PHD

# Non-singular actions of countable groups

Jarrett, Kieran

*Award date:*
2018

*Awarding institution:*
University of Bath

Link to publication

# Non-singular actions of countable groups

submitted by

Kieran Jarrett

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

May 2018

## Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Kieran Jarrett

## Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, with the exception of Chapter 3, which contains part of a research article that originated from collaboration with my supervisor Anthony Dooley.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Kieran Jarrett

# Summary

In this thesis we study actions of countable groups on measure spaces under the assumption that the dynamics are non-singular, with particular reference to pointwise ergodic theorems and their relationship to the critical dimensions of the action.

# Acknowledgements

Firstly I want to thank Tony for introducing me to ergodic theory during my masters, and taking me on as a PhD student. I am really grateful for the opportunity to work in Sydney, the hours of mathematical discussion and, of course, the many cups of tea!

I would also like to thank Ali and Veronique, who provided me with much appreciated support throughout my time here. In particular, I am very grateful to Ali for encouraging me to apply to do a PhD as this was precisely the push I needed at the time.

I am also really thankful to my friends, especially 'Wednesday group' and my office mates, who listened to me moan (a lot) through the hard times, introduced me to so many new things and made my time here the best I have had.

Finally, thank you to my family who have, as always, been there for me throughout.

# Contents

# Chapter 1

# Introduction

Ergodic theory is the study of dynamics on spaces equipped with a measure, in particular on the consequences of interactions between the motion and the measure. Discrete time systems are traditionally formulated as a $\sigma$-finite measure space $(X, \mathcal{B}, \mu)$ equipped with a measurable transformation $T : X \to X$. The dynamics are then described by repeatedly applying $T$ to the points in $X$ which generates forward orbits $\{T^n x\}_{n=1}^{\infty}$ for each point $x \in X$.

The usual assumption governing the above interaction is that the transformation preserves the measure, meaning that for all $A \in \mathcal{B}$ the measure satisfies $\mu(T^{-1}A) = \mu(A)$. In this case the dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be *measure preserving*. This is a natural condition both because physical systems which are in equilibrium often preserve quantities, and for many classical transformations one can find preserved measures which contain useful information about the system as a whole.

Another natural way a transformation can interact with a measure is for the transformation to preserve its measure zero (or *null*) sets, i.e. for all $A \in \mathcal{B}$ we have $\mu(T^{-1}A) = 0$ if and only if $\mu(A) = 0$. Such a system $(X, \mathcal{B}, \mu, T)$ is called *non-singular*. This weaker condition is of interest because it models non-equilibrium systems. In addition, from a mathematical perspective null sets are often of particular interest and this condition is the minimum necessary for the dynamics to respect their structure. There are many non-singular systems $(X, \mathcal{B}, \mu, T)$ for which there is no $\sigma$-finite measure $\nu$ on $X$ which is both preserved by $T$ and has the same null sets as $\mu$ (see [KW91], for example). This means, if one is interested in how the structure of null sets interact with the dynamics of a non-singular system it is not possible, in general, to consider another system with the same dynamics but where the measure is preserved.

The classical theory of measure preserving systems has been extended in a variety of directions, one of which is to generalise the notion of time. This was motivated by the observation that when the transformation $T$ has a measurable inverse it induces a measurable action of $\mathbb{Z}$ on $X$ via the map $(n, x) \mapsto T^n x$. Similarly, one can consider measurable actions of an arbitrary (countable) group $G$ on measure spaces $(X, \mathcal{B}, \mu)$ and call the system $(X, \mathcal{B}, \mu, G)$ measure preserving (non-singular) exactly when each of the transformations $x \mapsto gx$ is measure preserving (non-singular). Many aspects of the integer measure preserving theory have been extended to other groups, and amenable groups in particular. One example of this is Birkhoff's fundamental pointwise ergodic theorem which, with suitable modifications, has been shown to hold by Lindenstrauss for every amenable group [Lin01]. Another is how Ornstein and Weiss broadened the classical entropy theory to include the measure preserving actions of amenable groups [OW87].

This development of the measure preserving theory raises the following motivating question: to what extent can the theory of non-singular integer actions be generalised to non-singular actions of amenable groups? An example of a classical result with a non-singular amenable counterpart is the classical Rokhlin lemma also due to Ornstein and Weiss [OW80], which we discuss in detail later. In the course of this thesis we will focus on two other, related, problems in this area.

The first addresses the development of the pointwise ergodic theorem in the non-singular setting. Birkhoff's ergodic theorem has a non-singular counterpart in the Hurewicz ergodic theorem [Hur44], but the extension of the latter to other group actions is far more limited than the former.

The second problem is generalising the critical dimension theory of Dooley and Mortiss [Mor03, DM09, DM06, DM07] to other group actions. In the integer case the critical dimensions are invariants of a natural isomorphism between non-singular systems, due to the Hurewicz ergodic theorem.

In this chapter we begin by introducing the concepts needed to study these problems, then discuss them in the classical setting of non-singular transformations. We then move on to discuss what it means to consider them in the context of group actions, and the progress which has been made up to this work.

## 1.1 Preliminaries

The fundamental objects of study in this work are the actions of a countable (usually amenable) group $G$ on measure spaces $(X, \mathcal{B}, \mu)$.

As is standard in ergodic theory the $(X, \mathcal{B}, \mu)$ will be a *standard measure space*, meaning $X$ is a Polish space, i.e. a complete separable metric space, equipped with its Borel $\sigma$-algebra $\mathcal{B}$. Where there is no confusion over the $\sigma$-algebra we will exclude it, writing $(X, \mu)$. The measure $\mu$ will always be $\sigma$-finite, and will usually be a probability measure.

In this section we first discuss our assumptions on the group $G$ and, before addressing how its actions interact with measure, we will recall some useful terminology and results from measure theory.

### 1.1.1 The group

Throughout this work we assume that $G$ is countable, and this will play a role in some of the arguments which follow. We would like to note that (at the cost of additional technicality) some results in the area for countable groups have often been shown to hold in a corresponding manner for locally compact second countable groups. Checking whether this is the case for the results presented here could be a subject of future work.

Each of the groups we consider, motivated by developments in the measure preserving theory, will be amenable. Amenable groups have become objects of interest to mathematicians in different contexts, and many equivalent definitions have been produced. As stated in [Pat88], the concept was introduced by von Neumann in his study of the Banach-Tarski paradox. His definition of amenability was "infinitary" in the sense that it asserted the existence of an invariant mean on the space bounded functions on the group. We will be using an equivalent "finitary" definition. The equivalence between these definitions is non-constructive in the sense that the axiom of choice is required for the transition. The definition we will use asserts the existence of a Følner sequence in the group.

**Definition 1.1.1.** A *Følner sequence* in $G$ is a sequence $(F_n)_{n=1}^{\infty}$ of finite subsets such that for all $g \in G$

$$\lim_{n\to\infty} \frac{|F_n \triangle g F_n|}{|F_n|} = 0.$$

This is called the *Følner condition.* A general countable discrete group $G$ is called *amenable* if it has a Følner sequence.

The basic examples of amenable groups are the finite groups and the integers under addition. The constant Følner sequence $F_n = G$ works for any finite group and in the case of the integers one can take $F_n = [0, n] \cap \mathbb{Z}$.

The fact that $[0, n] \cap \mathbb{Z}$ is a Følner sequence for the integers is a major motivation for this generalisation. The set $[0, n] \cap \mathbb{Z}$ appears somewhat subtly in various places in integer ergodic theory as range of sums, for example in the Birkhoff and Hurewicz ergodic theorems as we will see later. It is thought that the Følner condition is a key property of the sequence $[0, n] \cap \mathbb{Z}$ which is implicitly used in the proofs of results such as the ergodic theorems above, and we will see examples of this later.

The following proposition lays out some basic facts about amenable groups, and is useful for producing further examples. Proofs of (i)-(iii) from the "infinitary" definition of amenability, which is often more efficient, can be found in [KM04] and the proof of (iv) is natural using the Følner condition.

**Proposition 1.1.2.** *Let $G$ be a countable discrete group.*

(i) *If $G$ is amenable and $H \leq G$ then $H$ is amenable.*

(ii) *If $N \triangleleft G$ then $G$ is amenable if and only if both $N$ and $G/N$ are amenable.*

(iii) *$G$ is amenable if and only if every finitely generated subgroup of $G$ is amenable.*

(iv) *If $G_1 \subset G_2 \subset ...$ is a sequence of amenable groups with $G = \bigcup_{i=1}^{\infty} G_i$ then $G$ is amenable.*

In particular, note that (ii) means semidirect products of amenable groups are amenable and homomorphic images of amenable groups are amenable. Since direct products of amenable groups are amenable we see that $\mathbb{Z}^d$ for any $d \in \mathbb{N}$ is also amenable; more generally any finitely generated abelian group is amenable. Hence, by (iii), every countable abelian group is amenable. It then follows from (ii) that any solvable group is amenable.

We will usually consider particular classes of amenable groups, rather than amenable groups as a whole, for reasons we will explain later.

To complete this introduction of amenable groups let us state that there are examples of non-amenable groups. The simplest is the free group on two generators $\mathbb{F}_2$, and by proposition 1.1.2 any group containing $\mathbb{F}_2$ as a subgroup is non-amenable. The proof is slightly easier using the "infinitary" definition, but still not difficult using Følner sequences. Since it is well known and we will not be using it, we do not include it.

### 1.1.2 Measure theory

A measurable set $A$ is called *positive* if $\mu(A) > 0$, *null* if $\mu(A) = 0$ and *finite* if $\mu(A) < \infty$. We add the prefix $\mu$- when we need to specify a measure, e.g. $\mu$-null. We use the word *co-null* to describe a set which has null complement. When we say that two sets $A$ and $B$ are *equal up to a null set* or *essentially equal* we mean $\mu(A \triangle B) = 0$, where $A \triangle B$ is the symmetric difference of $A$ and $B$. For simplicity, we will just say $A$ and $B$ are equal and write $A = B$ in this situation, adding the phrase 'equal as sets' if that distinction is necessary or beneficial. We say that a measure $\mu$ is *absolutely continuous* with respect to $\nu$ (on the same space) if for all measurable sets $A$ we have $\nu(A) = 0 \Rightarrow \mu(A) = 0$ and we write $\mu \ll \nu$ in this case. If both $\mu \ll \nu$ and $\nu \ll \mu$ then the measures are said to be *equivalent*, denoted $\mu \sim \nu$. Equivalent measures have the same null and co-null sets.

A positive measurable set $A$ is called an *atom* if for every $B \subseteq A$ either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. The space $(X, \mu)$ is said to be *atomic* if it has an atom, and *non-atomic* otherwise.

The following useful result is a corollary of a theorem due to Sierpinski, see [Sie22]. The proof makes use of the axiom of choice.

**Proposition 1.1.3.** *Given a non-atomic measure space $(X, \mu)$, a measurable set $A$ and some real number $b$ such that. $0 < b < \mu(A) < \infty$, there exists a measurable set $B \subseteq A$ such that $\mu(B) = b$.*

As Polish spaces are second countable and Hausdorff, it will also be useful to be aware of the following lemma, a proof of which can be found in [AB06].

**Lemma 1.1.4.** *Any atom of a Borel measure on a second countable Hausdorff space includes a singleton of positive measure. In particular, a Borel measure on a second countable Hausdorff space is non-atomic if and only if every singleton has measure zero.*

For reasons we will discuss shortly, typically $(X, \mathcal{B}, \mu)$ will be non-atomic.

### 1.1.3 Dynamics

Our primary assumption on the dynamics is non-singularity.

#### Non-singularity

A *transformation* $T$ on a measure space $(X, \mu)$ is a function $T : X \to X$ which we assume to be *bimeasurable*, i.e. it is measurable and has a measurable inverse $T^{-1}$. A transformation on $(X, \mu)$ is called *non-singular* (n.s.) if $\mu \circ T^{-1} \sim \mu$ or, equivalently, $\mu \circ T \sim \mu$. Intuitively, if $T$ is non-singular then $T$ cannot map null sets to positive sets or vice versa. If $\mu \circ T^{-1} = \mu$ then we call $T$ *measure preserving* (m.p.).

An action of $G$ on the space $X$ is called *non-singular* if for each $g \in G$ the map $x \mapsto gx$ is a non-singular transformation, and similarly for the m.p. case. For m.p. actions we say that $\mu$ is *invariant* with respect to the action of $G$. For n.s. actions we say that $\mu$ is *quasi-invariant* with respect to the action of $G$. For a n.s. action of $G$ on $X$ we will use each element $g \in G$ to also denote its associated n.s. transformation on $X$, as above. The quadruple $(X, \mathcal{B}, \mu, G)$ along with an implicit action of $G$ on $X$, will be called a *n.s. (m.p.) system* if the action is n.s. (m.p.). When unambiguous, the notation $(X, \mu, G)$ will be used.

As discussed earlier, the non-singular transformations $T$ on $X$ are in a natural correspondence with the non-singular integer actions on $X$. Each transformation induces a $\mathbb{Z}$-action via map $(n, x) \mapsto T^n x$, and each $\mathbb{Z}$-action can be associated with the transformation given by $1 \in \mathbb{Z}$. This also happens in the m.p. case. As such, in the case of integer actions we will make the transformation $T$, corresponding to 1, explicit by writing $T^n x$ rather than $nx$.

The following proposition is well known and it has a natural proof.

**Proposition 1.1.5.** *If $(X, \mu)$ is a $\sigma$-finite measure space then there is a probability measure $\nu$ on $X$ with $\nu \sim \mu$.*

In particular, when $G$ has a n.s. action on $X$ we can replace $\mu$ by an equivalent probability measure with respect to which the action of $G$ is also n.s.. Hence without loss of generality we will assume that $\mu(X) = 1$ for actions which are just n.s., unless stated otherwise. Be aware that given a m.p. action on the space $X$ one cannot, in general, find an equivalent invariant probability measure and so when we are using m.p. actions the measure may well be infinite).

Fundamental to the study of non-singular systems is the Radon-Nikodým theorem, see [Roy63], for example.

**Theorem 1.1.6** (Radon-Nikodým)**.** *Given a measurable space $(X, \mathcal{B})$ with $\sigma$-finite measures $\mu$ and $\nu$ on $(X, \mathcal{B})$ such that $\mu \ll \nu$ there is a non-negative measurable function $f$ on $X$ such that, for all $A \in \mathcal{B}$, $\mu(A) = \int_A f \, d\nu$. Moreover, this function is unique $\nu$ a.e..*

*Conversely, if we have such measures satisfying $\mu(A) = \int_A f \, d\nu$ then $\mu$ is absolutely continuous with respect to $\nu$.*

The function $f$ will be called the *Radon-Nikodým derivative* or simply the *derivative* of $\mu$ with respect to $\nu$ and will be denoted by $\frac{d\mu}{d\nu}$. Radon-Nikodým derivatives are unique up to sets of measure $\nu$-measure zero and satisfy a number of properties satisfied by the classical derivative, including the chain rule.

For a non-singular transformation $T$ and $i \in \mathbb{Z}$, let

$$\omega_i = \frac{d\mu \circ T^i}{d\mu}.$$

The properties of the Radon-Nikodým derivative ensure that $\omega_i$ must satisfy the cocycle identity: $\omega_{i+j}(x) = \omega_i(T^j x)\omega_j(x)$ a.e.. Similarly for a non-singular action of $G$ on $X$ denote $\frac{d\mu \circ g}{d\mu}$ by $\omega_g$, which will satisfy the corresponding cocycle identity.

### Ergodicity

A secondary assumption we will make on the dynamics, wherever justified, is that they are ergodic.

We say that a measurable set $A$ is *$G$-invariant* if $\mu(A \triangle gA) = 0$ for all $g \in G$. Similarly, we say a measurable function $f$ is *$G$-invariant* if for each $g \in G$ we have $f \circ g = f$ a.e.. Note that a set $A$ is $G$-invariant if and only if $1_A$ is $G$-invariant.

We say that a group action of $G$ on $X$ is *ergodic* if each $G$-invariant set is either null or conull, i.e. if $\mu(A \triangle gA) = 0$ for all $g \in G$ then either $\mu(A) = 0$ or $\mu(A^c) = 0$. Equivalently, an action is ergodic if each $G$-invariant measurable function $f : X \to \mathbb{R}$ is essentially constant.

Ergodicity is an irreducibility property in ergodic theory. For example, the ergodic decomposition theorem ensures that ergodic systems are building blocks for more complex systems.

**Theorem 1.1.7** (Ergodic decomposition theorem)**.** *Let $G$, a countable amenable group, act non-singularly on the standard probability space $(X, \mathcal{B}, \mu)$. Then there is a probability space $(Y, \mathcal{C}, \nu)$ and a family of probability measures $\{\mu_y : y \in Y\}$ on $(X, \mathcal{B})$ such that*

(i) *For each $A \in \mathcal{B}$ the map $y \mapsto \mu_y(A)$ is measurable and*

$$\mu(A) = \int_Y \mu_y(A)\, d\nu(y).$$

(ii) *The measures $\mu_y$ and $\mu_{y'}$ are mutually singular for $y \neq y'$.*

(iii) *For each $y \in Y$ the action of $G$ on $(X, \mathcal{B}, \mu_y)$ is ergodic, non-singular and for all $g \in G$*

$$\frac{d\mu \circ g}{d\mu} = \frac{d\mu_y \circ g}{d\mu_y} \qquad \mu_y\text{-a.e..}$$

(iv) *For any other probability space $(Y', \mathcal{C}', \nu')$ and family of probability measures $\{\mu'_{y'} : y' \in Y'\}$ on $(X, \mathcal{B})$ satisfying* (i)-(iii) *there exists a measure preserving isomorphism $\phi : Y \to Y'$ such that $\mu_y = \mu'_{\phi(y)}$ for $\nu$-a.e. $y \in Y$.*

The proof case where $G = \mathbb{Z}$ and the action is *conservative*, meaning any measurable set $W$ for which the collection $\{T^n W\}_{n \in \mathbb{Z}}$ is pairwise disjoint must be null, can be found in [Aar97, 1.0.8 & 2.2.8]. One can remove the assumption of conservativity by applying a result and approach we will discuss in Chapter 3, Remark 3.1.7. The above result then follows from the fact that for a non-singular action of a countable amenable group $G$ there is a non-singular transformation $T$ on the same space for which $\{gx : g \in G\} = \{T^n x : n \in \mathbb{Z}\}$ for a.e. $x \in X$, which is a consequence of a theorem due to Connes, Feldman and Weiss [CFW81]. We include the details of how to do this in Appendix A.

The ergodic decomposition theorem can frequently be used to generalise results from ergodic systems to more general non-singular systems. For this reason we will usually assume that the group actions are ergodic.

### The standard formulation

For the reasons described above we will be considering actions of a countable amenable group $G$ on a standard probability space $(X, \mathcal{B}, \mu)$ which are non-singular and usually ergodic, though we will not always need the latter assumption. We will also assume that the measure $\mu$ is non-atomic because otherwise, by Lemma 1.1.4, there is a singleton $\{x\}$ of positive measure and by the ergodicity the orbit of $x$ under $G$ then has full measure. In this scenario $\mu$ is equivalent to the counting measure restricted to $Gx$, which is not very interesting from an ergodic theoretic perspective. By assuming that $\mu$ is non-atomic we also exclude the possibility that $G$ is finite: if not, we can find a set $A$ such that for each $g \in G$ we have $0 < \mu(gA) < \frac{1}{2}|G|^{-1}$ and it follows that $GA$ is a $G$ invariant set with measure in $(0, \frac{1}{2})$ contradicting ergodicity.

In summary, unless specified otherwise we will take $G$ to be a countably infinite amenable group and consider its non-singular and ergodic actions on non-atomic standard probability spaces $(X, \mathcal{B}, \mu)$.

### Non-singular isomorphism

There are a number of ways two non-singular actions of a group $G$ can be compared. We will consider a relatively strong one: non-singular (or metric) isomorphism.

We say that a non-singular system $(Y, \mathcal{C}, \nu, G)$ is a (*non-singular*) factor of another system $(X, \mathcal{B}, \mu, G)$ if there exist measurable $G$-invariant sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$, and a measurable map $\phi : X_0 \to Y_0$ such that

(1) $\mu(X \setminus X_0) = 0 = \nu(Y \setminus Y_0)$,

(2) $\nu \ll \mu \circ \phi^{-1}$ and

(3) $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X_0$.

The map $\phi$ is called a *factor map* from $X$ into $Y$. We say that the systems are *(non-singularly) isomorphic* if there also exists a factor map $\psi : Y_0' \to X_0'$

from $X$ to $Y$ such that $\psi \circ \phi = \mathrm{Id}_{X_0 \cap X_0'}$ and $\phi \circ \psi = \mathrm{Id}_{Y_0 \cap Y_0'}$, and we call $\phi : X_0 \cap X_0' \to Y_0 \cap Y_0'$ a non-singular isomorphism and denote its inverse by $\phi^{-1}$. We use the same notation as we have the freedom to neglect null sets.

Intuitively, two systems are isomorphic when there is a map between them which respects the action of $G$ and preserves null sets. Note that, when $\phi$ is an isomorphism we have $\nu \sim \mu \circ \phi^{-1}$. Additionally, it is worth emphasising that this definition relies on the fact that the same group (up to group isomorphism) is acting on each of the two measure spaces.

## 1.2 Non-singular integer actions

In this section, we review aspects of the theory of non-singular integer actions. As stated earlier, our first problem is to consider extending the Hurewicz ergodic theorem to actions of other groups. A modern proof of this result can be found in [Aar97].

**Theorem 1.2.1** (Hurewicz ergodic theorem, [Hur44])**.** *Let $T$ be a (not necessarily invertible) conservative non-singular transformation of a probability space $(X, \mathcal{B}, \mu)$. Then given $f \in L^1(\mu)$ there exists a $T$-invariant function $\bar{f} \in L^1(\mu)$ such that*

$$\lim_{n \to \infty} \frac{\sum_{i=0}^{n} f(T^i x)\omega_i(x)}{\sum_{i=0}^{n} \omega_i(x)} = \bar{f}(x) \quad a.e.$$

*and $\int_A \bar{f}\, d\mu = \int_A f\, d\mu$ for all $T$-invariant sets $A \in \mathcal{B}$. In particular, $\bar{f} = \mathbb{E}(f|\mathcal{J})$ a.e. where $\mathcal{J}$ is the $\sigma$-algebra of $T$-invariant sets in $\mathcal{B}$.*

Note that if $T$ is ergodic and invertible it is necessarily conservative, so Hurewicz's theorem applies. Ergodicity ensures that $\bar{f}$ is a.e. constant and hence $\bar{f} = \int_X f\, d\mu$. The theorem then says that the weighted time average (i.e. over an orbit) of $f$ converges to the space average of $f$. Moreover, if the action is measure-preserving then Hurewicz's theorem reduces down to Birkhoff's classical ergodic theorem. This is because a measure preserving transformation $T$ is necessarily conservative and every $\omega_i = 1$ a.e., so the ratio in Hurewicz's theorem reduces to $\frac{1}{n} \sum_{i=1}^{n} f \circ T^i$ as in Birkhoff's result.

It is a consequence of Halmos' recurrence theorem, see [Aar97], that $T$ is conservative if and only if for every $f \in L^1$ with $f > 0$ we have

$$\sum_{i=0}^{\infty} f(T^i x)\omega_i(x) = \infty$$

almost everywhere. Therefore, if $f > 0$ both the numerator and the denominator in Hurewicz's theorem must diverge to infinity, but weighting by the Radon-Nikodým derivatives ensures that both sums diverge at the same rate. This suggests that the nature of the divergence of $\sum_{i=0}^{n} \omega_i(x)$ may contain information about some intrinsic behaviour of the system. This was a motivating factor for a rigorous study of the growth rate of $\sum_{i=0}^{n} \omega_i(x)$, and the creation of new invariants called the upper and lower critical dimensions which are the subject of our second main problem.

The critical dimensions were conceived of and studied by Dooley and Mortiss over the papers [DM06, DM07, DM09, Mor03].

For a non-singular transformation $T$ on a probability space $(X, \mu)$ and for $t \in \mathbb{R}$ let

$$L_t = \left\{ x \in X : \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=0}^{n} \omega_i(x) > 0 \right\}$$

and

$$U_t = \left\{ x \in X : \limsup_{n\to\infty} \frac{1}{n^t} \sum_{i=0}^{n} \omega_i(x) = 0 \right\}.$$

These sets are easily seen to be $T$-invariant and in particular must have measure 0 or 1 when $T$ is ergodic. Additionally, if $r < s$ then $L_r \supseteq L_s$ and $U_r \subseteq U_s$ (we say that $L_t$ and $U_t$ are decreasing and increasing respectively) and for all $t$ we have $L_t \cap U_t = \emptyset$.

With these properties in mind can make the following definitions.

**Definition 1.2.2.** The *lower and upper critical dimensions* of $(X, \mu, T)$ are

$$\alpha = \sup\{t : \mu(L_t) = 1\} \qquad \text{and} \qquad \beta = \inf\{t : \mu(U_t) = 1\}$$

respectively. In the case where $\alpha = \beta$ we will denote the value by $\gamma$ and refer to it as the *critical dimension*.

*Remark* 1.2.3. When $T$ is ergodic

$$\alpha = \liminf_{n\to\infty} \frac{\log\left(\sum_{i=0}^{n} \omega_i(x)\right)}{\log(n)} \qquad \text{and} \qquad \beta = \limsup_{n\to\infty} \frac{\log\left(\sum_{i=0}^{n} \omega_i(x)\right)}{\log(n)}$$

almost everywhere.

Intuitively, the lower critical dimension gives the slowest growth rate of all the subsequences of $\sum_{i=0}^{n} \omega_i(x)$, and the upper critical dimension the fastest. The critical dimension, when it exists, captures the growth rate of $\sum_{i=0}^{n} \omega_i(x)$ relative to $n$.

In principle we allow $\alpha$ and $\beta$ to take the values $\pm\infty$. However, there are a number of simple bounds for $\alpha$ and $\beta$. Since $\omega_1(x) > 0$ a.e., because $T$ is non-singular, when $t < 0$ we must have $\mu(L_t) = 1$ and hence $\alpha \geq 0$. Furthermore, as $L_t$ and $U_t$ are disjoint for all $t$ we must have $\alpha \leq \beta$. Additionally, $\alpha \leq 1$ since, by Fatou's Lemma, for all $t > 1$

$$\int \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=0}^{n} \omega_i \, d\mu \leq \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=0}^{n} \int \omega_i \, d\mu = \liminf_{n\to\infty} n^{1-t} = 0$$

and hence the limit inferior in the leftmost integral is almost surely 0. In particular, this latter bound suggests that comparing the growth rate of the sum to $n$ (rather than some other function e.g. $e^n$) is a sensible choice.

The critical dimensions are examples of new objects arising from the non-singularity of the action, they are trivially 1 for measure preserving actions and so do not reduce down to an object of interest in the traditional setting.

The intrinsic nature of the critical dimensions is reflected in the fact that it is an invariant of non-singular isomorphism.

**Proposition 1.2.4** ([Mor03]). *Let $(X, \mathcal{B}, \mu, T)$ and $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{T})$ be isomorphic conservative systems. Then $\mu(L_t) = 1$ if and only if $\tilde{\mu}(\tilde{L}_t) = 1$, and $\mu(U_t) = 1$ if and only if $\tilde{\mu}(\tilde{U}_t) = 1$. In particular, $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$.*

*Proof.* Let $\phi$ be an isomorphism from $X$ into $\tilde{X}$, which we may assume are its range and domain. Then $\mu$-a.s.

$$\frac{d\tilde{\mu}\circ\phi}{d\mu}(T^i x)\omega_i(x) = \frac{d\tilde{\mu}\circ\phi}{d\mu}(x)\tilde{\omega}_i(\phi(x)).$$

Let $A_m = \left\{x \in X : m^{-1} \leq \frac{d\tilde{\mu}\circ\phi}{d\mu}(x) \leq m\right\}$. For almost every $x \in A_m$

$$\sum_{i=0}^{n} \mathbf{1}_{A_m}(T^i x)\,\omega_i(x) \leq m^2 \sum_{i=0}^{n} \mathbf{1}_{\phi A_m}(\tilde{T}^i\phi(x))\,\tilde{\omega}_i(\phi(x)) \leq m^2 \sum_{i=0}^{n} \tilde{\omega}_i(\phi(x)).$$

We aim to remove the indicator function from the left hand side by applying the ergodic theorem.

Observe that for any $m$ the set $B_m = \{x \in X : \mathbb{E}(1_{A_m}|\mathcal{J})(x) = 0\}$ satisfies $\mu(A_m \cap B_m) = \int_{B_m} \mathbb{E}(1_{A_m}|\mathcal{J})d\mu = 0$ and hence (since $\mathbb{E}(1_{A_m}|\mathcal{J}) \geq 0$ a.e.) we have $\mu(A_m \cap \bigcup_l C_m(l)) = \mu(A_m)$ where $C_m(l) = \{x \in X : \mathbb{E}(1_{A_m}|\mathcal{J})(x) > l^{-1}\}$.

It follows from the ergodic theorem and the above that for a.e. $x \in A_m \cap C_m(l)$ there is some $N = N(x)$ such that for all $n \geq N$

$$\frac{1}{2l}\sum_{i=0}^{n}\omega_i(x) \leq \left(\mathbb{E}(1_{A_m}|\mathcal{J})(x) - \frac{1}{2l}\right)\sum_{i=0}^{n}\omega_i(x) \leq m^2\sum_{i=0}^{n}\tilde{\omega}_i(\phi(x)).$$

In particular, it follows that for a.e. $x \in A_m \cap C_m(l)$ we have $x \in L_t \Rightarrow \phi(x) \in \tilde{L}_t$ and $\phi(x) \in \tilde{U}_t \Rightarrow x \in U_t$. Since $\mu(\bigcup_m \bigcup_l A_m \cap C_m(l)) = 1$ the same must hold for almost every $x \in X$, and up to null sets $\phi(L_t) \subseteq \tilde{L}_t$ and $\tilde{U}_t \subseteq \phi(U_t)$. By symmetry, these sets must be essentially equal and the result follows. □

Note that the only reason we require the two systems to be conservative is to ensure that the ergodic theorem holds, and that we use the ergodic theorem once in each system.

The critical dimensions for non-ergodic systems can be described in terms of those of its ergodic components.

**Proposition 1.2.5.** *Let $(X, \mathcal{B}, \mu, T)$ be a non-singular system and $(Y, \mathcal{C}, \nu)$ and $\{\mu_y : y \in Y\}$ describe its ergodic decomposition. Then*

$$\alpha = \sup\left\{t : \nu(\{y : t < \alpha_y\}) = 1\right\} \quad \textit{and} \quad \beta = \inf\left\{t : \nu(\{y : t > \beta_y\}) = 1\right\}$$

*where $\alpha_y$ and $\beta_y$ are the lower and upper critical dimensions of $T$ with respect to $\mu_y$, respectively.*

*Remark* 1.2.6. The maps $y \mapsto \alpha_y$ and $y \mapsto \beta_y$ are measurable because, e.g., since $L_t$ is decreasing and $\mu_y(L_t) \in [0,1]$

$$\alpha_y = \sup\{t : \mu_y(L_t) = 1\} = \sup_{t\in\mathbb{Q}} t \lfloor \mu_y(L_t) \rfloor$$

and the map $y \mapsto \mu_y(L_t)$ is measurable for all $t$. The upper case is similar.

*Proof.* We consider only the lower critical dimension, as the proof for the upper dimension is similar.

First let $s > \sup\big\{t : \nu(\{y : t < \alpha_y\}) = 1\big\}$, and choose $\epsilon > 0$ so the same remains true for $s - \epsilon$. Then $\nu(\{y : s - \epsilon < \alpha_y\}) < 1$ and so there is some measurable $A \subset Y$ on which $\alpha_y \leq s - \epsilon < s$. Hence $\mu_y(L_s) < 1$ on $A$ and so $\mu(L_s) = \int_Y \mu_y(L_s) d\nu < 1$. This means $\alpha < s$, from which it follows that $\alpha \leq \sup\big\{t : \nu(\{y : t < \alpha_y\}) = 1\big\}$.

Now let $s < \sup\big\{t : \nu(\{y : t < \alpha_y\}) = 1\big\}$. This time $\nu(\{y : s < \alpha_y\}) = 1$ so for $\nu$-a.e. $y \in Y$ we have $\mu_y(L_s) = 1$ (as $L_t$ decreases with $t$), and then $\mu(L_s) = \int_Y \mu_y(L_s) d\nu = 1$. Hence $s \leq \alpha$ and therefore

$$\alpha \geq \sup\big\{t : \nu(\{y : t < \alpha_y\}) = 1\big\}$$

as required. □

In part due to this result, we will focus on ergodic systems when studying the critical dimensions.

A class of ergodic systems the critical dimensions the critical dimensions have been calculated for are the product odometers. Recall that the product odometer on the space

$$X = \prod_{i=1}^{\infty} \mathbb{Z}_2$$

with topology generated by the cylinder sets

$$[y_1, ..., y_n] = \{x \in X : x_1 = y_1, ..., x_n = y_n\}$$

and product measure $\mu = \prod_{i=1}^{\infty} \mu_i$, with each $\mu_i$ a non-trivial measure on $\{0,1\}$, is given by

$$(Tx)_i = \begin{cases} 0 & \text{if } i < l(x) \\ 1 & \text{if } i = l(x) \\ x_i & \text{if } i > l(x) \end{cases}$$

when $l(x) = \min\{i : x_i = 0\} < \infty$ and $T(1,1,1,...) = (0,0,0,...)$. It is well known that $(X, \mu, T)$ describes an ergodic non-singular system.

Dooley and Mortiss calculated the critical dimension for product odometers.

**Theorem 1.2.7** (see [DM09]). *Let $T$ denote the odometer transformation on the space $(\prod_{i=1}^{\infty} \mathbb{Z}_2, \prod_{i=1}^{\infty} \mu_i)$. Then the lower and upper critical dimensions are given by*

$$\alpha = \liminf_{n\to\infty} -\frac{1}{n}\sum_{i=1}^{n} \log_2 \mu_i(x_i) = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} H(\mu_i)$$

*and*

$$\beta = \limsup_{n\to\infty} -\frac{1}{n}\sum_{i=1}^{n} \log_2 \mu_i(x_i) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} H(\mu_i)$$

*a.e., where $H(\mu_i) = -\sum_{j=0}^{1} \mu_i(j)\log_2(\mu_i(j))$, the entropy of the measure $\mu_i$.*

The entropy $H(\mu)$ of the measure $\mu$ on $\{0,1\}$ can be chosen to take any value between 0 and 1, by varying $p \in (0,1)$ where $\mu(0) = p$. It is clear that for many choices of product measure $\prod_{i=1}^{\infty} \mu_i$ the sequence $\frac{1}{n}\sum_{i=1}^{n} H(\mu_i)$ converges as $n \to \infty$. In this case the upper and lower critical dimensions are equal. Moreover any value in $(0,1)$ can be achieved by the dimensions.

A consequence of this theorem is that for an odometer action $T$ the inverse $T^{-1}$ has the same upper and lower critical dimensions as $T$. This follows from how $T^{-1}$ can also be considered as an odometer on the same space, with the roles of 0 and 1 reversed, and the fact that $H(\mu_i) = H(\nu_i)$ where $\nu_i(0) = 1 - \mu_i(0)$.

## 1.3 Groups beyond the integers

In this section we first examine one of the first non-singular amenable versions of a classical result: Ornstein and Weiss' Rokhlin lemma. We then start to look at the necessary steps in generalising the critical dimensions, and the non-singular ergodic theorem.

### 1.3.1 The Rohklin lemma

The classical Rokhlin lemma says that for a free m.p. $\mathbb{Z}$-action on a standard probability space, any $\epsilon > 0$ and $n \in \mathbb{N}$ you can find a set $B \subset X$ with $B, TB, ..., T^{n-1}B$ pairwise disjoint and $\mu\left(B \cup TB \cup ... \cup T^{n-1}B\right) > 1 - \epsilon$. We call the collection $B, TB, ..., T^{n-1}B$ a *tower* or *Rokhlin tower* with *base* $B$ and *levels* $T^iB$. Conceptually the lemma says that you cut your space into an arbitrarily large number of pieces, thought of as levels in a tower, which can be chosen to cover as much of the space as desired.

The construction of such towers is useful in many arguments in ergodic theory, and so the lemma has been generalised in numerous directions. For non-singular integer actions a very similar result can be proved, but for the lemma to be useful one must also show that given a fixed $m \in \mathbb{N}$ and $\delta > 0$ one can choose the base and $n$ much larger than $m$ to ensure that the top and bottom $m$ levels contain a small amount of mass, i.e.

$$\mu\left(\bigcup_{i=0}^{m-1} T^iB \cup \bigcup_{i=n-m}^{n-1} T^iB\right) < \delta.$$

As stated in the AMS bulletin [OW80] Ornstein and Weiss proved the Rokhlin lemma for non-singular amenable actions. They published the proof of the measure preserving case in [OW87]. This generalisation required one to use multiple towers of varying heights to cover the space.

The non-singular amenable Rokhlin lemma incorporates both the requirement for the boundary of towers to be small and the use of multiple, mostly disjoint, towers. However, as far as the author is aware, the proof of the non-singular amenable version has not been published. The result was announced in [OW80] but its main application, that equivalence relations induced by non-singular amenable group actions are hyperfinite, was then extended by Connes, Feldman and Weiss [CFW81] to show that any non-singular amenable equivalence relation is hyperfinite. As such, Ornstein and Weiss only published a full proof of the m.p. case in [OW87] to use in their amenable entropy theory. However, they did add a short note to suggest how one would go about generalising it. We include a full proof of their result developed from these remarks in Appendix B both for completeness and because the proof exhibits an interesting interaction between the group and its non-singular action.

### 1.3.2 Generalising the critical dimensions

In order to study either the critical dimensions or the ergodic theorem for non-singular actions of other groups we need to choose how to sum over the derivatives. In the case of the integers the sums ranged over the sets $\{0, 1, ..., n\}$. This choice was largely motivated by how, in general, the transformation $T$ need not be invertible, and in that context it is natural to sum over the forward orbits. However for other groups, and particularly when considering non-singular actions, there is not necessarily an obvious natural choice. In fact, as we will discuss below even the sets $\{0, 1, ..., n\}^d$ will be insufficient for our purposes when $G = \mathbb{Z}^d$.

With this in mind, let us make the following definition. A *summing sequence* $(B_n)_{n=1}^{\infty}$ in our group $G$ is a sequence of finite sets such that $e \in B_1 \subseteq B_2 \subseteq ... \subseteq G$. The assumption that the identity lies in each set will be convenient, and is not restrictive. It will usually be the case that $\bigcup_{n=1}^{\infty} B_n = G$, but we will say if and when this is actually used.

Clearly for any given $G$ there are many suitable choices for $(B_n)$, but regardless of that choice the definition of the critical dimensions has a natural generalisation. For $t \in \mathbb{R}$ write

$$L_t = \left\{ x \in X : \liminf_{n \to \infty} \frac{1}{|B_n|^t} \sum_{g \in B_n} \omega_g(x) > 0 \right\}$$

and

$$U_t = \left\{ x \in X : \limsup_{n \to \infty} \frac{1}{|B_n|^t} \sum_{g \in B_n} \omega_g(x) = 0 \right\}.$$

As before $L_t$ and $U_t$ are decreasing and increasing respectively with $t$, and are disjoint. It is no longer clear whether the sets are invariant under the action of $G$. We can, however, deduce the following.

**Lemma 1.3.1.** *Suppose that for every $h \in G$ and a.e. $x \in X$*

$$\liminf_{n\to\infty} \frac{\sum_{g\in B_n h} \omega_g(x)}{\sum_{g\in B_n} \omega_g(x)} > 0$$

*then the sets $L_t$ and $U_t$ are essentially invariant under the action of $G$.*

*Proof.* We first consider $L_t$. As $G$ is countable it is enough to fix $h \in G$ and show that $\mu(L_t \triangle hL_t) = 0$. Let $x \in L_t \cap \{0 < \omega_h < \infty\}$, which is all of $L_t$ but a set of measure 0. Then for almost every such $x$

$$\sum_{g\in B_n h} \omega_g(x) = \omega_h(x) \sum_{g\in B_n} \omega_g(hx)$$

and hence

$$\liminf_{n\to\infty} \frac{1}{|B_n|^t} \sum_{g\in B_n} \omega_g(hx) > 0 \qquad \Longleftrightarrow \qquad \liminf_{n\to\infty} \frac{1}{|B_n|^t} \sum_{g\in B_n h} \omega_g(x) > 0.$$

Since

$$\liminf_{n\to\infty} \frac{1}{|B_n|^t} \sum_{g\in B_n h} \omega_g(x) \geq \left(\liminf_{n\to\infty} \frac{1}{|B_n|^t} \sum_{g\in B_n} \omega_g(x)\right) \left(\liminf_{n\to\infty} \frac{\sum_{g\in B_n h} \omega_g(x)}{\sum_{g\in B_n} \omega_g(x)}\right)$$

our assumption on the limit infimum ensures that $x \in L_t \cap \{0 < \omega_h < \infty\}$ which implies $hx \in L_t$, and hence $\mu(hL_t \setminus L_t) = 0$. As we can use the same argument to deduce that $\mu(h^{-1}L_t \setminus L_t) = 0$ and the action is non-singular we are done.

For $U_t$ note that if $y = hx$ the for a.e. $x \in X$

$$\frac{\sum_{g\in B_n h} \omega_g(x)}{\sum_{g\in B_n} \omega_g(x)} = \frac{\sum_{g\in B_n} \omega_g(y)}{\sum_{g\in B_n h^{-1}} \omega_g(y)}$$

from which it follows that for a.e. $y \in X$ and $h \in G$

$$\limsup_{n\to\infty} \frac{\sum_{g\in B_n h^{-1}} \omega_g(y)}{\sum_{g\in B_n} \omega_g(x)} < \infty.$$

This can be used to show that $U_t$ is invariant using an argument symmetrical to that for the $L_t$. $\square$

This uncertainty over the invariance fortunately does not prevent us making the natural changes to the previous definitions.

**Definition 1.3.2.** The *lower critical dimension* of $(X, \mu, G)$ with respect to summing sequence $\mathcal{S} = (B_n)_{n=1}^{\infty}$ is defined by

$$\alpha = \alpha(\mathcal{S}) = \sup\{t : \mu(L_t) = 1\}.$$

The *upper critical dimension* of $(X, \mu, G)$ with respect to $\{B_n\}_{n=1}^{\infty}$ is defined by

$$\beta = \beta(\mathcal{S}) = \inf\{t : \mu(U_t) = 1\}.$$

When $\alpha$ and $\beta$ coincide we let $\gamma = \alpha = \beta$ and call it the *critical dimension*.

When the action is ergodic the formulae in Remark 1.2.3 extend just as one would expect from the integer case. The simple bounds on the dimensions and ergodic decomposition formulae in the form of Proposition 1.2.5 also extend to the dimensions as defined above.

However, the changes do raise two questions which motivate most of this work.

- How do the critical dimensions depend on the choice of summing sequence?
- How do we choose the summing sequence to ensure that critical dimension is an invariant of metric isomorphism?

The first question we will address in later chapters, when we consider actions of specific groups. The latter, as remarked in the previous section, is governed by whether the ergodic theorem holds.

### 1.3.3 The ergodic theorem

The ergodic theorem played a key role in the proof of Proposition 1.2.4, which showed that the critical dimension was invariant under metric isomorphism. In fact the rest of the proof easily translates from the integer case to that of other countable group actions. This means the invariance of the critical dimension for a given summing sequence follows from the highly non-trivial problem of whether the ergodic theorem holds for the said summing sequence for the actions under consideration.

Let $(X, \mathcal{B}, \mu, G)$ be a non-singular system, for each $g \in G$ we can define an isometric linear isomorphism from $L^1(\mu)$ to itself via $\hat{g}f(x) = f(gx)\omega_g(x)$. Note that this is not the usual transfer operator, which in the context of group actions is given by $\phi(g^{-1}x)d(\mu \circ g^{-1})/d\mu$, but fulfills essentially the same role and simplifies notation significantly.

**Definition 1.3.3.** Let $G$ be a countable group. Given a summing sequence $(B_n)$ we say that *the (pointwise) ergodic theorem is satisfied for the sequence* $(B_n)$ if for every non-singular system $(X, \mathcal{B}, \mu, G)$ and integrable function $f$, we have

$$\lim_{n\to\infty} \frac{\sum_{g\in B_n} \hat{g}f}{\sum_{g\in B_n} \hat{g}1} = \mathbb{E}(f|\mathcal{J})$$

almost everywhere, where $\mathcal{J}$ is the $\sigma$-algebra of $G$-invariant sets.

*Remark* 1.3.4. The convergence in the general case, as above, can also be deduced from the ergodic case using the decomposition theorem.

We have already discussed that Birkhoff's ergodic theorem ensures that when $G = \mathbb{Z}$ and $B_n = \{0, 1, ..., n\}$ the ergodic theorem holds for all measure preserving actions, and Hurewicz's ergodic theorem ensures the same for all non-singular conservative actions. Far more recently, Lindenstrauss showed that if $G$ is an amenable group and $B_n$ is a *tempered* Følner sequence, meaning for some constant $C > 0$ and for all $n \in \mathbb{N}$

$$\left| \bigcup_{k<n} B_k^{-1} B_k \right| \leq C|B_n|,$$

then the ergodic theorem is satisfied for $(B_n)$ for every measure preserving action of $G$ [Lin01]. Moreover, since every Følner sequence has a tempered subsequence, every amenable group has a sequence with this property.

This is a remarkably general result, and of particular interest because it cannot be replicated in the non-singular case. The group $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}$ of integer sequences with finite support is an abelian amenable group. In [Hoc13], Hochman shows that when for any choice of summing sequence $(B_n)$ in $G$ there is an ergodic measure preserving action on an infinite non-atomic measure space $(X, \mathcal{B}, \nu)$ and

$\varphi, \psi \in L^1(\nu)$ with $\int_X \psi \, d\nu \neq 0$ such that

$$\frac{\sum_{g \in B_n} \varphi(gx)}{\sum_{g \in B_n} \psi(gx)}$$

diverges almost everywhere. Any probability measure $\mu$ on $X$ which is equivalent to $\nu$ (one exists by Proposition 1.1.5) is quasi-invariant and ergodic with respect to the action, we have $\varphi \frac{d\nu}{d\mu}, \psi \frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$\begin{aligned}
\frac{\sum_{g \in B_n} \hat{g}[\varphi \frac{d\nu}{d\mu}]}{\sum_{g \in B_n} \hat{g}[\psi \frac{d\nu}{d\mu}]}(x) &= \frac{\sum_{g \in B_n} \varphi(gx) \frac{d\nu}{d\mu}(gx) \frac{d\mu \circ g}{d\mu}(x)}{\sum_{g \in B_n} \psi(gx) \frac{d\nu}{d\mu}(gx) \frac{d\mu \circ g}{d\mu}(x)} \\
&= \frac{\sum_{g \in B_n} \varphi(gx) \frac{d\nu \circ g}{d\nu}(x) \frac{d\nu}{d\mu}(x)}{\sum_{g \in B_n} \psi(gx) \frac{d\nu \circ g}{d\nu}(x) \frac{d\nu}{d\mu}(x)} \\
&= \frac{\sum_{g \in B_n} \varphi(gx)}{\sum_{g \in B_n} \psi(gx)}
\end{aligned}$$

so the left hand side also diverges almost everywhere. In particular, this means that the ergodic theorem must fail for this action, else this ratio would converge to

$$\frac{\int \varphi \frac{d\nu}{d\mu} \, d\mu}{\int \psi \frac{d\nu}{d\mu} \, d\mu} = \frac{\int \varphi \, d\nu}{\int \psi \, d\nu}$$

for almost every $x \in X$. Therefore the ergodic theorem cannot be extended to all amenable groups in the non-singular case. This suggests that the framework of amenable groups is not the most suitable for generalising the non-singular theory of integer actions.

There are also problems with natural summing sets in other, simpler, groups. There is an example due to Brunel and Krengel which shows that the ergodic theorem is not satisfied when $G = \mathbb{Z}^d$ and $B_n = \{0, 1, ..., n\}^d$ [Kre85]. Hochman has also shown that it fails when $G$ is the discrete Heisenberg group, the group of upper triangular $3 \times 3$ matrices with integer coefficients, and $(B_n)$ is any subsequence of $(B^n)$ where $B$ is a finite symmetric generating set for $G$ [Hoc13].

The non-singular ergodic theorem has, however, been extended beyond the integer case. Feldman proved that it is satisfied when $G = \mathbb{Z}^d$ and $B_n = \{-n, ..., n\}^d$, under the additional assumption that each of the standard generators $e_1, ..., e_d$ of $\mathbb{Z}^d$ act conservatively on the measure space [Fel07]. The foremost result in the area is also due to Hochman. He also worked with $\mathbb{Z}^d$, but showed that the theorem is satisfied with $B_n = \{u \in \mathbb{Z}^d : \|u\| \leq n\}$ where $\|\cdot\|$ is any norm on $\mathbb{Z}^d$ so long as the action is free, and without the conservativity assumption [Hoc10].

The techniques used to prove the latter result took advantage of the geometry of $\mathbb{R}^d$ equipped with a norm in order to counter the lack of control one has in non-singular systems, which manifests itself in the Radon-Nikodým derivatives. Looking at groups with similar properties may help establish a more effective setting in which to consider extending the non-singular theory.

## Layout of thesis

In Chapter 2 we identify a number of geometric properties which played a key role in the Hochman's proof of the ergodic theorem for $\mathbb{Z}^d$, and use extensions of the techniques used in the proof to show that the theorem holds for any such group equipped with a sufficiently well behaved metric.

In Chapter 3 we apply this result to countable abelian groups. In particular we find further summing sequences for which the ergodic theorem holds. We then examine the critical dimensions of product actions of $\mathbb{Z}^d$ for these actions.

In Chapter 4 show that the discrete Heisenberg groups also fit into the framework established in Chapter 2, and so have summing sequences satisfying the ergodic theorem.

In Chapter 5 we calculate the critical dimensions for a natural action of the Lamplighter group, for a special class of measures on the underlying space. We will see that there a similarities between this action and integer odometer action.

We finish off the thesis with two appendices, the first containing the details on how to extend the ergodic decomposition theorem from the integer to countable group case and the second a proof of Ornstein and Weiss's non-singular Rohklin lemma.

# Chapter 2

# Ergodic Theorem

In this chapter we will show that, under suitable assumptions on the geometry of the group $G$, there is a summing sequence for which the non-singular ergodic theorem must hold. In Chapters 3 and 4 respectively we will apply this general ergodic theorem to the subgroups of $\mathbb{Q}^d$, in addition to some other abelian groups, and the discrete Heisenberg groups. An earlier version of the work in this chapter, along with the work in Chapter 4, have been submitted to a journal and is currently under consideration.

Throughout we will be working with a countable discrete group $G$ which is a subgroup of a larger group $\tilde{G}$ equipped with a right invariant metric $\rho$ with good geometric properties. We will also denote the restriction of $\rho$ to $G$ by $\rho$ and call $(\tilde{G}, \rho)$ an *extension* of $G$ in this situation. Our intention is that $\tilde{G}$ will be to $G$ what $\mathbb{R}^d$ is to $\mathbb{Z}^d$ or $\mathbb{Q}^d$, a natural 'continuous' group in which $G$ resides. Many of the properties we require of $G$ will either be described in terms of $\tilde{G}$ or (in some examples) naturally inherited from it.

To make use of the metric geometry of $\tilde{G}$ we would like to define the summing sequence $B_n$ in terms of the metric, as Hochman did with norms on $\mathbb{Z}^d$. However, this approach needs some refinement as, for example, if we take $G = \mathbb{Q}$ non-trival balls of norms contain infinitely many points. To handle this issue we will also assume that $G = \bigcup_{k=1}^{\infty} G_k$ where $G_1 \subset G_2 \subset ...$ is an increasing sequence of subgroups of $G$ such that $|G_k \cap B_n(e)| < \infty$ for all $k, n \in \mathbb{N}$, where $B_n(e)$ is the closed $\rho$-ball of radius $n$ about the identity in $\tilde{G}$. In this situation we say that $G$ has *finite levels* with respect to $(\tilde{G}, \rho)$. This assumption means we may define our summing sequence via $B_n = G_n \cap B_n(e)$, a sequence of finite sets which makes use of the metric geometry of $\tilde{G}$. In the case of $\mathbb{Q}$, for example, we will end up taking $G_k = (k!)^{-1}\mathbb{Z} = \{q \in \mathbb{Q} : k!\, q \in \mathbb{Z}\}$. For group metric spaces without cluster points, such as $\mathbb{Z}^d$ or the discrete Heisenberg group, we can just take every $G_k$ to be the group itself.

The overall structure of the proof is standard. First one shows that there is a dense subset of elements $f \in L^1$ for which the ergodic theorem holds, so

$$\lim_{n\to\infty} \frac{\sum_{g\in B_n} \hat{g} f}{\sum_{g\in B_n} \hat{g} 1} = \mathbb{E}(f|\mathcal{J})$$

almost everywhere, and then one extends this convergence to the whole of $L^1$ by using a maximal inequality. We draw on the ideas of [Aar97] for its elegant exposition of the overall method for non-singular $\mathbb{Z}$-actions, and [Fel07] for its simple proof of a suitable maximal inequality, and adapt these to our setting. The approach to the parts particularly specialised to the non-singular setting are significantly developed from Hochman's work with $\mathbb{Z}^d$ in [Hoc10]. We also include a new proof of the main technical result in that paper, and extend it to this wider context. A version of this work has been submitted to a journal and a preprint can be found at [Jar17].

## 2.1 The dense subset

The candidate for the dense subset is

$$S = \operatorname{span}\{c + h - \hat{\sigma} h : c \in I, \sigma \in G, h \in L^\infty\}$$

where $I$ is the set of $G$ invariant $L^1$-functions on $X$. A function of the form $h - \hat{\sigma} h$, as above, is called a coboundary. This is the natural analogue the set used in the traditional approach, and can be seen to be dense using the same techniques.

**Proposition 2.1.1.** *$S$ is a dense subset of $L^1$.*

*Proof.* We make use of the following corollary of the Hahn-Banach theorem, see Theorem 5.21 [RY08] for essentially the same result. If $V$ is a normed vector space, $W$ is a closed subspace and $v_0 \in V \setminus W$ then there exists an $F \in V^*$ such that $F = 0$ on $W$ and $F(v_0) = 1$.

Let us take $V = L^1(\mu)$ and $W = \bar{S}$, and assume for a contradiction that $L^1 \setminus \bar{S} \neq \emptyset$. Since $\mu$ is a probability measure, so $\sigma$-finite, the map taking $\psi \in L^\infty$ to the $L^1$-functional given by $f \mapsto \int f\psi \, d\mu$ is an isometric isomorphism. This means, by the corollary, there is some non-zero $\phi \in L^\infty$ such that $\int f\phi \, d\mu = 0$ for all $f \in S$.

In particular, by considering $f = h - \hat{\sigma}h$ for all $h \in L^\infty$ and $\sigma \in G$

$$\int \phi \, h \, d\mu = \int \phi \, \hat{\sigma}h \, d\mu = \int (\phi \circ \sigma) h \, d\mu$$

and hence for all $\sigma \in G$ we have $\phi = \phi \circ \sigma$ almost everywhere, i.e. $\phi \in I$. By taking $f = \phi \in S$ we then see that $\int \phi^2 \, d\mu = 0$ and hence $\phi = 0$ almost everywhere, which is a contradiction. □

Having seen that $S$ is dense, we now wish to show that the ergodic theorem holds on $S$. First consider functions $f = c + h - \hat{\sigma}h$ for some $c \in I$, $\sigma \in G$ and $h \in L^\infty$. Given an invariant set $A$ we have $\int_A \hat{\sigma}h \, d\mu = \int_A h \, d\mu$, and it follows that for every invariant set $\int_A f \, d\mu = \int_A c \, d\mu$. Therefore $c = \mathbb{E}(f|\mathcal{J})$ almost everywhere. In addition,

$$\frac{\sum_{g \in B_n} \hat{g}f}{\sum_{g \in B_n} \hat{g}1} = c + \frac{\sum_{g \in B_n \setminus \sigma B_n} \hat{g}h - \sum_{g \in \sigma B_n \setminus B_n} \hat{g}h}{\sum_{g \in B_n} \hat{g}1}$$

a.e. since $c$ is invariant and $\hat{g}(\hat{\sigma}h) = \widehat{\sigma g}h$. It follows that the ergodic theorem will hold for any $f$ of the above form so long as

$$\frac{\sum_{g \in B_n \setminus \sigma B_n} \hat{g}h - \sum_{g \in \sigma B_n \setminus B_n} \hat{g}h}{\sum_{g \in B_n} \hat{g}1} \to 0 \tag{2.1.1}$$

a.e. for all $\sigma \in G$ and $h \in L^\infty$. The required convergence follows for every other element of $S$ by linearity.

*Remark* 2.1.2. Observe that if (2.1.1) holds then

$$\lim_{n \to \infty} \frac{\sum_{g \in \sigma B_n} \omega_g}{\sum_{g \in B_n} \omega_g} = 1$$

a.e. for all $\sigma \in G$. Moreover, if $\sigma B_n = B_n \sigma$ for all $n$ and $\sigma$ then, by Lemma 1.3.1 the sets $U_t$ and $L_t$ used to the define the critical dimensions are invariant.

In practice, we are going to prove (2.1.1) by showing the actions under consideration satisfy the *non-singular Følner condition* (nsFC), i.e. for all $\sigma \in G$

$$\frac{\sum_{g \in B_n \triangle \sigma B_n} \omega_g}{\sum_{g \in B_n} \omega_g} \to 0 \qquad \text{a.e.,} \tag{2.1.2}$$

and using the boundedness of $h$. In the measure preserving amenable case (nsFC) reduces to the normal Følner condition from Definition 1.1.1. In contrast, in the proof for non-singular actions this condition takes a significant amount of work, and is the primary place we will use the geometric properties of the group.

*Remark* 2.1.3. Due to the above remark, if a summing sequence in an abelian group satisfies (nsFC) then the corresponding sets $U_t$ and $L_t$ are invariant.

Despite the Følner condition no longer being sufficient for the ergodic theorem in the non-singular case, Følner sequences are often still good candidates for summing sequences when trying to prove ergodic theorems in amenable groups. This is because it can still be beneficial for the number of terms in the numerator of (nsFC) to be small relative to the number in the denominator, particularly when one has control over the Radon-Nikodým derivatives.

### Well-separability and a crucial lemma

Though the summing sets are all subsets of $G$, most of the geometric structures we use will be obtained from the group $\tilde{G}$. Within $G$ and its subgroups, the $G_k$, this is done in a way which respects the group structure.

Now let us define some notation. Given $r > 0$ and $k \in \mathbb{N}$ let $B_r(e) = \{g \in \tilde{G} : \rho(g, e) \leq r\}$, $B_r^{(k)} = G_k \cap B_r(e)$, and $B_r^{(\infty)} = G \cap B_r(e)$. Then, with these definitions:

- The summing sequence $(B_n)$ is exactly the sequence $(B_n^{(n)})$.
- By the right invariance of $\rho$, for any $g \in G$ the right translate $B_r^{(k)} g$ is exactly $G_k g \cap B_r(g)$, the collection of points $h$ in $G$ within distance $r$ of $g$ for which $hg^{-1} \in G_k$.
- $B_r^{(k)} g$ is not, in general, equal to $G_k \cap B_r(g)$. Due to this we will not use the notation $B_r^{(k)}(g)$, to avoid confusion with $B_r^{(k)} g$.

- However for all $k$ large enough to ensure $g \in G_k$ we do have $B_r^{(k)}g = G_k \cap B_r(g)$.

- Note that $B_r^{(\infty)}g$ is exactly the restriction of the ball $B_r(g)$ to $G$.

We will call $B_r^{(k)}g$ the *levelled ball* with *radius* $r$, *centre* $g$ and *level* $k$. We assume that each levelled ball carries the information of its centre, radius and level along with it. In particular, given a levelled ball $B_r^{(k)}g$ this allows us to reconstruct the ball $B_r(g)$ in $\tilde{G}$ which contains it.

Due to our assumption on defining information, given a collection $\mathcal{V} = \{B_r^{(k)}g\}$ of levelled balls we can recover a corresponding collection $\tilde{\mathcal{V}} = \{B_r(g)\}$ of balls in $\tilde{G}$. In analogy with [Hoc10] we let $\operatorname{rmax}\mathcal{V}$ and $\operatorname{rmin}\mathcal{V}$ denote the maximum and minimum radii of the elements of $\mathcal{V}$, respectively. We will use the same notation for collections of balls in $\tilde{G}$. We say the collection $\mathcal{V}$ is *well-separated* if the distance between each of the balls in $\tilde{\mathcal{V}}$ is at least $\operatorname{rmin}\mathcal{V} = \operatorname{rmin}\tilde{\mathcal{V}}$, and use the same terminology for collections of standard balls in $\tilde{G}$.

Again in anology with [Hoc10] given a finite set $E \subset \tilde{G}$ a *carpet* over $E$ is a collection $\{B_{r(g)}(g) : g \in E\}$ of balls with their centres in $E$. Similarly, for a finite $E \subset G$ a *levelled carpet* over $E$ is a collection of levelled balls $\mathcal{U} = \{B_{r(g)}^{(k(g))}g : g \in E\}$ centred in $E$. A (*levelled*) *stack* of height $p$ over $E$ is a sequence of (levelled) carpets $\mathcal{U}_1, ..., \mathcal{U}_p$ over $E$. The first geometrical property we will require of the group metric space is the following.

**Definition 2.1.4.** $(\tilde{G}, \rho)$ is *well-separable* if there exists $\chi \in \mathbb{N}$ such that for every finite set $E \subset \tilde{G}$ and carpet $\mathcal{U}$ over $E$ there is a subcollection $\mathcal{V}$ of $\mathcal{U}$ which covers $E$ and can be partitioned into $\chi$ well-separated subcollections.

The motivation for this definition is that we immediately get the following statement, essentially by the pigeonhole principle.

**Lemma 2.1.5.** *Suppose that $(\tilde{G}, \rho)$ is well-separable with constant $\chi$ and assume there is a given finite measure $\nu$ supported in a finite set $E \subset G$. Then given any levelled carpet $\mathcal{U}$ over $E$ there is a subset of $\mathcal{V} \subset \mathcal{U}$ for which $\tilde{\mathcal{V}}$ is well-separated and satisifies $\nu\left(\bigcup \mathcal{U}\right) \geq (1/\chi)\nu(E)$.*

Now we define a concept of levelled thickened boundaries developed from the one given in [Hoc10], but which coincides in the context of that paper, i.e. $\mathbb{Z}^d$

with balls of norms. Given a ball $B_r(g)$ in $\tilde{G}$ the *t-boundary* $\partial_t B_r(g)$, where $t \geq 0$, is defined by

$$\partial_t B_r(g) = \{\tilde{g} \in \tilde{G} : \rho(\tilde{g}, \partial B_r(g)) \leq t\}.$$

where $\partial B_r(g) = \{\tilde{g} \in \tilde{G} : \tilde{d}(g, \tilde{g}) = r\}$ and is assumed to be non-empty. Given a levelled ball $B = B_r^{(k)} g$ its *levelled t-boundary* $\partial_t B$ is the set $(\partial_t B_r^{(k)})g$ where $\partial_t B_r^{(k)} = G_k \cap \partial_t B_r(e)$. As with balls, we assume levelled boundaries carry their defining information with them and for any $g \in G$ we have $\partial_t B_r^{(k)}\, g = G_k g \cap \partial_t B_r(g) \subset \partial_t B_r(g)$, and for all $k$ sufficiently large $\partial_t B_r^{(k)}\, g = G_k \cap \partial_t B_r(g)$.

Given a collection $\mathcal{V}$ of balls in $\tilde{G}$ we let $\partial \mathcal{V} = \{\partial B_r(x) : B_r(x) \in \mathcal{V}\}$, a collection of boundaries in $\tilde{G}$. We also call the collection $\partial\mathcal{V}$ *well-separated* if the distance between each of the boundaries in $\partial\mathcal{V}$ is at least $\operatorname{rmin}\mathcal{V}$. The distinction here to the case of balls is that some boundaries in $\partial\mathcal{V}$ may lie inside the balls corresponding to distinct boundaries in $\mathcal{V}$.

We can now prove the following crucial lemma.

**Lemma 2.1.6.** *Let $(\tilde{G}, \rho)$ be well-separable with constant $\chi$. Let $\epsilon, \delta \in (0,1)$, $t \geq 0$ and $p = \left\lceil \frac{2\chi}{\epsilon\delta} \right\rceil$. Suppose that*

(a) *$\nu$ is a finite measure on $G$,*

(b) *$F \subseteq G$ is finite and $\nu(F) > \delta\nu(G)$,*

(c) *$\mathcal{U}_1, ..., \mathcal{U}_p$ is a levelled stack over $F$ with*

$$\operatorname{rmin}\mathcal{U}_i > 4\operatorname{rmax}\mathcal{U}_{i-1} \qquad \text{and} \qquad \operatorname{rmin}\mathcal{U}_1 > 2t,$$

*and*

(d) *$\nu(\partial_t B) > \epsilon\,\nu(B)$ for each $B \in \bigcup_i \mathcal{U}_i$*

*then there is some integer $k \geq 2$ and a subcollection $\mathcal{V} \subseteq \bigcup_{i \geq k} \mathcal{U}_i$ such that*

(i) *for the corresponding collection $\tilde{\mathcal{V}}$ of balls in $\tilde{G}$ the collection $\partial\tilde{\mathcal{V}}$ of boundaries is well-separated and*

(ii) *$\nu\left(F \cap \bigcup_{B \in \mathcal{V}} \partial_{2r} B\right) > \frac{1}{2}\,\nu(F)$, where $r = \operatorname{rmax}\mathcal{U}_{k-1}$.*

*Proof.* We replicate [Hoc10] for levelled stacks. We can assume that $\nu(G) = 1$. We work up recursively from $l = 0$ and in stage $l$ we produce a collection $\mathcal{V} \subseteq \bigcup_{i>p-l} \mathcal{U}_i$ with $\partial\tilde{\mathcal{V}}$ well-separated and

$$\nu\left(\bigcup_{B\in\mathcal{V}} \partial_{2r(l)} B\right) \geq \nu\left(\bigcup_{B\in\mathcal{V}} \partial_t B\right) \geq \frac{\epsilon\delta}{2\chi} l$$

where $r(l) = \mathrm{rmax}\,\mathcal{U}_{p-l}$.

For $l = 0$ take $\mathcal{V} = \emptyset$. Assume we have completed stage $l$. If

$$\nu\left(F \cap \bigcup_{B\in\mathcal{V}} \partial_{2r(l)} B\right) > \frac{1}{2}\,\nu(F)$$

then take $k = p - l + 1$ and we are done. Otherwise let $E = F \setminus \bigcup_{B\in\mathcal{V}} \partial_{2r(l)} B$ and note that $\nu(E) \geq \frac{1}{2}\nu(F) \geq \frac{\delta}{2}$. By the previous lemma we may choose a well-separated subcollection of levelled balls $\mathcal{U}' \subseteq \mathcal{U}_{p-l}$ centred in $E$ with $\nu\left(\mathcal{U}'\right) > \frac{\delta}{2\chi}$. As $2t < r(l)$ it follows from (d) that

$$\nu\left(\bigcup_{B\in\mathcal{U}'} \partial_t B\right) > \epsilon\,\nu\left(\bigcup_{B\in\mathcal{U}'} B\right) \geq \frac{\epsilon\delta}{2\chi}.$$

The centres of each $B \in \mathcal{U}'$ are strictly more than $2\,r(l)$ from each element of $\partial\tilde{\mathcal{V}}$. As the radius of each element of $\mathcal{U}'$ is at most $r(l)$ this ensures $\partial\tilde{\mathcal{V}} \cup \partial\tilde{\mathcal{U}'}$ is well-separated. Since also $r(l) > 4\,r(l+1)$ the collection of thickenings $\{\partial_{2r(l+1)} B : B \in \mathcal{V} \cup \mathcal{U}'\}$ is disjoint and hence

$$\begin{aligned}
\nu\left(\bigcup_{B\in\mathcal{V}\cup\mathcal{U}'} \partial_{2r(l+1)} B\right) &= \nu\left(\bigcup_{B\in\mathcal{V}} \partial_{2r(l+1)} B\right) + \nu\left(\bigcup_{B\in\mathcal{U}'} \partial_{2r(l+1)} B\right) \\
&\geq \nu\left(\bigcup_{B\in\mathcal{V}} \partial_t B\right) + \nu\left(\bigcup_{B\in\mathcal{U}'} \partial_t B\right) \\
&> \frac{\epsilon\delta}{2\chi}(l+1)
\end{aligned}$$

and so we can complete the recursive step by adding $\mathcal{U}'$ to $\mathcal{V}$.

The process must terminate by stage $l = p - 1$, ensuring $k \geq 2$. If it does not then we can complete stage $p-1$. Then, as above, we see that

$$\nu\left(\bigcup_{B\in\mathcal{V}\cup\mathcal{U}'}\partial_t B\right) > \frac{\epsilon\delta}{2\chi}p \geq 1.$$

$\square$

**Finite intersection dimension and the technical theorem**

The reason Lemma 2.1.6 is crucial is that we are going to repeatedly apply it to produce a series of collections of thickened boundaries, each containing a not insignificant portion of a finite set $F$, and then seek to apply the following property.

**Definition 2.1.7.** We say $(\tilde{G},\rho)$ has *finite intersection dimension* if there is a positive integer $\kappa$ and an $R>1$ such that given

(a) $t(1),...,t(\kappa)\geq 1$,

(b) $r(1),...,r(\kappa)$ such that each $r(i)\geq t(1)...t(i)R$ and

(c) elements $g_1,...,g_\kappa\in\tilde{G}$ such that $g_i\in\bigcap_{j<i}\partial_{t(j)}B_{r(j)}(g_j)$ for all $i\leq\kappa$

then $\bigcap_{i=1}^{\kappa}\partial_{t(i)}B_{r(i)}(g_i)=\emptyset$. In this case, we say that $(\tilde{G},\rho)$ has *intersection dimension $\kappa$ at scale $R$.*

It is important to note here that the intersection dimension of a space is a minor reformulation of the quantity called the coarse dimension defined by Hochman in [Hoc10], it uses a different notion of boundary. The two quantities are in fact the same when considering norms on $\mathbb{Z}^d$. Our reason for instead calling the quantity the 'intersection dimension' is simply to avoid potential confusion with another quantity from the field of coarse geometry which is also called the coarse dimension (see e.g. [BD08]). There is no clear connection between the two quantities.

We can now give a new proof of Theorem 4.4 in [Hoc10], also extended to allow for levels, with a slight improvement in the bound for the height of the stack required.

**Theorem 2.1.8.** *Let $G$ have has finite levels $(G_k)$ with respect to extension $(\tilde{G}, \rho)$. Assume $(\tilde{G}, \rho)$ is well-separable with constant $\chi$ and has intersection dimension $\kappa \in \mathbb{N}_0$ at scales $R > 1$. Let $0 < \epsilon, \delta < 1$. Then the following holds for some positive integer $q \leq \kappa \left(\frac{4\sqrt{2}\chi}{\epsilon\delta}\right)^{\kappa} (\sqrt{2})^{\kappa^2}$. Suppose that*

(a) *$\nu$ is a finite measure on $G$,*

(b) *$F \subseteq G$ is finite,*

(c) *$\mathcal{U}_1, ..., \mathcal{U}_q$ is a levelled stack over $F$ with*

  (1) $\mathrm{rmin}\,\mathcal{U}_i > 2(\mathrm{rmax}\,\mathcal{U}_{i-1})^2$,

  (2) $\mathrm{rmin}\,\mathcal{U}_1 > 7 \max(t, R)$,

(d) *$\nu(\partial_t B) > \epsilon\, \nu(B)$ for each $B \in \bigcup_i \mathcal{U}_i$.*

*Then $\nu(F) \leq \delta\, \nu(G)$.*

*Proof.* Suppose for a contradiction that $\nu(F) > \delta\nu(M)$. Let $F_0 = F$, $p_i = \left\lceil \frac{2^{i+1}\chi}{\epsilon\delta} \right\rceil$, $q_\kappa = 0$ and set $q_i = p_i(1 + q_{i+1})$ for each $0 \leq i \leq \kappa - 1$. In particular $q = q_0$.

The idea behind this proof is to first apply Lemma 2.1.6 to find a collection of (levelled thickened) boundaries containing at least a half of the mass of $F$. We will then do this again with the portion of $F$ inside the first collection of boundaries to produce a second collection, with centres inside boundaries from the previous one, that contains at least a quarter of the mass in $F$. We will continue in this fashion until we have $\kappa$ such collections, with the last containing at least $2^{-\kappa}$ fraction of the mass of $F$. By taking care to control the radii of the boundaries at each stage we will ensure that any point in this portion of $F$ must lie in a sequence of $\kappa$ thickened boundaries whose associated boundaries in $\tilde{G}$ satisfying the definition of the intersection dimension, forcing a contradiction since the intersection of any such sequence must be empty.

More precisely, we construct a sequence of sets $F = F_0 \supset F_1 \supset ... \supset F_\kappa$ and select positive integers $n_1, ..., n_\kappa$ such that $1 \leq n_i \leq p_{i-1} - 1$ where for each $1 \leq i \leq \kappa$ we have $\nu(F_i) \geq \frac{1}{2}\nu(F_{i-1})$ and

$$F_i = F_{i-1} \cap \bigcup_{B \in \mathcal{V}_i} \partial_{t(i)} B$$

with $\mathcal{V}_i$ being a subcollection of levelled balls centred in $F_{i-1}$ from the levelled stack $\mathcal{U}_{N_i+q_i+1}, ..., \mathcal{U}_{N_{i-1}+q_{i-1}}$ and for which $\partial\tilde{\mathcal{V}}_i$ is well-separated. Here $N_i = \sum_{j=1}^{i} n_j(1+q_j)$ and $t(i) = 2\,\mathrm{rmax}\,\mathcal{U}_{N_i}$. Note that our assumptions ensure that $N_{i-1}+1 \leq N_i \leq N_i + q_i + 1 \leq N_{i-1} + q_{i-1}$. In particular if $i < j$ then $N_j + q_j \leq N_i + q_i$.

**How the sequences force a contradiction:**

From these conditions we are able to deduce that

$$\nu(F_\kappa) \geq \frac{1}{2^\kappa}\nu(F) \geq \frac{\delta}{2^\kappa}\nu(G) > 0$$

and so in particular $F_\kappa$ is non-empty.

Let

$$g \in F_\kappa = F \cap \bigcap_{i=1}^{\kappa} \bigcup_{B\in\mathcal{V}_i} \partial_{t(i)}B.$$

By definition of $F_\kappa$, and using that a levelled boundary is contained by its counterpart in $\tilde{G}$, there exist $g_1, ..., g_\kappa$ and $r(1), ..., r(\kappa)$ such that each $g_i \in F_{i-1}$ and $g \in \bigcap_{i=1}^{\kappa} \partial_{t(i)}B_{r(i)}(g_i)$. Suppose $i < j$ then $g_j \in F_{j-1} \subseteq F_i \subseteq \bigcup_{B\in\mathcal{V}_i} \partial_{t(i)}B$. Since the collection $\partial\tilde{\mathcal{V}}_i$ is well-separated its elements are a distance at least

$$\begin{aligned}\mathrm{rmin}\,\mathcal{V}_i \geq \mathrm{rmin}\,\mathcal{U}_{N_i+q_i+1} &> (\mathrm{rmax}\,\mathcal{U}_{N_i+q_i})^2\\ &> 7\,\mathrm{rmax}\,\mathcal{U}_{N_i+q_i}\\ &> 2t(i) + t(j) + \mathrm{rmax}\,\mathcal{U}_{N_{j-1}+q_{j-1}}\end{aligned}$$

apart, where we have applied properties (1) and (2). Since $g_j$ lies within distance $t(i)$ of some element of $\partial\tilde{\mathcal{V}}_i$ (which is well separated) this inequality means the ball of radius $t(j) + \mathrm{rmax}\,\mathcal{U}_{N_{j-1}+q_{j-1}}$ about $g_j$ can intersect at most one of the thickened boundaries $\{\partial_{t(i)}B : B \in \tilde{\mathcal{V}}_i\}$. Since $\mathrm{rmax}\,\mathcal{V}_j \leq \mathrm{rmax}\,\mathcal{U}_{N_{j-1}+q_{j-1}}$ and $\partial_{t(i)}B_{r(i)}(g_i) \cap \partial_{t(j)}B_{r(j)}(g_j)$ is non-empty we see that this sphere is $\partial_{t(i)}B_{r(i)}(g_i)$ and hence $g_j \in \partial_{t(i)}B_{r(i)}(g_i)$. Next note that given $1 \leq i \leq \kappa$

$$r(i) \geq \mathrm{rmin}\,\mathcal{U}_{N_i+q_i+1} > \mathrm{rmin}\,\mathcal{U}_{N_i+1} > 2(\mathrm{rmax}\,\mathcal{U}_{N_i})^2 \geq t(i)\,\mathrm{rmax}\,\mathcal{U}_{N_{i-1}+1}$$

and by recursion

$$\begin{aligned} r(i) &> t(i)t(i-1)...t(2)\,\mathrm{rmax}\,\mathcal{U}_{N_1+1} \\ &> t(i)t(i-1)...t(2)t(1)\,\mathrm{rmax}\,\mathcal{U}_{N_1} \\ &> t(i)t(i-1)...t(2)t(1)R. \end{aligned}$$

This means that the $g_i$, $r(i)$ and $t(i)$ satisfy the conditions in the definition of the intersection dimension and so $\bigcap_{i=1}^{\kappa} \partial_{t(i)} B_{r(i)}(g_i) = \emptyset$, a contradiction.

**Constructing the sequences:**

All that remains is to show such collections $\mathcal{V}_i$ and integers $n_i$ exist, and for this we will use Lemma 2.1.6. Given these up to a certain $0 \leq i \leq \kappa - 1$ we produce the $i+1$ set as follows: consider the (levelled) stack $\mathcal{U}_{N_i+1}, ..., \mathcal{U}_{N_i+q_i}$ over $F$. This can clearly be restricted to a stack $\mathcal{U}'_{N_i+1}, ..., \mathcal{U}'_{N_i+q_i}$ over $F_i$ by simply taking the balls with centres in $F_i$, and it inherits all the radii growth conditions from the original stack. In particular, we may apply Lemma 2.1.6 to the stack $\{\mathcal{U}'_{N_i+j(1+q_{i+1})}\}_{j=1}^{p_i}$ to find $1 \leq n \leq p_i - 1$ and find a subcollection $\mathcal{V} \subseteq \bigcup_{n+1 \leq j \leq p_i} \mathcal{U}_{N_i+j(1+q_{i+1})}$ for which $\partial\tilde{\mathcal{V}}$ is well separated and (if we take $n_{i+1} = n$)

$$\nu\left(F_i \cap \bigcup_{B \in \mathcal{V}} \partial_{t(i+1)} B\right) > \frac{1}{2}\nu(F_i)$$

since $N_{i+1} = N_i + n_{i+1}(1 + q_{i+1})$. By noting the range of $j$, we see that $\mathcal{V}$ consists of balls from the stack $\mathcal{U}_{N_{i+1}+q_{i+1}+1}, ..., \mathcal{U}_{N_i+q_i}$, and so we may take $\mathcal{V}_{i+1} = \mathcal{V}$.

**Proving the bound on $q$:**

To get the bound on $q$ observe that by unravelling the recursive definition $q_i = p_i(1 + q_{i+1})$ for each $0 \leq i \leq \kappa - 1$ and recalling that $p_i = \left\lceil \frac{2^{i+1}\chi}{\epsilon\delta} \right\rceil \leq \frac{4\chi}{\epsilon\delta} 2^i$ we see that

$$\begin{aligned} q = q_0 = p_0(1 + q_1) = p_0(1 + p_1(1 + ...(1 + p_{\kappa-1})...) &= \sum_{i=0}^{\kappa-1} \prod_{j=0}^{i} p_j \\ &\leq \sum_{i=0}^{\kappa-1} \left(\frac{4\chi}{\epsilon\delta}\right)^i 2^{\sum_{j=0}^{i} j} \leq \kappa \left(\frac{4\sqrt{2}\chi}{\epsilon\delta}\right)^{\kappa-1} \sqrt{2}^{(\kappa-1)^2} \end{aligned}$$

which completes the proof. □

With this result in hand we are now ready to show that for any $\sigma \in G$

$$\frac{\sum_{g \in B_n \triangle \sigma B_n} \omega_g}{\sum_{g \in B_n} \omega_g} \to 0 \qquad \text{a.s.}$$

under a some further assumptions on the metric and group structures.

### Doubling properties

A property which is frequently used in the proofs of ergodic theorems is the metric doubling property.

**Definition 2.1.9.** A metric space $(M, \rho)$ has the *metric doubling property* if there is a constant $D$ such that any open ball of radius $r$ and be covered by $D$ balls of radius $r/2$.

This is used to prove a doubling condition for (right) invariant group metric spaces. More specifically, let $(G, \rho)$ be a group equipped with a right invariant metric which satisfies the metric doubling property, and for which every closed ball contains finitely many points. Then the closed ball $B_{2n}(e)$ can be covered by a collection of $D$ closed balls with radius $n$. By the right invariance, each such ball has size $|B_n(e)|$. It follows that

$$|B_{2n}(e)| \le D|B_n(e)|,$$

which is a doubling condition. In this context, the summing sequence is exactly $\{B_n(e)\}_{n=1}^{\infty}$

In this work we use a modified version of this condition. As we will see below, the standard steps in proof of the ergodic theorem can be adapted to see that all one requires is the following property on the summing sequence.

**Definition 2.1.10.** Given a sequence $e \in B_1 \subset B_2 \subset ...$ of finite subsets of a countable group $G$ we say that it has the *multiplicative doubling property* (MDP) if there exists constants $D > 0$ and $N \in \mathbb{N}$ such that $|B_n B_n| \le D|B_n|$ for all $n \ge N$.

When $(G, \rho)$ is as described above the levels can all be taken to be $G$ itself, the induced summing sequence is $(B_n(e))$ and the MDP follows from the previous doubling condition because

$$B_n(e)B_n(e) \subseteq B_{2n}(e).$$

The benefits of instead using the mulplicative doubling property are twofold. Firstly it is a property we can demand of the summing sequence in the group, which is useful as our summing sets are not (in general) balls of the metric restricted to $G$. Secondly, and more crucially, in Section 3.1.2 we will consider metrics on $\mathbb{Z}^d$ for which the summing sequence $(B_n(e))$ do not satisfy the previous doubling condition but do satisfy the multiplicative doubling property. An example of such a sequence on $\mathbb{Z}^2$ is given by

$$\{-n, ..., n\} \times \{-\lfloor e^n - 1\rfloor, ..., \lfloor e^n - 1\rfloor\}.$$

Such sequences will assist us in our study of the critical dimensions.

### Voidlessness and the non-singular Følner condition

We say that a metric space $(M, \rho)$ is *voidless* if for all $x \in M$ and $r > 0$, every closed ball $B \subset M$ such that

$$B \cap \{y : \rho(x, y) < r\} \neq \emptyset \qquad \text{and} \qquad B \cap \{y : \rho(x, y) > r\} \neq \emptyset$$

satisfies $B \cap \{y : \rho(x, y) = r\} \neq \emptyset$.

*Remark* 2.1.11. If a metric space is such that every closed ball is path connected then the intermediate value theorem ensures it is voidless.

**Lemma 2.1.12.** *Suppose that $(\tilde{G}, \rho)$ is voidless. Then for all $\sigma \in G_k$ and $n \geq k$ we have we have $B_n \triangle \sigma B_n \subseteq \partial_t B_n$ where $t = \rho(\sigma, e)$.*

*Proof.* Let $g \in G$, then $\rho(\sigma^{-1}g, g) = \rho(\sigma^{-1}, e) = \rho(\sigma, e)$ i.e. $\sigma^{-1}g \in B_t(g)$. Now let $g \in G_n$ with $n \geq k$. Suppose $g \notin \partial_t B_n = G_n \cap \partial_t B_n(e)$, then $\rho(g, \partial B_n(e)) > t$. Hence $B_t(g)$ does not intersect $\partial B_n(e)$. Since $\tilde{G}$ is voidless it follows that either $B_t(g) \subseteq B_n(e)$ or $B_t(g) \subseteq B_n(e)^c$. Therefore, if $g \in B_n(e)$ then $\sigma^{-1}g \in B_n(e)$ and if $g \in B_n(e)^c$ then $\sigma^{-1}g \in B_n(e)^c$. Since $\sigma \in G_n$ this means either $g \in B_n \cap \sigma B_n$ or $g \notin B_n \cup \sigma B_n$, i.e. $g \notin \partial_t B_n$ implies $g \notin B_n \triangle \sigma B_n$. $\square$

So, under the conditions of the lemma given $\sigma$ it is enough for us to prove that

$$\frac{\sum_{g \in \partial_t B_n} \omega_g}{\sum_{g \in B_n} \omega_g} \to 0 \qquad \text{a.s.}$$

with for any $t > 0$.

**Theorem 2.1.13.** *Let $G$ be a countable group acting non-singularly on the probability space $(X, \mu)$ with extension $(\tilde{G}, \rho)$, and suppose that $G$ has finite levels $(G_k)$ with respect to $(\tilde{G}, \rho)$. If*

(a) *$(\tilde{G}, \rho)$ is well-separable,*

(b) *$(\tilde{G}, \rho)$ has finite interection dimension and*

(c) *the sequence $(B_n)$ given by $B_n = G_n \cap B_n(e)$ has the MDP*

*then for all $t > 0$*

$$\lim_{k \to \infty} \frac{\sum_{g \in \partial_t B_k} \omega_g}{\sum_{g \in B_k} \omega_g} = 0 \qquad \textit{a.s.}.$$

*Proof.* Here we return to the approach laid out in [Hoc10]. Let $\chi$ be the constant of well separability, the intersection dimension be $\kappa$ at scale $R$ and $D$ be the multiplicative doubling constant.

Suppose for a contradiction that for some $\epsilon > 0$

$$\limsup_{n \to \infty} F_n(x) > \epsilon$$

on some set $A_0$ with positive measure, where

$$F_n = \frac{\sum_{g \in \partial_t B_n} \omega_g}{\sum_{g \in B_n} \omega_g}.$$

Now we construct a sequence of integers $r_1^- < r_1^+ < r_2^- < r_2^+ < \ldots$ and sets $A_0 \supset A_1 \supset A_2 \supset \ldots$ as follows: we first let $r_0^+ = 0$ and ensure $r_1^- > 7 \max(t, R)$.

Then for each $i \geq 1$ given $r_{i-1}^+$ and $A_{i-1}$ with $\mu(A_{i-1}) > \frac{1}{2}(1+\frac{1}{i})\mu(A_0)$ we take $r_i^- > 2(r_{i-1}^+)^2$ and let

$$A_i = \left\{ x \in A_{i-1} : \max_{r_i^- \leq j \leq r_i^+} F_j(x) > \epsilon \right\}$$

where $r_i^+$ is chosen large enough to ensure that $\mu(A_i) > \frac{1}{2}(1+\frac{1}{i+1})\mu(A_0)$. In particular these properties ensure that the set $A = \bigcap_{i=0}^{\infty} A_i$ has measure at least $\frac{1}{2}\mu(A_0) > 0$, and that the $r_i^{\pm}$ satisfy the radii growth conditions for Theorem 2.1.8. We will use this latter property so show that we must have $\mu(A) = 0$, giving the contradiction.

Fix $\delta > 0$ and take $q = q(\chi, \kappa, \epsilon, \delta)$ as in Theorem 2.1.8. Fix $n > r_q^+ + t$ large enough to employ the MDP. Observe that

$$\mu(A) = \frac{1}{|B_n|} \int_X \sum_{g \in B_n} \hat{g} \mathbf{1}_A \, d\mu.$$

If we fix $x \in X$ then we may define a measure $\nu = \nu_{x,n}$ on $G$ by

$$\nu(E) = \sum_{g \in E \cap B_n^2} \omega_g(x)$$

where $B_n^2 = B_n B_n$. In addition for almost every such $x$ the measure $\nu$ is finite, since $B_n^2$ is, so it will suffice for us to consider only these $x$. Let

$$S = \{g \in B_n : gx \in A\},$$

so that

$$\nu(S) = \sum_{g \in B_n} \mathbf{1}_A(gx)\omega_g(x) = \sum_{g \in B_n} \hat{g}\mathbf{1}_A(x).$$

We can construct a levelled stack over $S$ as follows. If $h \in S$ then $hx \in A$ and so for each $1 \leq i \leq q$ there is $r_i^- \leq m = m(i,h) \leq r_i^+$ for which

$$\sum_{g \in \partial_t B_m} \omega_g(hx) > \epsilon \sum_{g \in B_m} \omega_g(hx)$$

and hence (as we may assume $x$ satisfies $\omega_h(x) > 0$ for all $h \in B_n$)

$$\sum_{g \in \partial_t B_m h} \omega_g(x) > \epsilon \sum_{g \in B_m h} \omega_g(x).$$

As $h \in B_n$, $m \leq r_q^+$ and $n > r_q^+ + t$ we have $B_m h \subseteq B_n^2$ and

$$\partial_t B_m h \subseteq B_{\lceil r_q^+ + t \rceil} h \subseteq B_n^2.$$

Hence $\nu(\partial_t B_m h) > \epsilon \nu(B_m h)$. It follows that given $1 \leq i \leq q$ we can let $\mathcal{U}_i = \{B_{m(i,h)} h : h \in F\}$, and this levelled stack satisfies all the requirements of Theorem 2.1.8.

Applying the theorem it follows that $\nu(S) \leq \delta \nu(B_n^2)$ for a.e. $x \in X$ and we may apply the multiplicative doubling condition to see that

$$\mu(A) \leq \frac{1}{|B_n|} \int_X \delta \sum_{g \in B_n^2} \omega_g \, d\mu = \frac{|B_n^2|}{|B_n|} \delta \leq D\delta.$$

Since $\delta > 0$ was arbitrary, we are done. □

## 2.2 The maximal inequality

In this section we follow the exposition given in [Fel07] to prove the maximal inequality. For the interested reader, [Fel07] also gives a concise account of the various authors who contributed to the approach.

### Besicovitch covering property

A geometrical assumption thought to be essential to the maximal inequality, see [Hoc10], is the Besicovitch covering property.

**Definition 2.2.1.** A sequence $(B_n)$ of subsets of a (not necessarily countable) group $G$ has the *Besicovitch covering property* (BCP) if there is a constant $C > 0$ such that for any finite set $E \subset G$ and any collection of translates $\mathcal{U} = \{B_{n(g)} g\}_{g \in E}$ we have a subcollection $\mathcal{V} \subseteq \mathcal{U}$ for which

$$\mathbf{1}_E \leq \sum_{B \in \mathcal{V}} \mathbf{1}_B \leq C.$$

In this situation we say $(B_n)$ has the BCP (in $G$) with constant $C$. A collection satisfying the second of the above inequalities is said to have *multiplicity* $C$.

As with the doubling property, this property has an analogue for metric spaces.

**Definition 2.2.2.** Let $(M,\rho)$ be a metric space, we say it has the *metric Besicovitch covering property* if there exists $C>0$ such that for any finite set $E\subseteq M$ and carpet $\mathcal{U}=\{B_{r(p)}(p):p\in E\}$ there is a subcollection $\mathcal{V}\subseteq\mathcal{U}$ for which

$$\mathbf{1}_E\leq\sum_{B\in\mathcal{V}}\mathbf{1}_B\leq C.$$

*Remark* 2.2.3. For a right invariant group metric space the metric Besicovitch covering property and the Besicovitch property we are using coincide for sequences of balls in the group.

The metric Besicovitch property has a useful reformulation in terms of incremental sequences. An *incremental sequence* in a metric space $(M,\rho)$ is a carpet $\{B_{r(i)}(p_i)\}_{i=1}^n$ such that $r(1)\geq r(2)\geq ...\geq r(n)$ and $p_j\notin B_{r(i)}(p_i)$ for all $i<j$.

**Proposition 2.2.4.** *$(M,\rho)$ has the metric Besicovitch covering property with constant $C>0$ if and only if every incremental sequence has multiplicity $\leq C$.*

A proof of this proposition can be found in [Hoc10] as Proposition 2.1.

The Besicovitch covering property plays a crucial role in the following lemma.

**Lemma 2.2.5.** *Let $\epsilon>0$ and $G$ be a countable group with an increasing sequence $(B_n)$ of finite subsets satisfying the BCP with constant $C$. For each function $a\in l^1(G)$ and $k\in\mathbb{N}$ let $s_ka(h)=\sum_{g\in B_k}a(gh)$ for all $h\in G$. Given $k\in\mathbb{N}$ and $a,b\in l^1(G)$ with $b\geq 0$ the set $H=H_k(a,b)=\bigcup_{i=1}^k H^{(i)}$, where*

$$H^{(i)}=\{h\in G:s_ia(h)>\epsilon s_ib(h)\},$$

*satisfies*

$$\|a\|_1\geq\epsilon C^{-1}\sum_{h\in H}b(h).$$

*Proof.* Let $E\subset G$ be finite and suppose that for each $h\in E\cap H$ there is a $1\leq m(h)\leq k$ for which $h\in H^{(m(h))}$. Consider the collection of translates $B_{m(h)}h$

with $h \in E \cap H$, by the BCP we can find a set $F \subset E \cap H$ for which

$$\mathbf{1}_{E\cap H} \leq \sum_{h\in F} \mathbf{1}_{B_{m(h)}h} \leq C.$$

It follows that

$$\sum_{h\in E\cap H} b(h) \leq \sum_{h\in F}\sum_{g\in B_{m(h)}h} b(g) = \sum_{h\in F} s_{m(h)} b(h)$$
$$< \epsilon^{-1}\sum_{h\in F} s_{m(h)} a(h) = \epsilon^{-1}\sum_{h\in F}\sum_{g\in B_{m(h)}h} a(g) \leq \epsilon^{-1} C\|a\|_1$$

and as $E$ was arbitrary the result follows. □

### Proof of the maximal inequality

Lemma 2.2.5 combines with the multiplicative doubling property to give the maximal inequality.

**Theorem 2.2.6** (The maximal inequality)**.** *Let $G$ be a countable group and with an increasing sequence $(B_n)$ of finite subsets satisfying the BCP with constant $C$. Suppose also that the sequence of integer balls $(B_n)$ has the MDP with constant $D$. Then for any $f \in L^1$ and $\epsilon > 0$*

$$\mu\left(\sup_{n\geq 1}\left|\frac{\sum_{g\in B_n}\hat{g}f}{\sum_{g\in B_n}\hat{g}1}\right| > \epsilon\right) \leq \frac{CD}{\epsilon}\|f\|_1.$$

*Proof.* For convenience let

$$F_n = F_n(x) = \frac{\sum_{g\in B_n}\hat{g}|f|}{\sum_{g\in B_n}\hat{g}1}.$$

Fix $N \in \mathbb{N}$ large enough to employ the MDP, it is enough to show that

$$\mu\left(\max_{1\leq n\leq N} F_k > \epsilon\right) \leq \frac{CD}{\epsilon}\|f\|_1.$$

We consider $f \geq 0$ without loss of generality. Now we fix a typical $x \in X$ and seek to apply Lemma 2.2.5 with $a_x(h) = \mathbf{1}_{B_N^2}(h)[\hat{h}f(x)] = \mathbf{1}_{B_N^2}(h)f(hx)\omega_h(x)$ and

$b_x(h) = \mathbf{1}_{B_N^2}(h)\omega_h(x)$. Observe that if $h \in B_n$ then since $\omega_{gh}(x) = \omega_g(hx)\omega_h(x)$ a.e.

$$s_n a_x(h) = \sum_{g\in B_n} \mathbf{1}_{B_N^2}(gh)f(ghx)\omega_{gh}(x) = \sum_{g\in B_n} f(ghx)\omega_{gh}(x) = \omega_h(x)\sum_{g\in B_n} \hat{g}f(hx)$$

and similarly

$$s_n b_x(h) = \omega_h(x)\sum_{g\in B_n} \hat{g}1(hx).$$

In particular, for almost every $x \in X$ we have $s_n a_x(h) > \epsilon s_n b_x(h)$ if and only if $F_n(x) > \epsilon$. Let $Y = \{x \in X : \max_{1\leq n\leq N} F_n(x) > \epsilon\}$ and $H_x = H_N(a_x, b_x)$ from Lemma 2.2.5. Then $gx \in Y$ if and only if $g \in H_x$, and hence

$$\begin{aligned}
\mu(Y) = \frac{1}{|B_N|}\int \sum_{g\in B_N} \hat{g}\mathbf{1}_Y\, d\mu &= \frac{1}{|B_N|}\int \sum_{g\in H_x} \mathbf{1}_{B_N}(g)\omega_g\, d\mu \\
&\leq \frac{C}{\epsilon|B_N|}\int \|a_x\|_1\, d\mu \\
&= \frac{C}{\epsilon|B_N|}\int \sum_{g\in B_N^2} \hat{g}f\, d\mu = \frac{C|B_N^2|}{\epsilon|B_N|}\|f\|_1
\end{aligned}$$

since $\mathbf{1}_{B_N}(g)\omega_g \leq b_x(g)$. The result then follows from the multiplicative doubling condition. $\square$

## 2.3 The ergodic theorem on $L^1$

We are now able to prove the ergodic theorem.

**Theorem 2.3.1** (The ergodic theorem)**.** *Let $G$ be a countable group acting non-singularly on the probability space $(X, \mu)$ with extension $(\tilde{G}, \rho)$, and that $G$ has finite levels $(G_k)$ with respect to $(\tilde{G}, \rho)$. Let $B_n = G_n \cap B_n(e)$. Suppose that:*

(a) *$(\tilde{G}, \rho)$ is well-separable,*

(b) *$(B_n)$ has the multiplicative doubling property,*

(c) *$(\tilde{G}, \rho)$ is voidless,*

(d) *$(\tilde{G}, \rho)$ has finite intersection dimension and*

(e) *$(B_n)$ has the Besicovitch covering property* *then for every $f \in L^1(\mu)$*

$$\lim_{n\to\infty} \frac{\sum_{g\in B_k} \hat{g}f}{\sum_{g\in B_k} \hat{g}1} = \mathbb{E}(f|\mathcal{J}) \qquad \textit{almost everywhere.}$$

*Proof.* Let $C$ and $D$ be the Besicovitch and doubling constants. We have already seen that the set

$$S = \operatorname{span}\{c + h - \hat{\sigma}h : c \in I, \sigma \in G, h \in L^\infty\}$$

is dense in $L^1$ and that given $\sigma \in G$ and $h \in L^\infty$

$$\frac{\sum_{g\in B_k} \hat{g}(c + h - \hat{\sigma}h)}{\sum_{g\in B_k} \hat{g}1} \to c \qquad \text{a.e.} \tag{2.3.1}$$

must occur if

$$\frac{\sum_{g\in B_k\triangle\sigma B_k} \omega_g}{\sum_{g\in B_k} \omega_g} \to 0 \qquad \text{a.e..}$$

This latter condition follows from first using (c) to apply Lemma 2.1.12 and then using (a), (b) and (d) to apply Theorem 2.1.13 with $t = \rho(0, \sigma)$. Therefore the result follows for any element of $S$ by linearity.

Now, for the general case, given $f \in L^1$ we may choose a sequence $f_m \in S$ such that $\|f - f_m\|_1 \leq \frac{1}{m}$ for all $m \geq 1$. Conditioning the generators of $S$ by the invariant $\sigma$-algebra gives $\mathbb{E}(c + h - \hat{\sigma}h|\mathcal{J}) = c$ a.e., and hence by linearity and (2.3.1) the ergodic theorem holds for each $f_m$.

Fix $\epsilon > 0$. By applying (b) and (e) via the maximal inequality to $f - f_m$ we see that

$$\mu\left(\limsup_{n\to\infty}\left|\frac{\sum_{g\in B_n} \hat{g}f}{\sum_{g\in B_n} \hat{g}1} - \mathbb{E}(f_m|\mathcal{J})\right| > \frac{\epsilon}{2}\right) \leq \frac{2CD}{m\epsilon}.$$

It follows from the properties of conditional expectations that

$$\mu\left(|\mathbb{E}(f|\mathcal{J}) - \mathbb{E}(f_m|\mathcal{J})| > \frac{\epsilon}{2}\right) \leq \frac{2}{\epsilon}\|\mathbb{E}(f|\mathcal{J}) - \mathbb{E}(f_m|\mathcal{J})\|_1 \leq \frac{2}{\epsilon}\|f - f_m\|_1.$$

Hence

$$\begin{aligned}
&\mu\left(\limsup_{n\to\infty}\left|\frac{\sum_{g\in B_n}\hat{g}f}{\sum_{g\in B_n}\hat{g}1}-\mathbb{E}(f|\mathcal{J})\right|>\epsilon\right)\\
&\leq\mu\left(\limsup_{n\to\infty}\left|\frac{\sum_{g\in B_n}\hat{g}f}{\sum_{g\in B_n}\hat{g}1}-\mathbb{E}(f_m|\mathcal{J})\right|>\frac{\epsilon}{2}\right)+\mu\left(|\mathbb{E}(f|\mathcal{J})-\mathbb{E}(f_m|\mathcal{J})|>\frac{\epsilon}{2}\right)\\
&\leq\frac{2(CD+1)}{m\epsilon}
\end{aligned}$$

for all $m$ sufficiently large. Since $\epsilon > 0$ was arbitrary the result follows. □

We will go on to apply this theorem to various abelian groups, in particular every subgroup of $\mathbb{Q}^d$ for any $d$, in Chapter 3 and to the discrete Heisenberg groups in Chapter 4.

There are, in certain circumstances, relationships between the five conditions in Theorem 2.3.1. We will use a result from [Hoc10] in the next chapter which shows that if $(\tilde{G},\rho)$ satisfies the metric Besicovitch covering property and the metric doubling condition then it is well-separable. Also in [Hoc10], Hochman identifies a property (analogous to one we will use in Proposition 3.1.13) which can be used to prove both the metric Besicovitch property and finite intersection property for norms on $\mathbb{R}^d$. This may suggest there is some potential to simplify these conditions, though whether this can be done in this level of generality is open.

In the work which follows, the least demanding of the conditions in Theorem 2.3.1 is the voidlessness of the space $(\tilde{G},\rho)$ since the topologies of the spaces we consider are very well-behaved. This, in particular the compactness of the unit ball in each case, is also helpful in showing the other properties hold.

In the examples we consider the most demanding of the conditions are the requirements that $(B_n)$ is Besicovitch and that $(\tilde{G},\rho)$ has finite intersection dimension. In fact, the proof that these properties hold for subgroups of $\mathbb{Q}^d$ account for the first half of Chapter 3, a proof of the former property for the Heisenberg group is the content of the paper [LDR17] and the latter property, for the same group, accounts for the most of work in Chapter 4.

The prior is a challenge in the case where $G$ is the discrete Heisenberg group, as many of the group's best known metrics such as the Korányi distance and the

Carnot-Carathéodory metric do not have the metric version of the property (see [Rig04, SW92]). Identifying another metric which does is the subject of the paper [LDR17] by Le Donne and Rigot. We define a summing sequence using the same metric which will have the BCP. In the same paper, Le Donne and Rigot show that the metric property is relatively unstable; they prove that if a metric space has the metric Besicovitch covering property and an accumulation point then there is a Lipschitz equivalent metric without the property. These facts suggest that in general it may not be an easy task to find a suitable metric.

The finite intersection dimension property is demanding in the sense that, in the situations we go on to consider, reasonably delicate arguments are required. The property is also far less well studied, since the Besicovitch covering property has other applications in (amoungst other things) measure differentiation theorems. In contrast the intersection dimension was essentially formulated by Hochman to prove the non-singular ergodic theorem for balls of norms on $\mathbb{Z}^d$.

# Chapter 3

# Countable Abelian Groups

In this chapter we consider the non-singular actions of countable abelian groups.

The first part of the chapter concerns the ergodic theorem. We start by considering $\mathbb{Q}^d$ and its subgroups then apply Theorem 2.3.1 to show that the ergodic theorem holds when the summing sequence is given by the balls of a norm on $\mathbb{R}^d$ (generalising Hochman's result) or by a class of metrics whose balls are rectangles. The corresponding summing sequence of rectangles need not satisfy the metric doubling condition and therefore also falls beyond the setting previously considered by Hochman.

Recall that an abelian group is *torsion-free* if no element has finite order and its *rank* is the cardinality of a (or, equivalently, any) maximal $\mathbb{Z}$-linearly independent subset. Motivated by how $\mathbb{Q}^d$ and its subgroups are exactly the torsion-free abelian groups with rank at most $d$ [Fuc15, p. 410] we move on to consider the problem for *torsion* groups (where every element has finite order), finitely generated groups and potential areas for further research.

The second part of the chapter is focussed on the critical dimensions of actions of $\mathbb{Z}^d$ with respect to balls of norms and the metrics considered in the first part. In both cases we will deduce the critical dimensions are invariants of metric isomorphism using the ergodic theorems. We show that, regardless of the choice of norm, the corresponding summing sequence will always give the same critical dimension. We then use rectangular metrics to establish that the critical dimensions do vary based on the choice of summing sequence, and so each summing sequences could lead to a different invariant.

The critical dimension work in this chapter, or an earlier version of it, has been submitted to a journal as part of a paper written with Anthony Dooley a preprint of which can be found at [DJ16].

## 3.1 Ergodic theorem for subgroups of $\mathbb{Q}^d$

In Chapter 2 we showed that if $G$ has an extension $\tilde{G}$ with good geometry, and satisfies some additional conditions, then there is a summing sequence for which the ergodic theorem holds. In Hochman's work with $\mathbb{Z}^d$ this role is played by $\mathbb{R}^d$, and we will use the same extension for $\mathbb{Q}^d$ and it rank $d$ subgroups. Let $G$ be a rank $d$ subgroup of $\mathbb{Q}^d$ throughout this subsection, and define its levels by $G_n = G \cap (n!)^{-1}\mathbb{Z}^d$.

Note that each $G_n$ can be considered as submodule of the finitely generated $\mathbb{Z}$-module $(n!)^{-1}\mathbb{Z}^d$, and since $\mathbb{Z}$ is a commutative Noetherian ring with 1 this means $(n!)^{-1}\mathbb{Z}^d$ is a Noetherian module (see e.g. [Rom08, p. 132-134]) and hence that $G_n$ is finitely generated. As it is torsion-free, the structure theorem for finite abelian groups tells us it must be isomorphic to $\mathbb{Z}^m$ for some $m \leq d$. In fact, as $G$ is rank $d$ we can see (by multiplying out by the denominators, if necessary) that $G_n$ contains a $\mathbb{Z}$-linearly independent subset of size $d$ and a subgroup isomorphic to $\mathbb{Z}^d$. Hence $G_n$ is isomorphic to $\mathbb{Z}^d$ itself.

In this part we will be considering the summing sequences induced by two different classes of metrics on $\mathbb{R}^d$: norms and rectangular metrics.

**Definition 3.1.1.** A metric $\rho$ on $\mathbb{R}^d$ is called *rectangular* if it is of the form

$$\rho(u, v) = \max_{1 \leq i \leq d} F_i(|u_i - v_i|) \tag{3.1.1}$$

where each $F_i : [0, \infty) \to [0, \infty)$ satisfies $F_i(0) = 0$, is subadditive and strictly increasing.

The first two properties of $F_i$ ensure $\rho$ is a metric and the latter guarantees that $F_i$ has an inverse, which we denote by $f_i$, and which is superadditive on $[0, \infty)$. Note that if $B_r(z)$ is a ball of a rectangular metric then $B_r(z) = z + B_r$ where

$$B_r = B_r(0) = \prod_{i=1}^{d} [-f_i(r), f_i(r)],$$

justifying the name rectangular.

We begin by considering summing sequences given by norms and then move on to rectangular metrics.

### 3.1.1 Summing sequences from norms

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ so our summing sequence is $B_n = G \cap (n!)^{-1}\mathbb{Z}^d \cap B_n(0)$ where $B_r(u) = \{v \in \mathbb{R}^d : \|v - u\| \leq n\}$. To show the ergodic theorem holds for this sequence we need only check that the conditions of Theorem 2.3.1 hold.

Balls in $(\mathbb{R}^d, \|\cdot\|)$ are convex, in particular path connected, and hence Remark 2.1.11 ensures that the space is voidless. The fact that the space has finite intersection dimension is precisely the content of Corollary 4.3 in [Hoc10], since for normed spaces our definition of thickened boundary coincides with the definition in that paper, and the intersection dimension reduces to the coarse dimension in the corollary.

The other properties require a little more work, partially as a result of the inclusion of levels, which are the feature which distinguishes this work from Hochman's. We will consider each of the remaining properties in turn. First, however, the following simple lemma will be quite useful to us.

**Lemma 3.1.2.** *Let $s > 0$. There exists $N(s) \in \mathbb{N}$ otherwise depending only on $(\mathbb{R}^d, \|\cdot\|)$ such that there are $N$ open balls of radius $s/2$ centred in $B_1(0)$ whose union covers $B_1(0)$. Consequently:*

(i) *if $u_1, ..., u_n \in B_1(0)$ and for all $i \neq j$ $\|u_i - u_j\| > s$ then $n \leq N$, and*

(ii) *if $u_1, ..., u_n \in B_1(0)$ with $n \geq kN$ for some $k \in \mathbb{N}$ then there is a subset $I \subset \{1, ..., n\}$ of size at least $k$ with $\|u_i - u_j\| < s$ for all $i, j \in I$.*

*Proof.* Since the closed unit ball $B_1(0)$ is compact the existence of such an $N$ follows from this compactness. Part (i) is due to the fact that if two points lie in the same ball in the cover then they are $< s$ apart, and part (ii) uses this along with the pigeon-hole principle. □

An easy corollary of this lemma is that $(\mathbb{R}^d, \|\cdot\|)$ has the *metric* doubling property. This can be seen by letting $s = 1$ in Lemma 3.1.2 and simply scaling and translating the $N(1)$ balls used to cover $B_1(0)$. We make use of this property later.

### Multiplicative doubling property

We can also apply the lemma to prove the multiplicative doubling property.

**Corollary 3.1.3.** *Let* $G \leq \mathbb{Q}^d$*,* $\|\cdot\|$ *be a norm on* $\mathbb{R}^d$ *and* $B_n = G \cap (n!)^{-1} \cap B_n(0)$*. Then the sequence* $B_n = G_n \cap B_n(0)$ *has the MDP.*

*Proof.* Recall that $G_n \leq \mathbb{Q}^d$ is isomorphic to $\mathbb{Z}^d$ and so we may fix a $\mathbb{Z}$-linearly independent generating set $\{u_1^{(n)}, ..., u_d^{(n)}\}$ of size $d$. This subset, considered now as a subset of $\mathbb{Q}^d$, must also be $\mathbb{Q}$-linearly independent and hence the standard basis is in its $\mathbb{Q}$-span. This in turn means it is a basis for $\mathbb{R}^d$ as a real vector space.

From this it follows that the set

$$D_n = \left\{a_1 u_1^{(n)} + ... + a_d u_d^{(n)} : a_1, ..., a_d \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)\right\}$$

is a fundamental domain for the natural action of $G_n$ on $\mathbb{R}^d$. Let

$$b_n = \sup\{\|u\| : u \in D_n\}$$

and note that $b_1$ is finite, since the closure of $D_1$ is compact, and $(b_n)$ is a decreasing sequence.

Let $\lambda$ be Lebesgue measure on $\mathbb{R}^d$, $r_n = 2\frac{n-b_n}{2n+b_n}$ and $n \in \mathbb{N}$, taken sufficiently large to ensure $r_n \geq \frac{2}{3}$. Then by applying Lemma 3.1.2 (between scaling by $(2n+b_n)^{-1}$ and $2n+b_n$) it follows that $B_n^2 = B_n + B_n$ satisfies

$$\begin{aligned}
|B_n^2|\lambda(D_n) = \lambda\left(\bigcup_{u \in B_n^2}(u + D_n)\right) &\leq \lambda\left(B_{2n+b_n}(0)\right) \\
&\leq N(r_n)\,\lambda(B_{n-b_n}(0)) \\
&\leq N\lambda\left(\bigcup_{u \in B_n}(u + D_n)\right) \leq N\lambda(D_n)|B_n|
\end{aligned}$$

where $N = N(2/3)$. For the third inequality we have used that each point in $B_{n-b_n}(0)$ lies in some unique $u + D_n$ with $u \in G_n$ which must then be in $B_n(0)$ and hence $B_n$. The result follows since $D_n$ has non-empty interior, so $\lambda(D_n) > 0$. $\square$

### Besicovitch covering property

Now let us show that the sequence $(B_n)$ satisfies the Besicovitch covering property in $G$. To do this we are going to make use of the metric version of the Besicovitch property on $\mathbb{R}^d$.

Normed spaces $(\mathbb{R}^d, \|\cdot\|)$ were shown to have the metric BCP by Morse [Mor47]. For completeness, and because are going to use similar ideas in Section 4.2.2 to show that the continuous Heisenberg groups have finite intersection dimension, we include a proof. The argument is different from that due to Morse, and may be original.

**Proposition 3.1.4.** $(\mathbb{R}^d, \|\cdot\|)$*, considered as a metric space, has the metric Besicovitch covering property.*

*Proof.* We make use of Proposition 2.2.4. Let $\{B_{r(i)}(v_i)\}_{i=1}^n$ be an incremental sequence, we show that there is some $C$, depending only on $(\mathbb{R}^d, \|\cdot\|)$, bounding the multiplicity of the sequence.

Fix a point in a maximal number of balls in the sequence, by translation invariance we may assume it is 0, and that it is in every ball in the sequence.

First suppose that $r(n) \geq \frac{1}{2}r(1)$, then (after scaling by a factor of $r(1)^{-1}$) we end up with $n$ balls of radius at least $\frac{1}{2}$ in $B_1(0)$, with the centre of each ball not lying in those preceding it. In particular, all the centres are strictly more than $\frac{1}{2}$ apart and by Lemma 3.1.2 we have that $n \leq N\left(\frac{1}{2}\right)$.

Now suppose that $n \geq m\,N\left(\frac{1}{2}\right)$, then we can restrict our incremental sequence to one of length $m$ such that $r(i+1) \leq \frac{1}{2}r(i)$ for all $1 \leq i \leq m-1$. We show that $m$ is bounded above by some $M$ depending only on $(\mathbb{R}^d, \|\cdot\|)$, and hence deduce that $n \leq MN$ which proves the claim.

Observe that for all $i < m$ we have $\|v_i\| > \frac{1}{2}r(i)$ else

$$\|v_i - v_{i+1}\| \leq \frac{1}{2}r(i) + r(i+1) \leq r(i)$$

and hence $v_{i+1} \in B_{r(i)}(v_i)$ contradicting incrementality. This means that

$$\|v_1\| > \|v_2\| > ... > \|v_m\|.$$

Now consider the radial lines $L_i = \{tv_i : t \geq 0\}$. Given $i < m$ and $j > i$ the point $v_i$ must be a distance more that $r(i)$ away from the intersection of $L_j$ and

$\partial B_{r(i)}(0)$. Otherwise $B_{r(i)}(v_i)$ contains both the (unique) point in the intersection and 0, then by convexity must contain $v_j$ giving a contradiction. After scaling down by $r(i)^{-1}$, we see that this means the unique point in $L_i \cap \partial B_1(0)$ must be at least distance 1 from the unique point in each $L_j \cap \partial B_1(0)$ where $j > i$. Hence we have a collection of $m$ distinct points in $\partial B_1(0) \subset B_1(0)$ all more than distance 1 from one another, and by Lemma 3.1.2 we must have $m \leq N(1)$ (and therefore $n \leq N(1)N\left(\frac{1}{2}\right)$). □

The idea from this proof we make use of in Section 4.2.2 is that to show the length of the sequence is bounded it is enough to first show, if one is able to take it arbitrarily long, then a subsequence of arbitrary length with additional properties exists. Then it suffices to prove that a sequence with these properties must have bounded length.

We can now use the metric BCP to prove that $(B_n)$ has the BCP for summing sequences.

**Corollary 3.1.5.** *Let $G \leq \mathbb{Q}^d$, $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and $B_n = G \cap (n!)^{-1} \cap B_n(0)$. Then $(B_n)$ satisfies the Besicovitch covering property in $G$.*

*Proof.* Let $\{g_1, g_2, ..., g_k\} \subseteq G$ and $n(1) \geq n(2) \geq ... \geq n(k)$ define a collection of translates $\{B_{n(i)}g_i\}_{i=1}^k$, which we aim to show has a subcollection with multiplicity $C$, where $C$ is the Besicovitch constant for $(\mathbb{R}^d, \|\cdot\|)$. Without loss of generality we may assume that $g_j \notin \bigcup_{i<j} B_{n(i)}g_i$ for all $j$, this can be done by recursively removing the first $B_{n(i)}g_i$ where $g_i$ is contained in the union of the preceding translates. It now suffices to show that $\{B_{n(i)}g_i\}_{i=1}^k$ has multiplicity $C$.

Suppose first that $i < j$, $B_{n(i)}g_i \cap B_{n(j)}g_j \neq \emptyset$ and $g_j \in B_{n(i)}(g_i)$. Then there are $\sigma \in G_{n(j)}$, $\tau \in G_{n(i)}$ such that $g_j = \sigma^{-1}\tau g_i \in G_{n(i)}g_i$ and hence $g_j \in B_{n(i)}g_i$. Therefore either $B_{n(i)}g_i \cap B_{n(j)}g_j = \emptyset$ or $g_j \notin B_{n(i)}(g_i)$.

Let $g$ be a point which is inside a maximal number of the $\{B_{n(i)}g_i\}_{i=1}^k$ and let $\mathcal{V} \subset \{B_{n(i)}g_i\}_{i=1}^k$ be the collection of translates which contain $g$. By the previous paragraph its associated collection $\tilde{\mathcal{V}}$ must be an incremental sequence in $(\mathbb{R}^d, \|\cdot\|)$, and hence has multiplicity at most $C$ by Propositions 3.1.4 and 2.2.4. It follows that $g$ is covered at most $C$ times, which proves the claim. □

#### Well-separability and the result

The last remaining property we need is well-separability. Lemma 3.1 from [Hoc10] says that if a (right invariant group) metric space satisfies the *metric* doubling property and the *metric* Besicovitch property then it is well-separable. We observed that the former property holds just after Lemma 3.1.2, and the latter is the content of Proposition 3.1.4.

We now have everything we need in order to apply Theorem 2.3.1 and deduce the following result, which extends Hochman's result to any subgroup of $\mathbb{Q}^d$.

**Theorem 3.1.6.** *Let $G$ be a subgroup of $\mathbb{Q}^d$ and $B_n = G \cap (n!)^{-1}\mathbb{Z}^d \cap B_n(0)$ where $B_n(0) = \{u \in \mathbb{R}^d : \|u\| \leq n\}$ for some norm $\|\cdot\|$ on $\mathbb{R}^d$. Then the non-singular ergodic theorem for $G$ holds for the summing sequence $(B_n)$.*

*Remark* 3.1.7. In particular when $G = \mathbb{Z}$ and $\|\cdot\|$ is the absolute value, so the summing sequence is given by $B_n = \{-n, -n+1, ..., n\}$, the ergodic theorem holds without any assumption on the conservativity of the action. This version of the ergodic theorem can be directly substituted in the proof the ergodic decomposition theorem in [Aar97] in order to remove the assumption of conservativity, as suggested after the statement of the result (1.1.7) in the introduction.

By applying essentially the same argument as used by Mortiss to prove Proposition 1.2.4 we can deduce that the critical dimensions for these summing sequences are invariants.

**Corollary 3.1.8.** *Let $G$ be a subgroup of $\mathbb{Q}^d$ and $B_n = G \cap (n!)^{-1}\mathbb{Z}^d \cap B_n(0)$ where $B_n(0) = \{u \in \mathbb{R}^d : \|u\| \leq n\}$ for some norm $\|\cdot\|$ on $\mathbb{R}^d$. Then the upper and lower critical dimensions with respect to $(B_n)$ are invariants of metric isomorphism.*

### 3.1.2 Summing sequences from rectangular metrics

There are a number of differences in the arguments when one considers rectangular metrics rather than norms. One of the most significant is that the metrics no longer need to respect scalar multiplication, a property we made extensive use of in the previous section. However, instead we are able to take advantage of the rectangular shape of the balls to address the issues which arise.

Again we consider each of the properties needed for Theorem 2.3.1 in turn. As with norms the rectangular balls in $\mathbb{R}^d$ are convex, and so the space is voidless.

### Multiplicative doubling property

In this section we show that rectangular metrics have the multiplicative doubling property.

Before this it should be noted that rectagular metrics need not satisfy the metric doubling condition; consider the $\mathbb{Z}^2$ case where $F_1(t) = t$ and $F_2(t) = \log(1+t)$, then

$$B_n(0) = \{-n, ..., n\} \times \{-\lfloor e^n - 1 \rfloor, ..., \lfloor e^n - 1 \rfloor\}.$$

Examples of this kind, and their applications in our study of the critical dimensions (see Section 3.3), were motivations for weakening the requirement for the metric doubling condition.

We use a similar argument to the case for norms, with necessary modifications.

**Lemma 3.1.9.** *Let $G$ be a subgroup of $\mathbb{Q}^d$ and $B_n = G \cap (n!)^{-1}\mathbb{Z}^d \cap B_n(0)$ where $B_n(0) = \{u \in \mathbb{R}^d : \rho(u, 0) \leq n\}$ for some rectangular metric $\rho$ on $\mathbb{R}^d$. Then the sequence $B_n = G_n \cap B_n(0)$ has the MDP.*

*Proof.* As in the proof of Corollary 3.1.3 we have can choose a $\mathbb{Z}$-linearly independent subset $\{u_1^{(n)}, ..., u_d^{(n)}\}$ of $G_n$ such that

$$D_n = \left\{a_1 u_1^{(n)} + ... + a_d u_d^{(n)} : a_1, ..., a_d \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)\right\}$$

is a fundamental domain of the action of $G_n$ on $\mathbb{R}^d$. Also as before let

$$b_n = \sup\{\rho(u, 0) : u \in D_n\} \leq b_1 < \infty.$$

Let $\lambda$ be Lebesgue measure on $\mathbb{R}^d$. Since $D_n \subseteq \prod_{i=1}^d [-f_i(b_n), f_i(b_n)]$ and

$B_n^2 \subseteq \prod_{i=1}^d [-2f_i(n), 2f_i(n)]$ we have

$$\begin{aligned}
|B_n^2|\lambda(D_n) &= \lambda\left(\bigcup_{u \in B_n^2} (u + D_n)\right) \\
&\leq \lambda\left(\prod_{i=1}^d [-2f_i(n) - f_i(b_n), 2f_i(n) + f_i(b_n)]\right) \\
&= \prod_{i=1}^d (2(2f_i(n) + f_i(b_n)) + 1) \\
&= \prod_{i=1}^d \frac{2(2f_i(n) + f_i(b_n)) + 1}{2(f_i(n) - f_i(b_n)) + 1} \lambda\left(\prod_{i=1}^d [-f_i(n) + f_i(b_n), f_i(n) - f_i(b_n)]\right) \\
&\leq 2^d \lambda\left(\bigcup_{u \in B_n} (u + D_n)\right) \\
&= 2^d |B_n| \lambda(D_n)
\end{aligned}$$

where we have used that if $v \in \prod_{i=1}^d [-f_i(n) + f_i(b_n), f_i(n) - f_i(b_n)]$ it is within distance $b_n$ of some $u \in G_n$ which must satisfy

$$u \in v + \prod_{i=1}^d [-f_i(b_n), f_i(b_n)] \subseteq \prod_{i=1}^d [-f_i(n), f_i(n)] = B_n(0).$$

As before, the result follows since $\lambda(D_n) > 0$. □

### Besicovitch covering property

To prove the Besicovitch covering property we apply the same overarching argument structure as with norms; we first show that that rectangular metrics satisfy the metric Besicovitch covering property and can then use the proof from the norm case to deduce the BCP. The distinction from that case is in the proof of the metric BCP. We provide a short proof here, for completeness. A proof of a more extensive version of the result can be found in [dG75].

**Lemma 3.1.10.** $(\mathbb{R}^d, \rho)$ *satisfies the metric Besicovitch covering property.*

*Proof.* Let $\mathcal{U} = \{B_{r(i)}(u_i)\}_{i=1}^N$ be an incremental sequence in $(\mathbb{R}^d, \rho)$. We can write each of the balls in this collection as a union of $2^d$ *orthants*

$$u_i + \prod_{l=1}^{d} (-1)^{m_l} [0, f_l(r_i)]$$

where each $m_l \in \{0, 1\}$.

Assume for a contradiction that there is $v \in \mathbb{Z}^d$ lying in $> 2^d$ elements of $\mathcal{U}$. Then by pigeonhole principle $v$ must lie in the same orthant of two elements of $\mathcal{U}$, corresponding to $u_i$ and $u_j$ say. Let $m_1, ..., m_d$ take the values determining this orthant. We may assume $i < j$. For the numbers $n_l = 1 - m_l \in \{0, 1\}$ we have

$$\begin{aligned} u_j \in v + \prod_{l=1}^{d} (-1)^{n_l} [0, f_l(r_j)] &\subseteq v + \prod_{l=1}^{d} (-1)^{n_l} [0, f_l(r_i)] \\ &\subseteq u_i + \prod_{l=1}^{d} [-f_l(r_i), f_l(r_i)] = B_{r(i)}(u_i) \end{aligned}$$

contradicting the fact that $\mathcal{U}$ is an incremental sequence. □

**Corollary 3.1.11.** *Let $G$ be a subgroup of $\mathbb{Q}^d$ and $B_n = G \cap (n!)^{-1}\mathbb{Z}^d \cap B_n(0)$ where $B_n(0) = \{u \in \mathbb{R}^d : \rho(u, 0) \leq n\}$ for some rectangular metric $\rho$ on $\mathbb{R}^d$. Then the sequence $(B_n)$ satisfies the Besicovitch covering property in $G$.*

*Proof.* Essentially the same as the proof of Corollary 3.1.5. □

### Well-separability

To show that well-separability holds we mimic the proof of [Hoc10, Lemma 3.3], and essentially check that the ideas involved still apply for rectangles.

**Lemma 3.1.12.** *Any rectangular metric space $(\mathbb{R}^d, \rho)$ is well-separable.*

*Proof.* Let $C = 2^d$ be the Besicovitch constant and $\chi = 4^d C + 1$. Let $u \in \mathbb{R}^d$ and $\mathcal{W}$ be a collection of balls of radius $r$ centred in $u + B_r(0)^3$ and suppose it has multiplicity $\leq C$. Let $\lambda$ be Lebesgue measure. Then $\bigcup \mathcal{W} \subseteq u + B_r(0)^4$, so

$$|\mathcal{W}| \lambda(B_r(0)) \leq C\lambda(u + B_r(0)^4) \leq 4^d C\lambda(B_r(0))$$

and hence $|\mathcal{W}| \leq \chi - 1$.

If instead $\mathcal{W}$ contains balls of radius $\geq r$ which all intersect $u + B_r(0)^2$, and multiplicity $\leq C$, then we may replace each ball $B \in \mathcal{W}$ with a ball of radius $r$ contained in $B$ and centred in $u + B_r(0)^3$. We deduce from above that again $|\mathcal{W}| \leq \chi - 1$.

As the metric Besicovitch covering property holds we can find an incremental sequence $\{U_i\}_{i=1}^n \subseteq \mathcal{U}$ covering $E$. We assign colours $1, 2, ..., \chi$ to the $U_i$ as follows. Colour $U_1$ as you like, and assume we have coloured $U_i$ for $i \leq k$ and consider $U_{k+1}$. Take $r$ to be the radius of $U_k$ and $u$ to be the centre of $U_{k+1}$, by assumption $U_{k+1} \subseteq u + B_r(0)$ and each $U_i$ with $i \leq k$ has radius at least $r$. Therefore, by the above, at most $\chi - 1$ intersect $u + B_r(0)^2$. Give $U_{k+1}$ one of the colours unused by those $U_i$.

Let $\mathcal{V}_k$ be the collection coloured $k$. To see each collection is well-separated note that the points within rectangular distance $r$ of $u + B_r(0)$ are exactly those in $u + B_r(0)^2$, combining this with the colouring process and the fact the radii of the $U_i$ is decreasing gives the result. □

**Finite intersection dimension**

The final property we need to consider is the intersection dimension of rectangular metric spaces. This is one of the areas in which we need to use somewhat different techniques to the case of norms, where the ability to scale was used. Instead we take advantage of the synergy between rectangular balls and the structure of $\mathbb{R}^d$.

We first show that these spaces have a property similar to having finite intersection dimension, which Hochman makes use of in his work [Hoc10]. Then we apply this property to sequences of thickened boundaries, as in the definition of the intersection dimension, and see that the sequence length must be bounded if the intersection of the boundaries is non-empty.

For $v \in \{\pm e_i : 1 \leq i \leq d\}$ let $F_{r,u}(v)$ be the face of $B_r(u)$ in direction $v$ from $u$, i.e. those points in $B_r(u)$ whose projection onto $v$ is maximal. The *face* of the thickened boundary $\partial_t B_r(u)$ in direction $v$ is the set of points in $\mathbb{R}^d$ within distance $t$ of $F_{r,u}(v)$ and is denoted by $\partial_t F_{r,u}(v)$.

**Proposition 3.1.13.** *Let $(\mathbb{R}^d, \rho)$ be a rectangular metric space. Then there are $R = R(\rho) > 4$ and $k \in \mathbb{N}$ with the following property: given $u_1, ..., u_k \in \mathbb{R}^d$, $t(1), ..., t(k) \geq 1$ and a decreasing sequence $r(1), ..., r(k)$ with $r(k) \geq t(1)...t(k)R$*

*such that* $u_i \in \bigcap_{j<i} \partial_{t(j)} B_{r(j)}(u_j)$ *then*

$$\bigcap_{i=1}^{k} \partial_{t(i)} B_{r(i)}(u_i) = \emptyset.$$

*Proof.* For notational clarity we write $r_i = r(i)$ and $t_i = t(i)$ in this proof.

Fix $R > 4$. We use induction on the $d$ to prove that there is $k = k(d)$ with the required property.

For $d = 1$ let $k = 2$. Let $f = f_1$. The set $\partial_{t(1)} B_{r(1)}(u_1)$ is a union of two closed intervals length $2f(t_1) + 1$ centred on $u_1 \pm f(r_1)$ respectively. These intervals are disjoint as $r(1) > t(1)$. We may assume $u_2$ lies in the interval centred on $u_1 - f(r_1)$. Now since $R > 4$ we have

$$f(r_2) - f(t_2) > f(2t_1 + t_2) - f(t_2) \geq 2f(t_1)$$

using superadditivity of $f$. In particular $\partial_{t(2)} B_{r(2)}(u_2)$ does not intersect the interval centred on $u_1 - f(r_1)$.

Also,

$$f(r_2) + f(t_2) < 2(f(r_1) - f(t_1))$$

else using $R > 4$ and the fact the $r(i)$ are decreasing

$$\begin{aligned} 2f(t_1) + f(t_2) &\geq 2f(r_1) - f(r_2) \\ &\geq f(r_1) > f(2t_1 + t_2) \geq 2f(t_1) + f(t_2). \end{aligned}$$

This means that $\partial_{t(2)} B_{r(2)}(u_2)$ also does not intersect the interval centred on $u_1 + f(r_1)$, and the claim follows.

Now, assume we have proved $k(d-1)$ exists. Suppose $k \geq 2\,d\,k(d-1) + 2$. By the pigeonhole principle the thickening of some face $F(v)$ of $B_{r(1)}(u_1)$ contains $k(d-1)+1$ of the points $u_2, ..., u_{k(d)}$. As these are the only points used from here we may assume they are $u_2, ..., u_{k(d-1)+2}$. Using essentially the same argument as in the initial step the thickened faces in directions $\pm v$ of each $\{\partial_{t(i)} B_{r(i)}(u_i)\}_{i=2}^{2k(d-1)+2}$ cannot intersect the thickened faces $F(\pm v)$ of $\partial_{t(1)} B_{r(1)}(u_1)$. Therefore the $\partial_{t(i)} B_{r(i)}(u_i)$ intersect in $\partial_t F(v)$ only if the projections of the sets

$$\partial_{t(i)} B_{r(i)}(u_i) \cap \partial_{t(1)} F(v)$$

along $v$ onto $F(v)$ intersect. These projections are exactly thick boxes for projection of the rectangular metric in direction $e$, so we may apply the previous case to deduce that

$$\partial_{t(1)}F(v) \cap \bigcap_{i=2}^{k(d-1)+1} \partial_{t(i)}B_{r(i)}(u_i) = \emptyset$$

but by assumption $u_{k(d-1)+2}$ lies in that intersection. Hence $k < 2\,d\,k(d-1) + 2$ and so $k(d) \leq 2\,d\,k(d-1) + 1$. $\square$

**Proposition 3.1.14.** $(\mathbb{R}^d, \rho)$ *has finite intersection dimension.*

*Proof.* Let $R = R(\rho)$ and $k' = k$ from the previous proposition. Let $k'' \in \mathbb{N}$, to be determined, and $k = k'k'' + 1$. Let

1. $t(1), ..., t(k) \geq 1$,
2. $r(1), ..., r(k)$ such that $r(i) \geq t(1)...t(k)R$ and
3. points $u_1, ..., u_k \in \mathbb{R}^d$ such that $u_i \in \bigcap_{j<i} \partial_{t(j)}B_{r(j)}(u_j)$ for $j < i$.

It is sufficient to show that $\bigcap_{i=1}^{k} \partial_{t(i)}B_{r(i)}(u_i) = \emptyset$. As before, we write $r_i = r(i)$ and $t_i = t(i)$ in this proof.

By the previous proposition it suffices to find a subsequence length $k'$ for which the radii are decreasing. Consider the points $u_2, ..., u_l$ $(l \geq 2)$ and suppose $r(j) > r(1)$ for each $2 \leq j \leq l$. Each of these points lies inside $\partial_{t(1)}B_{r(1)}(u_1)$, by assumption. Moreover if $i > j$ then

$$\begin{aligned} u_j \notin u_i + \prod_{m=1}^{d} (-f_m(r_i) + f_m(t_i), f_m(r_i) - f_m(t_i)) \\ \supseteq u_i + \prod_{m=1}^{d} (-f_m(r_1) + f_m(r_1/R), f_m(r_1) - f_m(r_1/R)). \end{aligned}$$

Let $A = \prod_{m=1}^{d}(-f_m(r_1) + f_m(r_1/R), f_m(r_1) - f_m(r_1/R))$. The final line implies that we also have $u_i \notin u_j + A$. Now, $u_2, ..., u_l$ is a collection of points contained by $B = \partial_{t(1)}B_{r(1)}(u_1) \cup B_{r(1)}(u_1)$ such that $u_i \notin u_j + A$ for all $i \neq j$. Then the sets $u_j + \frac{1}{2}A$ are disjoint and each $B \cap (u_j + \frac{1}{2}A)$ contains at least one orthant of

$u_j + \frac{1}{2}A$ each of which has Lebesgue measure greater than or equal to

$$\prod_{m=1}^{d} \frac{1}{2}(f_m(r_1) - f_m(r_1/R)).$$

By the disjointness we must have

$$(l-1)\prod_{m=1}^{d} \frac{1}{2}(f_m(r_1) - f_m(r_1/R)) \leq \prod_{m=1}^{d} 2(f_m(r_1) + f_m(t_1))$$

i.e.

$$l \leq 1 + 4^d \prod_{m=1}^{d} \frac{f_m(r_1) + f_m(r_1/R))}{f_m(r_1) - f_m(r_1/R)}.$$

Dividing through each fraction by $f_m(r_1)$ and recalling that $R > 4$ and

$$\frac{f_m(r_1/R)}{f_m(r_1)} \leq \frac{f_m(r_1/4)}{4f_m(r_1/4)} = \frac{1}{4}$$

we see that

$$l \leq 1 + 4^d \prod_{m=1}^{d} \frac{1+1/4}{1-1/4} \leq 7^d + 1.$$

Therefore if we take $k'' > 7^d + 1$ then some $r(j) \leq r(1)$ for $2 \leq j \leq k''$. We can then repeat this process with $r(j)$ and so on to find a subsequence with decreasing radii satisfying the conditions, which will have length at least $k'$ by our choice of $k$. □

With all the properties in hand we conclude from Chapter 2 that the ergodic theorem holds for the sequence $B_n$.

**Theorem 3.1.15.** *Let $G$ be a subgroup of $\mathbb{Q}^d$ and $B_n = G \cap (n!)^{-1}\mathbb{Z}^d \cap B_n(0)$ where $B_n(0) = \{u \in \mathbb{R}^d : \rho(u,0) \leq n\}$ for some rectangular metric $\rho$ on $\mathbb{R}^d$. Then the non-singular ergodic theorem for $G$ holds with the summing sequence $(B_n)$.*

As with the case of norms, we can deduce the invariance of the associated critical dimensions.

**Corollary 3.1.16.** *Let $G$ be a subgroup of $\mathbb{Q}^d$ and $B_n = G \cap (n!)^{-1}\mathbb{Z}^d \cap B_n(0)$ where $B_n(0) = \{u \in \mathbb{R}^d : \rho(u,0) \leq n\}$ for some rectangular metric $\rho$ on $\mathbb{R}^d$. Then the upper and lower critical dimensions with respect to $(B_n)$ are invariants of metric isomorphism.*

## 3.2 The ergodic theorem in other abelian groups

In the previous section we showed that there are summing sequences for which the ergodic theorem holds, in particular given by norms and rectangular metrics, for every subgroup of $\mathbb{Q}^d$ for any finite $d$. As referenced earlier, these are precisely the torsion-free abelian groups with rank in $\mathbb{N}$. It turns out that it does not take much to prove a similar result for the rank 0 groups, the torsion groups, where every element has finite order, or for finitely generated abelian groups.

### Torsion Groups

Let us assume $G$ is a torsion group. Since $G$ is countable we can write $G = \{g_1, g_2, ...\}$ and define $G_n = \langle g_1, ..., g_n \rangle$. As $G$ is a torsion group the sets $G_n$ are all finite, and so we may take $(G_n)$ as our summing sequence. Observe that as $G = \bigcup_{n=1}^{\infty} G_n$ given $\sigma \in G$ we have $G_n = \sigma G_n$ for all $n$ sufficiently large, and so the non-singular Følner condition (2.1.2) holds trivially. By applying the same techniques as found in the proof of Theorem 2.3.1, to show the ergodic theorem holds for $(G_n)$ it is enough to show that it has the BCP.

**Proposition 3.2.1.** *$(G_n)$ has the Besicovitch covering property.*

*Proof.* Let $E = \{h_1, ..., h_m\}$ be a finite subset of $G$ and $\{B_{n(i)}h_i\}_{i=1}^m$ a collection of translates. Assume, without loss of generality, that $n(1) \geq n(2) \geq ... \geq n(m)$. Consider $G_k$ with $k$ large enough to ensure $\bigcup_{i=1}^m B_{n(i)}h_i \subseteq G_k$.

Since the $B_{n(i)}$ are subgroups of $G_k$ the right cosets of each $B_{n(i)}$ in $G_k$ partition $G_k$. Moreover, if $i \leq j$ then the $j^{\text{th}}$ partition refines the $i^{\text{th}}$ and so the intersection of $B_{n(i)}h_i \cap B_{n(j)}h_j$ is either $B_{n(j)}h_j$ or the empty set.

We can use this to identify a subcollection of $\{B_{n(i)}h_i\}_{i=1}^m$ which covers $E$ exactly once and no point more than once by including each $B_{n(j)}h_j$ in the collection only if it is not contained by any $B_{n(i)}h_i$ with $i \leq j$.

It follows that $(G_n)$ has the BCP with constant $C = 1$. $\square$

### Finitely generated groups

A *mixed* abelian group is one which both contains elements with finite order and with infinite order. The simplest examples of such groups are the finitely generated abelian groups, which by the structure theorem for finitely generated abelian groups can be realised as a direct sum $T \oplus \mathbb{Z}^d$ for a finite torsion group $T$ and some $d \geq 0$.

**Proposition 3.2.2.** *Let $G$ be an abelian group which splits into a direct sum of subgroups $T$ and $H$ such that $T$ is a finite group and $H$ has a summing sequence $(B_n)_{n=1}^{\infty}$ for which the non-singular ergodic theorem holds. Then the non-singular ergodic theorem holds for $G$ with respect to the summing sequence $(TB_n)_{n=1}^{\infty}$.*

*Proof.* Since $G = T \oplus H$ the sets $TB_n$ and $T \times B_n$ are in bijection and therefore

$$\begin{aligned}
\frac{\sum_{g\in TB_n} \hat{g}f}{\sum_{g\in TB_n} \hat{g}1} &= \frac{\sum_{u\in B_n} \hat{u}\left(\sum_{t\in T} \hat{t}f\right)}{\sum_{u\in B_n} \hat{u}\left(\sum_{t\in T} \hat{t}1\right)} \\
&= \frac{\sum_{u\in B_n} \hat{u}\left(\sum_{t\in T} \hat{t}f\right)}{\sum_{u\in B_n} \hat{u}1} \frac{\sum_{u\in B_n} \hat{u}1}{\sum_{u\in B_n} \hat{u}\left(\sum_{t\in T} \hat{t}1\right)} \\
&\to \frac{\mathbb{E}(\sum_{t\in T} \hat{t}f|\mathcal{J})}{\mathbb{E}(\sum_{t\in T} \hat{t}1|\mathcal{J})}
\end{aligned}$$

almost surely, since the ergodic theorem holds for $H$ with respect to $(B_n)$. Now let $A \in \mathcal{J}$, then

$$\int_A \sum_{t\in T} \hat{t}f \, d\mu = \sum_{t\in T} \int_{tA} f \, d\mu = |T| \int_A f \, d\mu$$

and therefore we must have $\mathbb{E}(\sum_{t\in T} \hat{t}f|\mathcal{J}) = |T| \, \mathbb{E}(f|\mathcal{J})$. Since we may take $f \equiv 1$ it follows that

$$\frac{\mathbb{E}(\sum_{t\in T} \hat{t}f|\mathcal{J})}{\mathbb{E}(\sum_{t\in T} \hat{t}1|\mathcal{J})} = \mathbb{E}(f|\mathcal{J})$$

as required. $\square$

Therefore we have an ergodic theorem for any finitely generated abelian group.

**Open problems**

In summary, in this section we have shown that if $G$ is an abelian group and either torsion-free with finite rank, a torsion group or finitely generated then there are sequences for which the ergodic theorem holds.

In contrast, as we discussed in the introduction, Hochman has shown that the group $\bigoplus_{n=1}^{\infty} \mathbb{Z}$ has no summing sequence for which theorem holds [Hoc13]. Since any infinite rank group contains $\bigoplus_{n=1}^{\infty} \mathbb{Z}$ as a subgroup this suggests the result is unlikely to holds for such groups. It could be that Hochman's approach from [Hoc13] can be extended to such groups.

If that is the case, then the remaining abelian groups to study would be the mixed groups with finite rank which are not finitely generated. The simplest examples of such groups are direct sums of an infinite torsion group with a finite rank torsion-free group. However, mixed groups need not split in this way (see e.g. [Fuc15, p. 573]) and so new techniques would need to be developed.

## 3.3 Critical dimensions

For the second half of this chapter we consider the critical dimensions of certain actions of $\mathbb{Z}^d$. The work from the first half of the chapter provides a variety of summing sequences for which the associated critical dimensions are invariants of metric isomorphism. In this section we begin to address a problem raised in the introduction: does the critical dimension depend on the choice of summing sequence and, if so, how?

As discussed earlier, the summing sequence used in the theory of integer actions is $(\{1, ..., n\})_{n=1}^{\infty}$. In contrast, every sequence we considered in the preceding section is symmetric about the identity as they are defined using (right) invariant metrics. We begin by considering how results on the critical dimensions with respect to the sets $(\{1, ..., n\})_{n=1}^{\infty}$ relate to those with symmetric summing sets. We then move on to study critical dimensions of $\mathbb{Z}^d$ actions with respect to summing sequences of norms and, finally, calculate the critical dimensions of product actions on product measure spaces with respect to rectangular metrics in terms of the critical dimensions of the factors.

### 3.3.1 Symmetric summing sets in $\mathbb{Z}$

It will be useful to examine what the critical dimension of a $\mathbb{Z}$-action with respect to $\{1, ..., n\}$ says about the critical dimension with respect to $\{-n, ..., n\}$.

Let $T : X \to X$ be a non-singular transformation describing a $\mathbb{Z}$-action. We shall refer to the critical dimensions of $T$ with the summing sets $\{-n, ..., n\}$ as *standard* and denote the lower and upper standard critical dimensions by $\alpha_+$ and $\beta_+$ respectively. We will denote the lower and upper standard critical dimensions of $T^{-1}$ by $\alpha_-$ and $\beta_-$. Let $L_t^+$, $L_t^-$ denote $L_t$ for $T$ and $T^{-1}$ respectively, with the standard summing sets, and similarly with $U_t$.

**Lemma 3.3.1.** *Let $T : X \to X$ determine a non-singular $\mathbb{Z}$-action. Let $\alpha$ and $\beta$ be the critical dimensions with respect to $\{-n, ..., n\}$. Then*

$$\max(\alpha_+, \alpha_-) \leq \alpha \leq \beta \leq \max(\beta_+, \beta_-).$$

*Proof.* We first prove the result for the lower critical dimension. Observe that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{(2n+1)^t} \sum_{i=-n}^{n} \omega_i(x) &= \frac{1}{2^t} \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=-n}^{-1} \omega_i(x) + \frac{1}{n^t} \sum_{i=1}^{n} \omega_i(x) \\ &\geq \frac{1}{2^t} \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=-n}^{-1} \omega_i(x) + \frac{1}{2^t} \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=1}^{n} \omega_i(x).\end{aligned}$$

Hence $L_t \supseteq L_t^+ \cup L_t^-$ and the result follows. In the other case we get

$$\limsup_{n\to\infty} \frac{1}{(2n+1)^t} \sum_{i=-n}^{n} \omega_i(x) \leq \frac{1}{2^t} \limsup_{n\to\infty} \frac{1}{n^t} \sum_{i=-n}^{-1} \omega_i(x) + \frac{1}{2^t} \limsup_{n\to\infty} \frac{1}{n^t} \sum_{i=1}^{n} \omega_i(x).$$

Therefore $U_t \supseteq U_t^+ \cap U_t^-$ and we are done. □

In particular, if the standard upper and lower critical dimensions of $T$ agree and those of $T^{-1}$ do also then $\alpha = \max(\alpha_+, \alpha_-) = \beta$.

Theorem 1.2.7, due to Mortiss and Dooley, provides a number of situations where the upper and lower critical dimensions with respect to $\{1, ..., n\}$ of a transformation $T$, and those of its inverse, agree. These observations, combined with Lemma 3.3.1, ensure we can produce examples of transformations with a single critical dimension $\alpha = \beta = \gamma$ with respect to $\{-n, ..., n\}$ for any $\gamma \in (0, 1)$.

### 3.3.2 Balls of norms

In this part we show that the critical dimensions for balls of a norm are independent of the choice of norm.

Let $B_r = B_r(0) \cap \mathbb{Z}^d$ where $B_r(0)$ is the closed ball of radius $r$ with respect to a given norm $\|\cdot\|$, let $B'_r$ denote the corresponding set for another norm $\|\cdot\|'$. We consider the summing sequences $(B_n)$ and $(B'_n)$. The proof relies on essentially two properties of these sequences, which we will make precise below. The first is that any two sequences of balls are intertwined, in the sense that each ball is contained by a sufficiently large ball in the other sequence. The second property is that each ball is somewhat well approximated from above and below by balls in the other sequence.

The ideas used here make sense in a general countable group $G$ so we temporarily return to that setting.

Let each of $\{A_n\}_{n=1}^{\infty}$ and $\{A'_n\}_{n=1}^{\infty}$ be an increasing sequence of subsets of $G$. We say $\{A_n\}_{n=1}^{\infty}$ *overlays* $\{A'_n\}_{n=1}^{\infty}$ if for all $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $A'_n \subseteq A_N$. We say $\{A_n\}_{n=1}^{\infty}$ and $\{A'_n\}_{n=1}^{\infty}$ are *interweaving* if both $\{A_n\}_{n=1}^{\infty}$ overlays $\{A'_n\}_{n=1}^{\infty}$ and vice versa. In particular, this is the case if $\bigcup_n A_n = G = \bigcup_n A'_n$, as is the case for the sequences of balls in $\mathbb{Z}^d$ described above.

Suppose $\{A_n\}_{n=1}^{\infty}$ overlays $\{A'_n\}_{n=1}^{\infty}$. Let

$$m(n) = \max\left(k \geq 0 : A'_k \subseteq A_n\right) \qquad \text{and} \qquad M(n) = \min\left(k \geq 0 : A_n \subseteq A'_k\right)$$

where for technical reasons we take $A'_0 = \emptyset$. Then both $m(n)$ and $M(n)$ are increasing with $n$ and diverge as $n \to \infty$. We say $\{A_n\}_{n=1}^{\infty}$ *closely overlays* $\{A'_n\}_{n=1}^{\infty}$ if there exists $\delta \in (0,1)$ such that for all $n$ sufficiently large

$$\min\left\{\frac{|A'_{m(n)}|}{|A_n|}, \frac{|A_n|}{|A'_{M(n)}|}\right\} \geq \delta.$$

Similarly, we say two interweaving sequences $\{A_n\}_{n=1}^{\infty}$ and $\{A'_n\}_{n=1}^{\infty}$ are *closely interweaving* if $\{A_n\}_{n=1}^{\infty}$ closely overlays $\{A'_n\}_{n=1}^{\infty}$ and vice versa. This defines an equivalence relation between these sequences of subsets of $G$.

To see that two sequences of norm balls are closely interweaving take $\|\cdot\|$ to be the supremum norm and observe that, by equivalence of norms, for some

$k \in \mathbb{N}$ we have $B_{r/k} \subseteq B'_r \subseteq B_{kr}$ for all $r > 0$. It follows that

$$\frac{\left|B'_{m(n)}\right|}{|B_n|} \geq \frac{\left|B_{\lfloor n/k \rfloor}\right|}{|B_n|} = \left(\frac{2n/k+1}{2n+1}\right)^d \to k^{-d}$$

and

$$\frac{|B_{m'(n)}|}{|B'_n|} \geq \frac{|B_{\lfloor n/k \rfloor}|}{\left|B_{nk}\right|} \geq \left(\frac{2n/k+1}{2nk+1}\right)^d \to k^{-2d}$$

which deals with the conditions on $m(n)$ and its counterpart. A similar argument applies for $M(n)$, ensuring that every sequence of balls closely interweaves with those of the supremum norm, which suffices due to transitivity.

**Proposition 3.3.2.** *Let $G$ be a countable group with a non-singular ergodic action on a standard finite measure space $(X, \mu)$. Suppose that $\{A_n\}_{n=1}^\infty$ closely overlays $\{A'_n\}_{n=1}^\infty$. Then $L'_t \subseteq L_t$ and $U'_t \subseteq U_t$. Hence $\alpha' \leq \alpha \leq \beta \leq \beta'$ and, in particular, when the two sequences are closely interweaving they have the same upper and lower critical dimensions.*

*Proof.* We just tackle the lower case as the upper case is a similar argument involving the function $M(n)$ and $M'(n)$. Observe that with $N$ taken sufficiently large for all $n \geq N$

$$\begin{aligned}\frac{1}{|A_n|^t}\sum_{g\in A_n}\omega_g(x) &\geq \left(\frac{\left|A'_{m(n)}\right|}{|A'_n|}\right)^t \frac{1}{\left|A'_{m(n)}\right|^t}\sum_{g\in A'_{m(n)}}\omega_g(x)\\ &\geq \delta^{|t|}\frac{1}{\left|A'_{m(n)}\right|^t}\sum_{g\in A'_{m(n)}}\omega_g(x)\end{aligned}$$

and hence

$$\begin{aligned}\inf_{n\geq N}\frac{1}{|A_n|^t}\sum_{g\in A_n}\omega_g(x) &\geq \delta^{|t|}\inf_{n\geq N}\frac{1}{\left|A'_{m(n)}\right|^t}\sum_{g\in A'_{m(n)}}\omega_g(x)\\ &\geq \delta^{|t|}\inf_{n\geq m(N)}\frac{1}{\left|A'_n\right|^t}\sum_{g\in A'_n}\omega_g(x).\end{aligned}$$

By letting $N \to \infty$, and recalling that $m(N) \to \infty$ as $n \to \infty$ it follows that

$$\liminf_{n\to\infty} \frac{1}{|A_n|^t} \sum_{g\in A_n} \omega_g(x) \geq \delta^{|t|} \liminf_{n\to\infty} \frac{1}{|A'_n|^t} \sum_{g\in A'_n} \omega_g(x)$$

and hence $L'_t \subseteq L_t$. The same argument holds with the sequences exchanged. □

**Corollary 3.3.3.** *The upper and lower critical dimensions with respect to a summing sequence $B_n = \{u \in \mathbb{Z}^d : \|u\| \leq n\}$, where $\|\cdot\|$ is a norm on $\mathbb{R}^d$, are independent of the choice of norm.*

As one might expect it is not difficult to see that the fact the sequences are *closely* interweaving is necessary to the above argument. Consider, for example, the sequences $A'_n = \{-n, ..., n\}^2$ and

$$A_n = \{-\lfloor e^n - 1\rfloor, ..., \lfloor e^n - 1\rfloor\} \times \{-n, ..., n\} = [-(e^n - 1), e^n - 1] \times [-n, n] \cap \mathbb{Z}^2.$$

We have $m(n) = n$ and hence

$$\frac{\left|A'_{m(n)}\right|}{|A_n|} = \frac{(2n+1)^2}{(2n+1)(2\lfloor e^n - 1\rfloor + 1)} \to 0.$$

This means that the argument used in the above proof fails if one attempts to compare balls of arbitrary rectangular metrics to those of norms. Next we show that these sequences give rise to different critical dimensions for numerous actions.

### 3.3.3 Product actions with rectangular balls

We examine non-singular product actions, which are constructed as follows. Suppose that for each $1 \leq i \leq d$ we are given a non-singular transformation $T_i : X_i \to X_i$ on a probability space $(X_i, \mu_i)$, the factors of the product. We can define a non-singular $\mathbb{Z}^d$-action on the product measure space $X = X_1 \times ... \times X_d$ with measure $\mu = \mu_1 \times ... \times \mu_d$ via

$$(u_1, ..., u_d) \cdot (x_1, ..., x_n) = (T_1^{u_1} x_1, ..., T_d^{u_d} x_d).$$

**Lemma 3.3.4.** *A product action of $\mathbb{Z}^d$ on $(X, \mu)$, as defined above, is ergodic if and only if every $T_i$ is ergodic.*

*Proof.* Let us first show the ergodicity of the whole action implies that $T_1$ ergodic, a corresponding argument will work for each other $T_i$. Let $A \subseteq X_1$ be an $T_1$-invariant measurable set. Then $A \times X_2 \times ... \times X_d$ is invariant under the product action and hence by the ergodicity of the product action

$$\mu_1(A) = \mu_1(A)\mu_2(X_2)...\mu_d(X_d) = \mu(A \times X_2 \times ... \times X_d) \in \{0, 1\},$$

as required.

Now suppose each the dynamical systems $(X_i, \mu_i, T_i)$ is ergodic. Let $C \subseteq X$ be an invariant measurable set. Then

$$\mu(C) = \int_{X_d} ... \int_{X_1} \mathbf{1}_C \, d\mu_1 \, ... \, d\mu_d$$

and the maps $x_1 \mapsto \mathbf{1}_C(x_1, ..., x_d)$ are $T_1$ invariant and so is $\mu_1$ almost surely equal to either 0 or 1. Hence the maps $x_2 \mapsto \int_{X_1} \mathbf{1}_C(x_1, ..., x_d) \, d\mu_1(x_1)$ take values in 0 or 1, and are $T_2$-invariant so are $\mu_2$ almost surely equal to either 0 or 1. We can continue in this fashion to deduce that

$$x_d \mapsto \int_{X_{d-1}} ... \int_{X_1} \mathbf{1}_C(x_1, ..., x_d) \, d\mu_1 \, ... \, d\mu_{d-1}$$

is $\mu_d$ almost surely constant, and takes values in $\{0, 1\}$. Integrating over $\mu_d$ shows that $\mu(C) \in \{0, 1\}$. $\square$

We consider the upper and lower critical dimensions with respect to sequences of rectangles $B_n = B_n^1 \times ... \times B_n^d$ where each $B_n^i = \{-s_i(n), ..., s_i(n)\}$ for some increasing functions $s_i : \mathbb{N}_0 \to \mathbb{N}_0$. This setup includes the summing sequences induced by rectangular metrics. For each $1 \le i \le d$ we write $\alpha_i$ and $\beta_i$ for the lower and upper critical dimensions of $T_i$ with respect to $\{-n, ..., n\}$, taken in the space $(X_i, \mu_i)$.

Given two increasing functions $s, s' : \mathbb{N} \to \mathbb{N}_{>1}$ we write $s \lesssim s'$ and say $s$ is *controlled by* $s'$ if

$$\liminf_{n \to \infty} \frac{\log s'(n)}{\log s(n)} > 0.$$

The relation $\lesssim$ defines a preorder on the space such functions, and this preorder is total. We can use $\lesssim$ to define an equivalence relation by declaring that $s$ and

$s'$ have *equivalent growth*, denoted $s \approx s'$, if both $s \lesssim s'$ and $s' \lesssim s$, i.e. if

$$0 < \liminf_{n\to\infty} \frac{\log s'(n)}{\log s(n)} \leq \limsup_{n\to\infty} \frac{\log s'(n)}{\log s(n)} < \infty.$$

This definition ensures that all the function $\lfloor n^t \rfloor$ for $t > 0$ are in the same equivalence class, but $\lfloor e^n - 1 \rfloor$ is strictly greater.

Using the axiom of choice we may fix a representative of each equivalence class. Suppose that $\bar{s}$ is the representative of the equivalence class of $s$, then we set

$$a(s) = \liminf_{n\to\infty} \frac{\log s(n)}{\log \bar{s}(n)} \qquad \text{and} \qquad b(s) = \limsup_{n\to\infty} \frac{\log s(n)}{\log \bar{s}(n)}.$$

When referring to rectangles $B_n$ as above let us write $a_i = a(s_i)$ and $b_i = b(s_i)$ wherever there is no ambiguity.

Our main result of this part provides bounds for the critical dimensions with respect to the rectangles $B_n$ in terms of the critical dimensions of the product transformations and the growth rates of the rectangle sides.

**Theorem 3.3.5.** *Let $\mathbb{Z}^d$ act on a product space $(X, \mu)$ via a non-singular and ergodic product action, as described above. Let $D \subseteq \{1, ..., d\}$ such that for each $i \in D$ the function $s_i$ is a greatest element in $\{s_1, ..., s_d\}$ with respect to $\lesssim$. Then*

$$\frac{\sum_{i\in D} a_i \alpha_i}{\sum_{i\in D} b_i} \leq \alpha \leq \beta \leq \frac{\sum_{i\in D} b_i \beta_i}{\sum_{i\in D} a_i}.$$

Note that these bounds may depend on the choice of representative $\bar{s}$, but the inequalities remain the same if $\bar{s}$ is replaced by any $s$ for which the limit $\lim_{n\to\infty} \frac{\log s(n)}{\log \bar{s}(n)}$ exists and is non-zero. One usually chooses functions $s_i$ which are related to one another in this way, and then in addition the representative can then be chosen such that $a_i = b_i$ for all $i$. The benefit of the above more general formulation of the theorem is that it allows for some sides of the rectangles to grow rather slowly for periods of time but then 'catch up' later.

The inner bound is true by definition, the two outer bounds have slightly different proofs but both rely on two key ideas.

The first is that a small portion of the growth from the fastest growing sides of the rectangles can be used to dominate and hence neglect the behaviour from the slower growing sides. The second idea is that the rates of growth from the fastest growing sides can be compared using the representative of their equivalence class, resulting in the weighted average of critical dimensions seen above.

We first prove the lower bound, where growth from the slow growing sides is absorbed by the faster sides.

**Lemma 3.3.6.** *Let $\mathbb{Z}^d$ act on a product space $X$ via a non-singular and ergodic product action, as described above. Let $D \subseteq \{1, ..., d\}$ such that for each $i \in D$ the function $s_i$ is a greatest element in $\{s_1, ..., s_d\}$ with respect to $\lesssim$. Then*

$$\alpha \geq \frac{\sum_{i \in D} a_i \alpha_i}{\sum_{i \in D} b_i}.$$

*Proof.* Suppose

$$t = \frac{\sum_{i \in D}(a_i - \epsilon)(\alpha_i - 2\epsilon)}{\sum_{i \in D} b_i}$$

for some $\epsilon > 0$. It follows from considering cylinder sets and applying Fubini's theorem that for $u \in \mathbb{Z}^d$ we have $\omega_u(x) = \prod_{i=1}^d \omega^i_{u_i}(x)$ where

$$\omega^i_j(x) = \frac{d\mu_i \circ T_i^j}{d\mu_i}(x_i).$$

Then

$$\frac{1}{|B_n|^t} \sum_{u \in B_n} \omega_u = \frac{1}{2^{dt}} \frac{1}{(\prod_{i=1}^d s_i(n))^t} \prod_{i=1}^d \sum_{j \in B_n^i} \omega^i_j. \tag{3.3.1}$$

Let $\bar{s}$ be the representative of the growth equivalence class of the $s_i$ with $i \in D$ and fix a positive real number $\delta$. For $i \notin D$ we have

$$\liminf_{n \to \infty} \frac{\log s_i(n)}{\log \bar{s}(n)} = 0.$$

Hence for $i \notin D$ for all $n$ sufficiently large $s_i(n) \leq \bar{s}(n)^\delta$. By definition for $i \in D$ for large $n$ we must have $\bar{s}(n)^{a_i - \epsilon} \leq s_i(n) \leq \bar{s}(n)^{b_i + \delta}$. Therefore, for all sufficiently large $n$ we have

$$\prod_{i=1}^d s_i(n) \leq (\bar{s}(n))^{d\delta + \sum_{i \in D} b_i}$$

and so for some $\eta = O(\delta)$ we have

$$\left(\prod_{i=1}^{d} s_i(n)\right)^t \leq (\bar{s}(n))^{\sum_{i\in D}(a_i-\epsilon)(\alpha_i+\eta-2\epsilon)} \leq \prod_{i\in D}(s_i(n))^{\alpha_i+\eta-2\epsilon}.$$

As we retain the freedom to shrink $\delta$ we can assume that each $\eta < \epsilon$ to deduce that for large enough $n$

$$\frac{1}{|B_n|^t}\sum_{u\in B_n}\omega_u \geq \frac{1}{2^{dt}}\left(\prod_{i\notin D}\sum_{j\in B_n^i}\omega_j^i\right)\left(\prod_{i\in D}\frac{1}{s_i(n)^{\alpha_i-\epsilon}}\sum_{j\in B_n^i}\omega_j^i\right). \tag{3.3.2}$$

The first bracket is always at least 1 and each term of the latter product diverges to infinity. Hence we see that $\alpha \geq t$, but since $\epsilon > 0$ was arbitrary the lemma follows. □

For the upper bound a little of the growth from the fast growing sides of the rectangles is used to dominate the slower sides.

**Lemma 3.3.7.** *Let $\mathbb{Z}^d$ act on a product space $X$ via a non-singular and ergodic product action, as described above. Let $D \subseteq \{1, ..., d\}$ such that for each $i \in D$ the function $s_i$ is a greatest element in $\{s_1, ..., s_d\}$ with respect to $\lesssim$. Then*

$$\beta \leq \frac{\sum_{i\in D} b_i\beta_i}{\sum_{i\in D} a_i}.$$

*Proof.* The result is trivial if any $\beta_i = \infty$, so assume not. Suppose

$$t = \frac{\sum_{i\in D}(b_i+\epsilon)(\beta_i+2\epsilon)}{\sum_{i\in D} a_i}$$

for some $\epsilon > 0$. Let $\bar{s}$ be the representative of the $s_i$ with $i \in D$ and fix $\delta > 0$. By definition for $i \in D$ and $n$ sufficiently large $\bar{s}(n)^{a_i-\delta} \leq s(n) \leq \bar{s}(n)^{b_i+\epsilon}$. Hence for these $n$

$$\prod_{i=1}^{d} s_i(n) \geq \bar{s}(n)^{-|D|\delta+\sum_{i\in D} a_i}$$

and so for some $\eta = O(\delta)$ we have

$$\left(\prod_{i=1}^{d} s_i(n)\right)^t \geq \bar{s}(n)^{-\eta+\sum_{i\in D}(b_i+\epsilon)(\beta_i+2\epsilon)} \geq \bar{s}(n)^{-\eta+\epsilon\sum_{i\in D} b_i}\left(\prod_{i\in D} s_i(n)^{\beta_i+\epsilon}\right).$$

By shrinking $\delta$ we can assume that $c = \frac{1}{d-|D|}\left(\epsilon \sum_{i\in D} b_i - \eta\right) > 0$ and use (3.3.1) to deduce that for large $n$

$$\frac{1}{|B_n|^t}\sum_{u\in B_n}\omega_u \leq \frac{1}{2^{dt}}\left(\prod_{i\notin D}\frac{1}{\bar{s}(n)^c}\sum_{j\in B_n^i}\omega_j^i\right)\left(\prod_{i\in D}\frac{1}{s_i(n)^{\beta_i+\epsilon}}\sum_{j\in B_n^i}\omega_j^i\right).$$

For each $i \notin D$ eventually $\bar{s}(n)^c \geq s_i(n)^{\beta_i+\delta}$ and so each term in the first product tends to 0. Similarly with each of the terms in the second product. Hence we see that $\beta < t$, but since $\epsilon > 0$ was arbitrary the lemma follows. □

This completes the proof of Theorem 3.3.5. We can combine it with the integer theory to start to answer our question about the dependence of the critical dimensions on the summing sequence chosen.

We have seen that it is possible to produce transformations with any (single) critical dimension in $(0, 1)$, see Theorem 1.2.7. By constructing the product action using such $T_i$, and choosing the $s_i$ to ensure $a_i = b_i$ for all $i \in D$, by Theorem 3.3.5 the resulting actions will have critical dimension

$$\gamma = \frac{\sum_{i\in D} a_i\gamma_i}{\sum_{i\in D} a_i}.$$

We are now equipped to examine some specific examples which answer our earlier question.

### Values taken by the critical dimension

The simplest examples to consider are those where $s_1(n) = s_2(n) = ... = n$ which all satisfy $a(s_i) = 1$ with respect the natural choice of representative of their class, $\bar{s}(n) = n$. Then in the above circumstances there is a single critical dimension

$$\gamma = \frac{\gamma_1 + ... + \gamma_d}{d}.$$

This in turn means that for any $d$ and $r \in (0, 1)$ we can produce a $\mathbb{Z}^d$-action with critical dimension $r$.

**Dependence on the choice of summing set**

Consider a $\mathbb{Z}^2$-action, constructed via the method above, and its critical dimension with respect to

$$\{-n, ..., n\} \times \{-\lfloor e^n - 1 \rfloor, \lfloor e^n - 1 \rfloor\}.$$

Here $s_2$ grows strictly faster than $s_1$ and, with the sensible choice representatives, the critical dimension is seen to be $\gamma = \gamma_2$. This, taken with the last example, shows that the critical dimension very much depends on the choice of summing sequence. It also shows that critical dimensions of the factors can be deduced from those of the product action and vice-versa.

In fact, any desired weighting of the critical dimensions can be achieved. Suppose $t_i \in [0,1]$ such that $t_1 + ... + t_d = 1$. By taking $s_i(n) = n$ if $t_i = 0$ and $s_i(n) = \lfloor (e^n - 1)^{t_i} \rfloor$ otherwise. Then the critical dimension of the product action with respect to corresponding summing sequence is given by $\gamma = t_1\gamma_1 + ... + t_d\gamma_d$. Moreover, as each such summing sequence is induced by a rectangular metric, each of these weightings is an invariant of metric isomorphism.

**Open problems**

We have shown in the case of product actions on product spaces that the critical dimension for rectangles can be decomposed into a weighted average of the critical dimensions, for the projected measures, of maps corresponding to $e_1, ..., e_n$. It is an open question whether this extends more generally, for example the critical dimension of each $e_i$ can be calculated on $(X, \mu)$ as a $\mathbb{Z}$-action regardless of whether the $\mathbb{Z}^d$-action is a product action. Therefore it is reasonable to ask how the critical dimension of the $\mathbb{Z}^d$-action is related to those of the generators.

# Chapter 4

# The Heisenberg Groups

In this chapter we will show that a non-singular ergodic theorem holds for the discrete Heisenberg groups.

This case is of interest because the Heisenberg groups are amoungst the 'nicest' of non-abelian groups, in the sense that they are very close to being abelian and in particular have much in common with groups such as $\mathbb{Z}^d$. If the non-singular ergodic theorem can be extended to a large class of other non-abelian groups then it would be surprising for it not to include this family. However, as mentioned in the introduction, Hochman has proved that if $G$ is taken to be the discrete Heisenberg group and $B_n = B^n$, where $B$ is the collection of standard generators of $G$, then the ratio ergodic theorem fails for every subsequence of $(B_n)$ [Hoc13]. The key obstacle cited in the paper is the failure of the sequence $(B_n)$ to satisfy the Besicovitch covering property. This was thought to be serious obstacle because in the case of the Heisenberg group this property does not just fail for the sequence $(B^n)$ (balls of a word metric) but for the sequences of integer balls for the Korányi distance and the Carnot-Carathéodory metric (see [Rig04, SW92]); two of most the natural and well studied distances on the Heisenberg groups. This means that, in order to use techniques such as those in this thesis it is necessary to identify a new, or less well known, metric which respects the structure on the Heisenberg group and has the desired properties.

A candidate for such a metric was recently highlighted through work done by Le Donne and Rigot in [LDR17]. They showed that a metric $\rho$ identified by Hebisch and Sikora [HS90], which we will define shortly, has the metric Besicovitch covering property. As with norms, this will allow us to deduce its corresponding summing sequence has the BCP, thereby overcoming the obstacle identified by Hochman. The major content of this chapter is in proving that this candidate metric satisfies the other hypotheses of Theorem 2.3.1, a preprint of this work can be found at [Jar17].

This chapter has also been submitted to a journal as part of a paper also containing an earlier version of the work in Chapter 2.

## 4.1 Defining the group and basic properties

The classical discrete Heisenberg group is the matrix group

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

of $3 \times 3$ upper triangular matrices with integer entries and ones on the diagonal.

This definition can be extended into higher dimensions. The *$d$-dimensional discrete Heisenberg group* is as the matrix group

$$\left\{ \begin{pmatrix} 1 & a^T & c \\ 0 & I_d & b \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}^d, c \in \mathbb{Z} \right\}$$

which is generated by the collection

$$\left\{ \begin{pmatrix} 1 & e_j^T & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_d & e_j \\ 0 & 0 & 1 \end{pmatrix} : 1 \leq i \leq d \right\}$$

where $e_1, ..., e_d$ is the standard basis of $\mathbb{R}^d$.

This can be seen by noting

$$\begin{pmatrix} 1 & a^T & c \\ 0 & I_d & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a^T & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c - a^T b \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_d & b \\ 0 & 0 & 1 \end{pmatrix}$$

then observing that

$$\begin{pmatrix} 1 & 0 & m \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix}^m$$

and finally that

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e_1 & 0 \\ 0 & I_d & -e_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -e_1 & 0 \\ 0 & I_d & e_1 \\ 0 & 0 & 1 \end{pmatrix}$$

which suffices because

$$\begin{pmatrix} 1 & e_1^T & 0 \\ 0 & I_d & -e_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e_1^T & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_d & -e_1 \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly in the other case.

*Remark* 4.1.1. The discrete Heisenberg group is not abelian, for example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_d & e_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & e_1^T & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e_1^T & 0 \\ 0 & I_d & e_1 \\ 0 & 0 & 1 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & e_1^T & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_d & e_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e_1^T & 2 \\ 0 & I_d & e_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, it does not fit into the setting of Chapter 3.

The discrete group can be extended to the *d-dimensional continuous Heisenberg group* by replacing each occurance of $\mathbb{Z}$ with $\mathbb{R}$ in the definition of the matrix group.

However, we are going to make use of an alternative realisation, which emphasises the geometrical structures we will use and has the added benefit of being far more notationally efficient. The $d$-dimensional continuous Heisenberg group

$\mathbb{H}^d$ can also be realised as follows. As a set, take $\mathbb{H}^d = \mathbb{C}^d \times \mathbb{R}$ and equip it with the multiplication given by

$$(z, \tau) \cdot (w, \sigma) = \left( z + w, \tau + \sigma + \frac{1}{2} \mathrm{Im} \, \langle z, w \rangle \right)$$

where $z, w \in \mathbb{C}^d$, $\tau, \sigma \in \mathbb{R}$ and the inner product is the standard one on $\mathbb{C}^d$, given by $\langle z, w \rangle = \sum_{j=1}^d \overline{z_j} w_j$. This is essentially the same realisation as that used by Le Donne and Rigot in [LDR17] except we are using complex coordinates.

The map

$$\theta : \left\{ \begin{pmatrix} 1 & a^T & c \\ 0 & I_d & b \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}^d, c \in \mathbb{R} \right\} \to \mathbb{H}^d$$

given by

$$\begin{pmatrix} 1 & a^T & c \\ 0 & I_d & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \left( a + ib, c - \frac{1}{2} \langle a, b \rangle \right)$$

is an isomorphism from the matrix realisation to the one using complex coordinates.

The *d-dimensional discrete Heisenberg group* $\mathrm{H}^d$ is realised as the discrete subgroup generated by the elements of the form $(e_j, 0)$ or $(ie_j, 0)$. As a set

$$\mathrm{H}^d = \{(z, \tau) \in \mathbb{H}^d : z \in \mathbb{Z}^d + i\mathbb{Z}^d, \tau \in \tfrac{1}{2} \langle \mathrm{Re}\, z, \mathrm{Im}\, z \rangle + \mathbb{Z}\}.$$

We will, once we have considered the metric, end up taking

$$G_1 = G_2 = ... = G = \mathrm{H}^d$$

and $\tilde{G} = \mathbb{H}^d$ in Theorem 2.3.1.

For each $\lambda > 0$ there is a dilation map $\delta_\lambda : \mathbb{H}^d \to \mathbb{H}^d$ given by

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 \tau).$$

Each $\delta_\lambda$ is an automorphism of $\mathbb{H}^d$.

To describe the range of the sums in the ergodic theorem we will be using the balls of the metric

$$\rho\left(p,q\right)=\inf\left\{r>0:\delta_{1/r}(pq^{-1})\in B_{eucl}\right\}$$

where $p,q\in\mathbb{H}^d$ and $q^{-1}=(w,\sigma)^{-1}=(-w,-\sigma)$. This is one from the class of metrics identified by Hebisch and Sikora in [HS90] where $B_{eucl}$ denotes the closed euclidean unit ball in $\mathbb{C}^d\times\mathbb{R}$. It is the right invariant version of the metric given in [LDR17] with $\alpha=1$. It is *one-homogeneous* with respect to the dilation, meaning that for all $\lambda>0$ and $p,q\in\mathbb{H}^d$ we have $\rho(\delta_\lambda p,\delta_\lambda q)=\lambda\,\rho(p,q)$. By considering the case $q=0$ and using right invariance it is not difficult to show that for $p=(z,\tau)$ and $q=(w,\sigma)$

$$\rho(p,q)\leq r \qquad\Longleftrightarrow\qquad \frac{\|z-w\|^2}{r^2}+\frac{\left(\tau-\sigma-\frac{1}{2}\mathrm{Im}\left\langle z,w\right\rangle\right)^2}{r^4}\leq 1 \tag{4.1.1}$$

and

$$\rho(p,q)=r \qquad\Longleftrightarrow\qquad \frac{\|z-w\|^2}{r^2}+\frac{\left(\tau-\sigma-\frac{1}{2}\mathrm{Im}\left\langle z,w\right\rangle\right)^2}{r^4}=1 \tag{4.1.2}$$

where $\|\cdot\|$ is the euclidean norm on $\mathbb{C}^d$. In particular, taking $r=1$ and $q=0$ shows that the unit sphere of $\rho$ is exactly the Euclidean unit sphere, and similarly for the unit ball. This property is key to many of the coming arguments. It also follows from (4.1.2) that

$$\rho(p,q)=\frac{1}{\sqrt{2}}\left(\|z-w\|^2+\sqrt{\|z-w\|^4+4\left(\tau-\sigma-\frac{1}{2}\mathrm{Im}\left\langle z,w\right\rangle\right)^2}\right)^{\frac{1}{2}}. \tag{4.1.3}$$

This explicit expression can be used to show that $\rho$ is in fact a metric. In addition, as stated in [LDR17], $\rho$ induces the euclidean topology. Therefore $\rho$ defines a (right) *homogeneous distance* on $\mathbb{H}^d$, i.e. it induces the euclidean topology, is right invariant and one-homogeneous for the dilation.

Observe that we can use the dilations and right invariance to describe any ball in $\mathbb{H}^d$, explicitly for each $r>0$ and $p\in\mathbb{H}^d$ the closed ball $B_r(p)=\delta_r(B_{eucl})\cdot p$. Since the dilation is a linear map and right multiplication by $p$ is an affine map it follows that each ball is convex, in the euclidean sense.

It will be useful for us to note the following. Let $R_\theta$ be the $d \times d$ complex diagonal matrix with $R_\theta(j,j) = e^{i\theta_j}$ where $\theta = (\theta_j)_{j=1}^d \in \mathbb{R}^d$. Then the maps

$$(z, \tau) \mapsto (\overline{z}, -\tau) \qquad \text{and} \qquad (z, \tau) \mapsto (R_\theta z, \tau) \tag{4.1.4}$$

are isometries of $\rho$.

We will use $\rho$ to denote both the metric on $\mathbb{H}^d$ and its restriction to $\mathrm{H}^d$. Since the balls of the restriction of $\rho$ can be represented as $\mathrm{H}^d \cap B_r(p) = \mathrm{H}^d \cap B_r(0) \cdot p$, with $p \in \mathrm{H}^d$ and $B_r(0) \subseteq \{(z,\tau) : \|z\| \leq r, |\tau| \leq r^2\}$, it is clear from the set representation of $\mathrm{H}^d$ that the restriction of any ball to $\mathrm{H}^d$ is finite. This means we can define the summing sequence $(B_n)$ as in Chapter 2, with all the levels taken to be $\mathrm{H}^d$.

We are now ready to start checking the five conditions of Theorem 2.3.1 are satisfied and hence deduce the ergodic theorem.

**Theorem 4.1.2.** *Let $\mathrm{H}^d$ be the $d$-dimensional discrete Heisenberg group and $B_n = \mathrm{H}^d \cap B_n(0)$ where $B_n(0) = \{q \in \mathbb{H}^d : \rho(q, 0) \leq n\}$, with $\rho$ defined as above. Then the non-singular ergodic theorem for $\mathrm{H}^d$ holds with the summing sequence $(B_n)$.*

Since $\rho$ is right invariant and all the balls are euclidean convex (and so are path connected) this setup satisfies property (c), that $\mathbb{H}^d$ is voidless. It has already been mentioned that the central result of [LDR17] is that $\rho$ satisfies the metric Besicovitch covering property on $\mathbb{H}^d$. By applying the same argument as in Corollary 3.1.5 it follows that our summing sequence $B_n = \mathrm{H}^d \cap B_n(0)$ has the Besicovitch covering property (e). We will examine property (b), multiplicative doubling, just below. Properties (a) and (d), well-separability and the finite intersection dimension, are somewhat more complicated and will make use of the following lemma, analogous to Lemma 3.1.2.

**Lemma 4.1.3.** *Let $(\mathbb{H}^d, \rho)$ be as above and $\zeta > 0$. There exists $N(\zeta) \in \mathbb{N}$ such that there are $N$ open balls of radius $\zeta/2$ centred in $B_1(0)$ whose union covers $B_1(0)$. Suppose $p_1, ..., p_n \in B_1(0)$ then*

(i) *if for all $i \neq j$ $\rho(p_i, p_j) > \zeta$ then $n \leq N$, or*

(ii) *if* $n \geq kN$ *for some* $k \in \mathbb{N}$ *then there is a subset* $I \subset \{1, ..., n\}$ *of size at least* $k$ *with* $\rho(p_i, p_j) < \zeta$ *for all* $i, j \in I$.

*Proof.* Since the metric $\rho$ induces the euclidean topology the closed unit ball $B_1(0) = B_{eucl}$ is compact, and the existence of such an $N$ follows from this compactness. Part (i) is due to the fact that if two points lie in the same ball in the cover then they are $< \zeta$ apart, and part (ii) uses this along with the pigeon-hole principle. $\square$

In particular, if we let $r > 0$, $p \in \mathbb{H}^d$ and $\zeta = 1$ in Lemma 4.1.3 then $B_r(p) = \delta_r(B_{eucl}) \cdot p$ can be covered by $N(1)$ balls of radius $\frac{r}{2}$ (simply dilate and translate those used to cover $B_{eucl}$). This means exactly that $(\mathbb{H}^d, \rho)$ has the metric doubling property.

We can show that the summing sequence has the multiplicative doubling property using a similar technique to the one employed to prove Corollary 3.1.3, except here we can afford not to use fundamental domains, essentially since $\mathrm{H}^d$ has no accummulation points.

**Corollary 4.1.4.** *The sequence given by* $B_n = B_n(0) \cap \mathrm{H}^d$ *has the multiplicative doubling property.*

*Proof.* Fix $0 < r < \frac{1}{2} \inf \{\rho(p, q) : p, q \in \mathrm{H}^d, p \neq q\} = \frac{1}{2}$ and

$$s = \sup \{\rho(p, \mathrm{H}^d) : p \in \mathbb{H}^d\} \leq \frac{1}{2}\sqrt{d + \sqrt{d^2 + 4}}.$$

Let $\nu$ be the right invariant Haar measure on $\mathbb{H}^d$ and $\zeta = 2\frac{m-s}{2m+r}$ where $m \in \mathbb{N}$ is taken sufficiently large to ensure $\zeta \geq \frac{2}{3}$. Then by applying Lemma 4.1.3 (between dilating by $(2m + r)^{-1}$ and $2m + r$) it follows that

$$\begin{aligned} |B_m^2|\nu(B_r(0)) = \nu\left(\bigcup_{p \in B_m^2} B_r(p)\right) &\leq \nu\left(B_{2m+r}(0)\right) \\ &\leq N(\zeta)\,\nu(B_{m-s}(0)) \\ &\leq N\nu\left(\bigcup_{p \in B_m} B_s(p)\right) \leq N\nu(B_s(0))|B_m| \end{aligned}$$

where $N = N(2/3)$. The result follows since balls with strictly positive radius have positive Haar measure. $\square$

The remaining properties are (a), that $(\mathbb{H}^d, \rho)$ is well separable, and (d), that it has finite intersection dimension. These require a bit more work, and are tackled in the following sections.

## 4.2 Intersection dimension

We start with the intersection dimension. Recall that in order to prove the intersection dimension is $\kappa$ we must show that given a sequence of points $p_1, ..., p_m$ with $m \geq \kappa$ and thickened spheres about those points, with some conditions on the thickenings and radii, the intersection of these thickened spheres is empty. We will prove this in two stages. The first is to repeatedly apply the principle that if, by increasing $m$, we can find a subsequence of arbitrary length with an additional property then we can replace the original sequence with this subsequence (as we used to prove Proposition 3.1.4). The lemmas in this section will be used to impose these extra properties on the sequence. In the second stage we will use these to show that the resulting sequence of thickened spheres will have empty intersection if it is sufficiently long.

### 4.2.1 Separation lemmas

Given $p \in \mathbb{H}^d \setminus \{0\}$ let $\hat{p}$ be the unique element of the form $\delta_t p$ on the unit sphere, i.e. $\hat{p} = \delta_{1/\lambda} p$ where $\lambda = \rho(p, 0) > 0$. We will call $\hat{p}$ the *projection* (of $p$) onto the unit sphere.

The first lemma, below, will be used to show that if radii of the earlier spheres are not too large compared to later ones, and 0 is in their intersection, then their projections must be a fixed distance apart. This will allow us to assume that each radius is rather small compared to those preceding it.

**Lemma 4.2.1** (Large scale separation). *Let $p, q \in \mathbb{H}^d$ be non-zero and assume that $0 \in \partial_t B_r(p) \cap \partial_{\tilde{t}} B_{\tilde{r}}(q)$ and $q \in \partial_t B_r(p)$ where $t, \tilde{t} \geq 1$, $r \geq \tilde{r} \geq t\tilde{t}R$ and $R > 1$. Given $\epsilon \in (0, 1)$ such that $\tilde{r} \geq \epsilon r$ there exists $\bar{R}(\epsilon) > 0$ such that if $R > \bar{R}$ then*

$$\rho(\hat{p}, \hat{q}) \geq \frac{1}{2}\left(1 - \sqrt{1 - \frac{\epsilon^2}{4}}\right) > 0.$$

*Proof.* The triangle inequality ensures that $\rho(p,q) = r + s'$ and $\rho(p,0) = r + s$ for some $s, s'$ such that $|s|, |s'| \le t$, and $\rho(q,0) = \tilde{r} + \tilde{s}$ for some $\tilde{s}$ with $|\tilde{s}| \le \tilde{t}$. Therefore

$$\frac{r+s'}{r+s} = \rho(\hat{p}, \delta_\lambda \hat{q}) \le \rho(\hat{p}, \hat{q}) + \rho(\hat{q}, \delta_\lambda \hat{q})$$

where $\lambda = \frac{\tilde{r}+\tilde{s}}{r+s}$. Note that

$$\frac{\epsilon}{2} \le \frac{\epsilon - R^{-1}}{1+R^{-1}} \le \lambda = \frac{\tilde{r}/r + \tilde{s}/r}{1+s/r} \le \frac{1+R^{-1}}{1-R^{-1}}$$

for $R$ sufficiently large, depending on $\epsilon$. Since $|1-\lambda|^2 \le |1-\lambda^2|$ it follows from (4.1.3) that

$$\rho(\hat{q}, \delta_\lambda \hat{q}) \le \sqrt{|1-\lambda^2|}\rho(0,\hat{q}) = \sqrt{|1-\lambda^2|}.$$

Therefore if $\lambda \le 1$ then

$$\rho(\hat{p}, \hat{q}) \ge 1 - \frac{2R^{-1}}{1+R^{-1}} - \sqrt{1-\frac{\epsilon^2}{4}} \ge \frac{1}{2}\left(1 - \sqrt{1-\frac{\epsilon^2}{4}}\right)$$

for $R$ sufficiently large. Otherwise if $\lambda > 1$ then

$$\rho(\hat{p}, \hat{q}) \ge 1 - \frac{2R^{-1}}{1+R^{-1}} - \sqrt{\frac{2R^{-1}}{1-R^{-1}}} \ge \frac{1}{2}\left(1 - \sqrt{1-\frac{\epsilon^2}{4}}\right)$$

again for $R$ large enough. □

For the purposes of the remainder of this section it is useful to introduce a coordinate system on $\mathbb{H}^d$ which exploits the dilations and the fact that the unit sphere of $\rho$ is the Euclidean unit sphere. It is here that we are directly using properties of $(\mathbb{H}^d, \rho)$.

Given $p \in \mathbb{H}^d \setminus \{0\}$ let $\lambda_p = \rho(p,0) > 0$. Then $\hat{p} = \delta_{1/\lambda_p} p = (z_p, \tau_p)$ for some unique $z_p \in \mathbb{C}^d$ and $\tau_p \in \mathbb{R}$ with $\|z_p\|^2 + \tau_p^2 = 1$. In addition, using complex coordinates we have $\zeta(p) = (\zeta_i(p))_{j=1}^d \in \mathbb{R}^d_{\ge 0}$ and $\phi(p) = (\phi_j(p))_{j=1}^d \in (-\pi, \pi]^d$ such that $z_p = (\zeta_j(p)\exp[i\phi_j(p)])_{j=1}^d$. Given also $q \in \mathbb{H}^d \setminus \{0\}$ for each $1 \le j \le n$ let $\varphi_j(p,q) \in [0,\pi)$ denote the magnitude of the angle between $\exp[i\phi_p(j)]$ and

$\exp[i\phi_q(j)]$ in $\mathbb{C}$.

By applying Lemma 4.1.3 we will be able to assume that $\hat{p}_1, ..., \hat{p}_m$ are close on the unit sphere, and Lemma 4.2.1 will then allow us to assume that the radius of each sphere is small compared to the previous one. This is when we will use the following small scale separation lemmas to narrow down the possible positions of $\hat{p}_1, ..., \hat{p}_m$ relative to one another.

**Lemma 4.2.2** (Small scale separation 1)**.** *Given any $\bar{\tau} \in (0,1)$ there exist $\bar{\xi}, \bar{R}, \bar{\phi}, \bar{\epsilon} > 0$ for which the following holds. Let $p, q \in \mathbb{H}^d \setminus \{0\}$ with $q \in \partial_t B_r(p)$ and $0 \in \partial_t B_r(p) \cap \partial_{\tilde{t}} B_{\tilde{r}}(q)$ where $t, \tilde{t} \geq 1$ and suppose $r \geq \tilde{r} \geq t\tilde{t}R$ for some $R > 1$. Suppose also that $\tilde{r} \leq \epsilon r$. If $R > \bar{R}$, $\epsilon < \bar{\epsilon}$, $|\tau_p| \leq \bar{\tau}$ and $\max_{1 \leq i \leq d} \varphi_i(p,q) < \bar{\phi}$ then $\rho(\hat{p}, \hat{q}) > \bar{\xi}$.*

The condition that $\tau_p$ is bounded away from $\pm 1$ is the crucial feature distinguishing this lemma, and its proof, from the similar second small scale separation lemma which follows. This condition ensures that 0 and $q$, as in the statement, are not too close to the 'poles' of $B_r(p)$ where the first order euclidean behaviour (corresponding to $\|z_p\|$) becomes negligible. This means that to prove this lemma we are able to just use these lower order terms to control the size of $z_q$, and hence ensure $\tau_q$ is large enough for $\hat{p}$ and $\hat{q}$ to be separated by an appropriate distance $\bar{\xi}$.

*Proof.* Using the isometries of $\rho$, see (4.1.4), we may assume that $\tau_p \geq 0$ and $\phi(p) = 0$ without loss of generality. Setting $\phi = \phi(q)$ we therefore have

$$\max_{1 \leq i \leq d} |\phi_i| = \max_{1 \leq i \leq d} \varphi_i(p,q) < \bar{\phi},$$

and of course $z_p = \operatorname{Re} z_p$.

Our assumptions ensure that

$$p = \begin{pmatrix} (r+s)z_p \\ (r+s)^2 \tau_p \end{pmatrix} \qquad \text{and} \qquad q = \begin{pmatrix} (\tilde{r}+\tilde{s})z_q \\ (\tilde{r}+\tilde{s})^2 \tau_q \end{pmatrix}$$

for some $s, \tilde{s}$ with $|s| \leq t$ and $|\tilde{s}| \leq \tilde{t}$. Let $a = \frac{r+s}{r+s'}$ and $b = \frac{\tilde{r}+\tilde{s}}{r+s'}$. As $\rho(p,q) = r+s'$, for some $|s'| \leq t$, using equation (4.1.2) we know that

$$\begin{aligned} 1 &= \|az_p - bz_q\|^2 + \left(a^2\tau_p - b^2\tau_q - \frac{1}{2}ab\,\mathrm{Im}\,\langle z_p, z_q\rangle\right)^2 \\ &= a^2\|z_p\|^2 + b^2\|z_q\|^2 - 2ab\,\mathrm{Re}\,\langle z_p, z_q\rangle \\ &\qquad + a^4\tau_p^2 + b^4\tau_q^2 + \frac{1}{4}\left(ab\,\mathrm{Im}\,\langle z_p, z_q\rangle\right)^2 - \left(a^2\tau_p - b^2\tau_q\right)ab\,\mathrm{Im}\,\langle z_p, z_q\rangle \qquad (\dagger) \end{aligned}$$

where we have used the linearity properties of the inner product. Observe that

$$a - 1 = \frac{r+s}{r+s'} - 1 = \frac{\tilde{r}}{r}\frac{s/\tilde{r} - s'/\tilde{r}}{1 + s'/r} \quad \text{and} \quad b - \frac{\tilde{r}}{r} = \frac{\tilde{r}+\tilde{s}}{r+s'} - \frac{\tilde{r}}{r} = \frac{\tilde{r}}{r}\frac{\tilde{s}/\tilde{r} - s'/r}{1+s'/r}$$

and since $r \geq \tilde{r} \geq t\tilde{t}R$ the rightmost fraction in each of these equalities is $O(R^{-1})$, independent of all other variables, as $R \to \infty$. By recalling that $\|z_p\|^2 + \tau_p^2 = 1$, and similarly with $q$, we can use this observation to reduce ($\dagger$) to

$$1 = \|z_p\|^2 - 2\frac{\tilde{r}}{r}\mathrm{Re}\,\langle z_p, z_q\rangle + \tau_p^2 - \tau_p\frac{\tilde{r}}{r}\mathrm{Im}\,\langle z_p, z_q\rangle + O\left(\frac{\tilde{r}^2}{r^2}\right) + \frac{\tilde{r}}{r}E$$

where $E$ is also an $O(R^{-1})$ error term. We can now subtract $\|z_p\|^2 + \tau_p^2 = 1$ and divide by a factor of $\frac{\tilde{r}}{r}$ to see that

$$2\,\mathrm{Re}\,\langle z_p, z_q\rangle + \tau_p\,\mathrm{Im}\,\langle z_p, z_q\rangle = E + O\left(\frac{\tilde{r}}{r}\right).$$

Note that

$$\mathrm{Re}\,\langle z_p, z_q\rangle = \sum_{j=1}^{d}\zeta_j(p)\zeta_j(q)\cos\phi_j \quad \text{and} \quad \mathrm{Im}\,\langle z_p, z_q\rangle = \sum_{j=1}^{d}\zeta_j(p)\zeta_j(q)\sin\phi_j,$$

and so if we take $\bar{\phi}$ small enough to ensure for each $j$ we have

$$\cos\phi_j + \tau_p\sin\phi_j \geq \cos\phi_j - |\sin\phi_j| \geq 0$$

then

$$0 \leq \mathrm{Re}\,\langle z_p, z_q\rangle \leq 2\,\mathrm{Re}\,\langle z_p, z_q\rangle + \tau_p\,\mathrm{Im}\,\langle z_p, z_q\rangle$$

and hence $|\mathrm{Re}\,\langle z_p, z_q\rangle| \leq |E| + O\left(\frac{\tilde{r}}{r}\right)$.

Now suppose that $\rho(\hat{p}, \hat{q}) \le 1 - \bar{\tau}^2$ then $\|z_p - z_q\|^2 \le 1 - \bar{\tau}^2$ and

$$\begin{aligned} \tau_q^2 = 1 - \|z_q\|^2 = 1 + \|z_p\|^2 - \|z_p - z_q\|^2 - 2\operatorname{Re}\langle z_p, z_q\rangle \\ \ge 1 + (1 - \bar{\tau}^2) - (1 - \bar{\tau}^2) - 2\operatorname{Re}\langle z_p, z_q\rangle \\ > \frac{1 + \bar{\tau}^2}{2} > \bar{\tau}^2 \ge \tau_p^2 \end{aligned}$$

where we have ensured that $\bar{\epsilon}$ and $\bar{R}^{-1}$ are small enough for $4|\operatorname{Re}\langle z_p, z_q\rangle| < 1 - \bar{\tau}^2$. It follows that

$$\rho(\hat{p}, \hat{q}) > \frac{1}{2} d\left(\{h \in \partial B_1(0) : \tau_h^2 \le \bar{\tau}^2\}, \left\{h \in \partial B_1(0) : \tau_h^2 \ge \frac{1 + \bar{\tau}^2}{2}\right\}\right) > 0$$

and so we also take $\bar{\xi} > 0$ as the minimum of $1 - \bar{\tau}^2$ and this value to complete the proof. $\square$

**Lemma 4.2.3** (Small scale separation 2)**.** *There exists $\bar{\tau} \in (\frac{1}{2}, 1)$ and positive numbers $\bar{\xi}, \bar{R}, \bar{\phi}, \bar{\epsilon}$ for which the following holds. Let $p, q \in \mathbb{H}^d \setminus \{0\}$ with $q \in \partial_t B_r(p)$ and $0 \in \partial_t B_r(p) \cap \partial_{\tilde{t}} B_{\tilde{r}}(q)$ where $t, \tilde{t} \ge 1$ and suppose $r \ge \tilde{r} \ge TR$ for some $R > 1$ and $T \ge t\tilde{t}$. Let $I(p) = \left\{i : \zeta_i(p) < \frac{10T}{r}\right\}$, $\epsilon \in (0, 1)$ and assume that $\tilde{r} \le \epsilon r$. If $R > \bar{R}$, $\epsilon < \bar{\epsilon}$, $|\tau_p| \ge \bar{\tau}$ and $\max_{1 \le i \le n} \varphi_i(p, q) < \bar{\phi}$, then either there exists $i \notin I_p$ such that $\zeta_i(q) < \frac{10T}{\tilde{r}}$ or $\rho(\hat{p}, \hat{q}) > \bar{\xi}$.*

In this lemma we aim for the same conclusion as in the first small scale separation lemma but find an exceptional case. This we deal with later by using a slightly more sophisticated bounding argument.

As remarked above, in the setting of this lemma the first order argument used to prove Lemma 4.2.2 is not available to us; the argument stalls if $z_p$ can be made arbitrarily small. Instead we must make delicate use of the precise shape of $B_r(p)$ near the poles. This results in a somewhat more technical proof where special care must be paid to the thickenings, which in this case are large enough to easily throw off the estimates.

*Proof.* As before we may assume that $\tau_p \ge 0$ and $\phi(p) = 0$, so $z_p = \operatorname{Re} z_p$, without loss of generality. Again we set $\phi = \phi(q)$ so that

$$\max_{1 \le i \le d} |\phi_i| = \max_{1 \le i \le d} \varphi_i(p, q) < \bar{\phi}.$$

To keep track of a large quantity of error terms, we will slightly abuse the big $O$ and little $o$ notations. Throughout we shall write $O(x)$ for any function $f : \mathbb{R} \to \mathbb{R}$ (possibly depending on our variables) for which, by first taking $\bar{\tau}$ sufficiently close to 1, then $\bar{\epsilon}$ sufficiently small and $\bar{R}$ sufficiently large we can ensure $|f(x)| \leq K|x|$ for some $K > 0$ independent of all other variables. Similarly we will write $o(x)$ for any function $f(x)$ for which given any $\delta > 0$, with the same control over $\bar{\epsilon}, \bar{R}$ and $\bar{\tau}$, we can ensure $|f(x)| \leq \delta|x|$.

Our approach is to attempt to bound $\tau_q$ above by some constant $C < 1$. When successful, it will then suffice to take

$$\bar{\xi} < \frac{1}{2}\rho\left((0,0,1), \{h \in \partial B_1 : \tau_h \leq C\}\right)$$

since $\bar{\tau}$ can be increased to ensure $\hat{p}$ is arbitrarily close to $(0,0,1)$. We will encounter an exceptional case to account for the 'either' in the statement of the lemma. Firstly, if $\|z_q\| \geq \frac{1}{2}$ then $\tau_q^2 \leq \frac{3}{4}$ so we may assume that $\|z_q\| \leq \frac{1}{2}$.

**Step 1: Perturb $p$ and $q$ to suppress the thickenings**

We are now going to introduce some new points which incorporate the errors due to the thickenings; this enables us to keep the errors under sufficient control to be dealt with later. Let $\eta \in B_t(0)$ such that $\rho(\eta^{-1}, p) = r$ and $q' \in B_t(q)$ such that $\rho(q', p) = r$. Let $P = p\,\eta$ and $Q = q'\eta$. For notational simplicity we let $P = (z, \tau)$ and $Q = (w, \sigma)$. We can write these variables more explicitly using the coordinates of $p, q$ and $\eta$: for some $s, s_\eta$ with $|s|, |s_\eta| \leq t$ we have

$$z = (r+s)z_p + s_\eta z_\eta \text{ and } \tau = (r+s)^2\tau_p + s_\eta^2\tau_\eta + \frac{1}{2}\mathrm{Im}\,\langle (r+s)z_p, s_\eta z_\eta\rangle, \quad (4.2.1)$$

and (using $\rho(q, q') \leq t$) there are $\upsilon \in \mathbb{C}^d$ and $\upsilon_\tau \in \mathbb{R}$ such that $\|\upsilon\|, |\upsilon_\tau| \leq t$ for which

$$w = (\tilde{r} + \tilde{s})z_q + \upsilon + s_\eta z_\eta \quad (4.2.2)$$

and

$$\sigma = \left[ (\tilde{r}+\tilde{s})^2\tau_q + \frac{1}{2}\mathrm{Im}\,\langle(\tilde{r}+\tilde{s})z_q + \upsilon, (\tilde{r}+\tilde{s})z_q\rangle + \upsilon_\tau \right] \\ + s_\eta^2\tau_\eta + \frac{1}{2}\mathrm{Im}\,\langle(\tilde{r}+\tilde{s})z_q + \upsilon, s_\eta z_\eta\rangle.$$

This final expression is somewhat complicated, but by considering the dominant $\tilde{r}^2$ term and noting $t, \tilde{t} \leq R^{-1}\tilde{r}$ it becomes clear that

$$\sigma = \tau_q\tilde{r}^2 + o(\tilde{r}^2) = O(\tilde{r}^2).$$

Therefore, to bound $\tau_q$ above by some $C < 1$ it will suffice to do so for $\frac{\sigma}{\tilde{r}^2}$.

To do this we will use the fact that, by the right invariance of the metric, $\rho(P,Q) = r$ and $\rho(0,P) = r$. The first of these properties ensures that

$$\frac{\|z-w\|^2}{r^2} + \frac{(\tau - \sigma - \frac{1}{2}\mathrm{Im}\,\langle z,w\rangle)^2}{r^4} = 1$$

and hence that $\sigma$ is given by one of the roots

$$\tau - \frac{1}{2}\mathrm{Im}\,\langle z,w\rangle \pm r^2\sqrt{1 - r^{-2}\|z-w\|^2}.$$

**Step 2: Apply Taylor's theorem to the square root**

The fact that $\rho(0,P) = r$ means

$$\frac{\|z\|^2}{r^2} + \frac{\tau^2}{r^4} = 1$$

and so as long as $\tau > 0$, which we will see just below, we have

$$r^2\sqrt{1 - r^{-2}\|z-w\|^2} = \tau\sqrt{1 - \frac{r^2}{\tau^2}(\|w\|^2 - 2\mathrm{Re}\,\langle z,w\rangle)}.$$

Using the coordinates expressions in (4.2.1)

$$\frac{\tau}{r^2} = \left(1+\frac{s}{r}\right)^2\tau_p + \frac{s_\eta^2}{r^2}\tau_\eta + \frac{1}{2}\left(1+\frac{s}{r}\right)\frac{s_\eta}{r}\mathrm{Im}\,\langle z_p, z_\eta\rangle = \tau_p + o(1).$$

In particular we are able to ensure that $0 < \frac{1}{2}\bar{\tau} \leq r^{-2}\tau \leq \frac{3}{2}$. Additionally, we have that

$$\left\|\frac{w}{r}\right\| \leq \frac{\tilde{r}+|\tilde{s}|}{r} + \frac{t}{r} + \frac{|s_\eta|}{r} \leq \epsilon + \frac{3}{R}$$

and so $\left\|\frac{w}{r}\right\| = o(1)$. As

$$r^{-2}\left|\|w\|^2 - 2\mathrm{Re}\langle z, w\rangle\right| \leq \left\|\frac{w}{r}\right\|\left(1 + 2\left\|\frac{w}{r}\right\|\right)$$

we also have $r^{-2}(\|w\|^2 - 2\mathrm{Re}\langle z, w\rangle) = o(1)$. Combining these observations shows that

$$E = \frac{r^2}{\tau^2}\left(\|w\|^2 - 2\mathrm{Re}\langle z, w\rangle\right) = \frac{1}{(r^{-2}\tau)^2}r^{-2}\left(\|w\|^2 - 2\mathrm{Re}\langle z, w\rangle\right) = o(1).$$

With $\bar{\tau}$, $\bar{\epsilon}$ and $\bar{R}$ sufficiently well chosen we can therefore apply Taylor's theorem to see that

$$\tau\sqrt{1 - \frac{r^2}{\tau^2}(\|w\|^2 - 2\mathrm{Re}\langle z, w\rangle)} = \tau\sqrt{1-E} = \tau\left(1 - \frac{E}{2} + O\left(E^2\right)\right)$$

and hence that $\sigma$ is one of

$$\tau - \frac{1}{2}\mathrm{Im}\langle z, w\rangle \pm \left(\tau - \frac{\tau E}{2} + O\left(\tau E^2\right)\right).$$

In principle, as $\tau = r^2\tau_p + o(r^2)$ and $\tau_p \geq \bar{\tau} > \frac{1}{2}$, $\tau$ can be very large. This will cause problems bounding $\sigma$ if it is ever given by the positive root. However, we have already seen that $\sigma = O(\tilde{r}^2)$, $\tau = O(r^2)$ and $E = o(1)$ and so if $\sigma$ were given by the positive root then $\tau = o(r^2)$, contradicting $\tau_p > \frac{1}{2}$. Therefore

$$\sigma = -\frac{1}{2}\mathrm{Im}\langle z, w\rangle + \frac{\tau E}{2} + O\left(\tau E^2\right)$$

and we just need to find appropriate bounds for the remaining terms.

**Step 3: Bounding $\tau E^2$**

Observe that

$$
\begin{aligned}
\tau E^2 &= \frac{r^4}{\tau^3}\left(\|w\|^2 - 2\mathrm{Re}\,\langle z, w\rangle\right)^2 \\
&\le \frac{\tilde{r}^2}{(r^{-2}\tau)^3}\left(\left\|\frac{w}{r}\right\|\left\|\frac{w}{\tilde{r}}\right\| + 2\left|\mathrm{Re}\left\langle \frac{z}{r}, \frac{w}{\tilde{r}}\right\rangle\right|\right)^2 \\
&\le 2^6\left\|\frac{w}{\tilde{r}}\right\|^2\left(\left\|\frac{w}{r}\right\| + 2\left(1 + \frac{|s|}{r}\right)\|z_p\| + \frac{2|s_\eta|}{r}\right)^2\tilde{r}^2
\end{aligned}
$$

where we have used $r^{-2}\tau \ge \frac{1}{2}\bar{\tau} \ge \frac{1}{4}$ and the coordinate expression for $z$. Noting that

$$
\left\|\frac{w}{\tilde{r}}\right\| \le 1 + \frac{|\tilde{s}|}{\tilde{r}} + \frac{t}{\tilde{r}} + \frac{|s_\eta|}{\tilde{r}} \le 2
$$

for $\epsilon$ and $R^{-1}$ sufficiently small, and that $\|z_p\| = \sqrt{1 - \tau_p^2}$, we see that

$$
\tau E^2 \le 2^8\left(\epsilon + \frac{5}{R} + 2\left(1 + R^{-1}\right)\sqrt{1 - \bar{\tau}^2}\right)^2\tilde{r}^2.
$$

This means $\tau E^2 = o(\tilde{r}^2)$, which will be sufficient.

**Step 4: Bounding the explicit terms in the non-exceptional case**

This is the most technical step in the proof, but is not fundamentally difficult.

$$
\begin{aligned}
-\frac{1}{2}\mathrm{Im}\,\langle z, w\rangle + \frac{\tau E}{2} &= -\frac{1}{2}\mathrm{Im}\,\langle z, w\rangle + \frac{r^2}{2\tau}\left(\|w\|^2 - 2\mathrm{Re}\,\langle z, w\rangle\right) \\
&= \frac{1}{2}\left(-\mathrm{Im}\,\langle z, w\rangle - 2\frac{r^2}{\tau}\mathrm{Re}\,\langle z, w\rangle\right) + \frac{r^2}{2\tau}\|w\|^2.
\end{aligned}
$$

We aim to show the term inside the bracket is non-positive, modulo a small error term. We have

$$
\begin{aligned}
\langle z, w\rangle &= \langle (r+s)z_p, (\tilde{r} + \tilde{s})z_q + \upsilon + s_\eta z_\eta\rangle + o(\tilde{r}^2) \\
&= (r+s)\tilde{r}\left\langle z_p, \left(1 + \frac{\tilde{s}}{\tilde{r}}\right)z_q + \upsilon + \frac{s_\eta}{\tilde{r}}z_\eta\right\rangle + o(\tilde{r}^2).
\end{aligned}
$$

Next

$$
\begin{aligned}
&\operatorname{Im}\left\langle z_p, \left(1+\frac{\tilde{s}}{\tilde{r}}\right) z_q + \upsilon + \frac{s_\eta}{\tilde{r}} z_\eta \right\rangle \\
=&\sum_{j=1}^{d} \zeta_j(p) \left( \left(1+\frac{\tilde{s}}{\tilde{r}}\right) \zeta_i(q) \sin\phi_j + \operatorname{Im} \frac{\upsilon_i + s_\eta (z_\eta)_i}{\tilde{r}} \right) \\
=&\sum_{j\notin I(p)} \zeta_j(p) \left( \left(1+\frac{\tilde{s}}{\tilde{r}}\right) \zeta_i(q) \sin\phi_j + \operatorname{Im} \frac{\upsilon_i + s_\eta (z_\eta)_i}{\tilde{r}} \right) + o\left(\frac{\tilde{r}}{r}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Re}\left\langle z_p, \left(1+\frac{\tilde{s}}{\tilde{r}}\right) z_q + \upsilon + \frac{s_\eta}{\tilde{r}} z_\eta \right\rangle \\
=&\sum_{j=1}^{d} \zeta_j(p) \left( \left(1+\frac{\tilde{s}}{\tilde{r}}\right) \zeta_i(q) \cos\phi_j + \operatorname{Re} \frac{\upsilon_i + s_\eta (z_\eta)_i}{\tilde{r}} \right) \\
=&\sum_{j\notin I(p)} \zeta_j(p) \left( \left(1+\frac{\tilde{s}}{\tilde{r}}\right) \zeta_i(q) \cos\phi_j + \operatorname{Re} \frac{\upsilon_i + s_\eta (z_\eta)_i}{\tilde{r}} \right) + o\left(\frac{\tilde{r}}{r}\right).
\end{aligned}
$$

From earlier assumptions $\|\upsilon + s_\eta z_\eta\| \leq 2t$, and we can ensure that

$$
-\sin\phi_j - 2\frac{r^2}{\tau}\cos\phi_j \leq -1
$$

through further increasing $\bar{\tau}$ and then decreasing $\bar{\epsilon}$, $\bar{R}^{-1}$ and $\bar{\phi}$. So it will be enough for the magnitude of each $\zeta_i(q)$ to be large relative to $2t$. In the non-exceptional case we may assume that for all $j \notin I(p)$ we have $\zeta_j(q) \geq 10T\tilde{r}^{-1}$ so that

$$
\left(1+\frac{\tilde{s}}{\tilde{r}}\right)\zeta_j(q) \geq \zeta_j(q) \geq 10t\tilde{r}^{-1} \geq 2t\tilde{r}^{-1}\left(1+\frac{2r^2}{\tau}\right)
$$

since we ensured $r^{-2}\tau > \frac{1}{2}$. It follows that

$$
\begin{aligned}
-&\sum_{j\notin I_p} \zeta_j(p) \left( \left(1+\frac{\tilde{s}}{\tilde{r}}\right) \zeta_i(q) \sin\phi_j + \operatorname{Im} \frac{\upsilon_i + s_\eta (z_\eta)_i}{\tilde{r}} \right) \\
&\qquad - \frac{2r^2}{\tau} \sum_{j\notin I_p} \zeta_j(p) \left( \left(1+\frac{\tilde{s}}{\tilde{r}}\right) \zeta_i(q) \cos\phi_j + \operatorname{Re} \frac{\upsilon_i + s_\eta (z_\eta)_i}{\tilde{r}} \right) \leq 0.
\end{aligned}
$$

This means that the explicit terms are the sum of something non-positive and an error term with order

$$(r+s)\tilde{r}\, o\left(\frac{\tilde{r}}{r}\right) + o(\tilde{r}^2) = o(\tilde{r}^2).$$

**Step 5: Bound $\sigma$ in the non-exceptional case and complete the proof**

By combining the last two steps we see that

$$\begin{aligned}
\sigma &= \frac{1}{2}\left(-\mathrm{Im}\,\langle z, w\rangle - 2\frac{r^2}{\tau}\mathrm{Re}\,\langle z, w\rangle\right) + \frac{r^2}{2\tau}\|w\|^2 + O\left(\tau E^2\right) \\
&\leq \frac{r^2}{2\tau}\|w\|^2 + o(\tilde{r}^2) \\
&\leq \frac{\tilde{r}^2}{4\bar{\tau}} + o(\tilde{r}^2) \leq \frac{\tilde{r}^2}{2} + o(\tilde{r}^2)
\end{aligned}$$

which allows us to bound $\sigma$ in the required fashion unless we have some $j \notin I_p$ for which $\zeta_j(q) < \frac{10T}{\tilde{r}}$, which is the other option allowed by the statement. □

### 4.2.2 Finite intersection dimension

We can now fit these pieces together to show that property (d) of Theorem 2.3.1 holds.

**Theorem 4.2.4.** $(\mathbb{H}^d, \rho)$ *has finite intersection dimension.*

*Proof.* We need to show that there exists $R > 1$ and $\kappa \in \mathbb{N}$ such that if we are given

1. $t(1), ..., t(\kappa) \geq 1$,
2. $r(1), ..., r(\kappa)$ such that each $r(i) \geq t(1)...t(i)R$,
3. points $p_1, ..., p_\kappa \in \mathbb{H}^d$ such that $p_i \in \bigcap_{j<i} \partial_{t(j)} B_{r(j)}(p_j)$ for $j < i$,

then $\bigcap_{i=1}^{\kappa} \partial_{t(i)} B_{r(i)}(p_i) = \emptyset$. We assume that $0 \in \bigcap_{i=1}^{\kappa} \partial_{t(i)} B_{r(i)}(p_i)$, by using translation invariance, and show that $k$ must be bounded for $R$ sufficiently large.

The logical structure of the proof is to first apply a number of reductions of the form: we have a sequence of length $\kappa$ with a collection of properties $P$, we show that given $\kappa'$ there is $M(\kappa') \in \mathbb{N}$ such that if $\kappa \geq M$ then there is a subsequence of length $\kappa'$ with a property $Q$ in addition to those properties in $P$. It is then sufficient to show $\kappa'$ is bounded, because if so it follows that $\kappa < M(\kappa'+1)$ where $\kappa'$ is maximal with the properties in $P$ and $Q$ holding. We can then relabel and assume our sequence had property $Q$ in the first place. We finish off by using all the gathered properties to show $\kappa$ is bounded.

**Reduction 1:**

First we show that we can assume the $r(i)$ are decreasing, essentially as in [Hoc10]. Let $\kappa' \leq \kappa$ and assume that $r(i) \geq r(1)$ for all $2 \leq i \leq \kappa'$. By property 3 all these $p_i$ lie inside $\partial_{t(1)} B_{r(1)}(p_1) \subset B_{2r(1)}(p_1)$, this containment is due to property 2. Property 3 also ensures that for pair $i, j$ with $j > i$ there is a point $b \in \partial B_{r(i)}(p_i)$ with $\rho(b, p_j) \leq t(i)$, and hence by property 2

$$\rho(p_i, p_j) \geq |\rho(p_i, b) - \rho(p_j, b)| \geq r(i) - t(i) \geq r(1)(1 - R^{-1})$$

so for all $1 \leq i, j \leq k'$ with $i \neq j$,

$$\rho\left(p_1^{-1}\delta_{1/(2r(1))}p_i,\, p_1^{-1}\delta_{1/(2r(1))}p_j\right) \geq \frac{1 - R^{-1}}{2} > 0.$$

Since each $p_1^{-1}\delta_{1/(2r(1))}p_i \in B_1(0)$ by Lemma 4.1.3 part (i) $\kappa' \leq N_R$, where $N_R = N(\frac{1-R^{-1}}{2})$. Note that $N_R$ decreases as $R$ increases.

Clearly, this argument could be repeated with any chain of $\kappa'$ points satisfying the analogous conditions. Therefore if for some $\kappa'' \in \mathbb{N}$ we have $\kappa \geq \kappa''(N_R + 1)$ then there must be $i_1 = 1 < i_2 \leq ... < i_{\kappa''} \leq \kappa$ with $r(i_1) \geq r(i_2) \geq ... \geq r(i_{\kappa''})$. This means it suffices for us to prove the claim with the $r(i)$ assumed to be decreasing.

**Reduction 2:**

Next we use Lemma 4.1.3 and the large scale separation lemma, 4.2.1, to ensure that we can assume that $\hat{p}_1, ..., \hat{p}_\kappa$ are all within a distance

$$\xi(\epsilon) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{\epsilon^2}{4}}\right) > 0$$

of one another, here $\epsilon \in (0,1)$, and that for all $j > i$ we have $r(j) \leq \epsilon r(i)$. Note that $\xi(\epsilon)$ decreases as $\epsilon$ decreases.

Let $\kappa' \leq \kappa$, again. By Lemma 4.1.3 part (ii) if $\kappa \geq \kappa' N(\xi(\epsilon))$ then we have a subcollection $I \subset \{1, ..., \kappa\}$ of size at least $\kappa'$ with $\rho(\hat{p}_i, \hat{p}_j) < \xi(\epsilon)$ for all $i, j \in I$. By taking $R > \bar{R}(\epsilon)$ from Lemma 4.2.1, which is assumed to hold from here onwards, the lemma shows that for each pair $i, j \in I$ with $j < i$ we have $r(j) \leq \epsilon r(i)$. $I$ therefore gives the desired subsequence.

**Reduction 3:**

Before we do the final reduction, first take $\bar{\tau}$ as given by Lemma 4.2.3, and we take this as the input for $\bar{\tau}$ in Lemma 4.2.2. We can then decrease $\epsilon$ so that $\epsilon$ and $\xi(\epsilon)$ are small enough to apply Lemmas 4.2.2 and 4.2.3 with $\bar{\epsilon} = 2\epsilon$ and $\bar{\xi} = \xi(\epsilon)$. Similarly, we take $R$ large enough for both lemmas to hold.

It should be clear from an application of the pigeonhole principle that given $\kappa'$ by increasing $\kappa$ we can ensure that there is a subcollection $I \subset \{1, ..., \kappa\}$ of size $\kappa'$ such that for all $i, j \in I$ we have $\max_{1 \leq l \leq d} \varphi_l(p_i, p_j) < \bar{\phi}$, where $\bar{\phi}$ is small enough for both lemmas to hold. We can therefore assume the whole sequence also has this property.

**$\kappa$ is bounded:**

With all this in hand, we can apply Lemmas 4.2.2 and 4.2.3 to the sequence at will. Let $T = t_1 ... t_\kappa$ and for each $1 \leq i \leq \kappa$ set

$$I(p_i) = \left\{ m : \zeta_m(p_i) < \frac{10T}{r_i} \right\} \subseteq \{1, ..., d\}$$

as in Lemma 4.2.3. By assumption for all $i \neq j$ we have $\rho(\hat{p}_i, \hat{p}_j) \leq \bar{\xi}$ and so by Lemma 4.2.2 we must have $|\tau_{p_i}| > \bar{\tau}$ for all $i \leq \kappa - 1$. By applying this fact along with the same assumption Lemma 4.2.3 ensures that for each pair $i < j \leq \kappa$ there is some number in $I(p_j)$ which is not in $I(p_i)$. In particular, each of the sets $I(p_1), ..., I(p_\kappa) \subseteq \{1, ..., d\}$ are pairwise distinct, from which it follows that $\kappa \leq 2^d$. $\square$

Having completed this proof all that remains is property (a), well-separability.

## 4.3 Well-separability

We begin with a preliminary lemma.

**Lemma 4.3.1.** *Let $p, p' \in \mathbb{H}^d$ and $r > 0$. Then there exists $R > 0$, independent of $p$, $p'$ and $r$, such that if $\xi = \rho(p, p') > 2Rr$ then there is a point $q$ with $\rho(p', q) \leq 2r$ for which $B_r(q) \subseteq B_\xi(p)$.*

*Proof.* First of all, using the dilation and isometries of $d$ we may assume that $r = 1/2$ and $p' = 0$. Moreover we assume that $\tau_p \geq 0$ and all coefficients $z_p$ are non-negative reals.

By right invariance the points in $B_{1/2}(q)$ take the form

$$(w + q_z, \sigma + q_\tau + \frac{1}{2}\mathrm{Im}\,\langle w, q_z\rangle)$$

where $\|w\|^2 + 4\sigma^2 \leq \frac{1}{4}$. Therefore, by (4.1.1), it suffices to show that we can choose $R$ large enough such that given $(z_p, \tau_p)$ there is $q = (q_z, q_\tau)$ with $\|q_z\|^2 + q_\tau^2 \leq 1$ such that

$$\frac{\|w + q_z - \xi z_p\|^2}{\xi^2} + \frac{\left(\sigma + q_\tau + \frac{1}{2}\mathrm{Im}\,\langle w, q_z\rangle - \xi^2\tau_p - \frac{1}{2}\mathrm{Im}\,\langle w + q_z, \xi z_p\rangle\right)^2}{\xi^4} \leq 1$$

or equivalently (as $\rho(0, p) = \xi$) that

$$\begin{aligned} 0 \geq\ & \xi^3 \left(-2\mathrm{Re}\,\langle w + q_z, z_p\rangle + \tau_p \mathrm{Im}\,\langle w + q_z, z_p\rangle\right) \\ & + \xi^2 \left(\|w + q_z\|^2 - 2\tau_p\left(\sigma + q_\tau + \frac{1}{2}\mathrm{Im}\,\langle w, q_z\rangle\right) + \frac{1}{4}(\mathrm{Im}\,\langle w + q_z, z_p\rangle)^2\right) \\ & - \frac{\xi}{2}\left(\sigma + q_\tau + \frac{1}{2}\mathrm{Im}\,\langle w, q_z\rangle\right)\mathrm{Im}\,\langle w + q_z, \xi z_p\rangle + \left(\sigma + q_\tau + \frac{1}{2}\mathrm{Im}\,\langle w, q_z\rangle\right)^2. \end{aligned}$$

Notice that the coefficients of all powers of $\xi$ have bounds independent of all variables. Let $C > 0$ be strictly greater than the independent bound for the coefficient of $\xi^2$ and ensure that $R > C$. Consider the case when $\|z_p\| \geq \frac{2C}{\xi} > 0$, let us take $q_z = \lambda z_p$ where $\lambda > 0$ is chosen so that $\|q_z\| = 1$, and hence $q_\tau = 0$.

Then the coefficient of $\xi^3$ above satisfies

$$\begin{aligned}-2\mathrm{Re}\,\langle w+q_z, z_p\rangle + \tau_p \mathrm{Im}\,\langle w+q_z, z_p\rangle &= -2\langle q_z, z_p\rangle - \langle z_p, 2\mathrm{Re}\,w + \tau_p \mathrm{Im}\,w\rangle \\ &\le \|z_p\|(-2+\frac{3}{2}) \le -\frac{C}{\xi}.\end{aligned}$$

It follows that the polynomial above is bounded above by a quadratic in $\xi$ whose coefficients are independent of all variables, and the leading coefficient of which is negative. Hence we may take $R$ large enough, with the required independence, to ensure that the inequality holds for some appropriate $q$ regardless of the choice of $p$.

In the case where $\|z_p\| \le \frac{2C}{\xi}$ take $q_z = 0$ and $q_\tau = 1$. Then we have the bounds

$$-2\mathrm{Re}\,\langle w+q_z, z_p\rangle + \tau_p \mathrm{Im}\,\langle w+q_z, z_p\rangle \le \frac{3C}{\xi}$$

and

$$\|w+q_z\|^2 - 2\tau_p\left(\sigma + q_\tau + \frac{1}{2}\mathrm{Im}\,\langle w, q_z\rangle\right) + \frac{1}{4}(\mathrm{Im}\,\langle w+q_z, z_p\rangle)^2 \le \frac{1}{4} - \frac{3}{2}\tau_p + \frac{C^2}{4\xi^2}.$$

In particular, we can show that the above polynomial is bounded above by a quadratic with coefficients independent of all variables and has leading coefficient less than

$$\frac{3C}{\xi} + \frac{1}{4} - \frac{3}{2}\sqrt{1-\frac{C^2}{\xi^2}} + \frac{C^2}{4\xi^2} \le -1$$

where $R$ has been taken sufficiently large relative to $C$. So, as above we may increase $R$ to ensure the required inequality holds. □

Recall that we call a sequence of balls in a metric space *incremental* if the radii are non-increasing and the centre of each ball is not an element of any ball earlier in the sequence. In particular, each centre is only in one ball in the sequence.

**Proposition 4.3.2.** $(\mathbb{H}^d, \rho)$ *is well-separable.*

*Proof.* We mildly adapt a standard technique, see for example [Hoc10] or [dG75]. For the purposes of this proof we use $\nu$ to denote the right invariant Haar measure on $\mathbb{H}^d$.

Let $C$ be the constant of the Besicovitch covering property and $D$ be the constant for the *metric* doubling property of $\rho$. Furthermore, take $m \in \mathbb{N}$ large enough so that $2^m > R$, with $R$ as in Lemma 4.3.1. In particular, $m$ depends only on the metric $\rho$. Let $\chi = CD^{m+2} + 1$.

Let $E$ be a finite subset of $\mathbb{H}^d$ and $\mathcal{U}$ be a carpet covering $E$. By applying the BCP via, for example, Proposition 2.1 of [Hoc10] we can find an incremental sequence $U_1, ..., U_n$ of elements of $\mathcal{U}$ covering $E$. We assign colours $1, 2, ..., \chi$ to the $U_i$ as follows. Colour $U_1$ as you like, assume we have coloured the $U_i$ for $i \leq k$ and consider $U_{k+1}$. Take $r$ to be the radius of $U_k$ and $h$ to be the centre of $U_{k+1}$, by assumption $U_{k+1} \subseteq B_r(h)$ and each $U_i$ with $i \leq k$ has radius at least $r$.

Let $\mathcal{W}$ be the collection of balls $U_1, ..., U_k$ which are within distance $r$ of $U_{k+1}$, and hence of $B_r(h)$, and let $N = |\mathcal{W}|$. Each $U \in \mathcal{W}$ intersects nontrivially with $B_{2r}(h)$, so we may take $p'$ from Lemma 4.3.1 to be a point in this intersection. We may assume that $p'$ is on the boundary of $U$ because the straight line from $h$ to $p'$ is contained by $B_{2r}(h)$ (the balls are euclidean convex), $p' \in U$ and $h \notin U$ (by incrementality) so the intermediate value theorem implies there is a point on the boundary of $U$ inside $B_{2r}(h)$. The Lemma 4.3.1 then ensures that either the radius of $U$ is at most $2^{m+1}r$ we can replace $U$ with a ball of radius $r$ centred in $B_{4r}(h)$, call this new collection of balls $\mathcal{W}'$. Each ball in $\mathcal{W}'$, which is also of size $N$, has radius at least $r$ and is contained by the ball of radius $2^{m+2}r$ about $h$. Therefore by the Besicovitch and metric doubling properties

$$N\nu(B_r(0)) \leq C\nu(B_{2^{m+2}r}(h)) \leq CD^{m+2}\nu(B_r(h)) = CD^{m+2}\nu(B_r(0))$$

and so $N \leq CD^{m+2}$. Since $N \leq \chi - 1$ we assign a colour $U_k$ which is different to all those within distance $r$ of $U_k$.

Once the colouring is complete, the collection $\mathcal{V}_j$ of those balls coloured $j$ is well separated precisely because of this property combined with the fact that the radii are decreasing. □

This proposition completes the proof of the ergodic theorem.

**Theorem 4.3.3.** *Let* $\mathrm{H}^d$ *the discrete Heisenberg group and* $B_n = \mathrm{H}^d \cap B_n(0)$ *where* $B_n(0) = \{u \in \mathbb{R}^d : \rho(u, 0) \leq n\}$ *with* $\rho$ *given by 4.1.3. Then the non-singular ergodic theorem for* $G$ *holds with the summing sequence* $(B_n)$.

As in Chapter 3 this allows us to deduce the critical dimensions are invariants.

**Corollary 4.3.4.** *Let* $\mathrm{H}^d$ *the discrete Heisenberg group and* $B_n = \mathrm{H}^d \cap B_n(0)$ *where* $B_n(0) = \{u \in \mathbb{R}^d : \rho(u, 0) \leq n\}$ *with* $\rho$ *given by 4.1.3. Then the upper and lower critical dimensions with respect to* $(B_n)$ *are invariants of metric isomorphism.*

# Open questions

There are, of course, many other discrete subgroups $G$ of the continous Heisenberg groups. Having established that $(\mathbb{H}^d, \rho)$ is voidless, well separable and has finite intersection above, if one is interested in another discrete subgroup one only needs to check the Besicovitch covering property and multiplicative doubling condition hold for the corresponding summing sequence $(B_n)$. Assuming the group can be broken down into levels $G_n$, as in Chapter 2, the arguments in that chapter suffice to show the sequence is Besicovitch. Therefore multiplicative doubling is the last remaining property. The proof for $\mathrm{H}^d$ earlier this chapter used the fact that the distance between any two distinct points is bounded below; clearly this would not work for a dense subgroup of $\mathbb{H}^d$. In the proof of Lemma 3.1.3, where we considered subgroups of $\mathbb{Q}^d$ in $\mathbb{R}^d$, we used the fact that each $G_n$ had an associated fundamental domain with finite diameter (and positive Haar measure) to deduce multiplicative doubling. It is reasonable to expect that a similar construction will be possible for a large collection of discrete subgroups of $\mathbb{H}^d$, including for example the rational Heisenberg groups $\mathbb{H}^d(\mathbb{Q})$ (where $\mathbb{Z}$ is replaced by $\mathbb{Q}$ in the definition of the discrete Heisenberg group).

Recall that the metric $\rho$ used in this chapter can be defined by

$$\rho\,(p, q) = \inf\left\{r > 0 : \delta_{1/r}(pq^{-1}) \in B_{eucl}\right\}.$$

Hebisch and Sikora showed [HS90] that a similar construction can be done for any Carnot group (an introduction to which can be found in [LD17]) by embedding said group into some $\mathbb{R}^m$ and replacing $B_{eucl}$ with a Euclidean ball of sufficiently small radius. It may be that the arguments we use in this chapter can be extended to Carnot groups equipped with these metrics, possibly even more general stratified groups. As it currently stands part of the argument, in particular the proofs of the separation lemmas in Subsection 4.2.1 are very much described in terms of the explicit form of the metric $\rho$ used in this work and coordinate systems for the continuous Heisenberg groups. Whether it is possible to refine the arguments, and sift out these specifics (derived from equation 4.1.1), remains open.

# Chapter 5

# The Lamplighter Group

In this chapter we consider the natural action of the Lamplighter group $L$ on $\prod_{\mathbb{Z}} \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the group the additive group of integers with addition modulo 2, equipped with measures to ensure the action is non-singular. We define both the group and action below. Our reason for considering the Lamplighter group is that it is an amenable group and has a continuous extension, as with all the groups considered so far, but differs in some crucial respects. The group has exponential growth, meaning the size of its balls in the word metric grow at least exponentially. The natural Følner sequences, a starting point for candidate summing sequences, do not satisfy the multiplicative doubling condition, and therefore do not fit into the framework established in Chapter 2. The specific action we consider is in some ways similar to integer odometers, a comparison we will see reflected throughout the chapter.

The contents of this chapter are as follows. We first define the Lamplighter group and its natural action, and identify the class of measures we will place on the set being acted upon. We then identify conditions which will ensure that the action is non-singular and calculate the Radon-Nikodým derivatives for the action. Finally, we determine the values of the critical dimensions with respect to the natural summing sequence, given an additional condition on the measure.

## 5.1 The group and its natural action

The direct sum of copies of $\mathbb{Z}_2$ over the integers $\sum_{i\in\mathbb{Z}} \mathbb{Z}_2$ is the space of sequences of zeros and ones with finite support under pointwise addition. Let

$S : \sum_{i\in\mathbb{Z}} \mathbb{Z}_2 \to \sum_{i\in\mathbb{Z}} \mathbb{Z}_2$ be the left shift, given by $(Sa)_n = a_{n+1}$.

The Lamplighter group $L$ is given by the semidirect product $\mathbb{Z} \ltimes \sum_{i\in\mathbb{Z}} \mathbb{Z}_2$, with the multiplication

$$(i, a) \cdot (j, b) = (i + j, S^j a + b).$$

The group is generated by $(1, 0)$, the element corresponding to the shift, and $(0, \delta_0)$ where $\delta_0$ is the sequence which is all zeros except at 0 itself, where it is 1. We will frequently denote these elements by $S$ and $\delta_0$ respectively, with the meaning being clear.

Note that $\sum_{i\in\mathbb{Z}} \mathbb{Z}_2$ is a countable union of finite groups, so is amenable. Since $N = \sum_{i\in\mathbb{Z}} \mathbb{Z}_2$ is (isomorphic to) a normal subgroup of $L$ and $L/N = \mathbb{Z}$ and $\mathbb{Z}$ is amenable we see that $L$ is amenable.

It will be useful to have some Følner sequences for $L$ use as candidates for effective summing sequences. For each $i, j \in \mathbb{Z}$ with $i \leq j$ let $A(i, j)$ denote the set of elements in $\sum_{i\in\mathbb{Z}} \mathbb{Z}_2$ which are supported in $[i, j]$ when considered as a function on $\mathbb{Z}$. For $n \in \mathbb{N}$ we will write $A_n$ for $A(-n, n)$.

For notational convenience, for the remainder of this chapter we use the interval notation $[a, b]$ to denote the collection of integers $n$ with $a \leq n \leq b$.

**Proposition 5.1.1.** *Let $g : \mathbb{N} \to \mathbb{N}$ be an unbounded increasing function. For each $n \in \mathbb{N}$ let $B_n = [-g(n), g(n)] \times A_{n+g(n)}$. Then $(B_n)_{n=1}^{\infty}$ is a Følner sequence for the Lamplighter group.*

*Proof.* Let $\sigma = (k, a) \in L$ then elements of $\sigma B_n$ are of the form $(k + l, S^l a + b)$ with $l \in [-g(n), g(n)]$ and $b \in A_{n+g(n)}$. We can choose $n$ sufficiently large for $|k| \leq g(n)$ and for $a \in A_n$. This means that $S^l a \in A_{n+g(n)}$ for each $l \in [-n, n]$. In particular this means that $\sigma B_n = [k - g(n), k + g(n)] \times A_{n+g(n)}$. Then, for example when $k \geq 0$ (the other case is similar),

$$|B_n \cap \sigma B_n| = |[k - g(n), g(n)] \times A_{n+g(n)}| = (2g(n) + 1 - k)2^{2(n+g(n))+1}$$

Since $|B_n| = (2g(n) + 1)2^{2n+g(n)+1}$ we have

$$\lim_{n\to\infty} \frac{|B_n \cap \sigma B_n|}{|B_n|} = 1$$

which shows that $F_n$ is Følner. $\square$

The Lamplighter group $L$ has a natural action on the product measure space $X = \prod_{i \in \mathbb{Z}} \mathbb{Z}_2$, with $\sigma$-algebra generated by the cylinder sets, given by

$$(i, a) \cdot x = S^i x + a$$

where we have extended the shift map and addition to $X$ in the obvious manner. We equip $X$ with a product measure $\mu = \prod_{i \in \mathbb{Z}} \mu_i$ and each $\mu_i$ is a probability measure on $\mathbb{Z}_2$ with $\mu_i(\{0\}) = (1 + r_i)/2$ where each $r_i \in (-1, 1)$.

The relationship between this action of the Lamplighter group and that of the odometer map $T : \prod_{i=1}^{\infty} \mathbb{Z}_2 \to \prod_{i=1}^{\infty} \mathbb{Z}_2$ is that through repeated applications of $T$ to a sequence $y \in \prod_{i=1}^{\infty} \mathbb{Z}_2$ one can achieve any desired finite number of changes to the entries of $y$. More precisely, the orbit of a sequence under the natural action of $\sum_{i=1}^{\infty} \mathbb{Z}_2$ on $\prod_{i=1}^{\infty} \mathbb{Z}_2$ is the same as that under the odometer $T$. The Lamplighter groups is, essentially, what results from an attempt to include an invertible shift map into the odometer action.

One way this relationship manifests is in the similarity between the proofs that the action of $L$ described above and the odometer action on a space equipped with product measure are ergodic.

**Lemma 5.1.2.** *The natural action of* $L$ *on* $(X, \mu)$*, as described above, is ergodic.*

*Proof.* Let $\pi_i : X \to X_i$ be the projection map $x \mapsto x_i$. Then the $\sigma$-algebras $\mathcal{H}_k = \sigma(\pi_{-k}, \pi_k)$ for $k \in \mathbb{N}_0$ are mutually independent and by Kolmogorov's zero-one law any $\mathcal{F} = \bigcap_{k=0}^{\infty} \mathcal{F}_k$ measurable function, where $\mathcal{F}_k = \sigma\left(\bigcup_{i=k}^{\infty} \mathcal{H}_i\right)$, must be essentially constant. Therefore it suffices to show that any $L$ invariant function is $\mathcal{F}$ measurable. This can be see by observing that, if $f \in L^1$ is $L$ invariant then

$$f(x) = \frac{1}{2^{2k+1}} \sum_{a \in B_k} f\left((0, a) \cdot x\right) = \frac{1}{2^{2k+1}} \sum_{a \in B_k} f\left(..., x_{-k-1}, a_{-k}, ..., a_k, x_{k+1}, ...\right)$$

which is $\mathcal{F}_{k+1}$ measurable (and noting that $(\mathcal{F}_k)$ is a decreasing sequence of $\sigma$-algebras). $\square$

Before moving on to consider when the action is non-singular, it is worth commenting that we need not use $\mathbb{Z}_2$ in the above construction, we could instead have used $\mathbb{Z}_l$ for any $l \geq 3$. The study of such systems is usually very similar, but this restriction simplifies various formulae we will consider later in the chapter.

### Ensuring the $L$-action is non-singular

Recall that $L$ is generated by the elements $\delta_0$ and $S$ so the action is non-singular if the maps induced by $\delta_0$ and $S$ are non-singular. The prior is non-singular because $r_0 \in (-1, 1)$, so we only need to choose the $r_i$ in such a way to ensure that the shift is too. The following result, which is an application of Kakutani's criterion [Kak48], will allow us to determine exactly when this is the case.

**Proposition 5.1.3.** *Let $\prod_{n=1}^{\infty} \mathbb{Z}_2$ be equipped with the product $\sigma$-algebra. For a sequence $p = (p_n)_{n=1}^{\infty}$ in $(0, 1)$ let $\mu_p = \prod_{n=1}^{\infty} \mu_n$ where $\mu_n$ is the measure on $\mathbb{Z}_2$ with $\mu_n(0) = p_n$. Then for any two such sequences $p$ and $q$ we have $\mu_p \sim \mu_q$ if and only if*

$$\sum_{n=1}^{\infty} \left(1 - (p_n q_n)^{\frac{1}{2}} - (1 - p_n)^{\frac{1}{2}} (1 - q_n)^{\frac{1}{2}}\right) < \infty. \tag{5.1.1}$$

*If, in addition, there exists $\delta > 0$ such that $p_n, q_n \in [\delta, 1 - \delta]$ for all $n$ then* (5.1.1) *is equivalent to*

$$\sum_{n=1}^{\infty} (p_n - q_n)^2 < \infty. \tag{5.1.2}$$

Details on how to prove the above result can be found as an exercise in [HS69, p. 455] which also includes a proof of Kakutani's criterion.

This proposition will tell us exactly when the measures $\mu$ and $\mu \circ S$ on $X$ are equivalent. The fact that the proposition is for product measures over $\mathbb{N}$ rather than $\mathbb{Z}$ but is immaterial, it is not difficult to show the range of the sums and products can be changed to the integers. In fact, the proof of Kakutani's criterion says even more. Take $\nu = \prod_{i \in \mathbb{Z}} \nu_i$ and $\lambda = \prod_{i \in \mathbb{Z}} \lambda_i$ with each $\nu_i$ and $\lambda_i$ being non-trivial probability measures on $\{0, 1\}$, let $f : \mathbb{N} \to \mathbb{Z}$ be a bijection and assume that $\nu \sim \lambda$. Within the proof of the criterion one shows that the Radon-Nikodým derivative of the product measure is both the a.e. and $L^1$ limit of the derivatives of the measures in the product. In our context this means that

$$\frac{d\nu}{d\lambda}(x) = \frac{d\nu_f \circ F}{d\lambda_f \circ F}(x) = \frac{d\nu_f}{d\lambda_f}(Fx) = \lim_{n\to\infty}\prod_{i=1}^{n}\frac{d\nu_{f(i)}}{d\lambda_{f(i)}}\left(x_{f(i)}\right)$$

a.e. and in $L^1$. Crucially this happens regardless of the choice of $f$. We can therefore denote such products as $\prod_{\mathbb{Z}}$ without ambiguity.

For each $i \in \mathbb{Z}$ take $p_i = (1 + r_i)/2$ and note that

$$\mu(S\{x_i = j\}) = \mu_p(\{x_{i-1} = j\}) = (1 + r_{i-1})/2$$

we see that $\mu \circ S = \mu_q$ with $q_i = (1+r_{i-1})/2$. Therefore $\mu$ and $\mu \circ S$ are equivalent if and only if

$$\sum_{n\in\mathbb{Z}}\left(1 - \frac{1}{2}((1 + r_n)(1 + r_{n-1})^{\frac{1}{2}} - \frac{1}{2}(1 - r_n)^{\frac{1}{2}}(1 - r_{n-1})^{\frac{1}{2}}\right) < \infty$$

which, if there exists $\delta > 0$ such that $r_n \in [-1 + \delta, 1 - \delta]$ for all $n$, is equivalent to

$$\sum_{n\in\mathbb{Z}}(r_n - r_{n-1})^2 < \infty.$$

As this condition is rather simpler we will work in this context, ensuring that $S$ is also non-singular.

## Sums of Radon-Nikodým derivatives

To calculate the critical dimensions we first need to calculate the Radon-Nikodým derivatives for the action, and then sum these over the summing sets in question. We will take our summing sets to be a Følner sequence of the form $B_n = [-g(n), g(n)] \times A_{n+g(n)}$ as in Proposition 5.1.1. It follows from the arguments in the previous section that

$$\frac{d\mu \circ S}{d\mu}(x) = \prod_{i\in\mathbb{Z}}\frac{d\mu_{i-1}}{d\mu_i}(x) = \prod_{i\in\mathbb{Z}}\frac{1 + (-1)^{x_i}r_{i-1}}{1 + (-1)^{x_i}r_i}$$

then either using the cocycle identity or Kakutani's criterion directly we have

$$\frac{d\mu \circ S^m}{d\mu}(x) = \prod_{i \in \mathbb{Z}} \frac{1 + (-1)^{x_i} r_{i-m}}{1 + (-1)^{x_i} r_i}.$$

If $a \in A(k, l)$ then

$$\frac{d\mu \circ (+a)}{d\mu}(S^m x) = \prod_{i=k}^{l} \frac{\mu_i(x_{i+m} + a_i)}{\mu_i(x_{i+m})}$$

so

$$\sum_{a \in A(k,l)} \frac{d\mu \circ (+a)}{d\mu}(S^m x) = \prod_{i=k}^{l} \frac{1}{\mu_i(x_{i+m})} = \prod_{i=k}^{l} \frac{2}{1 + (-1)^{x_{i+m}} r_i}$$

and hence for $\tau = (l, b) \in L$ and $n = n(\tau)$ large enough we have

$$\sum_{\sigma \in B_n} \omega_\sigma(x) = \sum_{m=-n}^{n} \prod_{i=m-2g(n)}^{m+2g(n)} \frac{2}{1 + (-1)^{x_i} r_{i-m}} \prod_{j \in \mathbb{Z}} \frac{1 + (-1)^{x_j} r_{j-m}}{1 + (-1)^{x_j} r_j},$$

and

$$\sum_{\sigma \in B_n \tau} \omega_\sigma(x) = \sum_{m=l-n}^{l+n} \prod_{i=m-2g(n)-l}^{m+2g(n)-l} \frac{2}{1 + (-1)^{x_i} r_{i-m}} \prod_{j \in \mathbb{Z}} \frac{1 + (-1)^{x_j} r_{j-m}}{1 + (-1)^{x_j} r_j}.$$

With these formulae in hand we can prove the sets $U_t$ and $L_t$, in the definition of the critical dimensions, are invariant under the action of $L$, and hence have measure 0 or 1 when the action is ergodic.

## Invariance of $L_t$ and $U_t$

Using the same methods as for $S$, so long as $r_n \in [-1 + \delta, 1 - \delta]$ for all $n$, the transformation $S^m$ is non-singular if and only if

$$\sum_{n \in \mathbb{Z}} (r_n - r_{n-m})^2 < \infty.$$

Observe that

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(r_n - r_{n-m})^2 &= \sum_{n\in\mathbb{Z}}\left(1+(-1)^{x_n}r_n - (1+(-1)^{x_n}r_{n-m})\right)^2\\ &= \sum_{n\in\mathbb{Z}}(1+(-1)^{x_n}r_n)^2\left(1-R_n(m,x)\right)^2\end{aligned}$$

where

$$R_n = R_n(m,x) = \frac{1+(-1)^{x_n}r_{n-m}}{1+(-1)^{x_n}r_n}.$$

The product $\prod_{\mathbb{Z}} R_n$ is the derivative associated to $S^m$. The bounds on the $r_i$ mean that $1+(-1)^{x_n}r_n \in [\delta, 2-\delta]$ for all $n$ and hence that

$$\delta^2\sum_{n\in\mathbb{Z}}(1-R_n)^2 \le \sum_{n\in\mathbb{Z}}(r_n-r_{n-m})^2 \le (2-\delta)^2\sum_{n\in\mathbb{Z}}(1-R_n)^2.$$

In particular, $S^m$ is non-singular if and only if

$$\sum_{n\in\mathbb{Z}}(1-R_n)^2 < \infty.$$

We can use essentially the same argument to show that we can replace $R_n$ with $R_n^{-1}$ in the above expressions. These expressions hold for all $x \in X$.

This formulation is significant because it is a clear condition on the terms of the infinite product of the derivative associated to $S^m$. It would be useful to see whether this corresponds to some degree of control over the derivatives. One standard way of turning sums of the above form, but without the square, into products is using the exponential inequality $1+y \le e^y$ for all $y \in \mathbb{R}$. This motivates examining the special case where $\sum_n |r_n - r_{n-m}| < \infty$, which we take as an assumption from here. Then by similar arguments to the above we see that

$$\delta\sum_n |1-R_n| \le \sum_n |r_n - r_{n-m}| \le (2-\delta)\sum_n |1-R_n|$$

and similarly with $R_n$ replaced by $R_n^{-1}$. Here we have neglected to say the range of the sums because the statement holds any subset of $\mathbb{Z}$.

In contrast to the case with squares, however, we are able to make use of the exponential inequality. Observe that $R_n \leq e^{R_n - 1}$ and $R_n^{-1} \leq e^{R_n^{-1} - 1}$ so

$$\exp\left(\sum_n (1 - R_n^{-1})\right) \leq \prod_n R_n \leq \exp\left(\sum_n (R_n - 1)\right)$$

and hence

$$\exp\left(-\frac{\sum_n |r_n - r_{n-m}|}{2-\delta}\right) \leq \prod_n R_n \leq \exp\left(\frac{\sum_n |r_n - r_{n-m}|}{\delta}\right). \tag{5.1.3}$$

The key point here being that these bounds are not dependent on $x$, though they do still depend on $m$. This, however, will be enough to let us control the ratios of the Radon-Nikodým sums. Let us write $s_n(m,k) = \sum_{|i-k|>n} |r_i - r_{i-m}|$ and observe for $\tau = (l,b) \in L$ we have

$$\begin{aligned}
\frac{d\mu \circ S^l}{d\mu}(S^{m-l}x) &= \prod_{j\in\mathbb{Z}} \frac{1 + (-1)^{x_{j+m-l}} r_{j-l}}{1 + (-1)^{x_{j+m-l}} r_j} \\
&\leq \prod_{j=-2n}^{2n} \frac{1 + (-1)^{x_{j+m-l}} r_{j-l}}{1 + (-1)^{x_{j+m-l}} r_j} \exp\left(\delta^{-1} s_{2n}(l,0)\right) \\
&= \prod_{j=m-2n-l}^{m+2n-l} \frac{1 + (-1)^{x_j} r_{j-m}}{1 + (-1)^{x_j} r_{j+l-m}} \exp\left(\delta^{-1} s_{2n}(l,0)\right)
\end{aligned}$$

then

$$\begin{aligned}
\sum_{\sigma\in B_n\tau} \omega_\sigma(x) &\leq \exp\left(\delta^{-1} s_{2g(n)}(l,0)\right) \sum_{m=l-n}^{l+n} \prod_{i=m-2g(n)-l}^{m+2g(n)-l} \frac{2}{1 + (-1)^{x_j} r_{j+l-m}} \frac{d\mu \circ S^{m-l}}{d\mu}(x) \\
&= \exp\left(\delta^{-1} s_{2g(n)}(l,0)\right) \sum_{m=-n}^{n} \prod_{i=-2g(n)}^{2g(n)} \frac{2}{1 + (-1)^{x_j} r_{j-m}} \frac{d\mu \circ S^{m}}{d\mu}(x) \\
&= e^{s_{2g(n)}(l,0)/\delta} \sum_{\sigma\in B_n} \omega_\sigma(x)
\end{aligned}$$

and we can apply a symmetrical argument to get the lower estimate

$$e^{-s_{2g(n)}(l,0)/(2-\delta)} \leq \frac{\sum_{\sigma\in B_n\tau} \omega_\sigma(x)}{\sum_{\sigma\in B_n} \omega_\sigma(x)} \leq e^{s_{2g(n)}(l,0)/\delta}.$$

But note that $s_n(m,k) \to 0$ as $n \to \infty$ so

$$\lim_{n\to\infty} \frac{\sum_{\sigma\in B_n\tau} \omega_\sigma(x)}{\sum_{\sigma\in B_n} \omega_\sigma(x)} = 1$$

and by Lemma 1.3.1 the sets $U_t$ and $L_t$ must be invariant, and in particular have measure 0 or 1 because the action is ergodic.

## 5.2 The critical dimensions

In the introduction, Theorem 1.2.7, we saw that Dooley and Mortiss showed the upper and lower critical dimensions of product odometers are given by the limit superior and limit inferior of the averages of the entropies of the coordinate measures. Below we show that, under the assumption described above, the critical dimensions of the Lamplighter group with respect to the summing sequence described above are also determined by the sequence of average coordinate entropies.

**Theorem 5.2.1.** *Let $L$ act on $\prod_{\mathbb{Z}} \mathbb{Z}_2$ equipped with product measure $\mu = \prod_{\mathbb{Z}} \mu_i$ where $\mu_i(\{0\}) = \frac{1+r_i}{2}$. Let $\delta \in (0,1)$ and suppose that each $r_i \in [-(1-2\delta),(1-2\delta)]$ for some and that $\sum_{i\in\mathbb{Z}} |r_i - r_{i-1}| < \infty$. Let $g : \mathbb{N} \to \mathbb{N}$ be increasing, unbounded and $O(n^s)$ for some $s \in (0, \frac{1}{2})$. Let $B_n = [-g(n), g(n)] \times A_{n+g(n)}$. Then the critical dimensions with respect to summing sequence $(B_n)$ are given by*

$$\alpha = \liminf_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} H(\mu_i)$$

*and*

$$\beta = \limsup_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} H(\mu_i).$$

*Proof.* For notational simplicity we sometimes write $p_i = \frac{1+r_i}{2}$ and $h(n) = n + g(n)$. Observe that we may use the preceding inequalities in Equation 5.1.3 to produce the bounds

$$
\begin{aligned}
\sum_{\sigma \in B_n} \omega_\sigma(x) &= \sum_{m=-g(n)}^{g(n)} \prod_{i=m-h(n)}^{m+h(n)} \frac{2}{1+(-1)^{x_i} r_i} \prod_{|j-m|>h(n)} \frac{1+(-1)^{x_j} r_{j-m}}{1+(-1)^{x_j} r_j} \\
&\leq \sum_{m=-g(n)}^{g(n)} \prod_{i=m-h(n)}^{m+h(n)} \frac{2}{1+(-1)^{x_i} r_i} \exp_2 \left( \frac{C}{\delta} \sum_{|j|>h(n)} |r_j - r_{j-m}| \right) \\
&= \sum_{m=-g(n)}^{g(n)} \exp_2 \left( \sum_{i=h(n)}^{h(n)} Y_{i+m} + \frac{C}{\delta} \sum_{|j|>h(n)} |r_j - r_{j+m}| \right) \qquad (\star)
\end{aligned}
$$

and similarly

$$
\sum_{\sigma \in B_n} \omega_\sigma(x) \geq \sum_{m=-g(n)}^{g(n)} \exp_2 \left( \sum_{i=-h(n)}^{h(n)} Y_{i+m} - \frac{C}{2-\delta} \sum_{|j|>h(n)} |r_j - r_{j+m}| \right)
$$

where $C = \log_2(e)$ and

$$
Y_i(x) = -(1-x_i)\log_2 p_i - x_i \log_2 (1-p_i).
$$

We first show that sum of the $Y_j$ in $(\star)$ grows like the sum of coordinate entropies, and then we show that the sum involving the $r_j$ is $o(n)$.

Note that $(Y_i)_{i\in\mathbb{Z}}$ is a sequence of independent Bernoulli random variables such that

$$
-\log_2 (1-\delta) \leq Y_i \leq -\log_2(\delta)
$$

and with mean $\mu(Y_i) = H(\mu_i)$.

Let

$$
Z_n = \frac{1}{2h(n)+1} \sum_{i=-h(n)}^{h(n)} Y_i \quad \text{and} \quad H_n = \frac{1}{2h(n)+1} \sum_{i=-h(n)}^{h(n)} H(\mu_i).
$$

Let $\epsilon > 0$, we can apply Hoeffding's inequality [Hoe63, Theorem 2] to deduce that for each $m \in [-g(n), g(n)]$

$$
\mu\left(|Z_n \circ S^m - H_n \circ S^m| \geq g(n)^{-1}\right) \leq 2\exp\left( -\frac{2(2h(n)+1)}{g(n)^2 \log\left(\frac{1-\delta}{\delta}\right)^2} \right).
$$

It follows that

$$
\begin{aligned}
&\mu\left(\bigcup_{n\geq k}\bigcup_{m=-g(n)}^{g(n)}\{|Z_n\circ S^m - H_n\circ S^m|\geq g(n)^{-1}\}\right)\\
\leq &\sum_{n\geq k}\sum_{m=-g(n)}^{g(n)}\mu\left(|Z_n\circ S^m - H_n\circ S^m|\geq g(n)^{-1}\right)\\
\leq &\sum_{n\geq k}2(2g(n)+1)\exp\left(-\frac{2(2h(n)+1)}{g(n)^2\log\left(\frac{1-\delta}{\delta}\right)^2}\right).
\end{aligned}
$$

We show that this sum is finite. For some constant $D_1>0$ and for $n$ sufficiently large $g(n)\leq D_1 n^s$. Hence

$$
\frac{2h(n)+1}{g(n)^2}\geq\frac{2n}{g(n)^2}\geq\frac{2}{D_1}n^{1-2s}
$$

and hence the above sum is, up to a constant, controlled by one of the form

$$
\sum_{n\geq k}n^s\exp\left(-D_2 n^{1-2s}\right)
$$

for some constant $D_2>0$. Fix $t>\frac{1+s}{1-2s}$ and take $R>0$ large enough so that $x>R$ ensures $x^t<\exp(D_2x)$. Then for sufficiently large $k$ and all $n\geq k$ we have $n^{1-2s}>R$ and hence $\exp(-D_2n^{1-2s})<n^{-(1-2s)t}$. It follows that

$$
\sum_{n\geq k}n^s\exp\left(-D_2 n^{1-2s}\right)<\sum_{n\geq k}n^{s-(1-2s)t}<\infty.
$$

This means that as $k\to\infty$

$$
\mu\left(\bigcup_{n\geq k}\bigcup_{m=-g(n)}^{g(n)}\{|Z_n\circ S^m - H_n\circ S^m|\geq g(n)^{-1}\}\right)\to 0
$$

and the sets $\bigcup_{n\geq k}\bigcup_{m=-g(n)}^{g(n)}\{|Z_n\circ S^m - H_n\circ S^m|\geq g(n)^{-1}\}$ are decreasing with $k$. Therefore for a.e. $x\in X$ there exists $K\in\mathbb{N}$ such that for all $n\geq K$ we have

$$
\left|\frac{\sum_{i=-h(n)}^{h(n)}Y_{i+m}(x)}{2h(n)+1}-\frac{\sum_{i=-h(n)}^{h(n)}H(\mu_{i+m})}{2h(n)+1}\right|<g(n)^{-1}
$$

for each $m \in [-g(n), g(n)]$.

Returning attention to the sum $\sum_{|j|>h(n)} |r_{j+m} - r_j|$, regardless of the choice of $m \in [-g(n), g(n)]$ we can use the triangle inequality to show that

$$\sum_{|j|>h(n)} |r_{j+m} - r_j| \leq g(n) \sum_{|j|>n} |r_j - r_{j-1}| \leq \frac{\delta}{C} g(n)$$

for $n$ sufficiently large.

Combining these facts with the earlier derived inequalities we see that

$$\begin{aligned}
\sum_{\sigma \in B_n} \omega_\sigma(x) &\leq \sum_{m=-g(n)}^{g(n)} \exp_2 \left( \sum_{i=-h(n)}^{h(n)} H(\mu_{i+m}) + \frac{2h(n)+1}{g(n)} + g(n) \right) \\
&\leq (2g(n)+1) \exp_2 \left( \sum_{i=-n-2g(n)}^{n+2g(n)} H(\mu_i) + \frac{2h(n)+1}{g(n)} + g(n) \right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\sigma \in B_n} \omega_\sigma(x) &\geq \sum_{m=-g(n)}^{g(n)} \exp_2 \left( \sum_{i=-h(n)}^{h(n)} H(\mu_{i+m}) - \frac{2h(n)+1}{g(n)} - g(n) \right) \\
&\geq (2g(n)+1) \exp_2 \left( \sum_{i=-n}^{n} H(\mu_i) - \frac{2h(n)+1}{g(n)} - g(n) \right).
\end{aligned}$$

Now let $\epsilon > 0$, and take $n$ sufficiently large so that

$$\frac{1}{g(n)} + \frac{g(n)}{2h(n)+1} < \epsilon$$

then

$$\begin{aligned}
&\frac{1}{|B_n|^t} \sum_{\sigma \in B_n} \omega_\sigma(x) \\
\leq &(2g(n)+1)^{1-t} \exp_2 \left( (2h(n)+1) \left( \frac{\sum_{i=-n-2g(n)}^{n+2g(n)} H(\mu_i)}{2h(n)+1} - t + \epsilon \right) \right)
\end{aligned}$$

and if

$$t = 3\epsilon + \liminf_{n\to\infty} \frac{\sum_{i=-n-2g(n)}^{n+2g(n)} H(\mu_i)}{2h(n)+1}$$

then for almost every $x$ we can find a subsequence where the right hand side converges to 0. In particular, this means that $t > \alpha$ and since $\epsilon > 0$ was arbitrary

$$\alpha \leq \liminf_{n\to\infty} \frac{\sum_{i=-n-2g(n)}^{n+2g(n)} H(\mu_i)}{2h(n)+1} = \liminf_{n\to\infty} \frac{\sum_{i=-n}^{n} H(\mu_i)}{2n+1}$$

since $H(\mu_i) \in [0,1]$ for all $i$ and $g(n) = o(n)$. A similar argument, involving instead the limit superior and the entire sequence shows that

$$\beta \leq \limsup_{n\to\infty} \frac{\sum_{i=-n}^{n} H(\mu_i)}{2n+1}.$$

A symmetrical argument, but using the lower bound, can be used to show that equality must hold. □

## 5.3 Open problems

In Theorem 5.2.1 we assume that each $r_i \in [-(1-2\delta),(1-2\delta)]$ and that $\sum_{i\in\mathbb{Z}} |r_i - r_{i-1}| < \infty$. We saw earlier that the action is non-singular if and only if $\sum_{i\in\mathbb{Z}} (r_i - r_{i-1})^2 < \infty$. This condition is significantly weaker, for example it allows $|r_n - r_{n-1}| = \frac{1}{n}$ which in turn means the sequence $(r_n)$ could be chosen to oscillate between (essentially) $\pm(1-2\delta)$. We do not know whether the conclusion of Theorem 5.2.1 extends to such sequences, or more generally, as we directly used the property that $\sum_{i\in\mathbb{Z}} |r_i - r_{i-1}| < \infty$ in the proof. It would be of interest to know whether the conclusion can be extended further.

An issue we have not addressed in this chapter is the invariance of the critical dimensions from the summing sequences considered. In the previous chapters we used the ergodic theorem from Chapter 2 to show the ergodic theorem held for the summing sequence, and hence deduced the resulting critical dimension is an invariant of metric isomorphism. We cannot apply this formalism to the summing sequences from this chapter because

$$|B_n| = |[-g(n), g(n)] \times A_{n+g(n)}| = (2g(n)+1)2^{2n+2g(n)+1}$$

and hence $(B_n)$ does not satisfy the multiplicative doubling condition, which was crucial in the proof of Theorem 2.3.1. As such, ideally we would like establish by other means whether these sequences produce invariants. It may be possible to show these critical dimensions are invariant without proving a corresponding ergodic theorem. If the critical dimensions calculated here are not invariants then it may be that there are other viable summing sequences.

Lastly, the calculations we completed in this chapter are heavily action specific. If one can find summing sequences for which the critical dimensions are invariant, then it would be beneficial to have calculated the critical dimensions of other actions with respect to said summing sequences.

# Appendix A

# Ergodic Decomposition Theorem

In this appendix we give the details on how to extend the ergodic decomposition theorem, Theorem 1.1.7, from the integer case to that for countable amenable groups. For ease, we restate the theorem below.

**Theorem A.0.1** (Ergodic decomposition theorem)**.** *Let $G$, a countable amenable group, act non-singularly on the standard probability space $(X, \mathcal{B}, \mu)$. Then there is a probability space $(Y, \mathcal{C}, \nu)$ and a family of probability measures $\{\mu_y : y \in Y\}$ on $(X, \mathcal{B})$ such that*

(i) *For each $A \in \mathcal{B}$ the map $y \mapsto \mu_y(A)$ is measurable and*

$$\mu(A) = \int_Y \mu_y(A)\, d\nu(y).$$

(ii) *The measures $\mu_y$ and $\mu_{y'}$ are mutually singular for $y \neq y'$.*

(iii) *For each $y \in Y$ the action of $G$ on $(X, \mathcal{B}, \mu_y)$ is ergodic, non-singular and for all $g \in G$*

$$\frac{d\mu \circ g}{d\mu} = \frac{d\mu_y \circ g}{d\mu_y} \qquad \mu_y\text{-a.e.}.$$

(iv) *For any other probability space $(Y', \mathcal{C}', \nu')$ and family of probability measures $\{\mu'_{y'} : y' \in Y'\}$ on $(X, \mathcal{B})$ satisfying* (i)-(iii) *there exists a measure preserving isomorphism $\phi : Y \to Y'$ such that $\mu_y = \mu'_{\phi(y)}$ for $\nu$-a.e. $y \in Y$.*

As discussed in the introduction, the proof in the case where $G = \mathbb{Z}$ and the action is conservative can be found in [Aar97, 1.0.8 & 2.2.8]. One can remove the assumption of conservativity by applying our version of the ergodic theorem, as we discussed in Remark 3.1.7.

Here we will show how one goes about replacing $\mathbb{Z}$ with an arbitrary countable amenable group. The approach we take is to use the fact that, given a non-singular action of a countable amenable group $G$ there exists a non-singular transformation $T$ on the same space such that for almost every $x \in X$ the orbit of $x$ under $G$ is the same as its orbit under $T$, i.e.

$$\{gx : g \in G\} = \{T^n x : n \in \mathbb{Z}\}.$$

This is a consequence of a theorem due to Connes, Feldman and Weiss which can be found in [CFW81].

Let $G$ be a group acting on the measure space $(X, \mathcal{B}, \mu)$ as in the theorem, and $T$ be a transformation as given by the above result applied to the action. Without loss of generality, we can assume that every point has the same $G$ and $T$ orbit. Then we can apply the decomposition theorem to $T$ to achieve a decomposition $\{\mu_y : y \in Y\}$ with respect to the action generated by $T$. We now show essentially the same collection also serves as a decomposition with respect to the action of $G$.

Clearly, conditions (i) and (ii) are both still satisfied as they are action independent. Now let us consider condition (iii).

First we show that the measures $\mu_y$ are ergodic with respect to the action of $G$. Suppose that $A$ is a $G$ invariant set, then

$$\mu_y(GA \setminus A) \leq \sum_{g \in G} \mu_y(gA \setminus A) = 0$$

so has measure zero and, similarly, $\mu_y(A \setminus GA) = 0$. Hence

$$\mu_y(A) = \mu_y(GA) = \mu_y\left(\bigcup_{i \in \mathbb{Z}} T^i A\right) \in \{0, 1\}$$

as the latter set is evidently $T$-invariant, as required.

Next we need to show that the measures $\mu_y$ are non-singular for the action of

$G$. Given $g \in G$ we can partition $X$ using measurable sets

$$A_n = A_n(g) \subseteq \{x \in X : gx = T^n x\}.$$

Then

$$\begin{aligned}
\mu_y(gA) = 0 \qquad & \iff \qquad \forall n \quad \mu_y(g(A \cap A_n)) = 0 \\
& \iff \qquad \forall n \quad \mu_y(T^n(A \cap A_n)) = 0 \\
& \iff \qquad \forall n \quad \mu_y(A \cap A_n) = 0 \\
& \iff \qquad \mu_y(A) = 0
\end{aligned}$$

and since $g$ was arbitrary $\mu_y$ is non-singular with respect to the action of $G$.

The final part of point (iii) concerns the the Radon-Nikodým derivatives. Before proving this, first recall that for all measures $\lambda$ and $m$ on the same measurable space with $\lambda \ll m$ we have

$$\frac{d\lambda|_A}{dm} = \frac{d\lambda}{dm}\mathbf{1}_A$$

$m$-almost everywhere. It follows that for each $g \in G$ and $n \in \mathbb{Z}$

$$\frac{d\mu \circ g}{d\mu}\mathbf{1}_{A_n(g)} = \frac{d\mu \circ T^n}{d\mu}\mathbf{1}_{A_n(g)}$$

$\mu$-almost surely. Since any set of $\mu$ measure 1 must have $\mu_y$ measure 1 for $\nu$-almost all $y \in Y$, and $G$ (and $\mathbb{Z}$) are countable by removing a set of measure zero from $Y$ we can ensure that the above inequality holds $\mu_y$-almost surely for all $y \in Y$. Then for all $y \in Y$

$$\begin{aligned}
\frac{d\mu \circ g}{d\mu} = \sum_{n \in \mathbb{Z}} \mathbf{1}_{A_n(g)} \frac{d\mu \circ g}{d\mu} &= \sum_{n \in \mathbb{Z}} \mathbf{1}_{A_n(g)} \frac{d\mu \circ T^n}{d\mu} \\
&= \sum_{n \in \mathbb{Z}} \mathbf{1}_{A_n(g)} \frac{d\mu_y \circ T^n}{d\mu_y} = \sum_{n \in \mathbb{Z}} \mathbf{1}_{A_n(g)} \frac{d\mu_y \circ g}{d\mu_y} = \frac{d\mu_y \circ g}{d\mu_y}
\end{aligned}$$

$\mu_y$-almost everywhere, which shows that (iii) holds.

As for point (iv), if we instead start with any decomposition for the action of $G$ then we can find a $T$ using Connes-Feldman-Weiss as above independent of the particular decomposition. We can use the same techniques as above (albeit in reverse) to show that, up to a set of $Y$ measure zero, the decomposition for the action of $G$ can be used as one for the action of $T$, and then uniqueness follows from the uniqueness for the action of $T$.

# Appendix B

# The Rohklin Lemma

In this appendix we state and prove the non-singular and amenable version of the Rokhlin lemma due to Ornstein and Weiss, the proof of which (as far as the author is aware) has not been published, as we discussed in Subsection 1.3.1.

## B.1 Boundaries, interiors and set invariance

First it will be useful to introduce a notion of boundaries and interiors of sets in the measure space $(X, \mu)$ with respect to a given subset $K$ of $G$, which acts on $X$ in a non-singular fashion. These definitions are based on those in [Orn70], and some of notation and proofs are from [KL16].

**Definition B.1.1.** Let $K$ be a subset of $G$ and $A$ subset of $X$. The *$K$-boundary* of $A$ is given by

$$\mathrm{Bd}_K A = \{y \in X : Ky \cap A \neq \emptyset \text{ and } Ky \cap A^c \neq \emptyset\}.$$

The *$K$-interior* of $A$ is given by

$$\mathrm{Int}_K A = \{a \in A : Ka \subseteq A\}.$$

So the boundary consists of the elements of $X$ whose $K$-orbits intersect both $A$ and it's complement, and the interior those elements of $A$ whose $K$-orbit is contained in $A$.

*Remark* B.1.2. Note the following useful properties:

1. $\mathrm{Bd}_K A = \bigcup_{k\in K} k^{-1}A \setminus \bigcap_{k\in K} k^{-1}A = \bigcup_{s,t\in K}(s^{-1}A \triangle t^{-1}A)$ and perhaps more concisely $\mathrm{Bd}_K A = K^{-1}A \cap K^{-1}(X \setminus A)$.

2. If $e \in K$ then $\mathrm{Int}_K A = \bigcap_{k\in K} k^{-1}A$ and hence $K^{-1}A = \mathrm{Int}_K A \sqcup \mathrm{Bd}_K A$.

3. In general $\mathrm{Int}_K A \subseteq \bigcap_{k\in K} k^{-1}A$ and $\mathrm{Bd}_K A \subseteq K^{-1}A \setminus \mathrm{Int}_K A$.

Invariance of a set may be defined by requiring the set's boundary to be small relative to the size of the set. More precisely, let $\epsilon > 0$ and let $K$ and $A$ be finite subsets of $G$ and $X$ respectively, we say that $A$ is $(K, \epsilon)$-*invariant* if $\mu(\mathrm{Bd}_K A) \leq \epsilon\mu(A)$.

For the remainder of this section assume that either $\mu$ is a finite measure or it is invariant with respect to the action of $G$. In this context, when $e \in K$ a small set boundary easily implies that the interior is quite large.

**Lemma B.1.3.** *Let $K$ and $A$ be finite subsets of $G$ and $X$ respectively, with $e \in K$. If $A$ is $(K, \epsilon)$-invariant then $\mu(\mathrm{Int}_K A) \geq (1-\epsilon)\mu(A)$.*

*Proof.* Using the above formula if $\mu(\mathrm{Int}_K A) < (1-\epsilon)\mu(A)$ then

$$\mu(\mathrm{Bd}_K A) = \mu(K^{-1}A) - \mu(\mathrm{Int}_K A) > \mu(A) - (1-\epsilon)\mu(A) = \epsilon\mu(A). \qquad \square$$

For invariant measures $\mu$, the converse, i.e. that a large interior ensures a small boundary, also holds.

**Proposition B.1.4.** *Let $(G, |\cdot|)$ be a countable group acting on $(X, \mu)$ with $\mu$ invariant with respect to the action of $G$. Then the following are equivalent:*

(1) *For all finite sets $K \subseteq G$ and $\epsilon > 0$ there exists a finite set $A \subseteq X$ which is $(K, \epsilon)$-invariant.*

(2) *For all finite sets $K \subseteq G$ with $e \in K$ and $\epsilon > 0$ there exists a finite set $A \subseteq G$ such that $\mu(\mathrm{Int}_K A) \geq (1-\epsilon)\mu(A)$.*

*Proof.* (1) $\Rightarrow$ (2) is the content of the above lemma.

For the converse, let us assume (2) and let $K$ and $\epsilon$ be as in (1). Then $KK^{-1}$ is finite and contains $e$ so by (2) we may find a finite set $A$ such that

$$\mu(\mathrm{Int}_{KK^{-1}} A) \geq \left(1 - \frac{\epsilon}{2|K|^2}\right)\mu(A).$$

Then observe that

$$\mu(\mathrm{Bd}_K A) = \mu\bigg(\bigcup_{s,t\in K}(s^{-1}A\triangle t^{-1}A)\bigg) \le \sum_{s,t\in K}\mu(A\triangle st^{-1}A).$$

Then for any $r \in KK^{-1}$ we have

$$\mu(A\triangle rA) = \mu(A\setminus rA) + \mu(A\setminus r^{-1}A) \le 2\mu(A\setminus \mathrm{Int}_{KK^{-1}}A) \le \frac{\epsilon}{|K|^2}\mu(A)$$

and hence

$$\mu(\mathrm{Bd}_K A) \le \sum_{r\in KK^{-1}}\frac{\epsilon}{|K|^2}\mu(A) \le \epsilon\mu(A)$$

as required. □

In particular, we will use this for the case where $X = G$ and $\mu$ is counting measure on $G$.

## B.2 Statement and proof of the lemma

Let us first give formal definitions of some properties alluded to earlier and cite some results about their interactions from [OW87].

Let $H \subseteq G$ and $B \subseteq X$ such that the collection $\{hB : h \in H\}$ is pairwise disjoint. Then $B$ is called a *base of an $H$-tower*, namely $HB$. Note that unless $e \in H$ the base of the tower may not be in the tower. In the proof the freeness of the action will be used to produce collections of towers from which to construct cities, via the following lemma.

**Lemma B.2.1.** *If $G$ acts freely on $(X,\mu)$ then for any finite set $H \subseteq G$ there is a countable partition of $X$ (up to a $\mu$-null set) by bases of $H$-towers.*

For the remainder of this section $\mathcal{A} = \{A_i\}_{i\in I}$ is a collection of non-empty measurable subsets of $X$.

We will construct a collections of towers to prove the lemma, and we will require that these are mostly disjoint. More precisely, we will ensure for some small $\epsilon > 0$ the collection of towers is $\epsilon$-disjoint: the collection $\{A_i\}_{i\in I}$ is said to be *$\epsilon$-disjoint* if there exists a pairwise disjoint collection $\{\hat{A}_i\}_{i\in I}$ such that for each $i \in I$ we have $\hat{A}_i \subseteq A_i$ and $\mu(\hat{A}_i) \geq (1-\epsilon)\mu(A_i)$. Intuitively, this means that the proportion of each $A_i$ which touches any $A_j$ with $j \neq i$ is small.

For $A \subseteq X$ we will say that the collection $\mathcal{A}$ $\lambda$-*covers* $A$ if

$$\mu\left(A \cap \bigcup_{i \in I} A_i\right) \geq \lambda \mu(A).$$

So when $\lambda$ is close to 1 this condition says that the collection of the $A_i$ covers all but a small portion of $A$.

We say that $\mathcal{A}$ is an *even covering* of $A$ if there exists $M \in \mathbb{N}$ such that $\sum_{i \in I} \mathbf{1}_{A_i}(x) = M$ for a.e. $x \in A$. $M$ is called the *multiplicity* of the covering.

*Remark* B.2.2. The towers produced by lemma B.2.1 must be an even covering of $X$ of multiplicity $|H|$, since the bases partition $X$.

The collection $\mathcal{A}$ is a $\delta$-*even covering* of $A$ if there exists $M \in \mathbb{N}$ such that $\sum_{i \in I} \mathbf{1}_{A_i}(x) \leq M$ for a.e. $x \in A$ and $\sum_{i \in I} \mu(A_i) \geq (1-\delta)M\mu(A)$. $M$ is called the *multiplicity* of the covering in either case. The latter condition is, intuitively, a little more complex than the previous. The first part simply says that most $x \in A$ are in at most $M$ of the $A_i$. The second part says that, despite the first part, the measures of all the $A_i$ are almost $M$ times the measure of $A$ - so we would expect 'most' (in terms of mass) of the $x \in A$ to be in nearly $M$ of the $A_i$.

**Lemma B.2.3.** *Let $A \subseteq X$, $\delta, \epsilon \in (0,1)$ and $\{A_i\}_{i \in I}$ be a $\delta$-even covering of $A$. Then there is an $\epsilon$-disjoint subcollection of $\{A_i\}$ which $\epsilon(1-\delta)$-covers $A$.*

Finally before stating and proving the lemma, it will be useful to introduce some terminology. Let $A \subseteq X$ and $K \subseteq G$, we will call elements of $A \cap \mathrm{Bd}_K A$ *interior boundary points* and elements of $A^c \cap \mathrm{Bd}_K A$ *exterior boundary points*. An *almost even covering* is a collection which $\delta$-*even covering* for some $\delta < 1$.

**Theorem B.2.4** (Rokhlin's Lemma)**.** *Let $\epsilon > 0$ and $F_0 = K$ be a finite subset of $G$ with $e \in K$. Fix $k \in \mathbb{N}$ such that $(1-\epsilon/2)^k < \epsilon$. Assume that $F_1, ..., F_k$ are finite subsets of $G$ with $S_i = F_i F_i^{-1} \subseteq F_{i+1}$ and $F_{i+1}$ sufficiently invariant with respect to $S_i$ for each $0 \leq i \leq k-1$.*

*Then for any non-singular free action of $G$ on $(X, \mu)$ and each $1 \leq i \leq k$ we can find a finite collection of sets $\{V_i^l\}_{l=1}^{L_i}$ such that*

(i) *every $F_i V_i^l$ is an $F_i$-tower;*

(ii) *the* $F_iV_i^l$ *for* $1 \leq l \leq L_i$ *are* $\epsilon$*-disjoint;*

(iii) *for* $i \neq j$*, and any* $l$*,*$l'$*,* $F_iV_i^l \cap F_jV_j^{l'} = \emptyset$*;*

(iv) $\mu\left(\bigcup_{i=1}^k F_iV_i\right) \geq 1-\epsilon$ *where* $V_i = \bigcup_{l=1}^{L_i} V_i^l$*;*

(v) $\bigcup_{i=1}^k F_iV_i$ *is* $(K,\epsilon)$*-invariant.*

(vi) *each* $F_iV_i^l$ *satisfies* $\mu\left(\mathrm{Int}_K(F_iV_i^l)\right) \geq (1-\epsilon)\mu(F_iV_i^l)$*.*

*Remark* B.2.5. If $G$ is amenable then the existence of subsets $F_1, ..., F_k$ follows from the existence of the Følner sequence, and so the lemma can always be applied to a non-singular free action of an amenable group.

*Proof.* Without loss of generality $\epsilon < 1$. For notational simplicity, let us fix numbers $0 < \delta < 1/4$ and $0 < \eta < \epsilon\delta/2$. Our invariance assumption ensures that we can take each $F_{i+1}$ to be $(S_i, \delta_i)$-invariant, with $\delta_i$ small and to be determined. Note that $\delta_i$ can depend on $F_i$.

**The strategy:** we will produce the tower bases $V_j^l$ satisfying (i)-(iii) recursively working down from $j = k$. Additionally, when selecting the $V_j^l$'s we will ensure that

1. the set $F_jV_j$ contains at least $\epsilon/2$ of $X_j = X \setminus \bigcup_{i=j+1}^k F_iV_i$, the portion of the space uncovered by the previous steps, and

2. the $V_j^l$ are 'good' in the sense that they satisfy

$$\mu\left(\mathrm{Bd}_{S_{j-1}}(F_jV_j^l) \cap X_j\right) \leq \eta\, \mu(F_jV_j^l). \tag{$\star$}$$

The first property will mean that once the step $j = 1$ is complete the proportion of the space left uncovered by the towers is less than $(1-\epsilon/2)^k < \epsilon$ by assumption, giving us (iv). The second property is crucial in the inductive step, and at the end of the proof will give us properties (v) and (vi).

**The initial step:** we begin with $F_k$ and use Lemma B.2.1 to evenly cover $X$ by $F_k$-towers $F_kU_k^n$ with $n \in \mathbb{N}$. We restrict to the good sub-collection of these towers with property ($\star$), i.e. the $F_kU_k^m$ such that

$$\mu\left(\mathrm{Bd}_{S_{k-1}}(F_kU_k^m) \cap X_k\right) = \mu\left(\mathrm{Bd}_{S_{k-1}}(F_kU_k^m)\right) \leq \eta\, \mu(F_kU_k^m),$$

where $X_k = X$. We aim to use lemma B.2.3 on this subcollection of towers, and so need to prove it is an almost even covering. Since the whole collection is an even covering with multiplicity $|F_k|$ it suffices to show the good towers have enough mass to cover the space almost $|F_k|$ times. Let $B$ index the collection of the corresponding 'bad' towers and observe

$$\sum_{b\in B}\mu(F_kU_k^b) \leq \eta^{-1}\sum_{b\in B}\mu\left(\mathrm{Bd}_{S_{k-1}}(F_kU_k^b)\right) = \eta^{-1}\int_X\sum_{b\in B}\mathbf{1}_{\mathrm{Bd}_{S_{k-1}}(F_kU_k^b)}\,d\mu.$$

This integrand is counting the number of boundaries (of bad towers) each $x \in X$ lies in. So if we can bound this appropriately we can control the amount of mass in the bad towers.

In actuality we are going to bound the larger sum

$$\sum_{n\in\mathbb{N}}\mathbf{1}_{\mathrm{Bd}_{S_{k-1}}(F_kU_k^n)}.$$

Since $F_k$ is $(S_{k-1}, \delta_{k-1})$-invariant we have that

$$|\mathrm{Int}_{S_{k-1}}F_k| \geq (1-\delta_{k-1})|F_k|,$$

so for most $t \in F_k$ we have $S_{k-1}t \subseteq F_k$. Hence the $S_{k-1}$ orbits most levels of the tower $F_kU_k^n$ are contained by $F_kU_k^n$ itself. In particular, if $x$ is an interior boundary point for the tower $F_kU_k^n$ then $x \in tU_k^n$ for some

$$t \in F_k \setminus \mathrm{Int}_{S_{k-1}}F_k \subseteq \mathrm{Bd}_{S_{k-1}}F_k.$$

Since for each such $t$ the $tU_k^n$ partition $X$ each $x \in X$ is an interior boundary point for at most $|\mathrm{Bd}_{S_{k-1}}F_k|$ towers.

$S_{k-1}$ is symmetric so each $x \in X$ is in the $S_{k-1}$ orbit of at most $|S_{k-1}|$ points including itself, let $y$ be one such. Consider each tower $y$ lies in, any tower both $x$ and $y$ lie in will have $x$ in its boundary if and only if it is an interior boundary point, which were counted above. Therefore we need only count the towers $y$ lies in but $x$ does not. Since $x$ is in the $S_{k-1}$ orbit of $y$ we have that $y$ is an interior boundary point for each of these towers and so there are at most $|\mathrm{Bd}_{S_{k-1}}F_k|$ of them. Hence in total each $x \in X$ lies in at most $|S_{k-1}||\mathrm{Bd}_{S_{k-1}}F_k|$

tower boundaries.

It follows that

$$\sum_{b\in B}\mu(F_kU_k^b)\leq \eta^{-1}|S_{k-1}||\mathrm{Bd}_{S_{k-1}}F_k|\leq \frac{\delta_{k-1}}{\eta}|S_{k-1}||F_k|<\eta|F_k|,$$

by taking $\delta_{k-1}<\frac{\eta^2}{|S_{k-1}|}$. We can then deduce that the sum over 'good' towers satisfies

$$\sum_{m\notin B}\mu(F_kU_k^m)\geq \sum_n \mu(F_kU_k^n)-\eta|F_k|=(1-\eta)\,|F_k|\mu(X),$$

because

$$\sum_n \mu(F_kU_k^n)=\sum_n\sum_{t\in F_k}\mu(tU_k^n)=|F_k|,$$

since the $U_k^n$ partition $X$.

Now, as $\eta<1/2$, we can then apply Lemma B.2.3 to find a finite $\epsilon$-disjoint sub-collection of these towers which $\epsilon/2$-cover $X$, given by the bases $V_k^l$ for $1\leq l\leq L_k$.

**The inductive step:** Let $1\leq j<k$ suppose we have found $V_i^l$ for each $i>j$ such that

$$\mu\left(\mathrm{Bd}_{S_{i-1}}(F_iV_i^l)\cap X_i\right)\leq \eta\,\mu(F_iV_i^l). \tag{$\star$}$$

We may assume that for each $i\geq j$ we have $\mu(X_i)\geq \epsilon/2$, if not we could take each $V_j^l=\emptyset$ for each $j<i$ and conditions (i)-(iv) would be proved.

Find a covering of $X$ by $F_j$ towers $F_jU_j^n$, $n\in\mathbb{N}$. We may assume that the $U_j^n$ refine the partition of $X$ given by the set

$$Y_j=\mathrm{Int}_{F_j}X_j=\bigcap_{t\in F_j}t^{-1}X_j,$$

and its complement. Recall that $Y_j$ is the portion of $x\in X_j$ for which $F_jx\subseteq X_j$. We aim to show that the good towers $F_jU_j^m$ with $U_j^m\subseteq Y_j$ form an almost even covering of $X_j$, and hence by Lemma B.2.3 we may find a sub-collection which is $\epsilon$-disjoint and $\epsilon/2$-covers $X_j$.

We know that each $x\in X$ and hence each $x\in X_j$ lies in fewer than $|F_j|$ good $F_j$-towers, so it will suffice to show the sum of masses of good towers is at a minimum almost $|F_j|\mu(X_j)$. Now, it will be useful for us to consider the $F_j^{-1}$ interior of $Y_j$

$$Z_j = \mathrm{Int}_{F_j^{-1}} Y_j = \bigcap_{t\in F_j} tY_j = \bigcap_{t\in S_j} t^{-1}X_j,$$

which satisfies

$$\begin{aligned}
\mu\left(Z_j\right) = \mu\left(X \setminus \bigcup_{i=j+1}^{k} S_j^{-1}F_iV_i\right) &= \mu\left(X_j \setminus \bigcup_{i=j+1}^{k} S_j^{-1}F_iV_i\right) \\
&= \mu\left(X_j \setminus \bigcup_{i=j+1}^{k} \mathrm{Bd}_{S_j}F_iV_i\right) \\
&\geq \mu\left(X_j\right) - \sum_{i=j+1}^{k} \mu\left(X_j \cap \mathrm{Bd}_{S_j}F_iV_i\right).
\end{aligned}$$

By using the property $(\star)$, $\epsilon$-disjointness and recalling $\eta < \epsilon\delta/2$

$$\begin{aligned}
\sum_{i=j+1}^{k} \mu\left(X_j \cap \mathrm{Bd}_{S_j}F_iV_i\right) &\leq \sum_{i=j+1}^{k}\sum_{l=1}^{L_i} \mu\left(\mathrm{Bd}_{S_j}(F_iV_i^l) \cap X_j\right) \\
&\leq \sum_{i=j+1}^{k}\sum_{l=1}^{L_i} \mu\left(\mathrm{Bd}_{S_{i-1}}(F_iV_i^l) \cap X_i\right) \\
&\leq \sum_{i=j+1}^{k}\sum_{l=1}^{L_i} \eta\mu(F_iV_i^l) \\
&\leq \sum_{i=j+1}^{k} \frac{\eta}{1-\epsilon}\mu(F_iV_i) \\
&\leq \frac{\epsilon\delta}{2(1-\epsilon)}\mu\left(\bigcup_{i=j+1}^{k} F_iV_i\right) \\
&\leq \frac{\epsilon\delta}{2}.
\end{aligned}$$

Therefore $\mu\left(Z_j\right) \geq (1-\delta)\mu(X_j)$.

Now let us show that the good towers approximately even cover $X_j$. Like in the initial step we will need to control the measure of the bad towers. As before

$$\sum_{b\in B}\mu(F_jU_j^b) \leq \eta^{-1}\sum_{b\in B}\mu\left(\mathrm{Bd}_{S_{j-1}}(F_jU_j^b)\cap X_j\right) = \eta^{-1}\int_{X_j}\sum_{b\in B}\mathbf{1}_{\mathrm{Bd}_{S_{j-1}}(F_jU_j^b)}\,d\mu,$$

but this time we need to be a little more careful, because the collection of towers with bases in $Y_j$ may not cover all of $X_j$. However, for points in $X_j$ covered by the towers the same argument as before will apply since points in no tower cannot contribute to the number of tower boundaries other points sit in. For points not in any tower, the points in their $S_{j-1}$ orbit must lie in the boundary levels in towers or no towers at all. So, as before, we see the integrand is bounded above by $|S_{j-1}||\mathrm{Bd}_{S_{j-1}}F_j|$. Hence

$$\sum_{b\in B}\mu(F_jU_j^b) \leq \eta^{-1}|S_{j-1}||\mathrm{Bd}_{S_{j-1}}F_j|\mu(X_j) \leq \eta|F_j|\mu(X_j) \leq \delta|F_j|\mu(X_j),$$

where we have taken $\delta_{j-1} < \frac{\eta^2}{|S_{j-1}|}$. It follows that

$$\begin{aligned}\sum_{m\notin B}\mu\left(F_jU_j^m\right) &\geq \sum_{n}\mu\left(F_jU_j^n\right) - \delta|F_j|\mu(X_j)\\ &= \sum_{t\in F_j}\mu\left(tY_j\right) - \delta|F_j|\mu(X_j)\\ &\geq |F_j|\mu\left(Z_j\right) - \delta|F_j|\mu(X_j) \geq (1-2\delta)|F_j|\mu(X_j)\end{aligned}$$

and so the collection of good towers $F_jU_j^m$ with $U_j^m \subseteq Y_j$ forms a $2\delta$-even covering of $X_j$. As $\delta < 1/4$ we can find a finite sub-collection $F_jV_j^l$ which is $\epsilon$-disjoint and $\epsilon/2$-covers $X_j$. Since each $V_j^l \subseteq Y_j$ each $F_jV_j^l \subseteq X_j$ and so is disjoint from all previous $F_iV_i^l$. Therefore sub-collection satisfies (i)-(iii) and contains at least $\epsilon/2$ of $\mu(X_j)$ as required. Recur to achieve (iv).

**Property (v):** note that in the above proof it is said what happens in the final step. We do not actually need to make any restrictions on which $F_1$ towers to use if we simply want the conclusions (i)-(iv). However, we can use this flexibility to eke out (v) and (vi). We placed the same assumptions on $F_0 = K$ as we did the other $F_i$ and so we can, as above, achieve good towers satisfying

$$\mu\left(\mathrm{Bd}_{S_0}(F_1V_1^l)\cap X_i\right) \leq \eta\,\mu(F_1V_1^l),$$

and by following the argument above we can see that

$$\mu\left(\mathrm{Bd}_K\left(\bigcup_i F_iV_i\right)\right) \leq \sum_i \mu\left(\mathrm{Bd}_{S_0}(F_iV_i)\cap X_i\right) \leq \frac{\epsilon}{2(1-\epsilon)}\delta\mu\left(\bigcup_i F_iV_i\right),$$

but $\delta$ was allowed to be arbitrarily small, so we are able to ensure the $K$ boundary of the $\bigcup_i F_iV_i$ is arbitrarily small relative to the mass of the collection itself.

**Property (vi):** we observe that

$$\begin{aligned}
\mu\left(\mathrm{Int}_{S_0}(F_iV_i^l)\right) &= \mu\left(\mathrm{Int}_{S_0}(F_iV_i^l) \cap X_i\right) \\
&= \mu\left(X_i \cap S_0^{-1}F_iV_i^l \setminus \mathrm{Bd}_{S_0}(F_iV_i^l)\right) \\
&\geq \mu\left(S_0^{-1}F_iV_i^l \cap X_i\right) - \mu\left(\mathrm{Bd}_{S_0}(F_iV_i^l) \cap X_i\right) \\
&\geq (1-\eta)\mu(F_iV_i^l),
\end{aligned}$$

using the fact that our towers are good. Recall $\eta < \epsilon$. □

One of the most interesting parts of this proof is the fact that it uses a counting argument to control the total mass of the bad towers. It allows one to use the size of a subset of the group, in this case a boundary we can control, to determine the behaviour of the action. This is a common feature in measure preserving arguments, but in non-singular arguments one usually has to make some reference to the Radon-Nikodým derivatives. It is perhaps no coincidence that this appeared as one of the first significant non-singular and amenable results.

It would be of interest to know whether some of the ideas behind this proof can be adapted into techniques for managing other aspects of non-singular systems.

# Bibliography


[Aar97] J. Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[AB06] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis.* Springer, Berlin, third edition, 2006. A hitchhiker's guide.

[BD08] N. Brodskiy and J. Dydak. Coarse dimensions and partitions of unity. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 102(1):1–19, 2008.

[CFW81] A. Connes, J. Feldman, and B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergodic Theory Dynamical Systems*, 1(4):431–450 (1982), 1981.

[dG75] M. de Guzmán. *Differentiation of integrals in $R^n$*. Lecture Notes in Mathematics, Vol. 481. Springer-Verlag, Berlin-New York, 1975. With appendices by Antonio Córdoba, and Robert Fefferman, and two by Roberto Moriyón.

[DJ16] A. H. Dooley and K. Jarrett. Non-singular $\mathbb{Z}^d$-actions: an ergodic theorem over rectangles with application to the critical dimensions. *ArXiv e-prints*, June 2016.

[DM06] A. H. Dooley and G. Mortiss. On the critical dimension and AC entropy for Markov odometers. *Monatsh. Math.*, 149(3):193–213, 2006.

[DM07] Anthony H. Dooley and Genevieve Mortiss. The critical dimensions of Hamachi shifts. *Tohoku Math. J. (2)*, 59(1):57–66, 2007.

[DM09] A. H. Dooley and G. Mortiss. On the critical dimensions of product odometers. *Ergodic Theory Dynam. Systems*, 29(2):475–485, 2009.

[Fel07] Jacob Feldman. A ratio ergodic theorem for commuting, conservative, invertible transformations with quasi-invariant measure summed over symmetric hypercubes. *Ergodic Theory Dynam. Systems*, 27(4):1135–1142, 2007.

[Fuc15] László Fuchs. *Abelian groups.* Springer Monographs in Mathematics. Springer, Cham, 2015.

[Hoc10] M. Hochman. A ratio ergodic theorem for multiparameter non-singular actions. *J. Eur. Math. Soc. (JEMS)*, 12(2):365–383, 2010.

[Hoc13] M. Hochman. On the ratio ergodic theorem for group actions. *J. Lond. Math. Soc. (2)*, 88(2):465–482, 2013.

[Hoe63] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.

[HS69] E. Hewitt and K. Stromberg. *Real and abstract analysis. A modern treatment of the theory of functions of a real variable.* Second printing corrected. Springer-Verlag, New York-Berlin, 1969.

[HS90] Waldemar Hebisch and Adam Sikora. A smooth subadditive homogeneous norm on a homogeneous group. *Studia Math.*, 96(3):231–236, 1990.

[Hur44] Witold Hurewicz. Ergodic theorem without invariant measure. *Ann. of Math. (2)*, 45:192–206, 1944.

[Jar17] K. Jarrett. An ergodic theorem for non-singular actions of the Heisenberg groups. *ArXiv e-prints*, February 2017.

[Kak48] S. Kakutani. On equivalence of infinite product measures. *Ann. of Math. (2)*, 49:214–224, 1948.

[KL16] David Kerr and Hanfeng Li. *Ergodic theory.* Springer Monographs in Mathematics. Springer, Cham, 2016. Independence and dichotomies.

[KM04] A. S. Kechris and B. D. Miller. *Topics in orbit equivalence*, volume 1852 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[Kre85] U. Krengel. *Ergodic theorems*, volume 6 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.

[KW91] Y. Katznelson and B. Weiss. The classification of nonsingular actions, revisited. *Ergodic Theory Dynam. Systems*, 11(2):333–348, 1991.

[LD17] Enrico Le Donne. A primer on Carnot groups: homogenous groups, Carnot-Carathéodory spaces, and regularity of their isometries. *Anal. Geom. Metr. Spaces*, 5:116–137, 2017.

[LDR17] Enrico Le Donne and Séverine Rigot. Besicovitch covering property for homogeneous distances on the Heisenberg groups. *J. Eur. Math. Soc. (JEMS)*, 19(5):1589–1617, 2017.

[Lin01] E. Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.

[Mor47] Anthony P. Morse. Perfect blankets. *Trans. Amer. Math. Soc.*, 61:418–442, 1947.

[Mor03] Genevieve Mortiss. An invariant for non-singular isomorphism. *Ergodic Theory Dynam. Systems*, 23(3):885–893, 2003.

[Orn70] D. Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Advances in Math.*, 4:337–352 (1970), 1970.

[OW80] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980.

[OW87] Donald S. Ornstein and Benjamin Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987.

[Pat88] A.L.T. Paterson. *Amenability*, volume 29. American Mathematical Soc., 1988.

[Rig04] Séverine Rigot. Counter example to the Besicovitch covering property for some Carnot groups equipped with their Carnot-Carathéodory metric. *Math. Z.*, 248(4):827–848, 2004.

[Rom08] Steven Roman. *Advanced linear algebra*, volume 135 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008.

[Roy63] H. L. Royden. *Real analysis*. The Macmillan Co., New York; Collier-Macmillan Ltd., London, 1963.

[RY08] Bryan P. Rynne and Martin A. Youngson. *Linear functional analysis*. Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, second edition, 2008.

[Sie22] W. Sierpinski. Sur les fonctions d'ensemble additives et continues. *Fundamenta Mathematicae*, 3:240–246, 1922.

[SW92] E. Sawyer and R. L. Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.*, 114(4):813–874, 1992.