PHD

Tilting Bundles and Toric Fano Varieties

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Tilting Bundles And Toric Fano Varieties

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ABSTRACT

This thesis constructs tilting bundles obtained from full strong exceptional collections of line bundles on all smooth toric Fano fourfolds. The tilting bundles lead to a large class of explicit Calabi-Yau-5 algebras, obtained as the corresponding rolled-up helix algebra. We provide two different methods to show that a collection of line bundles is full, whilst the strong exceptional condition is checked using the package *QuiversToricVarieties* for the computer algebra system *Macaulay2*, written by the author. A database of the full strong exceptional collections can also be found in this package.

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CHAPTER

ONE

INTRODUCTION

Let X be a smooth variety over $\mathbb{C}$ and let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X. A tilting object $\mathcal{T} \in \mathcal{D}^b(X)$ is an object such that $\mathrm{Hom}^i(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$ and $\mathcal{T}$ generates $\mathcal{D}^b(X)$. If such a $\mathcal{T}$ exists, then tilting theory provides an equivalence of triangulated categories between $\mathcal{D}^b(X)$ and the bounded derived category $\mathcal{D}^b(A)$ of finitely generated right modules over the algebra $A = \mathrm{End}(\mathcal{T})$ via the adjoint functors

$$\mathcal{D}^b(X) \underset{(-)\overset{\mathbb{L}}{\otimes}_A \mathcal{T}}{\overset{\mathbf{R}\,\mathrm{Hom}_X(\mathcal{T}, -)}{\rightleftarrows}} \mathcal{D}^b(A)$$

If X is also projective then one can use a full strong exceptional collection to obtain a tilting object; a full strong exceptional collection of sheaves $\{E_i\}_{i \in I}$ defines a tilting sheaf $\mathcal{T} := \bigoplus_{i \in I} E_i$ and conversely, the non-isomorphic summands in a tilting sheaf determine a full strong exceptional collection. The classical example of a tilting sheaf was provided by Beĭlinson [Beĭ78], who showed that $\mathcal{O} \oplus \mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(n)$ is a tilting bundle for $\mathbb{P}^n$.

The combinatorial nature of toric varieties makes it feasible to check whether a collection of line bundles on a smooth projective toric variety is full strong exceptional, in which case one can construct the resulting endomorphism algebra explicitly. Smooth toric Fano varieties are of particular interest; there are a finite number of these varieties in each dimension and they have been classified in dimension 3 by Watanabe–Watanabe and Batyrev [WW82, Bat82b], dimension 4 by Batyrev and Sato [Bat99, Sat00], dimension 5 by Kreuzer–Nill [KN09], whilst Øbro [Øb07] provided a general classification algorithm. King [Kin97] has exhibited full strong exceptional collections of line bundles for the 5 smooth toric Fano surfaces, and by building on work by Bondal [Bon06], Costa–Miró-Roig [CMR04] and Bernardi–Tirabassi [BT09], Uehara [Ueh14] provided full strong exceptional collections of line bundles for the 18 smooth toric Fano threefolds. The main theorem of this thesis is as follows:

Theorem 7.4. *Let X be one of the* 124 *smooth toric Fano fourfolds. Then one can explicitly construct a full strong exceptional collection of line bundles on X, a database of which is contained in the computer package* QuiversToricVarieties *[PN15a] for* Macaulay2 *[GS].*

In addition to low-dimensional smooth toric Fano varieties, other classes of toric varieties have been shown to have full strong exceptional collections of line bundles – for example, see [CMR04, DLM09, LM11]. Kawamata [Kaw06] showed that every smooth toric Deligne-Mumford stack has a full exceptional collection of sheaves, but we note that these collections are not shown to be strong, nor do they consist of bundles. It is important to note that the existence of full strong exceptional collections of line bundles is rare; Hille–Perling [HP06] constructed smooth toric surfaces that do not have such collections. Even when only considering smooth toric Fano varieties, there exist examples in dimensions ≥ 419 that do not have full strong exceptional collections of line bundles, as demonstrated by Efimov [Efi10].

The tilting bundle we construct on each smooth toric Fano variety determines a tilting bundle on the total space of the canonical bundle ω_X:

Theorem 7.7. *Let X be an n-dimensional smooth toric Fano variety for $n \leq 4$, $\mathcal{L} = \{L_0, \ldots, L_r\}$ be the full strong exceptional collection on X from the database and $\pi\colon Y := \mathrm{tot}(\omega_X) \to X$ be the bundle map. Then Y has a tilting bundle that decomposes as a sum of line bundles, given by $\bigoplus_{i=0}^{r} \pi^*(L_i)$.*

1.1 Structure of the Thesis

This thesis comprises of 8 chapters and 3 appendices. It is the combination of two papers [PN15b, PN15a] by the author, with additional explanations and examples.

Chapter 2

Chapter 2 is divided into four sections and recalls most of the background material needed. In the first section, we introduce the construction of toric varieties via the theory of lattice polytopes. A fan Σ associated to a lattice polytope gives a combinatorial model for the toric variety and can be fully described by *primitive collections and relations* (see Definition 2.4). We also consider the blowup $X_0 \to X_1$ between two toric varieties, the corresponding change to the fan for X_1 and the resulting map of Picard lattices $\gamma\colon \mathrm{Pic}(X_0) \to \mathrm{Pic}(X_1)$.

The second section recalls some properties of line bundles on toric varieties, as well as Batyrev's classification of the smooth toric Fano fourfolds [Bat99]. A description of the combinatorial change to the fan of a toric variety after a blowup (see (2.1.6)) was used by Sato [Sat00] to complete the classification of the smooth toric Fano fourfolds; we call the maximal smooth toric Fano varieties with regard to these blowups *birationally maximal*. The map γ resulting from a blowup becomes important when we consider how to produce full strong exceptional collections of line bundles on all smooth toric Fano fourfolds from collections on the birationally maximal examples.

The third section of Chapter 2 introduces the reader to the definitions of a full strong exceptional collection and a tilting object for the bounded derived category

$\mathcal{D}^b(X)$ of coherent sheaves on a smooth variety X. We recall that the existence of a tilting object implies that $\mathcal{D}^b(X)$ is equivalent to the bounded derived category of a module category (see (2.3.1)), and that tilting objects on varieties Y and Z determine a tilting object on the product $Y \times Z$ (see Lemma 2.24).

In the final section, we recall Ginzburg's definition of a Calabi-Yau algebra [Gin06], as well as Bridgeland–Stern's definition of a geometric helix [BS10]. The condition for a helix to be geometric is central to the proof of Theorem 7.7.

Chapter 3

Chapter 3 focuses on how we can show that a given collection of line bundles on a toric variety X is strong exceptional, by utilising the construction of the *not-necessarily non-vanishing cohomology cones* (*nnnvc*-cones) in the Picard lattice for X as introduced by Eisenbud–Mustaţă–Stillman [EMS00]. The strong exceptional condition then becomes a computational exercise, which has been implemented into *QuiversToricVarieties* [PN15a]. In the second section of Chapter 3, we show that these *nnnvc*-cones behave well under the map $\gamma\colon \operatorname{Pic}(X_0) \to \operatorname{Pic}(X_1)$ corresponding to a blowup $X_0 \to X_1$. This simplifies the process of finding a strong exceptional collection on a smooth toric Fano variety such that the collection is the image under γ of a strong exceptional collection on a birationally maximal smooth toric Fano variety – see Propositions 3.13 and 3.14 for more details.

Chapter 4

The procedure to check whether a given strong exceptional collection $\mathcal{L} = \{L_0, L_1, \dots, L_r\}$ on X generates $\mathcal{D}^b(X)$ is less straightforward. We use one of two methods to show that $\mathcal{L}$ is full, the first of which is introduced in Chapter 4 and is similar to the method used by Uehara for the toric Fano threefolds [Ueh14]. This approach uses the Frobenius morphism $F_m\colon X \to X$, where m is some fixed positive integer; Thomsen [Tho00] has shown that the Frobenius pushforward $(F_m)_*(L)$ of a line bundle L on a toric variety splits into a direct sum of line bundles. We use the Frobenius pushforward to obtain a set of line bundles that are known to generate $\mathcal{D}^b(X)$ and then show that $\mathcal{L}$ generates this set by using exact sequences of line bundles.

Chapter 5

The second method we use to show that $\mathcal{L}$ is full uses the line bundles in $\mathcal{L}$ to obtain a resolution of $\mathcal{O}_\Delta$, the structure sheaf of the diagonal embedding of X into $X \times X$. Chapter 5 and Chapter 6 focus on how we produce this resolution. Chapter 5 begins by recalling how to construct a quiver of sections Q and the moduli space of quiver representations $\mathcal{M}_\theta(Q, J)$ corresponding to $\mathcal{L}$, for some stability parameter θ and ideal of relations J. We introduce the map $d_1\colon \mathcal{E}_1 \to \mathcal{E}_0$ (5.2.1) of vector bundles on $X \times X$ constructed from $\mathcal{L}$ and show that if there is a closed embedding of X into $\mathcal{M}_\theta(Q, J)$ such that the tautological bundles on $\mathcal{M}_\theta(Q, J)$ pull back to the line bundles in $\mathcal{L}$ on X, then the cokernel of d_1 is $\mathcal{O}_\Delta$ (Proposition 5.4). We finish the chapter by giving conditions as to when X embeds into $\mathcal{M}_\theta(Q, J)$ such that the tautological bundles restrict to the line bundles; the conditions given depend on whether all of the line bundles in $\mathcal{L}$ are nef or not.

Chapter 6

Chapter 6 explains how we compute the rest of the resolution

$$0 \to \mathcal{E}_k \to \cdots \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0$$

of $\mathcal{O}_\Delta$ and is motivated by the work of King [Kin97] on the smooth toric Fano surfaces. Setting A to be the endomorphism algebra $\mathrm{End}(\bigoplus_i L_i^{-1})$ and $\mathcal{T} = \bigoplus_i L_i^{-1}$, he constructs the object $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T} \in \mathcal{D}^b(X \times X)$ from a minimal projective A, A-bimodule resolution $P^\bullet$ of A and shows that if $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T}$ is quasi-isomorphic to $\mathcal{O}_\Delta$ in $\mathcal{D}^b(X \times X)$, then $\mathcal{L}$ generates $\mathcal{D}^b(X)$ (see Lemma 6.3). The final map in the minimal projective A, A-bimodule resolution of A is determined by d_1 and as King was working with 2-dimensional varieties, $P^\bullet$ could be calculated explicitly by only knowing the vertices and arrows in the quiver for $\mathcal{L}$ (see Lemma 6.2); however, in general it is not known how to compute $P^\bullet$ for the algebra A. Our method utilises the idea of a *toric cell complex*, introduced by Craw–Quintero-Vélez [CQV12], to guess a minimal projective A, A-bimodule resolution of A.

For the smooth toric Fano fourfolds in particular, by considering the pullback of $\mathcal{L}$ to the total space of the canonical bundle $Y := \mathrm{tot}(\omega_X)$, the resulting rolled-up helix algebra is expected to be a Calabi–Yau-5 (CY5) algebra for which we know the 0^{th}, 1^{st} and 2^{nd} terms of its minimal projective bimodule resolution. The natural duality inherent in a CY5 algebra then gives clues, via our guess as to what the corresponding toric cell complex is, as to what the 3^{rd}, 4^{th} and 5^{th} terms are. We sheafify the result, restrict to X and then check that the resulting exact sequence of sheaves $S^\bullet$ is indeed a resolution of $\mathcal{O}_\Delta$ by using quiver moduli as explained in Chapter 5. Similarly for the smooth toric Fano threefolds, we obtain an algebra expected to be CY4, in which case the 0^{th} term in its minimal projective bimodule resolution determines the 4^{th} term, the 1^{st} term determines the 3^{rd} term and the 2^{nd} term is self-dual. Using this method for the collections of line bundles from the database contained in *QuiversToricVarieties* [PN15a], we obtain resolutions of the diagonal for 88 of the smooth toric Fano fourfolds and all 18 of the smooth toric Fano threefolds.

The final section in the chapter gives the framework for the second method we use to show that a collection of line bundles on a smooth toric Fano fourfold is full, by bringing together the concepts in Chapters 5 and 6.

Chapter 7

We present the main theorems of the thesis in Chapter 7. By considering the birational geometry of the smooth toric Fano fourfolds (see Figure B.1) and choosing collections $\mathcal{L}$ from a special set of line bundles on X_0 as Uehara did for the toric Fano threefolds, the pushforward of $\mathcal{L}$ onto a torus-invariant divisorial contraction X_1 is automatically full if $\mathcal{L}$ is full, and the pushforward coincides with the image of $\mathcal{L}$ under the map $\gamma\colon \mathrm{Pic}(X_0) \to \mathrm{Pic}(X_1)$ (see Proposition 7.2 and Lemma 7.1). We can then check that the collection on X_1 is strong exceptional by ensuring that the necessary tensor products of $\mathcal{L}$ avoid the preimage of the *nnnvc*-cones for X under the map γ, in addition to the *nnnvc*-cones for X_0; as outlined in Chapter 3, these preimages have a simple description. Using this process, we obtain full strong exceptional collections on many of the toric Fano fourfolds from the pushforward of collections on the birationally maximal examples. With an additional computation, we then show that the tilting bundles we obtain on the smooth toric Fano fourfolds as well as the tilting bundles Uehara exhibits on the smooth toric Fano threefolds pull back to give tilting bundles that decompose as a direct sum of line bundles on the total space of the canonical bundle for the variety.

Chapter 8

This chapter presents unanswered questions that have arisen from this thesis. For the 88 smooth toric Fano fourfolds and 18 smooth toric Fano threefolds for which we compute a resolution of $\mathcal{O}_\Delta$ in Chapter 6, it is not known whether the toric cell complex exists in any of these cases. Our calculations therefore lead us to the following conjecture:

Conjecture 8.1. *Let X be a smooth toric Fano threefold or one of the 88 smooth toric Fano fourfolds such that the given full strong exceptional collection $\mathcal{L}$ in the database [PN15a] has a corresponding exact sequence of sheaves $S^\bullet \in \mathcal{D}^b(X \times X)$. Let B denote the rolled up helix algebra of $A = \mathrm{End}(\bigoplus_{L \in \mathcal{L}} L^{-1})$. Then the toric cell complex of B exists and is supported on a real four or five-dimensional torus respectively. Moreover,*

- *the cellular resolution exists in the sense of [CQV12], thereby producing the minimal projective bimodule resolution of B;*
- *the object $S^\bullet$ is quasi-isomorphic to $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T} \in \mathcal{D}^b(X \times X)$, where $\mathcal{T} := \bigoplus_{L \in \mathcal{L}} L^{-1}$ and $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T}$ is the exterior tensor product over A of $\mathcal{T}^\vee$ and $\mathcal{T}$.*

The chapter also provides an example of a smooth toric Fano fourfold for which we have failed to find a resolution of $\mathcal{O}_\Delta$ using the method in Chapter 6.

The Appendices

Appendix A consists of the article accompanying the *Macaulay2* package *QuiversToricVarieties* [PN15a]. This package contains a database of the full strong exceptional collections on n-dimensional smooth toric Fano varieties for $1 \leq n \leq 4$, as well as many of the computational tools used in the proofs of the theorems in the thesis. Appendix B details how the the full strong exceptional collections on the smooth toric Fano varieties of dimension ≤ 4 are obtained and has the divisorial contraction diagram for the smooth toric Fano fourfolds, whilst Appendix C contains examples of smooth toric Fano fourfolds for which we use the second method of generation to show that a given strong exceptional collection of line bundles generates $\mathcal{D}^b(X)$.

CHAPTER

TWO

BACKGROUND

This chapter provides the background material for the thesis and is divided into four sections. The first section introduces toric geometry; given a lattice polytope, we construct its corresponding fan Σ and show that this information can be used to create a toric variety X_Σ. Primitive collections and toric morphisms are defined and we introduce the map $\gamma\colon \operatorname{Pic}(X_0) \to \operatorname{Pic}(X_1)$ induced from a torus-invariant divisorial contraction $X_0 \to X_1$.

In the second section we detail some basic properties of line bundles and recall the classification of smooth toric Fano varieties in low dimensions. The third section introduces the reader to the notions of a full strong exceptional collection and a tilting object for the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on a smooth variety X. We recall that the existence of a tilting object implies that $\mathcal{D}^b(X)$ is equivalent to the bounded derived category of a module category, and that tilting objects on varieties Y and Z determine a tilting object on the product $Y \times Z$.

The final section describes the construction of helices of sheaves on a variety and gives Ginzburg's definition of a Calabi-Yau algebra [Gin06].

2.1 Toric Geometry

For $n \geq 0$, let M be a rank n lattice and define $N := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ to be its dual lattice. The realifications $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ are real vector spaces which contain the underlying lattices and there exists a natural pairing $\langle\ ,\ \rangle\colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. The convex hull of a finite set of lattice points in M defines a *lattice polytope* $P \subset M_{\mathbb{R}}$ and its *facets* are the codimension 1 faces of P. We will assume that the dimension of P is equal to the rank of M.

Definition 2.1. A *convex polyhedral cone* σ in $N_{\mathbb{R}}$ is the set

$$\left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \right\} \subset N_{\mathbb{R}}$$

for a given finite set $S \subset N_{\mathbb{R}}$. We say that σ is *rational* if additionally, $S \subset N$.

The theory of polytopes (see for example Cox–Little–Schenck [CLS11]) states that every facet F in P has an inward-pointing normal n_F that defines a one-dimensional cone $\{\lambda n_F \mid \lambda \in \mathbb{R}_{\geq 0}\}$ in $N_{\mathbb{R}}$. The cone is rational as P is a lattice polytope, so it has a unique generator $u_F \in N$. Given $a \in \mathbb{R}$ and a non-zero vector $u \in N_{\mathbb{R}}$ we have the *affine hyperplane* $H_{u,a} := \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle = a\}$ and the *closed half-space* $H^+_{u,a} := \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq a\}$. As P is full-dimensional, each facet F defines a unique number $a_F \in \mathbb{R}$ such that $F = H_{u_F,a_F} \cap P$ and $P \subset H^+_{u_F,a_F}$. We can therefore use the generators to completely describe P, using its unique *facet presentation*

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F \text{ for all facets } F \text{ in } P\}.$$

Example 2.2. Let $M = \mathbb{Z}^2$ and $P \subset M_{\mathbb{R}} = \mathbb{R}^2$ be the convex hull of the vertices

$$\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}.$$

The polygon is shown in Figure 2.1, and its facet presentation is

$$P = \left\{ m \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} \langle m, e_1\rangle \geq -1 \\ \langle m, e_2\rangle \geq -1 \\ \langle m, -e_1 - e_2\rangle \geq -1 \\ \langle m, -e_2\rangle \geq -1 \end{array} \right\}$$

where $\{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$.

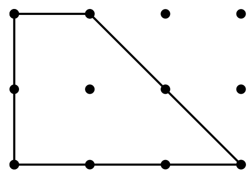


Figure 2.1: *A lattice polytope in* $\mathbb{R}^2$

If the origin of $M_{\mathbb{R}}$ is an interior lattice point of P, then P has a dual polytope P° which is defined to be the convex hull of the generators for the inward-pointing normal rays of P:

$$P^\circ = \mathrm{Conv}(u_F \mid F \text{ is a facet of } P) \subset N_{\mathbb{R}}$$

The dual polytope determines a *fan* in $N_{\mathbb{R}}$:

Definition 2.3. Let F be a proper face of P° with vertices $\{u_{i_1}, \ldots, u_{i_k}\}$. The cone $\sigma(F)$ is given by

$$\sigma(F) := \{\lambda_1 u_{i_1} + \ldots + \lambda_k u_{i_k} \in N_{\mathbb{R}} \mid \lambda_j \geq 0, 1 \leq j \leq k\}. \quad (2.1.1)$$

The fan $\Sigma(P^\circ) \subset N_{\mathbb{R}}$ associated to P° is given by the collection of cones

$$\Sigma = \Sigma(P^\circ) := \{0\} \cup \{\sigma(F)\}_{F \subsetneq P^\circ} \quad (2.1.2)$$

where F runs over all proper faces of P°.

Let $\Sigma(k)$ denote the set of k-dimensional cones in a fan Σ and we write $\tau \preceq \sigma$ when a cone τ is a face of a cone σ. The *rays* of Σ are the one-dimensional cones which, by construction, are generated by the vectors u_F for each facet $F \subset P$. We can use the ray generators to define *primitive collections* and *primitive relations* which describe Σ combinatorially.

Definition 2.4. A subset $\mathcal{P} = \{u_{i_1}, \ldots, u_{i_k}\}$ of the set of ray generators $\mathcal{V} = \{u_F \in N \mid F \text{ is a facet of } P\}$ for Σ is a *primitive collection* if

(i) there does not exist a cone in Σ that contains every element of $\mathcal{P}$ and

(ii) any proper subset of $\mathcal{P}$ is contained in some cone of Σ.

The integral element $s(\mathcal{P}) = u_{i_1} + \ldots + u_{i_k}$ is contained in some cone $\sigma \in \Sigma$ with ray generators $\{u_{j_1}, \ldots, u_{j_m}\}$ and so can be uniquely written as a sum of the generators:

$$s(\mathcal{P}) = c_1 u_{j_1} + \ldots + c_m u_{j_m}, c_i > 0, c_i \in \mathbb{Z}.$$

The linear relation

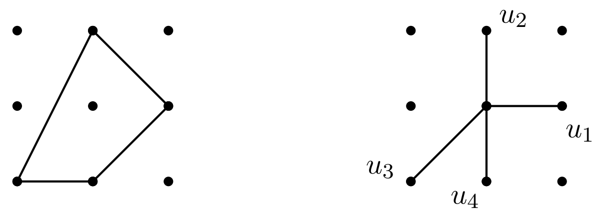
$$u_{i_1} + \ldots + u_{i_k} - (c_1 u_{j_1} + \ldots + c_m u_{j_m}) = 0$$

between the ray generators of Σ is the *primitive relation* associated to the primitive collection $\mathcal{P}$.

Example 2.5. The dual polytope P° to the lattice polytope in Example 2.2 has vertices

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ u_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \ u_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

and is shown along with its corresponding fan $\Sigma(P^\circ)$ in Figure 2.5.



(a) The dual polytope P°

(b) The fan

Figure 2.2: *The dual polytope in $N_\mathbb{R}$ and its corresponding fan*

The primitive relations for $\Sigma(P^\circ)$ are

- $u_1 + u_3 - u_4 = 0$;
- $u_2 + u_4 = 0$.

Toric geometry (see e.g. Fulton [Ful93] or Cox–Little–Schenck [CLS11]) associates to each fan Σ a toric variety X_Σ such that M is the character lattice of the dense torus

$T \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ in X_Σ. For a cone $\sigma \subset N_{\mathbb{R}}$, its *dual cone* $\sigma^\vee$ is the set

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Each cone $\sigma \in \Sigma$ gives an affine variety

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]),$$

and for $\tau \preceq \sigma$ we have $U_\tau \subset U_\sigma$. The affine varieties U_σ for $\sigma \in \Sigma$ are glued together according to the arrangement of Σ, in which case we obtain the toric variety X_Σ. If the fan Σ is constructed from a polytope $P \subset M_{\mathbb{R}}$ as above, then we use X_P to denote the corresponding toric variety.

For a cone $\sigma \in \Sigma$ define $\sigma^\perp := \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle = 0 \text{ for all } u \in \sigma\}$. Each cone σ determines a torus orbit $O(\sigma) \subset X_\Sigma$ and there is an isomorphism $O(\sigma) \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$ [CLS11, Lemma 3.2.5].

Proposition 2.6 (The Orbit-Cone Correspondence [CLS11])**.** *Let X_Σ be the toric variety corresponding to a fan $\Sigma \subset N_{\mathbb{R}}$ and $n = \dim N_{\mathbb{R}}$. Then*

1. *There is a bijective correspondence*

$$\begin{aligned} \{\textit{cones in } \Sigma\} &\longleftrightarrow \{T\textit{-orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma) \end{aligned}$$

2. *For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.*

3. *The affine open subset U_σ is the union of the orbits*

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

4. *$\tau \preceq \sigma$ if and only if $O(\sigma) \subset \overline{O(\tau)}$, and*

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma).$$

where $\overline{O(\tau)}$ is the closure of $O(\tau)$ in both the classical and Zariski topologies.

The Orbit-Cone Correspondence implies that for each ray $\rho \in \Sigma(1)$, the closure of the T-orbit $O(\rho)$ is a torus-invariant divisor D_ρ in X_Σ. The lattice of torus-invariant divisors in X_Σ will therefore be denoted $\mathbb{Z}^{\Sigma(1)}$ and the class group will be denoted $\mathrm{Cl}(X_\Sigma)$. We now have an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Cl}(X_\Sigma) \longrightarrow 0 \tag{2.1.3}$$

where the injective map is $m \mapsto \Sigma_{\rho\in\Sigma(1)}\langle m, u_\rho\rangle D_\rho$ and the map *deg* sends the divisor D to the isomorphism class of the rank one reflexive sheaf $\mathcal{O}_{X_\Sigma}(D)$. Henceforth, all of the varieties that we consider in this thesis will be smooth, in which case every rank one reflexive sheaf is invertible and so the class group $\mathrm{Cl}(X_\Sigma)$ is isomorphic to the

Picard group $\mathrm{Pic}(X_\Sigma)$. Note that X_Σ is smooth if and only if for every cone $\sigma \in \Sigma$, the minimal generators for σ form part of a $\mathbb{Z}$-basis for N.

Example 2.7. The variety determined by the fan in Example 2.5 is the Hirzebruch surface $\mathscr{H}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. From the fan, we see that $\mathscr{H}_1$ has four torus-invariant points and four torus-invariant divisors. The exact sequence (2.1.3) for this variety is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0, \tag{2.1.4}$$

The *Cox ring* for X_Σ is the semigroup ring $S_X := \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ of $\mathbb{N}^{\Sigma(1)} \subset \mathbb{Z}^{\Sigma(1)}$. The map *deg* induces a $\mathrm{Cl}(X_\Sigma)$-grading on S_X, where the degree of a monomial $\prod_{\rho\in\Sigma(1)} x_\rho^{a_\rho} \in S_X$ is the divisor class $[\sum_{\rho\in\Sigma(1)} a_\rho D_\rho]$. For $\alpha \in \mathrm{Cl}(X_\Sigma)$, we let $(S_X)_\alpha = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]_\alpha$ denote the α-graded piece. Cox [Cox95, Proposition 3.1] defines an exact functor from the category of $\mathrm{Cl}(X_\Sigma)$-graded S_X-modules to the category of quasi-coherent sheaves on X_Σ:

$$\{\mathrm{Cl}(X_\Sigma)\text{-graded } S_X\text{-modules}\} \longrightarrow \mathrm{Qcoh}(X_\Sigma) \,:\, M \mapsto \widetilde{M}. \tag{2.1.5}$$

As X_Σ is smooth, every coherent sheaf on X_Σ is isomorphic to $\widetilde{M}$ for some finitely generated $\mathrm{Pic}(X_\Sigma)$-graded S_X-module M and two finitely generated $\mathrm{Pic}(X_\Sigma)$-graded S_X-modules determine isomorphic coherent sheaves if and only if they agree up to saturation by the *irrelevant ideal* $B_X := \left(\prod_{\rho\not\preceq\sigma} x_\rho \mid \sigma \in \Sigma\right)$ [Cox95, Propositions 3.3, 3.5]. For $\alpha \in \mathrm{Pic}(X_\Sigma)$, we have the $\mathrm{Pic}(X_\Sigma)$-graded S_X-module $S_X(\alpha)$, where $(S_X(\alpha))_\beta = (S_X)_{\alpha+\beta}$ for $\beta \in \mathrm{Pic}(X_\Sigma)$.

Morphisms between two toric varieties can be described by maps between their associated fans that preserve the cone structure. For example, consider the blowup of a torus-invariant subvariety. By the Orbit-Cone Correspondence, a k-codimensional torus-invariant subvariety of a toric variety X_Σ corresponds to a cone $\sigma \in \Sigma(k)$, and the blowup of this subvariety is the toric variety whose fan is the *star subdivision* of σ. The star subdivision is a combinatorial process that introduces a new ray x with generator $u_\sigma = \sum_{\rho\in\sigma(1)} u_\rho$ and replaces Σ with

$$\Sigma^*_{\sigma,x} := \{\tau \in \Sigma \mid \sigma \not\preceq \tau\} \cup \bigcup_{\sigma\preceq\tau} \Sigma^*_\tau(\sigma) \tag{2.1.6}$$

where $\Sigma^*_\tau(\sigma) := \{\mathrm{Cone}(A) \mid A \subseteq \{u_\sigma\} \cup \tau(1), \sigma(1) \not\subseteq A\}$. The map between fans $\Sigma^*_{\sigma,x} \to \Sigma$ determines the blowup

$$\varphi\colon X_0 := X_{\Sigma^*_{\sigma,x}} \longrightarrow X_1 := X_\Sigma \tag{2.1.7}$$

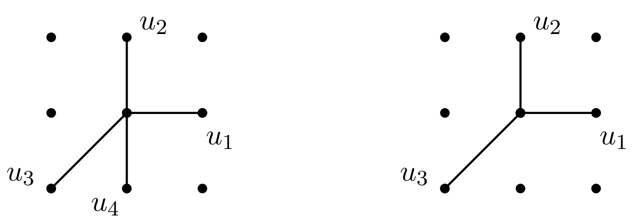
and induces a commutative diagram between the corresponding exact sequences (2.1.3) for the varieties:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Sigma^*_{\sigma,x}(1)} & \xrightarrow{\deg_{X_0}} & \mathrm{Pic}(X_0) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Sigma(1)} & \xrightarrow{\deg_{X_1}} & \mathrm{Pic}(X_1) & \longrightarrow & 0 \end{array} \tag{2.1.8}$$

where β projects away from the coordinate corresponding to the exceptional divisor and γ is such that $\gamma \circ \deg_{X_0} = \deg_{X_1} \circ \beta$.

Example 2.8. The Hirzebruch surface $\mathscr{H}_1$ is the blowup of $\mathbb{P}^2$ at a torus-invariant point. The corresponding fans are given in Figure 2.3, where the cone with rays $\{u_1, u_3\}$ is star-subdivided and the exceptional divisor corresponds to ray u_4. In this example, the commutative diagram (2.1.8) is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}} & \mathbb{Z}^4 & \xrightarrow{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} & \mathbb{Z}^2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 \end{bmatrix}} & & \\ 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow[\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}]{} & \mathbb{Z}^3 & \xrightarrow[\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}]{} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$



(a) The fan for $\mathscr{H}_1$

(b) The fan for $\mathbb{P}^2$

Figure 2.3: *The fans in the blowup* $\mathscr{H}_1 \to \mathbb{P}^2$

2.2 Smooth Toric Fano Varieties

From (2.1.3), a line bundle L on a smooth toric variety X is determined by some torus-invariant divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$.

Definition 2.9. A divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ is *effective* if $a_\rho \geq 0$, for all $\rho \in \Sigma(1)$.

Given an effective divisor D, we can consider whether $L = \mathcal{O}_X(D)$ can be used to embed X into a projective space. The space of global sections $W := \Gamma(X, L)$ is *basepoint free* if for every point $p \in X$, there is a section $s \in W$ such that $s(p) \neq 0$. The dual space $W^\vee$ determines the projective space $\mathbb{P}(W^\vee)$. Given a fixed $p \in X$ and a nonzero element v_p in the fibre of p, there exists $\lambda_s \in \mathbb{C}$ for each section $s \in W$ such that $s(p) = \lambda_s v_p$. By defining the map $l_p \in W^\vee$ by $l_p(s) := \lambda_s$, we obtain the morphism

$$\begin{aligned} \phi_L \colon X &\to \mathbb{P}(W^\vee) \\ p &\mapsto l_p \end{aligned}$$

Definition 2.10. The divisor D and line bundle L are *very ample* if D is basepoint free and $\phi_L \colon X \to \mathbb{P}(W^\vee)$ is a closed embedding. If kD is very ample for some integer $k > 0$, then D and L are *ample*.

Lemma 2.11. *[CLS11, Theorem 6.1.15] On a smooth complete toric variety X, a line bundle is ample if and only if it is very ample.*

For an irreducible complete curve C on X and normalisation $\phi \colon \overline{C} \to C$, the intersection product $D \cdot C$ of D and C is defined as the degree of the line bundle $D \cdot C := \deg(\phi^* \mathcal{O}_X(D))$.

Definition 2.12. The divisor D is *nef* if $D \cdot C \geq 0$ for every irreducible complete curve $C \subseteq X$.

Every basepoint free divisor is nef, and when the fan of X has convex support of full dimension, then a divisor is basepoint free if and only if it is nef [CLS11, Theorem 6.3.12]. The classes of nef divisors generate a cone $\mathrm{Nef}(X)$ in $\mathrm{Pic}(X)_\mathbb{R}$.

Lemma 2.13. *[CLS11, Theorem 6.3.22] Let X be a projective toric variety. Then the divisor D is ample if and only if its class in* $\mathrm{Pic}(X)_\mathbb{R}$ *is in the interior of Nef(X).*

On a variety X, the *canonical bundle* ω_X is the line bundle that is the top exterior power of the cotangent bundle. It is determined by the *canonical divisor* K_X; on a toric variety, $K_X = -\sum_{\rho\in\Sigma(1)} D_\rho$ [CLS11, Theorem 8.2.3]. The dual to ω_X is the *anticanonical bundle* ω_X^{-1} and hence the anticanonical divisor is $-K_X = \sum_{\rho\in\Sigma(1)} D_\rho$.

Definition 2.14. A smooth toric variety X whose anticanonical divisor $-K_X$ is ample is called a *smooth toric Fano variety.*

A lattice polytope P in $M_\mathbb{R}$ is *reflexive* if its facet presentation is

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_F \rangle \geq -1 \text{ for all facets } F \text{ in } P\}.$$

If P is reflexive then the origin of $M_\mathbb{R}$ is the only interior lattice point of P and its dual polytope is also a reflexive polytope. A polytope is *smooth* if its dual polytope determines a smooth fan, and two reflexive polytopes $P_1, P_2 \subset M_\mathbb{R}$ are *lattice equivalent* if P_1 is the image of P_2 under an invertible linear map of $M_\mathbb{R}$ induced by an isomorphism of M. Batyrev [Bat99] uses smooth reflexive polytopes to classify smooth toric Fano varieties:

Theorem 2.15. *[Bat99, Theorem 2.2.4] If P is an n-dimensional smooth reflexive polytope then X_P is an n-dimensional smooth toric Fano variety. Conversely, If X is an n-dimensional smooth toric Fano variety then there exists an n-dimensional smooth reflexive polytope P such that $X_P \cong X$. Moreover, if P_1 and P_2 are two smooth reflexive polytopes then $X_{P_1} \cong X_{P_2}$ if and only if P_1 and P_2 are lattice equivalent.*

Example 2.16. The lattice polygon in Example 2.2 is smooth and reflexive, hence $\mathscr{H}_1$ is a smooth toric Fano surface. Choosing the basis $\{[D_1],[D_4]\}$ for $\mathrm{Pic}(\mathscr{H}_1)$, the effective cone $\mathrm{Eff}(\mathscr{H}_1)$ is generated by $\{[D_1],[D_4]\}$ whilst the nef cone is generated by $\{[D_1],[D_1+D_4]\}$. The class of the anticanonical bundle $\mathcal{O}(D_1+D_2+D_3+D_4) \cong \mathcal{O}(3D_1+2D_4)$ is contained in the interior of $\mathrm{Nef}(\mathscr{H}_1)$, so it is ample. The cones in $\mathrm{Pic}(\mathscr{H}_1)$ are shown in Figure 2.4.

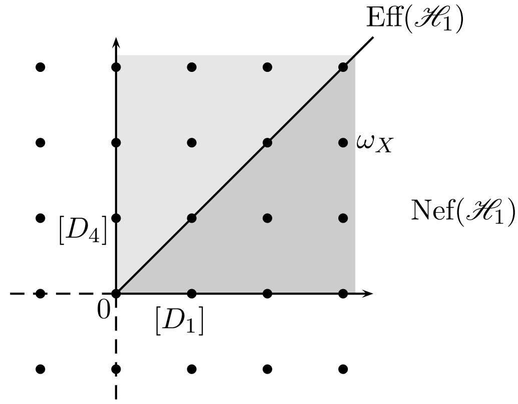


Figure 2.4: *The effective cone and nef cone in* $\mathrm{Pic}(\mathscr{H}_1)$

We therefore refer to smooth reflexive lattice polytopes as *Fano polytopes* and as Batyrev observed, there are finitely many Fano polytopes up to lattice equivalence in each dimension [Bat82a]. There are five corresponding smooth toric Fano varieties in dimension 2 that were known classically, whilst Watanabe-Watanabe [WW82] and Batyrev [Bat82b] classified the 18 smooth toric Fano varieties in dimension 3. In dimension 4, Batyrev [Bat99] used primitive collections and relations to classify the Fano polytopes and Sato [Sat00] completed the classification using toric blowups, bringing the total number of 4-dimensional smooth toric Fano varieties to 124. Kreuzer and Nill [KN09] calculated that there are 866 5-dimensional Fano polytopes up to lattice equivalence, while Øbro [Øb07] presented an algorithm that has classified Fano polytopes in dimensions up to 9.

Sato [Sat00] records the birational geometry between the smooth toric Fano fourfolds by computing toric divisorial contractions in terms of the primitive relations for each variety. Figure B.1 in Appendix B is a diagram of the divisorial contractions between the smooth toric Fano fourfolds. There are 29 maximal toric Fano fourfolds with regard to these divisorial contractions, and we call these varieties *birationally maximal.* A diagram showing the divisorial contractions between the smooth toric Fano threefolds can be found in [Oda88, page 92] [WW82].

Remark 2.17. In [Sat00, Table 1], Sato states that there is a contraction from variety K_2 to variety H_{10}. This contraction should be from variety K_3 to H_{10}.

2.3 Full Strong Exceptional Collections and Tilting Objects

For a set of objects $\mathcal{S} = \{\mathcal{S}_i\}$ in a triangulated category $\mathcal{D}$, define $\langle \mathcal{S} \rangle$ to be the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{S}$, closed under isomorphisms, taking cones of morphisms and direct summands, and $\langle \mathcal{S} \rangle^{\perp}$ to be the full triangulated subcategory of $\mathcal{D}$ containing objects $\mathcal{F}$ such that $\mathrm{Hom}(S, \mathcal{F}) = 0$ for all $S \in \mathcal{S}$.

Definition 2.18. For a set of objects $\mathcal{S} = \{\mathcal{S}_i\}$ in $\mathcal{D}$,

(i) $\mathcal{S}$ *classically generates* $\mathcal{D}$ if $\langle \mathcal{S} \rangle = \mathcal{D}$,

(ii) $\mathcal{S}$ *generates* $\mathcal{D}$ if $\langle \mathcal{S} \rangle^{\perp} = 0$.

Let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on a variety X. If X is projective, Van den Bergh provides us with a set of objects that generate $\mathcal{D}^b(X)$:

Lemma 2.19. *[VdB04, Lemma 3.2.2] Let X be a projective variety of dimension n, L an ample line bundle on X generated by global sections and $a \in \mathbb{Z}$. If M in $\mathcal{D}^b(X)$ is such that* $\mathrm{Hom}^i(L^{a+j}, M) = 0$ *for all i and for $0 \leq j \leq n$, then $M = 0$.*

Definition 2.20. Let $(\mathcal{E}_0, \dots, \mathcal{E}_r)$ be an ordered set of objects in $\mathcal{D}^b(X)$.

(i) The set $(\mathcal{E}_0, \dots, \mathcal{E}_r)$ is a *strong exceptional collection* if $\mathrm{Hom}(\mathcal{E}_k, \mathcal{E}_k) = \mathbb{C}$ for all $k \in \{0, \dots, r\}$ and

$$\mathrm{Hom}^i(\mathcal{E}_k, \mathcal{E}_j) = 0 \text{ when } \begin{cases} k > j, & i = 0, \\ \forall\, k, j, & i \neq 0. \end{cases}$$

(ii) A strong exceptional collection $(\mathcal{E}_0, \dots, \mathcal{E}_r)$ in $\mathcal{D}^b(X)$ is *full* if $\langle \mathcal{E}_0, \dots, \mathcal{E}_r \rangle = \mathcal{D}^b(X)$.

Remark 2.21. The distinction between classical generation and generation becomes irrelevant when using strong exceptional collections. To show that a strong exceptional collection $(\mathcal{E}_0, \dots, \mathcal{E}_r)$ is full, it is enough to show that $\langle \mathcal{E}_0, \dots, \mathcal{E}_r \rangle^{\perp} = 0$ as observed by Bridgeland–Stern [BS10, Lemma C.1].

Definition 2.22. An object $\mathcal{T}$ in $\mathcal{D}^b(X)$ is a *tilting object* if $\mathrm{Hom}^i(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$ and $\langle \mathcal{T} \rangle = \mathcal{D}^b(X)$. If additionally $\mathcal{T}$ is a sheaf or vector bundle, then it is called a *tilting sheaf* or *tilting bundle* respectively.

Given a full strong exceptional collection $(\mathcal{E}_0, \dots, \mathcal{E}_r)$ of non-isomorphic objects in $\mathcal{D}^b(X)$, its sum $\bigoplus_{i=0}^{r} \mathcal{E}_i$ is a tilting object.

For a tilting object $\mathcal{T}$, let $A = \mathrm{End}(\mathcal{T})$ and $\mathcal{D}^b(A)$ be the bounded derived category of finitely generated right A-modules. It was shown by Baer [Bae88] and Bondal [Bon90] that in the case when X is a smooth projective variety, if the tilting object $\mathcal{T}$ exists then we obtain an equivalence of categories

$$\mathbf{R}\,\mathrm{Hom}_X(\mathcal{T}, -) \colon \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(A). \tag{2.3.1}$$

Note that when $\mathcal{T} = \bigoplus_{i=0}^{r} \mathcal{E}_i$ is the direct sum of a full strong exceptional collection, the Grothendieck group $K_0(X)$ of X is isomorphic to a rank $r+1$ lattice; the equivalence of derived categories above induces an isomorphism $K_0(X) \cong K_0(A)$, and the classes of indecomposable projective A-modules corresponding to $[\mathcal{E}_i]$ for $0 \leq i \leq r$ freely generate $K_0(A)$.

Example 2.23. King [Kin97] showed that the set of line bundles $\{\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_4), \mathcal{O}(2D_1 + D_4)\}$ is a full strong exceptional collection on $\mathscr{H}_1$. In Example 3.3 we show that the collection is strong exceptional, whilst Examples 4.6 and 6.10 give two different methods to prove that the collection is full.

For two smooth projective varieties Y and Z, let $\mathcal{E} \in \mathcal{D}^b(Y)$ and $\mathcal{F} \in \mathcal{D}^b(Z)$. Define

$$\mathcal{E} \overset{\mathbf{L}}{\boxtimes} \mathcal{F} := \mathbf{L}p_1^*(\mathcal{E}) \overset{\mathbf{L}}{\otimes} \mathbf{L}p_2^*(\mathcal{F}) \in \mathcal{D}^b(Y \times Z)$$

where p_1 and p_2 are the natural projections of $Y \times Z$ onto its components. If we have tilting objects on two varieties, then we immediately obtain a tilting object on the product of the varieties:

Lemma 2.24. *[Ueh14, Lemma 5.2] Let Y and Z be as above. If $\mathcal{E} \in \mathcal{D}^b(Y)$ and $\mathcal{F} \in \mathcal{D}^b(Z)$ are tilting objects, then $\mathcal{E} \overset{\mathbf{L}}{\boxtimes} \mathcal{F}$ is a tilting object for $\mathcal{D}^b(Y \times Z)$.*

2.4 Helices and Calabi-Yau Algebras

An algebra B is *homologically smooth* if, viewed as a bimodule over itself, it has a bounded resolution by finitely generated projective B, B-bimodules. We have the contravariant functor on the derived category of B, B-bimodules that maps objects:

$$M \mapsto M^! := \mathbf{R}\operatorname{Hom}_{B,B\text{-mod}}(M, B \otimes B).$$

Using the outer bimodule structure on $B \otimes B$ when taking $\mathbf{R}\operatorname{Hom}$ results in $M^!$ being a B, B-bimodule using the inner structure. Any morphism $f \colon M \to N$ in the derived category then induces a morphism $f^! \colon N^! \to M^!$. The following definition is due to Ginzburg [Gin06]:

Definition 2.25. A homologically smooth algebra B is a *Calabi-Yau algebra* (CYd) of dimension d if there exists a B, B-bimodule quasi-isomorphism

$$f \colon B \overset{\cong}{\to} B^![d] \text{ such that } f = f^![d],$$

where $[d]$ is the shift by d functor on the derived category of B, B-bimodules.

Proposition 2.26. *[Gin06, Proposition 3.3.1] Let X be a smooth connected variety which is projective over an affine variety and let $\mathcal{E} \in \mathcal{D}^b(X)$ be a tilting object. Then the endomorphism algebra* $\operatorname{End}(\mathcal{E})$ *is a CYd algebra if and only if X is a Calabi-Yau manifold of dimension d.*

Following Bridgeland and Stern [BS10], we recall the definition of a geometric helix, which can be used to give examples of CYd algebras.

Definition 2.27. A sequence of coherent sheaves $\mathbb{H} = (E_i)_{i\in\mathbb{Z}}$ on a variety X is a *helix* if

- for each $i \in \mathbb{Z}$ the *thread* $(E_{i+1}, \ldots, E_{i+k})$ is a full exceptional collection,
- for each $i \in \mathbb{Z}$, we have $E_{i-k} = E_i \otimes \omega_X$.

If $\mathbb{H}$ satisfies the additional condition that for all $s < t$,

$$\mathrm{Hom}^j(E_s, E_t) = 0 \text{ unless } j = 0 \tag{2.4.1}$$

then $\mathbb{H}$ is *geometric*. If $\mathbb{H}$ satisfies the weaker condition that each thread is a strong exceptional collection, then $\mathbb{H}$ is said to be *strong*.

Remark 2.28. [BS10, Remark 3.2] If $\{E_0, \ldots, E_{k-1}\}$ in a helix $\mathbb{H}$ is a full exceptional collection, then each thread of $\mathbb{H}$ is a full exceptional collection.

The *helix algebra* $A(\mathbb{H})$ associated to $\mathbb{H}$ is the graded algebra

$$A(\mathbb{H}) := \bigoplus_{t\geq 0} \prod_{j-i=t} \mathrm{Hom}(E_i, E_j).$$

Twisting by ω_X induces a $\mathbb{Z}$-action

$$\mathrm{Hom}(E_i, E_j) \to \mathrm{Hom}(E_{i-k}, E_{j-k})$$

and the subalgebra of invariant elements is known as the *rolled-up helix algebra* $B(\mathbb{H})$.

Lemma 2.29. *[BS10, Theorem 3.6] Let $B = B(\mathbb{H})$ be the rolled-up helix algebra of a geometric helix on an n-dimensional variety X and $Y := \mathrm{tot}(\omega_X)$ be the total space of the canonical bundle. Then B is a graded CY$(n+1)$ algebra. Given a thread $\mathbb{E} \subset \mathbb{H}$ and the bundle map $\pi\colon Y \to X$, there is an equivalence*

$$\Phi_{\mathbb{E}} \colon \mathcal{D}^b(B) \to \mathcal{D}^b(Y)$$

sending B to the object $\pi^*(E)$, where $E = \bigoplus_{E_j\in\mathbb{E}} E_j$. In particular, $\pi^*(E)$ is a tilting object for Y.

If X is a smooth toric Fano variety then the total space of its canonical bundle $Y = \mathrm{tot}(\omega_X)$ is a smooth toric Calabi-Yau variety. The fan Σ_Y for Y can be constructed from the fan Σ_X for X; given a cone $\sigma \in \Sigma_X$, define

$$\tilde{\sigma} := \mathrm{Cone}((0,1), (u_\rho, 1) \mid \rho \in \sigma(1)) \subset N_{\mathbb{R}} \times \mathbb{R}$$

Then the set of cones $\tilde{\sigma}$ for $\sigma \in \Sigma_X$ and their faces form Σ_Y.

Example 2.30. The full strong exceptional collection on $\mathscr{H}_1$ given in Example 2.23 forms the geometric helix

$$\mathbb{H} = \{\ldots, \mathcal{O}(-D_1 - D_4), \mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_4), \mathcal{O}(2D_1 + D_4), \mathcal{O}(3D_1 + 2D_4), \ldots\}$$

(see Example 7.8). By Lemma 2.29, the decomposable vector bundle $\pi^*(\mathcal{O} \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_1 + D_4) \oplus \mathcal{O}(2D_1 + D_4))$ is therefore a tilting bundle on $\mathrm{tot}(\omega_{\mathscr{H}_1})$. The slice at height 1 of the fan for $\mathrm{tot}(\omega_{\mathscr{H}_1})$ is given in Figure 2.5.

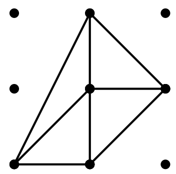


Figure 2.5: *The slice at height* 1 *of the fan for* $\mathrm{tot}(\omega_{\mathscr{H}_1})$

CHAPTER

THREE

STRONG EXCEPTIONAL COLLECTIONS ON SMOOTH TORIC VARIETIES

The combinatorics of the fan Σ for a toric variety X_Σ allow us to computationally determine whether a collection of line bundles on X_Σ is strong exceptional. The first section in this chapter explains the construction of the *not-necessarily non-vanishing cohomology cones* (*nnnvc*-cones) in the Picard lattice for X_Σ, described by Eisenbud, Mustaţă and Stillman [EMS00], which we utilise to achieve this goal.

The second section considers how the *nnnvc*-cones behave under torus-invariant divisorial contractions and the effect these contractions have on strong exceptional collections. We find that for a chain of contractions $X_0 \to X_1 \to \cdots \to X_t$, the preimage in $\mathrm{Pic}(X_0)$ of the *nnnvc*-cones for $\{X_1, \ldots, X_t\}$ under the induced maps between the Picard lattices have a simple description in terms of the *nnnvc*-cones for X_0 – see Proposition 3.13. Proposition 3.14 then shows that this simplicity has a consequence for the image of strong exceptional collections on X_0 under the Picard lattice maps when the varieties considered are smooth toric Fano fourfolds.

3.1 *nnnvc*-Cones

To check that a collection of effective line bundles $\{L_0 := \mathcal{O}_X, L_1, \ldots, L_r\}$ on a smooth toric variety X is strong exceptional, one needs to check that $H^i(X, L_s^{-1} \otimes L_t) \cong \mathrm{Hom}^i(L_s, L_t) = 0$ for $i > 0$ and $0 \le s, t \le r$. Eisenbud, Mustaţă and Stillman [EMS00] introduced a method to determine when the cohomology of a line bundle on X vanishes by considering whether the line bundle avoids certain affine cones constructed in $\mathrm{Pic}(X)_\mathbb{R}$. We recall the construction of these cones below.

Let X be an n-dimensional toric variety with fan Σ, $|\Sigma|$ be the support of the fan in $N_\mathbb{R}$ and recall that $\Sigma(1)$ denotes the set of rays in Σ. For $I \subseteq \Sigma(1)$, let Y_I be the union of the cones in Σ having all edges in the complement of I. Using reduced cohomology with coefficients in $\mathbb{C}$ we have

$$H^i_{Y_I}(|\Sigma|) := H^i(|\Sigma|, |\Sigma| \backslash Y_I) = H^{i-1}(|\Sigma| \backslash Y_I), \tag{3.1.1}$$

where the last equality holds for $i > 0$ as $|\Sigma|$ is contractible.

An element of

$$H_\Sigma := \{I \subseteq \Sigma(1) \mid H^i_{Y_I}(|\Sigma|) \neq 0 \text{ for some } i > 0\} \tag{3.1.2}$$

is called a *forbidden set.* Define

$$\mathbf{p}_I \in \mathbb{Z}^{\Sigma(1)}, \text{ where } (\mathbf{p}_I)_\rho = \begin{cases} -1 \text{ if } \rho \in I \\ \;\;0 \text{ if } \rho \notin I \end{cases} \tag{3.1.3}$$

and

$$C_I = \left\{\mathbf{x} = (x_\rho) \in \mathbb{Z}^{\Sigma(1)} \mid x_\rho \leq 0 \text{ if } \rho \in I, x_\rho \geq 0 \text{ if } \rho \notin I\right\}. \tag{3.1.4}$$

Setting $L_I := C_I + \mathbf{p}_I \subseteq \mathbb{Z}^{\Sigma(1)}$ we see that $L_I \subset C_I$ and $L_I = \{\mathbf{x} \in \mathbb{Z}^{\Sigma(1)} \mid \text{neg}(\mathbf{x}) = I\}$, where $\text{neg}(\mathbf{x}) = \{\rho \in \Sigma(1) \mid x_\rho < 0\} \subseteq \Sigma(1)$.

Eisenbud, Mustaţă and Stillman show that for $i \geq 1$, the cohomology of all twists of the structure sheaf

$$H^i_*(\mathcal{O}_X) \;:=\; \bigoplus_{\alpha \in \text{Pic}(X)} H^i(X, \mathcal{O}_X(\alpha))$$

is isomorphic as a graded S_X-module to the local cohomology $H^i_{B_X}(S_X)$ of the Cox ring [EMS00, Proposition 2.3(a)]. The ring S_X has a finer grading by $\mathbb{Z}^{\Sigma(1)}$ that is compatible with the $\text{Pic}(X)$-grading, and this descends to give a grading on $H^i_{B_X}(S_X)$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{\Sigma(1)}$ such that $\text{neg}(\mathbf{x}) = \text{neg}(\mathbf{y})$, we have $H^i_*(\mathcal{O}_X)_\mathbf{x} \cong H^i_*(\mathcal{O}_X)_\mathbf{y}$ [EMS00, Theorem 2.4].

Lemma 3.1. *[EMS00, Theorem 2.7] Let* $\mathbf{x} \in \mathbb{Z}^{\Sigma(1)}$ *and* $I = \text{neg}(\mathbf{x})$. *Then*

$$H^i_*(\mathcal{O}_X)_\mathbf{x} \cong H^i_{Y_I}(|\Sigma|).$$

Now, if $D = \sum_{\rho \in \Sigma(1)} x_\rho D_\rho$ is the toric divisor that corresponds to $\mathbf{x} \in \mathbb{Z}^{\Sigma(1)}$, then $H^i_*(\mathcal{O}_X)_\mathbf{x} \cong H^i(X, \mathcal{O}_X(D))$. It therefore follows that $\mathbf{x}$ lies in L_I for some $I \in H_\Sigma$ if and only if

$$H^i(X, \mathcal{O}_X(D)) \neq 0, \text{ for some } i > 0. \tag{3.1.5}$$

The convex hull of the set of lattice points L_I forms an affine cone in $\mathbb{R}^{\Sigma(1)}$.

Definition 3.2. Let $I \in H_\Sigma$ and consider the cone in $\mathbb{R}^{\Sigma(1)}$ determined by the convex hull of L_I. The image in $\text{Pic}(X)_\mathbb{R}$ of this cone under the map deg is a *not-necessarily non-vanishing cohomology cone* (*nnnvc*-cone) and is denoted by Λ_I.

We say that Λ_I is a not-necessarily non-vanishing cohomology cone as the semigroup corresponding to the image of L_I under the map deg may not be saturated. In particular, if $\alpha \in \Lambda_I$ then it is not necessarily the case that $H^i(X, \mathcal{O}_X(\alpha)) \neq 0$ for some $i > 0$, but if α is not in Λ_I for any $I \in H_\Sigma$ then $H^i(X, \mathcal{O}_X(\alpha)) = 0$ for all $i > 0$. Given a collection of line bundles $\{L_0, L_1, \ldots, L_r\}$ on X, it follows that if $L_s^{-1} \otimes L_t$ avoids all of the *nnnvc*-cones for all $0 \leq s, t \leq r$, then the collection is strong exceptional.

Example 3.3. Using the fan for $\mathscr{H}_1$ in Example 2.5, we see that the forbidden sets are $\{\{1,3\},\{2,4\},\{1,2,3,4\}\}$. The corresponding sets of lattice points in $\mathbb{Z}^{\Sigma(1)}$ are

$$\begin{aligned} L_{\{1,3\}} &= \{\mathbf{x} \in \mathbb{Z}^{\Sigma(1)} \mid x_1, x_3 \leq -1,\ x_2, x_4 \geq 0\} \\ L_{\{2,4\}} &= \{\mathbf{x} \in \mathbb{Z}^{\Sigma(1)} \mid x_2, x_4 \leq -1,\ x_1, x_3 \geq 0\} \\ L_{\{1,2,3,4\}} &= \{\mathbf{x} \in \mathbb{Z}^{\Sigma(1)} \mid x_1, x_2, x_3, x_4 \leq -1\}. \end{aligned}$$

The *nnnvc*-cones are given by the equations

$$\begin{aligned} \Lambda_{\{1,3\}} &= \{\mathbf{a} \in \mathrm{Pic}(\mathscr{H}_1)_{\mathbb{R}} \mid a_1 - a_2 \leq -2,\ -a_2 \leq 0\} \\ \Lambda_{\{2,4\}} &= \{\mathbf{a} \in \mathrm{Pic}(\mathscr{H}_1)_{\mathbb{R}} \mid -a_1 + a_2 \leq -1,\ a_2 \leq -2\} \\ \Lambda_{\{1,2,3,4\}} &= \{\mathbf{a} \in \mathrm{Pic}(\mathscr{H}_1)_{\mathbb{R}} \mid a_1 \leq -3,\ a_2 \leq -2\}. \end{aligned}$$

Let $L_0 = \mathcal{O}$, $L_1 = \mathcal{O}(D_1)$, $L_2 = \mathcal{O}(D_1 + D_4)$ and $L_3 = \mathcal{O}(2D_1 + D_4)$. Then each line bundle $L_i \otimes L_j^{-1}$, $0 \leq i, j \leq 3$ is denoted by a '$\times$' in Figure 3.1, which also displays the *nnnvc*-cones. As the line bundles avoid the *nnnvc*-cones, the collection $\mathcal{L} = \{L_0, \ldots, L_3\}$ is strong exceptional.

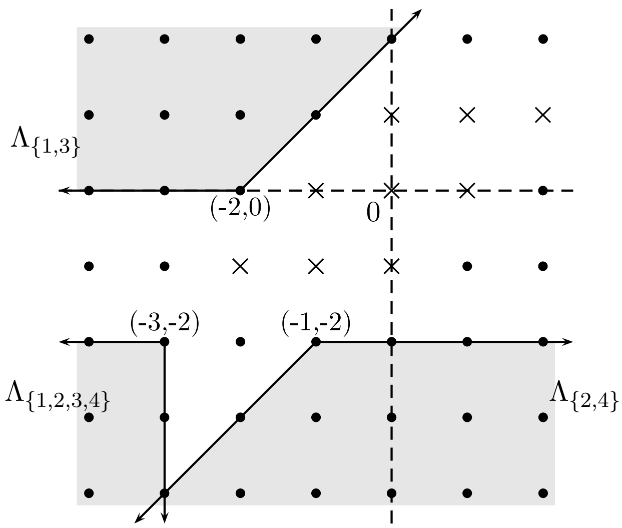


Figure 3.1: *The* nnnvc-*cones in* $\mathrm{Pic}(\mathscr{H}_1)$

3.2 Cones Affected by Blow Ups

Assume that we have a chain of torus-invariant divisorial contractions $X := X_0 \to X_1 \to \cdots \to X_t$ between smooth toric varieties and let $\mathcal{L} = \{L_0, L_1, \ldots, L_r\}$ be a collection of non-isomorphic line bundles on X with corresponding vectors $\{v_0, \ldots, v_r\}$ in $\mathrm{Pic}(X)_{\mathbb{R}}$. By (2.1.8) we have maps between the Picard lattices

$$\mathrm{Pic}(X_0) \xrightarrow{\gamma_1} \mathrm{Pic}(X_1) \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_t} \mathrm{Pic}(X_t). \tag{3.2.1}$$

For ease of notation we set:

- $\gamma_{(i\to j)}$ to be the composition of maps $\gamma_j \circ \gamma_{j-1} \circ \cdots \circ \gamma_{i+1}$ for $0 \leq i < j \leq t$;
- $\mathcal{L}_{X_k}$ to be the set of non-isomorphic line bundles on X_k in the image of $\gamma_{(0\to k)}(\mathcal{L})$, for $1 \leq k \leq t$;
- $\tilde{\Lambda}_{I,X_k}$ to be the preimage in $\mathrm{Pic}(X_0)_{\mathbb{R}}$ of the *nnnvc*-cone Λ_I for X_k under the map $\gamma_{(0\to k)}$, for $1 \leq k \leq t$;
- $\mathfrak{C}_k \subset \mathrm{Pic}(X_0)_{\mathbb{R}}$ to be the preimage of all *nnnvc*-cones for X_k under the map $\gamma_{(0\to k)}$ for $1 \leq k \leq t$, and $\mathfrak{C}_0 \subset \mathrm{Pic}(X_0)_{\mathbb{R}}$ to be the *nnnvc*-cones for X_0.

By the construction of the sets $\mathfrak{C}_k$, we have the following result:

Lemma 3.4. *If*

$$v_i - v_j \notin \bigcup_{k=0}^{t} \mathfrak{C}_k$$

for all $0 \leq i, j \leq r$ *then* $\mathcal{L}$ *is strong exceptional on* X *and* $\mathcal{L}_{X_k}$ *is strong exceptional on* X_k*, for* $1 \leq k \leq t$*.*

It will be shown in this section that the preimage of the *nnnvc*-cones for X_k under $\gamma_{(0\to k)}$ is closely related to the *nnnvc*-cones for X_0.

Lemma 3.5 (Forbidden sets duality)**.** *Let* $I \subsetneq \Sigma(1)$ *and set* $I^\vee = \Sigma(1)\backslash I$*. If* $I \in H_\Sigma$*, then* $I^\vee \in H_\Sigma$*.*

Proof. It is enough to show that the line bundle $\mathcal{O}(-\sum_{\rho\in I^\vee} D_\rho)$ corresponding to $\mathbf{p}_{I^\vee}$ has non-vanishing higher cohomology. Let $D := -\sum_{\rho\in I} D_\rho$ be the torus-invariant Weil divisor corresponding to $\mathbf{p}_I$. By assumption, $H^i(X, \mathcal{O}(D)) \neq 0$ for some $0 < i < n$. By Serre duality and the fact that the canonical divisor is $K_X = -\sum_{\rho\in\Sigma(1)} D_\rho$,

$$0 \neq H^i(X, \mathcal{O}(D))^\vee \cong H^{n-i}\left(X, \mathcal{O}(K_X - D)\right) = H^j(X, \mathcal{O}(\sum_{\rho\in\Sigma(1)} b_\rho D_\rho)) \tag{3.2.2}$$

where $b_\rho = -1 - a_\rho$. But

$$a_\rho = \begin{cases} -1 \text{ if } \rho \in I \\ 0 \text{ if } \rho \notin I \end{cases} \Rightarrow\ b_\rho = \begin{cases} 0 \text{ if } \rho \in I \\ -1 \text{ if } \rho \notin I. \end{cases} \tag{3.2.3}$$

Therefore, $(b_\rho) = \mathbf{p}_{I^\vee}$ and as $H^j(X, \mathcal{O}(\sum_{\rho\in\Sigma(1)} b_\rho D_\rho)) \neq 0$ for some $0 < j < n$, we have $I^\vee \in H_\Sigma$. ☐

Remark 3.6. In the following lemmas, we change convention by setting Y_I to be the union of the cones in Σ having all edges in $I \subsetneq \Sigma(1)$. Due to the duality statement of Lemma 3.5, this does not affect the outcome of Proposition 3.13.

Continuing with the notation in (2.1.7), it is clear that for

$$C_\sigma := \bigcup_{\sigma\preceq\tau\in\Sigma} \tau \tag{3.2.4}$$

we have

$$\Sigma\backslash C_\sigma = \Sigma^*\backslash \bigcup_{\sigma\preceq\tau} \Sigma^*_\tau(\sigma) \tag{3.2.5}$$

and so we only need to consider C_σ when determining how the cones of Σ change after the blow up of $\sigma \in \Sigma$.

Lemma 3.7. *Let $\emptyset \neq I \subseteq \Sigma(1)\backslash C_\sigma(1)$. Then $I \cup \{x\} \in H_{\Sigma^*}$.*

Proof. Firstly, assume for some $I \subset \Sigma(1)$ that there exists a ray $\tau \subsetneq Y_I$ such that $\tau \cap \sigma = \{0\}$ for all cones $\tau \neq \sigma \subset Y_I$. By considering $Y_I \subset |\Sigma| \cong \mathbb{R}^n$ for $n > 2$, we can construct a loop around τ that is not contractible in $|\Sigma|\backslash Y_I$; if $n = 2$, then $|\Sigma|\backslash Y_I$ is a disconnected space. Thus $I \in H_\Sigma$.

Now assume $\emptyset \neq I \subseteq \Sigma(1)\backslash C_\sigma(1)$. By the construction of Σ^* we have $x \cap \sigma = \{0\}$ for any cone $x \neq \sigma \subset Y^{\Sigma^*}_{I\cup\{x\}}$ and $x \neq Y^{\Sigma^*}_{I\cup\{x\}}$, so $I \cup \{x\} \in H_{\Sigma^*}$ by the observation above. □

By Lemma 3.5, $(I \cup \{x\})^\vee \in H_{\Sigma^*}$ for $\emptyset \neq I \subseteq \Sigma(1)\backslash C_\sigma(1)$. But $(I \cup \{x\})^\vee = J \cup C_\sigma(1)$ for some $J \subsetneq \Sigma(1)\backslash C_\sigma(1)$, so we have the corollary:

Corollary 3.8. *If $I \subsetneq \Sigma(1)\backslash C_\sigma(1)$, then $I \cup C_\sigma(1) \in H_{\Sigma^*}$.*

Lemma 3.9. *If $I \in H_\Sigma$ and $I \cap C_\sigma(1) = \emptyset$ then $I, I \cup \{x\} \in H_{\Sigma^*}$.*

Proof. Let $I \in H_\Sigma$ such that $I \cap C_\sigma(1) = \emptyset$. By Lemma 3.7, $I \cup \{x\} \in H_{\Sigma^*}$. As $I \cap C_\sigma(1) = \emptyset$, then $Y^{\Sigma^*}_I = Y^\Sigma_I$ and $|\Sigma^*| = |\Sigma|$, so $H^i_{Y^{\Sigma^*}_I}(|\Sigma^*|) = H^i_{Y^\Sigma_I}(|\Sigma|)$ for all i. Thus $I \in H_{\Sigma^*}$. □

Again by duality, we have the corollary:

Corollary 3.10. *If $I \in H_\Sigma$ is such that $C_\sigma(1) \subseteq I$, then $I, I \cup \{x\} \in H_{\Sigma^*}$.*

Lemma 3.11. *If $I \in H_\Sigma$ then either $I \in H_{\Sigma^*}$ or $I \cup \{x\} \in H_{\Sigma^*}$.*

Proof. We have shown that the statement holds if $I \cap C_\sigma(1) = \emptyset$ and dually if $C_\sigma(1) \subseteq I$. Therefore, assume that $I \cap C_\sigma(1) \neq \emptyset$ and $C_\sigma(1) \nsubseteq I$. There are two cases to consider:

Case 1: ($\sigma(1) \nsubseteq I$). Any subset $S \subseteq \tau(1)$ of any cone $\tau \subset \Sigma$ forms a cone in Σ as Σ is a smooth fan. From this and the fact that $\sigma(1), \{x\} \nsubseteq I$ we see that $Y^\Sigma_I = Y^{\Sigma^*}_I$ by the construction of Σ^*. Therefore $I \in H_\Sigma \Rightarrow I \in H_{\Sigma^*}$.

Case 2: ($\sigma(1) \subseteq I$). By duality $I^\vee \in H_\Sigma$ and $I^\vee \cap \sigma(1) = \emptyset$, so $I^\vee \in H_{\Sigma^*}$ by Case 1. Applying duality again we have $I \cup \{x\} = (I^\vee)^\vee \in H_{\Sigma^*}$.

□

Remark 3.12. It is not always the case that $I \in H_\Sigma \Rightarrow I, I \cup \{x\} \in H_{\Sigma^*}$ (see Example 3.17).

Recalling the chain of linear maps (3.2.1), we have a simple description of the preimage in $\mathrm{Pic}(X)_\mathbb{R}$ of the *nnnvc*-cones for the variety X_t using the *nnnvc*-cones for X. Let $\{E_1, \ldots, E_t\}$ be the exceptional divisors from the blow ups in (3.2.1). The list can be extended to give a basis $\{[E_1], \ldots, [E_t], y_1, \ldots, y_s\}$ of $\mathrm{Pic}(X)_\mathbb{R}$.

Proposition 3.13. *Let $\Lambda_I \subseteq \mathrm{Pic}(X_t)_{\mathbb{R}}$ be a* nnnvc-*cone for X_t in* (3.2.1)*. There exists a* nnnvc-*cone $\Lambda_{I'} \subseteq \mathrm{Pic}(X)_{\mathbb{R}}$ for X with the following property: describe $\Lambda_{I'}$ by the intersection of closed half-spaces in $\mathrm{Pic}(X)_{\mathbb{R}}$ given by equations $a_1^i[E_1] + \ldots + a_t^i[E_t] + a_{t+1}^i y_1 + \ldots + a_{t+s}^i y_s \leq a^i$ where $a_1^i, \ldots, a_{t+s}^i, a^i \in \mathbb{R}$ are fixed and i is in an indexing set S. Then the preimage $\tilde{\Lambda}_{I,X_t}$ is the intersection of the closed half spaces $a_{t+1}^i y_1 + \ldots + a_{t+s}^i y_s \leq a^i$, $i \in S$.*

Proof. We first show the statement for the blowup $\varphi\colon X_{\Sigma^*_{\sigma,x}} \longrightarrow X_\Sigma$ from (2.1.7). Let Λ_I be a *nnnvc*-cone for X_Σ determined by the forbidden set $I \subset \Sigma(1)$. By Lemma 3.11 there exists a *nnnvc*-cone $\Lambda_{I'} \subseteq \mathrm{Pic}(X_{\Sigma^*_{\sigma,x}})_{\mathbb{R}}$ for $X_{\Sigma^*_{\sigma,x}}$ such that its defining forbidden set I' is either $I \cup \{x\} \subset \Sigma^*_{\sigma,x}(1)$ or $I \subset \Sigma^*_{\sigma,x}(1)$. By construction of $L_I \subseteq \mathbb{Z}^{\Sigma(1)}$ and $L_{I'} \subseteq \mathbb{Z}^{\Sigma^*_{\sigma,x}(1)}$, the closed half spaces in $\mathrm{Pic}(X_{\Sigma^*_{\sigma,x}})_{\mathbb{R}}$ describing $\Lambda_{I'}$ are given by equations $a_0^i[E] + a_1^i y_1 + \ldots + a_s^i y_s \leq a^i$ for fixed $a_0^i, \ldots, a_s^i, a^i \in \mathbb{R}$ and $i \in S$, whilst those in $\mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$ describing Λ_I are $a_1^i y_1 + \ldots + a_s^i y_s \leq a^i$. The map β in (2.1.8) is a projection away from the coordinate corresponding to the exceptional divisor E, hence the map γ is a projection away from the exceptional divisor class $[E]$ as (2.1.8) is a commutative digram. Therefore, the preimage $\tilde{\Lambda}_{I,X_\Sigma}$ is given by the the intersection of halfspaces with equations $a_1^i y_1 + \ldots + a_s^i y_s \leq a^i$. By repeated application of Lemma 3.11, we obtain the required result for a chain of blowups (3.2.1). □

The simplicity of the preimage of *nnnvc*-cones under blowups can help explain why the following proposition holds. Recall that a smooth toric Fano variety X is called *birationally maximal* if there does not exist a smooth toric Fano variety X' with blowup $X' \to X$.

Proposition 3.14. *Let X be a birationally maximal smooth toric Fano fourfold and $r + 1 = \mathrm{rank}(K_0(X))$. There exists a strong exceptional collection of line bundles $\mathcal{L} = \{L_0, \ldots, L_r\}$ on X such that for every chain of torus-invariant divisorial contractions $X \to X_1 \to \cdots \to X_t$ from Figure B.1, the set of line bundles $\mathcal{L}_{X_i}$ on X_i is strong exceptional, for $1 \leq i \leq t$. A database of these collections can be found in [PNb].*

Proof. Given a birationally maximal smooth toric Fano fourfold X and a chain of divisorial contractions between $\{X_0 := X, X_1, \ldots, X_t\}$, we construct the preimage $\mathfrak{C}_i$ in $\mathrm{Pic}(X)_{\mathbb{R}}$ of the *nnnvc*-cones for each contraction X_i using the *QuiversToricVarieties* package [PN15a]. A computer search then finds line bundles $\{L_0, L_1, \ldots, L_r\}$ on X with corresponding vectors $\{v_0, \ldots, v_r\}$ in $\mathrm{Pic}(X)_{\mathbb{R}}$ such that $v_j - v_k$ avoids $\mathfrak{C}_i$ for all $0 \leq j, k \leq r$ and $0 \leq i \leq t$. □

Remarks 3.15.

(i) The collections given in Proposition 3.14 are not necessarily the same collections given by Theorem 7.4. In particular, not all of them have been shown to be full.

(ii) If two toric varieties X_1 and X_2 have the same primitive collections, then they have the same forbidden sets up to a suitable ordering of the rays of Σ_{X_1} and Σ_{X_2}. It is therefore often the case that given a suitable basis of $\mathrm{Pic}(X_1)_{\mathbb{R}}$ and $\mathrm{Pic}(X_2)_{\mathbb{R}}$, if the line bundles corresponding to a list of integral points $\{v_j\}_{j\in J} \subset \mathbb{R}^d \cong \mathrm{Pic}(X_1)_{\mathbb{R}}$ is strong exceptional on X_1, then the collection of line bundles corresponding to the same list $\{v_j\}_{j\in J} \subset \mathbb{R}^d \cong \mathrm{Pic}(X_2)_{\mathbb{R}}$ is strong exceptional on X_2.

Example 3.16. Example 2.8 describes the fans for the varieties in the blowup $\mathscr{H}_1 \to \mathbb{P}^2$ and the map γ. The only forbidden set for $\mathbb{P}^2$ is $\{1,2,3\}$, determining the *nnnvc*-cone

$$\Lambda_{\{1,2,3\}} = \{\mathbf{a} \in \mathrm{Pic}(\mathbb{P}^2)_\mathbb{R} \mid a_1 \leq -3\}.$$

The preimage under γ of this cone in $\mathrm{Pic}(\mathscr{H}_1)$ is

$$\tilde{\Lambda}_{\{1,2,3\},\mathbb{P}^2} = \{\mathbf{a} \in \mathrm{Pic}(\mathscr{H}_1)_\mathbb{R} \mid a_1 \leq -3\}$$

and is pictured in Figure 3.2 along with the *nnnvc*-cones for $\mathscr{H}_1$. In particular, $\tilde{\Lambda}_{\{1,2,3\},\mathbb{P}^2}$ is a supporting half-space of $\Lambda_{\{1,2,3,4\}}$. Using the same collection of line bundles $\mathcal{L}$ as in Example 3.3, Figure 3.2 shows each $L_i \otimes L_j^{-1}$, $0 \leq i,j \leq 3$ as a '$\times$'. We see that these line bundles avoid $\tilde{\Lambda}_{\{1,2,3\},\mathbb{P}^2}$ as well as the *nnnvc*-cones for $\mathscr{H}_1$, hence $\mathcal{L}$ is strong exceptional on $\mathscr{H}_1$ and $\mathcal{L}_{\mathbb{P}^2}$ is strong exceptional on $\mathbb{P}^2$.

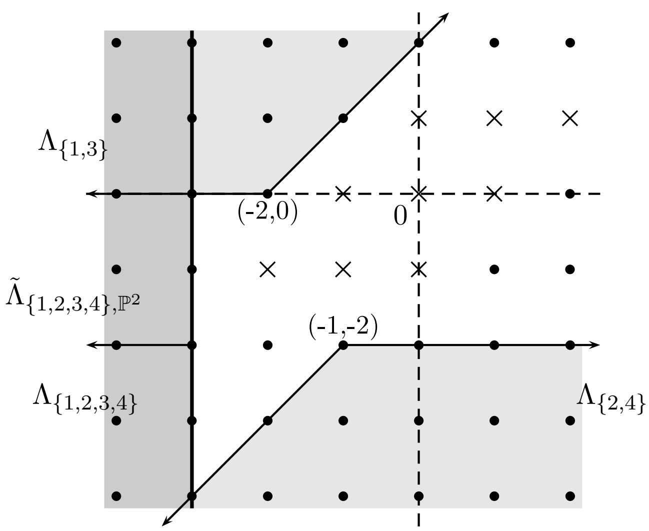


Figure 3.2: *The* nnnvc*-cones for the blowup* $\mathscr{H}_1 \to \mathbb{P}^2$

Example 3.17. The smooth toric Fano fourfold $X_0 := E_1$ has ray generators

$$u_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, u_1 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, u_2 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, u_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, u_4 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, u_5 = \begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix}, u_6 = \begin{bmatrix}2\\-1\\-1\\-1\end{bmatrix}$$

for its fan Σ_{X_0}. The blowup $\phi\colon E_1 \to B_1$ of the smooth toric Fano fourfold B_1 (see Figure B.1) has the exceptional divisor $E = D_6$ labelled by the ray generator u_6. Note that $X_1 := B_1$ has the fan Σ_{X_1} with ray generators $\{u_0,\ldots,u_5\}$. We take the corresponding divisor classes $\{[D_0],[D_1],[E]\}$ to be a basis for $\mathrm{Pic}(X_0)$, and the linear equivalences between the divisors for X_0 are $D_1 \sim D_2 \sim D_3$, $D_4 \sim D_0 + 3D_1 - E$, $D_5 \sim D_1 - E$. The linear equivalences between the divisors for X_1 are $D_1' \sim D_2' \sim D_3' \sim D_5'$, $D_4' \sim D_0' + 3D_1'$. The forbidden sets for X_0 are

nnnv i-th Cohomology Cones	Forbidden Sets
1	$\{0,4\},\{4,5\},\{0,4,5\},$ $\{0,6\},\{0,4,6\}$
2	
3	$\{1,2,3,5\},\{1,2,3,4,5\},$ $\{1,2,3,6\},\{0,1,2,3,6\},$ $\{1,2,3,5,6\}$
4	$\{0,1,2,3,4,5,6\}$

and the forbidden sets for X_1 are

nnnv i-th Cohomology Cones	Forbidden Sets
1	$\{0,4\}$
2	
3	$\{1,2,3,5\}$
4	$\{0,1,2,3,4,5\}$

In this example we see that for the forbidden set $I \in \{\{0,4\},\{1,2,3,5\}\}$ for X_1, both I and $I \cup \{6\}$ are forbidden sets for X_0, whilst for the forbidden set $I = \{0,1,2,3,4,5\}$ for X_1, only $I \cup \{6\}$ is a forbidden set for X_0. Now

$$\tilde{\Lambda}_{\{0,4\},X_1} \cap \operatorname{Pic}(X_0) = (\Lambda_{\{0,4\}} \cup \Lambda_{\{0,4,6\}}) \cap \operatorname{Pic}(X_0)$$

and

$$\tilde{\Lambda}_{\{1,2,3,5\},X_1} \cap \operatorname{Pic}(X_0) = (\Lambda_{\{1,2,3,5\}} \cup \Lambda_{\{1,2,3,5,6\}}) \cap \operatorname{Pic}(X_0).$$

Thus for a strong exceptional collection of line bundles $\mathcal{L}$ on X_0, only

$$\tilde{\Lambda}_{\{0,1,2,3,4,5\},X_1}$$

provides a restriction for the distinct line bundles in the image of $\gamma(\mathcal{L})$ to be strong exceptional on X_1. The cone $\Lambda_{\{0,1,2,3,4,5,6\}}$ is given by the system of equations

$$\begin{cases} a_1 \leq -2 \\ a_2 \leq -7 \\ a_2 + a_3 \leq -6 \end{cases}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \operatorname{Pic}(X_0)_{\mathbb{R}}$$

in $\operatorname{Pic}(X_0)_{\mathbb{R}}$, whilst $\tilde{\Lambda}_{\{0,1,2,3,4,5\},X_1}$ is given by the system of equations

$$\begin{cases} a_1 \leq -2 \\ a_2 \leq -7 \end{cases}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \operatorname{Pic}(X_0)_{\mathbb{R}}$$

as expected by Proposition 3.13.

CHAPTER

FOUR

GENERATION OF $\mathcal{D}^b(X)$: THE FROBENIUS MORPHISM (METHOD 1)

Let X be an n-dimensional smooth toric variety and $\mathcal{L}$ a strong exceptional collection on X. In this thesis we present two different methods to show that $\mathcal{L}$ is full. The first method depends on the Frobenius morphism and follows Uehara's approach [Ueh14] to generation of the derived category by line bundles on the smooth toric Fano threefolds. The first section in this chapter recalls the construction of the Frobenius morphism and how the Frobenius pushforward of a line bundle on a toric variety can be computed. The second section introduces the framework we use to show that a given collection of line bundles is full using the Frobenius morphism; we call this approach ***Method 1***.

4.1 The Frobenius Morphism

Fix a positive integer m and let N' be the lattice $N' := \frac{1}{m}N$ with dual M'. The *Frobenius morphism* is the finite surjective toric morphism $F_m \colon X \to X$ induced from the natural inclusion $f_m \colon N \hookrightarrow N'$ which maps a cone in $N_{\mathbb{R}}$ to one in $N'_{\mathbb{R}}$ with the same support. Thomsen [Tho00] shows that in characteristic $p > 0$, the Frobenius pushforward $(F_m)_*(L)$ of a line bundle L on X splits into a finite direct sum of line bundles. He provides an algorithm to compute these line bundles, which we explain below in the characteristic 0 setting by following [LM11] and [Ueh14].

Let Σ be the fan for X and set $d := |\Sigma(1)|$. From (2.1.3), a vector $\mathbf{w} \in \mathbb{Z}^{\Sigma(1)}$ determines the line bundle $L = \mathcal{O}_X(\sum w_i D_i)$. To compute $(F_m)_*(L)$, fix a maximal cone $\sigma \in \Sigma$ and set

$$P_m^p := \{\mathbf{v} \in \mathbb{Z}^p \mid 0 \le v_i < m\}. \tag{4.1.1}$$

Define $A := (u_\rho)_{\rho \in \Sigma(1)} \in M(d, n)$ to be the matrix whose rows are the ray generators u_ρ in Σ. As σ is maximal and Σ is smooth, the corresponding matrix $A_\sigma := (u_\rho)_{\rho \in \sigma(1)} \in M(n, n)$ is invertible. Define the restriction $\mathbf{w}$ to σ as $\mathbf{w}_\sigma := (w_\rho)_{\rho \in \sigma(1)} \in \mathbb{Z}^n$. For $\mathbf{v} \in P_m^n$, the vectors $\mathbf{q}^m(\mathbf{v}, \mathbf{w}, \sigma) \in \mathbb{Z}^{\Sigma(1)}$ and $\mathbf{r}^m(\mathbf{v}, \mathbf{w}, \sigma) \in P_m^d$ are uniquely determined by the equation

$$AA_\sigma^{-1}(\mathbf{v} - \mathbf{w}_\sigma) + \mathbf{w} = m\mathbf{q}^m(\mathbf{v}, \mathbf{w}, \sigma) + \mathbf{r}^m(\mathbf{v}, \mathbf{w}, \sigma). \tag{4.1.2}$$

Note that if we set

$$\mathbf{x} := \frac{AA_\sigma^{-1}(\mathbf{v} - \mathbf{w}_\sigma) + \mathbf{w}}{m},$$

then the vector $\mathbf{q}^m(\mathbf{v}, \mathbf{w}, \sigma)$ is given by $\lfloor \mathbf{x} \rfloor$; that is, the vector whose entries $\lfloor x_\rho \rfloor \in \mathbb{Z}$ are given by the round-down $x_\rho - 1 < \lfloor x_\rho \rfloor \leq x_\rho$. Finally, define the Weil divisor $D^m_{\mathbf{v},\mathbf{w},\sigma} := \sum_{\rho \in \Sigma(1)} q^m_\rho(\mathbf{v}, \mathbf{w}, \sigma) D_\rho$.

Lemma 4.1. *[LM11, Proposition 3.1] The Frobenius push-forward of $L = \mathcal{O}_X(\sum w_\rho D_\rho)$ is*

$$(F_m)_*(L) = \bigoplus_{\boldsymbol{v} \in P^n_m} \mathcal{O}_X(D^m_{\boldsymbol{v},\boldsymbol{w},\sigma}). \tag{4.1.3}$$

Proof. Fix an n-dimensional cone $\sigma \in \Sigma(n)$. Set $\mathbb{C}[S_\sigma] := \mathbb{C}[x_\rho \mid \rho \in \sigma(1)]$ and consider the torus-invariant open affine subvariety $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$. The map F_m induces a map of $\mathbb{C}$-algebras

$$F_m^\#\colon \mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma], \quad \mathbf{x}^{\mathbf{u}} \mapsto \mathbf{x}^{m\mathbf{u}}.$$

We have the M-graded decomposition

$$\mathbb{C}[S_\sigma] \cong \bigoplus_{m \in M} \mathbb{C}[S_\sigma]_m$$

and the embedding from $F_m^\#$ extends this grading to a finer M-grading of $\mathbb{C}[S_\sigma]$.

Shifting $\mathbb{C}[S_\sigma]$ by a degree $\mathbf{w}_\sigma$ gives a rank one M-graded $\mathbb{C}[S_\sigma]$-module and with respect to the refined grading we have the decomposition via $F_m^\#$

$$\mathbb{C}[S_\sigma](\mathbf{w}_\sigma) \cong \bigoplus_{\mathbf{v} \in P^n_m} \mathbb{C}[S_\sigma](\mathbf{v} - \mathbf{w}_\sigma),$$

as P^n_m gives a set of representatives of classes in M/mM. Now, in order to obtain a standard M-grading, we use the round-down to choose suitable representatives, in which case

$$\mathbb{C}[S_\sigma](\mathbf{v} - \mathbf{w}_\sigma) \cong \mathbb{C}[S_\sigma]\left(\left\lfloor \frac{AA_\sigma^{-1}(\mathbf{v} - \mathbf{w}_\sigma) + \mathbf{w}}{m} \right\rfloor\right) \cong \bigoplus_{\mathbf{u} \in M} \mathbb{C}[S_\sigma]_{\mathbf{u} + \left\lfloor \frac{AA_\sigma^{-1}(\mathbf{v} - \mathbf{w}_\sigma) + \mathbf{w}}{m} \right\rfloor}.$$

This isomorphism can be globalised as follows. For every maximal cone $\sigma \in \Sigma(n)$, there exists $\mathbf{w}_\sigma$ such that $\mathcal{O}_X(D)$ is represented by the M-graded $\mathbb{C}[S_\sigma]$-module $\Gamma(U_\sigma, \mathcal{O}_X(D)) \cong \mathbb{C}[S_\sigma](\mathbf{w}_\sigma)$. As before, we have

$$\mathbb{C}[S_\sigma](\mathbf{w}_\sigma) \cong \bigoplus_{\mathbf{v} \in P^n_m} \mathbb{C}[S_\sigma]\left(\left\lfloor \frac{AA_\sigma^{-1}(\mathbf{v} - \mathbf{w}_\sigma) + \mathbf{w}}{m} \right\rfloor\right).$$

Division by m and rounding down lead to compatible M-graded decompositions on each affine piece. Now, the pushforward $(F_m)_*(\mathcal{O}_X(D))$ is locally free and

$$(F_m)_*(\mathcal{O}_X(D)) \cong \bigoplus_{\chi \in M/mM} \mathcal{O}_\chi$$

where the $\mathcal{O}_\chi$ are invertible sheaves. As $\left\lfloor \frac{AA_\sigma^{-1}(\mathbf{v} - \mathbf{w}_\sigma) + \mathbf{w}}{m} \right\rfloor$ is independent of the representative for the class in M/mM, we can choose one fixed representative $\mathbf{v} \in P_m^n$ for each $\chi \in M/mM$. Hence $\mathcal{O}_\chi \cong \mathcal{O}_X(D^m_{\mathbf{v},\mathbf{w},\sigma})$ and the result follows. □

Following Thomsen [Tho00], Uehara notes that $(F_m)_*(L)$ does not depend on the choice of the maximal cone σ [Ueh14, Lemma 3.4]. We can assume the primitive ray generators of σ form the standard basis of $\mathbb{Z}^n$, in which case

$$q_\rho^m(\mathbf{v}, \mathbf{w}) = \lfloor \frac{u_\rho(\mathbf{v} - \mathbf{w}_\sigma) + w_\rho}{m} \rfloor. \tag{4.1.4}$$

Set

$$\mathfrak{D}(\mathcal{O}_X(D))_m := \{L \in \operatorname{Pic}(X) \mid L \text{ is a direct summand of } (F_m)_*(\mathcal{O}_X(D))\}. \tag{4.1.5}$$

Uehara [Ueh14, Lemma 3.5] also shows that the set

$$\mathfrak{D}(\mathcal{O}_X(D)) := \bigcup_{m>0} \mathfrak{D}(\mathcal{O}_X(D))_m \tag{4.1.6}$$

is finite. For brevity, we denote $\mathfrak{D}_m := \mathfrak{D}(\mathcal{O}_X)_m$, the set of line bundles in $(F_m)_*(\mathcal{O}_X)$. Note that we can use $\mathfrak{D}_m$ to find strong exceptional collections of line bundles on X:

Lemma 4.2. *[Ueh14, Lemma 3.8(i)] For any fixed positive integer m, the set of line bundles $\{L \in \mathfrak{D}_m \mid L^{-1} \text{ is nef}\} \subseteq \mathfrak{D}_m$ is a strong exceptional collection on X.*

Example 4.3. Set $m = 10$ and let $\mathbf{v} = (x, y) \in P_m^2$. Using the rays of the fan given in Example 2.5 for $\mathscr{H}_1$, the solutions to

$$\mathbf{q}^m(\mathbf{v}, \mathbf{0}) = \begin{bmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{-x-y}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lfloor \frac{-x-y}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \end{bmatrix}$$

determine the line bundles in the Frobenius pushforward of the structure sheaf for $\mathscr{H}_1$. We find that

$$\mathfrak{D}_m = \{\mathcal{O},\ \mathcal{O}(-D_3),\ \mathcal{O}(-D_3 - D_4),\ \mathcal{O}(-2D_3 - D_4)\}.$$

The inverse of every line bundle in this collection is nef, so $\mathfrak{D}_m$ is strong exceptional by Lemma 4.2. Note that this collection is dual to the collection given in Example 3.3.

4.2 Method 1

We can use the Frobenius morphism to find sets of line bundles that generate $\mathcal{D}^b(X)$.

Lemma 4.4. *[Ueh14, Lemma 5.1] Let $f\colon X \to Y$ be a proper morphism between smooth varieties. Assume that $\mathcal{E}$ generates $\mathcal{D}^b(X)$ and $\mathcal{O}_Y$ is a direct summand of $\mathbb{R}f_*\mathcal{O}_X$. Then $\mathbb{R}f_*\mathcal{E}$ generates $\mathcal{D}^b(Y)$.*

Proposition 4.5. *Let X be a smooth toric Fano variety of dimension n and $\mathcal{L}$ be a strong exceptional collection of line bundles on X. If the set of line bundles*

$$\mathfrak{D}_m^{gen} := \bigcup_{0\le i\le n} \mathfrak{D}(\omega_X^{-i})_m \tag{4.2.1}$$

is contained in $\langle\mathcal{L}\rangle$ for some positive integer m, then $\mathcal{L}$ is full.

Proof. As X is Fano, the anticanonical bundle ω_X^{-1} is ample and so Lemma 2.19 implies that $\bigoplus_{i=0}^n \omega_X^{-i}$ is a generator for $\mathcal{D}^b(X)$. The Frobenius morphism F_m is proper so $\bigcup_{0\le i\le n}\mathfrak{D}(\omega^{-i})_m$ generates $\mathcal{D}^b(X)$ by Lemma 4.4. By Remark 2.21, it follows that $\mathcal{L}$ classically generates $\mathcal{D}^b(X)$ and hence is full. ☐

To show that $\mathfrak{D}_m^{gen} \subset \langle\mathcal{L}\rangle$ for some $m > 0$, we use exact sequences of line bundles to generate objects in $\langle\mathcal{L}\rangle$; for examples of these calculations on the toric Fano threefolds see [Ueh14] or [BT09]. This process is easier when the line bundles in $\mathfrak{D}_m^{gen}$ are close together in $\mathrm{Pic}(X)$, which occurs when the value of m is large. However, the larger the value of m, the longer it takes to compute $\mathfrak{D}_m^{gen}$, so in practice m is often chosen by trial and error.

Example 4.6. Fix $m = 10$ and let $\mathbf{v} = (x, y) \in P_m^2$. Recall that the anticanonical divisor $-K_X$ for $\mathscr{H}_1$ corresponds to $\mathbf{w} = (1, 1, 1, 1) \in \mathbb{Z}^{\Sigma(1)}$. The set $\mathfrak{D}_m$ for $\mathscr{H}_1$ was calculated in Example 4.3; to find $\mathfrak{D}(\omega^{-1})_m$ and $\mathfrak{D}(\omega^{-2})_m$, we calculate

$$\mathbf{q}^m(\mathbf{v}, \mathbf{w}) = \begin{bmatrix} \lfloor\frac{(x-1)+1}{m}\rfloor \\ \lfloor\frac{(y-1)+1}{m}\rfloor \\ \lfloor\frac{(-x-y)+1}{m}\rfloor \\ \lfloor\frac{(-y)+1}{m}\rfloor \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lfloor\frac{-x-y+1}{m}\rfloor \\ \lfloor\frac{-y+1}{m}\rfloor \end{bmatrix}$$

and

$$\mathbf{q}^m(\mathbf{v}, 2\mathbf{w}) = \begin{bmatrix} \lfloor\frac{(x-2)+2}{m}\rfloor \\ \lfloor\frac{(y-2)+2}{m}\rfloor \\ \lfloor\frac{(-x-y)+2}{m}\rfloor \\ \lfloor\frac{(-y)+2}{m}\rfloor \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lfloor\frac{-x-y+2}{m}\rfloor \\ \lfloor\frac{-y+2}{m}\rfloor \end{bmatrix}$$

respectively. The result is $\mathfrak{D}(\omega^{-2})_m = \mathfrak{D}(\omega^{-1})_m = \mathfrak{D}_m$ and Example 4.3 shows that $\mathfrak{D}_m = \{\mathcal{O},\ \mathcal{O}(-D_3),\ \mathcal{O}(-D_3 - D_4),\ \mathcal{O}(-2D_3 - D_4)\}$ is a strong exceptional collection. As $\mathfrak{D}_m^{gen} = \mathfrak{D}_m$, then $\mathfrak{D}_m$ is a full strong exceptional collection for $\mathscr{H}_1$ by Proposition 4.5.

Example 4.7. We use ***Method 1*** to show that a given collection of line bundles generates $\mathcal{D}^b(X)$ when X is the smooth toric Fano fourfold I_1. The variety X has ray generators

$$u_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, u_1 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, u_2 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, u_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, u_4 = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, u_5 = \begin{bmatrix}2\\0\\-1\\-1\end{bmatrix}, u_6 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix} u_7 = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

and the linear equivalences between the toric divisors are $D_0 \sim -2D_5 + D_6 + D_7,\ D_1 \sim D_4,\ D_2 \sim -D_4 + D_5 - D_7,\ D_3 \sim D_5$. Set $m = 10$ and let $\mathbf{v} = (x, y, z, w) \in P_m^4$. The anticanonical divisor $-K_X$ corresponds to $\mathbf{w} = (1, \ldots, 1) \in \mathbb{Z}^8$. By (4.1.4), the solution to

$$\mathbf{q}^m(\mathbf{v}, \mathbf{w}) = \begin{bmatrix} \lfloor \frac{(x-1)+1}{m} \rfloor \\ \lfloor \frac{(y-1)+1}{m} \rfloor \\ \lfloor \frac{(z-1)+1}{m} \rfloor \\ \lfloor \frac{(w-1)+1}{m} \rfloor \\ \lfloor \frac{(-y+z)+1}{m} \rfloor \\ \lfloor \frac{(2x-z-w)+1}{m} \rfloor \\ \lfloor \frac{(-x+1)+1}{m} \rfloor \\ \lfloor \frac{(-x+z)+1}{m} \rfloor \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \lfloor \frac{-y+z+1}{m} \rfloor \\ \lfloor \frac{2x-z-w+1}{m} \rfloor \\ \lfloor \frac{-x+2}{m} \rfloor \\ \lfloor \frac{-x+z+1}{m} \rfloor \end{bmatrix}$$

for $\mathbf{v} \in P^4_m$ is an element of $\mathfrak{D}(\omega^{-1})_m$ and similarly we can calculate $\mathfrak{D}(\omega^{-i})_m$ by determining $\mathbf{q}^m(\mathbf{v}, i\mathbf{w})$, for $0 \leq i \leq 4$. It follows that $|\mathfrak{D}_m| = 18$, $|\mathfrak{D}(\omega^{-1})_m| = 18$ and $|\mathfrak{D}^{gen}_m| = 46$. For each line bundle L in the collection

$$\mathcal{L} = \left\{ \begin{array}{c} \mathcal{O}_X(-iD_4 - jD_5 - kD_6),\ \mathcal{O}_X(-D_6 - D_7), \\ \mathcal{O}_X(-D_4 - D_6 - D_7),\ \mathcal{O}_X(-D_5 - D_6 - D_7), \\ \mathcal{O}_X(-D_4 - D_5 - D_6 - D_7) \end{array} \middle| \begin{array}{c} 0 \leq i, k \leq 1 \\ 0 \leq j \leq 2 \\ \end{array} \right\} \subset \mathfrak{D}_m,$$

L^{-1} is nef, so $\mathcal{L}$ is a strong exceptional collection by Lemma 4.2. A list of rays $\{\rho_{i_1}, \ldots, \rho_{i_j}\}$ forms a cone in Σ if and only if $D_{i_1} \cap \ldots \cap D_{i_j} \neq \emptyset$, so we can use the primitive collections of X to determine which divisors do not intersect. For example, the primitive collection $\{u_0, u_7\}$ for X implies that $D_0 \cap D_7 = \emptyset$ and so we obtain the exact sequence

$$0 \to \mathcal{O}_X(-D_0 - D_7) \to \mathcal{O}_X(-D_0) \oplus \mathcal{O}_X(-D_7) \to \mathcal{O}_X \to 0.$$

Using the basis $\{[D_4], [D_5], [D_6], [D_7]\}$ for $\mathrm{Pic}(X)$, rewrite the exact sequence as

$$0 \to \mathcal{O}_X(2D_5 - D_6 - 2D_7) \to \mathcal{O}_X(2D_5 - D_6 - D_7) \oplus \mathcal{O}_X(-D_7) \to \mathcal{O}_X \to 0. \quad (4.2.2)$$

We can use the exact sequences determined by the primitive collections to show that $\mathfrak{D}^{gen}_m \subset \langle \mathcal{L} \rangle$. For example, the tensor of $\mathcal{O}_X(-2D_5 + D_7) \in \mathfrak{D}^{gen}_m \setminus \mathcal{L}$ with (4.2.2) gives the exact sequence

$$0 \to \mathcal{O}_X(-D_6 - D_7) \to \mathcal{O}_X(-D_6) \oplus \mathcal{O}_X(-2D_5) \to \mathcal{O}_X(-2D_5 + D_7) \to 0. \quad (4.2.3)$$

All of the line bundles in (4.2.3) except $\mathcal{O}_X(-2D_5 + D_7)$ are in $\langle \mathcal{L} \rangle$, hence so is $\mathcal{O}_X(-2D_5 + D_7)$. By the same method and using the exact sequences of line bundles determined by the primitive collections for X, every line bundle in $\mathfrak{D}^{gen}_m$ is contained in $\langle \mathcal{L} \rangle$ and so $\mathcal{L}$ is full by Proposition 4.5.

CHAPTER

FIVE

QUIVER MODULI AND THE STRUCTURE SHEAF OF THE DIAGONAL

Let X be a smooth toric variety and $\mathcal{L} = \{L_0, \dots, L_r\}$ be a collection of line bundles on X. The aim of this chapter is to introduce a map d_1 of sheaves on $X \times X$ constructed from the line bundles in $\mathcal{L}$ (see (5.2.1)) and consider when the cokernel of d_1 is the structure sheaf $\mathcal{O}_\Delta$ of the diagonal embedding into $X \times X$; this is a necessary step in the second method that we use to show that $\mathcal{L}$ generates $\mathcal{D}^b(X)$.

In the first section we use the *quiver of sections* to encode the endomorphism algebra $A = \operatorname{End}\left(\bigoplus_{i=0}^{r} L_i\right)$ and recall the construction of the moduli space of quiver representations $\mathcal{M}_\theta(Q, J)$ along with its tautological bundles. We show in the second section that if X embeds into $\mathcal{M}_\theta(Q, J)$ in such a way that the tautological bundles pull back to the line bundles in $\mathcal{L}$, then the cokernel of d_1 is $\mathcal{O}_\Delta$ (see Proposition 5.4). The final section in this chapter describes two different approaches to showing that this embedding does exist, depending on whether all of the line bundles in $\mathcal{L}$ are nef or not (see Propositions 5.6 and 5.8).

5.1 Quivers of Sections and Moduli Spaces of Quiver Representations

A quiver Q consists of a vertex set Q_0, an arrow set Q_1 and maps $\mathbf{h},\mathbf{t} \colon Q_1 \to Q_0$ giving the vertices at the head and tail of each arrow. We assume that Q is connected, acyclic and rooted at a unique source. A non-trivial path in Q is a sequence of arrows $p = a_1 \dots a_k$ such that $\mathbf{h}(a_i) = \mathbf{t}(a_{i+1})$ for $1 \leq i \leq k-1$, in which case $\mathbf{t}(p) := \mathbf{t}(a_1)$, $\mathbf{h}(p) := \mathbf{h}(a_k)$ and $\operatorname{supp}(p) = \{a_1, \dots, a_k\}$. Each vertex $i \in Q_0$ gives a trivial path e_i with $\mathbf{h}(e_i) = \mathbf{t}(e_i) = i$. By taking the paths as a generating set and defining multiplication to be concatenation of paths when possible and zero otherwise, we obtain the path algebra $\mathbb{C}Q$. A *relation* is a $\mathbb{C}$-linear combination of paths in Q that share the same head and tail and are of length at least two. If J is the two-sided ideal in $\mathbb{C}Q$ generated by a finite set of relations, then we obtain the quotient algebra $\mathbb{C}Q/J$ and we use the notation (Q, J) to denote the *quiver with relations*.

A *representation* W of a quiver Q assigns a $\mathbb{C}$-vector space W_i to each vertex $i \in Q_0$ and a $\mathbb{C}$-linear map $w_a \colon W_{\mathbf{t}(a)} \to W_{\mathbf{h}(a)}$ to each arrow $a \in Q_1$. The dimension of each vector space W_i determines the *dimension vector* $\mathbf{v}$. A morphism ϕ between two representations W and V is a collection of $\mathbb{C}$-linear maps $\phi_i\colon W_i \to V_i$ such that for any arrow $a \in Q_1$, the following square commutes:

$$\begin{array}{ccc} W_{\mathbf{t}(a)} & \xrightarrow{w_a} & W_{\mathbf{h}(a)} \\ {\scriptstyle\phi_{\mathbf{t}(a)}}\big\downarrow & & \big\downarrow{\scriptstyle\phi_{\mathbf{h}(a)}} \\ V_{\mathbf{t}(a)} & \xrightarrow[v_a]{} & V_{\mathbf{h}(a)} \end{array} \tag{5.1.1}$$

For the quiver with relations (Q, J), we can consider representations of Q that respect the relations in J. More precisely, a representation of (Q, J) is a representation W of Q such that for any relation generating J, the corresponding $\mathbb{C}$-linear combination of maps between the vector spaces $(W_i)_{i\in Q_0}$ is set to be the zero map.

Let $\mathbb{Z}^{Q_0}$ be the free abelian group of functions from Q_0 to $\mathbb{Z}$ and $\mathbb{Z}^{Q_1}$ be the free abelian group of functions from Q_1 to $\mathbb{Z}$. Define $\mathrm{Wt}(Q) := \{\theta \in \mathbb{Z}^{Q_0} \mid \theta(\mathbf{v}) = 0\}$ to be the *weight space* for Q. Each $\theta \in \mathrm{Wt}(Q)$ determines a stability parameter, where a representation W is θ-(semi)stable if for every non-zero proper subrepresentation $W' \subset W$ we have $\theta(W') := \sum_{\{i \mid W'_i \neq 0\}} \theta_i > (\geq)\ 0$. A parameter θ is *generic* if every θ-semistable representation is θ-stable. A generic stability parameter θ can then be used to construct the *fine moduli space of θ-stable representations* $\mathcal{M}_\theta(Q)$, as introduced by King [Kin94]. The space $\mathcal{M}_\theta(Q)$ is a projective variety as Q is acyclic [Kin94, Proposition 4.3].

Now fix the dimension vector $\mathbf{v}$ to be $(1, \dots, 1) \in (\mathbb{Z}^{Q_0})^\vee$, in which case Hille [Hil98, Section 1.3] has shown that $\mathcal{M}_\theta(Q)$ is a smooth toric variety. Note that the *special parameter* $\vartheta := (-r, 1, 1, \dots, 1)$ is generic for this dimension vector, where $r = |Q_0| - 1$. To construct $\mathcal{M}_\theta(Q)$ explicitly, let the characteristic functions $\chi_i\colon Q_0 \to \mathbb{Z}$ for $i \in Q_0$ and $\chi_a\colon Q_1 \to \mathbb{Z}$ for $a \in Q_1$ form bases for $\mathbb{Z}^{Q_0}$ and $\mathbb{Z}^{Q_1}$ respectively. The incidence map $\mathrm{inc}\colon \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}$ defined by $\mathrm{inc}(\chi_a) = \chi_{\mathbf{h}(a)} - \chi_{\mathbf{t}(a)}$ determines the exact sequence

$$0 \longrightarrow \tilde{M} \longrightarrow \mathbb{Z}^{Q_1} \xrightarrow{\ \mathrm{inc}\ } \mathrm{Wt}(Q) \longrightarrow 0. \tag{5.1.2}$$

For a fixed generic stability parameter $\theta \in \mathrm{Wt}(Q)$, let $\mathbb{C}[y_a \mid a \in Q_1]_\theta = \mathbb{C}[\mathbb{N}^{Q_1} \cap \mathrm{inc}^{-1}(\theta)]$ denote the θ-graded piece. Then $\mathcal{M}_\theta(Q)$ is the GIT quotient

$$\mathcal{M}_\theta(Q) = \mathbb{C}^{Q_1} /\!\!/_\theta T = \mathrm{Proj}\left(\bigoplus_{j\geq 0} \mathbb{C}\left[y_a \mid a \in Q_1\right]_{j\theta}\right) \tag{5.1.3}$$

where the action of $T := \mathrm{Hom}_\mathbb{Z}(\mathrm{Wt}(Q), \mathbb{C}^*)$ is induced from the action of $(\mathbb{C}^*)^{Q_0} \cong \prod_{i\in Q_0} \mathrm{GL}(W_i)$ on $\mathbb{C}^{Q_1}$ determined by the incidence map. By choosing a group isomorphism between T and $\{(g_0, \dots, g_r) \in (\mathbb{C}^*)^{Q_0} \mid g_0 = 1\}$ we obtain a T-equivariant vector bundle $\bigoplus_{i\in Q_0} \mathcal{O}_{\mathbb{C}^{Q_1}}$ on $\mathbb{C}^{Q_1}$ which descends to the universal family $\bigoplus_{i\in Q_0} F_i$ on $\mathcal{M}_\theta(Q)$ [Kin94, Proposition 5.3]. The summands F_i are called the *tautological line bundles* on $\mathcal{M}_\theta(Q)$ and F_0 is the trivial line bundle, where $0 \in Q_0$ labels the source of Q, as T acts trivially on the summand given by $i = 0$ in $\bigoplus_{i\in Q_0} \mathcal{O}_{\mathbb{C}^{Q_1}}$. The dimension of $\mathcal{M}_\theta(Q)$ is $|Q_1| - |Q_0| + 1$ and $\mathrm{Pic}(\mathcal{M}_\theta(Q)) \cong \mathrm{Wt}(Q)$ [Hil98, Theorem 2.3].

If we are considering a quiver with ideal of relations J, we denote the fine moduli space of θ-stable representations of Q that respect the relations in J by $\mathcal{M}_\theta(Q,J)$. By sending a path $p = a_1 \dots a_k$ to the monomial $y_{a_1} \cdots y_{a_k} \in \mathbb{C}[y_a \mid a \in Q_1]$ and extending linearly, we obtain a $\mathbb{C}$-linear map from $\mathbb{C}Q$ to $\mathbb{C}[y_a \mid a \in Q_1]$. We let I_J be the ideal in $\mathbb{C}[y_a \mid a \in Q_1]$ generated by the image of J under this map, in which case $\mathcal{M}_\theta(Q,J)$ is given by the GIT quotient

$$\mathcal{M}_\theta(Q,J) = \mathbb{V}(I_J)/\!\!/_\theta T = \operatorname{Proj}\left(\bigoplus_{j\geq 0} \left(\mathbb{C}[y_a \mid a \in Q_1]/I_J\right)_{j\theta}\right). \tag{5.1.4}$$

Let $\mathcal{L} = \{L_0, \dots, L_r\}$ be a collection of non-isomorphic effective line bundles on a projective normal toric variety X with $L_0 := \mathcal{O}_X$. As X is projective and irreducible, if $\operatorname{Hom}(L_i, L_j) \neq 0$ then $\operatorname{Hom}(L_j, L_i) = 0$ and so we can assume $\mathcal{L}$ is ordered such that $i < j$ whenever $\operatorname{Hom}(L_i, L_j) \neq 0$. The endomorphism algebra $\operatorname{End}(\bigoplus_i L_i)$ can be conveniently described by its *quiver of sections* Q, whose vertices $Q_0 = \{0, \dots, r\}$ are the line bundles in $\mathcal{L}$ and the number of arrows from vertex i to j for $i < j$ is given by the dimension of the cokernel of the map

$$\bigoplus_{i<k<j} \operatorname{Hom}(L_i, L_k) \otimes \operatorname{Hom}(L_k, L_j) \longrightarrow \operatorname{Hom}(L_i, L_j). \tag{5.1.5}$$

A torus-invariant section $s \in \operatorname{Hom}(L_i, L_j)$ is *irreducible* if it is not in the image of this map. Each section in a basis of the irreducible sections determines a divisor of zeroes, and these divisors label the arrows between vertex i and j; we therefore denote $\operatorname{div}(a)$ for the divisor that labels the arrow $a \in Q_1$, and $\operatorname{div}(p) := \sum_{a \in \operatorname{supp}(p)} \operatorname{div}(a)$ for a path p. The corresponding labelling monomial is $x^{\operatorname{div}(p)} := \prod_{a\in\operatorname{supp}(p)} x^{\operatorname{div}(a)} \in \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$. Note that the quiver is acyclic and as the collection is effective, the quiver is connected and rooted at 0. The arrow labels determine the two-sided ideal of relations J, generated by the set

$$\{p_i - p_j \mid p_i, p_j \text{ paths in } Q,\ \mathbf{t}(p_i) = \mathbf{t}(p_j), \mathbf{h}(p_i) = \mathbf{h}(p_j),\ \operatorname{div}(p_i) = \operatorname{div}(p_j)\}. \tag{5.1.6}$$

Lemma 5.1. *[CS08, Proposition 3.3] Let Q be the quiver of sections for the collection $\mathcal{L}$ above, with ideal of relations J. Then $\mathbb{C}Q/J \cong \operatorname{End}(\bigoplus_i L_i)$.*

Each line bundle L_i is isomorphic to $\mathcal{O}_X(D_i')$ for some Cartier divisor D_i' and we construct Q explicitly by computing the vertices of the polyhedron $\operatorname{conv}(\mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(D_i' - D_j'))$ for each $i \neq j \in Q_0$. The vertices correspond to the torus-invariant generators of $\operatorname{Hom}(L_i, L_j)$, from which we pick the irreducible sections.

Example 5.2. Figure 5.1(a) displays the quiver of sections for the full strong exceptional collection on $\mathscr{H}_1$ given in Example 2.23, whilst Figure 5.1(b) lists the arrows. The ideal of relations is

$$J = (a_3a_6 - a_1a_5, a_3a_7 - a_2a_5, a_2a_4a_6 - a_1a_4a_7).$$

Example 5.3. Let X be the smooth toric Fano fourfold E_1 in Example 3.17 and fix $m \gg 0$. Choose $\{[D_4], [D_5], [D_6]\}$ to be the basis of $\operatorname{Pic}(X)$; the exact sequence (2.1.3)

(a) Quiver of sections (b) List of arrows

Figure 5.1: *The quiver of sections of a full strong exceptional collection on* $\mathscr{H}_1$

for X is

$$0 \longrightarrow M \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 3 & -1 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{bmatrix}} \mathbb{Z}^7 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & 1 & 1 & 0 & 1 & 0 \\ -2 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}} \operatorname{Pic}(X) \longrightarrow 0.$$

Every line bundle L_i in the collection

$$\mathcal{L} = \{\mathcal{O}_X(iD_5+iD_6),\ \mathcal{O}_X(D_4+iD_5+iD_6),\ \mathcal{O}_X(D_4+jD_5+(j+1)D_6) \mid 0 \leq i \leq 3,\ 0 \leq j \leq 2\}$$

is nef and $L_i^{-1} \in \mathfrak{D}_m$, so $\mathcal{L}$ is a strong exceptional collection by Lemma 4.2. The quiver of sections Q for this collection is given in Figure 5.2.

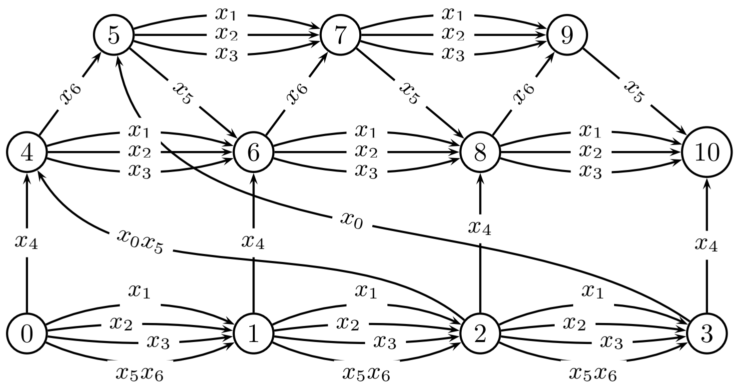


Figure 5.2: *A quiver of sections on the smooth toric Fano fourfold* E_1

5.2 Quiver Moduli and the Structure Sheaf of the Diagonal

Let $\iota\colon \Delta \hookrightarrow X \times X$ be the diagonal embedding and for two line bundles L_1 and L_2 on X define

$$L_1 \boxtimes L_2 := p_1^*(L_1) \otimes p_2^*(L_2)$$

where p_1 and p_2 are the projections from $X \times X$ onto the first and second component respectively. We define the map d_1 of vector bundles on $X \times X$ as follows. Let d_1 have domain and codomain:

$$d_1 \colon \bigoplus_{a \in Q_1} L_{\mathbf{t}(a)} \boxtimes L^{-1}_{\mathbf{h}(a)} \longrightarrow \bigoplus_{i \in Q_0} L_i \boxtimes L_i^{-1}. \tag{5.2.1}$$

The summands of the vector bundles are line bundles and so are given by twists of the $\mathrm{Pic}(X \times X)$-graded module $S_{X\times X}$. We write $S_{X\times X} = \mathbb{C}[x_1, \ldots, x_d, w_1, \ldots, w_d]$ where $d = |\Sigma_X(1)|$ to distinguish sections x_i on the first copy of X in $X \times X$ from sections w_i on the second copy. For line bundles L_i and L_j on X, denote $S_{X\times X}(L_i, L_j)$ to be a free $S_{X\times X}$-module generated by $\mathbf{e}_{L_i,L_j}$ corresponding to the line bundle $L_i \boxtimes L_j$ on $X \times X$ by (2.1.5). Then our map d_1 sends

$$S_{X\times X}(L_{\mathbf{t}(a)}, L^{-1}_{\mathbf{h}(a)}) \to S_{X\times X}(L_{\mathbf{h}(a)}, L^{-1}_{\mathbf{h}(a)}) \oplus S_{X\times X}(L_{\mathbf{t}(a)}, L^{-1}_{\mathbf{t}(a)})$$

$$\mathbf{e}_{L_{\mathbf{t}(a)},L^{-1}_{\mathbf{h}(a)}} \mapsto x^{\mathrm{div}(a)}\mathbf{e}_{L_{\mathbf{h}(a)},L^{-1}_{\mathbf{h}(a)}} - w^{\mathrm{div}(a)}\mathbf{e}_{L_{\mathbf{t}(a)},L^{-1}_{\mathbf{t}(a)}}.$$

The following proposition provides a condition as to when the cokernel of d_1 is $\mathcal{O}_\Delta$. Note that our choice of θ in the proposition will depend on our collection $\mathcal{L}$, as explained in the following section.

Proposition 5.4. *Suppose that there exists a generic stability parameter θ and a closed immersion $\phi \colon X \hookrightarrow \mathcal{M}_\theta(Q, J)$ such that $L_i \cong \phi^*(F_i)$ for $0 \leq i \leq r$. Then the cokernel of d_1 in* (5.2.1) *is $\mathcal{O}_\Delta$.*

Proof. We follow the arguments made in [Kin97]. Assume that θ and ϕ satisfy the conditions in the proposition. For the opposite quiver with relations (Q^{op}, J^{op}), the stability parameter $-\theta$ is generic and $\mathcal{M}_\theta(Q, J) \cong \mathcal{M}_{-\theta}(Q^{op}, J^{op})$. In addition, we have a closed immersion of X into $\mathcal{M}_{-\theta}(Q^{op}, J^{op})$ such that the tautological line bundles on $\mathcal{M}_{-\theta}(Q^{op}, J^{op})$ restrict to the line bundles L_i^{-1} on X. A θ-stable representation $W = (W_i, \psi_a)$ of (Q, J) determines a $(-\theta)$-stable representation $W^* = (W_i^*, \psi_a^*)$ of (Q^{op}, J^{op}) and so for a point $(x_1, x_2) \in X \times X \hookrightarrow \mathcal{M}_\theta(Q, J) \times \mathcal{M}_{-\theta}(Q^{op}, J^{op})$, the fibre over x_1 parametrises the isomorphism class of a θ-stable representation $V := (V_i, \phi_a)$, whilst the fibre over x_2 parametrises the isomorphism class of a $(-\theta)$-stable representation W^*. Therefore, the map d_1 of vector bundles on $X \times X$ from (5.2.1) restricted to the fibre over $(x_1, x_2) \in X \times X$ is given by

$$D \colon \bigoplus_{a \in Q_1} V_{\mathbf{t}(a)} \otimes W^*_{\mathbf{h}(a)} \longrightarrow \bigoplus_{i \in Q_0} V_i \otimes W_i^*.$$

The map D is dual to the map:

$$D^* \colon \bigoplus_{i \in Q_0} \mathrm{Hom}_{\mathbb{C}}(V_i, W_i) \longrightarrow \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{\mathbf{t}(a)}, W_{\mathbf{h}(a)}) \tag{5.2.2}$$

given by $(\beta_i) \mapsto (\beta_{\mathbf{h}(a)}\phi_a - \psi_a\beta_{\mathbf{t}(a)})$. The kernel $\ker(D^*)$ of this map is precisely the morphisms from V to W. As V and W are θ-stable, we have $\theta(V) = \theta(W) = 0$.

If f is a morphism in $\ker(D^*)$ then the image $\operatorname{im}(f)$ of f is a quotient of V, hence $\theta(\operatorname{im}(f)) \leq 0$. However, $\operatorname{im}(f)$ also injects into W implying that $\theta(\operatorname{im}(f)) \geq 0$, so $\theta(\operatorname{im}(f)) = 0$. Therefore, f is either an isomorphism or the zero morphism. It follows that when $W = V$, the kernel of D^* is canonically a copy of $\mathbb{C}$. Using this observation, we see that away from the diagonal of $X \times X$ the cokernel of d_1 is rank zero as the representations V and W are not isomorphic, whilst at each point on the diagonal the cokernel restricts to a canonical copy of $\mathbb{C}$. Therefore the cokernel of d_1 is $\mathcal{O}_\Delta$.

□

5.3 Nef And Non-Nef Collections

Our choice of the generic stability parameter used in Proposition 5.4 will depend on whether our chosen line bundles are nef or not. Firstly, assume that $\mathcal{L}$ is a collection of nef line bundles on X and recall the special stability parameter $\vartheta = (-r, 1, 1, \dots, 1)$. Craw and Smith [CS08] associate to Q a projective toric variety $|\mathcal{L}| \cong \mathcal{M}_\vartheta(Q)$ called the *multigraded linear series* of $\mathcal{L}$. They define the morphism $\phi_\mathcal{L} \colon X \to |\mathcal{L}|$ which factors into

$$X \longrightarrow \mathcal{M}_\vartheta(Q, J) \hookrightarrow |\mathcal{L}| \tag{5.3.1}$$

and [CS08, Corollary 4.10] present criteria as to when $\phi_\mathcal{L}$ is a closed embedding.

Lemma 5.5. *[CS08, Corollary 4.10] Let $\mathcal{L} = \{\mathcal{O}_X, L_1, \dots, L_r\}$ be a collection of basepoint-free line bundles and set $L = \bigotimes_{L_i \in \mathcal{L}} L_i$. Assume that the map $H^0(X, L_1) \otimes \cdots \otimes H^0(X, L_r) \to H^0(X, L)$ is surjective. Then the morphism $\phi_\mathcal{L} \colon X \to |\mathcal{L}|$ is a closed embedding if and only if L is very ample.*

For a line bundle L on X_Σ, there is a natural bijection between $\deg^{-1}(L) \cap \mathbb{N}^{\Sigma(1)}$ and the sections that generate $\Gamma(X_\Sigma, L)$. Define P_L to be the polytope in $\mathbb{R}^{\Sigma(1)}$ that is the convex hull of the lattice points $\deg^{-1}(L) \cap \mathbb{N}^{\Sigma(1)}$. As the tensor operation on sections corresponds to addition of lattice points in $\mathbb{N}^{\Sigma(1)}$ and the Minkowski sum of two polytopes adds each lattice point of the first polytope to every lattice point of the second polytope, the proposition below follows immediately from Lemma 5.5 and [CS08, Theorem 4.15]:

Proposition 5.6. *Let $\mathcal{L}$ be a collection of nef line bundles. If $L := \bigotimes_{L_i \in \mathcal{L}} L_i$ is very ample and the Minkowski sum of the polytopes $\{P_{L_i} \mid L_i \in \mathcal{L}\}$ is equal to P_L, then the morphism $\phi_\mathcal{L} \colon X \to |\mathcal{L}|$ is a closed embedding. In this case, we can recover the line bundles in $\mathcal{L}$ as the restriction of the tautological bundle on $|\mathcal{L}|$ to X.*

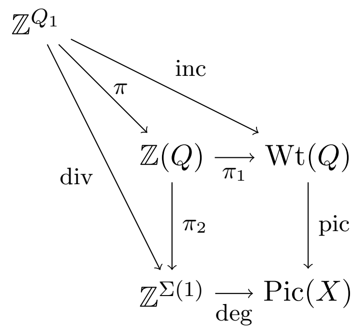
Note that as the varieties we are considering are smooth and toric, any ample line bundle is very ample by Lemma 2.11.

If the collection $\mathcal{L}$ contains a line bundle that is not nef then, for the special stability parameter ϑ, the multigraded linear series construction only gives a rational map from X to $\mathcal{M}_\vartheta(Q, J)$. In this case, we need to choose a different generic stability parameter θ in order to obtain a closed embedding of X into $\mathcal{M}_\theta(Q, J)$ via the map defined in [CS08], such that the tautological bundle on $\mathcal{M}_\theta(Q, J)$ restricts to $\bigoplus_{i \in Q_0} L_i$ on X. To achieve this, we recall the construction of the toric variety $Y_\theta \subset \mathcal{M}_\theta(Q, J)$ from [CS08], (see also [CMT07] and [CQV12]).

Define the map

$$\pi := (\mathrm{inc}, \mathrm{div})\colon \mathbb{Z}^{Q_1} \to \mathrm{Wt}(Q) \oplus \mathbb{Z}^{\Sigma(1)}$$

with image $\mathbb{Z}(Q) := \pi(\mathbb{Z}^{Q_1})$ and subsemigroup $\mathbb{N}(Q) := \pi(\mathbb{N}^{Q_1})$. The projections $\pi_1\colon \mathbb{Z}(Q) \to \mathrm{Wt}(Q)$ and $\pi_2\colon \mathbb{Z}(Q) \to \mathbb{Z}^{\Sigma(1)}$ fit in to the commutative diagram



where $\mathrm{pic}(\chi_i) := L_i$ for $i \in Q_0$ is a group homomorphism. Let $\mathbb{C}[\mathbb{N}(Q)]$ and $\mathbb{C}[\mathbb{N}^{Q_1}]$ be the semigroup algebras defined by $\mathbb{N}(Q)$ and $\mathbb{N}^{Q_1}$ respectively. The surjective map of semigroup algebras $\pi_*\colon \mathbb{C}[\mathbb{N}^{Q_1}] \to \mathbb{C}[\mathbb{N}(Q)]$ induced by π has kernel I_Q that defines an affine toric subvariety $\mathbb{V}(I_Q) \subset \mathbb{C}^{Q_1}$. We obtain a T-action on $\mathbb{V}(I_Q)$ via restriction of the T-action on $\mathbb{C}^{Q_1}$. For a generic weight $\theta \in \mathrm{Wt}(Q)$, we have the categorical quotient

$$Y_\theta := \mathbb{V}(I_Q)/\!/_\theta T = \mathrm{Proj}\left(\bigoplus_{j \geq 0} \mathbb{C}[\mathbb{N}(Q)]_{j\theta}\right)$$

where $\mathbb{C}[\mathbb{N}(Q)]_\theta$ is the θ-graded piece. The variety Y_θ is toric and is a closed subvariety of $\mathcal{M}_\theta(Q, J)$.

Lemma 5.7. *Let $\theta \in \mathrm{Wt}(Q)$ and consider the variety Y_θ. If each representation corresponding to a torus-invariant point in Y_θ is θ-stable, then θ is generic.*

Proof. We follow the first part of the proof for [BCQV14, Lemma 4.2]. Let $y \in Y_\theta$ and $V_y = ((V_y)_i, \phi_a)$ be the corresponding θ-semistable representation. A subrepresentation $S = (S_i, s_a)$ of V_y determines a set of vertices $I \subset Q_0$ such that $i \in I$ if and only if $S_i = \mathbb{C}$. Let y' be the distinguished point in the torus orbit containing y with corresponding representation $V_{y'}$. Then S determines a submodule $S' = (S'_i, s'_a) \subset V_{y'}$ where $S'_i = \mathbb{C}$ if and only if $i \in I$. Likewise, we obtain a submodule $S_0 \subset V_0$ of the module V_0 corresponding to the torus-invariant point in any toric chart containing the torus orbit, as V_0 is obtained from $V_{y'}$ by setting certain maps to zero. Now $\theta(S) = \theta(S') = \theta(S_0)$, but $\theta(S_0) > 0$ as V_0 is assumed to be θ-stable. As this holds for any subrepresentation of V_y, then V_y is θ-stable and hence θ is generic as $y \in Y_\theta$ was arbitrary. □

Proposition 5.8. *Fix a generic $\theta \in \mathrm{Wt}(Q)$ such that $L := \mathrm{pic}(\theta)$ is an ample line bundle on X. If*

$$\deg^{-1}(L) \cap \mathbb{N}^{\Sigma(1)} \subset \pi_2\left(\pi_1^{-1}(\theta) \cap \mathbb{N}(Q)\right) \tag{5.3.2}$$

then the homomorphism of graded rings

$$(\pi_2)_*\colon \bigoplus_{j\geq 0} \mathbb{C}[\mathbb{N}(Q)]_{j\theta} \to \bigoplus_{j\geq 0} \mathbb{C}[x_\rho \mid \rho \in \Sigma_X(1)]_{jL}$$

induces an isomorphism $X \cong Y_\theta$. Furthermore, if θ and ϑ are in the same open GIT-chamber for the T-action on $\mathbb{V}(I_Q)$, then the tautological bundles on $\mathcal{M}_\theta(Q)$ restrict to the line bundles L_i on X.

Proof. The morphism $X \to Y_\theta$ is equivariant under the action of T and $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic}(X), \mathbb{C}^*)$ on $\mathbb{V}(I_Q)$ and $\mathbb{C}^{\Sigma(1)}$ respectively as the diagram of lattice maps

$$\begin{array}{ccc} \mathbb{Z}(Q) & \xrightarrow{\pi_1} & \operatorname{Wt}(Q) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \operatorname{pic}} \\ \mathbb{Z}^{\Sigma(1)} & \xrightarrow[\deg]{} & \operatorname{Pic}(X) \end{array} \tag{5.3.3}$$

commutes, hence we obtain a rational map from X to Y_θ. As L is ample, we have $X = \operatorname{Proj}\left(\bigoplus_{j\geq 0} \mathbb{C}[x_\rho \mid \rho \in \Sigma_X(1)]_{jL}\right)$ and so X is a closed subvariety of Y_θ when the homomorphism of graded rings

$$(\pi_2)_*\colon \bigoplus_{j\geq 0} \mathbb{C}[\mathbb{N}(Q)]_{j\theta} \to \bigoplus_{j\geq 0} \mathbb{C}[x_\rho \mid \rho \in \Sigma_X(1)]_{jL}$$

induced from π_2 is surjective. The bundle L is very ample as X is smooth and toric, so $\bigoplus_{j\geq 0} \mathbb{C}[x_\rho \mid \rho \in \Sigma_X(1)]_{jL}$ is generated in the first graded piece and thus it is enough to check surjectivity on this piece, which follows from (5.3.2). By construction we have $\pi_2\left(\pi_1^{-1}(\theta) \cap \mathbb{N}(Q)\right) \subset \deg^{-1}(L) \cap \mathbb{N}^{\Sigma(1)}$ and given any two points $p_1, p_2 \in \pi_1^{-1}(\theta) \cap \mathbb{N}(Q)$ such that $p_1 \neq p_2$, then $\pi_2(p_1) \neq \pi_2(p_2)$. Therefore, $(\pi_2)_*$ induces an isomorphism $X \cong Y_\theta$.

Now assume that θ and ϑ are in the same open GIT-chamber for the T-action on $\mathbb{V}(I_Q)$ and denote $\mathcal{M}_{\theta'}(Q)$ by $\mathcal{M}_{\theta'}$, for $\theta' \in \operatorname{Wt}(Q)$. Following the proof of [CS08, Theorem 4.15], we can identify $\operatorname{Wt}(Q)$ with $\mathbb{Z}^r$ by choosing $(\chi_1 - \chi_0, \ldots, \chi_r - \chi_0)$ to be a basis of $\operatorname{Wt}(Q)$, in which case we obtain a group isomorphism between T and $\{(g_0, \ldots, g_r) \in (\mathbb{C}^*)^{Q_0} \mid g_0 = 1\} \subset (\mathbb{C}^*)^{Q_0}$ via the projection map $\mathbb{Z}^{Q_0} \to \mathbb{Z}^r$. The i-th summand in the the T-equivariant vector bundle $\bigoplus_{i\in Q_0} \mathcal{O}_{\mathbb{C}^{Q_1}}$ therefore corresponds to the $S_{\mathcal{M}_\vartheta}$-module $S_{\mathcal{M}_\vartheta}(\chi_i - \chi_0)$, and so the tautological line bundles on $\mathcal{M}_\vartheta$ are $\mathcal{O}_{\mathcal{M}_\vartheta}, \mathcal{O}_{\mathcal{M}_\vartheta}(\chi_1 - \chi_0), \ldots, \mathcal{O}_{\mathcal{M}_\vartheta}(\chi_r - \chi_0)$. These bundles restrict to the line bundles on $\mathbb{V}(I_Q)$ corresponding to the modules $(S_{\mathcal{M}_\vartheta}/I_Q), (S_{\mathcal{M}_\vartheta}/I_Q)(\chi_1 - \chi_0), \ldots, (S_{\mathcal{M}_\vartheta}/I_Q)(\chi_r - \chi_0)$. As the chosen weight θ' varies, the restriction of the tautological line bundles will change if and only if the θ'-stable representations parametrised by points of $\mathbb{V}(I_Q)$ change; as θ and ϑ are in the same open GIT-chamber this is not the case, so they correspond to the line bundles given by $(S_{\mathcal{M}_\theta}/I_Q), (S_{\mathcal{M}_\theta}/I_Q)(\chi_1 - \chi_0), \ldots, (S_{\mathcal{M}_\theta}/I_Q)(\chi_r - \chi_0)$ on Y_θ. From the isomorphism $X \cong Y_\theta$ induced by $(\pi_2)_*$, it follows that the module $(S_{\mathcal{M}_\theta}/I_Q)(\chi_i - \chi_0)$ corresponds to $\operatorname{pic}(\chi_i - \chi_0) = L_i$ on X. □

CHAPTER

SIX

GENERATION OF $\mathcal{D}^b(X)$: RESOLUTION OF $\mathcal{O}_\Delta$ (METHOD 2)

Chapter 5 introduced a map of vector bundles d_1 on $X \times X$ determined by the line bundles in $\mathcal{L}$ and gave methods to determine if the cokernel of d_1 is $\mathcal{O}_\Delta$. The first section of this chapter justifies why we would want to consider this problem; we show that if d_1 forms part of a resolution of $\mathcal{O}_\Delta$, then our collection $\mathcal{L}$ generates $\mathcal{D}^b(X)$ (see Proposition 6.1). Motivated by King (Lemmas 6.3 and 6.2), we attempt to find a resolution of $\mathcal{O}_\Delta$ by guessing a minimal projective bimodule resolution of the endomorphism algebra of $\bigoplus_{L\in\mathcal{L}} L$ and sheafifying the result. The second section gives the framework as to how we guess the resolution, based on the concept of a *toric cell complex* introduced by Craw–Quintero-Vélez [CQV12]. The final section in this chapter brings together this construction along with the results in Chapter 5 to present our second method for showing that a collection of line bundles on X is full.

6.1 Resolution of $\mathcal{O}_\Delta$

Let X be a smooth projective toric variety and $\mathcal{L} = \{L_0, \dots, L_r\}$ be a collection of line bundles on X. For $\mathcal{E} \in \mathcal{D}^b(X \times X)$, denote $\Phi^{\mathcal{E}}(-) := \mathbf{R}(p_1)_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_2^*(-)) \colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ to be the Fourier-Mukai transform with kernel $\mathcal{E}$.

Proposition 6.1. *If there exists an exact sequence of sheaves on $X \times X$ of the form:*

$$0 \to \mathcal{E}_k \to \cdots \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \to \mathcal{O}_\Delta \to 0$$

where

$$\mathcal{E}_0 = \bigoplus_{i\in Q_0} L_i \boxtimes L_i^{-1},$$
$$\mathcal{E}_1 = \bigoplus_{a\in Q_1} L_{\boldsymbol{t}(a)} \boxtimes L^{-1}_{\boldsymbol{h}(a)}$$

and

$$\mathcal{E}_t = \bigoplus_{L_i, L_j \in \mathcal{L}} L_i^{r_{i,t}} \boxtimes L_j^{-s_{j,t}}, \text{ for } 2 \le t \le k \text{ and some fixed } r_{i,t}, s_{j,t} \in \mathbb{Z}_{\ge 0},$$

then $\mathcal{L}$ classically generates $\mathcal{D}^b(X)$.

Proof. Assume that we have a resolution of $\mathcal{O}_\Delta$ as given in the Proposition. It follows from the projection formula that $\Phi^{\mathcal{O}_\Delta}$ is naturally isomorphic to the identity functor on $\mathcal{D}^b(X)$. Therefore for any object $\mathcal{F} \in \mathcal{D}^b(X)$, the object $\Phi^{\mathcal{O}_\Delta}(\mathcal{F}) \cong \mathcal{F}$ is classically generated by $\{\Phi^{\mathcal{E}_0}(\mathcal{F}), \dots, \Phi^{\mathcal{E}_k}(\mathcal{F})\}$. As $\mathbf{R}(p_1)_* \circ p_2^*(-) \cong \mathbf{R}\Gamma(-) \otimes \mathcal{O}_X$ [Huy06, page 86] we have

$$\Phi^{\mathcal{E}_t}(\mathcal{F}) \cong \bigoplus_{L_i, L_j \in \mathcal{L}} \mathbf{R}\Gamma(X, \mathcal{F} \otimes L_j^{-s_{j,t}}) \otimes L_i^{r_{i,t}}$$

which is an object in $\langle \bigoplus_{L_i \in \mathcal{L}} L_i^{r_{i,t}} \rangle$ for all $0 \le t \le k$. As $\bigoplus_{L_i \in \mathcal{L}} L_i^{r_{i,t}} \in \langle \mathcal{L} \rangle$ for all $0 \le t \le k$, $\mathcal{L}$ classically generates $\mathcal{D}^b(X)$. □

In order to find a resolution of the diagonal sheaf as in Proposition 6.1, we first recall the approach taken by King [Kin97]. For the locally free sheaf $\mathcal{T} = \bigoplus_{L \in \mathcal{L}} L^{-1}$ on X such that $\mathrm{Hom}^i_X(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$, define $A := \mathrm{End}(\mathcal{T})$ and $\mathcal{T}^\vee := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{O}_X)$. Note that

$$p_1^*(\mathcal{T}^\vee) = p_1^*(\bigoplus_{L \in \mathcal{L}} L) = \bigoplus_{L \in \mathcal{L}} p_1^*(L)$$

and

$$p_2^*(\mathcal{T}) = p_2^*(\bigoplus_{L \in \mathcal{L}} L^{-1}) = \bigoplus_{L \in \mathcal{L}} p_2^*(L^{-1}).$$

By Lemma 5.1, A is isomorphic to $\mathbb{C}Q/J$ for some quiver with relations (Q, J). The following gives the final part of a minimal projective A, A-bimodule resolution of A [Kin97].

Lemma 6.2. *Let $A = \mathbb{C}Q/J$ and $\{e_i \mid i \in Q_0\}$ be the indecomposable orthogonal idempotents. The following complex of A, A-bimodules gives the final part of the minimal projective resolution of A.*

$$\bigoplus_{a \in Q_1} Ae_{\boldsymbol{t}(a)} \otimes [a] \otimes e_{\boldsymbol{h}(a)} A \longrightarrow \bigoplus_{i \in Q_0} Ae_i \otimes [i] \otimes e_i A \tag{6.1.1}$$

where $[a]$ and $[i]$ are formal symbols. The map in the sequence is determined by

$$e_{\boldsymbol{t}(a)} \otimes [a] \otimes e_{\boldsymbol{h}(a)} \quad \mapsto \quad a \otimes [\boldsymbol{h}(a)] \otimes e_{\boldsymbol{h}(a)} - e_{\boldsymbol{t}(a)} \otimes [\boldsymbol{t}(a)] \otimes a$$

and the map onto A is $e_i \otimes [i] \otimes e_i \mapsto e_i$.

Given a minimal projective A, A-bimodule resolution $P^\bullet$ of A, define $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T}$ to be the object

$$p_1^*(\mathcal{T}^\vee) \otimes_A P^\bullet \otimes_A p_2^*(\mathcal{T}) \tag{6.1.2}$$

in $\mathcal{D}^b(X \times X)$. Using Lemma 6.2, the final map in this chain complex is the map d_1 from (5.2.1).

Lemma 6.3. *[Kin97, Theorem 1.2] If the cokernel of the map d_1 in the chain complex $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T}$ is $\mathcal{O}_\Delta$, then $\mathcal{T}$ is a classical generator of $\mathcal{D}^b(X)$.*

Although the final part of the minimal projective A, A-bimodule resolution is given by Lemma 6.2, the full resolution is not known in general and so one cannot compute $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T}$. What we do instead is guess what the resolution of A is and then consider the sheafified version of the resolution as a chain complex

$$S^\bullet := 0 \to S_k \to \cdots \to S_2 \to S_1 \overset{d_1}{\to} S_0 \tag{6.1.3}$$

of $\mathrm{Pic}(X \times X)$-graded $S_{X\times X}$-modules, where S_0 is a $S_{X\times X}$ module corresponding to

$$\bigoplus_{i \in Q_0} L_i \boxtimes L_i^{-1}$$

and S_1 is a $S_{X\times X}$ module corresponding to

$$\bigoplus_{a \in Q_1} L_{\mathbf{t}(a)} \boxtimes L_{\mathbf{h}(a)}^{-1}.$$

If the homology groups of the chain complex $S^\bullet$ are zero after saturation by the irrelevant ideal $B_{X\times X}$, we say that $S^\bullet$ is *exact up to saturation by $B_{X\times X}$*, in which case it determines an exact sequence of sheaves on $X \times X$ by (2.1.5).

We guess the construction of $S^\bullet$ by using the concept of the *toric cell complex* introduced by Craw–Quintero-Vélez [CQV12]. This is a combinatorial geometric structure that encodes the minimal projective bimodule resolution for certain classes of algebras; in particular, Calabi-Yau algebras in dimension 3 obtained from consistent (see Definition 8.2) dimer models and abelian skew group algebras. Given a collection $\mathcal{E} = \{E_0, \ldots, E_r\}$ of rank one reflexive sheaves on a Gorenstein affine toric variety Y, the associated *toric algebra* is $\mathrm{End}(\bigoplus_{i=0}^r E_i)$. Craw–Quintero-Vélez state the following conjecture for consistent toric algebras:

Conjecture 6.4. *[CQV12, Conjecture 6.4] Assume that the toric algebra associated to $\mathcal{E}$ is consistent. If the global dimension of the algebra equals the dimension of Y, then the toric cell complex exists and is constructed as in [CQV12], from which the minimal projective bimodule resolution of the toric algebra can be recovered.*

Although the endomorphism algebra of a tilting bundle $\mathcal{T}$ on a toric Fano variety X is not Calabi-Yau, the endomorphism algebra of the pullback $\pi^*(\mathcal{T})$ on the total space $\mathrm{tot}(\omega_X)$ of the canonical bundle is, so we guess the resolution on $\mathrm{tot}(\omega_X)$ and then restrict it to X.

In what follows, we define a combinatorial method to guess the resolution of the diagonal sheaf by $\mathcal{L}$ based on the construction in [CQV12]. Although the calculations are lengthy and tedious, many of the steps can be achieved using a computer algorithm, the results of which are contained in [PN15a].

6.2 The Toric Cell Complex

For a smooth n-dimensional Fano toric variety X, set $Y := \mathrm{tot}(\omega_X)$ to be the total space of the canonical bundle on X. A collection of line bundles $\mathcal{L}$ on X defines a collection of line bundles $\mathcal{L}_Y$ on Y by pulling back along $\mathrm{tot}(\omega_X) \to X$, and the Picard lattice $\mathrm{Pic}(Y)$ is isomorphic to $\mathrm{Pic}(X)$ under this map. Let Q' be the quiver of sections associated to $\mathcal{L}_Y$ and $B = \mathrm{End}(\bigoplus_{L \in \mathcal{L}_Y} L)$. The quiver Q' is cyclic and naturally embeds into $\mathrm{Pic}(Y)_{\mathbb{R}}$. As Y is a toric variety, it has a fan Σ' and we have the exact sequence

$$0 \longrightarrow M' \longrightarrow \mathbb{Z}^{\Sigma'(1)} \xrightarrow{\deg} \mathrm{Pic}(Y) \longrightarrow 0.$$

Definition 6.5. Let Q' be the quiver above. Define $\tilde{Q}'_0$ to be the set $\bigcup_{i \in Q'_0} \deg^{-1}(i) \subset \mathbb{Z}^{\Sigma(1)}$ and for every arrow $a \in Q'_1$ from i to j and each vertex $u \in \deg^{-1}(i)$, define the arrow $\tilde{a}$ in the set $\tilde{Q}'_1$ to be the arrow from u to $u + \mathrm{div}(a) \in \deg^{-1}(j)$. The *covering quiver* $\tilde{Q}'$ is the quiver in $\mathbb{R}^{\Sigma'(1)}$ with vertex set $\tilde{Q}'_0$ and arrow set $\tilde{Q}'_1$.

The embedding $M' \hookrightarrow \mathbb{Z}^{\Sigma'(1)}$ induces a projection $f \colon \mathbb{R}^{\Sigma'(1)} \to M'_{\mathbb{R}} \cong \mathbb{R}^{n+1}$ which restricts to $f|_{\mathbb{Z}^{\Sigma'(1)}} \colon \mathbb{Z}^{\Sigma'(1)} \to \mathbb{R}^{n+1}$. This map fits into the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & \mathbb{Z}^{\Sigma'(1)} & \xrightarrow{\deg} & \mathrm{Pic}(Y) & \longrightarrow & 0 \\
 & & \| & & \big\downarrow f|_{\mathbb{Z}^{\Sigma'(1)}} & & & & \\
0 & \longrightarrow & M' & \longrightarrow & \mathbb{R}^{n+1} & \longrightarrow & \mathbb{T}^{n+1} & \longrightarrow & 0
\end{array}$$

where $\mathbb{T}^{n+1} := \mathbb{R}^{n+1}/M'$ is a real $(n+1)$-torus.

If $\mathcal{L}$ is a full strong exceptional collection, Craw–Quintero-Vélez [CQV12, Conjecture 6.4] conjecture that the image of the arrows $a \in \tilde{Q}'_1$ in $\mathbb{T}^{n+1}$ under the map f decomposes $\mathbb{T}^{n+1}$ into a toric cell complex, comprising of *k-cells* for $0 \leq k \leq n+1$. The minimal B, B-bimodule projective resolution of B that is expected to be encoded by the toric cell complex has maps determined by differentiating k-cells with respect to $(k-1)$-cells, for $1 \leq k \leq n+1$. To any cell η in the toric cell complex, there is a well-defined divisor $\mathrm{div}(\eta)$ and monomial $x^{\mathrm{div}(\eta)} \in S_Y$ associated to it. By considering how the maps determined by cell differentiation produce ring homomorphisms on $S_{Y \times Y}$, we attempt to construct the exact sequence (6.1.3).

An *anticanonical cycle* in Q' is a path p such that $x^{\mathrm{div}(p)} = \prod_{\rho \in \Sigma'(1)} x_\rho$. Following [CQV12], define the superpotential W to be the sum of all anticanonical cycles in Q'; note that this is similar to the superpotential defined in [BSW10] but without the use of signs, the reason for which is given in [CQV12, Section 6.3]. For two paths p and q in Q', the partial left derivative of p with respect to q is

$$\partial_q p := \begin{cases} r & \text{if } p = rq, \\ 0 & \text{otherwise} \end{cases}$$

which can be extended by $\mathbb{C}$-linearity to determine partial derivatives in $\mathbb{C}Q'$. Let

$$\mathcal{P} := \left\{ q \text{ a path in } Q' \;\middle|\; \begin{array}{c} \partial_q W \text{ is the sum of precisely two paths} \\ \text{that share neither initial nor final arrow} \end{array} \right\}$$

and

$$\mathcal{J} := \{(p^+, p^-) \mid p^\pm \in \mathbb{C}Q', \exists q \in \mathcal{P} \text{ such that } \partial_q W = p^+ + p^-\}. \tag{6.2.1}$$

Assume now that the dimension of X is 4. We define the following sets:

$$\Gamma'_0 := Q'_0, \quad \Gamma'_1 := Q'_1, \quad \Gamma'_2 := \mathcal{J}.$$

For

- $(p^+, p^-) \in \Gamma'_2$, define $D_{p^+p^-} := \{p \text{ a path in } Q' \mid p \text{ is a summand in } \partial_{p^+}W \text{ or } \partial_{p^-}W\}$,
- $a \in \Gamma'_1$, define $D_a := \{p \text{ a path in } Q' \mid p \text{ is a summand in } \partial_a W\}$,
- $i \in \Gamma'_0$, define $D_i := \{p \text{ a path in } Q' \mid p \text{ is a summand in } \partial_{e_i} W\}$.

Then let

$$\Gamma'_3 := \{D_{p^+p^-} \mid (p^+, p^-) \in \Gamma'_2\}, \quad \Gamma'_4 := \{D_a \mid a \in \Gamma'_1\}, \quad \Gamma'_5 := \{D_i \mid i \in \Gamma'_0\}.$$

Remark 6.6. A set of paths $P \in \Gamma'_k$ is expected to be the 1-skeleton contained in a k-cell in the toric cell complex for Q', if the toric cell complex exists. The construction of Γ'_3, Γ'_4 and Γ'_5 follow from the conjecture on duality between k-cells and $(n-k)$-cells in [CQV12, Conjecture 6.5]. For brevity we will therefore refer to P as a *k-cell.*

Let $P \in \Gamma'_k$ for $1 \leq k \leq 5$ and $p \in P$ be a path. We define the *head*, *tail* and *label* of P as $\mathbf{h}(P) := \mathbf{h}(p) \in \Gamma'_0$, $\mathbf{t}(P) := \mathbf{t}(p) \in \Gamma'_0$ and $\operatorname{div}(P) := \operatorname{div}(p)$, and note that the definitions do not depend on our choice of p. For $P' \in \Gamma'_{k-i}, P \in \Gamma'_k$, $0 \leq i < k \leq 5$ we write $P' \subset P$ if for every path $p \in P'$, there is a path $q \in P$ such that $p \subset q$. If $P' \in \Gamma'_{k-1}, P \in \Gamma'_k$ and $P' \subset P$, then a path $q \in P$ containing a path $p \in P'$ defines a monomial $\overleftarrow{\partial}_p q := x^{\operatorname{div}(p')} \in \mathbb{C}[x_0, \ldots, x_d] \cong S_Y$ given by the label of the subpath $p' \subset q$ from $\mathbf{t}(P)$ to $\mathbf{t}(P')$, and a monomial $\overrightarrow{\partial}_p q = w^{\operatorname{div}(p'')} \in \mathbb{C}[w_0, \ldots, w_d] \cong S_Y$ given by the label of the subpath $p'' \subset q$ from $\mathbf{h}(P')$ to $\mathbf{h}(P)$. Let $\mathcal{R}_{P',P}$ be the set of equivalence classes

$$\{[(p,q)] \mid (p,q) \in P' \times P, \ p \subset q\}$$

where

$$(p,q) \sim (p',q') \Leftrightarrow \overleftarrow{\partial}_p q = \overleftarrow{\partial}_{p'} q'.$$

As $\operatorname{div}(p) = \operatorname{div}(p') = \operatorname{div}(P')$ and $\operatorname{div}(q) = \operatorname{div}(q') = \operatorname{div}(P)$, it follows that if $(p,q) \sim (p',q')$, then the label of the subpath in q from $\mathbf{t}(q)$ to $\mathbf{h}(p)$ is the same as the label of the subpath in q' from $\mathbf{t}(q')$ to $\mathbf{h}(p')$. Therefore, the label of the subpath in q from $\mathbf{h}(p)$ to $\mathbf{h}(q)$ is the same as the label of the subpath in q' from $\mathbf{h}(p')$ to $\mathbf{h}(q')$ and so

$$\overleftarrow{\partial}_p q = \overleftarrow{\partial}_{p'} q' \Leftrightarrow \overrightarrow{\partial}_p q = \overrightarrow{\partial}_{p'} q'.$$

For $P' \in \Gamma'_{k-1}$ and $P \in \Gamma'_k$, define

$$\partial_{P'} P := \sum_{[(p,q)] \in \mathcal{R}_{P',P}} (\overleftarrow{\partial}_p q, -\overrightarrow{\partial}_p q) \in S_Y \times S_Y \cong S_{Y \times Y}$$

if $P' \subset P$ and 0 otherwise. The definition of $\partial_{P'}P$ does not depend on the choice of representatives for the equivalence classes in $\mathcal{R}_{P',P}$. We now have the maps

$$d'_k := (\partial_{P'}P)_{\left\{\substack{P'\in\Gamma'_{k-1}\\ P\in\Gamma'_k}\right\}} : (S_{Y\times Y})^{\Gamma'_k} \longrightarrow (S_{Y\times Y})^{\Gamma'_{k-1}}, \quad 0 < k \leq 5,$$

$$\mathbf{e}_P \mapsto \bigoplus_{P'\in\Gamma'_{k-1}} (\partial_{P'}P)\mathbf{e}_{P'}.$$

Remark 6.7. The derivatives $\partial_{P'}P$ are defined differently to how they are defined in [CQV12] as a cell P' may 'appear' more than once in P (see Example 6.11). Consequently, the property [CQV12, (4.3)] does not hold and we do not immediately obtain an incidence function ε (see [CQV12, (4.4)]) that determines signs in the differentiations of cells.

The construction above can be restricted to the toric Fano variety X as follows. For any cone $\sigma \in \Sigma_X \subset N_{\mathbb{R}}$, define

$$\sigma' := \mathrm{Cone}((0,1),(u_\rho,1) \mid \rho \in \sigma(1)) \subset N_{\mathbb{R}} \times \mathbb{R}.$$

The fan $\Sigma' \subset N_{\mathbb{R}} \times \mathbb{R}$ that has cones given by σ' for all $\sigma \in \Sigma_X$ is the fan for Y [CLS11, Proposition 7.3.1]. The toric divisors for X are in one-to-one correspondence with the divisors for Y minus the divisor determined by the ray with generator $(0,1)$, which we label ρ_{tot}. Define the subsets $\Gamma_k := \{P \in \Gamma'_k \mid \text{for all } p \in P, x^{\rho_{\mathrm{tot}}} \nmid x^{\mathrm{div}(p)}\} \subset \Gamma'_k$, for $0 \leq k \leq 5$. Then the maps d'_k restrict to

$$d_k := (\partial_{P'}P)_{\left\{\substack{P'\in\Gamma_{k-1}\\ P\in\Gamma_k}\right\}} : (S_{X\times X})^{\Gamma_k} \longrightarrow (S_{X\times X})^{\Gamma_{k-1}}, \quad 0 < k \leq 4, \tag{6.2.2}$$

$$\mathbf{e}_P \mapsto \bigoplus_{P'\in\Gamma_{k-1}} (\partial_{P'}P)\mathbf{e}_{P'}.$$

The $(S_{X\times X})$-modules $(S_{X\times X})^{\Gamma_0}$ and $(S_{X\times X})^{\Gamma_1}$ are graded as follows: for $i \in \Gamma_0$ and $a \in \Gamma_1$, let $S^i_{X\times X} \subset (S_{X\times X})^{\Gamma_0}$ be given by $S_{X\times X}(L_i, L_i^{-1})$ and $S^a_{X\times X} \subset (S_{X\times X})^{\Gamma_1}$ be given by $S_{X\times X}(L_{\mathbf{t}(a)}, L^{-1}_{\mathbf{h}(a)})$. Then $(S_{X\times X})^{\Gamma_0}$ and $(S_{X\times X})^{\Gamma_1}$ correspond to the bundles $\bigoplus_{i\in Q_0} L_i \boxtimes L_i^{-1}$ and $\bigoplus_{a\in Q_1} L_{\mathbf{t}(a)} \boxtimes L^{-1}_{\mathbf{h}(a)}$ respectively, so the map d_1 in (6.2.2) is the map given in (5.2.1). Similarly for $2 \leq k \leq 4$, the modules $(S_{X\times X})^{\Gamma_k}$ are graded so that they correspond to $\bigoplus_{L_i,L_j\in\mathcal{L}} L_i^{r_{i,k}} \boxtimes L_j^{-s_{j,k}}$, for some fixed $r_{i,k}, s_{j,k} \in \mathbb{Z}_{\geq 0}$. We attempt to add signs to the terms $(\partial_{P'}P)\mathbf{e}_{P'}$ in the maps d_k for $2 \leq k \leq 4$ so that we get a $\mathrm{Pic}(X \times X)$-graded chain complex of $S_{X\times X}$-modules

$$0 \longrightarrow (S_{X\times X})^{\Gamma_4} \xrightarrow{d_4} (S_{X\times X})^{\Gamma_3} \xrightarrow{d_3} (S_{X\times X})^{\Gamma_2} \xrightarrow{d_2} (S_{X\times X})^{\Gamma_1} \xrightarrow{d_1} (S_{X\times X})^{\Gamma_0} \tag{6.2.3}$$

In order to show that the chain complex determines an exact sequence of sheaves on $X \times X$, it then needs to be checked that the chain complex is exact up to saturation by the irrelevant ideal $B_{X\times X}$.

Remark 6.8. A similar construction can be obtained for a smooth toric Fano threefold X. In this case, $\mathrm{tot}(\omega_X)$ is of dimension 4, and so the 4-cells in Γ'_4 are given as the dual cells to the 0-cells in Γ'_0, the 3-cells in Γ'_3 are computed from the 1-cells in Γ'_1 and the set of 2-cells is self-dual (see Remark 6.6).

Using the method above on the database of full strong exceptional collections of line bundles in [PN15a], the exact sequence of sheaves $S^\bullet$ from (6.1.3) has been computed for all smooth toric Fano threefolds and 88 of the 124 smooth toric Fano fourfolds [PN15a].

Example 6.9. Given the collection of line bundles $\mathcal{L}$ on $X = \mathscr{H}_1$ from Example 3.3, the quiver of sections Q' for the bundle $\pi^*(\bigoplus_{L\in\mathcal{L}} L)$ on $Y = \mathrm{tot}(\omega_{\mathscr{H}_1})$ is shown in Figure 6.1. Vertices in the quiver with the same label are identified and the monomial for the extra divisor is $x^{\rho_{\mathrm{tot}}} = x_4$. It follows that

$$\begin{aligned} W =& a_1a_4a_7a_8 + a_1a_5a_{10} + a_2a_4a_6a_8 + a_2a_5a_9 + a_3a_6a_{10} \\ &+ a_3a_7a_9 + a_4a_7a_8a_1 + a_4a_6a_8a_2 + \cdots + a_9a_3a_7. \end{aligned}$$

As Y is of dimension 3, the 0-cells are dual to the 3-cells and the 1-cells are dual to the 2-cells, hence

$$\begin{aligned} \Gamma_0' &= \{0,1,2,3\} \\ \Gamma_1' &= \{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}\} \\ \Gamma_2' &= \{(a_4a_7a_8, a_5a_{10}), (a_4a_6a_8, a_5a_9), (a_6a_{10}, a_7a_9), (a_7a_8a_1, a_6a_8a_2), (a_9a_2, a_{10}a_1), \\ &\quad (a_8a_2a_4, a_{10}a_3), (a_8a_1a_4, a_9a_3), (a_1a_4a_7, a_2a_4a_6), (a_2a_5, a_3a_7), (a_1a_5, a_3a_6)\} \\ \Gamma_3' &= \{(a_1a_4a_7a_8,\ a_1a_5a_{10},\ a_2a_4a_6a_8,\ a_2a_5a_9,\ a_3a_6a_{10},\ a_3a_7a_9),\ \ldots, \\ &\quad (a_8a_1a_4a_7,\ a_8a_2a_4a_6,\ a_9a_2a_5,\ a_9a_3a_7, a_{10}a_1a_5,\ a_{10}a_3a_6)\} \end{aligned}$$

Each arrow in $\{a_8, a_9, a_{10}\}$ has a label divisible by $x^{\rho_{\mathrm{tot}}}$ and so the sets Γ_k contain sets of paths that do not traverse any of these arrows. It follows that $|\Gamma_0| = 4,\ |\Gamma_1| = 7,\ |\Gamma_2| = 3,\ |\Gamma_3| = 0$ and

$$d_1 = \begin{bmatrix} -w_0 & -w_2 & -w_1 & 0 & 0 & 0 & 0 \\ x_0 & x_2 & 0 & -w_3 & -w_1 & 0 & 0 \\ 0 & 0 & x_1 & x_3 & 0 & -w_0 & -w_2 \\ 0 & 0 & 0 & 0 & x_1 & x_0 & x_2 \end{bmatrix}$$

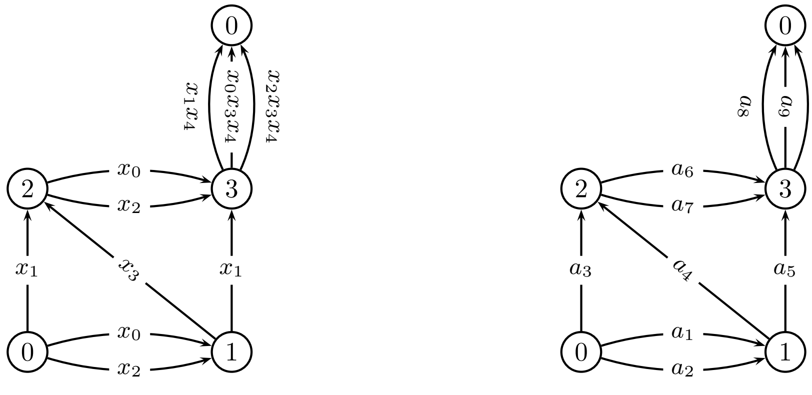
$$d_2 = \begin{bmatrix} -w_2w_3 & 0 & -w_1 \\ w_0w_3 & -w_1 & 0 \\ 0 & w_2 & w_0 \\ x_2w_0 - x_0w_2 & 0 & 0 \\ 0 & -x_2 & -x_0 \\ x_2x_3 & 0 & x_1 \\ -x_0x_3 & x_1 & 0 \end{bmatrix}$$

Note that these maps correspond to those given in [Kin97, Case (ii)]. The resulting chain complex

$$0 \longrightarrow (S_{X\times X})^{\Gamma_2} \xrightarrow{d_2} (S_{X\times X})^{\Gamma_1} \xrightarrow{d_1} (S_{X\times X})^{\Gamma_0}$$

is exact and hence we obtain an exact sequence of sheaves

$$
0 \longrightarrow 3\left(L_0 \boxtimes L_3^{-1}\right) \xrightarrow{d_2} \begin{matrix} 2\left(L_0 \boxtimes L_1^{-1}\right) \\ \oplus \\ L_0 \boxtimes L_2^{-1} \\ \oplus \\ L_1 \boxtimes L_2^{-1} \\ \oplus \\ L_1 \boxtimes L_3^{-1} \\ \oplus \\ 2\left(L_2 \boxtimes L_3^{-1}\right) \end{matrix} \xrightarrow{d_1} \begin{matrix} L_0 \boxtimes L_0^{-1} \\ \oplus \\ L_1 \boxtimes L_1^{-1} \\ \oplus \\ L_2 \boxtimes L_2^{-1} \\ \oplus \\ L_3 \boxtimes L_3^{-1} \end{matrix}
$$



(a) Quiver of sections (b) List of arrows

Figure 6.1: *A quiver of sections on* $\mathrm{tot}(\omega_{\mathscr{H}_1})$

6.3 Method 2

Given a strong exceptional collection $\mathcal{L}$ of line bundles on X with associated quiver Q, we can thus proceed as follows to show that $\mathcal{L}$ generates $\mathcal{D}^b(X)$:

Step 1: Using the method described in Section 6.2, construct the chain complex of $\mathrm{Pic}(X \times X)$-graded $S_{X\times X}$-modules (6.1.3) such that S_t determines the sheaf $\mathcal{E}_t$, where

$$\mathcal{E}_t = \bigoplus_{L_i, L_j \in \mathcal{L}} L_i^{r_{i,t}} \boxtimes L_j^{-s_{j,t}}, \text{ for } 2 \leq t \leq 4 \text{ and some fixed } r_{i,t}, s_{j,t} \in \mathbb{Z}_{\geq 0}.$$

Check that this chain complex is exact up to saturation by $B_{X\times X}$.

Step 2: If $\mathcal{L}$ is a collection of nef line bundles then

- check that the line bundle $L = \bigotimes_{L_i \in \mathcal{L}} L_i$ is ample;
- show that the Minkowski sum of the polytopes $\{P_{L_i} \mid L_i \in \mathcal{L}\}$ is equal to P_L.

Then Propositions 5.6 and 5.4 imply that the exact sequence of sheaves computed in *Step 1* is a resolution of $\mathcal{O}_\Delta$, so $\langle \mathcal{L} \rangle = \mathcal{D}^b(X)$ by Proposition 6.1.
If $\mathcal{L}$ contains non-nef line bundles then

- choose a weight $\theta \in \mathrm{Wt}(Q)$ such that $\mathrm{pic}(\theta)$ is ample and construct Y_θ;
- check that θ is generic (by Lemma 5.7, it is enough to show that the representations corresponding to each torus-invariant point of Y_θ are θ-stable) and confirm that θ and ϑ are in the same open GIT-chamber for Y_θ;
- show $\deg^{-1}(L) \cap \mathbb{N}^{\Sigma(1)} \subset \pi_2\left(\pi_1^{-1}(\theta) \cap \mathbb{N}(Q)\right)$.

Then Propositions 5.8 and 5.4 imply that the exact sequence of sheaves computed in *Step 1* is a resolution of $\mathcal{O}_\Delta$, so $\langle \mathcal{L} \rangle = \mathcal{D}^b(X)$ by Proposition 6.1.

An example of this construction for a collection of nef line bundles on the birationally maximal smooth toric Fano fourfold E_1 is given in Example 6.11, whilst a collection that contains non-nef line bundles on J_1 is shown to be full using this method in Example 6.12.

Example 6.10. Example 6.9 completes Step 1 of ***Method 2*** for the strong exceptional collection $\mathcal{L}$ of line bundles on $\mathscr{H}_1$ given in Example 3.3. Each line bundle is nef and $L = \bigotimes_{L_i \in \mathcal{L}} L_i = \mathcal{O}(4D_1 + 2D_4)$ is ample. For every matrix below, each column is a vertex of the convex polytope in $\mathbb{R}^{\Sigma(1)}$ corresponding to L_i:

$$L_1 : \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; L_2 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \; L_3 : \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Adding the vertices of P_{L_1} to P_{L_2}, the polytope $P_{L_1} + P_{L_2}$ is the convex hull of the columns in

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

This polytope has vertices given by the columns of

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, $P_{L_1} + P_{L_2} + P_{L_3}$ is obtained by taking the convex hull of the sum of the vertices for $P_{L_1} + P_{L_2}$ and P_{L_3}; the resulting polytope has vertices

$$\begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 4 \\ 2 & 0 & 0 & 2 \end{bmatrix}.$$

Note that this polytope is equal to P_L, so Step 2 of ***Method 2*** is complete and hence $\mathcal{L}$ is a full strong exceptional collection for $\mathscr{H}_1$.

Example 6.11. Using the variety X and the collection of line bundles $\mathcal{L}$ in Example 5.3, let $Y = \operatorname{tot}(\omega_X)$ and $\mathcal{L}_Y$ be the corresponding collection of line bundles on Y. The quiver of sections Q' for $\mathcal{L}_Y$ is given in Figure 6.2, where vertices with the same labels are identified. Where the arrows coincide, the labels of Q' are given by the labels of Q in Figure 5.2, whilst the labels of the 7 extra arrows are given in Table 6.1.

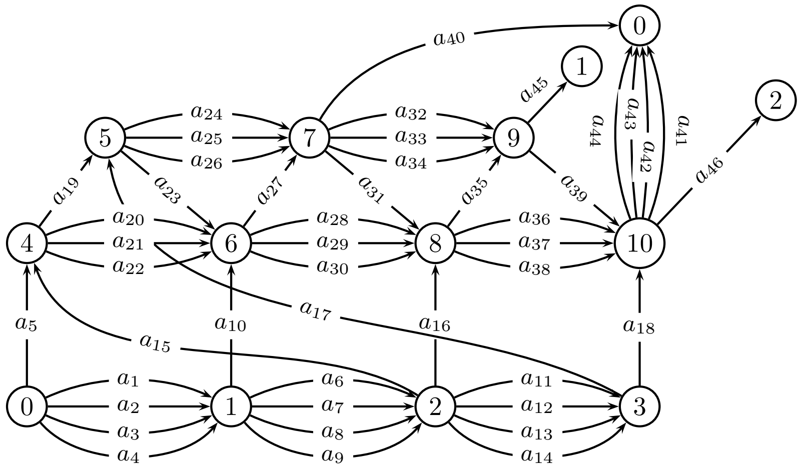


Figure 6.2: *A quiver of sections on* $\operatorname{tot}(\omega_X)$

a	$\mathbf{t}(a), \mathbf{h}(a)$	$\operatorname{div}(a)$
40	7,0	x_4x_7
41	10,0	$x_0x_5x_6x_7$

42	10,0	$x_0x_3x_7$
43	10,0	$x_0x_2x_7$
44	10,0	$x_0x_1x_7$

45	9,1	x_4x_7
46	10,2	$x_4x_6x_7$

Table 6.1: *The additional arrows in a quiver of sections for* $\operatorname{tot}(\omega_X)$

The set $\mathcal{J}$ is

$$\mathcal{J} = \left\{ \begin{array}{c} (a_1a_7, a_2a_6), (a_1a_8, a_3a_6), (a_1a_9, a_4a_6), (a_1a_{10}, a_5a_{20}), (a_{40}a_1, a_{32}a_{45}), \\ (a_{41}a_1, a_{44}a_4), (a_{42}a_1, a_{44}a_3), (a_{43}a_1, a_{44}a_2), (a_2a_8, a_3a_7), (a_2a_9, a_4a_7), \\ \vdots \\ (a_{36}a_{42}, a_{38}a_{44}), (a_{36}a_{43}, a_{37}a_{44}), (a_{37}a_{42}, a_{38}a_{43}) \end{array} \right\}$$

As $|Q'_0| = 11, |Q'_1| = 46$ and $|\mathcal{J}| = 83$, we have $|\Gamma'_0| = |\Gamma'_5| = 11, |\Gamma'_1| = |\Gamma'_4| = 46$ and $|\Gamma'_2| = |\Gamma'_3| = 83$. Note that $P' = a_{17} \in \Gamma'_1$ appears twice in $P = (a_{11}a_{17}a_{25}, a_{12}a_{17}a_{24}) \in \Gamma'_2$ (see Remark 6.7). In this case, $\partial_{P'}P = -(x_1w_2 + x_2w_1)$.

The monomial for the extra divisor in Y is $x^{\rho_{\text{tot}}} = x_7$, so the sets Γ_k are composed of sets of paths that do not contain any of the arrows in $\{a_{40}, a_{41}, \ldots, a_{46}\}$. Via this restriction, we obtain the chain complex of $\operatorname{Pic}(X \times X)$-graded $(S_{X\times X})$-modules

$$0 \to (S_{X\times X})^7 \xrightarrow{d_4} (S_{X\times X})^{31} \xrightarrow{d_3} (S_{X\times X})^{52} \xrightarrow{d_2} (S_{X\times X})^{39} \xrightarrow{d_1} (S_{X\times X})^{11}. \tag{6.3.1}$$

This complex is exact up to saturation by $B_{X\times X}$ [PN15a] and so is an exact sequence of sheaves on $X \times X$.

We now check that the cokernel of this sequence is $\mathcal{O}_\Delta$. For this collection, the line bundle $L = \bigotimes_{L_i\in\mathcal{L}} L_i = \mathcal{O}_X(7D_4 + 15D_5 + 18D_6)$ is ample. Each column of the matrices below is a vertex of the convex polytope in $\mathbb{R}^7$ for the corresponding line bundle in $\mathcal{L}$:

$$\mathcal{O}_X(iD_5+iD_6):\begin{bmatrix}0&0&0&0\\ i&0&0&0\\ 0&i&0&0\\ 0&0&i&0\\ 0&0&0&0\\ 0&0&0&i\\ 0&0&0&i\end{bmatrix},\ \mathcal{O}_X(D_4+iD_5+iD_6):\begin{bmatrix}0&0&0&1&1&1&0&1\\ i&0&0&i{+}2&0&0&0&0\\ 0&i&0&0&i{+}2&0&0&0\\ 0&0&i&0&0&i{+}2&0&0\\ 1&1&1&0&0&0&1&0\\ 0&0&0&1&1&1&i&i{+}3\\ 0&0&0&0&0&0&i&i{+}2\end{bmatrix},$$

$$\mathcal{O}_X(D_4+jD_5+(j+1)D_6):\begin{bmatrix}1&1&1&0&0&0&0&1\\ j{+}3&0&0&j&0&0&0&0\\ 0&j{+}3&0&0&j&0&0&0\\ 0&0&j{+}3&0&0&j&0&0\\ 0&0&0&1&1&1&1&0\\ 0&0&0&0&0&0&j&j{+}3\\ 0&0&0&1&1&1&j{+}1&j{+}3\end{bmatrix},\ i=0,1,2,3,\ j=0,1,2.$$

The vertices for the polytope corresponding to L are

$$\mathcal{O}_X(7D_4+15D_5+18D_6):\begin{bmatrix}3&3&3&7&7&7&0&0&0&0&7\\ 24&0&0&32&0&0&15&0&0&0&0\\ 0&24&0&0&32&0&0&15&0&0&0\\ 0&0&24&0&0&32&0&0&15&0&0\\ 4&4&4&0&0&0&7&7&7&7&0\\ 0&0&0&4&4&4&0&0&0&15&36\\ 0&0&0&0&0&0&3&3&3&18&32\end{bmatrix}.$$

Using *Macaulay2* [GS, PNa] to compute the Minkowski sum of the polytopes $\{P_{L_i} \mid L_i \in \mathcal{L}\}$, we find that it is equal to the polytope corresponding to L and so (6.3.1) is a resolution of $\mathcal{O}_\Delta$ by Propositions 5.6 and 5.4. Therefore, $\mathcal{L}$ is full by Proposition 6.1.

Example 6.12. Let X be the birationally maximal smooth toric Fano fourfold J_1. The primitive generators for the rays of X are

$$u_0=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},u_1=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},u_2=\begin{bmatrix}-1\\-1\\-1\\0\end{bmatrix},u_3=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},u_4=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},u_5=\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix},u_6=\begin{bmatrix}-1\\0\\0\\0\end{bmatrix},u_7=\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

The collection of line bundles on X

$$\mathcal{L}=\left\{\begin{array}{c|c}\mathcal{O}_X((2+i)D_2+(2+j-i)D_5+2D_6+(k+1)D_7), & \\ \mathcal{O}_X((1+i)D_2+(k+i)D_5+(1+j-i)D_6+(1+j-i)D_7), & 1\leq i\leq j\leq 2\\ \mathcal{O}_X(kD_7),\ \mathcal{O}_X(D_2+kD_5+D_6+D_7), & 0\leq k\leq 1\\ \mathcal{O}_X(3D_2+D_5+2D_6+2D_7) & \end{array}\right\}$$

is strong exceptional and contains the non-nef line bundle $\mathcal{O}_X(D_7)$. We obtain the chain complex of $\operatorname{Pic}(X\times X)-$graded $(S_{X\times X})-$modules

$$0\to(S_{X\times X})^{12}\xrightarrow{d_4}(S_{X\times X})^{38}\xrightarrow{d_3}(S_{X\times X})^{59}\xrightarrow{d_2}(S_{X\times X})^{50}\xrightarrow{d_1}(S_{X\times X})^{17}\qquad(6.3.2)$$

from this collection, which is exact up to saturation by $B_{X\times X}$ [PN15a]. The quiver of sections Q corresponding to $\mathcal{L}$ is shown in Figure 6.3, whilst Table 6.2 lists the labels of its arrows.

As $|Q_0|=17$ and $|\Sigma(1)|=8$, we let $\{\mathbf{e}_i \mid i\in Q_0\}\cup\{\mathbf{e}_\rho \mid \rho\in\Sigma(1)\}$ be the standard basis of $\mathbb{Z}^{17+8}$ and define the lattice points $c_a:=\mathbf{e}_{\mathbf{h}(a)}-\mathbf{e}_{\mathbf{t}(a)}+\mathbf{e}_{\operatorname{div}(a)}$ for each arrow $a\in Q_1$. The map π is then given by the matrix $C\colon\mathbb{Z}^{50}\to\mathbb{Z}^{17+8}$, where the columns of C are given by c_a for $a\in Q_1$ and the semigroup $\mathbb{N}(Q)$ is given by the lattice points generated by positive linear combinations of the c_a. Our choice of basis for $\operatorname{Pic}(X)$ and

a	$\mathbf{t}(a), \mathbf{h}(a)$	$\mathrm{div}(a)$
1	0,1	x_7
2	0,2	x_0
3	1,2	x_1x_6
4	1,2	x_2x_6
5	1,3	$x_3x_6x_7$
6	1,4	x_0x_3
7	2,3	x_4
8	2,3	x_5
9	2,4	x_3x_7
10	3,4	x_1
11	3,4	x_2
12	3,5	x_3x_7
13	3,6	x_0
14	4,5	x_4
15	4,5	x_5
16	4,6	x_6x_7
17	4,7	x_3x_7
18	4,9	x_0
19	5,7	x_1
20	5,7	x_2
21	5,8	x_6x_7
22	5,11	x_3x_7
23	5,12	x_0
24	6,8	x_4
25	6,8	x_5
26	6,9	x_1
27	6,9	x_2
28	6,10	x_3
29	7,10	x_6
30	7,11	x_4
31	7,11	x_5
32	7,16	x_0
33	8,12	x_1
34	8,12	x_2
35	8,13	x_3
36	9,12	x_4
37	9,12	x_5
38	9,14	x_3
39	10,12	x_7
40	10,13	x_4
41	10,13	x_5
42	10,14	x_1
43	10,14	x_2
44	11,13	x_6
45	12,15	x_4
46	12,15	x_5
47	12,16	x_1
48	12,16	x_2
49	13,15	x_7
50	14,16	x_7

Table 6.2: *The arrows in a quiver of sections for the smooth toric Fano fourfold* J_1

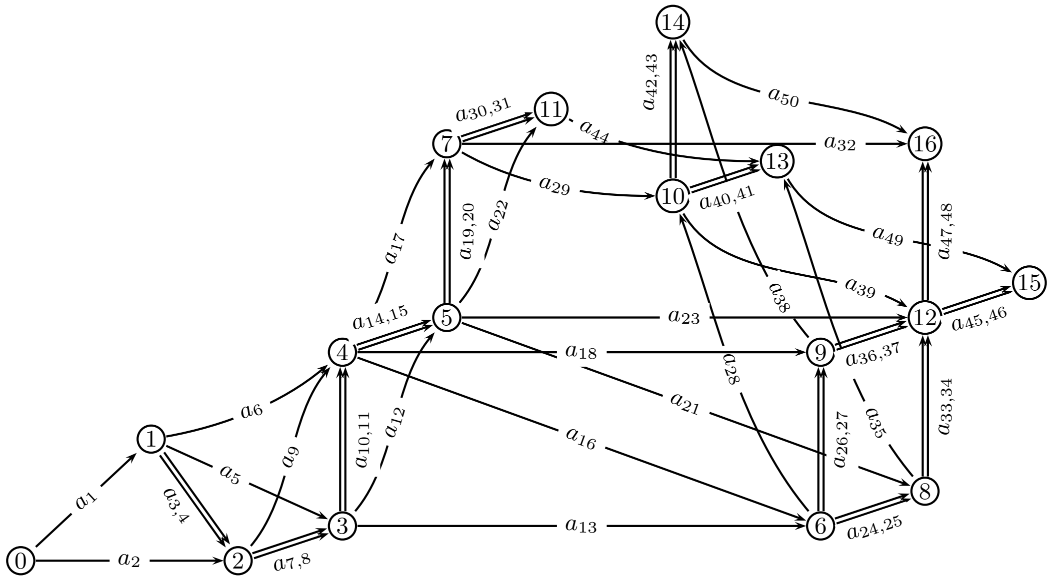
$\mathrm{Wt}(Q)$ imply that the lattice maps deg and pic are given by the matrices:

$$\mathrm{deg}\colon \begin{bmatrix} 1&1&1&1&0&0&0&0 \\ 0&0&0&1&1&1&0&0 \\ 1&0&0&0&0&0&1&0 \\ 1&0&0&-1&0&0&0&1 \end{bmatrix}, \quad \mathrm{pic}\colon \begin{bmatrix} 0&0&1&1&2&2&2&3&2&3&3&3&3&3&4&3&4 \\ 0&0&0&1&1&2&1&2&2&1&2&3&2&3&2&3&2 \\ 0&0&1&1&1&1&2&1&2&2&2&1&2&2&2&2&2 \\ 0&1&1&1&1&1&2&1&2&2&1&1&2&1&1&2&2 \end{bmatrix}$$

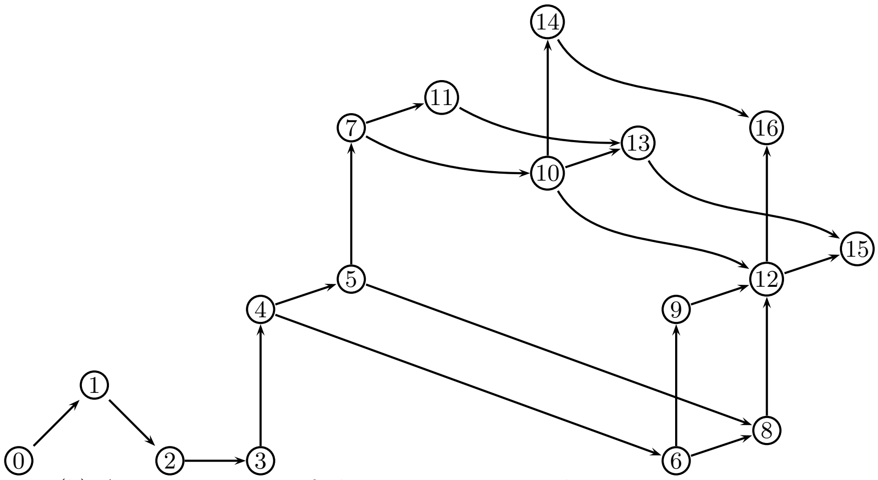
Fix θ to be the weight that assigns -6 to the vertex 0 in the quiver, 1 to the vertices $\{11, 12, \dots, 16\}$ and 0 to every other vertex. We note that $\mathrm{pic}(\theta)$ is the ample line bundle $L = \mathscr{O}_X(20D_2+15D_5+11D_6+9D_7)$. For this choice of θ, $\pi_2\left(\mathbb{N}(Q) \cap (\pi_1)^{-1}(\theta)\right)$ surjects onto $\mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(L)$ [GS, PNa] and so Y_θ is isomorphic to X. Any weight θ' such that $\theta'_i > 0$ for $i > 0$ is generic, for the same reason that the special parameter ϑ is generic; any proper subrepresentation $V' \subseteq V$ of a θ'-semistable representation V has $V'_0 = 0$ as Q is connected and rooted at 0, hence $\theta'(V') > 0$ and so V is θ'-stable. As our choice of weight θ has $\theta_i \geq 0$ for $i > 0$, it is immediate that θ is in the same closed GIT-chamber for the T-action on $\mathbb{V}(I_Q)$ as ϑ and therefore they are in the same open chamber if θ is generic. To check that θ is generic, it is enough to check that for each torus-invariant point on Y_θ, the corresponding representation is θ-stable. Recall that each maximal cone corresponds to a torus-invariant point and that the point is in the intersection of the divisors labelled by the rays of the cone – the list below gives the 17 maximal cones in the fan for Y_θ:

$$\{\rho_0,\rho_1,\rho_3,\rho_4\} \quad \{\rho_0,\rho_1,\rho_3,\rho_5\} \quad \{\rho_0,\rho_1,\rho_4,\rho_5\} \quad \{\rho_0,\rho_2,\rho_3,\rho_4\} \quad \{\rho_0,\rho_2,\rho_3,\rho_5\}$$
$$\{\rho_0,\rho_2,\rho_4,\rho_5\} \quad \{\rho_1,\rho_2,\rho_4,\rho_5\} \quad \{\rho_1,\rho_2,\rho_4,\rho_6\} \quad \{\rho_1,\rho_2,\rho_5,\rho_6\} \quad \{\rho_1,\rho_3,\rho_4,\rho_7\}$$
$$\{\rho_1,\rho_3,\rho_5,\rho_7\} \quad \{\rho_1,\rho_4,\rho_6,\rho_7\} \quad \{\rho_1,\rho_5,\rho_6,\rho_7\} \quad \{\rho_2,\rho_3,\rho_4,\rho_7\} \quad \{\rho_2,\rho_3,\rho_5,\rho_7\}$$
$$\{\rho_2,\rho_4,\rho_6,\rho_7\} \quad \{\rho_2,\rho_5,\rho_6,\rho_7\}$$

For the representation (V, ϕ) corresponding to the torus-invariant point with rays $\{\rho_{i_1}, \rho_{i_2}, \rho_{i_3}, \rho_{i_4}\}$, the map ϕ_a is 0 if for any $x_i \in \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$, x_i divides $\mathrm{div}(a)$, whilst $\phi_a = 1$ otherwise. For example, consider the maximal cone $\{\rho_0, \rho_1, \rho_3, \rho_4\}$. The



(a) The quiver of sections



(b) A representation of the quiver corresponding to a torus-invariant point in Y_θ

Figure 6.3: *A quiver of sections on the smooth toric Fano fourfold* J_1

corresponding representation $V = (V, \phi)$ has $\phi_a = 0$ for

$$a \in \left\{ \begin{array}{c} a_2,\ a_3,\ a_5,\ a_6,\ a_7,\ a_9,\ a_{10},\ a_{12},\ a_{13},\ a_{14},\ a_{17},\ a_{18},\ a_{19},\ a_{22},\ a_{23},\ a_{24},\ a_{26}, \\ a_{28},\ a_{30},\ a_{32},\ a_{33},\ a_{35},\ a_{36},\ a_{38},\ a_{40},\ a_{42},\ a_{45},\ a_{47} \end{array} \right\}$$

and is displayed in Figure 6.3. Specifying a subrepresentation (V', ϕ') of V is equivalent to setting $\phi'_a = \phi_a$ for all $a \in Q_1$ and choosing a subset $I \subset Q_0$ so that $V'_i = \mathbb{C}$ for $i \in I$, and $V'_i = 0$ otherwise. In our example, for any subrepresentation V' with $V'_0 = \mathbb{C}$, we have $V' = V$ as there is a non-zero map from V'_0 to every other V'_i. It is also clear from Figure 6.3 that for any $i \in Q_0$, there is a non zero map from V_i to V_j for some $j \in \{11, 12, \ldots, 16\}$. As a result, the corresponding nonzero proper subrepresentation V' of V must have $V'_j = \mathbb{C}$ and so $\theta(V') > 0$ by the choice of θ. By considering the subrepresentations of the representation corresponding to each of the 17 torus-invariant points on Y_θ, we see that θ is generic – the calculations for this example can be found in the file [PNa]. Therefore, (6.3.2) is a resolution of $\mathcal{O}_\Delta$ by Propositions 5.8 and 5.4, so the collection $\mathcal{L}$ on J_1 is full by Proposition 6.1.

CHAPTER
SEVEN

FULL STRONG EXCEPTIONAL COLLECTIONS ON TORIC VARIETIES

In this chapter we present the main theorems of this thesis: we construct full strong exceptional collections of line bundles on all smooth toric Fano fourfolds X and then show that they pull back to give tilting bundles on $\text{tot}(\omega_X)$.

We achieve the first result by choosing collections $\mathcal{L}$ from a certain set of line bundles on X so that the pushforward of $\mathcal{L}$ onto a torus-invariant divisorial contraction X_1 is automatically full if $\mathcal{L}$ is full, and the pushforward coincides with the image of $\mathcal{L}$ under the Picard map $\gamma\colon \text{Pic}(X) \to \text{Pic}(X_1)$ (see Proposition 7.2 and Lemma 7.1). We then check that the collection on X_1 is strong exceptional computationally by utilising the preimage of the *nnnvc*-cones for X_1 under γ as outlined in Chapter 3; these computations are achieved using the package *QuiversToricVarieties*. Using this process, we obtain full strong exceptional collections on many of the toric Fano fourfolds from the pushforward of collections on the birationally maximal examples (see Theorem 7.4 and Table B.3), with the rest constructed individually. To obtain tilting bundles on $\text{tot}(\omega_X)$, we show that the pullback of a full strong exceptional collection of line bundles is tilting on $\text{tot}(\omega_X)$ if it satisfies a computational condition on the vanishing of the higher cohomology of certain line bundles on X (see Theorem 7.7); again, this computation is performed in *QuiversToricVarieties*.

7.1 Tilting Bundles Comprising of Line Bundles

For a divisorial contraction $(f,\phi)\colon (X_0,\Sigma_{X_0}) \to (X_1,\Sigma_{X_1})$, the Frobenius morphism can be used to find examples of when the pushforward of a line bundle L via f and the image of L under the map γ from (2.1.8) are equal. Recall that for a toric variety X, the canonical bundle is $\omega_X = -\sum_{\rho\in\Sigma_X(1)} D_\rho$.

Lemma 7.1. *Fix an integer $m > 0$ and let $(f,\phi)\colon (X_0,\Sigma_{X_0}) \to (X_1,\Sigma_{X_1})$ be a torus-equivariant extremal birational contraction between smooth n-dimensional projective toric varieties. Let $\sigma \subset \Sigma_{X_0}$ be a maximal cone such that $\phi(\sigma)$ is a cone in Σ_{X_1}, and $\boldsymbol{w} = \boldsymbol{0}$ or $\boldsymbol{w} = (-1,\dots,-1) \in \mathbb{Z}^{\Sigma_{X_0}(1)}$. Then for any $\boldsymbol{v} \in P^n_m$,*

$$f_*\mathcal{O}_{X_0}(D^{X_0}_{\boldsymbol{v},\boldsymbol{w},\sigma}) = \mathcal{O}_{X_1}(D^{X_1}_{\boldsymbol{v},\boldsymbol{w},\phi(\sigma)}) \tag{7.1.1}$$

and

$$\mathfrak{D}(\mathcal{O}_{X_1})_m = \{f_* L_{X_0} \mid L_{X_0} \in \mathfrak{D}(\mathcal{O}_{X_0})_m\}, \tag{7.1.2}$$

$$\mathfrak{D}(\omega_{X_1})_m = \{f_* L_{X_0} \mid L_{X_0} \in \mathfrak{D}(\omega_{X_0})_m\}. \tag{7.1.3}$$

In particular, the maps f_* and γ coincide for $\mathcal{O}_{X_0}(D^{X_0}_{\mathbf{v},\mathbf{w},\sigma})$.

Proof. The result [Ueh14, Lemma 6.1] gives the case $\mathbf{w} = \mathbf{0}$. Noting that $f_*(\omega_{X_0}) \cong \omega_{X_1}$ [CLS11, Theorem 9.3.12], the proof can also be applied to $\mathbf{w} = (-1, \dots, -1)$. The algorithm to compute $F_m(L)$ demonstrates the equality between f_* and γ for the line bundles considered. □

Proposition 7.2. *With the same assumptions as in Lemma 7.1, choose a collection of line bundles $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_{X_0})_m$. If $\mathcal{L}$ generates $\mathcal{D}^b(X_0)$ then the line bundles in the image of $\gamma(\mathcal{L})$ generate $\mathcal{D}^b(X_1)$.*

Proof. Note that $\mathbb{R}f_*\mathcal{O}_{X_0} = \mathcal{O}_{X_1}$ and $\mathbb{R}f_*\omega_{X_0} = \omega_{X_1}$ [CLS11, Theorem 9.3.12]. We have the equality $F^{X_1}_m \circ f = f \circ F^{X_0}_m$; indeed, consider the morphism of affine varieties $f\colon \operatorname{Spec}\mathbb{C}[S_\sigma] \to \operatorname{Spec}\mathbb{C}[S_{\sigma'}]$ for two cones σ and σ'. A $\mathbb{C}[S_\sigma]$-module N becomes a $\mathbb{C}[S_{\sigma'}]$-module via the operation $\mathbf{x}^\mathbf{u} \cdot n := f^\#(\mathbf{x}^\mathbf{u}) \cdot n$ for $\mathbf{x}^\mathbf{u} \in \mathbb{C}[S_{\sigma'}]$ and $n \in N$. It follows that

$$(f^\# \circ (F^{U_{\sigma'}}_m)^\#)(\mathbf{x}^\mathbf{u}) \cdot n = f^\#(\mathbf{x}^{m\mathbf{u}}) \cdot n = (f^\#(\mathbf{x}^\mathbf{u}))^m \cdot n = ((F^{U_\sigma}_m)^\# \circ f^\#)(\mathbf{x}^\mathbf{u}) \cdot n$$

and this globalises to give $F^{X_1}_m \circ f = f \circ F^{X_0}_m$.

Consequently, $\mathbb{R}f_*(F^{X_0}_m)_*\mathcal{O}_{X_0} = (F^{X_1}_m)_*\mathcal{O}_{X_1}$ and $\mathbb{R}f_*(F^{X_0}_m)_*\omega_{X_0} = (F^{X_1}_m)_*\omega_{X_1}$. The result then follows by Lemmas 4.4 and 7.1. □

Remark 7.3. A collection of line bundles $\mathcal{L}$ on X is full strong exceptional if and only if the dual collection $\mathcal{L}^{-1} := \{L^{-1} \mid L \in \mathcal{L}\}$ is full strong exceptional. In the following theorem when we choose $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$ and use ***Method 2*** to show that $\mathcal{L}$ is full, we actually compute the $S_{X\times X}$-module chain complex using $\mathcal{L}^{-1}$, as $\mathcal{L}^{-1}$ will be an effective collection whilst $\mathcal{L}$ will not be effective.

Theorem 7.4. *Let X be a smooth toric Fano fourfold. There exists a full strong exceptional collection comprising of line bundles for X. A database of these collections can be found in [PN15a].*

Proof. The algorithm to construct full strong exceptional collections of line bundles on the smooth toric Fano fourfolds works in conjunction with Table B.3 and is as follows:

Step 1: For the fourfolds that are products of smooth toric Fano varieties of a lower dimension, a full strong exceptional collection of line bundles is provided by Lemma 2.24 and [Kin97, Ueh14]. This accounts for 28 of the 124 fourfolds. Beĭlinson's collection $\{\mathcal{O}_{\mathbb{P}^4}, \mathcal{O}_{\mathbb{P}^4}(1), \dots, \mathcal{O}_{\mathbb{P}^4}(4)\}$ provides a full strong exceptional collection for $\mathbb{P}^4$ [Beĭ78].

Step 2: List every fourfold that does not have a full strong strong exceptional collection constructed and is either

- birationally maximal, or

– blows up once to a fourfold that has a full strong exceptional collection constructed.

Step 3: Set $m = 10$ and let X be a fourfold in the list created in Step 2. The collection $\mathcal{L}_{\text{nef}} := \{L \in \mathfrak{D}_m \mid L^{-1} \text{ is nef}\} \subseteq \mathfrak{D}_m$ is strong exceptional by Lemma 4.2. For each X, check whether $|\mathcal{L}_{\text{nef}}|$ is equal to the number $|\Sigma(4)|$ of maximal cones in the fan for X; if this is the case, then $\mathcal{L}_{\text{nef}}$ is a candidate to be a full strong exceptional collection. If $|\mathcal{L}_{\text{nef}}| < |\Sigma(4)|$ then perform a computer search using the implementation of the *nnnvc*-cones in *QuiversToricVarieties* [PN15a, GS] to find a strong exceptional collection $\mathcal{L} \subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$ such that $|\mathcal{L}| = |\Sigma(4)|$, with preference for collections such that $\mathcal{L}^{-1}$ is a nef collection. If no such collection can be found, continue to search for a strong exceptional collection $\mathcal{L}$ not contained in $\mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$ but with $|\mathcal{L}| = |\Sigma(4)|$. The program *QuiversToricVarieties* [PN15a] contains a database of these collections, whilst Table B.3 states how each collection was found.

Step 4: Attempt to construct the sequence (6.2.3) for $\mathcal{L}$ as outlined in the first step of ***Method 2***; if this is possible, then continue with the steps in ***Method 2*** to show that $\mathcal{L}$ is full (see Chapter 6). If the sequence (6.2.3) cannot be constructed then use ***Method 1*** (see Proposition 4.5 and Example 4.7) to show that $\mathcal{L}$ is full. The result of this step is that we have now constructed a full strong exceptional collection of line bundles for every fourfold listed in Step 2.

Step 5: If the chosen collection $\mathcal{L}$ on X is contained in $\mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$ then construct the Picard lattice maps (3.2.1) from each chain of divisorial contractions $X_0 := X \to X_1 \to \cdots \to X_t$ (see Figure B.1). Check that $\{L_i \otimes L_j^{-1} \mid L_i, L_j \in \mathcal{L}\}$ avoids the preimages $\tilde{\Lambda}_{I,X_k}$ of the *nnnvc*-cones for $1 \leq k \leq t$ and all forbidden sets I for X_k, as explained in Chapter 3. This process, together with the efficient construction of the preimages $\tilde{\Lambda}_{I,X_k}$ given in Proposition 3.13, is implemented in *QuiversToricVarieties* [PN15a, GS]. If $\{L_i \otimes L_j^{-1} \mid L_i, L_j \in \mathcal{L}\}$ satisfies this condition, then the collection of line bundles $\mathcal{L}_{X_k}$ is full strong exceptional for each $1 \leq k \leq t$ by Lemma 3.4 and Proposition 7.2. Table B.3 details the full strong exceptional collections obtained in this way.

Step 6: If all the smooth toric Fano fourfolds have a full strong exceptional collection of line bundles constructed, then the algorithm finishes; otherwise, return to Step 2.

The algorithm stops after two iterations. In the first iteration, Step 1 determines full strong exceptional collections for the 28 fourfolds that are products of lower dimensional smooth toric Fano varieties, as well as for $\mathbb{P}^4$. Step 2 then lists the 26 birationally maximal fourfolds that do not arise as products of lower dimensional smooth toric Fano varieties. Of these, 21 have a collection $\mathcal{L}$ chosen from $\mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$, whilst for the birationally maximal variety R_3 we construct a full strong exceptional collection contained in $\mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$ from a second collection as detailed in Example 7.6. Examples 6.12 and C.2 provide more details on the construction of the resolution of the diagonal sheaf using the non-nef collections for J_1 and $\tilde{V}^4$ respectively. We then obtain full strong exceptional collections for 64 of the fourfolds via Step 5.

In the second iteration, Step 2 lists the 4 non-birationally maximal fourfolds H_{10}, M_1, M_2 and M_3. Steps 3 and 4 construct full strong exceptional collections for these varieties, with Example C.1 providing more details on the construction of the resolution of the diagonal sheaf using the non-nef collection for M_1. The only remaining fourfold without a full strong exceptional collection constructed is D_{16}, but there is a divisorial contraction from H_{10} to D_{16} and the collection for H_{10} is chosen from $\mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Hence, after completing Step 5 we obtain a full strong exceptional collection for D_{16} from the collection on H_{10} and so the algorithm terminates in Step 6.

The calculations in *Macaulay2* [GS] and *Sage* [S$^+$15] that are required for this proof can be found in the file [PNa]. □

Remark 7.5. The algorithm in the proof above can be adapted to provide a new proof that there exist full strong exceptional collections on n-dimensional smooth toric Fano varieties for $n \leq 3$. In particular, a resolution of $\mathcal{O}_\Delta$ using the line bundles in a collection $\mathcal{L}$ has been constructed for each birationally maximal Fano threefold X.

Example 7.6. The birationally maximal smooth toric Fano fourfold $X := R_3$ has ray generators

$$u_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, u_1 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, u_2 = \begin{bmatrix}0\\-1\\1\\-1\end{bmatrix}, u_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, u_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, u_5 = \begin{bmatrix}0\\0\\-1\\0\end{bmatrix}, u_6 = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix},$$

$$u_7 = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, u_8 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}$$

for its fan Σ_X. We take the corresponding divisors $\{[D_2], [D_5], [D_6], [D_7], [D_8]\}$ to be a basis for $\mathrm{Pic}(X)$. The collection of nef line bundles on X

$$\mathcal{L} = \left\{ \begin{array}{c} \mathcal{O}_X(iD_2 + jD_5 + D_6 + D_7 + D_8), \\ \mathcal{O}_X((j-1)D_5 + (i-1)D_7 + (i-1)D_8), \\ \mathcal{O}_X((i-1)D_2 + 2D_5 + jD_7 + jD_8),\ \mathcal{O}_X(D_2 + D_5 + iD_7 + jD_8), \\ \mathcal{O}_X(D_2 + 2D_5 + (i-1)D_6 + D_7 + jD_8), \\ \mathcal{O}_X(2D_2 + iD_5 + jD_7 + 2D_8), \\ \mathcal{O}_X(2D_2 + (j-i+1)D_5 + (1-j+i)D_6 + D_7 + jD_8) \end{array} \middle| \ 1 \leq i \leq j \leq 2 \right\}$$

is strong exceptional and is shown to be full using ***Method 2***, but $\mathcal{L}^{-1} \not\subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. However, we can use $\mathcal{L}$ to construct the helix $\mathbb{H}_\mathcal{L}$ for X and by Remark 2.28, each thread of $\mathbb{H}_\mathcal{L}$ is a full exceptional collection. We can therefore choose a thread in $\mathbb{H}_\mathcal{L}$ and twist it by the line bundle $\mathcal{O}_X(-D_5 - D_7 - D_8)$ to obtain the following full strong exceptional collection:

$$\mathcal{L}' = \left\{ \begin{array}{c} \mathcal{O}_X(jD_5 + iD_7 + iD_8),\ \mathcal{O}_X(D_2 + iD_7 + jD_8), \\ \mathcal{O}_X(D_2 + jD_5 + D_6 + iD_8),\ \mathcal{O}_X(D_2 + D_5 + iD_7 + jD_8), \\ \mathcal{O}_X(2D_2 + iD_5 + jD_7 + D_8),\ \mathcal{O}_X(2D_2 + iD_5 + D_6 + jD_8), \\ \mathcal{O}_X(2D_2 + (i+1)D_5 + jD_6 + (i-j+1)D_8) \end{array} \middle| \ 0 \leq i \leq j \leq 1 \right\}$$

Now $(\mathcal{L}')^{-1}$ is a non-nef collection that is contained in $\mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$ for some $m > 0$, which we can use to obtain a full strong exceptional collection on the divisorial contraction M_4 via the method outlined in the proof of Theorem 7.4.

The full strong exceptional collections on each smooth toric Fano variety X determine tilting bundles on the total space of ω_X.

Theorem 7.7. *Let $Y = tot(w_X)$ be the total space of the canonical bundle on an n-dimensional smooth toric Fano variety X, for $n \leq 4$. Then Y has a tilting bundle that decomposes as a direct sum of line bundles.*

Proof. Let $\pi : Y \to X$ be the bundle map and $\mathcal{L} = \{L_0, \ldots, L_r\}$ be a full strong exceptional collection of line bundles on X from Theorem 7.4, [Ueh14] or [Kin97]. The collection defines a helix

$$\mathbb{H}_{\mathcal{L}} = (\ldots,\ L_0 \otimes \omega_X, \ldots,\ L_r \otimes \omega_X,\ L_0, \ldots,\ L_r,\ L_0 \otimes \omega_X^{-1}, \ldots,\ L_r \otimes \omega_X^{-1}, \ldots).$$

By Lemma 2.29, the pullback $\pi^*(E)$ of the bundle $E := \bigoplus_{i=0}^r L_i$ is a tilting bundle on Y if $\mathbb{H}_{\mathcal{L}}$ is geometric, which in this case is the condition that

$$\mathrm{Hom}^k(L_i \otimes \omega_X^{t_1}, L_j \otimes \omega_X^{t_2}) = 0 \text{ unless } k = 0$$

for $0 \leq i, j \leq r$ and $t_1 \geq t_2$. This is equivalent to the condition that

$$H^k(X, L_i^{-1} \otimes L_j \otimes \omega_X^{-t}) = 0 \text{ unless } k = 0$$

for $0 \leq i, j \leq r$ and $t \geq 0$. As ω_X^{-1} is ample, there is some positive integer T such that for all $t \geq T$, $L_i^{-1} \otimes L_j \otimes \omega_X^{-t}$ is nef for all $0 \leq i, j \leq r$, in which case $H^k(X, L_i^{-1} \otimes L_j \otimes \omega_X^{-t}) = 0$ for $k > 0$ by Demazure vanishing [CLS11, Theorem 9.2.3]. Hence $\pi^*(E)$ is a tilting bundle if

$$H^k(X, L_i^{-1} \otimes L_j \otimes \omega_X^{-t}) = 0 \text{ for } k \neq 0,\ 0 \leq i, j \leq r,\ 0 \leq t < T. \qquad (7.1.4)$$

The *nnnvc*-cones in $\mathrm{Pic}(X)$ can be used to show that the line bundle $L_i^{-1} \otimes L_j \otimes \omega_X^{-t}$ has vanishing higher cohomology for $0 \leq i, j \leq r,\ 0 \leq t < T$, as implemented in *QuiversToricVarieties* [PN15a].

Now let $X =: X_0 \to X_1 \to \cdots X_d$ be a chain of divisorial contractions between smooth toric Fano varieties and assume that the collection $\mathcal{L}$ determines a full strong exceptional collection on each $X_k,\ 0 \leq k \leq d$ via the divisorial contractions, as detailed in Theorem 7.4. For each variety X_k with collection $\mathcal{L}_{X_k}$, we have an integer $T_k \geq 0$ such that $L_i^{-1} \otimes L_j \otimes \omega_{X_k}^{-T_k}$ is nef for all $L_i, L_j \in \mathcal{L}_{X_k}$. Define $T = \max(T_0, \ldots, T_d)$. Then we can check simultaneously that each $\mathcal{L}_{X_k}$ determines a tilting bundle on $\mathrm{tot}(\omega_{X_k})$ by considering whether the line bundles $L_i^{-1} \otimes L_j \otimes \omega_X^{-t}$ avoid the preimage in $\mathrm{Pic}(X)_{\mathbb{R}}$ of all *nnnvc*-cones for $X_1, \ldots, X_d$ via the Picard lattice maps, for all $0 \leq i, j \leq r,\ 0 \leq t < T$. Again, this calculation can be performed in *QuiversToricVarieties* [PN15a]. ☐

Example 7.8. The full strong exceptional collection $\mathcal{L}$ on $\mathscr{H}_1$ given in Example 2.23 determines the helix

$$\mathbb{H} = (\ldots, \mathcal{O}(-D_1 - D_4),\ \mathcal{O},\ \mathcal{O}(D_1),\ \mathcal{O}(D_1 + D_4),\ \mathcal{O}(2D_1 + D_4),\ \mathcal{O}(3D_1 + 2D_4), \ldots).$$

The line bundles $L_s^{-1} \otimes L_t$ for $L_s, L_t \in \mathcal{L}$ are

$$\left\{ \begin{array}{c} \mathcal{O},\ \mathcal{O}(D_1),\ \mathcal{O}(D_1+D_4),\ \mathcal{O}(2D_1+D_4),\ \mathcal{O}(D_4), \\ \mathcal{O}(-D_1),\ \mathcal{O}(-D_4),\ \mathcal{O}(-D_1-D_4),\ \mathcal{O}(-2D_1-D_4) \end{array} \right\}$$

and Figure 3.1 shows them with the *nnnvc*-cones in $\mathrm{Pic}(\mathscr{H}_1)$. Using Figures 2.4 and 3.1, we see that $L_s^{-1} \otimes L_t \otimes \omega_X^{-1}$ is nef for all $L_s, L_t \in \mathcal{L}$; hence, following the proof of Theorem 7.7, it is enough to show that each $L_s^{-1} \otimes L_t$ avoids the *nnnvc*-cones in $\mathrm{Pic}(\mathscr{H}_1)$ for $\mathbb{H}$ to be geometric. This immediately follows as $\mathcal{L}$ is a full strong exceptional collection, so $\mathbb{H}$ is geometric and $\pi^*(\mathcal{O} \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_1+D_4) \oplus \mathcal{O}(2D_1+D_4))$ is a tilting bundle on $\mathrm{tot}(\omega_{\mathscr{H}_1})$.

Remark 7.9. Using the full strong exceptional collections of line bundles given by King [Kin97], Uehara [Ueh14] and Theorem 7.4, we find that the minimal T such that $L_i^{-1} \otimes L_j \otimes \omega_X^{-T}$ is nef for all smooth toric Fano n-folds and all collections $\mathcal{L}$ on the n-folds is $n-1$, for $n \leq 4$. By Proposition 2.26, the endomorphism algebra of the tilting bundle obtained on $\mathrm{tot}(\omega_X)$ from $\mathcal{L}$ is $\mathrm{CY}(n+1)$.

CHAPTER

EIGHT

FUTURE DIRECTIONS

This thesis concludes with a problem that arises when using ***Method 2*** to show that a strong exceptional collection of line bundles $\mathcal{L}$ on a smooth toric Fano variety X is full.

Chapter 6 describes a method to construct a resolution of the diagonal sheaf $\mathcal{O}_\Delta$ from $\mathcal{L}$. The construction is based on concept of the toric cell complex, introduced by Craw–Quintero-Vélez [CQV12]. They proved that the toric cell complex exists for Calabi-Yau algebras in dimension 3 obtained from consistent dimer models and abelian skew group algebras, and that it encodes the minimal projective bimodule resolution of the algebra in these cases. Additionally, they conjecture that the toric cell complex exists for consistent toric algebras obtained from varieties in higher dimensions - see Conjecture 6.4. Using the method outlined in Chapter 6 and for the database of full strong exceptional collections of line bundles in [PN15a], the exact sequence of sheaves $S^\bullet$ from (6.1.3) has been computed for all smooth toric Fano threefolds and 88 of the 124 smooth toric Fano fourfolds. These exact sequences are contained in a database in [PN15a], and the fact that they can be computed leads us to pose the following conjecture:

Conjecture 8.1. *Let X be a smooth toric Fano threefold or one of the* 88 *smooth toric Fano fourfolds such that the given full strong exceptional collection $\mathcal{L}$ in the database [PN15a] has a corresponding exact sequence of sheaves $S^\bullet \in \mathcal{D}^b(X \times X)$. Let B denote the rolled up helix algebra of $A = \mathrm{End}(\bigoplus_{L\in\mathcal{L}} L^{-1})$. Then the toric cell complex of B exists and is supported on a real four or five-dimensional torus respectively. Moreover,*

- *the cellular resolution exists in the sense of [CQV12], thereby producing the minimal projective bimodule resolution of B;*
- *the object $S^\bullet$ is quasi-isomorphic to $\mathcal{T}^\vee \overset{\mathbf{L}}{\boxtimes}_A \mathcal{T} \in \mathcal{D}^b(X \times X)$, for $\mathcal{T} := \bigoplus_{L\in\mathcal{L}} L^{-1}$.*

It is interesting to note that the method fails to generate a resolution of $\mathcal{O}_\Delta$ for 36 of the 124 smooth toric Fano fourfolds. An example of a problem that arises in these cases is given in the next section.

8.1 The Full Strong Exceptional Collection on D_1

We continue with the notation introduced in Chapter 5 and 6, where:

- (Q, J) was the quiver of sections encoding the endomorphism algebra of the bundle $\bigoplus_{L\in\mathcal{L}} L$ on X;
- (Q', J') was the quiver of sections encoding the endomorphism algebra of the pullback $\pi^*(\bigoplus_{L\in\mathcal{L}} L)$ of this bundle to $\mathrm{tot}(\omega_X)$;
- W was the superpotential given by the sum of the anticanonical cycles in Q'.

The set $\mathcal{J}$ from (6.2.1) was defined by Craw–Quintero-Velez [CQV12], who use it to generate the *ideal of superpotential relations* J_W, an ideal in $\mathbb{C}Q'$. They then generalise the notion of consistency from dimer models as follows.

Definition 8.2. The algebra $\mathbb{C}Q'/J' \cong \mathrm{End}(\pi^*(\bigoplus_{L\in\mathcal{L}} L))$ is *consistent* if J' and J_W coincide.

In the following example, we see that the set $\mathcal{J}$ does not contain enough relations to generate J'.

Example 8.3. The smooth toric Fano fourfold $X := D_1$ has primitive ray generators

$$u_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, u_1 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, u_2 = \begin{bmatrix}-1\\-1\\0\\2\end{bmatrix}, u_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, u_4 = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}, u_5 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, u_6 = \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$$

and a full strong exceptional collection

$$\mathcal{L} = \{\mathcal{O}_X(iD_0 + jD_3 + kD_6) \mid 0 \le i \le 2,\ 0 \le j, k \le 1\}$$

by Theorem 7.4. Table 8.1 lists the arrows in the quiver Q' that encodes the algebra $\mathrm{End}(\pi^*(\bigoplus_{L\in\mathcal{L}} L))$. The set of relations $\mathcal{J}$ is given in Figure 8.1. The relation $(a_{24}a_{26}a_{34}a_{44}, a_{25}a_{50})$, or alternatively $(a_{24}a_{29}a_{36}a_{44}, a_{25}a_{50})$, is an example of a relation in the ideal J' that cannot be generated by the relations in $\mathcal{J}$, and so the sequence of sheaves (6.1.3) cannot be computed from the method outlined in Chapter 6. Note that if $p = a_2a_9a_{18}$, these relations appear in the derivative of the superpotential W as

$$\partial_p W = a_{24}a_{26}a_{34}a_{44} + a_{24}a_{29}a_{36}a_{44} + a_{25}a_{50}.$$

If we extend p to be the path $p' = a_{44}a_2a_9a_{18}a_{24}$, then we obtain the minimal relation $(a_{26}a_{34}, a_{29}a_{36}) \in \mathcal{J}$ from $\partial_{p'}W$.

As Example 8.3 shows, an important step in extending the toric cell complex to higher dimensions is to define a set that fully contains the minimal generators of the ideal J'.

Besides the resolutions of $\mathcal{O}_\Delta$ for the 88 smooth toric Fano fourfolds and all of the smooth toric Fano threefolds, the smooth toric Fano fivefolds provide us with further evidence that the toric cell complex exists in higher dimensions. There are 866 smooth toric Fano fivefolds, of which 288 are birationally maximal. Of these

a	$\mathbf{t}(a), \mathbf{h}(a)$	$\mathrm{div}(a)$	a	$\mathbf{t}(a), \mathbf{h}(a)$	$\mathrm{div}(a)$	a	$\mathbf{t}(a), \mathbf{h}(a)$	$\mathrm{div}(a)$
1	0,1	x_0	18	3,5	x_4	35	7,10	x_4
2	0,1	x_1	19	3,9	x_6	36	8,10	x_0
3	0,1	x_2	20	4,5	x_0	37	8,10	x_1
4	0,2	x_3	21	4,5	x_1	38	8,10	x_2
5	0,2	x_4	22	4,5	x_2	39	9,11	x_3
6	0,6	x_6	23	4,10	x_6	40	9,11	x_4
7	1,3	x_0	24	5,6	x_5	41	10,11	x_0
8	1,3	x_1	25	5,11	x_6	42	10,11	x_1
9	1,3	x_2	26	6,7	x_0	43	10,11	x_2
10	1,4	x_3	27	6,7	x_1	44	10,0	x_6x_7
11	1,4	x_4	28	6,7	x_2	45	11,0	$x_2x_4x_5x_7$
12	1,7	x_6	29	6,8	x_3	46	11,0	$x_2x_3x_5x_7$
13	2,4	x_0	30	6,8	x_4	47	11,0	$x_1x_4x_5x_7$
14	2,4	x_1	31	7,9	x_0	48	11,0	$x_1x_3x_5x_7$
15	2,4	x_2	32	7,9	x_1	49	11,0	$x_0x_4x_5x_7$
16	2,8	x_6	33	7,9	x_2	50	11,0	$x_0x_3x_5x_7$
17	3,5	x_3	34	7,10	x_3	51	11,1	x_6x_7

Table 8.1: *The arrows in a quiver of sections for* $\mathrm{tot}(\omega_{D_1})$

$$\left\{\begin{matrix}
(a_1a_8, a_2a_7), (a_1a_9, a_3a_7), (a_1a_{10}, a_4a_{13}), (a_1a_{11}, a_5a_{13}), (a_1a_{12}, a_6a_{26}), \\
(a_1a_{44}, a_{41}a_{51}), (a_1a_{45}, a_3a_{49}), (a_1a_{46}, a_3a_{50}), (a_1a_{47}, a_2a_{49}), (a_1a_{48}, a_2a_{50}), \\
(a_2a_9, a_3a_8), (a_2a_{10}, a_4a_{14}), (a_2a_{11}, a_5a_{14}), (a_2a_{12}, a_6a_{27}), (a_2a_{44}, a_{42}a_{51}), \\
(a_2a_{45}, a_3a_{47}), (a_2a_{46}, a_3a_{48}), (a_3a_{10}, a_4a_{15}), (a_3a_{11}, a_5a_{15}), (a_3a_{12}, a_6a_{28}), \\
(a_3a_{44}, a_{43}a_{51}), (a_4a_{16}, a_6a_{29}), (a_4a_{35}a_{44}, a_5a_{34}a_{44}), (a_4a_{45}, a_5a_{46}), (a_4a_{47}, a_5a_{48}), \\
(a_4a_{49}, a_5a_{50}), (a_5a_{16}, a_6a_{30}), (a_7a_{17}, a_{10}a_{20}), (a_7a_{18}, a_{11}a_{20}), (a_7a_{19}, a_{12}a_{31}), \\
(a_8a_{17}, a_{10}a_{21}), (a_8a_{18}, a_{11}a_{21}), (a_8a_{19}, a_{12}a_{32}), (a_9a_{17}, a_{10}a_{22}), (a_9a_{18}, a_{11}a_{22}), \\
(a_9a_{19}, a_{12}a_{33}), (a_{10}a_{23}, a_{12}a_{34}), (a_{10}a_{40}a_{51}, a_{11}a_{39}a_{51}), (a_{11}a_{23}, a_{12}a_{35}), \\
(a_{13}a_{21}, a_{14}a_{20}), (a_{13}a_{22}, a_{15}a_{20}), (a_{13}a_{23}, a_{16}a_{36}), (a_{14}a_{22}, a_{15}a_{21}), (a_{14}a_{23}, a_{16}a_{37}), \\
(a_{15}a_{23}, a_{16}a_{38}), (a_{17}a_{24}a_{30}, a_{18}a_{24}a_{29}), (a_{17}a_{25}, a_{19}a_{39}), (a_{18}a_{25}, a_{19}a_{40}), \\
(a_{20}a_{24}a_{27}, a_{21}a_{24}a_{26}), (a_{20}a_{24}a_{28}, a_{22}a_{24}a_{26}), (a_{20}a_{25}, a_{23}a_{41}), \\
(a_{21}a_{24}a_{28}, a_{22}a_{24}a_{27}), (a_{21}a_{25}, a_{23}a_{42}), (a_{22}a_{25}, a_{23}a_{43}), (a_{26}a_{32}, a_{27}a_{31}), \\
(a_{26}a_{33}, a_{28}a_{31}), (a_{26}a_{34}, a_{29}a_{36}), (a_{26}a_{35}, a_{30}a_{36}), (a_{27}a_{33}, a_{28}a_{32}), (a_{27}a_{34}, a_{29}a_{37}), \\
(a_{27}a_{35}, a_{30}a_{37}), (a_{28}a_{34}, a_{29}a_{38}), (a_{28}a_{35}, a_{30}a_{38}), (a_{31}a_{39}, a_{34}a_{41}), (a_{31}a_{40}, a_{35}a_{41}), \\
(a_{32}a_{39}, a_{34}a_{42}), (a_{32}a_{40}, a_{35}a_{42}), (a_{33}a_{39}, a_{34}a_{43}), (a_{33}a_{40}, a_{35}a_{43}), (a_{36}a_{42}, a_{37}a_{41}), \\
(a_{36}a_{43}, a_{38}a_{41}), (a_{37}a_{43}, a_{38}a_{42}), (a_{39}a_{45}, a_{40}a_{46}), (a_{39}a_{47}, a_{40}a_{48}), (a_{39}a_{49}, a_{40}a_{50}), \\
(a_{41}a_{45}, a_{43}a_{49}), (a_{41}a_{46}, a_{43}a_{50}), (a_{41}a_{47}, a_{42}a_{49}), (a_{41}a_{48}, a_{42}a_{50}), (a_{42}a_{45}, a_{43}a_{47}), \\
(a_{42}a_{46}, a_{43}a_{48})
\end{matrix}\right\}$$

Figure 8.1: *The set of relations* $\mathcal{J}$ *for the quiver on* $\mathrm{tot}(\omega_{D_1})$

birationally maximal fivefolds, 99 are such that the number of line bundles in the collection $\mathcal{L}_{\mathrm{nef}} \subseteq \mathfrak{D}_m$ for $m \gg 0$ is equal to the number of maximal cones $|\Sigma(5)|$ in the fan for that variety; these collections automatically become candidates for a full strong exceptional collection. By adapting the method given in Chapter 6, the author has constructed a resolution of $\mathcal{O}_\Delta$ for 26 of these varieties using $\mathcal{L}_{\mathrm{nef}}$. Further, it appears that the only obstruction to providing a resolution of $\mathcal{O}_\Delta$ for the rest of the 99 fivefolds is the lack of a sufficient set of minimal generators for the ideal J'.

APPENDIX

A

QUIVERSTORICVARIETIES: A PACKAGE TO CONSTRUCT QUIVERS OF SECTIONS ON COMPLETE TORIC VARIETIES

A.1 Introduction

For a collection of non-isomorphic line bundles $\mathcal{L} = \{\mathcal{L}_0 := \mathcal{O}_X, \mathcal{L}_1, \dots, \mathcal{L}_r\}$ on a complete normal toric variety X, the endomorphism algebra $\operatorname{End}(\bigoplus_i \mathcal{L}_i)$ can be described as the quotient of the path algebra of its quiver of sections by an ideal of relations determined by labels on the arrows in the quiver [CS08]. The vertices of the quiver correspond to the line bundles and there is a natural order on the vertices defined by $i < j$ if $\operatorname{Hom}(L_j, L_i) = 0$. For $i < j$, the number of arrows from i to j is equal to the dimension of the cokernel of the map

$$\bigoplus_{i<k<j} \operatorname{Hom}(L_i, L_k) \otimes \operatorname{Hom}(L_k, L_j) \longrightarrow \operatorname{Hom}(L_i, L_j). \tag{A.1.1}$$

A torus-invariant section $s \in \operatorname{Hom}(L_i, L_j)$ is *irreducible* if it is not in the image of this map. We label each arrow by the toric divisors corresponding to the sections in a basis of the irreducible sections. Using the given order on $\mathcal{L}$, the collection is *strong exceptional* if

$$\operatorname{Ext}^i(L_j, L_k) = 0, \forall j,\ k,\ \text{and } i \neq 0. \tag{A.1.2}$$

Let $\mathcal{D}^b(X)$ denote the bounded derived category of coherent sheaves on X. The collection $\mathcal{L}$ is *full*, or *generates* $\mathcal{D}^b(X)$, if the smallest triangulated full subcategory of $\mathcal{D}^b(X)$ containing $\mathcal{L}$ is $\mathcal{D}^b(X)$ itself. A tilting bundle T on X is a vector bundle such that T generates $\mathcal{D}^b(X)$ and $\operatorname{Ext}^i(T,T) = 0$ for $i > 0$; given a full strong exceptional collection of line bundles $\mathcal{L}$ on X, the direct sum $\bigoplus_{L_i \in \mathcal{L}} L_i$ is a tilting bundle. The following theorem by Baer and Bondal allows us to understand $\mathcal{D}^b(X)$ in terms of the module category of a finite dimensional algebra.

Theorem A.1. *[Bae88, Bon90] Let T be a tilting bundle on X, $A = \operatorname{End}(T)$ and $\mathcal{D}^b(\text{mod–}A)$ be the bounded derived category of finitely generated right A-modules. Then*

$$\mathbf{R}\operatorname{Hom}(T, -) \colon \mathcal{D}^b(X) \to \mathcal{D}^b(\text{mod–}A) \tag{A.1.3}$$

is an equivalence of triangulated categories.

A complete normal toric variety induces a short exact sequence of abelian groups

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Cl}(X) \longrightarrow 0, \tag{A.1.4}$$

where M is the character lattice of the dense torus in X, $\Sigma(1)$ is the set of rays in the fan Σ of X, and the map deg sends a toric divisor $D \in \mathbb{Z}^{\Sigma(1)}$ to the isomorphism class of the rank 1 reflexive sheaf $\mathcal{O}_X(D)$ in the class group $\mathrm{Cl}(X)$ (see for example [Ful93]). Showing that $\mathcal{L}$ is strong exceptional in this situation is equivalent to checking that $H^i(X, L_j^{-1} \otimes L_k) = 0$ for $i > 0,\ 0 \leq j, k \leq r$. Using a theorem of Eisenbud, Mustaţă and Stillman [EMS00], we can determine if the cohomology of $\mathcal{O}_X(D)$ vanishes by considering when $\mathcal{O}_X(D)$ avoids certain affine cones constructed in $\mathrm{Cl}(X)$, which we call *not-necessarily non-vanishing cohomology cones* (*nnnvc*-cones) – see [PN15b]. The purpose of the package *QuiversToricVarieties* for *Macaulay2* [GS] is to construct the quiver of sections for a collection of line bundles on a complete toric variety and check if the collection is strong exceptional. We note that there does exist computer programs that check if a collection of line bundles on a toric variety is strong exceptional; see for example Perling's *TiltingSheaves* [Per].

Restricting our attention to smooth toric Fano varieties, toric divisorial contractions give the collection of n-dimensional toric Fano varieties a poset structure, described for $n = 3$ by [Oda88] and $n = 4$ by [Sat00] (see also [PN15b, Remark 2.4]). The contractions induce lattice maps between the short exact sequences (A.1.4) determined by the varieties and these lattice maps are an essential ingredient in the proof that each smooth toric Fano variety of dimension ≤ 4 has a full strong exceptional collection of line bundles [PN15b, Theorem 6.4]. The package *QuiversToricVarieties* contains a database of these lattice maps and of full strong exceptional collections of line bundles on all smooth toric Fano varieties of dimension ≤ 4.

In the case when X is a smooth toric Fano variety, let $Y = \mathrm{tot}(\omega_X)$ be the total space of the canonical bundle on X. The package *QuiversToricVarieties* contains methods to check if the pullback of a full strong exceptional collection of line bundles on X along the morphism $Y \to X$ is a tilting bundle on Y.

QuiversToricVarieties depends on the package *NormalToricVarieties* for the construction of toric varieties and for the database of smooth toric Fano varieties. All varieties are defined over $\Bbbk = \mathbb{C}$.

A.2 Overview of the Package

Let X be a complete normal toric variety constructed in *NormalToricVarieties* with a torsion-free class group. The class group lattice of X has a basis determined by `fromWDivToCl` and the function `fromPicToCl` can be used to determine which vectors in the lattice correspond to line bundles. The input for the method `quiver` is a complete normal toric variety with a torsion-free class group, together with a list of vectors v_i in the class group lattice that correspond to the line bundles L_i. The vectors are ordered by `quiver` and the basis of $\mathrm{Hom}(L_i, L_j)$ is calculated by determining the basis of the multidegree $v_j - v_i$ over the Cox ring of the variety. From this basis, the irreducible maps are chosen and listed as arrows, with the corresponding toric divisors as labels. If some of the vectors do not correspond to line bundles then a quiver is still constructed but the resulting path algebra modulo relations may not be isomorphic to $\mathrm{End}(\bigoplus_{i \in Q_0} E_i)$, where E_i are the rank 1 reflexive sheaves corresponding to v_i. Alternatively, we can produce a quiver by explicitly listing the vertices, the arrows with labels and the variety. The methods `source`, `target`, `label` and `index` return the specific details of an arrow in the quiver, a list of which can be accessed by inputting `Q_1`.

Besides the method `quiver`, the method `doHigherSelfExtsVanish` forms the core of the package. The primary input is a quiver of sections. The method creates the *nnnvc*-cones in the class group lattice for X and determines if the vectors $v_i - v_j$ avoid these cones. The cones are determined by certain subsets I of the rays of the fan Σ for X; if the complement of the supporting cones for I in Σ has non-trivial reduced homology, then I is called a *forbidden set* and it determines a cone in $\mathbb{Z}^{\Sigma(1)}$. The forbidden sets can be calculated using the function `forbiddenSets`, and the image of a cone determined by a forbidden set under the map `fromWDivToCl X` is a *nnnvc*-cone in $\mathrm{Cl}(X)$.

A database in *NormalToricVarieties* contains the smooth toric Fano varieties up to dimension 6 and can be accessed using `smoothFanoToricVariety`. The divisorial contractions between the smooth toric Fano varieties up to dimension 4 are listed under the `contractionList` command, and the induced maps between their respective short exact sequences (A.1.4) are recalled from a database in *QuiversToricVarieties* using the `tCharacterMap`, `tDivisorMap` and the `picardMap` commands. Note that as each variety considered is smooth, its class group is isomorphic to its Picard group.

The database containing full strong exceptional collections of line bundles for smooth Fano toric varieties in dimension ≤ 4 can be accessed using `fullStrExcColl`. The collections for the surfaces were calculated by King [Kin97], the threefolds by Costa–Miró-Roig [CMR04], Bernardi–Tirabassi [BT09] and Uehara [Ueh14] and the fourfolds by Prabhu-Naik [PN15b].

A.3 An Example

We illustrate the main methods in *QuiversToricVarieties* using the blowup of $\mathbb{P}^2$ at three points, the birationally maximal smooth toric Fano surface. It is contained in the toric Fano database in *NormalToricVarieties*, which is loaded by the *QuiversToricVarieties* package.

```
i1 : loadPackage "QuiversToricVarieties";
i2 : X = smoothFanoToricVariety(2,4);
```

A full strong exceptional collection $\mathcal{L}$, first considered by King [Kin97], can be recalled from the database and its quiver of sections can be created.

```
i3 : L = fullStrExcColl(2,4);
o3 = {{0,0,0,0},{0,0,1,1},{0,1,0,0},{0,1,1,0},{1,0,0,0},{1,0,0,1}}
i4 : Q = quiver(L,X);
```

We can view the details of the quiver, either by displaying the arrows at each vertex, or by listing all of the arrows and considering their source, target and label.

```
i5 : Q#0
o5 = HashTable{1 => {x_0x_1 , x_3x_4 }      }
               2 => {x_1x_2 , x_4x_5 }
               3 => {x_2x_3 , x_0x_5 }
               degree => {0, 0, 0, 0}

i6 : first Q_1
o6 = arrow_1

i7 : source oo, target oo, label oo
o7 = (0, 1, x_0x_1 )
```

The forbidden sets of rays can be computed and the collection of line bundles can be checked to be strong exceptional. The method **doHigherSelfExtsVanish** creates a copy of the *nnnvc*-cones in the cache table for X, where the cones are given by a vector and a matrix $\{w, M\}$ encoding the supporting closed half spaces of the cone, in which case the lattice points of the cone are $\{v + w \in \mathrm{Cl}(X) \mid M^T v \leq 0\}$. The non-vanishing cone for H^2 is displayed below.

```
i8 : peek forbiddenSets X
o8 = MutableHashTable{1 => {{0,2},{0,3},{1,3},{0,1,3},{0,2,3},{0,4},
                            {1,4},...}
                      2 => {{0,1,2,3,4,5}}

i9 : doHigherSelfExtsVanish Q
o9 = true

i10 : X.cache.cones#2
o10 = {{| -1 |, | 1 1 1 0 1 |}}
        | -1 |  | 1 0 1 1 1 |
        | -1 |  | 0 1 1 0 0 |
        | -1 |  | 0 0 0 1 1 |
```

Consider the chain of divisorial contractions $X =: X_4 \to X_3 \to X_2 \to X_0$ from X to the toric Fano surfaces numbered 3, 2 and 0 in the database. The contractions induces lattice maps $\mathrm{Pic}(X_4) \to \mathrm{Pic}(X_3) \to \mathrm{Pic}(X_2) \to \mathrm{Pic}(X_0)$ and the method **doHigherExtsVanish** can check if the non-isomorphic line bundles in the image of $\mathcal{L}$ under these lattice maps are strong exceptional for each contraction.

```
i11 : doHigherSelfExtsVanish(Q,{4,3,2,0})
o11 = true
```

Now consider the morphism $\pi\colon \mathrm{tot}(\omega_X) \to X$. The pullback $\pi^*(\bigoplus_{L \in \mathcal{L}} L)$ is a tilting bundle on $Y = \mathrm{tot}(\omega_X)$ if

$$H^k(X, L_i \otimes L_j^{-1} \otimes \omega_X^{-m}) = 0$$

for all $k > 0$, $m \geq 0$ and $L_i, L_j \in \mathcal{L}$ (see for example [PN15b, Theorem 6.7]). As ω_X^{-1} is ample, there exists a non-negative integer n such that $L_i \otimes L_j^{-1} \otimes \omega_X^{-m}$ is nef for $0 \leq i,j \leq r$ and $m \geq n$, and hence $H^k(X, L_i \otimes L_j^{-1} \otimes \omega_X^{-m}) = 0$ for all $k > 0$ by Demazure vanishing. The method `bundlesNefCheck` checks for a given integer n whether $L_i \otimes L_j^{-1} \otimes \omega_X^{-n}$ is nef for all $L_i, L_j \in \mathcal{L}$.

```
i12 : n=2;
i13 : bundlesNefCheck(Q,n)
o13 = true
```

If an integer p is included as an additional input in `doHigherSelfExtsVanish`, then the method checks that for all $0 \leq m \leq p$, whether the line bundles $L_i \otimes L_j^{-1} \otimes \omega^{-m}$ avoid the *nnnvc*-cones. Note that for our example, the computation above implies that it is enough to use the integer $n-1$.

```
i14 : doHigherSelfExtsVanish(Q,n-1)
o14 = true
```

For $t \in \{4,3,2,0\}$, let $\{L_{i,t}\}_{i \in I_t}$ denote the list of non-isomorphic line bundles in the image of $\mathcal{L}$ under the map given by `picardMap` from $\mathrm{Pic}(X) \to \mathrm{Pic}(X_t)$, where I_t is an index set. By including the list of divisorial contractions as an input in `doHigherSelfExtsVanish`, we can check that

$$H^k(X_t, L_{i,t} \otimes (L_{j,t})^{-1} \otimes \omega_{X_t}^{-m}) = 0$$

for $k > 0$, $0 \leq m \leq n-1$, $t \in \{4,3,2,0\}$ and all $i,j \in I_t$.

```
i15 : doHigherSelfExtsVanish(Q,{4,3,2,0},n-1)
o15 = true
```

For all n-dimensional smooth toric Fano varieties, $1 \leq n \leq 3$, and 88 of the 124 smooth toric Fano fourfolds, the database contains a chain complex of modules over the Cox ring for the variety. The chain complexes are used in [PN15b] to show that the collections of line bundles in the database for these varieties are full.

```
i16 : C = resOfDiag(2,4);
i17 : SS = ring C;
i18 : C
         6        12        6
o18 = SS  <-- SS    <-- SS
```

APPENDIX

B

TABLES OF RESULTS AND THE FOURFOLD CONTRACTION DIAGRAM

The tables in this appendix contain details on the construction of the full strong exceptional collections of line bundles $\mathcal{L}$ on smooth n-dimensional toric Fano varieties, for $2 \leq n \leq 4$. If $X = \mathbb{P}^n$ then the full strong exceptional collection is provided by Beĭlinson [Beĭ78], whilst if X is a product of smooth toric Fano varieties then a full strong exceptional collection is given for X by Lemma 2.24.

A *maximal* variety is a variety that is birationally maximal, as explained in Section 2.2. The sets $\mathfrak{D}_m$ and $\mathfrak{D}(\omega_X)_m$ are the Frobenius pushforwards of $\mathcal{O}_X$ and ω_X respectively as defined in Section 4.1, for some integer $m > 0$. ***Method 1*** and ***Method 2*** are described in Section 4.2 and Section 6.3 respectively, and are used where stated to show that the collection $\mathcal{L}$ is full. The description "collection from (j)" for an n-dimensional variety X means that there is a chain of torus-invariant divisorial contractions $X_0 \to X_1 \to \cdots \to X_t := X$ between smooth n-dimensional toric Fano varieties, where X_0 is the j^{th} n-dimensional variety. The full strong exceptional collection on X is then given by the non-isomorphic line bundles in the image of the full strong exceptional collection for X_0 under the induced Picard lattice map $\gamma_{(0\to t)}\colon \operatorname{Pic}(X_0) \to \operatorname{Pic}(X_t)$ (see Section 3.2 and Theorem 7.4 for details).

The full strong exceptional collections on the smooth toric Fano surfaces were originally described by King [Kin97]. He checked for each variety that the collections chosen were strong exceptional, and provided resolutions of the diagonal for each collection to show generation of $\mathcal{D}^b(X)$. Uehara [Ueh14] observed that these collections $\mathcal{L}$ were chosen from $\mathfrak{D}_m$, and the pushforward $f_*(\mathcal{L})$ is full strong exceptional for any torus-invariant divisorial contraction $f : X_1 \to X_2$ between two smooth toric Fano surfaces X_1 and X_2. Table B.1 indicates that we can use ***Method 2*** to recover King's resolution of the diagonal on the maximal surface and by using Lemma 7.1, the subsequent collections given for varieties (2) and (3) coincide with King's result [Kin97] and Uehara's calculations [Ueh14, Theorem 6.3].

The full strong exceptional collections on the smooth toric Fano threefolds were found by Uehara [Ueh14], having built on work by Bondal [Bon06], Costa–Miró-Roig [CMR04] and Bernardi–Tirabassi [BT09] – for the collections $\mathcal{L}$ he exhibits, he shows that $f_*(\mathcal{L})$ is full strong exceptional for any torus-invariant divisorial contraction $f :$ $X_1 \to X_2$ between two smooth toric Fano threefolds X_1 and X_2. We choose the same collections as Uehara from $\mathfrak{D}_m$ for each maximal variety and again note that by Lemma 7.1, our collections on the other threefolds coincide with Uehara's collections. Our method of generation differs from Uehara; he uses ***Method 1*** to show that the collections on the maximal threefolds are full, whilst we use ***Method 2*** to find a resolution of the diagonal.

Figure B.1 pictorially encodes the torus-invariant divisorial contractions between the smooth toric Fano fourfolds as listed in [Sat00, Table 1] by Sato. Each line

$$\boxed{i}\longrightarrow\!\!\!\!\!-\!\!\!-\boxed{j}$$

implies that there is a torus-invariant divisorial contraction from variety i to variety j, where i and j are smooth toric Fano fourfolds listed in Table B.3. The oval nodes indicate fourfolds that are products of lower dimensional smooth toric Fano varieties. Each level in the diagram corresponds to the rank of the Picard lattice of a fourfold in that level.

B.1 Toric del Pezzo Surfaces

	Variety	Details of the Full Strong Exceptional Collection
(0)	$\mathbb{P}^2$	Beĭlinson's collection
(1)	$\mathbb{P}^1 \times \mathbb{P}^1$	product of smooth toric Fano varieties
(2)	S_1, $Bl_1(\mathbb{P}^2)$	collection from (4)
(3)	S_2, $Bl_2(\mathbb{P}^2)$	collection from (4)
(4)	S_3, $Bl_3(\mathbb{P}^2)$	Maximal variety. $\mathcal{L} = \mathfrak{D}_m$. Generation: ***Method 2***

Table B.1: *Tilting bundles on smooth toric Fano surfaces*

B.2 Smooth Toric Fano Threefolds

	Variety	Details of the Full Strong Exceptional Collection
(0)	$\mathbb{P}^3$	Beĭlinson's collection
(1)	$\mathcal{B}_1$	collection from (10)
(2)	$\mathcal{B}_2$	collection from (17)
(3)	$\mathcal{B}_3$	collection from (17)
(4)	$\mathcal{B}_4$, $\mathbb{P}^2 \times \mathbb{P}^1$	product of smooth toric Fano varieties
(5)	$\mathcal{C}_1$	collection from (17)
(6)	$\mathcal{C}_2$	collection from (17)
(7)	$\mathcal{C}_3$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	product of smooth toric Fano varieties
(8)	$\mathcal{C}_4$, $S_1 \times \mathbb{P}^1$	product of smooth toric Fano varieties
(9)	$\mathcal{C}_5$	collection from (17)
(10)	$\mathcal{D}_1$	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 2***
(11)	$\mathcal{D}_2$	collection from (17)

(12)	$\mathcal{E}_1$	collection from (17)
(13)	$\mathcal{E}_2$	collection from (17)
(14)	$\mathcal{E}_3$, $S_2 \times \mathbb{P}^1$	product of smooth toric Fano varieties
(15)	$\mathcal{E}_4$	collection from (17)
(16)	$\mathcal{F}_1$, $S_3 \times \mathbb{P}^1$	Maximal, product of smooth toric Fano varieties
(17)	$\mathcal{F}_2$	Maximal variety. $\mathcal{L} = \mathfrak{D}_m$. Generation: ***Method 2***

Table B.2: *Tilting bundles on smooth toric Fano threefolds*

B.3 Smooth Toric Fano Fourfolds

	Variety	Details of the Full Strong Exceptional Collection
(0)	$\mathbb{P}^4$	Beĭlinson's collection
(1)	B_1	collection from (10)
(2)	B_2	collection from (65)
(3)	B_3	collection from (114)
(4)	B_4, $\mathbb{P}^1 \times \mathbb{P}^3$	product of smooth toric Fano varieties
(5)	B_5	collection from (114)
(6)	C_1	collection from (100)
(7)	C_2	collection from (114)
(8)	C_3	collection from (80)
(9)	C_4, $\mathbb{P}^2 \times \mathbb{P}^2$	product of smooth toric Fano varieties
(10)	E_1	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 2***
(11)	E_2	collection from (65)
(12)	E_3	collection from (109)
(13)	D_1	collection from (100)
(14)	D_2	collection from (100)
(15)	D_3	collection from (101)
(16)	D_4	collection from (65)
(17)	D_5, $\mathbb{P}^1 \times \mathcal{B}_1$	product of smooth toric Fano varieties
(18)	D_6	collection from (109)
(19)	D_7	collection from (115)
(20)	D_8	collection from (109)
(21)	D_9	collection from (107)
(22)	D_{10}	collection from (114)
(23)	D_{11}	collection from (114)
(24)	D_{12}, $\mathbb{P}^1 \times \mathcal{B}_2$	product of smooth toric Fano varieties
(25)	D_{13}, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	product of smooth toric Fano varieties
(26)	D_{14}, $\mathbb{P}^1 \times \mathcal{B}_3$	product of smooth toric Fano varieties
(27)	D_{15}, $S_1 \times \mathbb{P}^2$	product of smooth toric Fano varieties

(28)	D_{16}	collection from (47)
(29)	D_{17}	collection from (114)
(30)	D_{18}	collection from (100)
(31)	D_{19}	collection from (114)
(32)	G_1	collection from (81)
(33)	G_2	collection from (75)
(34)	G_3	collection from (82)
(35)	G_4	collection from (80)
(36)	G_5	collection from (82)
(37)	G_6	collection from (114)
(38)	H_1	collection from (100)
(39)	H_2	collection from (101)
(40)	H_3	collection from (100)
(41)	H_4	collection from (109)
(42)	H_5	collection from (109)
(43)	H_6	collection from (101)
(44)	H_7	collection from (100)
(45)	H_8, $S_2 \times \mathbb{P}^2$	product of smooth toric Fano varieties
(46)	H_9	collection from (109)
(47)	H_{10}	Non-maximal. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(48)	L_1	collection from (108)
(49)	L_2	collection from (108)
(50)	L_3	collection from (109)
(51)	L_4	collection from (109)
(52)	L_5, $\mathbb{P}^1 \times \mathcal{C}_1$	product of smooth toric Fano varieties
(53)	L_6, $\mathbb{P}^1 \times \mathcal{C}_2$	product of smooth toric Fano varieties
(54)	L_7, $S_1 \times S_1$	product of smooth toric Fano varieties
(55)	L_8, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	product of smooth toric Fano varieties
(56)	L_9, $S_1 \times \mathbb{P}^1 \times \mathbb{P}^1$	product of smooth toric Fano varieties
(57)	L_{10}	collection from (114)
(58)	L_{11}, $\mathbb{P}^1 \times \mathcal{C}_5$	product of smooth toric Fano varieties
(59)	L_{12}	collection from (114)
(60)	L_{13}	collection from (115)
(61)	I_1	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 1***
(62)	I_2	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 1***
(63)	I_3	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 1***
(64)	I_4	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 1***
(65)	I_5	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 2***
(66)	I_6	collection from (114)
(67)	I_7, $\mathbb{P}^1 \times \mathcal{D}_1$	Maximal, product of smooth toric Fano varieties
(68)	I_8	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(69)	I_9	collection from (115)
(70)	I_{10}	collection from (110)

(71)	I_{11}	collection from (107)
(72)	I_{12}	collection from (114)
(73)	I_{13}, $\mathbb{P}^1 \times \mathcal{D}_2$	product of smooth toric Fano varieties
(74)	I_{14}	collection from (109)
(75)	I_{15}	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 2***
(76)	M_1	Non-maximal. $\mathcal{L} \not\subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(77)	M_2	Non-maximal. $\mathcal{L} \not\subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 1***
(78)	M_3	Non-maximal. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(79)	M_4	collection from (106)
(80)	M_5	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(81)	J_1	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(82)	J_2	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(83)	Q_1	collection from (108)
(84)	Q_2	collection from (109)
(85)	Q_3	collection from (108)
(86)	Q_4	collection from (110)
(87)	Q_5	collection from (109)
(88)	Q_6, $\mathbb{P}^1 \times \mathcal{E}_1$	product of smooth toric Fano varieties
(89)	Q_7	collection from (114)
(90)	Q_8, $\mathbb{P}^1 \times \mathcal{E}_2$	product of smooth toric Fano varieties
(91)	Q_9	collection from (110)
(92)	Q_{10}, $S_1 \times S_2$	product of smooth toric Fano varieties
(93)	Q_{11}, $\mathbb{P}^1 \times \mathbb{P}^1 \times S_2$	product of smooth toric Fano varieties
(94)	Q_{12}	collection from (114)
(95)	Q_{13}	collection from (108)
(96)	Q_{14}	collection from (109)
(97)	Q_{15}, $\mathbb{P}^1 \times \mathcal{E}_4$	product of smooth toric Fano varieties
(98)	Q_{16}	collection from (115)
(99)	Q_{17}	collection from (114)
(100)	K_1	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 2***
(101)	K_2	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 2***
(102)	K_3	collection from (109)
(103)	K_4, $\mathbb{P}^2 \times S_3$	product of smooth toric Fano varieties
(104)	R_1	Maximal variety. $\mathcal{L} \not\subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(105)	R_2	Maximal variety. $\mathcal{L} \not\subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 1***
(106)	R_3	Maximal variety. See Example 7.6. Generation: ***Method 2***

(107)		Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m$. Generation: ***Method 1***
(108)	U_1	Maximal variety. $\mathcal{L} = \mathfrak{D}_m$. Generation: ***Method 1***
(109)	U_2	Maximal variety. $\mathcal{L} = \mathfrak{D}_m$. Generation: ***Method 2***
(110)	U_3	Maximal variety. $\mathcal{L} = \mathfrak{D}_m$. Generation: ***Method 1***
(111)	U_4, $S_1 \times S_3$	product of smooth toric Fano varieties
(112)	U_5, $\mathbb{P}^1 \times \mathbb{P}^1 \times S_3$	product of smooth toric Fano varieties
(113)	U_6, $\mathbb{P}^1 \times \mathcal{F}_2$	Maximal, product of smooth toric Fano varieties
(114)	U_7	Maximal variety. $\mathcal{L} = \mathfrak{D}_m$. Generation: ***Method 2***
(115)	U_8	Maximal variety. $\mathcal{L} = \mathfrak{D}_m$. Generation: ***Method 2***
(116)	V^4	Maximal variety. $\mathcal{L} \not\subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(117)	$\widetilde{V}^4$	Maximal variety. $\mathcal{L} \not\subseteq \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 2***
(118)	$S_2 \times S_2$	product of smooth toric Fano varieties
(119)	$S_2 \times S_3$	product of smooth toric Fano varieties
(120)	$S_3 \times S_3$	Maximal, product of smooth toric Fano varieties
(121)	Z_1	collection from (123)
(122)	Z_2	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 1***
(123)	W	Maximal variety. $\mathcal{L} \subset \mathfrak{D}_m \cup \mathfrak{D}(\omega_X)_m$. Generation: ***Method 1***

Table B.3: *Tilting bundles on smooth toric Fano fourfolds*

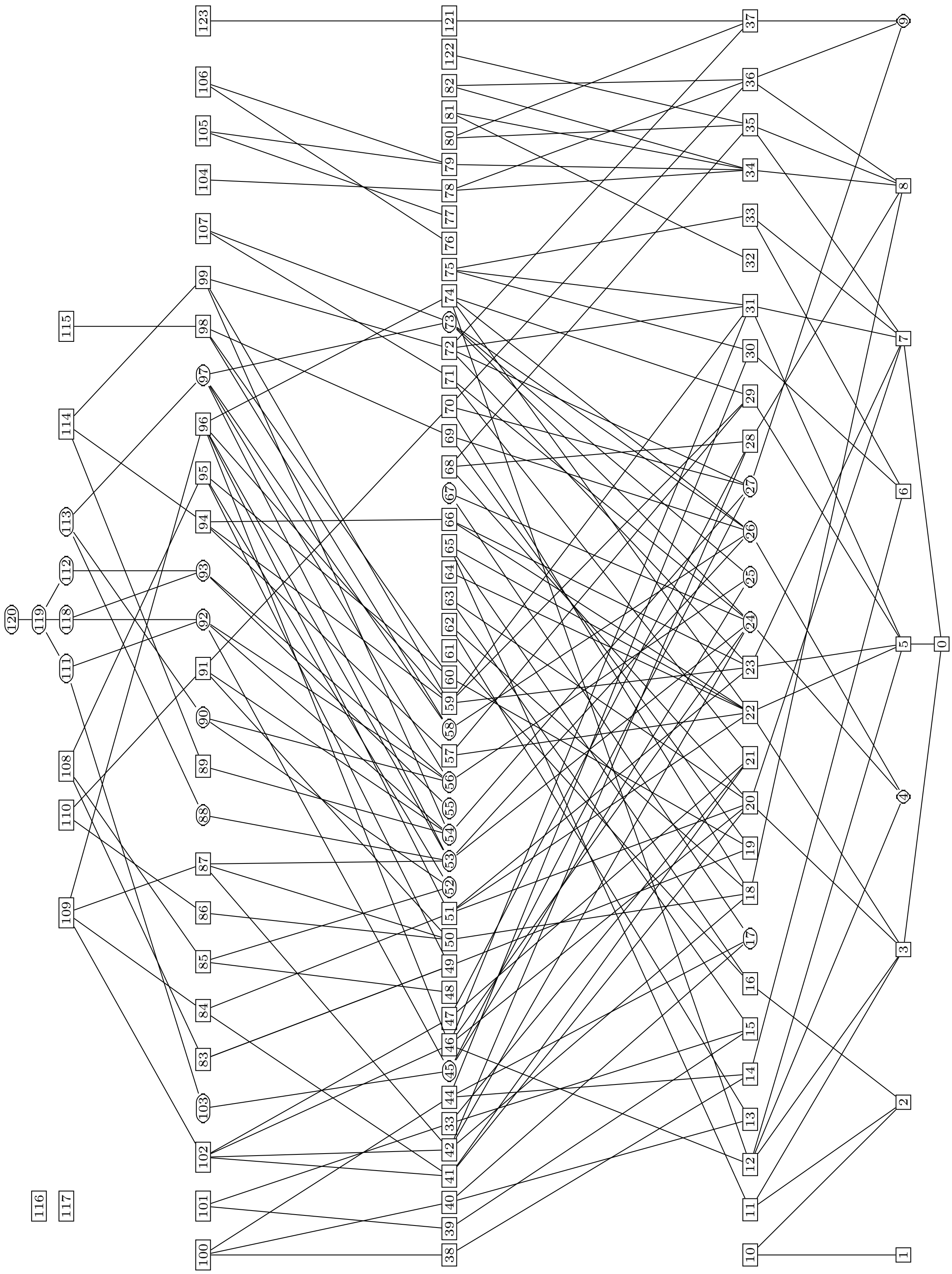


Figure B.1: *The torus-invariant divisorial contractions between the smooth toric Fano fourfolds*

APPENDIX

C

FURTHER EXAMPLES

The two examples below prove that the non-nef collections of line bundles on the smooth toric Fano fourfolds M_1 and $\tilde{V}^4$ are full strong exceptional, using ***Method 2***.

Example C.1. Let X be the smooth toric Fano fourfold M_1. The primitive generators for the rays of X are

$$u_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, u_1 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, u_2 = \begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}, u_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, u_4 = \begin{bmatrix}0\\0\\-1\\0\end{bmatrix}, u_5 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, u_6 = \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}, u_7 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}$$

The collection of line bundles on X

$$\mathcal{L} = \left\{ \begin{array}{c} \mathcal{O}_X(kD_2 + iD_6 + jD_7), \mathcal{O}_X(jD_2 + D_4 + iD_6 + D_7), \\ \mathcal{O}_X((k-1)D_2 + (j-1)D_4 + (1+i-j)D_6), \\ \mathcal{O}_X((k+1)D_4 + (k+1)D_6 + (k+1)D_7) \end{array} \middle| \begin{array}{c} 0 \leq i \leq j \leq 1 \\ 0 \leq k \leq 1 \end{array} \right\}$$

is strong exceptional and contains the non-nef line bundles $\{\mathcal{O}_X(-D_2+D_6), \mathcal{O}_X(-D_2+D_4), \mathcal{O}_X(-D_2+D_4+D_6), \mathcal{O}_X(D_7), \mathcal{O}_X(D_6+D_7), \mathcal{O}_X(D_4+D_7), \mathcal{O}_X(D_2)\}$. We obtain the chain complex of $\mathrm{Pic}(X \times X)$−graded $(S_{X\times X})$−modules

$$0 \to (S_{X\times X})^{10} \xrightarrow{d_4} (S_{X\times X})^{43} \xrightarrow{d_3} (S_{X\times X})^{76} \xrightarrow{d_2} (S_{X\times X})^{60} \xrightarrow{d_1} (S_{X\times X})^{17}$$

from this collection, which is exact up to saturation by $B_{X\times X}$ [PN15a, GS]. Table C.1 lists the arrows in the quiver of sections Q corresponding to $\mathcal{L}$.

As $|Q_0| = 17$ and $|\Sigma(1)| = 8$, we let $\{\mathbf{e}_i \mid i \in Q_0\} \cup \{\mathbf{e}_\rho \mid \rho \in \Sigma(1)\}$ be the standard basis of $\mathbb{Z}^{17+8}$ and define the lattice points $c_a := \mathbf{e}_{\mathbf{h}(a)} - \mathbf{e}_{\mathbf{t}(a)} + \mathbf{e}_{\mathrm{div}(a)}$ for each arrow $a \in Q_1$. The map π is then given by the matrix $C\colon \mathbb{Z}^{60} \to \mathbb{Z}^{17+8}$ where the columns of C are given by $c_a, a \in Q_1$, and the semigroup $\mathbb{N}(Q)$ is given by the lattice points generated by positive linear combinations of the c_a. Our choice of bases for $\mathrm{Pic}(X)$ and $\mathrm{Wt}(Q)$ imply that the lattice maps deg and pic are given by the matrices:

$$\deg\colon \begin{bmatrix} 1&1&1&-1&0&-1&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&1&1&0 \\ 1&0&0&0&0&0&0&1 \end{bmatrix}, \quad \mathrm{pic}\colon \begin{bmatrix} 0&-1&-1&0&1&0&0&1&-1&0&0&0&1&1&0&1&0 \\ 0&0&1&0&0&0&1&0&1&0&1&1&0&1&1&1&2 \\ 0&1&0&0&0&1&0&0&1&1&0&1&1&0&1&1&2 \\ 0&0&0&1&0&0&0&1&0&1&1&0&1&1&1&1&2 \end{bmatrix}$$

Fix θ to be the weight that assigns -2 to the vertex 0 in the quiver, 1 to the vertices

a	$\mathbf{t}(a), \mathbf{h}(a)$	div(a)
1	0,1	x_5
2	0,2	x_3
3	0,3	x_7
4	0,4	x_1
5	0,4	x_2
6	0,5	x_6
7	0,6	x_4
8	0,7	x_0
9	1,5	x_1
10	1,5	x_2
11	1,8	x_4
12	1,9	x_0
13	2,6	x_1
14	2,6	x_2
15	2,8	x_6
16	2,10	x_0
17	3,7	x_1
18	3,7	x_2
19	3,9	x_6
20	3,10	x_4
21	4,5	x_5
22	4,6	x_3
23	4,7	x_7
24	5,8	x_3
25	5,9	x_7
26	5,11	x_4
27	5,12	x_0
28	6,8	x_5
29	6,10	x_7
30	6,11	x_6
31	6,13	x_0
32	7,9	x_5
33	7,10	x_3
34	7,12	x_6
35	7,13	x_4
36	8,11	x_1
37	8,11	x_2
38	8,14	x_0
39	9,12	x_1
40	9,12	x_2
41	9,14	x_4
42	10,13	x_1
43	10,13	x_2
44	10,14	x_6
45	11,14	x_7
46	11,15	x_0
47	12,14	x_3
48	12,15	x_4
49	13,14	x_5
50	13,15	x_6
51	14,15	x_1
52	14,15	x_2
53	14,16	$x_0x_4x_5$
54	14,16	$x_0x_3x_6$
55	14,16	$x_4x_6x_7$
56	15,16	$x_0x_3x_5$
57	15,16	$x_1x_3x_5x_7$
58	15,16	$x_2x_3x_5x_7$
59	15,16	$x_4x_5x_7$
60	15,16	$x_3x_6x_7$

Table C.1: *The arrows in a quiver of sections for the smooth toric Fano fourfold M_1*

15 and 16 and 0 to every other vertex. We note that pic(θ) is the ample line bundle $L = \mathscr{O}_X(D_2 + 3D_4 + 3D_6 + 3D_7)$. For this choice of θ, $\pi_2\left(\mathbb{N}(Q) \cap (\pi_1)^{-1}(\theta)\right)$ surjects onto $\mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(L)$ and so Y_θ is isomorphic to X. As $\theta_i \geq 0$ for $i > 0$, θ is in the same closed GIT-chamber for the T-action on $\mathbb{V}(I_Q)$ as ϑ and so they are in the same open chamber if θ is generic. To check that θ is generic, it is enough to check that for each torus-invariant point on Y_θ, the corresponding representation is θ-stable. There are 17 maximal cones in the fan for Y_θ – recall that each maximal cone corresponds to a torus-invariant point.

For each quiver that describes a torus-invariant representation in Y_θ, we need to specify a path from the vertex 0 to vertices 15 and 16, and a path from every other vertex to the vertex 15 or 16. Examples of these paths are given in Table C.2 and are computed in [PNa]. As a result, every torus-invariant θ-semistable representation of Q is θ-stable, so θ is generic and the collection $\mathcal{L}$ on X is full by Propositions 5.8, 5.4 and 6.1.

Example C.2. Let X be the smooth toric Fano fourfold $\tilde{V}^4$. The primitive generators are

$$u_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, u_1 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, u_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, u_3 = \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}, u_4 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, u_5 = \begin{bmatrix}0\\0\\-1\\0\end{bmatrix}, u_6 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, u_7 = \begin{bmatrix}0\\0\\0\\-1\end{bmatrix},$$

$$u_8 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

The strong exceptional collection of line bundles $\mathcal{L}$ is given by the columns of the

Torus-Invariant Point	$(0 \to 15$, via $\{i_1, \dots, i_{j_1}\})$, $(0 \to 16$, via $\{i_1, \dots, i_{j_2}\})$	(vertex i, $i \to 15$ or $i \to 16$, via vertices $\{i_1, \dots, i_{j_3}\})$
$\{\rho_0, \rho_1, \rho_3, \rho_5\}$	$(a_3a_{18}a_{34}a_{48}, \{3, 7, 12\})$, $(a_3a_{19}a_{41}a_{55}, \{3, 9, 14\})$	$(1, a_{10}a_{25}a_{40}a_{48}, \{5, 9, 12\})$, $(2, a_{14}a_{29}a_{43}a_{50}, \{6, 10, 13\})$, $(4, a_{23}a_{34}a_{48}, \{7, 12\}), (8, a_{37}a_{45}a_{52}, \{11, 14\})$
$\{\rho_0, \rho_1, \rho_3, \rho_6\}$	$(a_1a_{10}a_{25}a_{40}a_{48}a_{59}$, $\{1, 5, 9, 12, 15\})$	$(2, a_{14}a_{28}a_{37}a_{45}a_{52}, \{6, 8, 11, 14\})$, $(3, a_{18}a_{32}a_{40}a_{48}, \{7, 9, 12\})$, $(4, a_{21}a_{25}a_{40}a_{48}, \{5, 9, 12\})$, $(10, a_{43}a_{49}a_{52}, \{13, 14\})$
$\{\rho_0, \rho_1, \rho_4, \rho_5\}$	$(0, a_2a_{14}a_{29}a_{43}a_{50}a_{60}$, $\{2, 6, 10, 13, 15\})$	$(1, a_{10}a_{24}a_{37}a_{45}a_{52}, \{5, 8, 11, 14\})$, $(3, a_{18}a_{33}a_{43}a_{50}, \{7, 10, 13\})$, $(4, a_{22}a_{29}a_{43}a_{50}, \{6, 10, 13\})$, $(9, a_{40}a_{47}a_{52}, \{12, 14\})$
$\{\rho_0, \rho_1, \rho_4, \rho_6\}$	$(0, a_1a_{10}a_{24}a_{37}a_{45}a_{52}a_{58}$, $\{1, 5, 8, 11, 14, 15\})$	$(2, a_{14}a_{29}a_{43}a_{49}a_{52}, \{6, 10, 13, 14\})$, $(3, a_{18}a_{32}a_{40}a_{47}a_{52}, \{7, 9, 12, 14\})$, $(4, a_{21}a_{24}a_{37}a_{45}a_{52}), \{5, 8, 11, 14\})$
$\{\rho_0, \rho_2, \rho_3, \rho_5\}$	$(0, a_3a_{17}a_{34}a_{48}, \{3, 7, 12\})$, $(0, a_3a_{19}a_{41}a_{55}, \{3, 9, 14\})$	$(1, a_9a_{25}a_{39}a_{48}, \{5, 9, 12\})$, $(2, a_{13}a_{29}a_{42}a_{50}, \{6, 10, 13\})$, $(4, a_{23}a_{34}a_{48}, \{7, 12\}), (8, a_{36}a_{45}a_{51}, \{11, 14\})$
$\vdots$	$\vdots$	$\vdots$
$\{\rho_2, \rho_4, \rho_6, \rho_7\}$	$(0, a_1a_9a_{24}a_{36}a_{46}a_{56}$, $\{1, 5, 8, 11, 15\})$	$(2, a_{13}a_{28}a_{36}a_{46}, \{6, 8, 11\})$, $(3, a_{17}a_{32}a_{39}a_{47}a_{51}, \{7, 9, 12, 14\})$, $(4, a_{21}a_{24}a_{36}a_{46}, \{5, 8, 11\})$, $(10, a_{42}a_{49}a_{51}, \{13, 14\})$

Table C.2: *Paths in the quiver associated to each torus-invariant representation in* $Y_\theta \cong M_1$

matrix pic below, where we choose the divisors $\{[D_1], [D_3], [D_4], [D_6], [D_8]\}$ as a basis of $\mathrm{Pic}(X)$. This collection contains the non-nef line bundle $\mathcal{O}_X(D_8)$. We obtain the chain complex of $\mathrm{Pic}(X \times X)-$graded $(S_{X\times X})-$modules

$$0 \to (S_{X\times X})^{18} \xrightarrow{d_4} (S_{X\times X})^{78} \xrightarrow{d_3} (S_{X\times X})^{124} \xrightarrow{d_2} (S_{X\times X})^{87} \xrightarrow{d_1} (S_{X\times X})^{23}$$

from this collection, which is exact up to saturation by $B_{X\times X}$ [PN15a, GS]. The lattice maps deg and pic are given by the matrices:

$$\deg\colon \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad \mathrm{pic}\colon \begin{bmatrix} 0&0&1&1&1&1&1&2&1&1&1&2&2&2&1&2&2&2&1&2&2&2&2 \\ 0&0&1&1&1&1&2&1&1&2&2&1&1&2&2&1&2&2&2&1&2&2&2 \\ 0&0&1&1&1&2&1&1&2&1&2&1&2&1&2&2&1&2&2&2&1&2&2 \\ 0&0&1&1&2&1&1&1&2&2&1&2&1&1&2&2&2&1&2&2&2&1&2 \\ 0&1&0&1&1&1&0&0&2&1&1&1&1&0&1&1&0&0&2&2&1&1&0 \end{bmatrix}$$

Fix θ to be the weight that assigns -9 to vertex 0, 1 to vertices $\{14,\ 15, \dots, 22\}$ and 0 to all other vertices. We note that $\mathrm{pic}(\theta)$ is the ample line bundle $L = \mathcal{O}_X(16D_1 + 16D_3 + 16D_4 + 16D_6 + 8D_8)$. For this choice of θ, $\pi_2\left(\mathbb{N}(Q) \cap (\pi_1)^{-1}(\theta)\right)$ surjects onto $\mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(L)$ and so Y_θ is isomorphic to X. As $\theta_i \geq 0$ for $i > 0$, θ is in the same closed GIT-chamber for the T-action on $\mathbb{V}(I_Q)$ as ϑ and so they are in the same open chamber if θ is generic. For each quiver that describes a torus-invariant representation in Y_θ, we need to specify paths from the vertex 0 to the vertices $\{14,\ 15, \dots, 22\}$, and a path from every other vertex to one of the vertices in $\{14,\ 15, \dots, 22\}$ to show that θ is generic. These paths, as well as the other necessary computations for this example are found in [PNa]. As a result, every torus-invariant θ-semistable representation of Q is θ-stable, so θ is generic and the collection $\mathcal{L}$ on X is full by Propositions 5.8, 5.4 and

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