That is  $R \otimes \omega_\Gamma^{-1}$  is canonical in  $\text{CM}_\Gamma(R)$ . By the adjunction formula  $S \otimes \omega_\Gamma^{-1} \otimes \chi_f$  is canonical for  $S$ .  $\square$

Let  $R$  be Gorenstein with a  $\Gamma$ -action. As a full extension-closed subcategory of  $\text{mod}_\Gamma(R)$ ,  $\text{CM}_\Gamma(R)$  inherits an exact structure which coincides with the structure induced by forgetting equivariance.

**Lemma 2.3.7.** The projective objects in the forgetful exact structure on  $\text{CM}_\Gamma(R)$  are those that are projective when equivariance is forgotten (so they look like  $P \otimes V$  for  $P \in \text{CM}(R)$  projective and  $V$  a representation of  $\Gamma$ ).

*Proof.* Every projective  $P \in \text{CM}(R)$  as a direct summand of a free module carries a natural equivariant structure; these objects are trivially projective in  $\text{CM}_\Gamma(R)$ . Furthermore, tensoring any such with a representation of  $\Gamma$  preserves projectivity.



Any projective  $P \in \text{CM}_\Gamma(R)$  is also projective in  $\text{mod}_\Gamma(R)$ , which is to say it is projective as a module over the twisted group ring  $R \rtimes \Gamma$ . But  $R \rtimes \Gamma$  is projective as a module over  $R$ , so forgetting equivariance,  $P$  is a projective  $R$ -module. □

**Proposition 2.3.8.** Let  $R$  be a Gorenstein ring,  $\Gamma$  an extended McKay group acting on  $R$ . Then  $\text{CM}_\Gamma(R)$  is a Frobenius category.

*Proof.* The proof is as in the classical case, relying on the fact that the  $\Gamma$ -equivariant canonical module  $K_R$  is trivial up to a twist, so  $\Gamma$ -equivariant Cohen-Macaulay modules are reflexive up to a twist and the standard facts (see for example [10] 3.3.10)

- $\text{Hom}_{\text{CM}_\Gamma(R)}(M, K_R) \in \text{CM}_\Gamma(R)$
- $\text{Ext}_{\text{CM}_\Gamma(R)}^i(M, K_R) = 0 \quad \forall i > 0$

hold for all  $M \in \text{CM}_\Gamma(R)$ . Hence in particular  $R$  has finite injective dimension and  $\text{Ext}_{\text{CM}_\Gamma(R)}^i(M, P) = 0$  for all projectives  $P$  and  $i > 0$ . So all projectives are injective.

Now  $\text{CM}_\Gamma(R)$  certainly has enough projectives. Let  $M \in \text{CM}_\Gamma(R)$  and take a projective resolution  $P_\bullet \rightarrow M^* \otimes V$  of an appropriate twist of its  $K_R$ -dual. Dualizing this resolution embeds  $M$  in a projective-injective; so  $\text{CM}_\Gamma(R)$  has enough injectives and every injective, being therefore a summand of a projective is projective. □

Beware: a category can carry more than one exact structure and may therefore be or fail to be Frobenius in more than one way.

**Example 2.3.9.** Consider again the category  $\text{Vect } \mathbb{P}^1$ . As a subcategory of  $\text{Coh } \mathbb{P}^1$  this inherits an exact structure (Example 2.3.3). However it is not Frobenius with respect to this structure: there are no projectives (so in particular there aren't enough).

But  $\text{Vect } \mathbb{P}^1$  is equivalent to the category of  $\mathbb{Z}$ -graded Cohen-Macaulay modules on  $R = \mathbb{C}[x, y]$  (Grothendieck). This category consists of all shifts of the free module  $R$ . It therefore inherits an exact structure as a subcategory of  $\text{mod-}R$ . It is Frobenius with respect to this structure: everything is projective-injective.

One may also think of  $\mathbb{P}^1$  as the  $\mathbb{C}^*$  quotient of the Veronese cone, that is as  $\text{Proj } \mathbb{C}[x^2, xy, y^2]$ . So  $\text{Vect } \mathbb{P}^1 \cong \text{CM}^{\mathbb{Z}} \mathbb{C}[x^2, xy, y^2]$ , where  $x^2$ ,  $xy$  and  $y^2$  are in degree two. Now  $A = \mathbb{C}[[x^2, xy, y^2]]$  is an  $A_1$  singularity; it has two indecomposable CM modules, the free one and the maximal ideal  $\mathfrak{m}$ . The category  $\text{CM}^{\mathbb{Z}} A$  is Frobenius with respect to the inherited exact structure. The projective-injectives are the shifts of  $A$ .

**Definition 2.3.10.** For a Frobenius category  $\mathcal{F}$ , define its stable category,  $\underline{\mathcal{F}}$  to be the category whose objects are the objects of  $\mathcal{F}$  and whose morphisms are given by

$$\mathrm{Hom}_{\underline{\mathcal{F}}}(M, M') := \mathrm{Hom}_{\mathcal{F}}(M, M') / \mathbf{P}(M, M')$$

for  $M, M' \in \mathcal{F}$ , where  $\mathbf{P}(M, M')$  is the subspace of morphisms which factor through a projective-injective.

**Lemma 2.3.11.**  $\mathrm{MF}(R, f)$  carries a natural exact structure. The conflations are precisely the sequences  $M' \rightarrow M \rightarrow M''$  for which the corresponding sequences of free  $R$  modules  $P'_i \rightarrow P_i \rightarrow P''_i$  ( $i = 0, 1$ ) are short exact.

*Proof.* Consider first the category of 2-cycles on  $\mathrm{mod}\text{-}R$  over  $f$ , in the sense of Lenzing [35]. The objects of this category are 2-periodic sequences of  $R$ -modules for which consecutive morphisms compose to  $f$ . It is abelian, with kernels and cokernels formed pointwise. The category  $\mathrm{MF}(R, f)$  appears as the full extension-closed subcategory of complexes of free modules. It therefore carries the exact structure as claimed (see for example [11]).  $\square$

**Proposition 2.3.12.** With respect to its natural exact structure,  $\mathrm{MF}(R, f)$  is Frobenius.

*Proof.* See [14] Example 3.4, also [31] Example 3.5 for the case of (untwisted) complexes; the argument is essentially the same. Check first that for any free module  $P$ , the matrix factorizations  $(\mathrm{id}_P, f \cdot \mathrm{id}_P)$  and  $(f \cdot \mathrm{id}_P, \mathrm{id}_P)$  are projective and injective as follows. For projectivity of  $P := (\mathrm{id}_P, f \cdot \mathrm{id}_P)$  we need to show that morphisms from  $P$  lift along deflations. So let  $\phi : B \rightarrow C$  be a deflation and  $\gamma : P \rightarrow C$  a morphism. Since the underlying modules are all free,  $\gamma$  lifts pointwise. In particular there is a  $\beta_0 : P \rightarrow B_0$  with  $\gamma_0 = \phi_0 \beta_0$ . A diagram chase shows that the composition  $b_0 \beta_0$  lifts  $\gamma_1$  along  $\phi_1$ . This defines the desired morphism  $\beta : P \rightarrow C$ . Injectivity is similar.

Now let

$$A = \left( A_0 \begin{array}{c} \xrightarrow{a_0} \\ \xleftarrow{a_1} \end{array} A_1 \right)$$

be any matrix factorization. By the above,  $IA := (\mathrm{id}_A, f \cdot \mathrm{id}_A) \oplus (f \cdot \mathrm{id}_A, \mathrm{id}_A)$  is projective and injective. Now the pointwise sequences

$$A_0 \xrightarrow{\begin{pmatrix} a_0 \\ 1 \end{pmatrix}} A_1 \oplus A_0 \xrightarrow{\begin{pmatrix} -1 & a_0 \end{pmatrix}} A_1$$

and

$$A_1 \xrightarrow{\begin{pmatrix} 1 \\ a_1 \end{pmatrix}} A_1 \oplus A_0 \xrightarrow{\begin{pmatrix} a_1 & -1 \end{pmatrix}} A_0$$

are short exact and define a conflation

$$A \longrightarrow IA \longrightarrow TA,$$

where  $T$  denotes the shift functor which moves all diagrams one place to the left and negates the differentials. Therefore  $\text{MF}(R, f)$  has enough injectives. Projectives are similar.

Observe that this conflation splits if and only if  $A$  is injective if and only if  $A$  is projective if and only if  $A$  is homotopic to zero. So  $\text{MF}(R, f)$  is Frobenius, and its class of projective-injectives is all of those matrix factorizations which are homotopic to zero or equivalently direct summands of matrix factorizations of the form  $(\text{id}_P, f \cdot \text{id}_P) \oplus (f \cdot \text{id}_P, \text{id}_P)$ .  $\square$

**Corollary 2.3.13.**  $\underline{\text{MF}}(R, f) = \text{CM}(R/f)$  as triangulated categories.

*Proof.* Theorem 2.2.4 above.  $\square$

### 2.3.1 The Chen quotient.

Let  $\mathcal{A}$  be a Frobenius category and  $\mathcal{P}$  be its subcategory of projective-injectives.

**Definition 2.3.14.** Given some subcategory  $\mathcal{S} \subset \mathcal{A}$ , a right  $\mathcal{S}$ -approximation of  $X \in \mathcal{A}$  is a morphism  $\phi : S \longrightarrow X$  with  $S \in \mathcal{S}$  such that any morphism from an object of  $\mathcal{S}$  to  $X$  factors through  $\phi$ . Define a left  $\mathcal{S}$ -approximation dually.

**Definition 2.3.15** (Chen [14]). Define a Chen subcategory of  $\mathcal{A}$  to be a subcategory  $\mathcal{F} \subset \mathcal{P}$  such that for any  $A \in \mathcal{A}$ ,

1. there exists  $F_A \in \mathcal{F}$  and  $P_A \in \mathcal{P}$  and a sequence of objects

$$A \xrightarrow{i_A} F_A \xrightarrow{p_A} P_A$$

such that  $i_A$  is a left  $\mathcal{F}$ -approximation of  $A$  and  $p_A$  is a pseudo-cokernel of  $i_A$  (that is:  $p_A \circ i_A = 0$  and any other such morphism factors through  $p_A$ , though not uniquely);

2. dually there exists  $F^A \in \mathcal{F}$  and  $P^A \in \mathcal{P}$  and a sequence of objects

$$P^A \xrightarrow{i^A} F^A \xrightarrow{p^A} A$$

such that  $p^A$  is a right  $\mathcal{F}$ -approximation of  $A$  and  $i^A$  is a pseudo-kernel of  $i_A$  (a notion dual to that of the pseudo-cokernel).

**Theorem 2.3.16** (Chen). Let  $\mathcal{A}$  be a Frobenius category with conflations  $\mathcal{E}$  and projective-injectives  $\mathcal{P}$ . Let  $\mathcal{F}$  be a Chen subcategory of  $\mathcal{A}$ . Then  $\mathcal{A}/\mathcal{F}$ , the stabilization of  $\mathcal{A}$  with respect to  $\mathcal{F}$  is Frobenius with induced conflations  $\mathcal{E}_{\mathcal{F}}$  and projective-injectives  $\mathcal{P}/\mathcal{F}$ .

*Proof.* [14] Theorem 3.1. □

We have seen the projective-injectives of  $\text{MF}_R(f)$  are the direct summands of objects of the form  $(\text{id}_P, f \cdot \text{id}_P) \oplus (f \cdot \text{id}_P, \text{id}_P)$  (Proposition 2.3.12), and that stabilizing with respect to this class yields the stable category  $\underline{\text{CM}}(S)$  of Cohen-Macaulay modules (Theorem 2.2.4).

## 2.4 Orbifolds

### 2.4.1 Basic properties

Geigle-Lenzing weighted projective lines were introduced in [18], though the exposition of this section owes a certain amount to that of [1].

A weighted projective line  $\mathbb{X}$  is a smooth, rational Deligne-Mumford stack of dimension one with no generic stabilizer. In other words it is an orbifold with a finite number of orbifold points of finite cyclic isotropy and coarse space  $X \cong \mathbb{P}^1$ . The order  $p_i$  of the isotropy group at an orbifold point  $x_i \in X$  is called its weight, and the sequence  $\mathbf{p} = (p_1, \dots, p_n)$  the weight sequence of  $\mathbb{X} = \mathbb{X}_{\mathbf{p}}$ . We will sometimes refer to weighted projective lines simply as (one-dimensional) orbifolds.

An orbifold  $\mathbb{X}$  carries a line bundle  $\mathcal{O}(\mathbf{1}_{\mathbb{X}})$ , sections  $s$  of which give rise to the torsion sheaves  $\mathcal{O}_x$  lying over coarse points  $x \in X$ .

$$\mathcal{O}(-\mathbf{1}_{\mathbb{X}}) \xrightarrow{s} \mathcal{O} \longrightarrow \mathcal{O}_x$$

These sheaves  $\mathcal{O}_x$  are generically simple, write  $\mathcal{S}_x = \mathcal{O}_x$ , except over orbifold points  $x_i \in X$  where they have a filtration of length  $p_i$ . Let  $\mathcal{S}_{x_i}$  denote the simple top in this case, and  $\mathcal{O}(-x_i)$  the kernel

$$\mathcal{O}(-x_i) \longrightarrow \mathcal{O} \longrightarrow \mathcal{S}_{x_i}.$$

The dual bundles  $\mathcal{O}(x_i)$  and  $\mathcal{O}(\mathbf{1}_{\mathbb{X}})$  generate the picard group  $L_{\mathbf{p}} := \text{Pic}(\mathbb{X}_{\mathbf{p}})$  of  $\mathbb{X}$ , subject to the relations

$$\mathcal{O}(p_i x_i) = \mathcal{O}(\mathbf{1}_{\mathbb{X}}).$$

Thus

$$L_{\mathbf{p}} \cong \langle \vec{x}_1, \dots, \vec{x}_n, \mathbf{1}_{\mathbb{X}} \mid p_1 \vec{x}_1 = \dots = p_n \vec{x}_n = \mathbf{1}_{\mathbb{X}} \rangle,$$

a rank-one abelian group. The  $L_{\mathbf{p}}$ -graded Cox ring of  $\mathbb{X}$  is by definition

$$R_{\mathbf{p}} := \bigoplus_{\ell \in L} H^0(\mathcal{O}(\ell)).$$

Write  $V$  for the two-dimensional space  $H^0(\mathcal{O}(\mathbf{1}_{\mathbb{X}}))$ , and let  $S(V)$  be its symmetric algebra. To each orbifold point  $x_i \in X$  there corresponds (up to scale) a section  $\lambda_i \in V$  and a section  $X_i \in H^0(\mathcal{O}_{x_i})$ . We have

$$R_{\mathbf{p}} \cong \frac{S(V)[X_1, \dots, X_n]}{X_1^{p_1} - \lambda_1, \dots, X_n^{p_n} - \lambda_n}$$

a finitely generated  $L$ -graded Gorenstein ring ([1] Section 1) and

$$\mathbb{X} = \left[ \frac{\text{Spec } R_{\mathbf{p}} - 0}{L^{\vee}} \right]$$

where  $L^{\vee}$  is the affine algebraic group  $\text{Spec } \mathbb{C}[L_{\mathbf{p}}]$  (this is essentially Geigle-Lenzing's Proj construction). In particular weighted projective lines are examples of Mori dream stacks in the sense of [22], see also [24].

### 2.4.2 Coherent sheaves

Geigle-Lenzing define  $\mathbb{X}_{\mathbf{p}}$  to be  $\text{Proj}^{L_{\mathbf{p}}} R_{\mathbf{p}}$ , that is, the collection of  $L_{\mathbf{p}}$ -homogeneous primes of  $R_{\mathbf{p}}$  not containing the irrelevant ideal, with the Zariski topology. The structure sheaf is given by

$$\mathcal{O}(D(f)) = (R_{\mathbf{p}})_f$$

over principal open sets  $D(f)$ ,  $f \in R_{\mathbf{p}}$ : contrast this with the classical Proj, in which sections are the degree zero parts of the localizations, rather than the whole  $L_{\mathbf{p}}$ -graded ring  $(R_{\mathbf{p}})_f$ .

Coherent sheaves on  $\mathcal{O}$  are now singled out as those  $L_{\mathbf{p}}$ -graded  $\mathcal{O}$ -modules  $\mathcal{F}$  which satisfy the classical condition

- for all  $x \in \mathbb{X}$  there exists an open set  $U \subset \mathbb{X}$  and an exact sequence

$$\bigoplus \mathcal{O}(\ell_j)|_U \longrightarrow \bigoplus \mathcal{O}(\ell_i)|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

As we will see, the structure of  $\text{Coh}(\mathbb{X})$  is closely analogous to that of the category of coherent sheaves on  $\mathbb{P}^1$  (this perspective is developed carefully in the survey

[15]). Geigle-Lenzing modify the classical construction of Serre to define a sheafification functor

$$\begin{aligned} \sim : \text{mod}^{L_{\mathbf{p}}}(R_{\mathbf{p}}) &\longrightarrow \text{Coh}(\mathbb{X}) \\ M &\longmapsto (D(f) \mapsto M_f) =: \widetilde{M}, \end{aligned}$$

left adjoint to the graded global sections functor  $\Gamma : \mathcal{F} \mapsto \bigoplus_{l \in L_{\mathbf{p}}} \mathcal{F}(l)(\mathbb{X})$ .

**Theorem 2.4.1.** Sheafification  $\sim : \text{mod}^{L_{\mathbf{p}}}(R_{\mathbf{p}}) \longrightarrow \text{Coh}(\mathbb{X})$  is an exact functor with a right adjoint  $\Gamma : \text{Coh}(\mathbb{X}) \longrightarrow \text{mod}^{L_{\mathbf{p}}}(R_{\mathbf{p}})$  which is a full embedding satisfying  $\mathcal{F} \cong \widetilde{\Gamma(\mathcal{F})}$  for all  $\mathcal{F} \in \text{Coh}(\mathbb{X})$ . Furthermore, sheafification induces an equivalence

$$\text{mod}^{L_{\mathbf{p}}}(R_{\mathbf{p}})/\text{mod}_0^{L_{\mathbf{p}}}(R_{\mathbf{p}}) \longrightarrow \text{Coh}(\mathbb{X})$$

where  $\text{mod}_0^{L_{\mathbf{p}}}$  denotes the Serre subcategory of finite length modules.

*Proof.* [18] Section 1.8. □

In his classification of vector bundles on  $\mathbb{P}^1$ , Grothendieck proved that the sheafification functor induces an additive equivalence between the category of  $\mathbb{Z}$ -graded projective  $\mathbb{C}[x, y]$ -modules and  $\text{Vect}(\mathbb{P}^1)$ . We have an analogue here.

**Theorem 2.4.2.** The graded global sections functor induces an equivalence

$$\Gamma : \text{Vect}(\mathbb{X}) \longrightarrow \text{CM}^{L_{\mathbf{p}}}(R_{\mathbf{p}}).$$

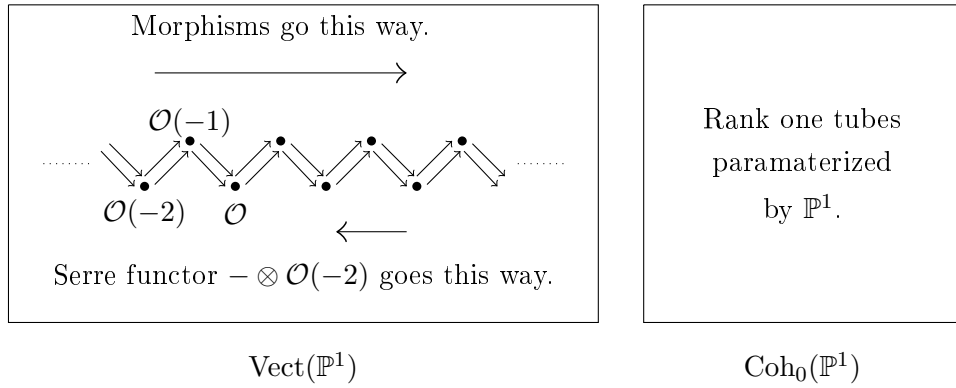
*Proof.* [18] Theorem 5.1 □

The explicit description of  $\text{Coh}(\mathbb{X})$  relies on the following fact, also directly analogous to the  $\mathbb{P}^1$  situation.

**Proposition 2.4.3.** Every coherent sheaf  $\mathcal{F}$  has a largest torsion subsheaf,  $t\mathcal{F}$ .  $\mathcal{F}/t\mathcal{F}$  is torsion-free and hence a vector bundle. Every  $\mathcal{F} \in \text{Coh}(\mathbb{X})$  decomposes as  $t\mathcal{F} \oplus \mathcal{E}$ , where  $\mathcal{E} \in \text{Vect}(\mathbb{X})$ .

*Proof.* [18] Proposition 2.4. □

Recall the structure of  $\text{Coh}(\mathbb{P}^1)$ . It is hereditary and Hom-finite and has Serre duality, with dualizing bundle  $\mathcal{O}(-2)$ .



The category  $\text{Coh}(\mathbb{P}^1)$  divides into two subcategories, the vector bundles  $\text{Vect}(\mathbb{P}^1)$  and the torsion sheaves  $\text{Coh}_0(\mathbb{P}^1)$ . The Auslander-Reiten quiver of  $\text{Vect}(\mathbb{P}^1)$  is a translation quiver with Serre functor (or Auslander-Reiten translate) given by tensoring with the canonical bundle  $\mathcal{O}(-2)$ . The category of torsion sheaves decomposes as

$$\text{Coh}_0(\mathbb{P}^1) = \coprod_{\lambda \in \mathbb{P}^1} \mathcal{T}_\lambda,$$

a disjoint collection of connected uniserial subcategories  $\mathcal{T}_\lambda$ , or (size one) tubes, parameterized by  $\mathbb{P}^1$ .

For a weighted projective line  $\mathbb{X}$  we have a similar picture.  $\text{Coh}(\mathbb{X})$  is hereditary and Hom-finite and has Serre duality ([18] Theorem 2.2) with dualizing bundle  $\mathcal{O}(\omega)$ , where

$$\omega := (n-2)\mathbf{1}_{\mathbb{X}} - \sum_{i=1}^n x_i \in L,$$

the dualizing element of  $L$ . The category of torsion sheaves  $\text{Coh}_0(\mathbb{X})$  decomposes as before into connected uniserial categories parameterized by the line  $\mathbb{X}$  itself. These tubes are generically size one, but over orbifold points  $x_i \in \mathbb{X}$  they have size  $p_i$ .

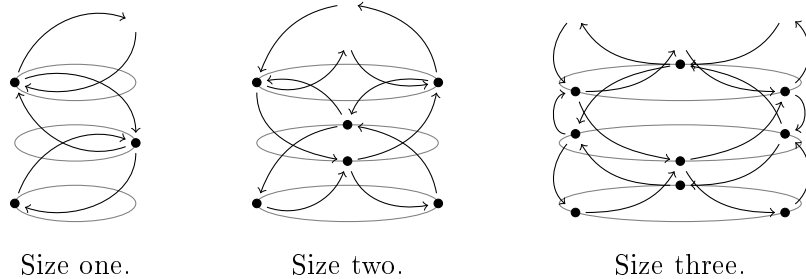


Figure 2-7: Some tubes.

Let  $p$  be the least common multiple of the  $p_i$ s. We define the degree map by

$$\begin{aligned} \deg : L &\longrightarrow \frac{1}{p}\mathbb{Z} \\ x_i &\longmapsto 1/p_i. \end{aligned}$$

Its kernel is the torsion of  $L$ . Describing  $\text{Vect}(\mathbb{X})$  depends essentially on the degree of the dualizing bundle

$$\deg(\mathcal{O}(\omega)) = (n - 2 - \sum_{i=1}^n \frac{1}{p_i}) \in \frac{1}{p}\mathbb{Z}.$$

There are three cases.

**Proposition 2.4.4** (Geigle-Lenzing [18] 5.4).

- If  $\deg(\mathcal{O}(\omega)) < 0$  then  $\mathbb{X}$  is Fano and  $\text{Vect}(\mathbb{X})$  is of tame domestic representation type, which is to say it has only finitely many orbits of indecomposables under the action of the AR translate. The Fano weight sequences are precisely those corresponding to the Dynkin diagrams  $A_n, D_n, E_6, E_7, E_8$  by counting the lengths of the branches. Indeed in these cases, the AR quiver of  $\text{Vect}(\mathbb{X})$  is a  $\mathbb{Z}$ -cover of the corresponding affine diagram.
- If  $\deg(\mathcal{O}(\omega)) = 0$  then  $\mathbb{X}$  is Calabi-Yau and  $\text{Vect}(\mathbb{X})$  is of tame tubular representation type: it contains one-parameter families of indecomposables. The description of  $\text{Vect}(\mathbb{X})$  in the four CY cases,  $\mathbf{p} = (2, 2, 2, 2), (3, 3, 3), (2, 4, 4)$  and  $(2, 3, 6)$  is closely analogous to Atiyah's classification of vector bundles on an elliptic curve, see [18] 5.4.2.
- If  $\deg(\mathcal{O}(\omega)) > 0$  then  $\mathbb{X}$  is of general type and  $\text{Vect}(\mathbb{X})$  is wild.

### 2.4.3 Tilting

A key motivating property of weighted projective lines is that they are derived equivalent to the canonical algebras.

**Proposition 2.4.5.**  $T = \bigoplus_{\ell \in [0, 1_{\mathbb{X}}]} \mathcal{O}(\ell)$  is a tilting bundle for  $\text{Coh}(\mathbb{X})$ .

*Proof.* [18] Proposition 4.1. □

Let  $\Lambda = \text{End}(T)$ . Then  $\Lambda$  is the path algebra of the quiver of Figure 2-8 with relations

$$X_i^{p_i} = X_1^{p_1} - \lambda_i X_2^{p_2}, \quad i = 3, \dots, n;$$

such algebras are called canonical algebras.



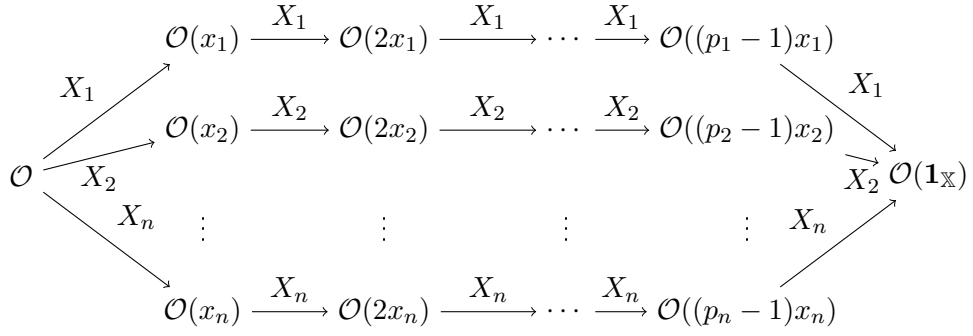


Figure 2-8: A tilting bundle for  $\text{Coh}(\mathbb{X})$

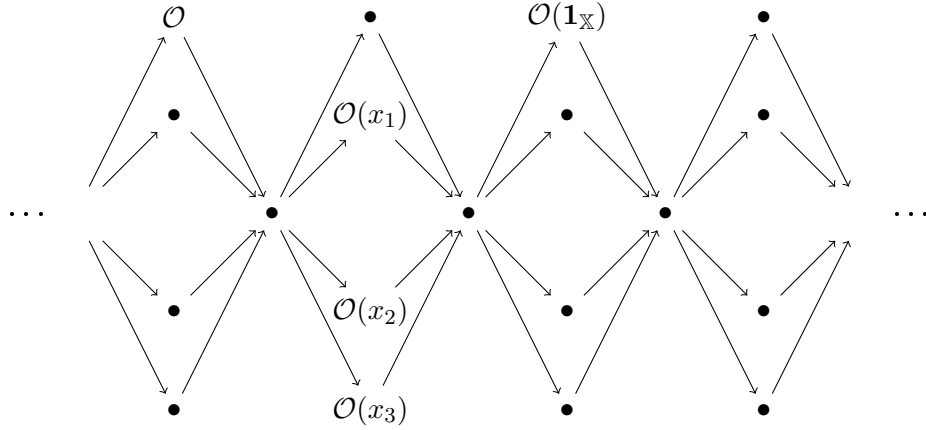
**Theorem 2.4.6.**

$$D^b(\text{Coh}(\mathbb{X})) \cong D^b(\text{mod}(\Lambda)).$$

*Proof.* See [18] Proposition 4.1 and [36] Theorem 4.10. □

When  $\mathbb{X}_{\mathbf{p}}$  is Fano so the weight sequence  $\mathbf{p} = (p_1, p_2, p_3)$  is the lengths of the branches of a Dynkin diagram  $\Delta$ ,  $\Lambda = \Lambda(p_1, p_2, p_3)$  is derived equivalent to a tame algebra of affine Dynkin type  $\tilde{\Delta}$  ([18] 5.4.1 and [41]). This justifies the first claim of Proposition 2.4.4.

**Example 2.4.7.** Let  $\mathbb{X} = \mathbb{X}_{2,2,2}$  of type  $D_4$ . Then  $T$  appears inside  $\text{Vect}(\mathbb{X})$  as follows.



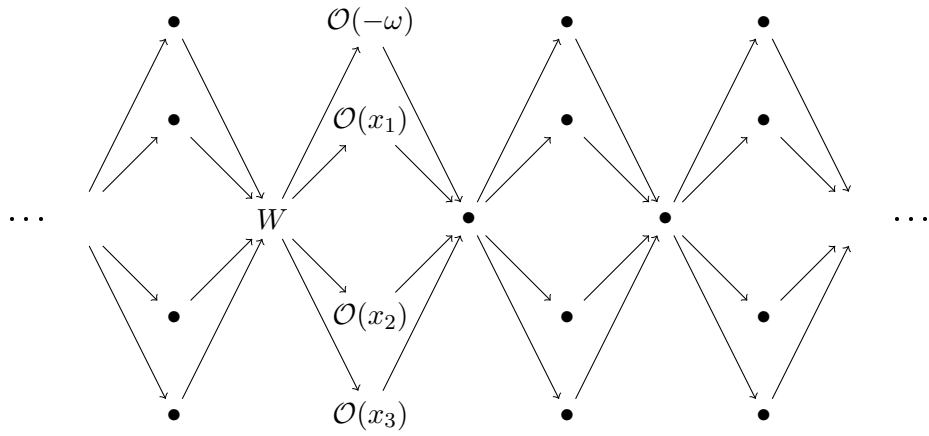
In the Fano cases however there actually appear tilting bundles that will be more useful to us.

**Definition 2.4.8.** The slope  $\mu(\mathcal{F})$  of a bundle  $\mathcal{F} \in \text{Vect}(\mathbb{X})$  is  $\text{deg}(\mathcal{F})/\text{rk}(\mathcal{F})$ .

**Proposition 2.4.9.** If  $\mathbb{X}_{\mathbf{p}}$  is Fano then the indecomposable bundles  $\mathcal{F}$  with slope in the range  $0 < \mu\mathcal{F} \leq -\deg(\mathcal{O}(\omega))$  form a tilting bundle  $\bar{T}$  for  $\text{Coh}(\mathbb{X})$ . Furthermore the bundles  $\mathcal{F}$ , as a transverse section of  $\text{Vect}(\mathbb{X})$  have underlying affine graph  $\tilde{\Delta}$ , where  $\Delta$  is the Dynkin diagram of type  $\mathbf{p} = (p_1, p_2, p_3)$ . In particular the AR quiver of  $\text{Vect}(\mathbb{X})$  is of the form  $\mathbb{Z}\tilde{\Delta}$ .

*Proof.* This is originally due to [25] but the version here is roughly as appears in [33] (Proposition 5.1). For a proof, see [38], Proposition 6.5.  $\square$

**Example 2.4.10.** Let  $\mathbb{X} = \mathbb{X}_{2,2,2}$  of type  $D_4$  again. Then the bundle  $\bar{T}$  appears inside  $\text{Vect}(\mathbb{X})$  as follows



where  $W$  is the indecomposable rank-two bundle of degree  $1/2$ .

#### 2.4.4 Exact structures and a theorem of Orlov

Let  $\mathbb{X}$  be a weighted projective line with Cox ring  $R$  generated by  $X_1, \dots, X_n$ . The category  $\text{Vect}(\mathbb{X})$ , as a full extension-closed subcategory of the abelian category  $\text{Coh}(\mathbb{X})$ , inherits an exact structure. This exact structure is not Frobenius. Observe that any vector bundle  $E$  is the target of  $n$  maps, corresponding to multiplication by  $X_1, \dots, X_n$ . Hence there is a surjection

$$\bigoplus_{i=1}^n E(-x_i) \longrightarrow E$$

which for reasons of degree cannot split. So no vector bundles are projective with respect to this exact structure, which is therefore not Frobenius.

On the other hand, Theorem 2.4.2 gives us an additive equivalence  $\text{Vect}(\mathbb{X}) \cong \text{CM}^L(R)$ . The category of  $L$ -graded Cohen-Macaulay modules is a full extension-closed subcategory of  $\text{mod}^L(R)$ , and therefore also inherits an exact structure. This structure

is different from the one induced by  $\text{Vect}(\mathbb{X}) \subset \text{Coh}(\mathbb{X})$ : in particular it is Frobenius (Proposition 2.3.8) with all rank-one modules (that is, those corresponding to the line bundles) projective. The observation that  $\text{Vect}(\mathbb{X})$  carries two different, very natural exact structures foreshadows a great deal of what is to come. For now let us exhibit a third.

**Proposition 2.4.11** (Geigle-Lenzing). Suppose  $\mathbb{X}$  is Fano with Cox ring  $R$  and write  $R'$  for the  $\mathbb{Z}$ -graded subring

$$R' = \bigoplus_{n \in \mathbb{Z}} R_{-n\omega}.$$

Then

$$\text{Vect}(\mathbb{X}) \cong \text{CM}^{\mathbb{Z}}(R').$$

*Proof.* This is essentially [19], Proposition 8.5 and [18] Theorem 5.1, though see also our Theorem 3.1.6 below.  $\square$

In the Fano case,  $R'$  is Gorenstein ([19], discussion preceding Proposition 8.5). The category  $\text{CM}^{\mathbb{Z}}(R')$ , is again Frobenius, but has only a single  $\tau$ -orbit of indecomposable projectives: all degree shifts of the trivial module  $R'$ . The associated stable category  $\underline{\text{CM}}^{\mathbb{Z}}(R')$  is of considerable theoretical interest. For one thing it is by definition equal to what Orlov introduces as the triangulated category of singularities of  $R'$ ,  $D_{\text{Sg}}^{\text{gr}}(R')$ , see for example [39]. We have the following important result of Orlov.

**Theorem 2.4.12** (Orlov.). Suppose  $\mathbb{X}$  is Fano. Then there is a fully faithful functor  $\Phi : \underline{\text{CM}}^{\mathbb{Z}}(R') \rightarrow D^b(\text{Coh}(\mathbb{X}))$  and a semiorthogonal decomposition

$$D^b(\text{Coh}(\mathbb{X})) = \langle \mathcal{O}, \Phi(\underline{\text{CM}}^{\mathbb{Z}}(R')) \rangle.$$

*Proof.* [39], Theorem 2.5 Case (i).  $\square$

The full version of this Theorem describes a trichotomy. However, the trichotomy we are interested in comes from the following  $L$ -graded generalization due to Kussin-Lenzing-Meltzer [33].

**Theorem 2.4.13** (Kussin-Lenzing-Meltzer). Let  $\mathbb{X}$  be an orbifold with three orbifold points, weight sequence  $(p_1, p_2, p_3)$  and Cox ring  $R$ .

- If  $\deg(\mathcal{O}(\omega)) < 0$  (that is,  $\mathbb{X}$  is Fano) then  $\underline{\text{CM}}^L(R)$  is a triangulated subcategory of  $D^b(\text{Coh}(\mathbb{X}))$  (right perpendicular to the bundle  $\bar{T}$  of Proposition 2.4.9 above).
- If  $\deg(\mathcal{O}(\omega)) = 0$  (that is,  $\mathbb{X}$  is Calabi-Yau) then  $\underline{\text{CM}}^L(R)$  is equivalent to  $D^b(\text{Coh}(\mathbb{X}))$ .

- If  $\deg(\mathcal{O}(\omega)) > 0$  (that is,  $\mathbb{X}$  is of general type) then  $D^b(\text{Coh}(\mathbb{X}))$  is a triangulated subcategory of  $\underline{\text{CM}}^L(R)$ .

## Part I

# Black and white diagrams

## Chapter 3

# Graded McKay correspondence

### 3.1 Graded McKay correspondence

We begin with a fundamental observation.

**Lemma 3.1.1.** Let  $R$  be a Cohen-Macaulay isolated surface singularity upon which a group  $G$  acts, fixing 0. Put  $X = \text{Spec } R$ ,  $U = X - 0$  and  $i : U \rightarrow X$  the inclusion. Then the composition  $i^* : \text{CM}_G(R) \rightarrow \text{Vect}_G(U)$  of sheafification with pullback is exact and an equivalence of additive categories.

*Proof.* This is essentially [13] Corollary 3.12. Our only modification is the equivariant structure, but this carries straight across since  $i^*$  is in any case fully faithful.  $\square$

**Remark 3.1.2.** The inverse equivalence  $i_* : \text{Vect}(U) \rightarrow \text{CM}(R)$  (see [21], Proposition 1.6) composing direct image with global sections is only left exact.

**Remark 3.1.3.** If in particular  $R$  is regular (for example  $R = \mathbb{C}[[x, y]]$ ) then  $j^* : \text{Vect}_G(X) \rightarrow \text{Vect}_G(U)$  is an equivalence, since Cohen-Macaulay modules on a regular local ring are free. This is the Hartogs phenomenon.

**Theorem 3.1.4** (Auslander [4]). Let  $G$  be a finite subgroup of  $\text{SL}(2, \mathbb{C})$  acting in the standard way on  $\hat{\mathbb{C}}^2 = \text{Spec } \mathbb{C}[[u, v]]$  and  $R = \mathbb{C}[[x, y, z]]/(f)$  its ring of invariants. Then the McKay graph of  $G$  (an extended Dynkin diagram  $\tilde{\Delta}$ ) is isomorphic to the Auslander-Reiten quiver of the category of Cohen-Macaulay modules on the singularity  $\hat{\mathbb{C}}^2/G = \text{Spec } R$ .

*Proof.* See [4], Section 2 for Auslander's proof. The proof we offer here is slightly different. By Lemma 3.1.1

$$i^* : \text{CM}(R) \rightarrow \text{Vect}(\hat{\mathbb{C}}^2/G - 0)$$

is an exact equivalence. Since the action of  $G$  is free here, bundles on the punctured quotient are the same thing as equivariant bundles on the punctured plane, that is  $\text{Vect}(\hat{\mathbb{C}}^2/G - 0) = \text{Vect}_G(\hat{\mathbb{C}}^2 - 0)$ . Since  $\mathbb{C}[[u, v]]$  is regular, Remark 3.1.3 gives an equivalence

$$j^* : \text{Vect}_G(\hat{\mathbb{C}}^2) \longrightarrow \text{Vect}_G(\hat{\mathbb{C}}^2 - 0).$$

The AR quiver of  $\text{Vect}_G(\hat{\mathbb{C}}^2)$  has nodes corresponding to the indecomposable  $G$ -equivariant bundles over  $\hat{\mathbb{C}}^2$ , which look like  $\mathcal{O}_{\hat{\mathbb{C}}^2} \otimes V_i$  for  $V_i$  the irreducible representations of  $G$ . The arrows correspond to irreducible morphisms  $\mathcal{O}_{\hat{\mathbb{C}}^2} \otimes V_i \longrightarrow \mathcal{O}_{\hat{\mathbb{C}}^2} \otimes V_j$  which in turn correspond to maps of representations  $V_i \longrightarrow \hat{\mathbb{C}}^2 \otimes V_j$ . Thus the AR quiver of  $\text{Vect}_G(\hat{\mathbb{C}}^2)$  is nothing other than the McKay graph of  $G$ .  $\square$

This theorem establishes a correspondence between a certain class of groups, namely the finite subgroups of  $\text{SL}(2, \mathbb{C})$ , and a class of categories, namely the categories of CM modules over the simple singularities. This amounts to the classical classification of both classes by the list of extended Dynkin diagrams (Table 3.1), see for example [46] Chapter 10.

$\tilde{\Delta}$	$G$	singularity $f$
$\tilde{A}_n$	$\mu_{n+1}$ (cyclic)	$x^{n+1} + yz$
$\tilde{D}_n$	$\text{BDih}_{n-2}$ (binary dihedral)	$x^2y + y^{n-1} + z^2$
$\tilde{E}_6$	$\text{BTet}$ (binary tetrahedral)	$x^3 + y^4 + z^2$
$\tilde{E}_7$	$\text{BOct}$ (binary octahedral)	$x^3 + xy^3 + z^2$
$\tilde{E}_8$	$\text{BIco}$ (binary icosahedral)	$x^3 + y^5 + z^2$

Table 3.1: Finite subgroups of  $\text{SL}(2, \mathbb{C})$  and simple singularities.

**Remark 3.1.5.** The additive equivalence  $i^*$  of Theorem 3.1.4 is exact but its inverse is not. This means that the collection of exact sequences defining the exact structure on  $\text{CM}(R)$  is a strict subcollection of the exact sequences of  $\text{Vect}(\mathbb{C}^2/G - 0)$ . The same remarks apply to  $j^* : \text{Vect}_G(\mathbb{C}^2) \longrightarrow \text{Vect}_G(\mathbb{C}^2 - 0)$ . We say that the exact structures on  $\text{CM}(R)$  and  $\text{Vect}_G(\mathbb{C}^2)$  are coarser than that on  $\text{Vect}(\mathbb{C}^2/G - 0)$ .

One consequence of this is that  $\text{Vect}(\mathbb{C}^2/G - 0)$  has fewer projective objects than both  $\text{CM}(R)$  and  $\text{Vect}_G(\mathbb{C}^2)$ . In fact,  $\text{CM}(R)$  has one projective object, the trivial module  $R$ ; in  $\text{Vect}_G(\mathbb{C}^2)$  everything is projective; while in  $\text{Vect}_G(\mathbb{C}^2 - 0)$  nothing is projective.

We now prove a graded version on Theorem 3.1.4 to obtain a correspondence analogous to that of Table 3.1 but between the categories of  $\mathbb{Z}$ -graded CM modules over the simple singularities and the extended McKay groups (Definition 2.1.12).

**Theorem 3.1.6** (Graded McKay correspondence). Let  $\Gamma = G \cdot \mathbb{C}^*$  be an extended McKay group and  $R = \mathbb{C}[x, y, z]/(f)$  the  $\mathbb{Z}$ -graded ring of  $G$ -invariants. Then the McKay graph of  $\Gamma$  is isomorphic to the Auslander-Reiten quiver of the category of  $\mathbb{Z}$ -graded Cohen-Macaulay modules on the singularity  $\mathbb{C}^2/G = \text{Spec } R$ .

*Proof.* The group  $\Gamma$  sits in a short exact sequence

$$G \longrightarrow \Gamma \xrightarrow{\det} \mathbb{C}^*$$

where  $G$  is the finite determinant-one part, a binary polyhedral group. The point is that the action of  $G$  on the punctured plane is again free, so we have an isomorphism of stacks

$$\mathbb{X} := \left[ \frac{\mathbb{C}^2 - 0}{\Gamma} \right] \cong \left[ \frac{(\mathbb{C}^2/G) - 0}{\mathbb{C}^*} \right].$$

By Lemma 3.1.1,  $\text{Vect}(\mathbb{X})$  is additively equivalent to  $\text{CM}^{\mathbb{Z}}(R)$ , where  $\mathbb{Z}$  is the character group of the determinant  $\mathbb{C}^*$ . By Remark 3.1.3 on the other hand,  $\text{Vect}(\mathbb{X})$  is additively equivalent to  $\text{Vect}_{\Gamma}(\mathbb{C}^2)$ , whose AR quiver is the McKay graph of  $\Gamma$ , as in the proof of Theorem 3.1.4. Note that this category has split idempotents and finite-dimensional Hom spaces so is Krull-Schmidt ([34] Corollary 4.4) and has a well-defined Auslander-Reiten quiver.  $\square$

**Remark 3.1.7.** We have again written down three categories which are additively but not exactly equivalent:  $\text{Vect}[\mathbb{C}^2 - 0/\Gamma]$  is not Frobenius, while both  $\text{CM}^{\mathbb{Z}}(R)$  and  $\text{Vect}[\mathbb{C}^2/\Gamma]$  are, though their classes of projective-injectives differ. Precisely how they differ is the subject of Chapter 4.

In order to get a classification analogous to table 3.1 we need to be explicit about the  $\mathbb{Z}$ -grading in the proof of Theorem 3.1.6. Let  $u, v$  be coordinates on  $\mathbb{C}^2$ . By definition the extended McKay group  $\Gamma$  contains a diagonal  $\mathbb{C}_{\alpha}^*$ , acting with weight  $(\alpha_1, \alpha_2)$  on  $u, v$ ;

$$\begin{array}{ccc} \mathbb{C}_{\alpha}^* & & \mathbb{C}_{\omega}^* \\ & \searrow & \nearrow \text{det} \\ & \Gamma & \\ & \nearrow G & \searrow \pi_1 \end{array}$$

we write:  $\Gamma = \mathbb{C}^*(\alpha_1, \alpha_2) \cdot G$ . When  $G \subset \Gamma$  is nonabelian the diagonal  $\mathbb{C}_{\alpha}^*$  must act with weights  $(\alpha_1, \alpha_2) = (1, 1)$ , because it commutes with  $G$  in  $\text{GL}(2, \mathbb{C})$ . When  $G$  is



cyclic,  $G = \mu_{p+q}$ , the weights can differ:  $(\alpha_1, \alpha_2) = (p/d, q/d)$ , where  $d = \gcd(p, q)$ .

The three fundamental invariants  $x, y, z$  for the  $G$ -action are semi-invariants for  $\Gamma$ ; as polynomials in  $u, v$  they correspond to characters of  $\mathbb{C}_\alpha^*$ . However we want to think of the ring  $R = \mathbb{C}[x, y, z]/(f)$  as graded by the characters of the determinant  $\mathbb{C}_\omega^*$ ; in the dual picture

$$\begin{aligned} \mathbb{Z}_\omega &\hookrightarrow \mathbb{Z}_\alpha \\ 1 &\longmapsto \alpha_1 + \alpha_2 =: \omega. \end{aligned}$$

One calculates the degrees of  $x, y$  and  $z$  by writing down their degrees as  $\mathbb{Z}_\alpha$ -homogeneous polynomials in  $u, v$  and then dividing this number by  $\omega = \alpha_1 + \alpha_2$ . Theorem 3.1.6 now yields a classification analogous to the classical case: Table 3.2.

$\tilde{\Delta}$	$\Gamma$	singularity $f$	degrees of $x, y, z$
$\tilde{A}_{p,q}$	$\mu_{p+q} \cdot \mathbb{C}^*(p/d, q/d)$	$x^{p+q} + yz$	$1, p, q$
$\tilde{D}_n$	$\text{BDih}_{n-2} \cdot \mathbb{C}^*(1, 1)$	$x^2y + y^{n-1} + z^2$	$(n-2), 2, (n-1)$
$\tilde{E}_6$	$\text{BTet} \cdot \mathbb{C}^*(1, 1)$	$x^3 + y^4 + z^2$	$4, 3, 6$
$\tilde{E}_7$	$\text{BOct} \cdot \mathbb{C}^*(1, 1)$	$x^3 + xy^3 + z^2$	$6, 4, 9$
$\tilde{E}_8$	$\text{BIco} \cdot \mathbb{C}^*(1, 1)$	$x^3 + y^5 + z^2$	$10, 6, 15$

Table 3.2: One dimensional subgroups of  $\text{GL}(2, \mathbb{C})$  with finite determinant-one part and graded singularities.

**Remark 3.1.8.** A version of this classification and Theorem 3.1.6 appears in [32], see Section 7 and the derived equivalence of Theorem 8.8.

Of course the result is in a sense folklore by now, since it is really nothing other than the observation that the category  $\text{CM}^{\mathbb{Z}_\omega}(R) \cong \text{Vect}_\Gamma(\mathbb{C}^2 - 0)$  is a category of vector bundles on a one-dimensional orbifold, in the sense of Section 2.4 above, compare also [19] Section 8. We will pursue this line in the next chapter.

Observe that an  $A_n$  singularity  $x^{n-1} + yz$  admits of more than one possible grading, that is  $\deg x = 1$  and  $y, z$  in degrees  $p$  and  $q$  with  $p+q = n-1$ . These different gradings yield nonequivalent categories of graded Cohen-Macaulay modules, corresponding to the different possible orientations of the cyclic quiver  $\tilde{A}_{n-1}$ . This is the meaning of the notation  $\tilde{A}_{p,q}$ : it is the cyclic quiver with  $p+q$  vertices (so extended Dynkin of type  $\tilde{A}_{p+q-1}$ ) with  $p$  arrows going in one direction and  $q$  in the other.

**Example 3.1.9.** The group  $\Gamma = \text{BDih}_2 \cdot \mathbb{C}^*(1, 1)$  is generated inside  $\text{GL}(2, \mathbb{C})$  by the matrices

$$G = \text{BDih}_2 = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle$$

and the commuting  $\mathbb{C}_\alpha^*$

$$\langle \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right), \lambda \in \mathbb{C}^* \rangle$$

which acts with weight 1 in both coordinates, so  $\omega = 2$ . Three fundamental invariants for the  $G$  action on  $\mathbb{C}[u, v]$  are given by the polynomials

$$x = \frac{1}{2i}(u^4 + v^4), \quad y = u^2v^2, \quad z = \frac{1}{2}uv(u^4 - v^4)$$

related by the equation  $x^2y + y^3 + z^2 = 0$ . These are  $\mathbb{C}_\alpha^*$  characters of weights 4, 4 and 6. Hence the  $\mathbb{Z}_\omega$  degrees of  $x$ ,  $y$  and  $z$  are 2, 2, and 3.

**Example 3.1.10.** Let  $G = \mu_{10}$ , generated inside  $\mathrm{GL}(2, \mathbb{C})$  by

$$\left\langle \left( \begin{array}{cc} \zeta & 0 \\ 0 & \zeta^{-1} \end{array} \right) \right\rangle$$

for  $\zeta$  a primitive tenth root of 1. The three fundamental invariants are

$$x = uv, \quad y = -u^{10}, \quad z = v^{10}$$

related by  $x^{10} + yz = 0$ .

In this case there are five diagonal  $\mathbb{C}^*(\alpha_1, \alpha_2)$ s to consider, corresponding to the five ways that  $\alpha_1 + \alpha_2 = 10$ . The  $\mathbb{Z}_\omega$ -degrees of the fundamental invariants in each case are as follows.

$\mathbb{C}^*(\alpha_1, \alpha_2)$	$\mathbb{C}^*(1, 1)$	$\mathbb{C}^*(1, 4)$	$\mathbb{C}^*(2, 3)$	$\mathbb{C}^*(3, 7)$	$\mathbb{C}^*(1, 9)$
$\omega$	2	5	5	10	10
$\deg x$	1	1	1	1	1
$\deg y$	1	2	4	3	1
$\deg z$	1	8	6	7	9

As mentioned above, the categories  $\mathrm{CM}^{\mathbb{Z}_\omega}(R)$  in each case are not equivalent, although as we will now see, the Auslander-Reiten quiver is always a  $\mathbb{Z}_\omega$ -sheeted cover of the extended Dynkin diagram  $\tilde{A}_{p,q}$ .

The category  $\mathrm{Vect}[\mathbb{C}^2/\Gamma] \cong \mathrm{CM}^{\mathbb{Z}_\omega}(\mathbb{C}[x, y, z]/f)$  carries an automorphism: tensoring by the canonical bundle on the left hand side, or degree shift by  $1 \in \mathbb{Z}_\omega$  on the right. This functor induces the inverse of the Auslander-Reiten translate on the stable category (Corollary 8.1.2 below), so we denote it by  $\tau^{-1}$ . Taking the quotient by  $\tau^{-1}$  of the AR quiver of  $\mathrm{CM}^{\mathbb{Z}_\omega}(\mathbb{C}[x, y, z]/f)$  yields the AR quiver of  $\mathrm{CM}(\mathbb{C}[[x, y, z]]/f)$ , the extended Dynkin diagram  $\tilde{\Delta}$  corresponding to  $G \subset \Gamma$ . To put it another way, there is a  $\mathbb{Z}$ -sheeted cover

$$\mathrm{McKay}(\Gamma) \longrightarrow \mathrm{McKay}(G).$$

Indeed a node on  $\text{McKay}(\Gamma)$  corresponds to a pair  $(\rho, \chi)$  where  $\rho$  is a representation of  $G$  and  $\chi$  is a character of  $\mathbb{C}^*(\alpha_1, \alpha_2)$ , such that  $\rho(g)$  is multiplication by  $\chi(g)$  for all  $g$  in the intersection  $G \cap \mathbb{C}^*(\alpha_1, \alpha_2)$ . The covering map takes  $(\rho, \chi)$  to  $\rho$ .

**Example 3.1.11.** Let  $\Gamma = \text{BDih}_2 \cdot \mathbb{C}^*(1, 1)$ , label the 2-dimensional faithful irreducible of  $G = \text{BDih}_2$  by  $V$ , and the four linear irreducibles  $L_1, \dots, L_4$ . Since  $V$  sends  $-\mathbb{I} \in \text{BDih}_2 \cap \mathbb{C}_\alpha^*$  to multiplication by  $-1$ , it pairs with the odd characters of  $\mathbb{C}_\alpha^*$ . The 1-dimensional representations, all of which contain  $-1$  in their kernels, pair with the even characters.

Since  $\mathbb{C}_\alpha^*$  is acting with weight one in both coordinates, the canonical representation of  $\Gamma$  corresponds to the pair  $(V, 1)$ , and it is with respect to this that we calculate the McKay graph. We have

$$(V, 1) \otimes (V, n) = (L_1, n+1) \oplus (L_2, n+1) \oplus (L_3, n+1) \oplus (L_4, n+1)$$

and

$$(V, 1) \otimes (L_i, n) = (V, n+1)$$

for all  $n$  and  $i$ . Thus we have arrows  $V \rightarrow L$  and  $L \rightarrow V$  going from left to right for each linear representation  $L$ .

By Theorem 3.1.6, interpreting Figure 3-1 as a picture of the AR quiver of  $\text{CM}^{\mathbb{Z}_\omega}(R)$ , one can see that  $\tau^{-1}$  is indeed given by degree shift by  $\omega = \alpha_1 + \alpha_2 = 2$ .

**Example 3.1.12.** Let  $\Gamma = \mu_{10} \cdot \mathbb{C}^*(3, 2)$ , and label the 10 irreducible representations of  $G = \mu_{10}$  by  $0, \dots, 9$ . The  $\mathbb{C}^*$  is acting by  $L = \begin{pmatrix} \lambda^3 & 0 \\ 0 & \lambda^2 \end{pmatrix}$  for  $\lambda \in \mathbb{C}^*$ , and  $\mu_{10}$  by  $Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$  for  $\zeta$  a tenth root of 1. Thus a pair  $\chi : L \mapsto \lambda^n$  and  $\rho : Z \mapsto \zeta^m$  agree on the intersection

$$\mu_{10} \cap \mathbb{C}^*(3, 2) = \mu_5 = \left\{ \begin{pmatrix} \zeta^{2k} & 0 \\ 0 & \zeta^{-2k} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} (\zeta^{4k})^3 & 0 \\ 0 & (\zeta^{4k})^2 \end{pmatrix} \right\}$$

when  $m = 2n \pmod{5}$ .

Since  $\Gamma$  in this case is abelian, the canonical two-dimensional representation is not irreducible; it corresponds to the direct sum  $(9, 2) \oplus (1, 3)$ . We have

$$((9, 2) \oplus (1, 3)) \otimes (m, n) = (m+9, n+2) \oplus (m+1, n+3)$$

so the arrows go  $(m, n) \rightarrow (m-1, n+2)$  and  $(m, n) \rightarrow (m+1, n+3)$ . See Figure 3-2. In terms of  $\mathbb{Z}_\omega$ -graded CM modules, the arrows coming out of a node can be thought of as multiplication by  $u$  and  $v$  on the underlying  $\mathbb{C}^2$ , with degrees 3 and 2 respectively. The automorphism  $\tau^{-1}$  is degree shift by  $\omega = \alpha_1 + \alpha_2 = 3 + 2 = 5$ .

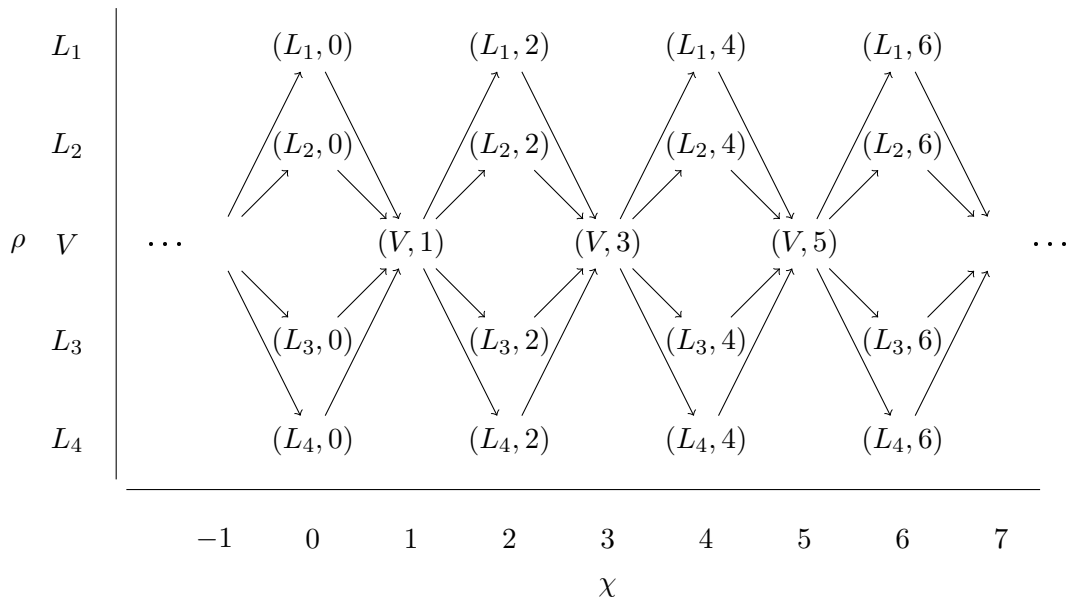


Figure 3-1: McKay graph of  $\Gamma = \text{BDih}_2 \cdot \mathbb{C}^*$ .

Compare Figure 3-3, the McKay graph of  $\Gamma = \mu_{10} \cdot \mathbb{C}^*(4, 1)$ ; here  $u$  and  $v$  have degrees 4 and 1. The canonical representation in this case corresponds to  $(9, 4) \oplus (1, 1)$  and the arrows go  $(m, n) \rightarrow (m - 1, n + 4)$  and  $(m, n) \rightarrow (m + 1, n + 1)$ . The automorphism  $\tau^{-1}$  is again degree shift by  $\omega = 5$ .

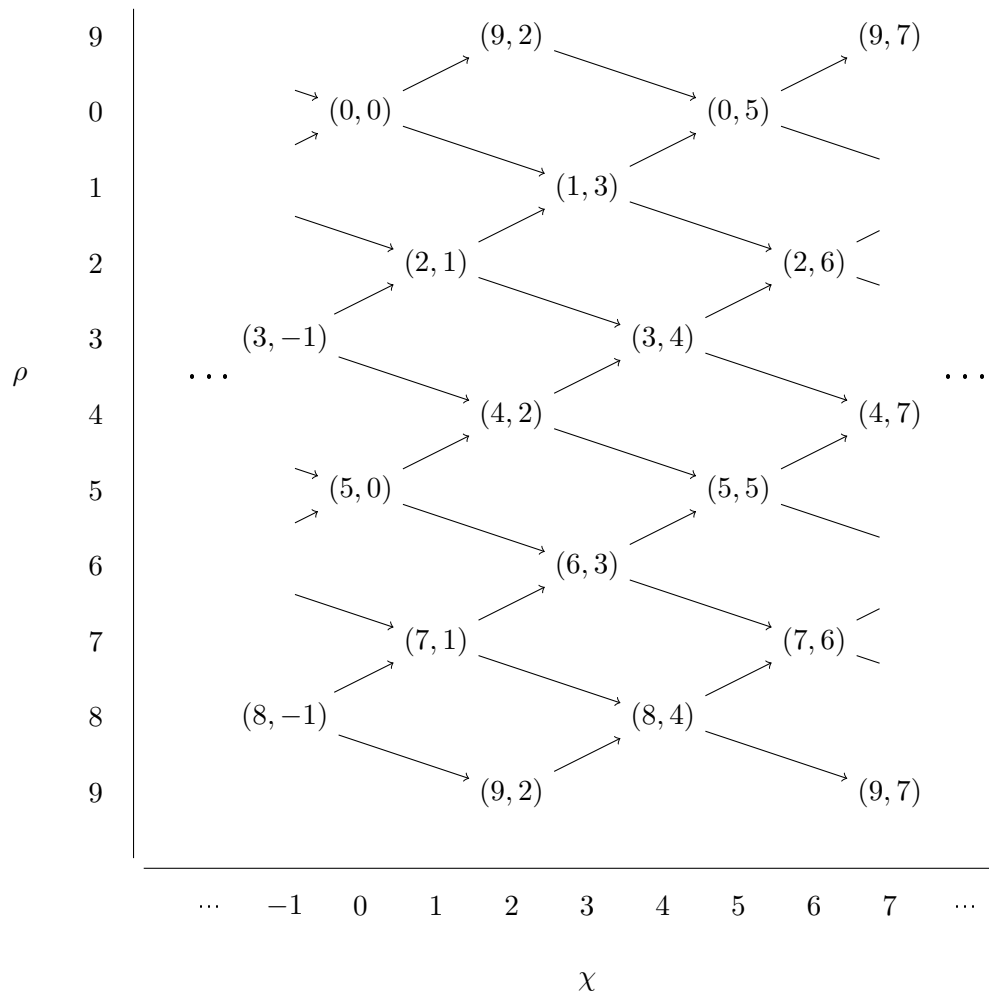


Figure 3-2: McKay graph of  $\Gamma = \mu_{10} \cdot \mathbb{C}^*(3, 2)$ .

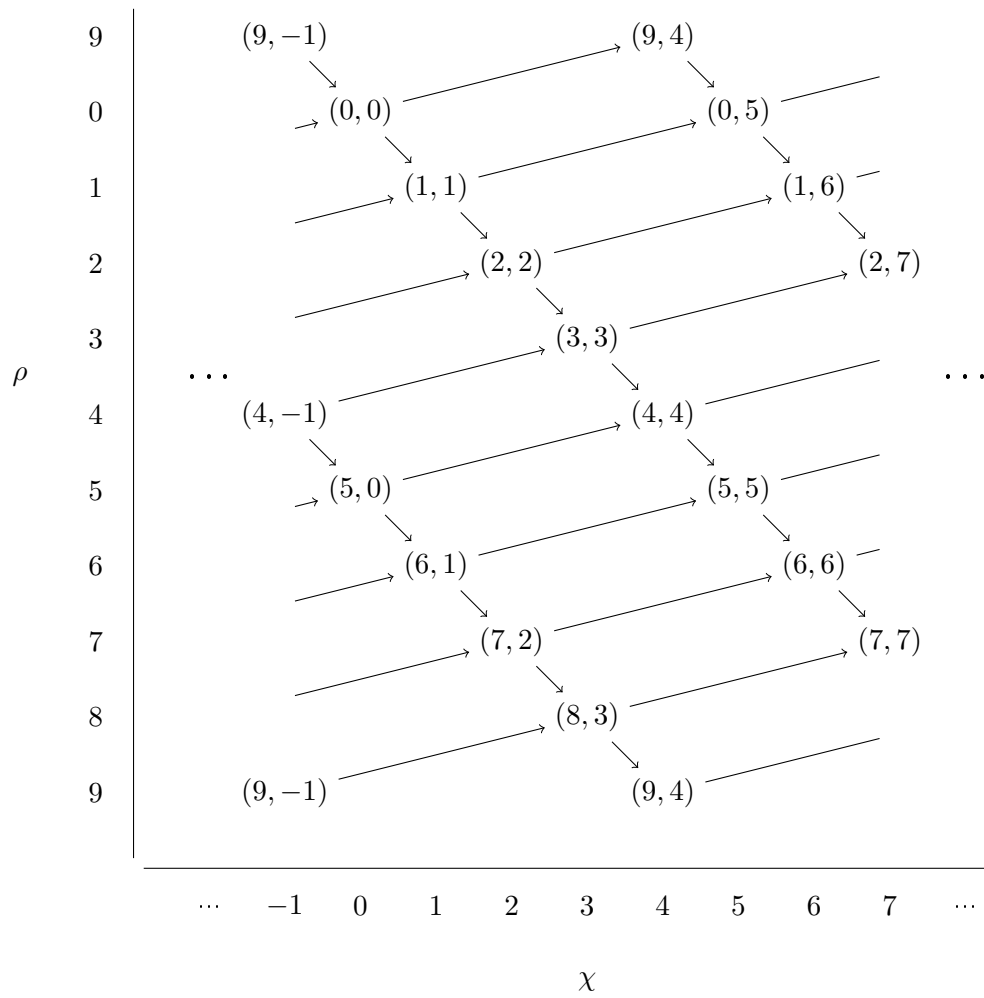


Figure 3-3: McKay graph of  $\Gamma = \mu_{10} \cdot \mathbb{C}^*(4, 1)$ .

## Chapter 4

# Frobenius structures for Fano orbifolds

### 4.1 Fano orbifolds

Let  $G \subset \mathrm{SL}(2, \mathbb{C})$  and  $\Gamma = G \cdot \mathbb{C}^*(\alpha_1, \alpha_2)$  as before. The stack

$$\mathbb{X} := [\mathbb{C}^2 - 0/\Gamma]$$

is a smooth, rational Deligne-Mumford stack with no generic stabilizers. Quotienting first by the freely acting diagonal  $\mathbb{C}_\alpha^*$  it is clear that

$$\mathbb{X} = \left[ \frac{(\mathbb{C}^2 - 0)/\mathbb{C}_\alpha^*}{\pi_1} \right]$$

is an orbifold sphere with a finite number of points of finite cyclic isotropy. Indeed  $(\mathbb{C}^2 - 0)/\mathbb{C}_\alpha^*$  is a  $\mathbb{P}^1$  when  $(\alpha_1, \alpha_2) = (1, 1)$  and an orbifold sphere with isotropy  $\mu_{\alpha_1}$  and  $\mu_{\alpha_2}$  at the poles otherwise. This possibly weighted  $\mathbb{P}^1$  is the universal cover of  $\mathbb{X}$  and  $\pi_1 = G/(G \cap \mathbb{C}_\alpha^*)$  is the fundamental group (see [45] Chapter 13).

**Example 4.1.1.** Let  $\Gamma = \mathrm{BDih}_2 \cdot \mathbb{C}^*(1, 1)$ , so  $(\mathbb{C}^2 - 0)/\mathbb{C}_\alpha^* = \mathbb{P}^1$  and  $\pi_1 = G/G \cap \mathbb{C}_\alpha^* = \mathrm{BDih}_2/\mu_2 = \mathrm{Dih}_2 = \mu_2 \times \mu_2$ , the Klein 4-group. As a subgroup of  $\mathrm{Aut}(\mathbb{P}^1)$ ,  $\pi_1$  consists of the rotations by  $180^\circ$  around the points  $0, 1$  and  $i$ . As such,  $\mathbb{X}$  is an orbifold sphere with order 2 isotropy in these three points.

**Example 4.1.2.** Let  $\Gamma = \mu_{10} \cdot \mathbb{C}^*(3, 2)$ , so  $(\mathbb{C}^2 - 0)/\mathbb{C}_\alpha^*$  is an orbifold sphere with isotropy of orders 3 and 2 at the poles. Since  $\pi_1 = G/G \cap \mathbb{C}_\alpha^* = \mu_2$ ,  $(\mathbb{C}^2 - 0)/\mathbb{C}_\alpha^*$  is a double cover of  $\mathbb{X}$ , an orbifold sphere with isotropy of orders 6 and 4 at the poles.

By contrast, when  $\Gamma = \mu_{10} \cdot \mathbb{C}^*(1, 4)$  the universal cover has a single orbifold point

of order four. The fundamental group is still  $\mu_2$ , rotating around this point and its antipode. So  $\mathbb{X}$  in this case has isotropy of order 8 and 2 at the poles.

An important feature of these orbifolds is that they may be understood as weighted projective lines in the sense of Geigle-Lenzing [18], from which description much of their geometry can be simply read off.

**Proposition 4.1.3.** If  $N \triangleleft G$  is the commutator subgroup, we can write  $\mathbb{X}$  as a GIT-quotient of a simple surface singularity,

$$\mathbb{X} = \left[ \frac{(\mathbb{C}^2/N) - 0}{(\Gamma/N)} \right] = \text{Proj}^L \left( \frac{\mathbb{C}[x_1, x_2, x_3]}{x_1^{p_1} + x_2^{p_2} + x_3^{p_3}} \right) = \mathbb{X}_{\mathbf{p}},$$

where  $L$  is the group of linear characters of the abelianization  $\Gamma/N$ . This is a Geigle-Lenzing weighted projective line of positive Euler characteristic in the sense of [18] with weight sequence  $\mathbf{p} = (p_1, p_2, p_3)$  given by the lengths of the branches of the (non-extended) Dynkin diagram  $\Delta$  associated to  $G$ .

*Proof.* This is a matter of checking, for each finite subgroup of  $\text{SL}(2, \mathbb{C})$ , that the simple singularity  $\mathbb{C}^2/N$  is given by the Geigle-Lenzing equation  $x_1^{p_1} + x_2^{p_2} + x_3^{p_3}$  and that the dual of  $\Gamma/N$  is isomorphic to the Geigle-Lenzing group  $L = \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 | p_1 \vec{x}_1 = p_2 \vec{x}_2 = p_3 \vec{x}_3 \rangle$  (see [18] §5.4.1).

For example, consider the group  $\Gamma = \mathbb{C}^* \cdot \text{BDih}_2$ . So  $G = \text{BDih}_2$  with associated Dynkin diagram  $D_4$ , and commutator subgroup  $N = \mu_2$ . The quotient of  $\mathbb{C}^2 = \text{Spec } \mathbb{C}[u, v]$  by  $N$  is a surface singularity of type  $A_1$

$$\mathbb{C}^2/N = \text{Spec } \mathbb{C}[u^2 + v^2, u^2 - v^2, uv] = \text{Spec } \mathbb{C}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2).$$

So

$$\mathbb{X} = \left[ \frac{\mathbb{C}^2 - 0}{\Gamma} \right] = \left[ \frac{(\mathbb{C}^2/N) - 0}{(\Gamma/N)} \right] = \text{Proj}^L \left( \frac{\mathbb{C}[x, y, z]}{x^2 + y^2 + z^2} \right),$$

where  $L$  is the group of characters of  $\Gamma/N$ . We need to check that  $L = \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 | 2\vec{x}_1 = 2\vec{x}_2 = 2\vec{x}_3 \rangle$ . The action of  $\Gamma$  on  $\mathbb{C}[u, v]$  is generated by the matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

$\lambda \in \mathbb{C}^*$ . So  $\Gamma/N$  acts on the fundamental semi-invariants  $x_1 = u^2 + v^2$ ,  $x_2 = u^2 - v^2$



and  $x_3 = uv$  by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix},$$

or equivalently by

$$\left\{ \begin{pmatrix} \pm\lambda & 0 & 0 \\ 0 & \pm\lambda & 0 \\ 0 & 0 & \pm\lambda \end{pmatrix} : \lambda \in \mathbb{C}^* \right\} = \{(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^3 : \lambda_1^2 = \lambda_2^2 = \lambda_3^2\},$$

which is indeed dual to  $L = \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 | 2\vec{x}_1 = 2\vec{x}_2 = 2\vec{x}_3 \rangle$ .  $\square$

## 4.2 Frobenius structures

The exact structure that  $\text{Vect}(\mathbb{X}_{\mathbf{p}}) = \text{Vect}[\mathbb{C}^2 - 0/\Gamma]$  inherits as a full extension-closed subcategory of  $\text{Coh}(\mathbb{X}_{\mathbf{p}})$  is not Frobenius: there are no projective objects. However, by exploiting the Hartogs phenomenon we have already exhibited two different exact structures on the same additive category (Theorem 3.1.6), both of which are Frobenius. To wit,

- $\text{Vect}[\mathbb{C}^2/\Gamma] \subset \text{Coh}[\mathbb{C}^2/\Gamma]$  in which everything is projective, and
- $\text{CM}^{\mathbb{Z}\omega}(R) \subset \text{mod}^{\mathbb{Z}\omega}(R)$  ( $\text{Spec } R = \mathbb{C}^2/G$ ) in which the projectives are the degree shifts of the module  $R$ .

Proposition 4.1.3 immediately suggests that we add a third to this list:

- $\text{CM}_{\Gamma/N}(R') \subset \text{mod}_{\Gamma/N}(R')$  with  $N \triangleleft G$  the derived subgroup and  $\text{Spec } R' = \mathbb{C}^2/N$ . Under this structure, the projectives are the line bundles.

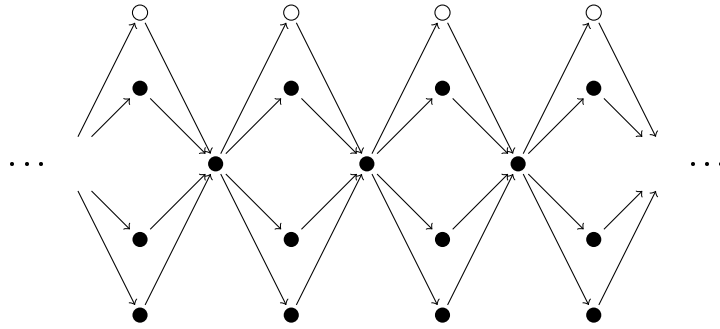
Of course there is nothing special about the derived subgroup of  $G$  here, we can do this with any normal subgroup.

**Theorem 4.2.1.** Let  $\Gamma = G \cdot \mathbb{C}^*(\alpha_1, \alpha_2)$  and  $N \triangleleft G$  a normal subgroup. The category  $\text{Vect}[(\mathbb{C}^2/N)/(\Gamma/N)]$  is additively equivalent to  $\text{Vect}(\mathbb{X}_{\mathbf{p}})$ . As a category of  $\Gamma/N$ -equivariant Cohen-Macaulay modules it carries an exact structure under which it is Frobenius. The indecomposable projective-injective objects are precisely those bundles corresponding to representations of  $\Gamma$  which factor through  $\Gamma/N$ .

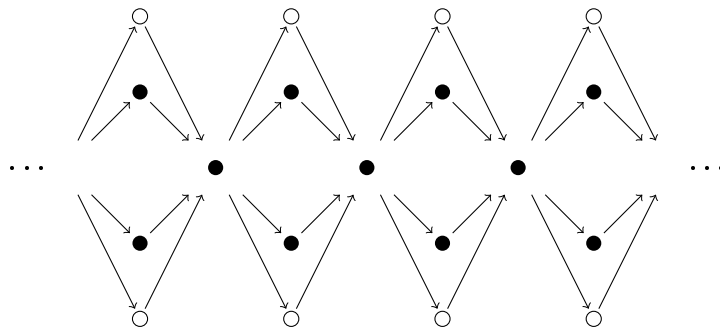
*Proof.* The additive equivalence  $\text{Vect}[(\mathbb{C}^2/N)/(\Gamma/N)] \cong \text{Vect}(\mathbb{X}_{\mathbf{p}})$  follows by the argument of Theorem 3.1.6 and Proposition 4.1.3. Write the simple singularity  $\mathbb{C}^2/N = \text{Spec } R$ . Then  $\text{Vect}[(\mathbb{C}^2/N)/(\Gamma/N)] = \text{CM}_{\Gamma/N}(R)$  by Lemma 3.1.1 and Remark 3.1.3, which is Frobenius since categories of equivariant Cohen-Macaulay modules over Gorenstein rings are always Frobenius (Lemma 2.3.8). The indecomposable projective-injectives are the orbits of the module  $R$  under the action of the monoid of representations of  $\Gamma/N$  (see Lemma 2.3.7); these correspond to the representations of  $\Gamma$  which are trivial on  $N$ , or in other words factor through  $\Gamma/N$ . □

**Example 4.2.2.** Let  $\Gamma = \text{BDih}_2 \cdot \mathbb{C}_\alpha^*$ . There are four isoclasses of normal subgroups of  $\text{BDih}_2$ :  $\{1\}$ ,  $\mu_2$ ,  $\mu_4$  and  $\text{BDih}_2$  itself.

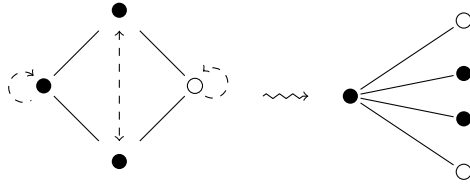
$N = \text{BDih}_2$  : This is the  $\mathbb{Z}_\omega$ -graded situation for modules over the  $D_4$  singularity. The projective-injectives are the degree shifts of the free  $\mathbb{C}[x, y, z]/(x^2y+y^3+z^2)$  module, corresponding to the representations of  $\Gamma$  trivial on  $\text{BDih}_2$  by Lemma 2.3.7, see also Example 3.1.11. Colouring projective nodes white, the AR quiver looks as follows.



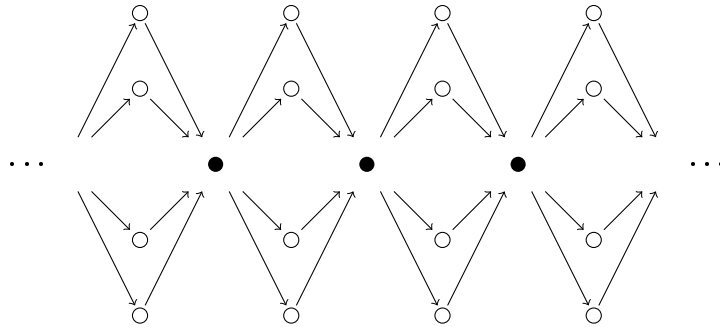
$N = \mu_4$  : This is the category of  $\Gamma/\mu_4$ -equivariant Cohen-Macaulay modules over the  $A_3$  singularity. There are two orbits of representations that are trivial on  $\mu_4$ .



Alternatively, one can view this picture as arising from the quotient of the McKay quiver of  $N$  by the action of the character group of  $\text{BDih}_2/N \cong \mu_2$ , which fixes the even representations and swaps the odd ones.



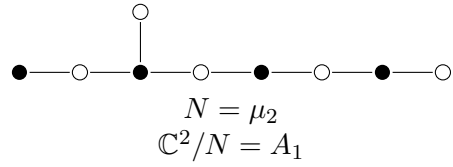
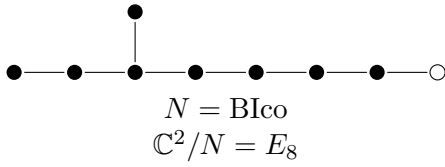
$N = \mu_2$  : Here  $N$  is the derived subgroup of  $\Gamma$ , and indeed all of the linear representations kill  $N = \{\pm 1\} \subset \Gamma$ . That is to say, the projective-injectives are the line bundles.



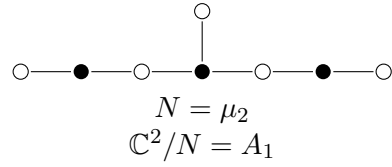
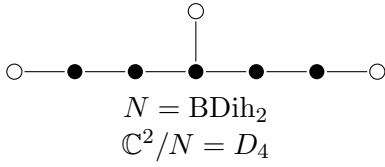
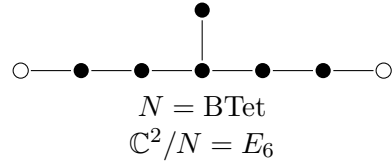
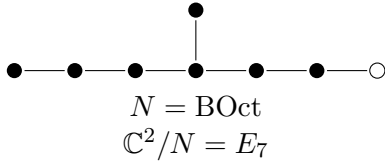
$N = \{1\}$  : Finally, when  $N$  is trivial, every representation factors through the quotient, so everything is projective.

Recall from Chapter 3 that the McKay graph of  $\Gamma$  is a  $\mathbb{Z}$ -cover of  $\text{McKay}(G)$ . Hence we can describe the Frobenius structure on  $\text{Vect}(\mathbb{X})$  by a picture of the extended Dynkin diagram  $\tilde{\Delta}$  associated to  $G$  with the nodes coloured white to indicate projective and black to indicate nonprojective objects.

$\mathbb{X}_{2,3,5}$  ( $\Gamma = \text{BIco} \cdot \mathbb{C}^*(1,1)$ )



$\mathbb{X}_{2,3,4}$  ( $\Gamma = \text{BOct} \cdot \mathbb{C}^*(1,1)$ )



$\mathbb{X}_{2,3,3}$  ( $\Gamma = \text{BTet} \cdot \mathbb{C}^*(1,1)$ )

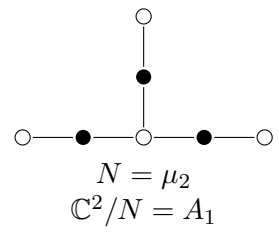
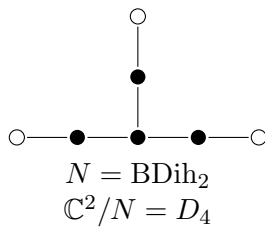
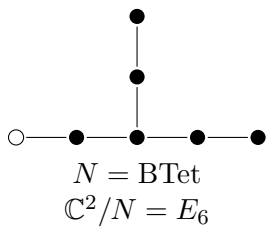
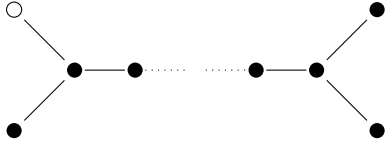
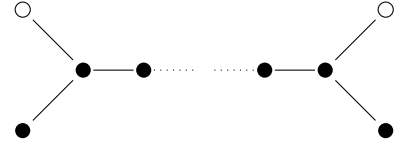


Table 4.1: Black and white diagrams for Fano orbifolds: I

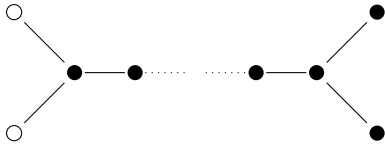
$\mathbb{X}_{2,2,n}$  ( $\Gamma = \text{BDih}_n \cdot \mathbb{C}^*(1,1)$ )



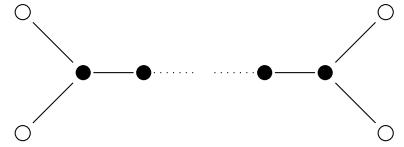
$N = \text{BDih}_n$   
 $\mathbb{C}^2/N = D_{n+2}$



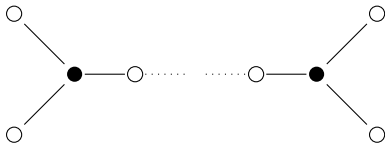
$N = \text{BDih}_{n/2}$   
 $\mathbb{C}^2/N = D_{n/2+2}$   
 (only appears for  $n$  even)



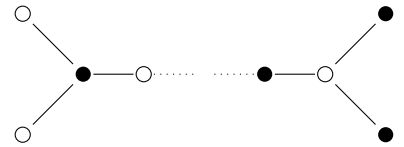
$N = \mu_{2n}$   
 $\mathbb{C}^2/N = A_{2n-1}$



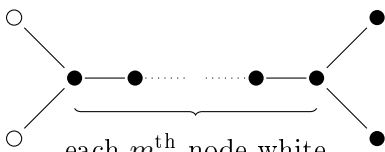
$N = \mu_n$   
 $\mathbb{C}^2/N = A_{n-1}$



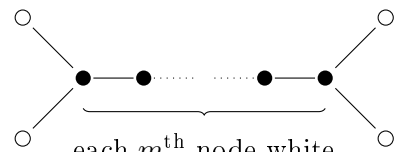
(bipartite colouring)  
 $N = \mu_2$   
 $\mathbb{C}^2/N = A_1$   
 ( $n$  even)



(bipartite colouring)  
 $N = \mu_2$   
 $\mathbb{C}^2/N = A_1$   
 ( $n$  odd)



each  $m^{\text{th}}$  node white  
 $N = \mu_m, 2n/m$  odd  
 $\mathbb{C}^2/N = A_{m-1}$



each  $m^{\text{th}}$  node white  
 $N = \mu_m, 2n/m$  even  
 $\mathbb{C}^2/N = A_{m-1}$

Table 4.2: Black and white diagrams for Fano orbifolds: II

# Chapter 5

## The black part

In the last chapter we associated to the data of an extended McKay group  $\Gamma = G \cdot \mathbb{C}^*$  and a normal subgroup  $N \triangleleft G$  a black and white diagram  $\Delta$ . This icon is a picture of the natural Frobenius structure on the category  $\text{CM}_{\Gamma/N}(R)$  of  $\Gamma/N$ -equivariant Cohen-Macaulay modules on the simple surface singularity  $\text{Spec } R = \mathbb{C}^2/N$ . Each node of  $\Delta$  corresponds to a  $\tau$ -orbit of indecomposable modules, and the white nodes correspond to orbits of projectives. The black part of the diagram  $\Delta^b \subset \Delta$  is therefore a picture of the stable category  $\underline{\text{CM}}_{\Gamma/N}(R)$ . Since  $\Delta^b$  is a strict subset of an extended Dynkin diagram, it is itself Dynkin and therefore has an associated derived category of representations,  $D^b(\Delta^b)$  (in fact an isomorphism class of such, but this subtlety doesn't concern us here).

### 5.1 Derived equivalence

The first aim of this chapter is the following theorem.

**Theorem 5.1.1.** Let  $\Gamma$  be an extended McKay group  $\Gamma = G \cdot \mathbb{C}^*$ ,  $N \triangleleft G$  a normal subgroup and  $\text{Spec } R = \mathbb{C}^2/N$ . Let  $\Delta$  be the black and white diagram associated to this data and  $\Delta^b$  its black part (a Dynkin diagram). Then

$$D^b(\Delta^b) \cong \underline{\text{CM}}_{\Gamma/N}(R)$$

as triangulated categories.

Before proving this, we need the following duality theorem of Buchweitz.

**Theorem 5.1.2** (Buchweitz Duality Theorem). Let  $S$  be an equi-codimensional Gorenstein isolated singularity of dimension  $d$  with an action of an extended McKay group  $\Gamma = \mathbb{C}^* \cdot G$ . So  $S$  carries a  $\Gamma$ -equivariant module  $K_S$  which is canonical. Then we have a

bifunctorial isomorphism

$$\mathrm{Ext}_{\underline{\mathrm{CM}}_{\Gamma}(S)}^{i+d-1}(M, N) \cong \mathrm{Ext}_{\underline{\mathrm{CM}}_{\Gamma}(S)}^{-i}(N, M \otimes_S K_S)^\vee$$

for all  $M, N \in \underline{\mathrm{CM}}_{\Gamma}(S)$  and  $i \in \mathbb{Z}$ , where  $^\vee$  denotes the  $\mathbb{C}^*$ -dual.

*Proof.* In [12] Proposition 10.1.5, Buchweitz proves the non-equivariant version of this, in the form of a perfect pairing

$$\mathrm{Ext}_{\underline{\mathrm{CM}}(S)}^{i+d-1}(M, N) \times \mathrm{Ext}_{\underline{\mathrm{CM}}(S)}^{-i}(N, M \otimes_S K_S) \longrightarrow \mathbb{C}.$$

The Example 10.1.6 which follows on from this proposition indicates how to proceed in the  $\mathbb{Z}$ -graded case. We are generalizing only very slightly to the case of a rank-one grading that possibly contains some torsion. Indeed the  $\Gamma$ -action on  $S$  induces  $\Gamma$ -actions on the Ext spaces, which decompose isotypically into  $\mathbb{Z}$ -graded pieces labelled by the characters of  $G$ . The perfect pairing matches a piece corresponding to  $\chi \in G^\vee$  with that corresponding to the inverse character  $\chi^{-1}$  and matches each  $\mathbb{Z}$ -degree with its negative, making the pairing homogeneous. □

**Remark 5.1.3.** Observe that under the additive equivalence  $\mathrm{CM}_{\Gamma/N}(R) \cong \mathrm{Vect}(\mathbb{X})$  for  $\mathbb{X}$  the weighted projective line  $\mathbb{X} := [\mathbb{C}^2 - 0/\Gamma]$ , the canonical module  $K_R$  corresponds to the canonical bundle  $\omega_{\mathbb{X}}$ . It is an immediate consequence of Theorem 5.1.2 that tensoring by the canonical module  $K_R$  is equivalent to the AR translate on  $\underline{\mathrm{CM}}_{\Gamma/N}(R)$ .

We are now ready to prove Theorem 5.1.1. Our proof is based on that of [33] Theorem 5.1, which this theorem generalises.

*Proof of Theorem 5.1.1.* Let  $\Gamma = G \cdot \mathbb{C}^*$  be an extended McKay group acting in the usual way on  $\mathbb{C}^2$  and put  $\mathbb{X} := [\mathbb{C}^2 - 0/\Gamma]$ , a weighted projective line. By Proposition 2.4.9 above,  $\mathrm{Coh}(\mathbb{X})$  has a tilting bundle  $T$  whose summands correspond to a transverse section of the action of the AR translate on  $\mathrm{Vect}(\mathbb{X})$ , that is to the nodes of the black and white diagram  $\Delta$ . The endomorphism ring  $\mathrm{End}(T)$  is isomorphic to the path algebra of some orientation of  $\Delta$  ([25], [33]).

We now add the data of a normal subgroup  $N \triangleleft G$  and put  $\mathbb{C}^2/N = \mathrm{Spec} R$ . The category  $\mathrm{CM}_{\Gamma/N}(R) \cong \mathrm{Vect}(\mathbb{X})$  carries a natural Frobenius structure (Proposition 2.3.8). We need to show that  $T$ , considered as an object of the stable category  $\underline{\mathrm{CM}}_{\Gamma/N}(R)$  is

- rigid, that is  $\mathrm{Hom}_{\underline{\mathrm{CM}}_{\Gamma/N}(R)}(T, T[i]) = 0$  for all nonzero  $i$ , and
- generating: the smallest thick triangulated subcategory of  $\underline{\mathrm{CM}}_{\Gamma/N}(R)$  containing  $T$  is  $\underline{\mathrm{CM}}_{\Gamma/N}(R)$  itself.

In  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$  the projective-injectives of  $T$  corresponding to the white nodes of  $\Delta$  are killed, so that the endomorphism ring of  $T$  in  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$  is isomorphic to the path algebra of an orientation of  $\Delta^b$ . Note that this algebra has finite global dimension. Standard results (see for example [30] and [20] Chapter III) then yield the equivalence  $D^b(\Delta^b) \cong \underline{\mathbf{CM}}_{\Gamma/N}(R)$ .

The category  $\mathbf{Vect}(\mathbb{X})$  has a well-defined slope function  $\mu : \mathbf{Vect}(\mathbb{X}) \rightarrow \mathbb{Q}$  with respect to which all indecomposable bundles are stable ([18] Proposition 5.5). This means that in  $\mathbf{Vect}(\mathbb{X})$  all morphisms between indecomposable bundles go in the direction of non-decreasing slope. Furthermore, along any non-isomorphism the inequality of slopes is strict. All of this remains true on the indecomposable objects of  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$ .

Recall from Proposition 2.4.9 that the summands  $T_j$  of  $T$  lie in the slope range

$$0 < \mu(T_j) \leq m$$

where  $-m$  denotes the slope of  $\omega_{\mathbb{X}}$ . We aim to show first that the negative shift takes any summand  $T_j$  out of this  $m$ -wide slice of  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$ . For all  $X \in \underline{\mathbf{CM}}_{\Gamma/N}(R)$  we have  $\mu(\tau(X)) = \mu(X) - m$  (see Remark 5.1.3). By Theorem 5.1.2,  $\mathrm{Hom}_{\underline{\mathbf{CM}}_{\Gamma/N}(R)}(Y[-1], \tau Y) \cong \mathrm{End}_{\underline{\mathbf{CM}}_{\Gamma/N}(R)}(Y)^\vee$  is nonempty for any indecomposable object  $Y$  of  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$ , so there is a nontrivial morphism  $Y[-1] \rightarrow \tau Y$ . Thus  $\mu(Y[-1]) \leq \mu(Y) - m$ . Now let  $T_k, T_j$  be indecomposable summands of the tilting bundle  $T$ . By construction we have  $|\mu(T_k) - \mu(T_j)| < m$ . It follows that for all  $i > 0$ ,  $\mu(T_k[-i]) < \mu(T_j)$  and hence  $\mathrm{Hom}_{\underline{\mathbf{CM}}_{\Gamma/N}(R)}(T_j, T_k[-i]) = 0$ .

Now consider  $\mathrm{Hom}_{\underline{\mathbf{CM}}_{\Gamma/N}(R)}(T_j, T_k[i])$  for  $i > 0$ . By Theorem 5.1.2 (and Remark 5.1.3),

$$\mathrm{Hom}_{\underline{\mathbf{CM}}_{\Gamma/N}(R)}(T_j, T_k[i]) \cong \mathrm{Hom}_{\underline{\mathbf{CM}}_{\Gamma/N}(R)}(T_k, \tau T_k[1-i])^\vee.$$

Since  $|\mu(T_k) - \mu(T_j)| < m$  we have  $\mu(\tau T_k[1-i]) < \mu(T_j)$  and so  $\mathrm{Hom}_{\underline{\mathbf{CM}}_{\Gamma/N}(R)}(T_j, T_k[i]) = 0$  and  $T$  is rigid as claimed.

By construction of  $T$ , every  $\tau$ -orbit in  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$  intersects  $T$  in one indecomposable summand. Let  $\mathcal{C}$  denote the smallest thick triangulated subcategory of  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$  containing  $T$ . Note that every indecomposable with slope in the interval  $(0, m]$  is a summand of  $T$  and hence contained in  $\mathcal{C}$ . Suppose that  $T$  is not generating, so there exists some collection of indecomposables of  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$  which are not contained in  $\mathcal{C}$ . Suppose further that some of these indecomposables have slope strictly greater than  $m$  and let  $X$  denote the one of these with least slope. Since  $\underline{\mathbf{CM}}_{\Gamma/N}(R)$  has almost split triangles (Proposition 2.1.16), there is a nonsplit triangle

$$\tau X \longrightarrow Y \longrightarrow X \longrightarrow .$$



Since this triangle is nonsplit none of the components of the maps are isomorphisms, so  $\tau X$  and all of the indecomposable summands of  $Y$  have slopes in the interval  $[\mu(X) - m, \mu(X))$ . It follows that all of these objects are contained in  $\mathcal{C}$ . But as triangulated subcategories are in particular closed under taking cones and extensions,  $X$  must be contained in  $\mathcal{C}$  too, a contradiction. The argument for those indecomposable objects  $X$  not contained in  $\mathcal{C}$  with  $\mu(X) \leq m$  (and hence  $\mu(X) \leq 0$ ) is similar.  $\square$

**Remark 5.1.4.** As already mentioned, this theorem is a direct generalization of Kussin-Lenzing-Meltzer [33] Theorem 5.1. It also generalizes Kajiuura-Saito-Takahashi [28] Theorem 3.1. It is worth pointing out that these two theorems deal with two different stable categories arising from two different Frobenius structures on  $\text{Vect}(\mathbb{X})$ . Kajiuura-Saito-Takahashi deal with the case of a single  $\tau$ -orbit of projectives and prove  $\underline{MF}^{\mathbb{Z}}(\mathbb{C}[x, y, z], f(x, y, z)) \cong D^b(\Delta)$  where  $\Delta$  is the Dynkin type of the singularity  $f$ .

Kussin-Lenzing-Meltzer on the other hand deal with the case where all line bundles are projective. Thus when they claim in the last part of the theorem that for a Fano orbifold with weight sequence of Dynkin type  $\Delta$  that  $\underline{\text{Vect}}(\mathbb{X})$  is of the form  $\mathbb{Z}\Delta$ , this is not actually correct. The black part of the black and white diagram for this Frobenius structure is not isomorphic to  $\Delta$ . See their figure immediately following the proof of [33] Theorem 5.1 which illustrates this.

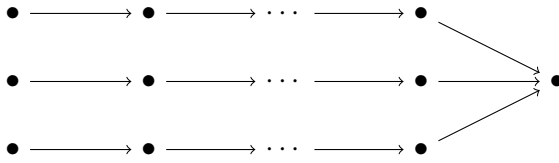
**Remark 5.1.5.** Our proof of Theorem 5.1.1 is essentially that of Kussin-Lenzing-Meltzer's [33] Theorem 5.1. In [28], Kajiuura-Saito-Takahashi offer two proofs of their Theorem 3.1. The first is a direct construction of an exceptional collection in the homotopy category of  $\mathbb{Z}$ -graded matrix factorizations  $\text{HMF}^{\mathbb{Z}}(\mathbb{C}[x, y, z], f(x, y, z))$  ([28] Corollary 3.12). The second, given in the Appendix A of [28] is due to Ueda, and is based upon two observations. The first is the derived equivalence

$$D^b(\text{Coh}(\mathbb{X})) \cong D^b(\text{mod}(\Lambda))$$

for  $\Lambda$  a canonical algebra (Theorem 2.4.6 above), coming from the tilting bundle

$$T = \bigoplus_{\ell \in [0, \mathbf{1}_{\mathbb{X}}]} \mathcal{O}(\ell)$$

(Proposition 2.4.5 above). The second is Orlov's Theorem 2.4.12 above. A tilting bundle for  $\text{HMF}^{\mathbb{Z}}(\mathbb{C}[x, y, z], f(x, y, z)) = \underline{\text{CM}}^{\mathbb{Z}}(R')$  is thus obtained by deleting the leftmost node (that is,  $\mathcal{O}$ ) from Figure 2-8. The resulting picture is a tree, the Dynkin diagram that Kajiuura-Saito-Takahashi-Ueda claim.



We expect that these ideas could be used to prove our generalization 5.1.1, by using a generalization of Orlov’s theorem. Note that Kussin-Lenzing-Meltzer have already generalized Orlov’s theorem to the  $L$ -graded situation (Theorem 2.4.13 above), however we would require an equivariant version allowing for the possibility of non-abelian  $\Gamma$ .

## 5.2 Discussion

Fix again  $\Gamma$  and  $N \triangleleft G$ . The simple singularity  $\text{Spec } R = \mathbb{C}^2/N$  has a Dynkin type, coming from the homology of its Milnor fibre. Inspection of the tables 4.1 and 4.2 of Chapter 4 shows that the relationship between this Dynkin type and the Dynkin type of the black part  $\Delta^b$  of the associated black and white diagram is not completely straightforward: sometimes they are the same, and sometimes they are not. Takahashi [44] has suggested that this is a mirror symmetry phenomenon, relating the symplectic (A-model) data of the Milnor fibre to the holomorphic (B-model) data of the stable category of Cohen-Macaulay modules.

By 4.2.1, the category  $\text{CM}_{\Gamma/N}(R) \cong \text{Vect}(\mathbb{X})$ , and hence its AR quiver, has an automorphism  $\tau$ , the AR translate, given by tensoring by the canonical bundle  $\omega_{\mathbb{X}}$  of  $\mathbb{X} = [\mathbb{C}^2 - 0/\Gamma]$ . As a node in the McKay graph of  $\Gamma$ ,  $\omega_{\mathbb{X}}$  corresponds to the determinant representation of  $\Gamma$ , with finite kernel  $G$  (Proposition 2.3.6). The quotient of the AR quiver of  $\text{CM}_{\Gamma/N}(R)$  by  $\tau$  is by definition the black and white diagram associated to the data  $N \triangleleft G \subset \Gamma$ . This black and white diagram may be thought of then as the McKay graph of  $G$ , with nodes coloured white if and only if they correspond to representations which are trivial on  $N$ . Or alternatively we can think of it as the AR quiver of the category

$$\text{CM}_{G/N}(\hat{R}).$$

The diagram is a picture of the Frobenius structure on  $G/N$ -equivariant Cohen-Macaulay  $\hat{R}$ -modules. Indecomposable projectives in this category arise by tensoring the single indecomposable projective  $\hat{R}$ -module, that is  $\hat{R}$  itself, with the irreducible representations of  $G/N$ . Which is to say they correspond to representations of  $G$  trivial on  $N$ , the white nodes on the diagram.

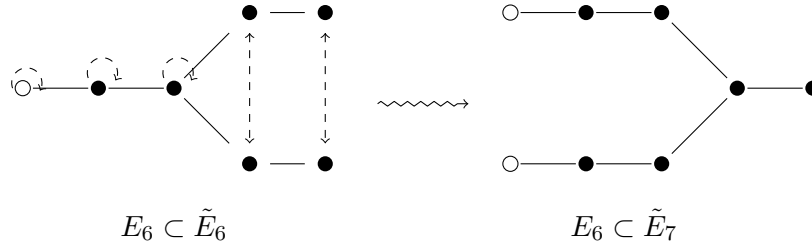
Describing the black nodes is a little more tricky, and the analysis here becomes somewhat case-by-case.

### 5.2.1 $G$ is a semidirect product $N \rtimes G/N$ .

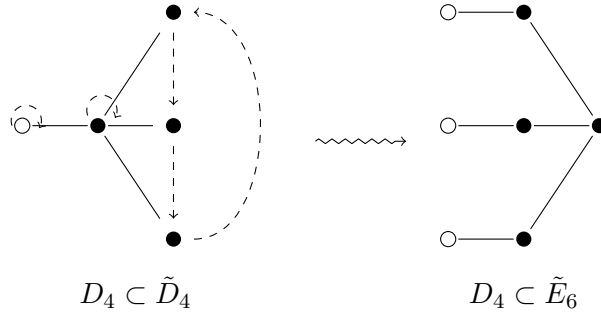
Since  $G$  is a semidirect product, there is an action of  $G/N$  on  $N$ , which pulls back to an action on  $\text{McKay}(N)$ . The orbifolding of this graph by this action of  $G/N$  gives  $\text{McKay}(G)$ .

Every group is trivially a (semi)direct product of itself with the identity. The other semidirect decompositions of the finite subgroups of  $\text{SL}(2, \mathbb{C})$  are  $\text{BOct} \cong \text{BTet} \rtimes \mu_2$ ,  $\text{BTet} \cong \text{BDih}_2 \rtimes \mu_3$  and  $\text{BDih}_n \cong \mu_p \rtimes \mu_{4n/p}$  for  $p$  an odd prime dividing  $n$ . Curiously enough, in all of these cases, the black part of the  $G/N$ -orbifolded  $\text{McKay}(N)$  is isomorphic to the black part of the non-orbifolded  $\text{McKay}(N)$ .

**Example 5.2.1** ( $\text{BOct} \cong \text{BTet} \rtimes \mu_2$ ). The pullback action of  $\mu_2$  on  $\text{McKay}(\text{BTet})$  fixes the trivial representation, the two dimensional quaternionic irrep and the unique three dimensional irrep. The two nontrivial linear representations are exchanged, as are the remaining pair of two dimensional representations.



**Example 5.2.2** ( $\text{BTet} \cong \text{BDih}_2 \rtimes \mu_3$ ). The  $\mu_3$  action fixes the trivial representation and the two dimensional representation and cycles the three nontrivial linear representations.



**Example 5.2.3** ( $\text{BDih}_n \cong \mu_r \rtimes \mu_s$ ). When  $n$  is not a power of 2,  $\text{BDih}_n$  decomposes as a semidirect product  $\mu_r \rtimes \mu_s$  for each odd prime  $p|4n$ , with  $r$  the highest power of  $p$  dividing  $4n$  and  $s = 4n/p$ . Generators of  $\mu_s = \text{BDih}_n/\mu_r$  act on  $\mu_r$  as inversion; since  $|\mu_r|$  is odd the pullback action on  $\text{McKay}(\mu_r)$  is free outside of the trivial representation.

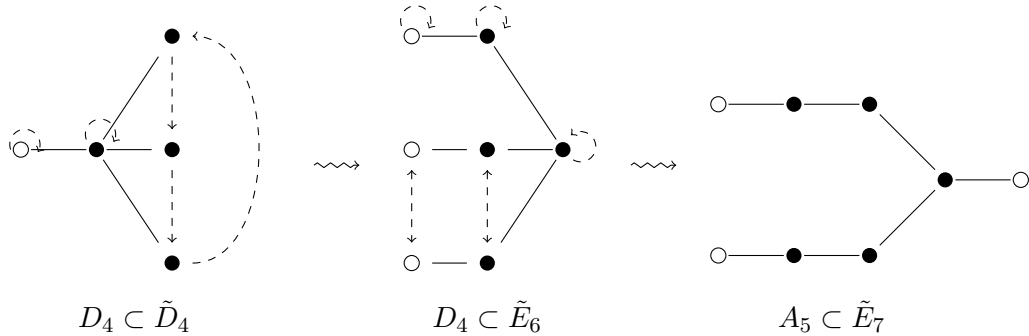
Hence the black part of the black and white diagram for  $\mu_r \triangleleft \text{BDih}_n$  is isomorphic to the black part of the black and white diagram of  $\mu_r$

Since they are exhaustive, we can collect these examples into a proposition.

**Proposition 5.2.4.** Let  $\Gamma$  be a one-dimensional group with finite determinant-one part  $G$  and  $N \triangleleft G$  a normal subgroup such that  $G \cong N \rtimes G/N$  is a semidirect product. Then the black part of the black and white diagram associated to this data is isomorphic to the McKay graph of  $N$ , minus the trivial node.

### 5.2.2 $G$ is not a semidirect product $N \rtimes G/N$ .

In this situation there is not a great deal we can say, except make the following observation. Along with a  $\text{BTet} \triangleleft \text{BOct}$ ,  $\text{BOct}$  also contains a normal  $\text{BDih}_2$ . But this subgroup is not a semidirect factor as  $\text{BOct}$  contains no subgroup isomorphic to the quotient  $\text{BOct}/\text{BDih}_2 \cong S_3$ . However we can in a sense compose the previous two examples. That is, we have  $\text{BOct} \cong \text{BTet} \rtimes \mu_2 \cong (\text{BDih}_2 \rtimes \mu_3) \rtimes \mu_2$ . If we take the McKay graph of  $\text{BDih}_2$  and act on it first by  $\mu_3$  and then by  $\mu_2$ , we get the black and white diagram for  $\text{BDih}_2 \triangleleft \text{BOct}$ .



## Part II

# Curves and cluster categories

# Chapter 6

## *p*-cycles

In this chapter we introduce Lenzing's *p*-cycle construction (see [35]). Section 6.1 is largely technical and not new, though we prove some facts that we could not find proofs of in the literature; in particular the fundamental example, that taking *p*-cycles on a category of coherent sheaves over an orbifold  $\mathbb{X}$  yields the category of coherent sheaves over a heavier orbifold  $\mathbb{X}^\sharp$  (Proposition 6.1.7). This rests on the observations that the category of *p*-cycles on a module category is itself a module category (Proposition 6.1.5) and that *p*-cycling behaves well with respect to Serre quotients (Lemma 6.1.2).

It is worth pointing out here that we do make one minor adjustment to Lenzing's construction which is of some importance to us. That is, we allow one to take *p*-cycles in multiple points simultaneously, see Proposition 6.1.7.

### 6.1 Abelian categories

Let  $\mathcal{A}$  be an abelian category with an automorphism  $\sigma_f$  and a natural transformation  $f : \text{Id} \rightarrow \sigma_f$ . In [35], Lenzing constructs the category of *p*-cycles on  $\mathcal{A}$  concentrated in  $f$ .

**Definition 6.1.1.** An *p*-cycle concentrated in  $f$  is a diagram

$$\dots \xrightarrow{a_{-1}} A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} \dots \xrightarrow{a_{p-2}} A_{p-1} \xrightarrow{a_{p-1}} A_p \xrightarrow{a_p} \dots ,$$

in  $\mathcal{A}$  indexed by  $\mathbb{Z}$ , which is *p*-periodic in the following sense:  $A_{i+p} = \sigma_f A_i$ ,  $a_{i+p} = \sigma_f a_i$  and  $a_{i+p-1} \circ \dots \circ a_i = f_{A_i}$  for each  $i \in \mathbb{Z}$ . We denote *p*-cycles by

$$A_0 \xrightarrow{a_0} \dots \xrightarrow{a_{p-2}} A_{p-1} \xrightarrow{a_{p-1}} \sigma_f A_0.$$

A morphism  $\phi : A \rightarrow B$  of *p*-cycles is a sequence of morphisms  $\phi_i : A_i \rightarrow B_i$ ,

periodic in the sense that  $\phi_{i+p} = \sigma_f \phi_i$  for all  $i$ , such that every square

$$\begin{array}{ccc} A_i & \xrightarrow{a_i} & A_{i+1} \\ \downarrow \phi_i & & \downarrow \phi_{i+1} \\ B_i & \xrightarrow{b_i} & B_{i+1} \end{array}$$

commutes.

Denote the abelian category of all  $p$ -cycles concentrated in the transformation  $f$  by  $p\text{-cyc}(\mathcal{A}, f)$ . This category comes with an embedding

$$\iota : \mathcal{A} \hookrightarrow p\text{-cyc}(\mathcal{A}, f), \quad A \mapsto A \xrightarrow{\text{id}} \dots \xrightarrow{\text{id}} A \xrightarrow{f} \sigma_f A$$

and a cycling autoequivalence

$$\sigma : p\text{-cyc}(\mathcal{A}, f) \rightarrow p\text{-cyc}(\mathcal{A}, f),$$

shifting  $p$ -cycles one place to the left; observe that  $\sigma^p A_\bullet = \sigma_f A_\bullet$ .

In practice our applications of this construction will be exclusively to categories of equivariant modules and of sheaves. Before turning to these more concrete cases though we need a further generality.

**Lemma 6.1.2.** Let  $\mathcal{A}$  be an abelian category with an automorphism  $\sigma_f$  and a natural transformation  $f : \text{Id} \rightarrow \sigma_f$ . Let  $\mathcal{S} \subset \mathcal{A}$  be a Serre subcategory fixed by  $\sigma_f$ . Then

1.  $p\text{-cyc}(\mathcal{S}, f) \subset p\text{-cyc}(\mathcal{A}, f)$  is a Serre subcategory,
2.  $\sigma_f$  and  $f$  descend to an autoequivalence  $\sigma_{[f]}$  on  $\mathcal{A}/\mathcal{S}$  and a natural transformation  $[f] : \text{Id}_{\mathcal{A}/\mathcal{S}} \rightarrow \sigma_{[f]}$ ,
3. the obvious map

$$p\text{-cyc}(\mathcal{A}, f) \longrightarrow p\text{-cyc}(\mathcal{A}/\mathcal{S}, [f])$$

is functorial and induces an equivalence

$$p\text{-cyc}(\mathcal{A}, f)/p\text{-cyc}(\mathcal{S}, f) \xrightarrow{\cong} p\text{-cyc}(\mathcal{A}/\mathcal{S}, [f]).$$

*Proof.* The first claim is immediate since any short exact sequence

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

of  $p$ -cycles of objects of  $\mathcal{A}$  implies exactness pointwise, so  $A_\bullet, C_\bullet \in p\text{-cyc}(\mathcal{S}, f)$  if and

only if  $B_\bullet \in p\text{-cyc}(\mathcal{S}, f)$ .

Now define an autoequivalence  $\sigma_{[f]}$  on  $\mathcal{A}/\mathcal{S}$  as follows. Since the objects of  $\mathcal{A}$  and  $\mathcal{A}/\mathcal{S}$  are the same, put

$$\sigma_{[f]}(X) := \sigma_f(X)$$

for  $X \in \mathcal{A}/\mathcal{S}$ . If  $\phi \in \text{Hom}_{\mathcal{A}/\mathcal{S}}(X, Y)$  then by definition there exist  $X' \subset X$  and  $Y' \subset Y$  such that  $X/X', Y' \in \mathcal{S}$  and a morphism  $\bar{\phi} \in \text{Hom}_{\mathcal{A}}(X', Y/Y')$  such that  $\phi = [\bar{\phi}]$ . I claim that  $[\sigma_f(\bar{\phi})] \in \text{Hom}_{\mathcal{A}/\mathcal{S}}(\sigma_{[f]}(X), \sigma_{[f]}(Y))$ . This is reasonably clear, since  $\sigma_f$  is an equivalence, so

$$\sigma_f(\bar{\phi}) : \sigma_f(X') \longrightarrow \sigma_f(Y/Y') = \sigma_f(Y)/\sigma_f(Y')$$

with on the one hand  $\sigma_f(X') \subset \sigma_f(X)$  and  $\sigma_f(Y') \subset \sigma_f(Y)$ , and on the other  $\sigma_f(Y') \in \mathcal{S}$  and  $\sigma_f(X)/\sigma_f(X') = \sigma_f(X/X') \in \mathcal{S}$ . So put

$$\sigma_{[f]}(\phi) := [\sigma_f(\bar{\phi})].$$

We need to check that this is well-defined. Suppose then that there exist  $X'' \subset X$  and  $Y'' \subset Y$  such that  $X/X'', Y'' \in \mathcal{S}$  and a morphism  $\bar{\bar{\phi}} \in \text{Hom}_{\mathcal{A}}(X'', Y/Y'')$  such that  $\phi = [\bar{\bar{\phi}}]$ . Then by definition there is a diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}(X', Y/Y') & & \text{Hom}_{\mathcal{A}}(X'', Y/Y'') \\ & \searrow \alpha & \swarrow \beta \\ & \text{Hom}_{\mathcal{A}}(X''', Y/Y''') & \end{array}$$

such that  $\alpha(\bar{\phi}) = \beta(\bar{\bar{\phi}})$ . Applying  $\sigma_f$  gives  $\alpha(\sigma_f(\bar{\phi})) = \beta(\sigma_f(\bar{\bar{\phi}}))$  (since  $\sigma_f$  is an equivalence), so  $\sigma_{[f]}$  is well-defined. Note that applying  $\sigma_f^{-1}$  to the corresponding diagram for  $\phi, \psi \in \text{Hom}_{\mathcal{A}/\mathcal{S}}(X, Y)$  with  $\sigma_{[f]}(\phi) = \sigma_{[f]}(\psi) \in \text{Hom}_{\mathcal{A}/\mathcal{S}}(\sigma_{[f]}(X), \sigma_{[f]}(Y))$  shows that  $\sigma_{[f]}$  is faithful. It is clearly full and essentially surjective, hence an autoequivalence as desired.

The natural transformation  $f$  gives us a map  $\text{Id}_{\mathcal{A}}(X) \longrightarrow \sigma_f(X)$  for all  $X$ . We therefore define

$$[f] : \text{Id}_{\mathcal{A}/\mathcal{S}}(X) \longrightarrow \sigma_{[f]}(X) = \sigma_f(X)$$

to be the class of  $f$  in  $\text{Hom}_{\mathcal{A}/\mathcal{S}}(X, \sigma_{[f]}(X))$ .

Finally, define a functor

$$F : p\text{-cyc}(\mathcal{A}, f) \longrightarrow p\text{-cyc}(\mathcal{A}/\mathcal{S}, [f])$$



by sending  $p$ -cycles

$$A_0 \xrightarrow{a_0} \dots \xrightarrow{a_{p-2}} A_{p-1} \xrightarrow{a_{p-1}} \sigma_f A_0$$

to

$$A_0 \xrightarrow{[a_0]} \dots \xrightarrow{[a_{p-2}]} A_{p-1} \xrightarrow{[a_{p-1}]} \sigma_{[f]} A_0$$

and morphisms  $(\phi_i)$  to  $([\phi_i])$ . Pointwise, this is simply the canonical quotient  $Q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ , hence it is exact. The kernel of  $F$  is precisely the Serre subcategory  $p\text{-cyc}(\mathcal{S}, f) \subset p\text{-cyc}(\mathcal{A}, f)$ . Hence by the universal property of the quotient, there exists a unique exact functor

$$G : p\text{-cyc}(\mathcal{A}, f)/p\text{-cyc}(\mathcal{S}, f) \rightarrow p\text{-cyc}(\mathcal{A}/\mathcal{S}, [f])$$

which is faithful and essentially surjective. This functor acts as the identity on objects and sends classes  $([\phi_i])$  of sequences of morphisms to the corresponding sequence  $([\phi_i])$  of classes. It is therefore surjective on  $\text{Homs}$ , and an equivalence of categories.  $\square$

**Example 6.1.3** (Modules). Let  $R$  be a ring and  $\mathcal{A} = \text{mod}_\Gamma(R)$  be the category of  $\Gamma$ -equivariant  $R$ -modules for some extended McKay group  $\Gamma$  (so in particular  $R$  is equivariant-local). Let  $f \in R$  be a semi-invariant for  $\Gamma$  with associated character  $\chi_f$ . Then  $\sigma_f := - \otimes_R \chi_f$  is a shift autoequivalence on  $\text{mod}_\Gamma(R)$  and multiplication by  $f$  gives a natural transformation  $f : \text{Id} \rightarrow \sigma_f$ . We can therefore take  $p$ -cycles over  $\text{mod}_\Gamma(R)$  concentrated in  $f$ . The category  $p\text{-cyc}(\text{mod}_\Gamma(R), f)$  can be thought of as a generalization of the category of  $\Gamma$ -equivariant matrix factorizations of  $f$  over  $R$ . Indeed  $\text{MF}_\Gamma(R, f) = 2\text{-cyc}(\text{Free}_\Gamma(R), f)$ , though be aware that  $\text{Free}_\Gamma(R)$  is not an abelian category; it is exact. We will have more to say about this shortly.

We now realise  $p\text{-cyc}(\text{mod}_\Gamma(R), f)$  itself as a module category.

**Definition 6.1.4.** Write

$$\mathcal{L}_p(R, f) := R[z]/(f - z^p)$$

and

$$\mathcal{L}_p(\Gamma, \chi_f) := \{(\gamma, u) \in \Gamma \times \mathbb{C}^* \mid \chi_f(\gamma) = u^p\}.$$

We will sometimes abuse this notation by omitting some of the arguments.

**Proposition 6.1.5.**

$$p\text{-cyc}(\text{mod}_\Gamma(R), f) \cong \text{mod}_{\mathcal{L}_p(\Gamma, \chi_f)}(\mathcal{L}_p(R, f)).$$

*Proof.* Define a functor

$$\mathcal{F} : p\text{-cyc}(\text{mod}_\Gamma(R), f) \longrightarrow \text{mod}_{\mathcal{L}_p(\Gamma, \chi_f)}(\mathcal{L}_p(R, f))$$

by sending a  $p$ -cycle

$$A_0 \xrightarrow{a_0} \dots \xrightarrow{a_{p-2}} A_{p-1} \xrightarrow{a_{p-1}} \sigma_f A_0$$

of equivariant  $R$ -modules to

$$A := \bigoplus_{i=0}^{p-1} A_i.$$

This receives the structure of an  $\mathcal{L}_p(\Gamma, \chi_f)$ -equivariant  $\mathcal{L}_p(R, f)$ -module by letting  $z$  act on  $A_i$  by  $a_i$ .

Now  $\mathcal{L}_p(\Gamma, \chi_f)$  contains a  $\mu_p$  generated by  $(1, e^{\frac{2\pi i}{p}})$ , so any  $B \in \text{mod}_{\mathcal{L}_p(\Gamma, \chi_f)}(\mathcal{L}_p(R, f))$  is  $\mu_p^\vee = \mathbb{Z}/p\mathbb{Z}$ -graded,  $B = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} B_i$ , with  $z \in \mathcal{L}_p(R, f)$  acting in degree one. Thus

$$B_\bullet := B_0 \xrightarrow{z} \dots \xrightarrow{z} B_{p-1} \xrightarrow{z} \sigma_f B_0$$

is a  $p$ -cycle of  $\text{mod}_\Gamma(R)$ -modules and  $\mathcal{F}(B_\bullet) = B$ . This shows that  $\mathcal{F}$  is essentially surjective. It is fully faithful by the definition of morphisms in  $p\text{-cyc}(\text{mod}_\Gamma(R), f)$ .  $\square$

**Example 6.1.6** (Sheaves). Let  $\mathbb{X}$  be an orbifold,  $L$  a line bundle on  $\mathbb{X}$  and  $f$  a rational section of  $L$ . We ask further that the associated divisor  $D_f$  is everywhere multiplicity one so its data is that of a collection of points of the coarse space  $X = \mathbb{P}^1$  of  $\mathbb{X}$ . Tensoring with  $L$  defines an autoequivalence  $\sigma_f$  on  $\text{Coh}(\mathbb{X})$  and multiplication by  $f$  a natural transformation  $\text{Id}_{\text{Coh}(\mathbb{X})} \longrightarrow \sigma_f$ . We can therefore take  $p$ -cycles in  $f$  over  $\text{Coh}(\mathbb{X})$ . Indeed, this construction was Lenzing's original motivation for the introduction of  $p$ -cycles in [35]. The result is an orbifold  $\mathbb{X}^\sharp$  with the same coarse space as  $\mathbb{X}$ , on which the points of  $D_f$  have weights  $p$  times what they were on  $\mathbb{X}$ .

**Proposition 6.1.7.** Let  $\mathbb{X}$  be an orbifold,  $L$  a line bundle on  $\mathbb{X}$  and  $f$  a rational section of  $L$  such that the associated divisor  $D_f$  is everywhere multiplicity one. Write  $(w_1, \dots, w_n)$  for the orbifold weights of the points of  $D_f$  (including points of weight one) and  $(q_1, \dots, q_k)$  for the weights of the orbifold points outside  $D_f$ . Let  $p > 1$  and  $\mathbb{X}^\sharp$  be the orbifold with weight sequence  $(pw_1, \dots, pw_n, q_1, \dots, q_k)$ . Then

$$p\text{-cyc}(\text{Coh}(\mathbb{X}), f) \cong \text{Coh}(\mathbb{X}^\sharp).$$

*Proof.* Let  $R$  be the Cox ring of  $\mathbb{X}$ , graded by  $L := \text{Pic}(\mathbb{X})$ , so

$$\text{Coh}(\mathbb{X}) = \text{mod}^L(R) / \text{mod}_0^L(R)$$

(Theorem 2.4.1). By Lemma 6.1.2 and Proposition 6.1.5 we have

$$\begin{aligned} p\text{-cyc}(\text{Coh}(\mathbb{X}), f) &\cong p\text{-cyc}(\text{mod}^L(R) / \text{mod}_0^L(R)) \\ &\cong p\text{-cyc}(\text{mod}^L(R)) / p\text{-cyc}(\text{mod}_0^L(R)) \\ &\cong \text{mod}_{\mathcal{L}_p(L^\vee, \chi_f)}(\mathcal{L}_p(R, f)) / (\text{mod}_{\mathcal{L}_p(L^\vee, \chi_f)})_0(\mathcal{L}_p(R, f)). \end{aligned}$$

When  $f$  cuts out a single point of  $\mathbb{X}$ , by [35] one can see that

$$\mathcal{L}_p(R, f) = R[z] / (f - z^p)$$

is isomorphic to the Cox ring of  $\mathbb{X}^\sharp$  and that

$$\mathcal{L}_p(L^\vee, \chi_f)$$

is dual to  $\text{Pic}(\mathbb{X}^\sharp)$ , given by

$$\langle L, x | px = \deg_L(f) \rangle$$

(this was Lenzing's original set-up in [35]). Our claim is that one can  $p$ -cycle simultaneously on multiple points of  $\mathbb{X}$ . By induction it is enough to show that when  $f$  cuts out two points, that is when  $f$  is a product  $f = f_1 f_2$  of two  $L$ -homogeneous factors, we have

$$\mathbb{X}^\sharp = \left[ \frac{\text{Spec } \mathcal{L}(\mathcal{L}(R, f_1), f_2) - 0}{\mathcal{L}(\mathcal{L}(L^\vee, \chi_{f_1}), \chi_{f_2})} \right] \cong \mathcal{L}(\mathbb{X}, f) := \left[ \frac{\text{Spec } \mathcal{L}(R, f) - 0}{\mathcal{L}(L^\vee, \chi_f)} \right]$$

where

$$\mathcal{L}(\mathcal{L}(R, f_1), f_2) = \frac{R[z_1, z_2]}{(f_1 - z_1^p, f_2 - z_2^p)} =: R^\sharp \quad \text{and} \quad \mathcal{L}(R, f) = \frac{R[z]}{(f_1 f_2 - z^p)}.$$

Observe that we have a surjection

$$\begin{aligned} \mathcal{L}(\mathcal{L}(L^\vee, \chi_{f_1})) &= \{(\gamma, u_1, u_2) | \chi_{f_1} = u_1^p, \chi_{f_2} = u_2^p\} \longrightarrow \mathcal{L}(L^\vee, \chi_f) = \{(\gamma, u) | \chi_f = u^p\} \\ (\gamma, u_1, u_2) &\longmapsto (\gamma, u_1 u_2) \end{aligned}$$

with kernel  $\{1, \zeta_p, \zeta_p^{-1}\} = \mu_p$ . This kernel acts freely on  $\text{Spec } R^\sharp - 0$ , with ring of invariants

$$\frac{R[z_1^p, z_2^p, z_1 z_2]}{(f_1 - z_1^p, f_2 - z_2^p)} \cong \frac{R[z]}{(f_1 f_2 - z^p)} = \mathcal{L}(R, f).$$

We have shown that

$$\mathcal{L}(\mathbb{X}, f) = \left[ \frac{\text{Spec } R^\sharp / \mu_p - 0}{\mathcal{L}(\mathcal{L}(L^\vee, \chi_{f_1}), \chi_{f_2}) / \mu_p} \right] \cong \mathbb{X}^\sharp$$

as claimed.  $\square$

Recall (Section 2.4 above) that the category of coherent sheaves on  $\mathbb{X}$  decomposes into the category of torsion-free sheaves  $\text{Vect}(\mathbb{X})$  and the category of finite length sheaves  $\text{Coh}_0(\mathbb{X})$ , each of whose indecomposable objects is supported at a single point of  $\mathbb{X}$ . The subcategory of finite length sheaves decomposes further as a collection  $\text{Coh}_0(\mathbb{X}) = \coprod_{\lambda \in \mathbb{X}} \mathcal{U}_\lambda$  of connected uniserial categories (or tubes) parameterized by the points of  $\mathbb{X}$ . The simple objects of  $\text{Coh}(\mathbb{X})$  lie at the bottom of these tubes, with a tube  $\mathcal{U}_\lambda$  having  $w_\lambda$  simples at its mouth, where  $w_\lambda$  is the orbifold weight of the point  $\lambda \in \mathbb{X}$ . Thus given  $\mathbb{X}$ ,  $f$  and  $p$  we can identify the simple objects of  $p\text{-cyc}(\text{Coh}(\mathbb{X}), f) = \text{Coh}(\mathbb{X}^\sharp)$ . However it will be useful to have the following more explicit description.

**Lemma 6.1.8** (Lenzing [35] Lemma 4.2). The simple objects of  $p\text{-cyc}(\text{Coh}(\mathbb{X}), f) = \text{Coh}(\mathbb{X}^\sharp)$  occur in two types:

1. a single simple object for each simple  $S \in \text{Coh}(\mathbb{X})$  supported at a point  $\lambda \in \mathbb{X} - D_f$ ,
2. for each simple  $S \in \text{Coh}(\mathbb{X})$  supported at a point  $\lambda \in D_f$ ,  $p$  simples

$$\begin{array}{cccccccc} S_1 & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & S & \longrightarrow & 0 \\ S_2 & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & S & \longrightarrow & 0 & \longrightarrow & 0 \\ \dots & & & & & \dots & & & & & & \\ S_{p-1} & 0 & \longrightarrow & S & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ S_p & S & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma_f S. \end{array}$$

Furthermore, if  $\mathcal{S} = \{S_1, S_2, \dots, S_{p-1} \mid S \in \text{Coh}(\mathbb{X}) \text{ simple and supported in } D_f\}$  then the left perpendicular category  ${}^\perp \mathcal{S} \subset p\text{-cyc}(\text{Coh}(\mathbb{X}), f)$  is equivalent to  $\text{Coh}(\mathbb{X})$ .

*Proof.* Notice first that the pointwise factors of any simple object  $S_\bullet \in p\text{-cyc}(\text{Coh}(\mathbb{X}), f)$  must

- be either simple or zero in  $\text{Coh}(\mathbb{X})$ , and
- all be supported at the same point  $\lambda \in \mathbb{X}$ .

Now let  $\lambda \in \mathbb{X} - D_f$  and  $S_\bullet$  be a  $p$ -cycle of simples supported at  $\lambda$ . Since  $f$  is invertible in the only nonzero stalk of any simple  $S \in \text{Coh}(\mathbb{X})$  supported at  $\lambda$ , it is an

isomorphism of sheaves. This means that the pointwise factors of  $S_\bullet$  are all nonzero and isomorphic. Thus up to isomorphism there is precisely one simple  $p$ -cycle,

$$S \xrightarrow{1} S \xrightarrow{1} \dots \xrightarrow{1} S \xrightarrow{f} S \xrightarrow{1} \sigma_f S,$$

for each  $S \in \text{Coh}(\mathbb{X})$  supported at  $\lambda$ .

On the other hand, if  $S$  is simple and supported at  $\lambda \in D_f$ , then  $f$  is the zero morphism on  $S$ . Thus

$$S \xrightarrow{1} S \xrightarrow{1} \dots \xrightarrow{1} S \xrightarrow{f} S \xrightarrow{1} \sigma_f S$$

has in this case a Jordan-Hölder filtration, with simple factors the  $S_i$ . This describes all the simples of  $\text{Coh}(\mathbb{X})$ .

Now let  $\mathcal{S} = \{S_1, S_2, \dots, S_{p-1} \mid S \in \text{Coh}(\mathbb{X}) \text{ simple and supported in } D_f\}$ . If  $A_\bullet \in p\text{-cyc}(\text{Coh}(\mathbb{X}, f))$  is (without loss of generality) indecomposable and such that the maps  $a_0, a_1, \dots, a_{p-2}$  are isomorphisms, then the pictures

$$\begin{array}{ccccc} A_{i-1} & \xrightarrow{\cong} & A_i & \longrightarrow & A_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} 0 & \longrightarrow & S & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ A_{i-2} & \longrightarrow & A_{i-1} & \xrightarrow{\cong} & A_i \end{array}$$

show that  $\text{Hom}(A_\bullet, \mathcal{S})$  and  $\text{Hom}(\tau(\mathcal{S}), A_\bullet)$  are zero, that is  $A_\bullet$  is left perpendicular to  $\mathcal{S}$ .

On the other hand, suppose some of the  $a_i$  are not isomorphisms. Then  $A_\bullet$  is either in  $\mathcal{S}$  or it is a  $p$ -cycle of bundles. Since every nonzero bundle has a nonzero morphism into a simple supported in  $D_f$ , in either case  $A_\bullet$  is not perpendicular to  $\mathcal{S}$ . □

## 6.2 Exact categories

Although  $p$ -cycles were first introduced for abelian categories, they behave well when restricted to exact subcategories.

**Theorem 6.2.1.** If  $\mathcal{A}$  is an exact category then  $p\text{-cyc}(\mathcal{A}, f)$  inherits an exact structure, by taking those sequences of  $p$ -cycles to be conflations which are conflations pointwise. Furthermore, if  $\mathcal{A}$  is Krull-Schmidt and Frobenius, then so is  $p\text{-cyc}(\mathcal{A}, f)$ , with

projective-injective objects

$$\text{Add}\left(\bigcup_{\substack{i \in \mathbb{Z} \\ P \text{ projective}}} \sigma^i \iota(P)\right).$$

*Proof.* One observes immediately that the class of pointwise conflations in  $p\text{-cyc}(\mathcal{A}, f)$  is closed under composition, pushout, pullback and contains the identity for all  $A_\bullet \in p\text{-cyc}(\mathcal{A}, f)$ . This proves the first claim.

Suppose  $\mathcal{A}$  is Frobenius and let  $P_\bullet$  be  $\sigma^i \iota(P)$  for some projective  $P$  and some  $i$ . So  $P_\bullet$  has  $f$  in the  $i^{\text{th}}$  position and 1 elsewhere. We want to show that maps from  $P_\bullet$  lift. So suppose

$$\begin{array}{ccccccc} A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & \dots & \xrightarrow{a_{p-2}} & A_{p-1} & \xrightarrow{a_{p-1}} & \sigma_f A_0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ B_0 & \xrightarrow{b_0} & B_1 & \xrightarrow{b_1} & \dots & \xrightarrow{b_{p-2}} & B_{p-1} & \xrightarrow{b_{p-1}} & \sigma_f B_0 \end{array}$$

is a deflation. Then any map  $\beta : (\bigoplus_i P_i)_\bullet \rightarrow B_\bullet$  lifts pointwise to maps

$$(\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{i(p-1)}) : \bigoplus_i P_i \rightarrow A_i.$$

These do not immediately glue together to give a map  $P_\bullet \rightarrow A_\bullet$ . To remedy this, simply put  $\alpha_{i+1}$  in the  $(i+1)^{\text{th}}$  position and then postcompose along the cycle with  $a_{i+1}$ ,  $a_{i+2}$  and so on. This produces the desired lift. A dual construction shows that these objects are injective. So all of  $\text{Add} \bigcup \sigma^i \iota(P)$  is projective-injective, as claimed.

Before showing that these are all the projectives, we show that  $p\text{-cyc}(\mathcal{A}, f)$  has enough. Let  $A_\bullet \in p\text{-cyc}(\mathcal{A}, f)$ . Since  $\mathcal{A}$  has enough projectives we have surjections  $\pi_i : P_i \rightarrow A_i$ . So  $\phi : (\bigoplus_i P_i)_\bullet \rightarrow A_\bullet$ ,

$$\begin{array}{ccccccc} \bigoplus_i P_i & \xrightarrow{\Pi_0} & \bigoplus_i P_i & \xrightarrow{\Pi_1} & \dots & \xrightarrow{\Pi_{p-2}} & \bigoplus_i P_i & \xrightarrow{\Pi_{p-1}} & \sigma_f \bigoplus_i P_i \\ \downarrow \phi_0 & & \downarrow \phi_1 & & & & \downarrow \phi_{p-1} & & \downarrow \\ A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & \dots & \xrightarrow{a_{p-2}} & A_{p-1} & \xrightarrow{a_{p-1}} & \sigma_f A_0 \end{array}$$

surjects onto  $A_\bullet$ , where  $\phi$  is given by

$$\phi_0 = (\pi_0, \pi_1 a_{p-1}, \pi_2 a_{p-1} a_{p-2}, \dots, \pi_{p-2} a_{p-1} \cdots a_2, \pi_{p-1} a_{p-1} \cdots a_1)$$

$$\phi_1 = (\pi_0 a_0, \pi_1 a_0 a_{p-1}, \pi_2 a_0 a_{p-1} a_{p-2}, \dots, \pi_{p-2} a_0 a_{p-1} \cdots a_2, \pi_{p-1})$$

$$\phi_2 = (\pi_0 a_1 a_0, \pi_1 a_1 a_0 a_{p-1}, \pi_2 a_1 a_0 a_{p-1} a_{p-2}, \dots, \pi_{p-2}, \pi_{p-1} a_1)$$

...

$$\phi_{p-1} = (\pi_0 a_{p-2} a_{p-1} \cdots a_0, \pi_1, \pi_2 a_{p-2}, \dots, \pi_{p-2} a_{p-2} a_{p-1} \cdots a_2, \pi_{p-1} a_{p-2} a_{p-1} \cdots a_1),$$

and  $\Pi_i$  is an identity matrix with  $f$  in position  $i$  along the diagonal (counting up from the bottom right).

A dual construction shows that  $p\text{-cyc}(\mathcal{A}, f)$  has enough injectives.

Finally observe that if  $A_\bullet \in p\text{-cyc}(\mathcal{A}, f)$  is projective then the surjection  $\phi : (\bigoplus_i P_i)_\bullet \rightarrow A_\bullet$  splits, making  $A_\bullet \in p\text{-cyc}(\mathcal{A}, f)$  a direct summand of  $(\bigoplus_i P_i)_\bullet$ , which proves the last claim. □

**Remark 6.2.2.** Suppose  $\mathcal{A}$  inherits an exact structure as a full extension-closed subcategory of some abelian category  $\mathcal{B}$ . Then  $p\text{-cyc}(\mathcal{B})$  is abelian,  $p\text{-cyc}(\mathcal{A}) \subset p\text{-cyc}(\mathcal{B})$  is full and extension-closed, and the exact structure  $p\text{-cyc}(\mathcal{A})$  inherits is that of Theorem 6.2.1.

The exact categories we are interested in are categories (of  $p$ -cycles of) equivariant Cohen-Macaulay modules  $\text{CM}_\Gamma(R) \subset \text{mod}_\Gamma(R)$  and categories of orbifold vector bundles  $\text{Vect}(\mathbb{X}) \subset \text{Coh}(\mathbb{X})$ . It is worth restating Theorem 6.1.5 for the Cohen-Macaulay situation.

From the data of a number  $p$ , a Cohen-Macaulay ring  $R$  with an extended McKay group action  $\Gamma$  and a semi-invariant  $f$ , we constructed a new ring  $\mathcal{L}_p(R, f) := R[z]/(f - z^p)$  and group  $\mathcal{L}_p(\Gamma, \chi_f) := \{(\gamma, u) \in \Gamma \times \mathbb{C}^* \mid \chi_f(\gamma) = u^p\}$ .

Then  $\mathcal{L}(\Gamma)$  acts naturally on  $\mathcal{L}(R)$  and  $\text{CM}_{\mathcal{L}(\Gamma)}(\mathcal{L}(R))$  acquires a Frobenius structure as a full extension-closed subcategory of  $\text{mod}_{\mathcal{L}(\Gamma)}(\mathcal{L}(R))$ . This is the structure obtained by forgetting the equivariance; a triangle is exact if and only if its image under the forgetful functor  $\text{CM}_{\mathcal{L}(\Gamma)}(\mathcal{L}(R)) \rightarrow \text{CM}(\mathcal{L}(R))$  is (see Proposition 2.3.8). The projective objects are those of the form  $\mathcal{L}(R) \otimes V$  for  $V$  a representation of  $\mathcal{L}(\Gamma)$  (see Lemma 2.3.7).

**Theorem 6.2.3.**

$$p\text{-cyc}(\text{CM}_\Gamma(R), f) = \text{CM}_{\mathcal{L}_p(\Gamma, \chi_f)}(\mathcal{L}_p(R, f)).$$

*Proof.* First of all,  $\mathcal{L}_p(R, f)$  is finitely generated and free as a module over  $R$  (it has rank  $p$ ). It is therefore Cohen-Macaulay over  $R$ , and hence a CM ring (see for example [46] Proposition 1.8). Since sums and summands of CM modules are CM, the functor

$$\begin{aligned} \mathcal{F} : p\text{-cyc}(\text{CM}_\Gamma(R), f) &\longrightarrow \text{CM}_{\mathcal{L}_p(\Gamma, \chi_f)}(\mathcal{L}_p(R, f)) \\ (A_0 \longrightarrow \dots \longrightarrow A_{p-1} \longrightarrow \sigma_f A_0) &\longmapsto \bigoplus_{i=0}^{p-1} A_i \end{aligned}$$

of Proposition 6.1.5, restricted to  $p\text{-cyc}(\text{CM}_\Gamma(R), f)$ , is an additive equivalence: the module  $\bigoplus_{i=0}^{p-1} A_i$  is CM over  $\mathcal{L}_p(R, f)$  by [46] Proposition 1.8 again. Moreover, since we are just summing and the exact structure on  $p\text{-cyc}(\text{CM}_\Gamma(R), f)$  is induced pointwise,  $\mathcal{F}$  is exact with exact inverse. □

**Corollary 6.2.4.** Theorem 6.2.3 induces an equivalence on the stable categories

$$\underline{p\text{-cyc}}(\text{CM}_\Gamma(R), f) = \underline{\text{CM}}_{\mathcal{L}_p(\Gamma, \chi_f)}(\mathcal{L}_p(R, f)).$$

In particular, when  $p = 2$  and  $R = \mathbb{C}[x, y]$  is trivial we have

$$\underline{MF}_\Gamma(\mathbb{C}[x, y], f) = \underline{MF}_{\mathcal{L}(\Gamma)}(\mathbb{C}[x, y, z], f + z^2)$$

or equivalently

$$\underline{\text{CM}}_\Gamma(R/f) = \underline{\text{CM}}_{\mathcal{L}(\Gamma)}(\mathcal{L}(R)).$$

*Proof.* This is immediate. □

Turning to vector bundles, it is also worth restating Proposition 6.1.7.

**Proposition 6.2.5.** Let  $\mathbb{X}$  be an orbifold,  $L$  a line bundle on  $\mathbb{X}$  and  $f$  a rational section of  $L$  such that the associated divisor  $D_f$  is everywhere multiplicity one. Write  $(w_1, \dots, w_n)$  for the orbifold weights of the points of  $D_f$  (including points of weight one). Let  $p > 1$  and  $\mathbb{X}^\sharp$  be the orbifold with weight sequence  $(pw_1, \dots, pw_n)$ . Then

$$p\text{-cyc}(\text{Vect}(\mathbb{X}), f) \cong \text{Vect}(\mathbb{X}^\sharp).$$

*Proof.* This follows from Theorem 2.4.2, Theorem 6.2.3 and Proposition 6.1.7. □



# Chapter 7

## Curves

In Part I we exploited the description of Fano orbifolds as stacky quotients of the punctured plane by the action of a one-dimensional group

$$\mathbb{X} = [\mathbb{C}^2 - 0/\Gamma]$$

to construct various exact structures on their categories of vector bundles. These exact structures arose as categories of equivariant Cohen-Macaulay modules

$$\mathrm{CM}_{\Gamma/N}(\mathbb{C}^2/N)$$

corresponding to normal subgroups of the finite determinant-one part of  $\Gamma$  and were catalogued by black and white diagrams.

We now extend this story by means of Lenzing's  $p$ -cycle construction (see [35]).

### 7.1 Frobenius structures for matrix factorizations

We begin with a Fano orbifold

$$\mathbb{X}^b = [\mathbb{C}^2 - 0/\Gamma^b]$$

where  $\Gamma^b = G^b \cdot \mathbb{C}^*$  is an extended McKay group and a normal subgroup  $N^b \triangleleft G^b$ . Let  $f(x, y)$  be a semi-invariant for the action of  $\Gamma^b$  on  $\mathbb{C}^2$ , invariant under the action of  $N^b$  and such that the associated divisor  $D_f$  on  $\mathbb{X}^b$  is everywhere multiplicity one. Write  $\mathrm{Spec} S^b$  for the surface  $\mathbb{C}^2/N^b$ , so the character  $\chi_f$  of  $\Gamma^b$  associated to  $f$  corresponds to a white node in the black and white diagram of the Frobenius category  $\mathrm{CM}_{\Gamma^b/N^b}(S^b) \cong$

$\text{Vect}(\mathbb{X}^b)$ . By Proposition 6.2.5 there is an additive equivalence

$$p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(S^b), f) \cong p\text{-cyc}(\text{Vect}(\mathbb{X}^b), D_f) \cong \text{Vect}(\mathbb{X}^\sharp)$$

where  $\mathbb{X}^\sharp$  is the orbifold obtained from  $\mathbb{X}^b$  by multiplying the orders of isotropy at the points of  $D_f$  by  $p$ .

Recall from Section 6.1 that the category of  $p$ -cycles comes with an inclusion

$$\iota : \text{CM}_{\Gamma^b/N^b}(S^b) \hookrightarrow p\text{-cyc}((\text{CM}_{\Gamma^b/N^b}(S^b), f))$$

and a cycling autoequivalence  $\sigma$ . By Theorem 6.2.1,  $p\text{-cyc}((\text{CM}_{\Gamma^b/N^b}(S^b), f))$  inherits a natural Frobenius structure from  $\text{CM}_{\Gamma^b/N^b}(S^b)$  under which the exact sequences of  $p$ -cycles are precisely those for which the underlying sequences of modules are exact. Under this structure the projective objects are the collection

$$\bigcup_{i \in \mathbb{Z}} \sigma^i(\iota(\mathcal{P}))$$

where  $\mathcal{P}$  is the subcategory of projective objects of  $\text{CM}_{\Gamma^b/N^b}(S^b)$ .

By Theorem 6.2.3, this exact structure is the one obtained from the equivalence

$$p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(S^b), f) = \text{CM}_{\mathcal{L}_p(\Gamma^b/N^b, \chi_f)}(\mathcal{L}_p(S^b, f))$$

where by definition (6.1.4)

$$\mathcal{L}_p(S^b, f) := S^b[z]/(f - z^p)$$

and

$$\mathcal{L}_p(\Gamma^b/N^b, \chi_f) := \{(\gamma, u) \in (\Gamma^b/N^b) \times \mathbb{C}^* \mid \chi_f(\gamma) = u^p\}.$$

So the  $p$ -cycle construction gives us another way of producing Frobenius structures on categories of bundles  $\text{Vect}(\mathbb{X}^\sharp)$ . The first thing to observe is that in case  $\mathbb{X}^\sharp$  is Fano, these are in fact nothing new.

**Proposition 7.1.1.** Let  $\Gamma^b, N^b \triangleleft G^b$ ,  $f(x, y)$  and  $p \in \mathbb{N}$  be such that the orbifold  $\mathbb{X}^\sharp$  obtained in

$$p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(S^b), f) \cong \text{Vect}(\mathbb{X}^\sharp)$$

is Fano. Then there exist  $\Gamma = G \cdot \mathbb{C}^*$  and  $N \triangleleft G$  and an exact equivalence

$$p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(S^b), f) = \text{CM}_{\Gamma/N}(S^\sharp)$$

where  $\text{Spec } S^\sharp = \mathbb{C}^2/N$ .

*Proof.* Tables 7.1-7.2 list all of the Fano orbifolds  $\mathbb{X}^\sharp$  by their weight sequences (column one). In column two we list every possible way of  $p$ -cycling up to  $\mathbb{X}^\sharp$  from a lighter orbifold  $\mathbb{X}^\flat$ . Column three gives the value of  $p$ , and column four the extended McKay group  $\Gamma^\flat = G^\flat \cdot \mathbb{C}^*$  with  $\mathbb{X}^\flat = \mathbb{C}^2/\Gamma^\flat$ . Column five gives the semiinvariant  $f$  whose associated divisor  $D_f$  cuts out the points to be cycled upon, and column six all possible normal subgroups of  $G$  that fix  $f$ . Finally column seven gives the black and white diagram describing the resulting Frobenius structure on  $\text{Vect}(\mathbb{X}^\sharp)$ . Therefore all  $\Gamma^\flat$ ,  $N^\flat \triangleleft G^\flat$ ,  $f(x, y)$  and  $p \in \mathbb{N}$  corresponding to a Fano  $\mathbb{X}^\sharp$  appear in Tables 7.1-7.2, and we observe that the Frobenius structure that arises is in all cases one we have encountered already in Chapter 4 as  $\text{CM}_{\Gamma/N}(S^\sharp)$  for some  $\Gamma$ ,  $N$  and simple singularity  $S^\sharp$ .  $\square$

**Remark 7.1.2.** In Table 7.3 we list every Frobenius structure arising in this way for  $\mathbb{X}^\sharp$  Calabi-Yau. These are not classified by black and white diagrams, as  $\text{Vect}(\mathbb{X})$  is rather more complicated in this case.

Now suppose we begin with a one-dimensional group  $\Gamma^\sharp$  and subgroup  $N^\sharp$  specifying a black and white diagram. It is natural to ask, when does this Frobenius structure arise as a category of  $p$ -cycles? More precisely, when do there exist  $p$ ,  $\Gamma^\flat$ ,  $N^\flat$  and  $f$  such that

$$\text{CM}_{\Gamma^\sharp/N^\sharp}(\mathbb{C}^2/N^\sharp) = p\text{-cyc}(\text{CM}_{\Gamma^\flat/N^\flat}(\mathbb{C}^2/N^\flat), f)?$$

The answer depends upon the following Conditions A and B.

**Condition A.** One can choose fundamental invariants  $x, y, z$  for the action of  $N^\sharp$  on  $\mathbb{C}^2$  that are related by a polynomial  $f(x, y) + z^p$  for some  $p$  in such a way that the natural representation of  $\Gamma^\sharp/N^\sharp$  on their span  $\langle x, y, z \rangle$  splits as  $\langle x, y \rangle \oplus \langle z \rangle$ .

Observe that Condition A only fails in six cases, namely the all-white and bipartite diagrams of types  $E_6$ ,  $E_7$  and  $E_8$  corresponding to the irreducible three-dimensional representations of the polyhedral and binary polyhedral groups.

In the following chapter we will be using the Frobenius structures we have constructed to derive cluster categories from matrix factorizations of curve singularities. We therefore need to know which black and white diagrams arise, not just as categories of the more general  $p$ -cycles but actually as matrix factorizations. More precisely we need to ask, given  $N^\sharp \triangleleft \Gamma^\sharp$  as before, do there exist  $\Gamma^\flat$  and  $f$  such that for  $p = 2$  (and  $N^\flat = \{1\}$ ),

$$\text{CM}_{\Gamma^\sharp/N^\sharp}(\mathbb{C}^2/N^\sharp) = 2\text{-cyc}(\text{CM}_{\Gamma^\flat}(\mathbb{C}^2), f) = \text{MF}_{\Gamma^\flat}(\mathbb{C}[x, y], f)?$$

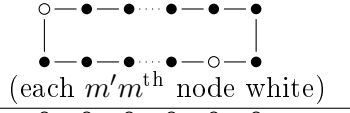
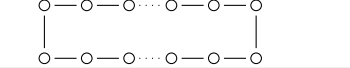
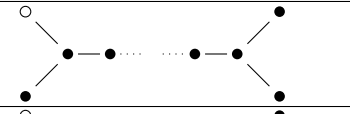
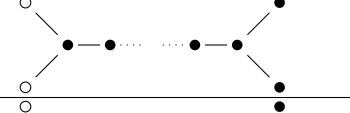
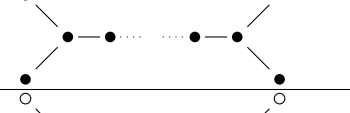
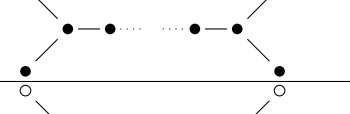
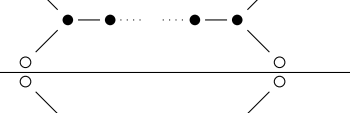
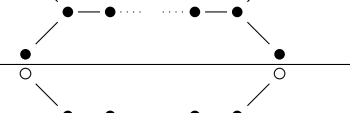
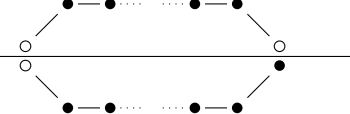

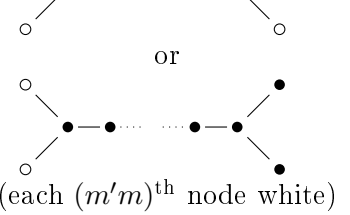
$\mathbb{X}^\sharp$	$\mathbb{X}^\flat$	$p$	$\Gamma^\flat$	$f + z^p$	$N^\flat$	bw diagram
$(p, q)$	$(\frac{p}{m}, \frac{q}{m})$ $m \mid p, q$	$m$	$\mathbb{C}^* \cdot \mu_{\frac{p+q}{m}}$	$xy + z^m$	$\mu_{m'}$ $m' \mid \frac{p+q}{m}$	
$(p, q)$	$(\frac{p}{m}, q)$	$m$	$\mathbb{C}^* \cdot \mu_{\frac{p}{m}+q}$	$x + z^m$	$\{1\}$	
$(2, 2, n)$	$(\frac{n}{2}, \mathbf{1}, \mathbf{1})$ $(n \text{ even})$	2	$\mathbb{C}^* \cdot \mu_{\frac{n}{2}+1}$	$y(x^2 + y^n) + z^2$	$\{1\}$	
	$(n, \mathbf{1}, \mathbf{1})$	2	$\mathbb{C}^* \cdot \mu_{n+1}$	$x^2 + y^{2n} + z^2$	$\{1\}$	
					$(\mu_2)$ $(n \text{ odd})$	
	$(2, \frac{n}{2}, \mathbf{1})$ $(n \text{ even})$	2	$\mathbb{C}^* \cdot \mu_{\frac{n}{2}+2}$	$y(x^2 + y^{\frac{n}{2}}) + z^2$	$\{1\}$	
	$(2, n, \mathbf{1})$	2	$\mathbb{C}^* \cdot \mu_{n+2}$	$x^2 + y^n + z^2$	$\{1\}$	
					$(\mu_2)$ $(n \text{ even})$	
	$(2, 2, \mathbf{1})$	$n$	$\mathbb{C}^* \cdot \mu_4$	$x^2 + y^2 + z^n$	$\{1\}$	
					$\mu_2$	
$(2, 2, \frac{n}{m})$ $(m \mid n)$	$m$	$\mathbb{C}^* \cdot \text{BDih}_{\frac{n}{m}}$	$xy + z^m$	$\mu_{m'}$ $(m' \mid \frac{2n}{m})$	or 	

Table 7.1: Frobenius structures from  $p$ -cycles, Fano types  $A$  and  $D$ .

$\mathbb{X}^\sharp$	$\mathbb{X}^\flat$	$p$	$\Gamma^\flat$	$f + z^p$	$N^\flat$	bw diagram	
(2,3,5)	(1,3,5)	2	$\mathbb{C}^* \cdot \mu_8$	$x^3 + y^5 + z^2$	{1}		
	(2,1,5)	3	$\mathbb{C}^* \cdot \mu_7$	$x^2 + y^5 + z^3$	{1}		
	(2,3,1)	5	$\mathbb{C}^* \cdot \mu_5$	$x^2 + y^3 + z^5$	{1}		
	-	-	-	-	-	-	
	-	-	-	-	-	-	
(2,3,4)	(1,2,3)	2	$\mathbb{C}^* \cdot \mu_5$	$x(x^2 + y^3) + z^2$	{1}		
	(1,3,4)	2	$\mathbb{C}^* \cdot \mu_7$	$x^3 + y^4 + z^2$	{1}		
	(2,3,1)	4	$\mathbb{C}^* \cdot \mu_5$	$x^2 + y^3 + z^4$	{1}		
	(2,1,4)	3	$\mathbb{C}^* \cdot \mu_6$	$x^2 + y^4 + z^3$	{1}		
					$\mu_2$		
	(2,2,3)	2	$\mathbb{C}^* \cdot \text{BDih}_3$	$x^3 + y^3 + z^2$	{1}		
	-	-	-	-	-	-	
	-	-	-	-	-	-	
(2,3,3)	(2,1,1)	3	$\mathbb{C}^* \cdot \mu_3$	$x^2 + y^4 + z^3$	{1}		
	(2,3,1)	3	$\mathbb{C}^* \cdot \mu_5$	$x^2 + y^4 + z^3$	{1}		
	(1,3,3)	2	$\mathbb{C}^* \cdot \mu_6$	$x^3 + y^3 + z^2$	{1}		
					$\mu_3$		
	-	-	-	-	-	-	
	-	-	-	-	-	-	

Table 7.2: Frobenius structures from  $p$ -cycles, Fano types  $E_6$ ,  $E_7$ ,  $E_8$ .

$\mathbb{X}^\sharp$	$\mathbb{X}^\flat$	$p$	$\Gamma^\flat$	$f + z^p$	
(2,2,2,2)	(1, 1, 1, 1)	2	$\mathbb{C}^* \cdot \mu_2$	$x^4 + y^4 + z^2$	$X_9$
	(2, 1, 1, 1)	2	$\mathbb{C}^* \cdot \mu_3$	$x^3 + y^6 + z^2$	$J_{10}$
	(2, 2, 1, 1)	2	$\mathbb{C}^* \cdot \mu_4$	$x^4 + y^4 + z^2$	$X_9$
	(2, 2, 2, 1)	2	$\mathbb{C}^* \cdot \text{BDih}_2$	$x^4 + y^4 + z^2$	$X_9$
(2,4,4)	(1, 2, 2)	2	$\mathbb{C}^* \cdot \mu_4$	$xy(x^2 + y^2) + z^2$	$X_9$
	(1, 2, 4)	2	$\mathbb{C}^* \cdot \mu_6$	$x(x^2 + y^4) + z^2$	$J_{10}$
	(1, 4, 4)	2	$\mathbb{C}^* \cdot \mu_8$	$x^4 + y^4 + z^2$	$X_9$
	(2, 2, 2)	2	$\mathbb{C}^* \cdot \text{BDih}_2$	$x^4 + y^4 + z^2$	$X_9$
	(2, 2, 4)	2	$\mathbb{C}^* \cdot \text{BDih}_4$	$x^4 + y^4 + z^2$	$X_9$
	(2, 1, 1)	4	$\mathbb{C}^* \cdot \mu_3$	$x^2 + y^4 + z^4$	$X_9$
	(2, 4, 1)	4	$\mathbb{C}^* \cdot \mu_6$	$x^2 + y^4 + z^4$	$X_9$
(3,3,3)	(1, 1, 1)	3	$\mathbb{C}^* \cdot \mu_2$	$x^3 + y^3 + z^3$	$P_8$
	(1, 1, 3)	3	$\mathbb{C}^* \cdot \mu_4$	$x^6 + y^2 + z^3$	$J_{10}$
	(3, 3, 1)	3	$\mathbb{C}^* \cdot \mu_6$	$x^3 + y^3 + z^3$	$P_8$
(2,3,6)	(1, 3, 3)	2	$\mathbb{C}^* \cdot \mu_6$	$x(x^3 + y^3) + z^2$	$X_9$
	(1, 3, 6)	2	$\mathbb{C}^* \cdot \mu_9$	$x^3 + y^6 + z^2$	$J_{10}$
	(2, 3, 3)	3	$\mathbb{C}^* \cdot \text{BTet}$	$x^4 + y^4 + z^2$	$X_9$
	(1, 2, 2)	3	$\mathbb{C}^* \cdot \mu_4$	$x(x^2 + y^2) + z^3$	$P_8$
	(2, 1, 6)	3	$\mathbb{C}^* \cdot \mu_8$	$x^2 + y^6 + z^3$	$J_{10}$
	(2, 2, 3)	3	$\mathbb{C}^* \cdot \text{BDih}_3$	$x^3 + y^3 + z^3$	$P_8$
	(2, 3, 1)	6	$\mathbb{C}^* \cdot \mu_5$	$x^2 + y^3 + z^6$	$J_{10}$

Table 7.3: Frobenius structures from  $p$ -cycles, Calabi-Yau type.

Suppose condition A is satisfied, so that we have invariants  $x, y, z$  for the action of  $N^\sharp$  on  $\mathbb{C}^2$  with  $f(x, y) + z^p = 0$ , corresponding to a splitting  $\langle x, y \rangle \oplus \langle z \rangle$  of the action of  $\Gamma^\sharp/N^\sharp$  on  $\langle x, y, z \rangle$ . Write  $\Gamma^\flat$  for the intersection of  $\Gamma^\sharp/N^\sharp$  with  $\mathrm{GL}(2, \mathbb{C})$ . There is a natural projection  $\mathcal{L}_2(\Gamma^\flat, \chi_f) \longrightarrow \Gamma^\flat$  with kernel  $\mu_p = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_p \end{pmatrix} \mid \zeta_p^p = 1 \right\}$ .

**Condition B.** The group  $\Gamma^\sharp/N^\sharp$  contains the kernel  $\mu_p = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_p \end{pmatrix} \right\}$  of the natural projection  $\mathcal{L}_2(\Gamma^\flat, \chi_f) \longrightarrow \Gamma^\flat$ .

**Proposition 7.1.3.** Let  $\Gamma^\sharp = G^\sharp \cdot \mathbb{C}^*$  and  $N^\sharp \triangleleft G^\sharp$  a normal subgroup specifying a Frobenius category  $\mathrm{CM}_{\Gamma^\sharp/N^\sharp}(S^\sharp)$ .

1. If  $\Gamma^\sharp, N^\sharp$  satisfy condition A then there exist  $p, \Gamma^\flat, N^\flat$  and  $f$  such that

$$\mathrm{CM}_{\Gamma^\sharp/N^\sharp}(\mathbb{C}^2/N^\sharp) = p\text{-cyc}(\mathrm{CM}_{\Gamma^\flat/N^\flat}(\mathbb{C}^2/N^\flat), f).$$

2. If  $\Gamma^\sharp, N^\sharp$  satisfy condition A and condition B for  $p = 2$  then there exist  $\Gamma^\flat$  and  $f$  such that

$$\mathrm{CM}_{\Gamma^\sharp/N^\sharp}(\mathbb{C}^2/N^\sharp) = 2\text{-cyc}(\mathrm{CM}_{\Gamma^\flat}(\mathbb{C}^2), f) = \mathrm{MF}_{\Gamma^\flat}(\mathbb{C}[x, y], f).$$

*Proof.* The first part of the proposition relies as in Proposition 7.1.1 on the exhaustiveness of Tables 7.1 and 7.2. There every Frobenius structure arising as  $\mathrm{CM}_{\Gamma^\sharp/N^\sharp}(\mathbb{C}^2/N^\sharp)$  is listed, in particular all those that satisfy Condition A. The desired  $p, \Gamma^\flat, N^\flat$  and  $f$  may then be read off.

Given now  $\mathrm{CM}_{\Gamma^\sharp/N^\sharp}(S^\sharp)$  satisfying conditions A and B for  $p = 2$ , we have  $S^\sharp = \mathbb{C}[x, y, z]/(f(x, y) + z^2)$ , so putting  $S^\flat := \mathbb{C}[x, y]$  gives

$$\mathcal{L}_2(S^\flat, f) = S^\flat[z]/(f - z^2) = S^\sharp.$$

On the other hand, with  $\Gamma^\flat := \Gamma^\sharp/N^\sharp \cap \mathrm{GL}(2, \mathbb{C})$ , we have a diagram

$$\begin{array}{ccccc} \ker & \longrightarrow & \Gamma^\sharp/N^\sharp & \longrightarrow & \Gamma^\flat \\ \downarrow \phi & & \downarrow \psi & & \parallel \\ \mu_2 & \longrightarrow & \mathcal{L}_2(\Gamma^\flat, \chi_f) & \longrightarrow & \Gamma^\flat. \end{array}$$

By Condition B,  $\phi$  and hence  $\psi$  are isomorphisms. Thus

$$\mathrm{MF}_{\Gamma^\flat}(\mathbb{C}[x, y], f) = 2\text{-cyc}(\mathrm{CM}_{\Gamma^\flat}(\mathbb{C}^2), f) = \mathrm{CM}_{\mathcal{L}_2(\Gamma^\flat, \chi_f)}(\mathcal{L}_2(S^\flat, f)) = \mathrm{CM}_{\Gamma^\sharp/N^\sharp}(S^\sharp)$$

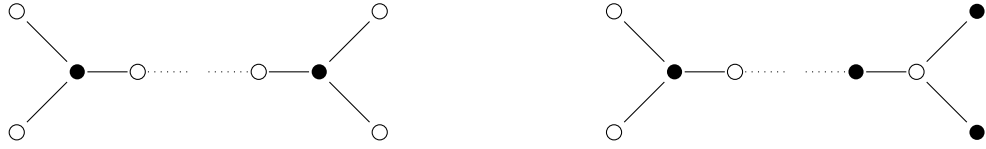
by Theorem 6.2.3. □

Of the pairs  $\Gamma^\sharp, N^\sharp$  satisfying condition A, there are four cases that fail condition B,

- $\text{BTet} \triangleleft \text{BTet}$ ,
- $\text{BDih}_n \triangleleft \text{BDih}_n$  for odd  $n$ ,
- $\mu_m \triangleleft \text{BDih}_n$  for  $m \nmid n$ , and
- $\mu_m \triangleleft \mu_{p+q}$  for  $m \mid p+q$  but  $m \nmid \gcd(p, q)$ .

In the the three examples that follow we illustrate how Condition B fails in each case.

**Example 7.1.4** ( $\mu_2 \triangleleft \text{BDih}_n$ ). The bipartite black and white diagrams



of type  $\tilde{D}_n$  arise with  $N^\sharp$  the central  $\mu_2 \triangleleft \text{BDih}_n$ . The quotient  $\Gamma^\sharp/N^\sharp$  is the dihedral group  $\text{Dih}_n$ , acting naturally as a subgroup of  $\text{SO}(3, \mathbb{C})$  on the space of invariants  $\langle x, y, z \rangle$  of  $\mu_2$ . This representation decomposes as  $\langle x, y \rangle \oplus \langle z \rangle$ , corresponding to the relation  $xy + z^2 = 0$ .

Since  $\Gamma^\flat = \text{Dih}_n \cdot \mathbb{C}^*$  contains a diagonal  $\mathbb{C}^*$  and in particular the elements  $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$  and  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ , we find that when  $n$  is even,  $\Gamma^\flat \cong \text{BDih}_{\frac{n}{2}} \cdot \mathbb{C}^*$  and when  $n$  is odd,  $\Gamma^\flat \cong \text{BDih}_n \cdot \mathbb{C}^*$ . In the first case the projection

$$\Gamma^\sharp/N^\sharp \longrightarrow \Gamma^\flat$$

has kernel  $\{\pm 1\}$  and condition B is satisfied. In the second case the kernel is trivial, and condition B is not satisfied.

This applies more generally. If  $N^\sharp = \mu_m$  with  $m \mid 2n$  then we have a diagram of groups

$$\begin{array}{ccccc} \mu_\alpha & \longrightarrow & \Gamma^\sharp/N^\sharp & \longrightarrow & \Gamma^\flat = \text{BDih}_\beta \\ & & \downarrow & & \parallel \\ \mu_m & \longrightarrow & \mathcal{L}_m(\Gamma^\flat) & \longrightarrow & \Gamma^\flat. \end{array}$$

When  $m \mid n$ ,  $\beta = n/m$  and  $\alpha = m$  so condition B is satisfied. When  $m \nmid n$ ,  $\beta = 2n/m$ ,  $\alpha = m/2$  and condition B is not satisfied.

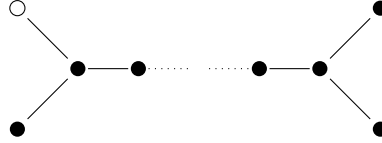
According to Proposition 7.1.3 part 1, the Frobenius structures





can be recovered as categories of  $p$ -cycles by taking  $\frac{m}{m'}$ -cycles on  $\text{CM}_{\Gamma^b/\mu_{m'}}(\mathbb{C}^2/\mu_{m'})$ , that is,  $\frac{m}{m'}$ -cycles of equivariant Cohen-Macaulay modules over an  $A_{m'-1}$  singularity. This amounts to a change of coordinates  $\bar{z} := z^{m'}$  in the ring  $\mathcal{L}_m(S^b)$ .

**Example 7.1.5** ( $\text{BDih}_n \triangleleft \text{BDih}_n$ ). Consider the Frobenius structure



with a single orbit of projectives, arising from  $N^\sharp = \text{BDih}_n \triangleleft \mathbb{C}^* \text{BDih}_n$ . The invariants  $x, y, z$  of  $N^\sharp$  have degrees  $n, 2$  and  $n+1$ , and related by  $x^2y + y^{n+1} + z^2$ . So  $\Gamma^\sharp/N^\sharp = \mathbb{C}^*(n, 2, n+1)$ , and aiming to take 2-cycles in  $f(x, y) = x^2y + y^{n+1}$ , we put  $\Gamma^b = \mathbb{C}^*(n, 2)$ .

When  $n$  is even,  $\Gamma^b \cong \mathbb{C}^*(\frac{n}{2}, 1)$ , and the projection  $\Gamma^\sharp/N^\sharp \rightarrow \Gamma^b$  has kernel  $\{\pm 1\}$ . Condition B is satisfied in this case and  $\text{CM}_{\Gamma^\sharp/N^\sharp}(\mathbb{C}^2/N^\sharp) = \text{MF}_{\mathbb{C}^*(\frac{n}{2}, 1)}(\mathbb{C}[x, y], y(x^2 + y^n))$ .

When  $n$  is odd,  $\Gamma^b = \mathbb{C}^*(n, 2)$ . The projection  $\Gamma^\sharp/N^\sharp \rightarrow \Gamma^b$  has trivial kernel in this case, and condition B fails. The construction  $\mathcal{L}_2(\Gamma^b)$  now picks up an extra  $\mu_2$ , corresponding to the extension

$$\text{BDih}_n \longrightarrow \text{BDih}_{2n} \longrightarrow \mu_2$$

and we end up producing an exact structure not on  $\text{Vect}[\mathbb{C}^2 - 0/\mathbb{C}^* \cdot \text{BDih}_n]$ , but on  $\text{Vect}[\mathbb{C}^2 - 0/\mathbb{C}^* \cdot \text{BDih}_{2n}]$ .

As in the previous example, this situation can be remedied by a change of variables. Put  $\bar{z} := x^2 + y^n$ , of degree  $2n$ . Then  $\Gamma^\sharp/N^\sharp$  acts on  $S^b = \mathbb{C}[y, z, \bar{z}]/y\bar{z} + z^2$  as  $\bar{\Gamma}^b = \mathbb{C}^*(2, n+1, 2n)$ . We now have a projection

$$\Gamma^\sharp/N^\sharp \longrightarrow \bar{\Gamma}^b$$

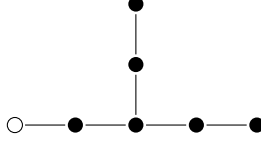
with kernel  $\{\pm 1\}$ . Observe that  $S^b$  is the ring of invariants for the action of a  $\mu_2$  on  $\mathbb{C}^2$ . Thus rewriting  $S^b$  as  $\mathbb{C}[\xi^2, \zeta^2, \xi\zeta]$  and taking 2-cycles in the defining equation of  $\bar{z} = \zeta^2$ , we have  $\mathcal{L}_2(S^b, \zeta^2 + \xi^{2n}) \cong \mathbb{C}^2/N^\sharp$  and  $\mathcal{L}_2(\bar{\Gamma}^b) \cong \Gamma^\sharp/N^\sharp$  as required.

This is the construction of the single white node black and white diagram as

$$2\text{-cyc}(\text{CM}_{\mathbb{C}^*(1, n) \cdot \mu_{n+1}/\mu_2}(\mathbb{C}^2/\mu_2), x^2 + y^{2n})$$

of table 7.1.

**Example 7.1.6** (BTet  $\triangleleft$  BTet). The subgroup  $N^\sharp = \text{BTet} \triangleleft \mathbb{C}^* \cdot \text{BTet}$  gives rise to the black and white diagram



with a single white node. The representation of  $\Gamma^\sharp/N^\sharp = \mathbb{C}^*(4, 3, 6)$  on the fundamental invariants  $x_1, x_2, x_3$  of  $N^\sharp$  splits as a direct sum of linear representations, and so there are three possibilities for  $\Gamma^\flat$ ;  $\Gamma_{12}^\flat = \mathbb{C}^*(4, 3)$ ,  $\Gamma_{23}^\flat = \mathbb{C}^*(3, 6) = \mathbb{C}^*(1, 2)$  and  $\Gamma_{13}^\flat = \mathbb{C}^*(4, 6) = \mathbb{C}^*(2, 3)$  corresponding to the three ways of parsing the relation  $x_1^3 + x_2^4 + x_3^2$  as  $f(x_i, x_j) + x_k^{pk}$ . Both  $\Gamma_{12}^\flat = \mathbb{C}^*(4, 3)$  and  $\Gamma_{13}^\flat = \mathbb{C}^*(2, 3)$  fail condition B, picking up an extra  $\mu_2$  in the  $\mathcal{L}_p$  construction corresponding to the extension

$$\text{BTet} \longrightarrow \text{BOct} \longrightarrow \mu_2.$$

In both cases we end up constructing the same Frobenius structure on  $\mathbb{X}_{2,3,4}$  rather than  $\mathbb{X}_{2,3,3}$  (see table 7.2).

Since  $\Gamma_{23}^\flat = \mathbb{C}^*(1, 2)$  satisfies condition A, we can construct the diagram we're aiming for by taking 3-cycles in  $x_2^4 + x_3^2$ . Alternatively, as before, a change of variables allows us to construct it by taking 2-cycles of Cohen-Macaulay modules over an  $A_2$  singularity in  $x^3 + y^3$ .

In Appendix A, Tables A.1-A.4 we reorganize the information contained in Tables 7.1-7.3, putting the singularity  $f + z^p$  first. The point is that given the following data:

- a curve  $f \in \mathbb{C}[x, y]$
- a one-dimensional group  $\Gamma^\flat$  with finite determinant-one part  $G^\flat$
- a normal subgroup  $N^\flat \triangleleft G^\flat$
- such that  $f$  is a semi-invariant for the natural action of  $\Gamma^\flat$  on  $\mathbb{C}^2$  which is fixed by  $N^\flat$ , and
- a number  $p \in \mathbb{N}$

there is an associated Frobenius category

$$p\text{-cyc}(\text{CM}_{\Gamma^\flat/N^\flat}(\mathbb{C}^2/N^\flat), f) \cong \text{Vect}(\mathbb{X})$$

for some orbifold  $\mathbb{X}$ . Tables A.1-A.4 classify all cases where  $\mathbb{X}$  is either Fano or Calabi-Yau. In particular when  $p = 2$  and  $N^b \triangleleft \Gamma^b$  is trivial,  $p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(\mathbb{C}^2/N^b), f)$  is the category of  $\Gamma^b$ -equivariant matrix factorizations of the curve singularity  $f$ ,

$$\text{MF}_{\Gamma^b}(\mathbb{C}[x, y], f).$$

So our Tables A.1-A.4 classify all Frobenius categories of equivariant matrix factorizations of finite or tame (tubular) type arising by the  $p$ -cycle construction. This includes various  $\mathbb{Z}$ -graded cases (including for example those studied by Araya [2]), as well as matrix factorizations graded by rank-one groups with torsion, and certain noncommutative examples.

We end this section with a final observation.

**Proposition 7.1.7.** The category

$$p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(\mathbb{C}^2/N^b), f)$$

has finite representation type if and only if the surface  $f + z^p$  is a simple singularity. It has tame (tubular) representation type if and only if  $f + z^p$  is one of the three unimodular singularities  $x^3 + y^3 + z^3$  ( $P_8$ ),  $x^4 + y^4 + z^2$  ( $X_9$ ),  $x^3 + y^6 + z^2$  ( $J_{10}$ ).

*Proof.* Tables 7.1-7.2 list all categories  $p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(\mathbb{C}^2/N^b), f)$  of finite representation type (that is, additively equivalent to vector bundles on a Fano orbifold). In all cases  $f + z^p$  is a simple singularity. Table 7.3 lists all of the tame tubular cases, and  $f + z^p$  is here always one of the three unimodular singularities  $x^3 + y^3 + z^3$  ( $P_8$ ),  $x^4 + y^4 + z^2$  ( $X_9$ ),  $x^3 + y^6 + z^2$  ( $J_{10}$ ).

Conversely Tables A.1-A.3 list all ways to take  $p$ -cycles on a simple singularity. The resulting category is always of finite type. Table A.4 lists all ways to take  $p$ -cycles on one of the three unimodular singularities  $x^3 + y^3 + z^3$  ( $P_8$ ),  $x^4 + y^4 + z^2$  ( $X_9$ ),  $x^3 + y^6 + z^2$  ( $J_{10}$ ). The resulting category is always tame tubular.

This result is hardly surprising: the difference between finite, tame and wild is the difference between having finitely many (isomorphism classes of) indecomposables, containing one-parameter families, and containing *everything*. The extra equivariant structure imparted by a one-dimensional group can only replace single things by countable collections and is not going to interfere with the trichotomy.  $\square$

In particular fix  $p = 2$  and  $N^b$  trivial so that  $p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(\mathbb{C}^2/N^b), f)$  is the category of  $\Gamma^b$ -equivariant matrix factorizations of the curve singularity  $f$ .

**Corollary 7.1.8.** The category of  $\Gamma^b$ -equivariant matrix factorizations

$$\mathrm{MF}_{\Gamma^b}(\mathbb{C}[x, y], f)$$

has finite representation type if and only if  $f$  is a simple singularity, tame if and only if  $f$  is unimodular of type  $X_9$  or  $J_{10}$  and wild otherwise.

## Chapter 8

# Construction of cluster categories

### 8.1 Cluster categories

We now apply the machinery and results of the previous two chapters to the special case of equivariant matrix factorizations of a curve singularity. The set-up is as follows. Let  $\Gamma^b$  be a one-dimensional group with finite determinant-one part acting on  $R^b = \mathbb{C}[x, y]$  and let  $f \in R$  be a semi-invariant for this action. Then

1. Let  $\mathbb{X}^b = [\mathbb{C}^2 - 0/\Gamma^b]$ , and  $\mathbb{X}^\sharp$  be the orbifold obtained from  $\mathbb{X}^b$  by multiplying the orbifold weights of the points of the divisor  $D_f$  by two. Then  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  is additively equivalent to  $2\text{-cyc}(\mathrm{Vect}(\mathbb{X}^b), f)$  and to  $\mathrm{Vect}(\mathbb{X}^\sharp)$  (Proposition 6.2.5).
2. The representation type of  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  corresponds to the type of singularity  $f$ : it is finite ( $\mathbb{X}^\sharp$  is Fano) when  $f$  is simple, tame ( $\mathbb{X}^\sharp$  is Calabi-Yau) when  $f$  is unimodular of type  $X_9$  or  $J_{10}$  and wild ( $\mathbb{X}^\sharp$  is of general type) otherwise (Corollary 7.1.8).
3.  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  carries a natural Frobenius structure under which the projective-injective objects are the matrix factorizations of the form  $M \xrightarrow{1} M \xrightarrow{f} M \otimes \chi_f$  and  $M \xrightarrow{f} M \xrightarrow{1} M \otimes \chi_f$  for  $M \in \mathrm{CM}_{\Gamma^b}(R)$  (Theorem 6.2.1). When  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  is Fano, this structure is described by a black and white diagram  $\Delta$  (see Chapter 4).
4. The Frobenius category  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  is exactly equivalent to  $\mathrm{CM}_{\mathcal{L}(\Gamma^b)}(\mathcal{L}(R^b))$ , the category of  $\mathcal{L}(\Gamma^b)$ -equivariant Cohen-Macaulay modules on  $\mathcal{L}(R^b)$  (Theorem 6.2.3, see also Definition 6.1.4)
5. The subcollection  $\mathcal{F}$  of the projective objects of  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  consisting of matrix factorizations of the form  $M \xrightarrow{f} M \xrightarrow{1} M$  ( $M \in \mathrm{CM}_{\Gamma^b}(R)$ ) is a Chen

subcategory ([14] Example 3.4). So the partial stabilization of  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  with respect to  $\mathcal{F}$  is Frobenius. Indeed it is equivalent to the category  $\mathrm{CM}_{\Gamma^b}(R^b/f)$  of  $\Gamma^b$ -equivariant Cohen-Macaulay modules over  $R^b/f$ , and taking cokernels yields an exact functor  $\mathrm{MF}_{\Gamma^b}(R^b, f) \rightarrow \mathrm{CM}_{\Gamma^b}(R^b/f)$ .

6. The full stabilization  $\underline{\mathrm{CM}}_{\Gamma^b}(R^b/f)$  is a triangulated category, which is, in the Fano case, equivalent to the bounded derived category  $D^b(\Delta^b)$  of the black part of the black and white diagram  $\Delta$  associated to  $\mathrm{MF}_{\Gamma^b}(R^b, f)$  (Theorem 5.1.1).

So we have the following diagram of categories

$$\begin{array}{c}
2\text{-cyc}(\mathrm{Vect}(\mathbb{X}^b), f) \cong \mathrm{MF}_{\Gamma^b}(R^b, f) = \mathrm{CM}_{\mathcal{L}(\Gamma^b)}(\mathcal{L}(R^b)) \cong \mathrm{Vect}(\mathbb{X}^\sharp) \\
\downarrow \\
\mathrm{CM}_{\Gamma^b}(R^b/f) \\
\downarrow \\
\underline{\mathrm{CM}}_{\Gamma^b}(R^b/f) = \underline{\mathrm{CM}}_{\mathcal{L}(\Gamma^b)}(\mathcal{L}(R^b))
\end{array}$$

where, as always, we use “=” to indicate an exact equivalence of exact categories, or a triangle equivalence of triangulated categories, and “ $\cong$ ” to indicate an equivalence which is only additive. The vertical arrows are stabilization with respect to the relevant Frobenius structure and Chen subcategory and the categories at the bottom are triangulated. As such, this category  $\underline{\mathrm{CM}}_{\Gamma^b}(R^b/f) = \underline{\mathrm{CM}}_{\mathcal{L}(\Gamma^b)}(\mathcal{L}(R^b))$  carries a triangle shift, denoted, as is customary by [1], and an Auslander-Reiten translate  $\tau$ .

The triangle shift [1] on  $\underline{\mathrm{MF}}_{\Gamma^b}(R^b, f)$  is obtained by applying the cycle automorphism  $\sigma$  coming from its structure as a category of 2-cycles, and then multiplying the factorization morphisms by -1. That is,

$$[1] : (M_0 \xrightarrow{m_0} M_1 \xrightarrow{m_1} M_0 \otimes \chi_f) \mapsto (M_1 \xrightarrow{-m_1} M_0 \otimes \chi_f \xrightarrow{-m_0} M_1 \otimes \chi_f),$$

where  $\chi_f$  is the character of  $\Gamma^b$  corresponding to  $f$ . The following lemma is obvious but important.

**Lemma 8.1.1.** On  $\mathrm{MF}_{\Gamma^b}(R^b, f)$ , twice the triangle shift is equivalent to tensoring everything by  $\chi_f$ .

The analysis of the AR translate is a little more delicate and depends essentially on the Buchweitz Duality Theorem, which we quote again for reference.

**Theorem 5.1.2** (Buchweitz Duality Theorem). Let  $S$  be an equi-codimensional Gorenstein isolated singularity of dimension  $d$  with an action of an extended McKay group  $\Gamma =$

$\mathbb{C}^* \cdot G$ . So  $S$  carries a  $\Gamma$ -equivariant module  $K_S$  which is canonical. Then we have a bifunctorial isomorphism

$$\mathrm{Ext}_{\underline{\mathrm{CM}}_\Gamma(S)}^{i+d-1}(M, N) \cong \mathrm{Ext}_{\underline{\mathrm{CM}}_\Gamma(S)}^{-i}(N, M \otimes_S K_S)^\vee$$

for all  $M, N \in \underline{\mathrm{CM}}_\Gamma(S)$  and  $i \in \mathbb{Z}$ , where  $^\vee$  denotes the  $\mathbb{C}^*$ -dual.

The Gorenstein rings we are interested in are  $R^b = \mathbb{C}[x, y]$ ,  $R^b/f$  and  $\mathcal{L}(R^b) = \mathbb{C}[x, y, z]/(f + z^2)$ .

**Corollary 8.1.2.** By Corollary 6.2.4, the category  $\underline{MF}_{\Gamma^b}(R^b, f)$  may be interpreted equivalently as  $\underline{\mathrm{CM}}_{\mathcal{L}(\Gamma^b)}(\mathcal{L}(R^b))$  or as  $\underline{\mathrm{CM}}_{\Gamma^b}(R^b/f)$ . In the first case, the Auslander-Reiten translate is given by

$$\tau_{\mathcal{L}(R^b)} = - \otimes_{\mathcal{L}(R^b)} K_{\mathcal{L}(R^b)}$$

and in the second by

$$\tau_{R^b/f} = - \otimes_{R^b/f} K_{R^b/f}[-1].$$

*Proof.* Immediate from the theorem. It is worth remarking that under the equivalence  $\underline{\mathrm{CM}}_{\mathcal{L}(\Gamma^b)}(\mathcal{L}(R^b)) \cong \mathrm{Vect}(\mathbb{X}^\sharp)$ ,  $K_{\mathcal{L}(R^b)}$  corresponds to the canonical bundle  $\omega_{\mathbb{X}^\sharp}$ , and  $- \otimes \omega_{\mathbb{X}^\sharp}$  is indeed the AR translate on  $\mathrm{Vect}(\mathbb{X}^\sharp)$ .  $\square$

Let us quote here Proposition 2.3.6 describing the canonical module in these cases.

**Proposition 2.3.6** (Equivariant canonical module). Let  $R = \mathbb{C}[x_1, \dots, x_n]$  be a polynomial ring acted on by an extended McKay group  $\Gamma$ ,  $f$  a semiinvariant and  $S = R/f$ . Then the equivariant canonical modules look like

$$K_R \cong R \otimes \omega_\Gamma^{-1}$$

and

$$K_S \cong S \otimes \omega_\Gamma^{-1} \otimes \chi_f$$

where  $\omega_\Gamma$  is the determinant representation and  $\chi_f$  the character of  $\Gamma$  associated to  $f$ .

So in particular

$$K_{R^b/f} = (R^b/f) \otimes \chi_f \otimes \omega_{\Gamma^b}^{-1}.$$

and

$$K_{\mathcal{L}(R^b)} = \mathcal{L}(R^b) \otimes \chi_{f+z^2} \otimes \omega_{\mathcal{L}(\Gamma^b)}^{-1}.$$

**Theorem 8.1.3.** The functor

$$- \otimes \omega_{\Gamma^b}$$

on  $\underline{MF}_{\Gamma^b}(R^b, f)$  is isomorphic to the cluster automorphism,  $\tau^{-1}[1]$ .

*Proof.* By Corollary 8.1.2 and Lemma 8.1.1

$$\tau = - \otimes \omega_{\Gamma^b}^{-1}[1]$$

and so  $- \otimes \omega_{\Gamma^b} = \tau^{-1}[1]$ . □

It follows that taking the orbit category of the triangulated category

$$\underline{MF}_{\Gamma^b}(R^b, f) = \underline{CM}_{\Gamma^b}(R^b/f)$$

with respect to this automorphism  $- \otimes \omega_{\Gamma^b}$  yields the associated cluster category. We claim that in the Fano case, this category is equivalent to the stable category of Cohen-Macaulay modules over the complete local ring  $\hat{R}^b$  equivariant for the action of the (finite) kernel of the determinant map  $\Gamma^b \rightarrow \mathbb{C}^*$ .

$$\begin{array}{ccc} \underline{MF}_{\Gamma^b}(R^b, f) & \longrightarrow & \underline{MF}_{G^b}(\hat{R}^b, f) \\ \downarrow & & \downarrow \\ \underline{CM}_{\Gamma^b}(R^b/f) & \longrightarrow & \underline{CM}_{G^b}(\hat{R}^b/f) \\ \downarrow & & \downarrow \\ \underline{CM}_{\Gamma^b}(R^b/f) & \longrightarrow & \underline{CM}_{G^b}(\hat{R}^b/f) \end{array}$$

As this diagram of categories implies, at each stage of stabilization we have a covering functor going from left to right, given by completion. More precisely, any  $\Gamma^b$ -equivariant object  $M$  has an isotypic decomposition by characters of the finite determinant-one part  $G^b$

$$M = \bigoplus_{\chi \in G^b \vee} M_\chi.$$

Each of these components  $M_\chi$  is then graded by the dual of the quotient  $\mathbb{C}^*$ ,

$$M_\chi = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda.$$

The completion functor replaces this direct sum by a direct product, yielding an  $G^b$ -



equivariant object

$$\bigoplus_{\lambda \in \mathbb{Z}} M_\lambda \longmapsto \prod_{\lambda \in \mathbb{Z}} M_\lambda.$$

It is important to note that this functor is not in general dense: its image is the subcategory of gradable objects. In the situation of a simple singularity with a  $\mathbb{C}^*$  action, everything is gradable.

**Theorem 8.1.4** (Auslander-Reiten [5] Theorem 5). If  $\hat{R}^b/f$  is of finite Cohen-Macaulay type then every module in  $\text{CM}(\hat{R}^b/f)$  is gradable.

We conjecture that the same remains true in our slightly more general Fano case, that is that if  $\hat{R}^b/f$  is of finite Cohen-Macaulay type then every module in  $\text{CM}_{G^b}(\hat{R}^b/f)$  is gradable.

## 8.2 Categorification of the Grassmannian

Let  $G_{k,n}$  denote the Grassmannian of  $k$ -planes in  $\mathbb{C}^n$ . By Fomin-Zelevinsky [17] and Scott [43] the homogeneous coordinate ring  $\mathbb{C}[G_{k,n}]$  has the structure of a cluster algebra, in which the Plücker coordinates are (some of) the cluster variables and the short Plücker are (some of) the exchange relations. The Grassmannian cluster algebras  $\mathbb{C}[G_{k,n}]$  of finite cluster type are precisely those for which  $k = 2$  or  $k = n - 2$  (type  $Q = A_{n-3}$ ); and  $k = 3$  or  $k = n - 3$  and  $n = 6, 7$  or  $8$  ( $Q$  is  $D_4, E_6, E_8$  respectively).

Let  $f = x^k - y^{n-k}$ ,  $R := \mathbb{C}[[x, y]]/f$ . This curve singularity has finite Cohen-Macaulay representation type  $Q$  precisely when  $k = 2$  or  $k = n - 2$  (type  $A_{n-3}$ ); and  $k = 3$  or  $k = n - 3$  and  $n = 6, 7$  or  $8$  (types  $D_4, E_6, E_8$ ). This is one of the observations that has led Jensen-King-Su [27] to propose the category

$$\text{CM}_G(R)$$

as an additive categorification of  $\mathbb{C}[G_{k,n}]$ , where  $G \subset \text{SL}(2, \mathbb{C})$  is the cyclic group of order  $n$ , acting on  $\mathbb{C}^2$  by  $(x, y) \mapsto (\zeta x, \zeta^{-1}y)$ .

**Theorem 8.2.1.** The stable category  $\underline{\text{CM}}_G(R)$  is equivalent to the cluster category of type  $Q$ , the Dynkin type of the singularity  $f$ .

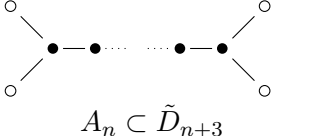
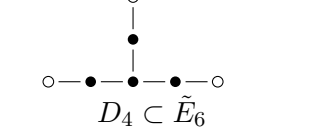
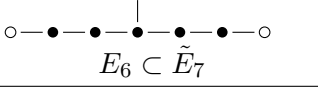
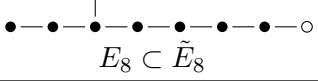
*Proof.* First take the product inside  $\text{GL}(2, \mathbb{C})$  of each of the finite groups  $G$  with a (weighted)  $\mathbb{C}^*$ , to produce the one-dimensional group  $\Gamma^b$  whose determinant-one part is  $G$ . Now consider the category  $\text{MF}_{\Gamma^b}(\mathbb{C}[x, y], f)$ . By Theorem 6.2.3 and Proposition 6.2.5,

$$\text{MF}_{\Gamma^b}(\mathbb{C}[x, y], f) = \text{CM}_{\mathcal{L}(\Gamma^b)}(\mathcal{L}(\mathbb{C}[x, y], f)) \cong \text{Vect}(\mathbb{X}^\sharp)$$

for some orbifold  $\mathbb{X}^\sharp$ . Furthermore, by Corollary 7.1.8,  $\mathbb{X}^\sharp$  is Fano. So (Proposition 7.1.1) we can find an extended McKay group  $\Gamma = G \cdot \mathbb{C}^*$  and normal subgroup  $N \triangleleft G$  such that

$$\mathrm{MF}_{\Gamma^\flat}(\mathbb{C}[x, y], f) = \mathrm{CM}_{\Gamma/N}(S^\sharp)$$

where  $\mathrm{Spec} S^\sharp = \mathbb{C}^2/N$  and  $\mathbb{X}^\sharp = \left[ \frac{\mathbb{C}^2/N - 0}{\Gamma/N} \right]$ . This gives us a black and white diagram from  $\mathrm{MF}_{\Gamma^\flat}(\mathbb{C}[x, y], f) = \mathrm{CM}_{\Gamma/N}(S^\sharp)$ . We reproduce the relevant parts of Tables A.1-A.3 here.

$f$	$\Gamma^\flat$	bw diagram for $\mathrm{MF}_{\Gamma^\flat}(\mathbb{C}[x, y], f)$
$A_n \quad x^2 + y^{n+1}$	$\mathbb{C}^*(2, n+1) \cdot \mu_{3+n}$	 $A_n \subset \tilde{D}_{n+3}$
$D_4 \quad x^3 + y^3$	$\mathbb{C}^*(1, 1) \cdot \mu_6$	 $D_4 \subset \tilde{E}_6$
$E_6 \quad x^3 + y^4$	$\mathbb{C}^*(3, 4) \cdot \mu_7$	 $E_6 \subset \tilde{E}_7$
$E_8 \quad x^3 + y^5$	$\mathbb{C}^*(3, 5) \cdot \mu_8$	 $E_8 \subset \tilde{E}_8$

Each of these Frobenius categories has a particular Chen subcategory, namely the matrix factorizations of the form  $(1, f)$  ([14] Example 3.4). Stabilization with respect to this Chen subcategory yields a new Frobenius category, a “frieze category” in which one half of each  $\tau$ -orbit of projectives has been deleted. Some examples are pictured in the figures below.

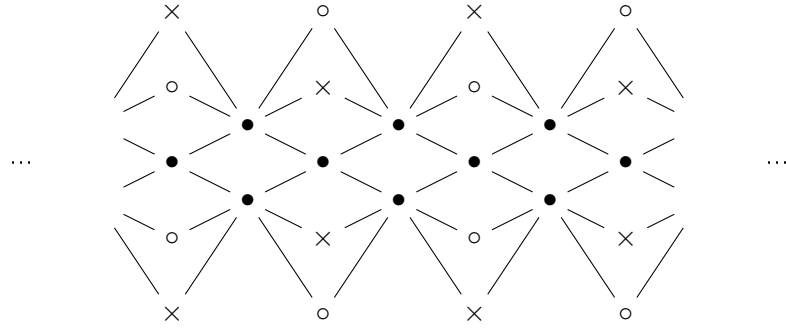


Figure 8-1: Frieze category for  $\text{MF}_{\Gamma^b}(\mathbb{C}[x, y], f)$ ,  $f = x^2 + y^4$ ,  $\Gamma^b = \mathbb{C}^*(2, 4) \cdot \mu_6$  (type  $A_3$ ). Projectives are white, Chen subcategory is crossed.

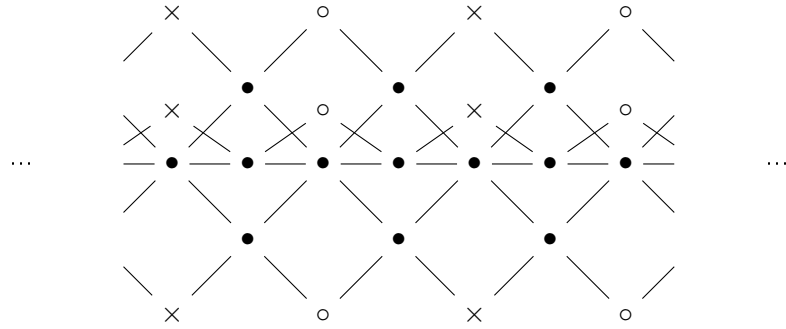


Figure 8-2: Frieze category for  $\text{MF}_{\Gamma^b}(\mathbb{C}[x, y], f)$ ,  $f = x^3 + y^3$ ,  $\Gamma^b = \mathbb{C}^*(1, 1) \cdot \mu_6$  (type  $D_4$ ).

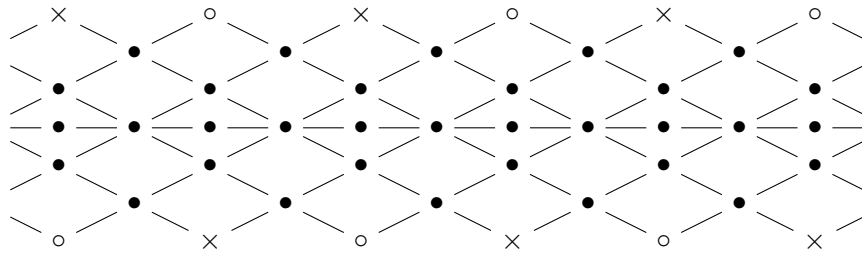


Figure 8-3: Frieze category for  $\text{MF}_{\Gamma^b}(\mathbb{C}[x, y], f)$ ,  $f = x^3 + y^4$ ,  $\Gamma^b = \mathbb{C}^*(3, 5) \cdot \mu_7$  (type  $E_6$ ).

By Corollary 6.2.4 we have an equivalence of triangulated categories

$$\underline{MF}_{\Gamma^b}(\mathbb{C}[x, y], f) = \underline{CM}_{\Gamma^b}(\mathbb{C}[x, y]/f) = \underline{CM}_{\Gamma/N}(S^\#)$$

and by Theorem 5.1.1 a derived equivalence

$$D^b(\Delta^b) = \underline{\text{CM}}_{\Gamma/N}(S^\sharp)$$

where  $\Delta^b$  is the black part of the black and white diagram associated to  $\Gamma$ ,  $N$ . Observe that in all cases under consideration,  $\Delta^b \cong Q$ , the Dynkin diagram associated to  $f$  (see Section 5.2 for a discussion of this, in particular Proposition 5.2.4).

Let  $G$  denote the finite determinant-one part of  $\Gamma^b$ . Putting everything together along with Theorem 8.1.3 we have that  $\underline{\text{CM}}_G(R)$  is equivalent to the cluster category of type  $Q$ , as claimed.

□

# Appendix A

## Tables

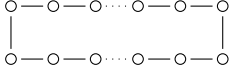
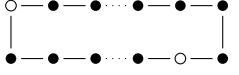

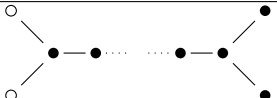
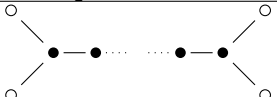

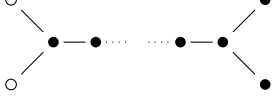


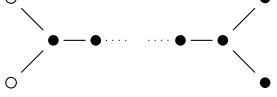
	$f + z^p$	$p$	$\Gamma^b$	$N^b$	bw diagram for $p$ -cyc( $\text{CM}_{\Gamma^b/N^b}(S^b)$ )
$A_0$	$x + z^n$	$n$	$\mathbb{C}^*(a, b) \cdot \mu_{r(a+b)}$	$\{1\}$	 $A_0 \subset \tilde{A}_{r(a+b)-1}$
$A_n$	$xy + z^{n+1}$	$n + 1$	$\mathbb{C}^*(a, b) \cdot \mu_{r(a+b)}$	$\mu_{n'}$ $n' \mid r(a+b)$	 $A_{n'n} \subset \tilde{A}_{r(n+1)(a+b)-1}$ $\frac{r(a+b)}{n'}$ times
			$\mathbb{C}^*(1, 1) \cdot \text{BDih}_r$	$\mu_{n'}$ $n' \mid 2r,$ $n' \mid r$	 $A_{n'n} \subset \tilde{D}_{r(n+1)+2}$ $\frac{r}{n'}$ times
				$n' \nmid r$	 $A_{n'n}(\frac{r}{n'} - \frac{1}{2})$ times $+D_{\frac{n'(n+1)}{2}+1} \subset \tilde{D}_{r(n+1)+2}$
$A_n$	$x^2 + y^{n+1} + z^2$	$2$	$\mathbb{C}^*(2, n+1) \cdot \mu_{3+n}$	$\{1\}$	 $A_n \subset \tilde{D}_{n+3}$
				$\mu_2$ $n$ odd	 $A_{n+2} \subset \tilde{D}_{n+3}$
			$\mathbb{C}^*(1, \frac{n+1}{2}) \cdot \mu_{1+\frac{n+1}{2}}$ $n$ odd	$\{1\}$	 $D_{\frac{n+3}{2}} \subset \tilde{D}_{\frac{n+5}{2}}$
				$\mu_2$ $\frac{n+1}{2}$ odd	 $D_{\frac{n+1}{2}} \subset \tilde{D}_{\frac{n+5}{2}}$
	$x^2 + y^2 + z^{n+1}$	$n + 1$	$\mathbb{C}^*(1, 1) \cdot \mu_4$	$\{1\}$	 $A_n \subset \tilde{D}_{n+3}$
				$\mu_2$	 $D_{\frac{n+3}{2}} \subset \tilde{D}_{\frac{n+5}{2}}$

Table A.1: The  $p$ -cycles on curve singularities of type  $A$  giving rise to Fano orbifolds.

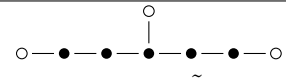
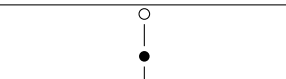
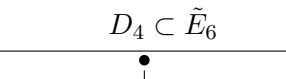
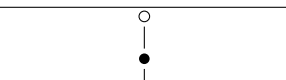
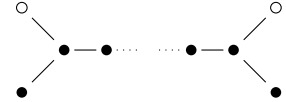
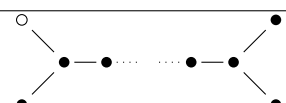
	$f + z^p$	$p$	$\Gamma^b$	$N^b$	bw diagram for $p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(S^b))$
$D_4$	$x^3 + y^3 + z^2$	2	$\mathbb{C}^* \cdot \text{BDih}_3$	$\{1\}$	 $A_5 \subset \tilde{E}_7$
			$\mathbb{C}^*(1, 1) \cdot \mu_6$	$\{1\}$	 $D_4 \subset \tilde{E}_6$
				$\mu_3$	 $E_6 \subset \tilde{E}_6$
	$x^2 + y^3 + z^3$	3	$\mathbb{C}^*(2, 3) \cdot \mu_5$	$\{1\}$	 $D_4 \subset \tilde{E}_6$
$D_n$	$y(x^2 + y^{n-2}) + z^2$	2	$\mathbb{C}^*(n-2, 2) \cdot \mu_n$	$\{1\}$	 $A_{2n-3} \subset \tilde{D}_{2n-2}$
			$\mathbb{C}^*(1, \frac{n-2}{2}) \cdot \mu_{\frac{n}{2}}$ $n$ even	$\{1\}$	 $D_n \subset \tilde{D}_n$

Table A.2: The  $p$ -cycles on curve singularities of type  $D$  giving rise to Fano orbifolds.

	$f + z^p$	$p$	$\Gamma^b$	$N^b$	bw diagram for $p\text{-cyc}(\text{CM}_{\Gamma^b/N^b}(S^b))$
$E_6$	$x^2 + y^4 + z^3$	3	$\mathbb{C}^*(1, 2) \cdot \mu_3$	$\{1\}$	 $E_6 \subset \tilde{E}_6$
	$x^3 + y^4 + z^2$	2	$\mathbb{C}^*(3, 4) \cdot \mu_7$	$\{1\}$	 $E_6 \subset \tilde{E}_7$
	$x^2 + y^3 + z^4$	4	$\mathbb{C}^*(2, 4) \cdot \mu_5$	$\{1\}$	
	$x^2 + y^4 + z^3$	3	$\mathbb{C}^*(1, 2) \cdot \mu_6$	$\{1\}$	
	$x^2 + y^4 + z^3$	3	$\mathbb{C}^*(1, 2) \cdot \mu_6$	$\mu_2$	 $E_7 \subset \tilde{E}_7$
$E_7$	$x(x^2 + y^3) + z^2$	2	$\mathbb{C}^*(2, 3) \cdot \mu_5$	$\{1\}$	 $E_7 \subset \tilde{E}_7$
$E_8$	$x^3 + y^5 + z^2$	2	$\mathbb{C}^*(3, 5) \cdot \mu_8$	$\{1\}$	 $E_8 \subset \tilde{E}_8$
	$x^2 + y^5 + z^3$	3	$\mathbb{C}^*(2, 5) \cdot \mu_7$	$\{1\}$	
	$x^2 + y^3 + z^5$	5	$\mathbb{C}^*(2, 3) \cdot \mu_5$	$\{1\}$	

Table A.3: The  $p$ -cycles on curve singularities of type  $E_6, E_7, E_8$  giving rise to Fano orbifolds.



	$f + z^p$	$p$	$\Gamma^b$	type of $p$ -cyc( $\text{CM}_{\Gamma^b/\Lambda^b}(S^b)$ )
$X_9$	$x^4 + y^4 + z^2$	2	$\mathbb{C}^*(1, 1) \cdot \mu_2$	(2,2,2,2)
			$\mathbb{C}^*(1, 1) \cdot \mu_4$	(2,2,2,2)
			$\mathbb{C}^*(1, 1) \cdot \text{BDih}_2$	(2,2,2,2)
			$\mathbb{C}^*(1, 1) \cdot \mu_8$	(2,4,4)
			$\mathbb{C}^*(1, 1) \cdot \text{BDih}_4$	(2,4,4)
	$xy(x^2 + y^2) + z^2$	2	$\mathbb{C}^*(1, 1) \cdot \mu_4$	(2,4,4)
			$\mathbb{C}^*(1, 1) \cdot \text{BDih}_2$	(2,4,4)
	$x(x^3 + y^3) + z^2$	2	$\mathbb{C}^*(1, 1) \cdot \mu_6$	(2,3,6)
	$x^4 + y^4 + z^2$	2	$\mathbb{C}^*(1, 1) \cdot \text{BTet}$	(2,3,6)
	$x^2 + y^4 + z^4$	4	$\mathbb{C}^*(1, 2) \cdot \mu_3$	(2,4,4)
			$\mathbb{C}^*(1, 2) \cdot \mu_6$	(2,4,4)
$J_{10}$	$x^3 + y^6 + z^2$	2	$\mathbb{C}^*(1, 2) \cdot \mu_3$	(2,2,2,2)
	$x(x^2 + y^4) + z^2$	2	$\mathbb{C}^*(1, 2) \cdot \mu_6$	(2,4,4)
	$x^3 + y^6 + z^2$	2	$\mathbb{C}^*(1, 2) \cdot \mu_9$	(2,3,6)
	$x^2 + y^6 + z^3$	3	$\mathbb{C}^*(1, 3) \cdot \mu_8$	(2,3,6)
	$x^2 + y^3 + z^6$	6	$\mathbb{C}^*(2, 3) \cdot \mu_5$	(2,3,6)
	$x^2 + y^6 + z^3$	3	$\mathbb{C}^*(1, 3) \cdot \mu_4$	(3,3,3)
$P_8$	$x^3 + y^3 + z^3$	3	$\mathbb{C}^*(1, 1) \cdot \mu_2$	(3,3,3)
			$\mathbb{C}^*(1, 1) \cdot \mu_6$	(3,3,3)
	$x(x^2 + y^2) + z^3$	3	$\mathbb{C}^*(1, 1) \cdot \mu_4$	(2,3,6)
	$x^3 + y^3 + z^3$	3	$\mathbb{C}^*(1, 3) \cdot \text{BDih}_3$	(2,3,6)

Table A.4: The  $p$ -cycles on curve singularities of type  $P_8$ ,  $X_9$ ,  $J_{10}$  giving rise to Calabi-Yau orbifolds.

# Bibliography

- [1] T. Abdelgadir and K. Ueda ‘Weighted projective lines as fine moduli spaces of quiver representations’. *Comm. Alg.* **43** 2 (2015), 636-649.
- [2] T. Araya ‘Exceptional Sequences Over Graded Cohen-Macaulay Rings’. *Math. J. Okayama Univ.* **41** (1999), 81-102.
- [3] M. Auslander ‘Isolated singularities and existence of almost split sequences’. *Proc. ICRA IV, Springer Lecture Notes in Math.* **1178** (1986) 194-241.
- [4] M. Auslander ‘Rational singularities and almost split sequences’. *Trans. Amer. Math. Soc.* **293** (1986), 511-531.
- [5] M. Auslander and I. Reiten ‘Cohen-Macaulay Modules for Graded Cohen-Macaulay Rings and their Completions’ in *Commutative Algebra: Proceedings of a Microprogram Held June 15-July 2, 1987*. Springer 1989.
- [6] M. Auslander and I. Reiten ‘Graded Modules and their Completions’ in *Topics in Algebra*. Banach Centre Publications volume 26 part 1. Warsaw 1990.
- [7] M. Auslander and I. Reiten ‘DTr-periodic modules and functors’ in *Representation Theory of Algebras*. Canad. Math. Soc. Conf. Proc. **18**, Amer. Math. Soc., Providence, RI (1996) 39-50.
- [8] M. Auslander and I. Reiten ‘Stable Equivalence of Dualizing  $R$ -Varieties’. *Adv. Math.* **12** (1974), 306-366.
- [9] M. Auslander, I. Reiten and S. O. Smalø *Representation Theory of Artin Algebras*. CUP 1997.
- [10] W. Bruns and J. Herzog *Cohen-Macaulay rings*. CUP 1993.
- [11] T. Bühler ‘Exact Categories’. *Expo. Math.* **28** (2010), 1-69.

- [12] R.-O. Buchweitz *Maximal Cohen-Macaulay modules and Tate-Cohomology over Gorenstein rings*, Preprint 1987.
- [13] I. Burban and Y. Drozd ‘Maximal Cohen-Macaulay modules over surface singularities’ in *Trends in Representations of Algebras and Related Topics*, EMS Publishing House (2008) 101-166.
- [14] X.-W. Chen ‘Three Results on Frobenius categories’. *Mathematische Zeitschrift* **270** (2012), 43-58.
- [15] X.-W. Chen and H Krause ‘Introduction to Coherent Sheaves on Weighted Projective Lines’. (arXiv:0911.4473)
- [16] D. Eisenbud ‘Homological algebra on a complete intersection, with an application to group representations’. *Trans. Amer. Math. Soc.* **260** (1980) 35-64.
- [17] S. Fomin and A. Zelevinsky, ‘Cluster algebras II: Finite type classification’. *Invent. Math.* **154** (2003) 63-121.
- [18] W. Geigle and H. Lenzing, ‘A class of weighted projective curves arising in representation theory of finite dimensional algebras’ in *Singularities, Representations of Algebras and Vector Bundles*, Lecture Notes in Math. **1273** 265-297, Springer (1987).
- [19] W. Geigle and H. Lenzing, ‘Perpendicular Categories with Applications to Representations and Sheaves’. *J. Algebra* **14** (1991) 273-343.
- [20] D. Happel *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*. CUP 1988.
- [21] R. Hartshorne ‘Stable Reflexive Sheaves’. *Math. Ann.* **254** (1980), 121-176.
- [22] A. Hochenegger and E. Martinego ‘MD-Stacks’. *Mathematische Zeitschrift* **280** 3 (2015), 1185-1202.
- [23] M. Hochster ‘Some Applications of the Frobenius in Characteristic 0’. *Bulletin of the AMS* **84** (1978), 886-912.
- [24] Y. Hu and S. Keel ‘Mori Dream Spaces and GIT’. *Michigan Math. J.* **48** (2000), 331-348.
- [25] T. Hübner ‘Classification of indecomposable vector bundles on weighted curves’. Diplomarbeit, Paderborn, 1989.

- [26] B. Iversen *Cohomology of Sheaves*. Springer 1986.
- [27] B. T. Jensen, A. King and X. Su ‘A Category for Grassmannian Cluster Algebras’. (arXiv:1309.7301)
- [28] H. Kajiura, K. Saito and A. Takahashi ‘Matrix Factorizations and Representations of Quivers II: Type ADE Case’. *Adv. Math.* **211** 1 (2007), 327-362.
- [29] D. E. Kedzierski, H. Lenzing and H. Meltzer ‘Matrix Factorizations for Domestic Triangle Singularities’. *Coll. Math.* **140** 2 (2015), 239-278.
- [30] B. Keller ‘Derived Categories and Tilting’ in *Handbook of Tilting Theory*, LMS Lecture Notes Series **332** 49-104, CUP 2007.
- [31] B. Keller ‘Derived Categories and their uses’ in *Handbook of Algebra* Vol. 1, 671-701, North Holland Elsevier, Amsterdam 1996.
- [32] A. Kirillov, Jr. ‘McKay Correspondence and Equivariant Sheaves on  $\mathbb{P}^1$ ’. *Moscow Math. J.* **6** (2006), 505-529.
- [33] D. Kussin, H. Lenzing and H. Meltzer, ‘Triangle singularities, ADE-chains, and weighted projective lines’. *Adv. Math.* **237** (2013), 194-251.
- [34] H. Krause, ‘Krull-Schmidt categories and projective covers’. *Exp. Math.* **33** 4 (2015), 535-549.
- [35] H. Lenzing ‘Representations of Finite Dimensional Algebras and Singularity Theory’. *CMS Conference Proceedings* **22** (1998), 71-97.
- [36] H. Lenzing ‘Curve singularities arising from the representation theory of tame hereditary Artin algebras’ in *Representation theory I. Finite dimensional Algebras*, Lecture Notes in Math. **1177** 199-231, Springer (1986).
- [37] H. Lenzing and H. Meltzer, ‘Sheaves on a weighted projective line of genus one, and representations of a tubular algebra’ in *Representations of algebras* CMS Conf. Proc. **14** (1993) 313-337.
- [38] H. Lenzing and I. Reiten ‘Hereditary Noetherian categories of positive Euler characteristic’. *Math. Z.* **254**(1) (2006) 133-171.
- [39] D. Orlov ‘Derived Categories of Coherent Sheaves and Triangulated Categories of Singularities’ in *Algebra, Arithmetic and Geometry* Prog. Math. **270** (2010) 503-531.

- [40] I. Reiten and C. Riedtmann ‘Skew Group Algebras in the Representation Theory of Artin Algebras’. *J. Algebra* **92** (1985) 224-282.
- [41] C. M. Ringel *Tame Algebras and Integral Quadratic Forms* Lecture Notes in Mathematics **1099**, Springer 1984.
- [42] K. Saito ‘Einfach-elliptische Singularitäten’. *Invent. Math.* **23** (1974) 289-325.
- [43] J. Scott, ‘Grassmannians and cluster algebras’. *Proc. London Math. Soc.* **92** (2006) 345-380.
- [44] A. Takahashi, ‘Mirror Symmetry of Singularities’, seminar at Kyoto July 24 2009, <http://www.mi.s.osakafu-u.ac.jp/~kiriko/seminar/09JulRIMS/lectures/takahashi.pdf>
- [45] W. Thurston *The Geometry and Topology of Three-Manifolds*, Princeton University lecture notes (1978-1981).
- [46] Y. Yoshino *Cohen-Macaulay modules over Cohen-Macaulay rings*. CUP 1990.