PHD

The numerical solution of Wiener-Hopf integral equations

Mendes, Wendy R.

Award date:
1988

Awarding institution:
University of Bath

THE NUMERICAL SOLUTION OF WIENER-HOPF INTEGRAL EQUATIONS

submitted by Wendy R. Mendes for the
degree of Ph.D. of the University of Bath

1988

CONTENTS

ABSTRACT

This thesis is concerned with the numerical solution of the Wiener-Hopf integral equation $u(x) - \int_0^\infty \kappa(x-t)u(t)dt = f(x)$, where $\kappa(x)$ has a logarithmic singularity at x=0, and decays exponentially or polynomially as $|x| \to \infty$. An approximate solution u_n is defined by introducing a mesh Π_n on $[0,\infty]$, and then approximating the truncated integral operator $\int_0^{\beta(n)} \kappa(x-t)u(t)dt$ (where $\beta(n)$ is some mesh point) by using a composite m-point quadrature rule. Because of the weak singularity in κ, product integration is employed near t=x. We call our numerical method the "Nystrom-Product Integration" method.

In Chapter 1 we discuss some practical applications of the Wiener-Hopf equation, and we describe in detail the Nystrom-Product Integration method based on an interpolation process and an associated quadrature rule defined on $\mathbf{R}^+$.

In Chapter 2 we investigate the convergence of the interpolation process and the quadrature rule for functions which are smooth on $(0,\infty)$ and which exhibit certain types of behaviour as $x \to 0$ and as $x \to \infty$. We show that if Π_n is suitably chosen then it is possible, in many cases, to obtain the same orders of convergence that occur for smooth functions on finite intervals.

In Chapter 3 we present regularity results for the solution, u, under detailed assumptions on κ and f.

In Chapter 4 we provide a complete convergence analysis for $u-u_n$ using specific meshes Π_n introduced in Chapter 2.

In Chapter 5 we solve two important examples of Wiener-Hopf equations arising in Radiative Transfer and provide numerical results.

At the end of the thesis there are two appendices which contain some technical proofs of results that are required in Chapters 3 and 4.

The description of the Nystrom-Product Integration method, together with the analysis of the method for the case when κ decays exponentially at ∞, has been published in [22].

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr Ivan Graham, for his advice and valuable guidance throughout the course of this work.

I would especially like to thank my mother for her unfailing support and encouragement throughout my academic career.

The research was funded by the Science and Engineering Research Council, and the University of Bath Research Council.

CHAPTER 1
INTRODUCTION

1.1 THE WIENER-HOPF EQUATION

In this thesis we will be concerned with the analysis and numerical solution of Wiener-Hopf equations of the form

$$u(x) - \int_0^\infty \kappa(x-t)u(t)dt = f(x) , \qquad x \in \mathbf{R}^+ := [0,\infty), \tag{1.1}$$

where the kernel, κ, and right hand side, f, are given, and u: $\mathbf{R}^+ \to \mathbf{C}$ is the unknown solution. With K denoting the integral operator, we abbreviate (1.1) as

$$u - Ku = f.$$

Such equations occur in a variety of applications. They arise as reformulations of the linear Transport Equation [3], when it is used to model the transfer of radiation in an homogeneous stellar atmosphere with infinite slab geometry, and also classically in radiative equilibrium [24] and radiative transfer [16]. Applications also arise in other fields, for example, in the refraction of plane electromagnetic waves [26], in the electromagnetic coastal effect [42], in traffic noise simulation [13], in crack problems in linear elasticity, and in general boundary integral methods [11]. In the last two examples, the equation of interest is not directly of the form (1.1), but may be reduced to (1.1) by an appropriate transformation.

We will now describe some of these examples more fully.

Example 1. The Transport Equation For Coherent Scattering

Consider the linear Transport Equation

$$\mu \frac{\partial}{\partial x} I(x,\mu) - I(x,\mu) = -S(x) , \qquad -1 \le \mu \le 1, \tag{1.2}$$

where

$$S(x) = \left[\frac{1-\varepsilon}{2}\right] \int_{-1}^{1} I(x,\eta)d\eta + \varepsilon B(x),$$

which arises in the modelling of both neutron and radiative transfer. In the astrophysical examples of radiative transfer, (see, for example, [8, 25]), I is the specific intensity of radiation at an optical depth, x, from the boundary of the medium, in a direction making an angle $\theta = cos^{-1}\mu$ with the normal to the boundary, ε is the probability that a photon is lost in a single scattering, and $B \in C$ (the space of continuous functions on $\mathbf{R}^+$ with a limit at infinity) is the Planck Function (see [1]). $S(x)$ is known as the "source function" and describes the internal radiation field. Note that, in the case of coherent scattering, the intensity is independent of the frequency of radiation. In Chapter 5 we investigate the case when the region being modelled is an infinite slab, $0 \le x < \infty$. We assume that there is no radiation incident on the boundary of the slab, and that I does not grow exponentially at infinity, i.e. $I(0,\mu)=0$ for all $\mu<0$ and $e^{-\alpha x} I(x,\mu) \to 0$ as $x \to \infty$, for all $\mu>0$ and $\alpha>0$. In many astrophysical examples, the quantity of interest is $S(x)$ rather than $I(x,\mu)$ ([8, pg.295], [25, pg.56]), and in Chapter 5 we show how to reformulate (1.2) as an integral equation of the form (1.1) for $u(x)=S(x)$ with

$$\kappa(x) = \left[\frac{1-\varepsilon}{2}\right] E_1(|x|) = \left[\frac{1-\varepsilon}{2}\right] \int_0^1 e^{-|x|/\mu} \frac{1}{\mu} d\mu \ , \qquad (1.3)$$

and

$$f(x) = \varepsilon B(x) \ ,$$

where E_1 is the first exponential integral, (see [1, (5.1.1)]). We show that there exists a unique solution, $S \in C$, to the integral equation problem, and hence a unique solution, I, to (1.2). We also demonstrate numerical solutions for (1.1) for this case.

Example 2. The Transport Equation For Non-Coherent Scattering

Example 1 considers the case of the Transport Equation in which I is frequency independent. In this example we consider the Transport Equation for non-coherent scattering, in which I is frequency dependent, but the source function is still frequency independent. In this case, the Transport Equation is given by

$$\mu \frac{\partial}{\partial x} I(x,\mu,s) - \varphi(s) I(x,\mu,s) = -\varphi(s) S(x), \qquad -1 \le \mu \le 1, \tag{1.4}$$

for some positive profile φ with

$$\int_{-\infty}^{\infty} \varphi = 1,$$

where s is a suitable measure of the frequency, and where this time the source function is given by

$$S(x) = \left[\frac{1-\varepsilon}{2}\right] \int_{-\infty}^{\infty} \varphi(s') \int_{-1}^{1} I(x,\eta,s') d\eta ds' + \varepsilon B(x).$$

In Chapter 5 we investigate the case when the region being modelled is an infinite slab, and the boundary conditions are $I(0,\mu,s)=0$ for $\mu<0$ and for all s, and $e^{-\alpha x} I(x,\mu,s) \to 0$ as $x \to \infty$, for all $\mu>0$, $\alpha>0$ and for all s.

We will show how to reformulate (1.4) as an integral equation of the form (1.1) for $u(x)=S(x)$ with

$$\kappa(x) = \left[\frac{1-\varepsilon}{2}\right] \int_{-\infty}^{\infty} \varphi^2(s) E_1(|x|\varphi(s)) ds \ , \tag{1.5}$$

and

$$f(x)=\varepsilon B(x) \ .$$

To be consistent with the notation in the literature (see for example [8]), we let

$$K_1(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} \varphi^2(s') E_1(x\varphi(s')) ds' \ , \tag{1.6}$$

and so in this example we have

$$\kappa(x)=(1-\varepsilon)K_1(|x|) \ .$$

Now the form of $\varphi(s)$ depends on the type of non-coherent scattering that is present. In [8], Doppler and impact broadening are discussed, as well as their combined effect. The respective profiles are

$$\varphi(s) = \frac{1}{\sqrt{\pi}} \exp(-s^2) \qquad \textit{DOPPLER} \tag{1.7a}$$

$$\varphi(s) = \frac{a}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{\exp(-y^2)}{a^2+(s-y)^2} dy \qquad \textit{VOIGT} \tag{1.7b}$$

$$\varphi(s) = \frac{1}{\pi}\frac{1}{1+s^2} \qquad \textit{LORENTZ} , \qquad (1.7c)$$

where a is the ratio of collision to Doppler widths (see [8]).

In Chapter 5 we will be interested in the case when Doppler broadening is dominant. We show that, in this case, there is a unique solution, $S \in C$, to the integral equation problem, and hence a unique solution, I, to (1.4). For this case, we also solve (1.1) numerically and give the results in Chapter 5.

Example 3. Electromagnetic Wave Refraction

In the study of the refraction of plane electromagnetic waves, Grinberg and Fok [23], (see also Ivanov [26]), obtained the integral equation (1.1) with

$$\kappa(x) = \frac{\lambda}{\pi}\int_0^\infty \frac{\cos(\nu|x|)}{(1+\nu^2)^{\frac{1}{2}}}\,d\nu , \qquad x>0,\ \ \lambda \notin (1,\infty) , \qquad (1.9)$$

and

$$f(x) = e^{-x} .$$

In Chapter 5 we present a more detailed analysis of (1.1) for this example.

The theory of Wiener-Hopf equations is discussed extensively in Krien [29]. In this thesis we shall be concerned with the analysis and numerical solution of (1.1). Atkinson [6], Sloan and Spence [40], and Chandler and Graham [12], have tackled this problem, each from slightly different points of view. The fundamental paper on the numerical solution of (1.1) is Atkinson [6]. In the most recent of these papers [12], the authors provide an analysis of a composite Nystrom (quadrature) method for (1.1) for the case when $\kappa(x)$ is smooth, and decaying sufficiently fast as $|x| \to \infty$. The Nystrom method essentially consists of approximating the integral term, Ku, in (1.1) by a quadrature rule and then setting the arbitrary point x (in the approximate equation) equal to each quadrature point in turn, (i.e collocation at the quadrature points), thus yielding a linear system for the values of u at the quadrature points. Quadrature methods for integral equations of the second kind over infinite intervals are discussed in [39] and [18]. In [5] numerical approximation schemes of quadrature type are investigated for (1.1) in the case where κ is bounded and uniformly continuous.

Sloan and Spence [40] have analysed a piecewise constant collocation method for (1.1). This method was proposed (but not analysed) by Finn and Jefferies [16] for the solution of a class of integral equations arising in astrophysics. Note that the integral equation reformulation of the Transport Equation (1.4), with kernel given by (1.5), arises in the astrophysical examples of Finn and Jefferies.

In many practical examples of (1.1), the kernel is not smooth. In fact, the kernels given by (1.3), (1.5) (with φ given by (1.7a), (1.7b) and (1.7c)) and (1.9), arising in Examples 1 to 3 above, have a logarithmic singularity at the origin. The main purpose of this thesis is to propose and analyse a numerical method for the solution of (1.1) which will cover a number of these important practical examples.

To introduce our numerical method (§1.4.2) we need to make some simple assumptions on (1.1). The analysis (Chapter 4) will be carried out under more detailed assumptions given in Chapter 3. So, suppose that κ is integrable on $\mathbf{R}$, continuous on $\mathbf{R}\backslash\{0\}$ and has a logarithmic singularity at $x=0$. More explicitly, assume

$$\kappa(x) = a(x)b(x), \tag{1.10}$$

where the factor $a(x)$ is smooth but may be complicated. The factor $b(x)$ will be assumed to have the form

$$b(x) = \log(|x|) + c(|x|), \tag{1.11}$$

where c is smooth and sufficiently simple so that certain integrals involving $b(x)$ are known analytically in $|x|<\delta$ for some δ (see (1.27) below). Suppose also that $f \in C$, and that (1.1) has a unique solution $u \in C$ for every right hand side $f \in C$. Finally, assume that $u(\infty) = f(\infty) = 0$. This assumption yields no loss of generality, since, if necessary, (1.1) can always be arranged to have this property (see Chapter 3, Remark 1).

Since we are allowing a weak singularity in the kernel, we cannot use the Nystrom method on $\mathbf{R}^+$. Now the standard way to deal with weakly singular integral equations defined on a finite interval is to use product integration which is discussed in Atkinson [7], Chandler [10] and Schneider [37]. Product integration treats the weakly singular factor in the kernel exactly, and approximates only the smooth part of the integrand by a suitable Lagrange interpolation polynomial, (see §1.4.1).

Monegato and Colombo [30] describe a product integration method for the discretisation of a linear Transport Equation, where the geometry considered is a multilayered system composed of a finite number of homogeneous slabs, infinite in both the y and z directions, but of finite extent in the x direction. This yields integral equations on a finite interval, and so their results are closely related to [7], [10] and [37].

The numerical method that we propose for the solution of (1.1) is a hybrid method consisting of a composite Nystrom method away from the singularity in the kernel, and product integration techniques in a region containing the singularity. Thus, we call our numerical method the "Nystrom-Product Integration" method. The Nystrom-Product Integration method for (1.1) has not previously been analysed, although Atkinson [7] and Thomas [41] have implemented a similar procedure for a simpler class of equations defined on a finite interval. However, the efficient handling of the infinite interval turns out to be a delicate issue and is one of the main points of the thesis.

The plan of the thesis then, is as follows: In §1.2 and §1.3 we review briefly some background material. In §1.4.1 we define an interpolation process and an associated quadrature rule for functions $v \in C \cap L^1$ (where L^1 is the space of Lebesgue integrable functions on $\mathbf{R}^+$), on a mesh $\Pi_n : 0 = x_0 < x_1 < \ldots < x_n = \infty$. Then in §1.4.2 we describe in detail the Nystrom-Product Integration method for the solution of (1.1), based on the interpolation process and associated quadrature rule introduced in §1.4.1.

In Chapter 2 we investigate the convergence of the interpolation process and quadrature rule described in § 1.4.1, for functions $v \in C \cap L^1$ which are smooth on $(0,\infty)$, and which exhibit certain types of behaviour as $x \to 0$ and as $x \to \infty$. We show that if Π_n is suitably chosen then it is possible, in many cases, to obtain the same orders of convergence that occur for smooth functions on finite intervals. The results of Chapter 2 constitute a considerable advance in the theory of interpolation and numerical integration on $\mathbf{R}^+$. The technical results of Chapter 2 will be used in the analysis of our numerical method for (1.1) (see Chapter 4).

In Chapter 3 we present regularity results for the solution, u, of (1.1) under detailed assumptions on κ and f. These assumptions can be easily verified for a number of important practical examples (see Chapter 5). The regularity results obtained in Chapter 3 are needed in order to analyse properly the rate of convergence of our numerical method. The analysis of Chapter 2 will cover the convergence of the interpolation process and quadrature rule for u under the assumptions of Chapter 3.

In Chapter 4 we use the regularity results of Chapter 3 and the technical results of Chapter 2 to provide a convergence analysis for $u - u_n$ in the space C (u_n is the Nystrom-Product Integration approximation to u, see (1.29) below), using specific meshes Π_n. Our analysis will cover the integral equations arising in Examples 1 to 3 above (see Chapter 5).

In Chapter 5 we implement our numerical method on a particular test example for which the solution is known, and we provide numerical results for this example. We give the details of the reformulations of the Transport Equations, in Examples 1 and 2 above, as integral equations of the form (1.1) for the source function, $S(x)$. We discuss the practical implementation of the Nystrom-Product Integration method for these examples and we present numerical results. Finally, we discuss the relevance of the results of this thesis to Example 3 above.

1.2 THE EMBEDDING METHOD

The embedding method is a method of transforming integral equations into initial value problems (or Cauchy systems). This approach is discussed extensively by Kagiwada and Kalaba [28] for a variety of cases. In particular, they derive an initial value problem from an integral equation of radiative transfer, first physically and then analytically, and present numerical results for the source function and certain auxiliary functions of physical interest. They also discuss the embedding method for solving integral equations with general displacement kernels. As an example, consider the system of integral equations :

$$J(x,\mu) = Ie^{-x/\mu} + \int_0^\infty K(|x-t|)J(t,\mu)dt\ , \qquad x\in\mathbf{R}^+,\ 0\le\mu\le 1, \qquad (1.12)$$

where K is a symmetric $n\times n$ matrix given by

$$K(x) = \int_0^1 e^{-x/\mu'}W(\mu')d\mu'\ , \quad x>0\ , \qquad (1.13)$$

and W is a symmetric $n\times n$ matrix, I is the unit $n\times n$ matrix and J, the solution to be found, is an $n\times n$ matrix. Further assume

$$K(\infty) = J(\infty,\mu) = 0\ , \quad 0\le\mu\le 1\ .$$

(1.12) arises in several branches of physics including optimal filtering and astrophysics. In fact, when μ in (1.12) is constant and n=1, (1.12) reduces to an equation of the form (1.1) with $\kappa(x)=K(|x|)$. In this case, the kernel (1.3) of Example 1 above is of the form (1.13). Kagiwada and Kalaba show that (1.12) is equivalent to the following Cauchy system

$$J_x(x,\mu) = -\mu^{-1}J(x,\mu) + \left[\int_0^1 J(x,\mu')W(\mu')d\mu'\right]H(\mu), \quad x\ge 0,$$

where

$$H(\mu) = J(0,\mu)\ , \quad 0\le\mu\le 1,$$

and $H(\mu)$ satisfies the non-linear equation

$$H(\mu) = I + \int_0^1 W(\mu')\frac{\mu'\mu}{\mu'+\mu}H^T(\mu')H(\mu)d\mu'\ , \quad 0\le\mu\le 1.$$

(Note J_x denotes differentiation with respect to x, and superscript T denotes transposition). The disadvantage of solving (1.12) (and certain other integral equations) using the embedding approach is that, in order to solve the initial value problem, it is necessary to solve a non-linear equation, which may prove quite difficult in practice. In general though, for many integral equations, certain functionals of the solution often have important physical significance, and the embedding approach can provide a simple algebraic means of determining these functionals. However, if the quantities of interest can be obtained directly from the solution of the integral equation, then the Nystrom-Product Integration method may be used efficiently. The embedding approach provides different techniques for solving integral equations, and so is of considerable comparative interest.

1.3 THE DISCRETE ORDINATES METHOD

The numerical solution of the Transport Equation using the Discrete ordinates method is widely discussed in the literature. In particular, a full account of this method for solving (1.2) and (1.4) is given in Chandrasekhar [14] and Hummer and Rybicki [25], respectively. Neither contains a complete error analysis.

As an illustration of the method, we describe a discrete ordinates method for the solution of (1.2) when the region being modelled is a finite slab of unit thickness, and there is no radiation incident on the boundaries. In this case the Transport Equation is given by

$$\mu \frac{\partial I}{\partial x}(x,\mu) - I(x,\mu) + \left[\frac{1-\varepsilon}{2}\right] \int_{-1}^{1} I(x,\eta)d\eta = -\varepsilon B(x), \quad 0 \le x \le 1 \ , \ -1 \le \mu \le 1, \tag{1.14}$$

with

$$I(0,\mu)=0 \text{ for } \mu<0, \quad I(1,\mu)=0 \text{ for} \mu>0,$$

(see, for example Pitkaranta and Scott [32] and Anselone [3]). The Discrete ordinates method essentially consists of introducing a quadrature rule on [-1,1],

$$\int_{-1}^{1} \varphi(\mu)d\mu \approx \sum_{|k|=1}^{n} w_k \varphi(\mu_k) \ ,$$

and then replacing (1.14) by the system

$$\mu_j \frac{\partial}{\partial x} I_j(x) - I_j(x) + \left[\frac{1-\varepsilon}{2}\right] \sum_{|k|=1}^{n} w_k I_k(x) = -\varepsilon B(x) , \quad 0 \le x \le 1, \ |j|=1,\ldots,n , \tag{1.15}$$

$$I_j(0)=0 \text{ for } \mu_j<0, \ I_j(1)=0 \text{ for } \mu_j>0,$$

where $I_j(x)$ approximates $I(x,\mu_j)$. The system of o.d.e.'s, (1.15), is called the semi-discrete approximation to (1.14). Upon further approximating (1.15) by various methods for systems of o.d.e.'s, a fully discrete system for solving (1.14) is obtained. This yields an approximation $S_n(x)$ to $S(x)$ for all x, using

$$S_n(x) = \left[\frac{1-\varepsilon}{2}\right] \sum_{|k|=1}^{n} w_k I_k(x) + \varepsilon B(x).$$

Recall that

$$S(x) = \left[\frac{1-\varepsilon}{2}\right] \int_{-1}^{1} I(x,\eta) d\eta + \varepsilon B(x) .$$

Anselone [3] only proves the convergence of the semi-discrete approximation to (1.14). However, Pitkaranta and Scott [32] give a complete error analysis for both the semi-discrete and the fully-discrete approximation of (1.14). They also provide stability criteria for the combined spatial and angular approximation for particular quadrature rules and spatial differencing schemes. (Related results, for the 2-D Transport Equation with finite geometry, may be found in [27]). In a subsequent paper by Pitkaranta [31], he studies the convergence of a family of projection schemes for solving the discrete-ordinate Transport equation. He assumes a fixed angular discretisation and gives special attention to quantities of physical interest.

The Discrete ordinates method for (1.4) in the case when Doppler scattering is dominant is discussed in [25]. In this case, it turns out that the Discrete ordinates solution of (1.4) has the incorrect functional form asymptotically and the incorrect finite slope at the origin, and it represents $S(x)$ closely only over a finite interval. It seems therefore, that in this case at least, the Transport Equation can be more effectively solved using the integral equation reformulation for $S(x)$. In general, it certainly seems that when the geometry of the problem is simple, our numerical method can be used to solve the integral equation reformulation of the Transport Equation. If, on the other hand, the geometry is more complicated, then the Discrete ordinates method is more favourable.

1.4 THE NYSTROM-PRODUCT INTEGRATION METHOD

1.4.1 PRELIMINARIES

We first describe an interpolation process and an associated quadrature rule for functions $v \in C[0,1]$ (the space of functions which are continuous on $[0,1]$). To this end, choose m points, $0 \le \xi_1 < \xi_2 < \ldots < \xi_m \le 1$, in $[0,1]$, and let Pv be the Lagrange polynomial of order m which interpolates v at ξ_j, $j=1,\ldots,m$. That is

$$Pv(x) = \sum_{j=1}^{m} v(\xi_j) L_j(x) ,$$

where

$$L_j(x) = \prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{(x-\xi_k)}{(\xi_j-\xi_k)} , \quad j=1,\ldots,m.$$

Standard results show that if $v \in C^m[0,1]$ (the space of functions which have m continuous derivatives on $[0,1]$) then for each $x \in [0,1]$, a number $\xi(x)$ in $[0,1]$ exists with

$$v(x) = Pv(x) + \frac{D^m v(\xi(x))}{m!} \prod_{j=1}^{m} (x-\xi_j) , \tag{1.16}$$

(see, for example, [9]), where the second term on the right hand side of (1.16) is known as the *truncation error*. Then, to approximate

$$\int_0^1 v(x)dx ,$$

we integrate $Pv(x)$ and the truncation error over $[0,1]$ to obtain the quadrature formula

$$\int_0^1 v(x)dx = \int_0^1 \sum_{j=1}^{m} v(\xi_j)L_j(x)dx + \frac{1}{m!}\int_0^1 D^m v(\xi(x))\prod_{j=1}^{m}(x-\xi_j)dx$$

$$= \sum_{j=1}^{m} w_j v(\xi_j) + \frac{1}{m!}\int_0^1 D^m v(\xi(x))\prod_{j=1}^{m}(x-\xi_j)dx \ , \qquad (1.17)$$

where w_j are the weights of the quadrature formula given by

$$w_j = \int_0^1 L_j(x)dx \ , \qquad j=1,\ldots,m \ . \qquad (1.18)$$

We now define an associated quadrature rule on $[0,1]$ by

$$\int_0^1 v(x)dx \approx \sum_{j=1}^{m} w_j v(\xi_j) \ . \qquad (1.19)$$

The error in this quadrature rule is given by the second term on the right hand side of (1.17). If this error term vanishes for polynomials of degree $R-1$, we say that the quadrature rule is of order R, where $m \le R \le 2m$. For example, $R=2$ for the Trapezoidal rule $(m=2,\ \xi_1=0,\ \xi_2=1)$, and $R=4$ for Simpson's rule $(m=3,\ \xi_1=0,\ \xi_2=\frac{1}{2},\ \xi_3=1)$.

We can now define an interpolation process and an associated quadrature rule for functions $v \in C \cap L^1$. So for $n \in \mathbf{N}$, introduce a mesh, $\Pi_n : 0=x_0<x_1<\ldots<x_n=\infty$, on $\mathbf{R}^+$. For $i=1,\ldots,n$ set $I_i=(x_{i-1},x_i)$ and define v_i by $v_i=v$ on I_i and $v_i=0$ on $\mathbf{R}^+ \setminus I_i$.

Recall the m points, $0 \le \xi_1 < \xi_2 < \ldots < \xi_m \le 1$, chosen above. Shift them linearly to each $[x_{i-1},x_i]$, $(i=1,\ldots,n-1)$ by the formula

$$x_{ij} = x_{i-1} + \xi_j h_i \ , \qquad j=1,\ldots,m \ ,$$

where $h_i = x_i - x_{i-1}$.

Now define $P_n v$ on each I_i for $i=1,\ldots,n-1$, to be the Lagrange polynomial of order m which interpolates v_i at x_{ij} for $j=1,\ldots,m$, and set $P_n v \equiv 0$ on I_n. That is

$$P_n v_i(x) = \sum_{j=1}^{m} \left[\prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{(x - x_{ik})}{(x_{ij} - x_{ik})} \right] v(x_{ij}) \ , \quad i=1,\ldots n-1 \ ,$$

(1.20)

$$P_n v_n \equiv 0 \ .$$

Finally extend $P_n v$ to all of $\mathbf{R}^+$ by requiring that $P_n v$ be right continuous at each x_i, $i=1,\ldots,n-1$.

We now define an associated quadrature rule on each I_i, $i=1,\ldots,n-1$, by

$$\int_{I_i} v \approx \int_{I_i} P_n v = \sum_{j=1}^{m} w_j v(x_{ij}) h_i \ , \qquad (1.21)$$

where w_j are the weights of the quadrature rule, (1.19), on $[0,1]$, with points ξ_j, $j=1,\ldots,m$. Note that this rule integrates exactly functions which reduce to polynomials of order R on each I_i, $i=1,\ldots,n-1$.

Thus, a perfectly well defined composite quadrature rule on $\mathbf{R}^+$ is given by

$$\int_0^\infty v \approx \int_0^\infty P_n v = \int_0^{x_{n-1}} P_n v \ . \qquad (1.22)$$

However, for the purposes of the analysis of our numerical method (Chapter 4), it will be necessary to allow a greater generality in the choice of *cut–off* point in (1.22). So for some $n > i* \geq 1$ (to be chosen), we define a composite quadrature rule on $\mathbf{R}^+$ by first introducing the approximation

$$\int_0^\infty v \approx \int_0^{x_{n-i*}} v \ .$$

Then we approximate the right hand integral by

$$\int_0^{x_{n-i*}} v \approx \int_0^{x_{n-i*}} P_n v \ , \qquad (1.23)$$

and overall this yields

$$\int_0^\infty v \approx \sum w_j v(x_{ij}) h_i \ , \qquad (1.24)$$

where the sum is over $i=1,\ldots,n-i*$, and $j=1,\ldots,m$.

1.4.2 THE NYSTROM-PRODUCT INTEGRATION METHOD

We shall approximate the integral term, $Ku(x)$, of (1.1) by a process similar to that in (1.24), except that, for t close to x, we shall employ a product integration technique, based on the singular expansion (1.10), (1.11). Accordingly for each $x \in \mathbf{R}^+$, $1 \le i* < n$, we divide $[0, x_{n-i*}]$ into two subregions, one "singular region" which may contain x, and the other region excluding x. It will be numerically convenient if the boundaries between these regions are mesh points.

So, suppose we have chosen some minimum radius $\delta > 0$ for our singular region. Then define

$$\underline{I} = \max\left\{ 1 \le i \le n-i* : x - x_{i-1} \ge \delta \right\},$$

if this maximum exists. Otherwise set

$$\underline{I} = 1.$$

Similarly define

$$\overline{I} = \min\left\{ 1 \le i \le n-i* : x_i - x > \delta \right\},$$

if this minimum exists, and otherwise set

$$\overline{I} = n-i*.$$

Then define the singular region by

$$\Omega^{[1]}(x) = \cup\left\{ I_i : \underline{I} \le i \le \overline{I} \right\}$$

and set

$$\Omega^{[2]}(x) = [0, x_{n-i*}] \setminus \Omega^{[1]}(x).$$

Finally set

$$Q^{[k]}(x) = \left\{ i : I_i \subseteq \Omega^{[k]}(x) \right\}, \text{ for } k=1,2.$$

Then let

$$K_{i*}u(x) = \int_0^{x_{n-i*}} \kappa(x-t)u(t)dt$$

$$= \int_{\Omega^{[1]}(x)} \kappa(x-t)u(t)dt + \int_{\Omega^{[2]}(x)} \kappa(x-t)u(t)dt$$

$$:= K_{i*}^{[1]}u(x) + K_{i*}^{[2]}u(x), \tag{1.25}$$

say. Now approximate K_{i*} (and hence K) by

$$K_{i*,n} := K_{i*,n}^{[1]} + K_{i*,n}^{[2]}, \tag{1.26}$$

where

$$K_{i*,n}^{[1]}\ u(x) = \int_{\Omega^{[1]}(x)} \left[b(x-t)P_n(a(x-\cdot)u(\cdot))(t) \right] dt \tag{1.27}$$

and

$$K_{i*,n}^{[2]}(x) = \int_{\Omega^{[2]}(x)} P_n(\kappa(x-\cdot)u(\cdot))(t)dt\ . \tag{1.28}$$

We assume that the factorisation (1.10), (1.11) is such that the integral $K_{i*,n}^{[1]}\ u(x)$ in (1.27) may be found analytically. Now we define the Nystrom-Product Integration approximation u_n to u by replacing K in (1.1) by $K_{i*,n}$. That is, u_n satisfies the approximate equation

$$u_n - K_{i*,n}\ u_n = f. \tag{1.29}$$

The unknowns in the function $K_{i*,n}\ u_n$ are the values of u_n at the quadrature points, x_{ij}, for all $i=1,\ldots,n-i*$, $j=1,\ldots,m$. Hence collocation of (1.29) at the quadrature points, i.e.

$$u_n(x_{kl}) - K_{i*,n}u_n(x_{kl}) = f(x_{kl}),\ \ k=1,\ldots,n-i*,\ l=1,\ldots,m\ , \tag{1.30}$$

yields a linear system of the form

$$(I-P)\mathbf{u} = \mathbf{f}, \tag{1.31}$$

where

$$\mathbf{u} = \{u_n(x_{ij}) : i=1,\ldots,n-i*,\ j=1,\ldots,m\}$$

$$\mathbf{f} = \{f(x_{ij}) : i=1,\ldots,n-i*,\ j=1,\ldots,m\},$$

where P is an order $(n-i*)\times m$ matrix and I is the identity matrix of order $(n-i*)m$.

In implementing (1.31), the elements of **u** must be ordered in a convenient fashion. Here we assume they are ordered as

$$(u_{11},\ldots,u_{1m},\ u_{21},\ldots,u_{2m},\ldots,u_{(n-i*)m}),$$

and that the rows and columns of P are ordered analogously. Then for $k=1,\ldots,n-i*$, $l=1,\ldots,m$, the row of P corresponding to collocation of (1.29) at x_{kl} consists of the coefficients of $u_n(x_{ij})$ in $K_{i*,n}u_n(x_{kl})$ (correctly ordered), for $i=1,\ldots,n-i*$, $j=1,\ldots,m$.

Consider, as an illustration, the calculation of P in the simple case where the underlying quadrature rule is the midpoint rule. Then $m=1$, $\xi_1=\frac{1}{2}$, $w_1=1$, and there is one quadrature point in each I_i, which we denote by $x_{i1} = x_{i-1} + (\frac{1}{2})h_i$. Then, for each k

$$K_{i*n}u_n(x_{k1}) = \sum_{i\in Q^{[1]}(x_{k1})} \left[\int_{I_i} b(x_{k1}-t)\,dt\right] a(x_{k1}-x_{i1})u_n(x_{i1})$$

$$+ \sum_{i\in Q^{[2]}(x_{k1})} h_i\kappa(x_{k1}-x_{i1})u_n(x_{i1}).$$

Let $p_{k1,i1}$ denote the coefficient of $u_n(x_{i1})$ in this expression. Then, if $i\in Q^{[1]}(x_{k1})$

$$p_{k1,i1} = \left[\int_{I_i} b(x_{k1}-t)\,dt\right] a(x_{k1}-x_{i1}), \tag{1.32}$$

and if $i\in Q^{[2]}(x_{k1})$

$$p_{k1,i1} = h_i\kappa(x_{k1}-x_{i1}). \tag{1.33}$$

Thus setting up (1.31), in this case, requires the calculation of (1.32), (1.33) and the assembly of P according to the appropriate ordering.

For higher order quadrature rules a similar but more involved procedure will be carried out. Once all the $u_n(x_{ij})$ have been found, $K_{i*,n}u_n(x)$ is known for all $x\in\mathbf{R}^+$, and hence $u_n(x)$ is also known for all $x\in\mathbf{R}^+$, via (1.29).

Now return to (1.1) and its approximation (1.29). Provided $(I - K_{i*n})^{-1}$ exists then

$$u - u_n = (I - K_{i*,n})^{-1}(K - K_{i*,n})(I - K)^{-1}f$$
$$= (I - K_{i*,n})^{-1}(K - K_{i*,n})u \ . \tag{1.34}$$

In Chapter 4 we give a convergence analysis for $u - u_n$ in the space C under certain assumptions (given in Chapter 3). This will consist of two parts. We first prove a stability result which gives a bound on $\| (I - K_{i*,n})^{-1} \|_\infty$. We then prove consistency results which show the order of convergence of $\| (K - K_{i*,n})u \|_\infty$ as $n \to \infty$. The convergence analysis will be carried out using specific meshes of the type studied in Chapter 2.

The Nystrom-Product Integration method introduced above is particularly useful for solving several practical examples of (1.1), including Examples 1 to 3 above (see Chapter 5). As described above, results related to the analysis of our method may be found in the literature. In some sense, our analysis will generalise the results of Atkinson [7], Chandler [10], Schneider [37], Sloan and Spence [40] and Chandler and Graham [12].

CHAPTER 2
GRADED MESHES

2.1 INTRODUCTION

In Chapter 1, §1.4.1 we introduced the interpolation operator, P_n, of order m, on a mesh $\Pi_n: 0=x_0<x_1<\ldots<x_n=\infty$, (see (1.20)), and the associated quadrature rule of order R, (see (1.24)), for functions $v \in C \cap L^1$. In this Chapter we investigate the convergence of the interpolation process and quadrature rule for functions $v:\mathbf{R}^+ \rightarrow \mathbf{C}$, which are smooth on $(0,\infty)$ and which exhibit certain types of behaviour as $x \rightarrow \infty$ and as $x \rightarrow 0$. For the convergence analysis of this Chapter, we will take $i*=1$ in (1.23). There is no loss in generality in choosing $i*=1$ since the analysis for $i*>1$ follows analogously. However, for the analysis of our numerical method for (1.1), we will allow greater generality in the choice of $i*$ (see Chapter 4). Our objective in this Chapter is to show that, provided Π_n is suitably chosen, it is possible to obtain the same orders of convergence for interpolation and quadrature of poorly behaved functions on $\mathbf{R}^+$ that occur for smooth functions on finite intervals. In order to show this we require the next two propositions which are necessary technical preliminaries.

Notation

For any $v:\mathbf{R} \rightarrow \mathbf{C}$, and any $J \subseteq \mathbf{R}$ define

$$\|v\|_{\infty,J} = \sup_{x \in J} |v(x)|, \quad \|v\|_{1,J} = \int_J |v| .$$

When $J = \mathbf{R}^+$ we abbreviate these by $\|v\|_\infty$, $\|v\|_1$. Let D denote differentiation.

Proposition 2.1

Let $v \in C$ and let Π_n be any mesh.

(i) *There exists a constant* C *independent of n,i and v, such that for all $1 \le i \le n$,*

$$\|P_n v_i\|_\infty \le \mathrm{C} \|v_i\|_\infty .$$

(ii) *For all* $1 \le l \le m$, *if* $\|(D^l v)_i\|_\infty < \infty$, *there exists a constant* C *independent of* n, i *and* v, *such that for* $1 \le i \le n-1$,

$$\|(I-P_n)v_i\|_\infty \le Ch_i^l \|(D^l v)_i\|_\infty \ .$$

Proof

(i) For $i=n$ the result follows trivially since $P_n v_n \equiv 0$ by definition. For $1 \le i \le n-1$ we have

$$\|P_n v_i\|_\infty = \max_{x \in I_i} \left| \sum_{j=1}^{m} \left[\prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{(x-x_{ik})}{(x_{ij}-x_{ik})} \right] v(x_{ij}) \right| \ .$$

Now note that if $x \in [x_{i-1}, x_i]$, then for some $\xi \in [0,1]$ we have

$$\begin{aligned} x - x_{ik} &= (x_{i-1} + \xi h_i) - (x_{i-1} + \xi_k h_i) \\ &= (\xi - \xi_k) h_i \ . \end{aligned}$$

Also note that

$$x_{ij} - x_{ik} = (\xi_j - \xi_k) h_i \ ,$$

and thus it follows that

$$\begin{aligned} \|P_n v_i\|_\infty &\le \max_{\xi \in [0,1]} \sum_{j=1}^{m} \left[\prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{|(\xi - \xi_k)|}{|(\xi_j - \xi_k)|} \right] \|v_i\|_\infty \\ &\le C \|v_i\|_\infty \ , \end{aligned}$$

as required.

(ii) If ψ is the Taylor polynomial for v of degree $l-1$ about x_i, then $(P_n \psi)_i = \psi_i$. Hence, (i) above and Taylor's Theorem give

$$\|(I-P_n)v_i\|_\infty = \|(I-P_n)(v-\psi)_i\|_\infty \le C \|(v-\psi)_i\|_\infty \le Ch_i^l \|(D^l v)_i\|_\infty \ . \ \square$$

Remarks

(i) Proposition 2.1(i) shows that $\|P_n\|_\infty$ is uniformly bounded. For the rest of this thesis we will use this result without explicit reference to Proposition 2.1(i).

(ii) The case of Proposition 2.1(ii) that we will be most interested in, is that in which $l=m$. In this case, if $\|(D^m v)_i\|_\infty<\infty$ then for $1 \le i \le n-1$ we have

$$\|(I-P_n)v_i\|_\infty \le Ch_i^m \|(D^m v)_i\|_\infty \ . \tag{2.1}$$

(iii) Note that by definition we have

$$\|(I-P_n)v_n\|_\infty = \|v_n\|_\infty \ . \tag{2.2}$$

Proposition 2.2

Let $v \in C$ and let Π_n be any mesh. For all $1 \le l \le R$, if $\|(D^l v)_i\|_1<\infty$, there exists a constant C *independent of n,i and v, such that for $1 \le i \le n-1$,*

$$\left| \int_{I_i} (I-P_n)v \right| \le Ch_i^l \|(D^l v)_i\|_1 \ .$$

Proof

Let ψ be the Taylor polynomial for v of degree $l-1$ about x_i. Then by Taylor's Theorem with integral remainder we have

$$\|(v-\psi)_i\|_\infty \le Ch_i^{l-1} \|(D^l v)_i\|_1 \ ,$$

and hence

$$\|(v-\psi)_i\|_1 \le Ch_i^l \|(D^l v)_i\|_1 \ .$$

Thus, as the quadrature rule is exact for polynomials of degree $l-1$, it follows, using (1.21), that

$$\begin{aligned} \left| \int_{I_i} (I-P_n)v \right| &= \left| \int_{I_i} v - \sum_j w_j v(x_{ij}) h_i \right| \\ &= \left| \int_{I_i} (v-\psi) - \sum_j w_j (v-\psi)(x_{ij}) h_i \right| \\ &\le C[h_i^l \|(D^l v)_i\|_1 + h_i^{l-1} h_i \sum_j |w_j| \ \|(D^l v)_i\|_1] \ , \end{aligned}$$

and the result follows. □

Remarks

(i) The case of Proposition 2.2 that we will be most interested in, is that in which $l=R$. In this case, if $\|(D^R v)_i\|_1<\infty$ then for $1\le i\le n-1$ we have

$$| \int_{I_i}(I-P_n)v \; | \le Ch_i^R \|(D^R v)_i\|_1 \; . \tag{2.3}$$

(ii) Note that by definition we have

$$|\int_{I_n}(I-P_n)v \; | = |\int_{I_n} v \; | \; . \tag{2.4}$$

To motivate the new results that are proved in this Chapter, consider the analogue of the interpolation operator (1.20), and the associated quadrature rule (1.24), for functions $v\in C[0,1]$. If $\Pi'_n: 0=x_0<x_1<\ldots<x_n=1$ is any mesh on [0,1], then Proposition 2.1 and Proposition 2.2 hold for all $I_i\subseteq[0,1]$. Now suppose that Π'_n is the uniform mesh. If $v\in C^m[0,1]$ then for all $1\le i\le n$, $\|(D^m v)_i\|_\infty<\infty$, and $h_i^m=O(1/n^m)$. Then (2.1) shows that

$$\|(I-P_n)v\|_\infty = O\left[\frac{1}{n^m}\right] .$$

Similarly, if $v\in C^R[0,1]$ then for all $1\le i\le n$, $\|(D^R v)_i\|_1<\infty$, and $h_i^R=O(1/n^R)$, and then (2.3) shows that

$$| \int_0^1 (I-P_n)v \; | \le C \sum_{i=1}^{n} h_i^R \|(D^R v)_i\|_1 = O\left[\frac{1}{n^R}\right] .$$

In fact, saturation results in [34] show that if there exists a mesh such that the interpolation error is $o(1/n^m)$ as $n\to\infty$, then v is a polynomial of order m. Thus, we call $O(1/n^m)$ convergence *optimal* for the interpolation process. Now even if $v\notin C^m[0,1]$, it is still possible, in many cases, to achieve optimal interpolation results. To see this consider (2.1). If we could grade the mesh so that h_i is small enough when $\|(D^m v)_i\|_\infty$ is large, so that for each I_i, $i=1,\ldots,n$, the error, given by (2.1), is of $O(1/n^m)$, then we have regained the optimal results for uniform meshes when $v\in C^m[0,1]$, (see for example [33], [37]). What we are effectively doing is placing more mesh points in regions where v is "badly" behaved. By a similar reasoning, $O(1/n^R)$ convergence may be possible for the quadrature even if $v \notin C^R[0,1]$. From now on we will call $O(1/n^R)$ convergence for the quadrature rule *optimal*.

For functions $v \in C \cap L^1$, a similar, but a considerably more involved procedure, is required to obtain optimal orders of convergence for the interpolation process (1.20) and the quadrature rule (1.24). This is because approximation techniques over the whole of $\mathbf{R}^+$ are required and the problem of regaining optimal convergence results becomes much more difficult. However, in this Chapter we show that for certain functions $v \in C \cap L^1$, categorised in the next section, we can also achieve $O(1/n^m)$ convergence for the interpolation process and $O(1/n^R)$ convergence for the quadrature rule, provided Π_n (still a mesh with n subintervals, but now over $\mathbf{R}^+$) is suitably chosen. The results of this Chapter will be very important in the analysis of our numerical method for (1.1), and are indeed a considerable advance in the theory of interpolation and numerical integration over infinite intervals.

Throughout this Chapter we illustrate some of our theoretical results by numerical experiment. So, for $k,N \in \mathbf{N}$ we let e_N be some measure of the interpolation or quadrature error, on a mesh with $n \le kN$ subintervals. Then, when we wish to demonstrate that $e_N = O(1/n^\lambda)$ for some $\lambda \in \mathbf{R}^+$, we estimate the order of convergence by

$$EOC = \frac{\log(e_N/e_{2N})}{\log 2} \tag{2.5}$$

Note that when $k=1$ and $n=N$, we let $e_N = e_n$.

2.2 FUNCTION SPACES

In this section we define certain spaces of functions in which the convergence analysis will be carried out.

Let $l \in \mathbf{N}$, $\mu > 0$ and $0 < \beta \leq l$. For any $v:\mathbf{R}^+ \to \mathbf{C}$ define

$$\|v\|_{l,\beta} = \max\left\{ \sup_{x\in[0,1]} |v(x)|,\ \sup_{x\in[0,1]} |x^{[j-\beta]} D^j v(x)|,\ j=1,\ldots,l \right\},$$

where

$$[j-\beta] = \begin{cases} j-\beta & \text{if } j \geq \beta \\ 0 & \text{if } j \leq \beta \end{cases}$$

Remarks

(i) If $\|v\|_{l,\beta} < \infty$ then v has l continuous derivatives on $(0,1]$.

(ii) If $\|v\|_{l,\beta} < \infty$ for $0 < \beta < 1$, then v is continuous on $[0,1]$ and $D^j v(x) = O(x^{\beta-j})$ as $x \to 0$, $j=1,\ldots,l$. That is $v(x)$ "behaves like" x^β as $x \to 0$.

(iii) If $\|v\|_{l,\beta} < \infty$ for $1 \leq \beta < l$, then for $j \leq \beta$, $D^j v(x)$ is continuous on $[0,1]$. For $l \geq j > \beta$, $D^j v(x) = O(x^{\beta-j})$ as $x \to 0$.

(iv) If $\|v\|_{l,l} < \infty$ then v has l continuous derivatives on $[0,1]$.

THE EXPONENTIAL CASE

For any $v:\mathbf{R}^+ \to \mathbf{C}$ define

$$\|v:e^{-\mu x}\|_l = \max\left\{ \sup_{x\in[1,\infty)} |e^{\mu x} D^j v(x)|,\ j=0,\ldots,l \right\}.$$

If $\|v:e^{-\mu x}\|_l < \infty$ then v has l continuous derivatives on $[1,\infty)$ and $D^j v(x) = O(e^{-\mu x})$ as $x \to \infty$, $j=0,\ldots,l$. That is v "behaves like" $e^{-\mu x}$ as $x \to \infty$.

Now set

$$\|v:e^{-\mu x}\|_{l,\beta} = \max\left\{ \|v\|_{l,\beta}\ ,\ \|v:e^{-\mu x}\|_l \right\}.$$

Definition

Let $C^l_\beta[e^{-\mu x}]$ denote the space of all functions $v:\mathbf{R}^+\to\mathbf{C}$ which have l continuous derivatives on $(0,\infty)$ and for which $\|v:e^{-\mu x}\|_{l,\beta}<\infty$.

THE POLYNOMIAL CASE

For any function $v:\mathbf{R}^+\to\mathbf{C}$ define

$$\|v:x^{-\mu}\|_l=\max\left\{\sup_{x\in[1,\infty)}|x^\mu D^j v(x)|,\ j=0,\ldots,l\right\}.$$

If $\|v:x^{-\mu}\|_l<\infty$ then v has l continuous derivatives on $[1,\infty)$ and $D^j v(x)=O(x^{-\mu})$ as $x\to\infty$, $j=0,\ldots,l$.

Now set

$$\|v:x^{-\mu}\|_{l,\beta}=\max\left\{\|v\|_{l,\beta}\ ,\ \|v:x^{-\mu}\|_l\right\}.$$

Definition

Let $C^l_\beta[x^{-\mu}]$ denote the space of all functions $v:\mathbf{R}^+\to\mathbf{C}$ which have l continuous derivatives on $(0,\infty)$ and for which $\|v:x^{-\mu}\|_{l,\beta}<\infty$.

Functions in $C^l_\beta[e^{-\mu x}]$ and $C^l_\beta[x^{-\mu}]$ have quite complex behaviour on $\mathbf{R}^+$, but cover a wide variety of cases which might arise in practice. We will investigate the convergence of the interpolation process, (1.20), and the associated quadrature rule, (1.24), for functions in these spaces. These results will be of direct relevance to the solution of (1.1) because in Chapter 3 it will be shown that the solution, u, of (1.1) will lie in some one of the spaces $C^l_\beta[e^{-\mu x}]$ or $C^l_\beta[x^{-\mu'}]$ for some $l\in\mathbf{N}$, μ or $\mu'>0$ and for all $0<\beta<1$, when certain assumptions on κ and f are satisfied. The technical results of this Chapter will be used in the convergence analysis (Chapter 4) of the Nystrom-Product Integration method which was introduced in Chapter 1 §1.4.2.

2.3 CONVERGENCE ANALYSIS FOR INTERPOLATION

In this section we investigate the convergence of the interpolation process, (1.20), for functions $v:\mathbf{R}^+\to\mathbf{C}$ which lie in the spaces defined in §2.2. We will restrict the convergence analysis for interpolation to the cases $0<\beta<1$ (unless explicitly stated otherwise). Thus we will be treating the worst singularities in $v(x)$ as $x\to 0$. The analysis for $1\le\beta\le l$ could be carried out using similar methods. We define specific meshes and we consider the interpolation error in each I_i, $i=1,\ldots,n$, (recall (2.1) and (2.2)).

For the rest of this Chapter let C be a generic constant independent of n,i, and v.

2.3.1 THE EXPONENTIAL CASE

The convergence analysis for functions $v\in C^l_\beta[e^{-\mu x}]$ for $0<\beta<1$ is very technical and so, in order to motivate the analysis for functions in this class, we first consider functions $v\in C^l_l[e^{-\mu x}]$ for which the analysis is technically easier.

Note that if $v\in C^l_l[e^{-\mu x}]$ then v has l continuous derivatives on $(0,\infty)$ and

$$\|v\|_{l,l}=\max\left\{\sup_{x\in[0,1]}|D^j v(x)|,\ \ j=0,\ldots,l\right\}<\infty\ ,$$

(that is $v(x)$ has no singularity at x=0). Hence, if $v\in C^l_l[e^{-\mu x}]$ then

$$|||v:e^{-\mu x}|||_l:=\max\left\{\sup_{x\in\mathbf{R}^+}|e^{\mu x}D^j v(x)|,\ \ j=0,\ldots,l\right\}<\infty\ .$$

We consider the interpolation error for functions $v\in C^l_l[e^{-\mu x}]$ using a mesh Π_n which consists of the points

$$x_i=q\log\left[\frac{n}{n-i}\right],\quad i=0,\ldots,n. \tag{2.6}$$

for some q>0. The mesh (2.6) was used by Sloan and Spence [40] and Chandler and Graham [12], when solving particular cases of (1.1) for which the solution was known to decay exponentially at infinity. Note that for $i=1,\ldots,n-1$, we have

$$h_i = q\left[\log\left(\frac{n}{n-i}\right) - \log\left(\frac{n}{n-i+1}\right)\right] = q\log\left(\frac{n-i+1}{n-i}\right)$$

$$= q\log\left(1+\frac{1}{n-i}\right) < q\frac{1}{(n-i)}\ , \tag{2.7}$$

where we have used the fact that $\log(1+x) < x$ for all $x > 0$.

The following proposition is required to prove convergence results for functions $v \in C_l^l[e^{-\mu x}]$.

Proposition 2.3

Let $v \in C_l^l[e^{-\mu x}]$, $0 \le j \le l$ and let Π_n consist of the points defined in (2.6). Then, for all $j \le t$ and for all $1 \le i \le n-1$, we have

$$h_i^t \| (D^j v)_i \|_\infty \le C\frac{1}{n^t} |||v : e^{-\mu x}|||_l\ ,$$

provided $q \ge t/\mu$.

Proof

For convenience set $|||v||| = |||v : e^{-\mu x}|||_l$. Then, using (2.6), we have for $i=1,\ldots,n-1$

$$\| (D^j v)_i \|_\infty \le C\exp(-\mu x_{i-1})\ |||v||| \tag{2.8}$$

$$\le C\exp\left[-\mu q\log\left(\frac{n}{n-i+1}\right)\right] |||v|||$$

$$\le C\left(\frac{n}{n-i+1}\right)^{-\mu q} |||v|||. \tag{2.9}$$

Hence, (2.7), (2.8) and (2.9) show that for all $1 \le i \le n-1$

$$h_i^t \| (D^j v)_i \|_\infty \le C\, h_i^t \exp(-\mu x_{i-1}) |||v||| \tag{2.10}$$

$$\le C\left(\frac{1}{n-i}\right)^t \left(\frac{n}{n-i+1}\right)^{-\mu q} |||v|||$$

$$\le C\frac{1}{n^t}\left(\frac{n-i}{n}\right)^{\mu q - t}\left(\frac{n-i+1}{n-i}\right)^{\mu q} |||v|||$$

$$\le C\frac{1}{n^t} |||v|||, \tag{2.11}$$

since $q \geq t/\mu$, and hence the result follows. □

The next Theorem shows that if $v \in C_m^m[e^{-\mu x}]$ then optimal convergence results can be achieved for the interpolation process using the mesh (2.6) with q appropriately chosen.

Theorem 2.4

Let $v \in C_m^m[e^{-\mu x}]$ and let Π_n be the mesh (2.6). Then

$$\|(I-P_n)v\|_\infty \leq C\frac{1}{n^m}|||v:e^{-\mu x}|||_m .$$

provided $q \geq m/\mu$.

Proof

(2.1) and Proposition 2.3 (with $j=t=m$) show that for $1 \leq i \leq n-1$

$$\|(I-P_n)v_i\|_\infty \leq C\frac{1}{n^m}|||v:e^{-\mu x}|||_m , \qquad (2.12)$$

by our choice of q. Also, from (2.2) and (2.6) we have

$$\begin{aligned} \|(I-P_n)v_n\|_\infty = \|v_n\|_\infty &\leq C\exp(-\mu x_{n-1})|||v:e^{-\mu x}|||_m \\ &\leq C\exp(-\mu q\log(n))|||v:e^{-\mu x}|||_m \\ &\leq C\frac{1}{n^m}|||v:e^{-\mu x}|||_m , \qquad (2.13) \end{aligned}$$

where (2.13) follows from our choice of q. Then, from (2.12) and (2.13), the result follows. □

Example 1

We consider $v(x) = e^{-x} \in C_l^l[e^{-x}]$ for all $l \in$ N. Table 2.1 of §2.3.1.1 gives the results for piecewise quadratic interpolation ($m=3$, $\xi_1=0$, $\xi_2=\frac{1}{2}$, $\xi_3=1$) using the mesh (2.6) with $q=3$. The interpolation error was calculated at 5 equally spaced points in each I_i, $i=1,\ldots,n-1$. We let

$$e_n = \max|(I-P_n)v(x)|,$$

taken over these points. The results indicate that $e_n \approx O(1/n^3)$, as predicted by Theorem 2.4

We now consider functions $v \in C_\beta^l[e^{-\mu x}]$ for $0<\beta<1$. Recall that if $v \in C_\beta^l[e^{-\mu x}]$ for $0<\beta<1$, then v "behaves like" x^β as $x \to 0$ and v "behaves like" $e^{-\mu x}$ as $x \to \infty$. Then, in view of the results of Rice [33] and Theorem 2.4, we consider the interpolation error for functions $v \in C_\beta^l[e^{-\mu x}]$, $0<\beta<1$, using a composite mesh, Π_n, which consists of the points

$$\left[\frac{i}{N}\right]^{q_1}, \quad i=0,\ldots,N\,, \tag{2.14}$$

for some $q_1 \geq 1$ and $N \in$ N, together with the points

$$\left\{ q_2 \log\left[\frac{rN}{rN-k}\right], \; k=1,\ldots,rN \right\} \cap (1,\infty]\,, \tag{2.15}$$

for some $q_2>0$ and $r \in$ N, where $n \leq (r+1)N$. Note that

$$n=O(N)\ ,\ N=O(n). \tag{2.16}$$

The points in (2.14) where used by Rice to approximate x^β for $\beta>0$, on $[0,1]$, using spline approximation techniques, and he proves optimal convergence results using these points. The parameter r, which is included in (2.15), allows that part of Π_n which lies in $(1,\infty]$ to contain more mesh points than are in $[0,1]$ if r is chosen to be greater than 1.

Note that for $i=1,\ldots,N$, the Mean Value Theorem and (2.16) give

$$h_i = \left[\frac{i}{N}\right]^{q_1} - \left[\frac{i-1}{N}\right]^{q_1} \leq q_1 \frac{1}{N}\left[\frac{i}{N}\right]^{q_1-1} = O\left[\frac{1}{n}\right] \tag{2.17}$$

Also, by observing that the set in (2.15) contains $rN-k_0$ points with $k_0 = k_0(N)$ and $0 \leq k_0 < rN$, it follows that $n = (r+1)N-k_0$. Then for $i=N+1,\ldots,(r+1)N-k_0-1$ $(=n-1)$, we have

$$h_i = q_2 \log\left[\frac{(r+1)N-k_0-i+1}{(r+1)N-k_0-i}\right] < q_2\left[\frac{1}{(r+1)N-k_0-i}\right]. \tag{2.18}$$

The following proposition is required to obtain convergence results for functions $v \in C^l_\beta[e^{-\mu x}]$, $0<\beta<1$.

Proposition 2.5

Let $v \in C^l_\beta[e^{-\mu x}]$, $0<\beta<1$, $0\le j\le l$ *and let* Π_n *consist of the points defined in (2.14) and (2.15). For any* $j\le t$, $2\le i\le n-1$ *we have*

$$h_i^t \|(D^j v)_i\|_\infty \le C\frac{1}{n^t}\|v:e^{-\mu x}\|_{l,\beta},$$

provided $q_1 \ge t/\beta$ *and* $q_2 \ge t/\mu$.

Proof

For convenience we set $\|v\| = \|v:e^{-\mu x}\|_{l,\beta}$. Let $0<\beta<1$ and first let $2\le i\le N$. If $j=0$ the result follows trivially from (2.17). So suppose $j\ge 1$, then using (2.14) we have

$$\|(D^j v)_i\|_\infty \le C\, x_{i-1}^{\beta-j}\, \|v\| \le C\left[\frac{i-1}{N}\right]^{q_1(\beta-j)} \|v\|. \tag{2.19}$$

Then from (2.17) and (2.19) we get the estimate

$$h_i^t \|(D^j v)_i\|_\infty \le C\, h_i^t x_{i-1}^{\beta-j}\|v\| \tag{2.20}$$

$$\le C\frac{1}{N^t}\left[\frac{i}{N}\right]^{t(q_1-1)+q_1(\beta-j)}\|v\|$$

$$= C\frac{1}{N^t}\left[\frac{i}{N}\right]^{q_1(t-j)+q_1\beta-t}\|v\|$$

$$\le C\frac{1}{N^t}\|v\|, \tag{2.21}$$

where (2.21) follows since $q_1 \ge t/\beta$ and $t\ge j$.

If $N+1 \le i \le (r+1)N-k_0-1$ $(=n-1)$, then (2.15) shows that for all $j \ge 0$

$$\|(D^j v)_i\|_\infty \le C \exp(-\mu x_{i-1})\|v\| \tag{2.22}$$

$$\le C\left[\frac{rN}{(r+1)N-k_0-i+1}\right]^{-\mu q_2}\|v\|. \tag{2.23}$$

Thus, from (2.18), (2.22) and (2.23), we get the estimate

$$h_i^t\|(D^j v)_i\|_\infty \le C\, h_i^t \exp(-\mu x_{i-1})\|v\| \tag{2.24}$$

$$\le C\left[\frac{1}{(r+1)N-k_0-i}\right]^t\left[\frac{rN}{(r+1)N-k_0-i+1}\right]^{-\mu q_2}\|v\|$$

$$= C\left[\frac{1}{rN}\right]^t\left[\frac{(r+1)N-k_0-i}{rN}\right]^{\mu q_2-t}\left[\frac{(r+1)N-k_0-i+1}{(r+1)N-k_0-i}\right]^{\mu q_2}\|v\|$$

$$\le C\frac{1}{N^t}\|v\|, \tag{2.25}$$

where (2.25) follows from our choice of q_2. Then from (2.21), (2.25) and (2.16) the result follows. □

The next Theorem shows that if $v \in C_\beta^m[e^{-\mu x}]$ for $0<\beta<1$, then optimal convergence results can be achieved for the interpolation process using the mesh defined by (2.14) and (2.15). Thus, we call this the *exponential mesh.*

Theorem 2.6

Let $v \in C_\beta^m[e^{-\mu x}]$, $0<\beta<1$ and let Π_n be the exponential mesh, (2.14) and (2.15). Then

$$\|(I-P_n)v\|_\infty \le C\frac{1}{n^m}\|v:e^{-\mu x}\|_{m,\beta}\ ,$$

provided $q_1 \ge m/\beta$ and $q_2 \ge m/\mu$.

Proof

(2.1) and Proposition 2.5 (with $t=j=m$) show that for $2 \le i \le n-1$

$$\|(I-P_n)v_i\|_\infty \le C\frac{1}{n^m}\|v:e^{-\mu x}\|_{m,\beta}\ , \tag{2.26}$$

by our choice of q_1 and q_2.

Now note that when $x \in I_1$ we have

$$|v_1(x) - v(0)| = | \int_0^x Dv_1(\sigma)d\sigma | \leq C\|v\|_{1,\beta} \int_0^x \sigma^{\beta-1} d\sigma$$

$$\leq C\, h_1^\beta \, \|v\|_{m,\beta} \tag{2.27}$$

$$\leq C \frac{1}{n^m} \|v : e^{-\mu x}\|_{m,\beta}, \tag{2.28}$$

by (2.14), our choice of q_1 and (2.16). Thus we have

$$\|(I-P_n)v_1\|_\infty = \|(I-P_n)(v-v(0))_1\|_\infty \leq C\|(v-v(0))_1\|_\infty \tag{2.29}$$

$$\leq C \frac{1}{n^m} \|v : e^{-\mu x}\|_{m,\beta} \, . \tag{2.30}$$

Also, from (2.2) and (2.15) we have

$$\|(I-P_n)v_n\|_\infty = \|v_n\|_\infty \leq C \exp(-\mu x_{n-1}) \, \|v : e^{-\mu x}\|_{m,\beta}$$

$$\leq C \exp(-\mu q_2 \log(rN)) \|v : e^{-\mu x}\|_{m,\beta}$$

$$\leq C \frac{1}{n^m} \|v : e^{-\mu x}\|_{m,\beta} \, , \tag{2.31}$$

where (2.31) follows from our choice of q_2 and (2.16). Then from (2.26), (2.30) and (2.31) the result follows. □

Example 2

We consider $v(x) = E_2(x)$, the second exponential integral, where

$$E_2(x) = e^{-x} - xE_1(x), \tag{2.32}$$

(recall that E_1 is the first exponential integral). By [1, (5.1.26), (5.1.12), (5.1.51)], $E_2 \in C_\beta^l[e^{-x}]$ for all $0<\beta<1$ and all $l \in \mathrm{N}$.

Table 2.2 of §2.3.1.1 gives the results for piecewise linear interpolation, ($m=2$, $\xi_1=0$, $\xi_2=1$), on the *exponential mesh*, (2.14) and (2.15), with $q_1=2=q_2$ and $r=1$. The interpolation error was calculated at 5 equally spaced points in each I_i, $i=1,\ldots,n-1$. We let

$$e_N = \max |(I-P_n)v(x)|$$

taken over these points. The results indicate that $e_N \approx O(1/n^2)$, as predicted by Theorem 2.6. In order to compute $E_2(x)$ for $x \in \mathbf{R}^+$, NAG routine S13aaf was used to calculate $E_1(x)$ for $x>0$, (at $x=0$, $E_2(x)=1$, see for example [1]).

2.3.1.1 NUMERICAL RESULTS

Example 1

$v(x)=e^{-x}$ - Piecewise quadratic interpolation.

n	e_n	EOC
2	0.232×10^{-1}	
		2.876
4	0.316×10^{-2}	
		2.950
8	0.409×10^{-3}	
		2.976
16	0.520×10^{-4}	
		2.991
32	0.654×10^{-5}	
		2.994
64	0.821×10^{-6}	

TABLE 2.1

Example 2

$v(x)=E_2(x)$ - Piecewise linear interpolation.

N	n	e_N	EOC
2	4	0.793×10^{-1}	
			1.609
4	7	0.260×10^{-1}	
			1.840
8	13	0.726×10^{-2}	
			1.949
16	26	0.188×10^{-2}	
			1.973
32	52	0.479×10^{-3}	
			1.985
64	104	0.121×10^{-3}	

TABLE 2.2

2.3.2 THE POLYNOMIAL CASE

The convergence analysis for functions $v \in C_\beta^l[x^{-\mu}]$ is very technical and so, in order to motivate the analysis, we will define a better behaved class of functions and we will consider the convergence of the interpolation process for functions in this class first.

Let $l \in \mathbf{N}$ and let $\mu > 0$. For any $v:\mathbf{R}^+ \to \mathbf{C}$ define

$$\|v:x^{-\mu}\|'_l = \max\left\{ \sup_{x\in[1,\infty)} |x^{\mu+j}D^j v(x)|,\ \ j=0,\ldots,l \right\}.$$

If $\|v:x^{-\mu}\|'_l < \infty$, then v has l continuous derivatives on $[1,\infty)$ and $D^j v(x) = O(x^{-\mu-j})$ as $x \to \infty$, $j=0,\ldots,l$.

Now for $0<\beta\le l$ set

$$\|v:x^{-\mu}\|'_{l,\beta} = \max\left\{ \|v\|_{l,\beta}\ ,\ \|v:x^{-\mu}\|'_l \right\}.$$

Definition

Let $C_\beta^l[x^{-\mu}]'$ denote the space of all functions $v:\mathbf{R}^+ \to \mathbf{C}$ which have l continuous derivatives on $(0,\infty)$ and for which $\|v:x^{-\mu}\|'_{l,\beta} < \infty$.

The convergence analysis for functions in $C_\beta^l[x^{-\mu}]'$ is technically easier than the convergence analysis for functions in $C_\beta^l[x^{-\mu}]$. This is because if $v \in C_\beta^l[x^{-\mu}]'$ then $D^j v(x) = O(x^{-\mu-j})$ as $x \to \infty$, $j=0,\ldots,l$, whereas, if $v \in C_\beta^l[x^{-\mu}]$ then $D^j v(x) = O(x^{-\mu})$ as $x \to \infty$, $j=0,\ldots,l$. Thus, the jth derivative of functions in $C_\beta^l[x^{-\mu}]'$ decays faster than the jth derivative of functions in $C_\beta^l[x^{-\mu}]$, for $j \ge 1$.

We investigate the interpolation error for functions $v \in C_\beta^l[x^{-\mu}]'$ for $0<\beta<1$. The analysis for $1\le\beta\le l$ could be dealt with by similar means. Again we are considering the worst possible case for β. The mesh that we use is a composite mesh, Π_n, which consists of the points

$$\left(\frac{i}{N}\right)^{p_1}\ ,\quad i=0,\ldots,N\ , \tag{2.33}$$

for some $p_1 \geq 1$ and $N \in \mathbf{N}$, together with the points

$$\left[\frac{rN}{rN-k} \right]^{p_2} , \quad k=1,\ldots,rN , \tag{2.34}$$

for some $p_2 > 0$ and $r \in \mathbf{N}$, where $n = (r+1)N$. Note that in this case we also have

$$n=O(N) \ , N=O(n). \tag{2.35}$$

Thus we are again using the Rice points in (2.33). The parameter r, which is in (2.34), is used for the same reason as in the *exponential mesh*, (2.14) and (2.15), (see discussion following (2.16)). Now for $i= 1,\ldots,N$, the Mean Value Theorem and (2.35) show

$$h_i \leq p_1 \left[\frac{1}{N} \right] \left[\frac{i}{N} \right]^{p_1 - 1} = O \left[\frac{1}{n} \right] \tag{2.36}$$

(see (2.17)).

For $i=N+1,\ldots,(r+1)N-1$ $(=n-1)$, the Mean Value Theorem gives

$$h_i = \left[\frac{rN}{n-i} \right]^{p_2} - \left[\frac{rN}{n-i+1} \right]^{p_2} \leq p_2 \left[\frac{rN}{n-i} - \frac{rN}{n-i+1} \right] \zeta^{p_2 - 1} , \tag{2.37}$$

for some

$$\frac{rN}{n-i+1} \leq \zeta \leq \frac{rN}{n-i} .$$

Now note that for $N < i \leq n-1$ we have

$$1 < \frac{n-i+1}{n-i} \leq 2 , \tag{2.38}$$

and so if $p_2 < 1$

$$\zeta^{p_2-1} \leq \left[\frac{rN}{n-i+1} \right]^{p_2-1} < \left[\frac{rN}{n-i} \right]^{p_2-1} , \tag{2.39}$$

and if $p_2 \geq 1$ we have

$$\zeta^{p_2-1} \leq \left[\frac{rN}{n-i} \right]^{p_2-1} . \tag{2.40}$$

Then, by substituting (2.39) or (2.40) in (2.37) and using (2.35), we obtain

$$h_i \leq C\left[1 - \frac{n-i}{n-i+1}\right]\left[\frac{rN}{n-i}\right]^{p_2} = C\left[\frac{1}{n-i+1}\right]\left[\frac{rN}{n-i}\right]^{p_2}$$

$$\leq C\left[\frac{1}{n-i}\right]\left[\frac{rN}{n-i}\right]^{p_2}. \tag{2.41}$$

The following proposition is required to prove convergence results for functions $v \in C^l_\beta[x^{-\mu}]'$, $0<\beta<1$.

Proposition 2.7

Let $v \in C^l_\beta[x^{-\mu}]'$, $0<\beta<1$, $0\leq j\leq l$ and let Π_n consist of the points defined in (2.33) and (2.34). For any $0\leq t-\mu<j\leq t$, $2\leq i\leq n-1$

$$h_i^t \| (D^j v)_i \|_\infty \leq C \frac{1}{n^t} \| v : x^{-\mu} \|'_{l,\beta} ,$$

provided $p_1 \geq t/\beta$ and $p_2 \geq t/(\mu-(t-j))$.

Proof

For convenience we set $\|v\|' = \|v:x^{-\mu}\|'_{l,\beta}$. Let $0<\beta<1$ and first let $2\leq i\leq N$. Then we have

$$h_i^t \| (D^j v)_i \|_\infty \leq C \frac{1}{N^t} \| v \|'. \tag{2.42}$$

(The proof of (2.42) is analogous to that of (2.21) in Proposition 2.5).

For $i=N+1,\dots,(r+1)N-1$ $(=n-1)$, (2.34) and (2.38) show

$$\| (D^j v)_i \|_\infty \leq C\, x_{i-1}^{-\mu-j} \| v \|' \tag{2.43}$$

$$\leq C\left[\frac{rN}{n-i+1}\right]^{-p_2(\mu+j)} \| v \|'$$

$$\leq C\left[\frac{rN}{n-i}\right]^{-p_2(\mu+j)} \| v \|'. \tag{2.44}$$

Then using (2.41), (2.43) and (2.44) we have

$$h_i^t \| (D^j v)_i \|_\infty \leq C\, h_i^t\, x_{i-1}^{-\mu-j} \| v \|' \tag{2.45}$$

$$\leq C\left[\frac{1}{n-i}\right]^t \left[\frac{rN}{n-i}\right]^{p_2 t} \left[\frac{n-i}{rN}\right]^{p_2(\mu+j)} \| v \|'$$

$$\leq C\left[\frac{1}{rN}\right]^t\left[\frac{n-i}{rN}\right]^{p_2(\mu+j-t)-t}\|v\|'$$

$$\leq C\frac{1}{N^t}\|v\|', \tag{2.46}$$

by our choice of p_2. Hence, the result follows from (2.42), (2.46) and (2.35). □

The following Theorem shows that if $v\in C^m_\beta[x^{-\mu}]'$, $0<\beta<1$, then optimal convergence for the interpolation process can be achieved using the mesh defined by the points in (2.33) and (2.34) with p_1 and p_2 appropriately chosen. Thus, we call this the *polynomial mesh.*

Theorem 2.8

Let $v\in C^m_\beta[x^{-\mu}]'$, $0<\beta<1$, *and let* Π_n *be the polynomial mesh, (2.33) and (2.34). Then*

$$\|(I-P_n)v\|_\infty\leq C\frac{1}{n^m}\|v:x^{-\mu}\|'_{m,\beta}\ ,$$

provided $p_1\geq m/\beta$ *and* $p_2\geq m/\mu$.

Proof

(2.1) and Proposition 2.7 (with $t=j=m$) show that for all $2\leq i\leq n-1$

$$\|(I-P_n)v_i\|_\infty\leq C\frac{1}{n^m}\|v:x^{-\mu}\|'_{m,\beta}\ , \tag{2.47}$$

by our choice of p_1 and p_2.

Now note that (2.29) and (2.27) still hold, and so from (2.29), (2.27) and (2.33) we have

$$\|(I-P_n)v_1\|_\infty\leq C\frac{1}{n^m}\|v:x^{-\mu}\|'_{m,\beta}\ , \tag{2.48}$$

by our choice of p_1 and (2.35). Also, from (2.2) and (2.34) we have

$$\|(I-P_n)v_n\|_\infty=\|v_n\|_\infty\ \leq C\,x_{n-1}^{-\mu}\|v:x^{-\mu}\|'_{m,\beta}$$

$$\leq C(rN)^{-\mu p_2} \| v:x^{-\mu} \|'_{m,\beta}$$

$$\leq C\frac{1}{n^m} \| v:x^{-\mu} \|'_{m,\beta}, \qquad (2.49)$$

by our choice of p_2 and (2.35). Then from (2.47), (2.48) and (2.49) the result follows. □

Example 3

We consider $v(x) = 1/(1+x^2) \in C_3^3[x^{-2}]'$. Table 2.3 of §2.3.2.1 gives the results for piecewise quadratic interpolation, ($m=3$, $\xi_1=0$, $\xi_2=\frac{1}{2}$, $\xi_3=1$), on the *polynomial mesh*, (2.33) and (2.34), with $p_1=1$, $p_2=3/2$ and $r=1$. The interpolation error was calculated at 5 equally spaced points in each I_i, $i=1,\ldots,n-1$. We let

$$e_N = \max |(I-P_n)v(x)|$$

taken over all these points. The results indicate that that $e_N \approx O(1/n^3)$, which is an optimal result.

We now consider the interpolation error for functions $v \in C_\beta^l[x^{-\mu}]$, $0<\beta<1$, using the *polynomial mesh*, (2.33) and (2.34). Thus again we are considering the worst possible case for β. The analysis for $1 \leq \beta \leq l$ could be carried out using similar methods. The next proposition is required to prove convergence results for functions $v \in C_\beta^l[x^{-\mu}]$, $0<\beta<1$.

Proposition 2.9

Let $v \in C_\beta^l[x^{-\mu}]$, $0<\beta<1$, $0 \leq j \leq l$, and let Π_n be the polynomial mesh, (2.33) and (2.34). For any $j \leq t < \mu$, $2 \leq i \leq n-1$

$$h_i^t \| (D^j v)_i \|_\infty \leq C\frac{1}{n^t} \| v:x^{-\mu} \|_{l,\beta} \quad ,$$

provided $p_1 \geq t/\beta$ and $p_2 \geq t/(\mu-t)$.

Proof

For convenience we set $\|v\|=\|v:x^{-\mu}\|_{l,\beta}$. Let $0<\beta<1$ and first let $2\le i\le N$. From (2.42) and (2.35) we have

$$h_i^t \|(D^j v)_i\|_\infty \le C\frac{1}{n^t}\|v\| \quad . \tag{2.50}$$

For $N+1\le i\le (r+1)N-1$ $(=n-1)$, (2.34) and (2.38) show

$$\|(D^j v)_i\|_\infty \le C\, x_{i-1}^{-\mu}\|v\| \tag{2.51}$$

$$\le C\left[\frac{rN}{n-i+1}\right]^{-\mu p_2}\|v\|$$

$$\le C\left[\frac{rN}{n-i}\right]^{-\mu p_2}\|v\| \quad . \tag{2.52}$$

Then using (2.41), (2.51) and (2.52) we have

$$h_i^t\|(D^j v)_i\|_\infty \le C\, h_i^t\, x_{i-1}^{-\mu}\|v\| \tag{2.53}$$

$$\le C\left[\frac{1}{n-i}\right]^t\left[\frac{rN}{n-i}\right]^{tp_2}\left[\frac{n-i}{rN}\right]^{\mu p_2}\|v\|$$

$$= C\left[\frac{1}{rN}\right]^t\left[\frac{n-i}{rN}\right]^{p_2(\mu-t)-t}\|v\|$$

$$\le C\frac{1}{n^t}\|v\| \;, \tag{2.54}$$

by our choice of p_2 and (2.35). The result then follows from (2.50) and (2.54). □

The crucial feature here is (2.51) where the factor $x_{i-1}^{-\mu}$ appears, as opposed to $x_{i-1}^{-\mu-j}$, which appears in (2.43) for functions in $C_\beta^l[x^{-\mu}]'$. This feature makes the convergence analysis for functions in $C_\beta^l[x^{-\mu}]$ more difficult. This can be seen in the next Theorem, which shows that if $v\in C_\beta^m[x^{-\mu}]$ for $0<\beta<1$, then optimal convergence results can be achieved on the *polynomial mesh*, (2.33) and (2.34), provided $m<\mu$, that is provided the decay of $v(x)$ is fast enough as $x\to\infty$.

Theorem 2.10

Let $v \in C^m_\beta[x^{-\mu}]$, $0<\beta<1$, *and let* Π_n *be the polynomial mesh, (2.33) and (2.34). If* $m<\mu$ *then*

$$\|(I-P_n)v\|_\infty \le C\frac{1}{n^m}\|v:x^{-\mu}\|_{m,\beta} ,$$

provided $p_1 \ge m/\beta$ *and* $p_2 \ge m/(\mu-m)$.

Proof

(2.1) and Proposition 2.9 (with $t=j=m<\mu$) show that for all $2\le i\le n-1$

$$\|(I-P_n)v_i\|_\infty \le C\frac{1}{n^m}\|v:x^{-\mu}\|_{m,\beta} , \tag{2.55}$$

by our choice of p_1 and p_2.

Now (2.29) and (2.27) still hold, and so using (2.29), (2.27) and (2.33) we have

$$\|(I-P_n)v_1\|_\infty \le C\frac{1}{n^m}\|v:x^{-\mu}\|_{m,\beta} , \tag{2.56}$$

by our choice of p_1 and (2.35). Also, from (2.2) and (2.34) we have

$$\|(I-P_n)v_n\|_\infty = \|v_n\|_\infty \le C\, x_{n-1}^{-\mu}\|v:x^{-\mu}\|_{m,\beta}$$

$$\le C(rN)^{-\mu p_2}\|v:x^{-\mu}\|_{m,\beta} \tag{2.57}$$

$$\le C\frac{1}{n^m}\|v:x^{-\mu}\|_{m,\beta} , \tag{2.58}$$

by our choice of p_2 and (2.35), since

$$p_2 \ge \frac{m}{\mu-m} > \frac{m}{\mu} .$$

Then from (2.55), (2.56) and (2.58) the result follows. □

2.3.2.1 NUMERICAL RESULTS

Example 3

$v(x)=1/(1+x^2)$ - Piecewise quadratic interpolation.

N	n	e_N	EOC
4	8	0.845×10^{-2}	
			2.852
8	16	0.117×10^{-2}	
			2.944
16	32	0.152×10^{-3}	
			2.970
32	64	0.194×10^{-4}	
			2.985
64	128	0.245×10^{-5}	
			2.992
128	256	0.308×10^{-6}	
			3
256	512	0.385×10^{-7}	

TABLE 2.3

2.4 CONVERGENCE ANALYSIS FOR THE QUADRATURE RULE

In this section we investigate the convergence of the quadrature rule (1.24) for functions v which lie in the spaces $C_\beta^l[e^{-\mu x}]$, $C_\beta^l[x^{-\mu}]'$ and $C_\beta^l[x^{-\mu}]$, for $0<\beta<1$. Thus again we are treating the worst singularities in $v(x)$ as $x \to 0$. The convergence analysis for $1 \le \beta \le l$ can be treated in a similar manner. We use the meshes defined in §2.3.

Recall that R is the order of the quadrature rule (1.21), with points $\xi_1, \xi_2, \ldots, \xi_m$. For the Trapezoidal rule, $m=2$, $\xi_1=0$, $\xi_2=1$ and $R=2$. For Simpson's rule, $m=3$, $\xi_1=0$, $\xi_2=\frac{1}{2}$, $\xi_3=1$ and $R=4$.

For $v \in C \cap L^1$, let

$$Iv = \int_0^\infty v \ , \ \bar{I}v = \int_0^{x_{n-1}} v \ , \ \bar{I}_n v = \int_0^{x_{n-1}} P_n v .$$

Observe that using the triangle inequality we have

$$| \ (I-\bar{I}_n)v \ | \le | \ (I-\bar{I})v \ | + | \ (\bar{I}-\bar{I}_n)v \ | . \tag{2.59}$$

Thus, the error in approximating Iv by $\bar{I}_n v$ can be bounded by the sum of two errors. One is the truncation error in approximating Iv by $\bar{I}v$, and the other is the quadrature error.

Now using (2.3) we have for any $k=0,\ldots,n-2$

$$| \int_{x_k}^{x_{n-1}} (I-P_n)v \ | \le \mathrm{C} \sum_{i=k+1}^{n-1} h_i^R \| (D^R v)_i \|_1 = \mathrm{C} \sum_{i=k+1}^{n-1} h_i^R \int_{I_i} |D^R v| \ . \tag{2.60}$$

Also, we can bound the truncation error by

$$| \ (I-\bar{I})v \ | \le \mathrm{C} \int_{x_{n-1}}^\infty |v| . \tag{2.61}$$

In the subsequent analysis we will use the above estimates extensively together with the following identity,

$$\sum_{i=2}^{n-1} h_i^R \int_{I_i} |D^R v| = \left[\sum_{i=2}^{N} + \sum_{i=N+1}^{n-1} \right] h_i^R \int_{I_i} |D^R v| \ . \tag{2.62}$$

2.4.1 THE EXPONENTIAL CASE.

In order to motivate the convergence analysis for functions $v \in C^l_\beta[e^{-\mu x}]$, $0<\beta<1$, we first investigate the convergence of the quadrature rule for functions $v \in C^R_R[e^{-\mu x}]$. The convergence analysis for this class of functions is technically easier. Recall that if $v \in C^R_R[e^{-\mu x}]$, then

$$|||v:e^{-\mu x}|||_R := \max\left\{ \sup_{x\in \mathbb{R}^+} |e^{\mu x} D^j v(x)|,\ j=0,\dots,R \right\} < \infty .$$

The next Theorem shows that if $v \in C^R_R[e^{-\mu x}]$ then optimal convergence rates can be achieved for the quadrature rule using the mesh (2.6) with q appropriately chosen.

Theorem 2.11

Let $v \in C^R_R[e^{-\mu x}]$ and let Π_n be mesh (2.6). Then

$$|(I-\bar{I}_n)v| \le C\frac{1}{n^R}|||v:e^{-\mu x}|||_R ,$$

provided $q > R/\mu$.

Proof

First observe that using (2.61) and (2.6) we have

$$\begin{aligned} |\ (I-\bar{I})v\ | &\le C\int_{x_{n-1}}^{\infty} |v(x)|\,dx \\ &\le C\int_{x_{n-1}}^{\infty} \exp(-\mu x)\,dx\ |||v:e^{-\mu x}|||_R \\ &\le C\exp(-\mu x_{n-1})|||v:e^{-\mu x}|||_R \\ &\le C\frac{1}{n^R}|||v:e^{-\mu x}|||_R , \end{aligned} \tag{2.63}$$

by our choice of q.

Now let $0<\varepsilon<\mu$ be sufficiently small so that $q \ge R/(\mu-\varepsilon)$. Then using (2.60) we have

$$|\ (\bar{I}-\bar{I}_n)v\ | = |\ \int_0^{x_{n-1}} (I-P_n)v(x)\,dx\ |$$

$$\le C\sum_{i=1}^{n-1} h_i^R \int_{I_i} |D^R v(x)|\,dx$$

$$= C\sum_{i=1}^{n-1} h_i^R \int_{I_i} e^{-(\mu-\varepsilon)x} e^{(\mu-\varepsilon)x} |D^R v(x)|\,dx$$

$$\le C\sum_{i=1}^{n-1} h_i^R e^{-(\mu-\varepsilon)x_{i-1}} |||v:e^{-\mu x}|||_R \int_{I_i} e^{-\varepsilon x}\,dx$$

$$\le C\frac{1}{n^R} |||v:e^{-\mu x}|||_R \ , \qquad (2.64)$$

where (2.64) follows by our choice of q, since for $i=1,...,n-1$

$$h_i^R \exp(-(\mu-\varepsilon)x_{i-1}) \le C\frac{1}{n^R} \ ,$$

(by (2.10) with $t=j=R$ and μ replaced by $(\mu-\varepsilon)$, and (2.11)). The result then follows from (2.63), (2.64) and (2.59). □

Example 4

We consider $v(x)=e^{-x^2} \in C_l^l[e^{-x}]$, for all $l \in \mathrm{N}$. It is known analytically that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

$v(x)$ is proportional to the probability density function of a normal random variable with mean zero and variance one half.

We use the mesh (2.6) and we let

$$e_n = |\sqrt{\pi}/2 - \bar{I}_n v|$$

Table 2.4 of §2.4.1.1 gives the results for Simpson's rule with $q=4$. The results indicate that $e_n \approx O(1/n^4)$, as predicted by Theorem 2.11.

Example 5

The integral

$$I_x := -\tfrac{1}{4}\int_x^\infty E_1(t)\,dt \qquad (2.65)$$

is the right hand side of the reformulation of the integral equation which arises in

Example 1 §1.1, when $B(x)=1$ and $\varepsilon=\frac{1}{2}$, and must be calculated for many $x\in\mathbf{R}^+$ (see Chapter 5 §5.2.2). By [1, (5.1.51), (5.1.11)]

$$E_1(t) = O(e^{-t}) \quad \text{as } t\to\infty,$$

and

$$E_1(t) = -\log t - \gamma - \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{k.k!} \quad , \quad |\arg t|\le\pi, \tag{2.66}$$

where γ is the Euler constant. Then, since $E_1(t)$ has a logarithmic singularity at $t=0$, we cannot use a quadrature rule which uses $\xi_1=0$ to approximate I_x when $x=0$. However, note that for small values of the argument, t, $E_1(t)$ can be approximated to a high degree of accuracy by truncating the infinite series in (2.66) to a reasonably small value of k. To see this, if we approximate the alternating infinite series in (2.66) by replacing $k=\infty$ with $k=10$, then the error in such an approximation is bounded by

$$\frac{|t|^{11}}{11.11!}.$$

Hence we find uniformly accurate answers for (2.65) for many $x\in\mathbf{R}^+$ as follows: For any mesh Π_n, let $r=\min\{\, i\colon 1\le i\le n-1,\ x_i \ge 1 \,\}$. Then, for general x, if $0\le x\le x_r$, we use the approximation

$$-\tfrac{1}{4}\int_x^{\infty} E_1 \approx -\tfrac{1}{4}\int_x^{x_r} E_1^* - \tfrac{1}{4}\int_{x_r}^{x_{n-1}} P_n E_1 := \tfrac{1}{4}\bar{I}_{n,x} E_1, \tag{2.67}$$

where E_1^* is the approximation to E_1 with the infinite series in (2.66) truncated at $k=10$. Then the first integral on the right hand side of (2.67) may be evaluated analytically.

If $x_r\le x\le x_i\le x_{n-1}$, we use the approximation

$$-\tfrac{1}{4}\int_x^{\infty} E_1 \approx -\tfrac{1}{4}\int_x^{x_i} \tilde{P}_n^i\, E_1 - \tfrac{1}{4}\int_{x_i}^{x_{n-1}} P_n\, E_1 \;:= \tfrac{1}{4}\bar{I}_{n,x}\, E_1. \tag{2.68}$$

where $\tilde{P}_n^i\, E_1$ is the polynomial of order m which interpolates E_1 at the points $x+\xi_j(x_i-x)$, $j=1,\ldots m$.

In this particular experiment, we use the mesh (2.6) to compute (2.65) for x at the quadrature points, (NAG routine S13aaf was used to calculate $E_1(x)$ for $x>0$). Now by [1, (5.1.26)], (2.65) actually defines $\frac{1}{4}E_2(x)$, where E_2 is the second exponential integral, (see (2.32)). Table 2.5 of §2.4.1.1 gives the results for the Trapezoidal rule with $q=2$. We let

$$e_n = \max|\tfrac{1}{4}E_2(x) - \tfrac{1}{4}\bar{I}_{n,x}E_1|$$

taken over all quadrature points. The results show that $e_n \approx O(1/n^2)$ which indicates that the quadrature rule, (2.67) and (2.68), is performing optimally, uniformly over all quadrature points x.

(To compute $E_2(x)$ for x in $\mathbf{R}^+$, (2.32) was used. NAG routine S13aaf was used to calculate $E_1(x)$ for $x>0$, (at $x=0$, $E_2(x)=1$, see [1])).

We now consider functions $v \in C^R_\beta[e^{-\mu x}]$, $0<\beta<1$. The next Theorem shows that if $v \in C^R_\beta[e^{-\mu x}]$, $0<\beta<1$, then optimal results can be achieved for the quadrature rule using the *exponential mesh*, (2.14) and (2.15), with q_1 and q_2 appropriately chosen.

Theorem 2.12

Let $v \in C^R_\beta[e^{-\mu x}]$, $0<\beta<1$, *and let* Π_n *be the exponential mesh, (2.14) and (2.15). Then*

$$|(I-\bar{I}_n)v| \leq C\frac{1}{n^R}\|v:e^{-\mu x}\|_{R,\beta} ,$$

provided $q_1 \geq R/\beta$ *and* $q_2 > R/\mu$.

Proof

First observe that

$$|\int_{I_1}(I-P_n)v| \leq Ch_1\|v_1\|_\infty \leq C\frac{1}{n^R}\|v:e^{-\mu x}\|_{R,\beta} , \qquad (2.69)$$

by (2.14), our choice of q_1 and (2.16). Also, using (2.61) and (2.15) we have

$$| (I-\bar{I})v | \le C\int_{x_{n-1}}^{\infty} |v(x)|dx \le C\int_{x_{n-1}}^{\infty} \exp(-\mu x)\, dx \, \|v{:}e^{-\mu x}\|_{R,\beta}$$

$$\le C \exp(-\mu x_{n-1})\|v{:}e^{-\mu x}\|_{R,\beta}$$

$$\le C\frac{1}{n^R}\|v{:}e^{-\mu x}\|_{R,\beta} \ , \tag{2.70}$$

where (2.70) follows by our choice of q_2 and (2.16).

We now give a bound for (2.62). For the first term on the right hand side of (2.62) we have

$$\sum_{i=2}^{N} h_i^R \int_{I_i} |D^R v(x)|dx = \sum_{i=2}^{N} h_i^R \int_{I_i} x^{\beta-R} x^{R-\beta} |D^R v(x)|dx$$

$$\le C\sum_{i=2}^{N} h_i^R x_{i-1}^{\beta-R} \|v\|_{R,\beta} \int_{I_i} dx$$

$$\le C\frac{1}{n^R}\|v\|_{R,\beta} \ , \tag{2.71}$$

where (2.71) follows by (2.14), our choice of q_1 and (2.16), since for i=2,...,N

$$h_i^R x_{i-1}^{\beta-R} \le C\frac{1}{n^R} \ ,$$

(by (2.20) with $t=j=R$, and (2.21)), and since

$$\sum_{i=2}^{N} \int_{I_i} dx < 1 \ .$$

Now we deal with the second term on the right hand side of (2.62). Let $0<\varepsilon<\mu$ be sufficiently small so that $q_2 \ge R/(\mu-\varepsilon)$. Then we have

$$\sum_{N+1}^{n-1} h_i^R \int_{I_i} |D^R v(x)|dx = \sum_{i=N+1}^{n-1} h_i^R \int_{I_i} e^{-(\mu-\varepsilon)x} e^{(\mu-\varepsilon)x} |D^R v(x)|dx$$

$$\le C \sum_{i=N+1}^{n-1} h_i^R e^{-(\mu-\varepsilon)x_{i-1}} \|v{:}e^{-\mu x}\|_R \int_{I_i} e^{-\varepsilon x} dx$$

$$\le C\frac{1}{n^R}\|v{:}e^{-\mu x}\|_R \ , \tag{2.72}$$

where (2.72) follows by (2.15), our choice of q_2 and (2.16), since for i=N+1,...,n−1

$$h_i^R e^{-(\mu-\varepsilon)x_{i-1}} \leq C\frac{1}{n^R} ,$$

(by (2.24) with μ replaced by $(\mu-\varepsilon)$ and $t=j=R$, and (2.25)).

Then from (2.60), (2.62), (2.71) and (2.72) we have

$$| \int_{x_1}^{x_{n-1}}(I-P_n)v \mid \leq C\sum_{i=2}^{n-1} h_i^R \int_{I_i} |D^R v| \leq C\frac{1}{n^R}\|v:e^{-\mu x}\|_{R,\beta} . \qquad (2.73)$$

Then (2.69) and (2.73) show that the quadrature error is bounded by

$$| (\bar{I}-\bar{I}_n)v \mid \leq C\frac{1}{n^R}\|v:e^{-\mu x}\|_{R,\beta} ,$$

and the result then follows from (2.70) and (2.59). □

In the next example, we consider the convergence of the interpolation process, (1.20), for a function $v\in C_\beta^1[x^{-1}]'$, which is defined by an integral for which the integrand decays exponentially. This composite process is thus a severe test for Theorem 2.8 and Theorem 2.12.

Example 6

We consider the function

$$v(x) = K_2(x) = 2\int_0^\infty \varphi(s)E_2(x\varphi(s))ds , \qquad (2.74)$$

where

$$\varphi(s)=e^{-s^2}/\sqrt{\pi}.$$

By [8], $K_2\in C_\beta^1[x^{-1}]'$ for all $0<\beta<1$. Also, $E_2(x)\in C_\beta^l[e^{-x}]$ for all $0<\beta<1$ and all $l\in$N. (see Example 2 §2.3.1)

For $x\in$R$^+$, we first approximate $K_2(x)$ by attacking the integral (2.74) using a high order Simpson's rule on the *exponential mesh*, (2.14) and (2.15), with $q_1=4=q_2$, $N=N_1=128$, $r=2$ and $n=n_1=327$, (that is we fix N_1 and hence n_1). We call this high order approximation $\tilde{K}_2$. We then interpolate $\tilde{K}_2(x)$ using piecewise linear interpolation (m=2, ξ_1=0, ξ_2=1), on the *polynomial mesh*, (2.33) and

(2.34), with $p_1=2=p_2$ and $r=1$. The interpolation error was calculated at 5 equally spaced points in each I_i, $i=1,\ldots,n-1$, and we let

$$e_N = \max|(I-P_n)\tilde{K}_2(x)|$$

taken over all these points. Table 2.6 of §2.4.1.1 gives the results which indicate that $e_N = O(1/n^2)$, which is an optimal result.

2.4.1.1 NUMERICAL RESULTS

Example 4

$\int_0^{\infty} e^{-x^2} dx$ - Simpson's rule.

n	e_n	EOC
8	0.585×10^{-4}	
		3.563
16	0.495×10^{-5}	
		3.924
32	0.326×10^{-6}	
		3.984
64	0.206×10^{-7}	
		3.997
128	0.129×10^{-8}	
		3.999
256	0.807×10^{-10}	

TABLE 2.4

Example 5

$-\frac{1}{4}\int_x^\infty E_1(t)\,dt$ - Trapezoidal rule

n	e_n	EOC
4	0.138×10^{-1}	
		0.756
8	0.817×10^{-2}	
		1.635
16	0.263×10^{-2}	
		1.755
32	0.779×10^{-3}	
		1.831
64	0.219×10^{-3}	
		1.887
128	0.592×10^{-4}	
		1.924
256	0.156×10^{-4}	
		1.931
512	0.409×10^{-5}	

TABLE 2.5

Example 6

$v(x)=K_2(x)$ - Piecewise linear interpolation.

N	n	e_N	EOC
2	4	0.708×10^{-1}	
			1.150
4	8	0.319×10^{-1}	
			1.772
4	8	0.934×10^{-2}	
			1.972
8	16	0.238×10^{-2}	
			1.986
16	32	0.601×10^{-3}	
			2.002
64	128	0.150×10^{-3}	
			1.996
128	256	0.376×10^{-4}	

TABLE 2.6

2.4.2 THE POLYNOMIAL CASE

We first consider the quadrature error for functions $v \in C_\beta^R[x^{-\mu}]'$, $0<\beta<1$. By dealing with $0<\beta<1$ we deal with the worst singularity in $v(x)$ as $x \to 0$. The next Theorem shows that if $v \in C_\beta^R[x^{-\mu}]'$, $0<\beta<1$, $\mu>1$, then optimal convergence results can be achieved for the quadrature rule using the *polynomial mesh,* (2.33) and (2.34), with p_1 and p_2 appropriately chosen.

Theorem 2.13

Let $v \in C_\beta^R[x^{-\mu}]'$, $0<\beta<1$, $\mu>1$ *and let* Π_n *be the polynomial mesh, (2.33) and (2.34). Then*

$$|(I-\bar{I}_n)v| \le C\frac{1}{n^R}\|v:x^{-\mu}\|'_{R,\beta} ,$$

provided $p_1 \ge R/\beta$ *and* $p_2 > R/(\mu-1)$.

Proof

First observe that

$$| \int_{I_1} (I-P_n)v(x)dx \, | \le Ch_1 \|v_1\|_\infty \le C\frac{1}{n^R}\|v:x^{-\mu}\|'_{R,\beta}, \tag{2.75}$$

by (2.33), choice of p_1 and (2.35). Also, using (2.61) we have

$$\begin{aligned} | \, (I-\bar{I})v(x) \, | &\le C\int_{x_{n-1}}^\infty |v(x)|dx \le C\int_{x_{n-1}}^\infty x^{-\mu}dx \, \|v:x^{-\mu}\|'_{R,\beta} \\ &\le C\, x_{n-1}^{-\mu+1}\|v:x^{-\mu}\|'_{R,\beta} \\ &= C(rN)^{p_2(-\mu+1)}\|v:x^{-\mu}\|'_{R,\beta} \\ &\le C\frac{1}{n^R}\|v:x^{-\mu}\|'_{R,\beta} , \end{aligned} \tag{2.76}$$

by (2.34), choice of p_2 and (2.35).

Now we bound (2.62). For the first term on the right hand side of (2.62) we have

$$\sum_{i=2}^N h_i^R \int_{I_i} |D^R v(x)|dx \le C\frac{1}{n^R}\|v\|_{R,\beta} \, . \tag{2.77}$$

(The proof of (2.77) follows analogously to the proof of (2.71) in Theorem 2.12)).

Now we bound the second term on the right hand side of (2.62). Let $0<\varepsilon<\mu-1$ be sufficiently small so that $p_2 \geq R/(\mu-1-\varepsilon)$. Then we have

$$\sum_{i=N+1}^{n-1} h_i^R \int_{I_i} |D^R v(x)|\,dx = \sum_{i=N+1}^{n-1} h_i^R \int_{I_i} x^{-\mu-R+1+\varepsilon} x^{\mu+R-1-\varepsilon} |D^R v(x)|\,dx$$

$$\leq C \sum_{i=N+1}^{n-1} h_i^R x_{i-1}^{-(\mu-1-\varepsilon)-R} \| v{:}x^{-\mu} \|'_R \int_{I_i} x^{-1-\varepsilon} dx$$

$$\leq C \frac{1}{n^R} \| v{:}x^{-\mu} \|_R \,, \qquad (2.78)$$

where the last inequality follows by (2.34), our choice of p_2 and (2.35), since for $i=N+1,\ldots,n-1$

$$h_i^R x_{i-1}^{-(\mu-1-\varepsilon)-R} \leq C \frac{1}{n^R} \,,$$

(by (2.45) with μ replaced by $(\mu-1-\varepsilon)$ and $j=t=R$, and (2.46)). Then (2.60), (2.62), (2.77) and (2.78) show that

$$\left| \int_{x_1}^{x_{n-1}} (I-P_n) v \right| \leq C \sum_{i=2}^{n-1} h_i^R \int_{I_i} |\, D^R v \,| \leq C \frac{1}{n^R} \| v{:}x^{-\mu} \|'_{R,\beta} \,. \qquad (2.79)$$

Thus, (2.75) and (2.79) show that the quadrature error is bounded by

$$| (\bar{I}-\bar{I}_n) v | \leq C \frac{1}{n^R} \| v{:}x^{-\mu} \|'_{R,\beta} \,,$$

and the result then follows from (2.76) and (2.59). □

Example 7

We take $v(x)=(1+x^2)^{-1} \in C_4^4[x^{-2}]'$. It is known analytically that

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2},$$

and so we let

$$e_N = \left| \frac{\pi}{2} - \bar{I}_n v \right|.$$

We use the *polynomial mesh,* (2.33) and (2.34). Table 2.7 of §2.4.2.1 gives the results for Simpson's rule with $p_1=1$, $p_2=4$ and $r=1$. The results indicate that $e_N \approx O(1/n^4)$, which is an optimal result.

In the next example we approximate the integral of a function which decays polynomially at ∞. This function is itself defined by an integral in which the integrand decays exponentially at ∞.

Example 8

Let K_1 be as in (1.6) with φ given by (1.7a). Then the integral

$$I_x := -\tfrac{1}{2}\int_x^\infty K_1(t)\,dt \qquad (2.80)$$

is the right hand side of the reformulation of the integral equation which arises in Example 2 §1.1, when Doppler broadening is dominant, with $B(x)=1$ and $\varepsilon=\frac{1}{2}$. (2.80) must be calculated for many $x \in \mathbf{R}^+$ (see Chapter 5 §5.3.2) and it is important to find uniformly accurate answers for (2.80) for many values of $x \in \mathbf{R}^+$.

Observe that with φ given by (1.7a) we have

$$K_1(t) = \tfrac{1}{2}\int_{-\infty}^{\infty} \varphi^2(s)E_1(t\varphi(s))\,ds = \int_0^{\infty} \varphi^2(s)E_1(t\varphi(s))\,ds \qquad (2.81)$$

Now for each $t \in \mathbf{R}^+$, the integrand in (2.81) decays exponentially at ∞.

By [8, I.23, I.14],

$$K_1(t) = O(t^{-2}) \quad \text{as } t \to \infty,$$

and

$$K_1(t) = \frac{1}{2\sqrt{2\pi}}\left[\frac{1}{4} - \gamma - \log\left[\frac{t}{\sqrt{\pi}}\right] - \sqrt{2}\sum_{k=1}^{\infty}\frac{(-t/\sqrt{\pi})^k}{k.k!\sqrt{(k+2)}}\right], \quad t \geq 0, \qquad (2.82)$$

where again γ is the Euler constant. Thus, $K_1(t)$ has a logarithmic singularity at $t=0$, and decays polynomially at ∞, and so in this experiment we use the *polynomial mesh,* (2.33) and (2.34), and compute (2.80) for x at the quadrature points, using the same method as in Example 5.

Thus, for $t>0$ we first approximate K_1 by attacking the integral in (2.81), using a high order Simpson's rule on the mesh (2.6) with $q=4$ and $n=n_1=512$, and we call this high order approximation $\tilde{K}_1$, (that is we are fixing n_1). Then for $0\le x\le 1$ we use the approximation

$$-\tfrac{1}{2}\int_x^\infty K_1 \approx -\tfrac{1}{2}\int_x^1 K_1^* - \tfrac{1}{2}\int_1^{x_{n-1}} P_n\tilde{K}_1 := \tfrac{1}{2}\bar{I}_{n,x}K_1 \ , \tag{2.83}$$

where K_1^* is given by (2.82) with the alternating infinite series truncated to $k=10$. The error in such an approximation is bounded by

$$\frac{1}{2\sqrt{\pi}}\,\frac{|t|^{11}}{11.11!.\sqrt{13}\sqrt{\pi}^{11}} .$$

The first integral on the right hand side of (2.83) is then calculated analytically. For $x_{n-1}\ge x_i\ge x\ge 1$ we use the approximation

$$-\tfrac{1}{2}\int_x^\infty K_1 \approx -\tfrac{1}{2}\int_x^{x_i} \tilde{P}_n^i\tilde{K}_1 - \tfrac{1}{2}\int_{x_i}^{x_{n-1}} P_n\tilde{K}_1 := \tfrac{1}{2}\bar{I}_{n,x}K_1 \ , \tag{2.84}$$

where $\tilde{P}_n^i\tilde{K}_1$ is the polynomial of order m which interpolates $\tilde{K}_1$ at the points $x+\xi_j(x_i-x)$, $j=1,\ldots,m$. Now by [8, (I.8)] we know that

$$-\tfrac{1}{2}\int_x^\infty K_1(t)dt = \tfrac{1}{4}K_2(x) \ ,$$

where K_2 is given by (2.74). Thus, we let

$$e_N = \max\left|\tfrac{1}{4}\tilde{K}_2(x) - \tfrac{1}{2}\bar{I}_{n,x}K_1\right| ,$$

taken over all quadrature points x, where $\tilde{K}_2$ is the high order approximation to K_2, determined as in in Example 6. Table 2.8 of §2.4.2.1 gives the results for Simpson's rule with $p_1=1$, $p_2=4$ and r=1. The results show that $e_N\approx O(1/n^4)$, which indicates that the quadrature rule, (2.83) and (2.84), is performing optimally, uniformly over all quadrature points x. These are excellent results considering the complexity of the task at hand.

We now consider functions $v\in C_\beta^R[x^{-\mu}]$, $0<\beta<1$, $\mu>1$. The next Theorem shows that if $v\in C_\beta^R[x^{-\mu}]$, $0<\beta<1$, $\mu>1$, then optimal convergence rates for the quadrature rule can be achieved on the *polynomial mesh,* (2.33) and (2.34), with p_1 and p_2 appropriately chosen, provided $R<\mu-1$, that is provided the decay of $v(x)$ as $x\to\infty$ is fast enough.

Theorem 2.14

Let $v\in C_\beta^R[x^{-\mu}]$, $0<\beta<1$, $\mu>1$ *and let* Π_n *be the polynomial mesh, (2.33) and (2.34). If* $R<\mu-1$ *then*

$$|(I-\bar{I}_n)v| \le C\frac{1}{n^R}\|v:x^{-\mu}\|_{R,\beta} ,$$

provided $p_1\ge R/\beta$ *and* $p_2>R/(\mu-1-R)$.

Proof

First observe that

$$| \int_{I_1}(I-P_n)v \;| \le Ch_1\|v_1\|_\infty \le C\frac{1}{n^R}\|v:x^{-\mu}\|_{R,\beta} , \tag{2.85}$$

by (2.33), our choice of p_1 and (2.35).

Also, using (2.61) and (2.34) we have

$$\begin{aligned} |\; (I-\bar{I})v(x) \;| &\le C\int_{x_{n-1}}^\infty |v(x)|dx \le C\int_{x_{n-1}}^\infty x^{-\mu}dx \;\|v:x^{-\mu}\|_{R,\beta} \\ &\le C\, x_{n-1}^{-\mu+1}\|v:x^{-\mu}\|_{R,\beta} \\ &= C\,(rN)^{p_2(-\mu+1)}\|v:x^{-\mu}\|_{R,\beta} \\ &\le C\frac{1}{n^R}\|v:x^{-\mu}\|_{R,\beta} , \end{aligned} \tag{2.86}$$

by (2.34), our choice of p_2 and (2.35), since

$$p_2>\frac{R}{\mu-1-R}>\frac{R}{\mu-1} .$$

We now bound (2.62). Using (2.77), we have for the first term on the right hand side of (2.62)

$$\sum_{i=2}^N h_i^R\int_{I_i}|D^Rv(x)|dx \le C\frac{1}{n^R}\|v\|_{R,\beta} . \tag{2.87}$$

Now let $0<\varepsilon<\mu-1-R$ be sufficiently small so that $p_2 \geq R/(\mu-1-\varepsilon-R)$. Then

$$\sum_{i=N+1}^{n-1} h_i^R \int_{I_i} |D^R v(x)| dx = \sum_{i=N+1}^{n-1} h_i^R \int_{I_i} x^{-\mu+1+\varepsilon} x^{\mu-1-\varepsilon} |D^R v(x)| dx$$

$$\leq C \sum_{i=N+1}^{n-1} h_i^R x_{i-1}^{-(\mu-1-\varepsilon)} \| v{:}x^{-\mu} \|_R \int_{I_i} x^{-1-\varepsilon} dx$$

$$\leq C \frac{1}{n^R} \| v{:}x^{-\mu} \|_R \;, \tag{2.88}$$

where (2.88) follows by (2.34), our choice of p_2 and (2.35), since for $i=N+1,\ldots,n-1$

$$h_i^R x_{i-1}^{-(\mu-1-\varepsilon)} \leq C \frac{1}{n^R} \;,$$

(by (2.53) with μ replaced by $(\mu-1-\varepsilon)$ and $t=j=R<\mu-1-\varepsilon$, and (2.54)). Then (2.60), (2.62), (2.87) and (2.88) show that

$$\left| \int_{x_1}^{x_{n-1}} (I-P_n)v \right| \leq C \sum_{i=2}^{n-1} h_i^R \int_{I_i} | D^R v | \leq C \frac{1}{n^R} \| v{:}x^{-\mu} \|_{R,\beta} \,. \tag{2.89}$$

Thus, (2.85) and (2.89) show that the quadrature error is bounded by

$$| (\bar{I}-\bar{I}_n) | \leq C \frac{1}{n^R} \| v{:}x^{-\mu} \|_{R,\beta} \;,$$

and the result then follows from (2.86) and (2.59). □

2.4.2.1 NUMERICAL RESULTS

Example 7

$$\int_0^\infty \frac{1}{1+x^2}\,dx \text{ - Simpson's rule.}$$

N	n	e_N	EOC
4	8	0.184×10^{-1}	
			3.716
8	16	0.140×10^{-2}	
			3.819
16	32	0.992×10^{-4}	
			3.884
32	64	0.672×10^{-5}	
			3.923
64	128	0.443×10^{-6}	
			3.948
128	256	0.287×10^{-7}	

TABLE 2.7

Example 8

$-\frac{1}{2}\int_x^{\infty} K_1(t)\,dt,\ x \in \mathbf{R}^+$ - Simpson's rule

N	n	e_n	EOC
4	8	0.567×10^{0}	
			2.538
8	16	0.976×10^{-1}	
			3.795
16	32	0.703×10^{-2}	
			3.854
32	64	0.486×10^{-3}	
			3.916
64	128	0.322×10^{-4}	
			3.952
128	256	0.208×10^{-5}	

TABLE 2.8

CHAPTER 3
REGULARITY ANALYSIS

3.1 INTRODUCTION

In this Chapter we present regularity results for the solution, u, of (1.1) under certain assumptions on κ and f. These assumptions allow κ to have a logarithmic singularity at the origin, and decay either exponentially or polynomially at infinity. We show that the behaviour of u is inherited from κ and f, and we determine this behaviour explicitly. Related results for a simpler class of weakly singular integral equations are found in [35], [10], [36] and [20]. In Chapter 4 a complete error analysis is given for the Nystrom-Product Integration method, which uses the results of this Chapter, and takes into account singularities contained in the kernel and the solution.

We shall make the following assumptions on κ.

A1. $\int_{\mathbf{R}} |\kappa| < 1$

A2. For all $l \in \mathbf{N}_0$, some $\rho > 0$ and all $\delta > 0$

$$\int_{\mathbf{R}\setminus[-\delta,\delta]} e^{\rho|x|} |D^l \kappa(x)| dx < \infty$$

or

A2′. For all $l \in \mathbf{N}_0$, some $\rho' > 1$ and all $\delta > 0$

$$\int_{\mathbf{R}\setminus[-\delta,\delta]} (1+|x|)^{\rho'-1} |D^l \kappa(x)| dx < \infty$$

A3. (1.10) and (1.11) hold with a a polynomial and c infinitely continuously differentiable on $\mathbf{R}^+$.

Remarks

1. A1 implies that K is bounded on C with $\|K\|_\infty = \int_{\mathbf{R}} |\kappa| < 1$, and that, if $u \in C$,

$$\lim_{x\to\infty} Ku(x) = \left(\int_{\mathbf{R}} \kappa(x) dx\right) u(\infty) = \hat{\kappa}(0) u(\infty),$$

where $\hat{\kappa}$ denotes the Fourier transform of κ. These well-known facts are proved in [29] (see also [19]). Thus if $f \in C$, (1.1) has a unique solution $u \in C$, and letting $x \to \infty$ in (1.1) yields $u(\infty) = f(\infty)/(1-\hat{\kappa}(0))$. Setting $\tilde{u} = u - u(\infty)$, we may then rearrange (1.1) as $\tilde{u} - K\tilde{u} = F$, another equation of the form (1.1), where

$$F(x) = f(x) - (1 - K1(x))f(\infty)/(1 - \hat{\kappa}(0)).$$

Then $\tilde{u}(\infty) = 0 = F(\infty)$, so there is no loss of generality in assuming $u(\infty) = 0 = f(\infty)$ from the outset in (1.1).

2. A2 implies that for all $l \in \mathbf{N}_0$, $D^l\kappa(x) = o(e^{-\rho|x|})$ as $x \to \pm\infty$, and A2′ implies that for all $l \in \mathbf{N}_0$, $D^l\kappa(x) = o(|x|^{-\rho'})$ as $x \to \pm\infty$.

3. A3 implies nothing about the behaviour of $c(t)$ as $t \to \infty$. Note that we expect the regularity re[illegible] of this chapter to hold with "a a polynomial" replaced with "a smooth on $\mathbf{R}$" in A3. However we provide technical details for the former case only.

3.2 REGULARITY ANALYSIS

Consider (1.1) and let κ satisfy A1, A2 and A3. If $f \in C$ and $e^{\mu x}f(x) \in C$ for some $\mu > 0$, then multiplying (1.1) through by $e^{\mu x}$, and rearranging slightly yields

$$u_\mu - K_\mu u_\mu = f_\mu, \tag{3.1}$$

where

$$u_\mu(x) = e^{\mu x}u(x), \quad f_\mu(x) = e^{\mu x}f(x)$$

and

$$K_\mu u_\mu(x) = \int_{\mathbf{R}^+} e^{\mu(x-t)}\kappa(x-t)u_\mu(t)\,dt \ .$$

Then (3.1) will have a unique solution, $u_\mu \in C$, if $\|K_\mu\|_\infty < 1$. Now note that

$$\|K_\mu\|_\infty = \int_{\mathbf{R}} e^{\mu x}|\kappa(x)|\,dx \ , \tag{3.2}$$

and so if we let $\mu \in [0,\rho)$, and choose ε so that $0 \le \mu + \varepsilon \le \rho$, then A1 and A2 show that for $n \in \mathbf{N}$,

$$\kappa_n(x) := e^{(\mu+\varepsilon/n)x}|\kappa(x)| < e^{(\mu+\varepsilon)x}|\kappa(x)| \in L^1(\mathrm{R}) . \tag{3.3}$$

Also we have

$$\kappa_n(x) \to e^{\mu x}|\kappa(x)| \text{ as } n\to\infty . \tag{3.4}$$

Then by (3.3), (3.4) and the Dominated Convergence Theorem [2] we have

$$\int_{\mathrm{R}} e^{(\mu+\varepsilon/n)x}|\kappa(x)|dx = \int_{\mathrm{R}} \kappa_n(x)dx \to \int_{\mathrm{R}} e^{\mu x}|\kappa(x)|dx \text{ as } n\to\infty ,$$

which shows that the function

$$\int_{\mathrm{R}} e^{\mu x}|\kappa(x)|dx$$

is a continuous function of $\mu\in[0,\rho)$. Hence by A1, the Intermediate Value Theorem and (3.2), $\|K_\mu\|_\infty<1$ for all μ sufficiently close to zero.

Now return to (1.1) and let κ satisfy A1, A2$'$ and A3. If $f\in C$ and $(1+x)^{\mu'}f(x)\in C$ for some $\mu'>0$, then multiplying (1.1) through by $(1+x)^{\mu'}$ and rearranging slightly yields

$$u_{\mu'} - K_{\mu'}u_{\mu'} = f_{\mu'} , \tag{3.5}$$

where

$$u_{\mu'}(x) = (1+x)^{\mu'}u(x), \quad f_{\mu'}(x) = (1+x)^{\mu'}f(x) ,$$

and

$$K_{\mu'}u_{\mu'}(x) = \int_{\mathrm{R}^+}(1+x)^{\mu'}\kappa(x-t)(1+t)^{-\mu'}u_{\mu'}(t)dt .$$

Then (3.5) will have a unique solution, $u_{\mu'}\in C$, if $\|K_{\mu'}\|_\infty<1$. Now note that, using the inequality

$$(1+x) \le (1+|x-t|)(1+t) , \quad x,t\in\mathrm{R}^+ , \tag{3.6}$$

(see for example [38]) we have

$$\|K_{\mu'}\|_\infty \le \sup_{x\in\mathrm{R}^+} \int_{\mathrm{R}^+}(1+|x-t|)^{\mu'}|\kappa(x-t)|dt \le \int_{\mathrm{R}}(1+|x|)^{\mu'}|\kappa(x)|dx , \tag{3.7}$$

and so if we let $\mu'\in[0,\rho'-1)$, and choose ε so that $0\le\mu'+\varepsilon\le\rho'-1$, then A1 and A2$'$ show that for $n'\in\mathrm{N}$,

$$\kappa_{n'}(x) := (1+|x|)^{\mu'+\varepsilon/n'}|\kappa(x)| < (1+|x|)^{\mu'+\varepsilon}|\kappa(x)| \in L^1(\mathbf{R}) \ . \qquad (3.8)$$

Also we have

$$\kappa_{n'}(x) \to (1+|x|)^{\mu'}|\kappa(x)| \text{ as } n'\to\infty \ . \qquad (3.9)$$

Then by (3.8), (3.9) and the Dominated Convergence Theorem [2] we have

$$\int_{\mathbf{R}}(1+|x|)^{\mu'+\varepsilon/n'}|\kappa(x)|dx = \int_{\mathbf{R}}\kappa_{n'}(x)dx \to \int_{\mathbf{R}}(1+|x|)^{\mu'}|\kappa(x)|dx \text{ as } n'\to\infty \ ,$$

which shows that the function

$$\int_{\mathbf{R}}(1+|x|)^{\mu'}|\kappa(x)|dx$$

is a continuous function of $\mu'\in[0,\rho'-1)$. Hence by A1, the Intermediate Value Theorem and (3.7), $\|K_{\mu'}\|_\infty<1$ for all μ' sufficiently close to zero.

Thus we have the following Lemma.

Lemma 3.1

(i) *If* κ *satisfies* A1, A2, A3, *then there exists* $0<\mu<\rho$ *such that, if* $f\in C$ *with* $e^{\mu x}f(x)\in C$, *the solution,* u, *of (1.1) also satisfies* $e^{\mu x}u(x)\in C$.

(ii) *If* κ *satisfies* A1, A2′, A3, *then there exists* $0<\mu'<\rho'-1$ *such that, if* $f\in C$ *with* $(1+x)^{\mu'}f(x)\in C$, *the solution,* u, *of (1.1) also satisfies* $(1+x)^{\mu'}u(x)\in C$.

For results related to Lemma 3.1(ii) see [38].

In general, μ and μ' in Lemma 3.1 will not be known analytically and will have to be approximated numerically. However note that (3.2) is $|\kappa|^\wedge(i\mu)$, where $|\kappa|^\wedge$ denotes the Fourier transform of $|\kappa|$. If this transform is known, μ in Lemma 3.1(i) may be computed by finding the smallest positive solution, $\mu*$, of the non-linear equation $|\kappa|^\wedge(i\mu)=1$, and then choosing $\mu<\mu*$.

Although Lemma 3.1 gives information about the behaviour of $u(x)$ as $x\to\infty$, it tells us nothing about $u(x)$ as $x\to0$, or about the derivatives of u. The next Lemma will enable us to determine such information.

Lemma 3.2

Let κ *satisfy* A1. *If* $f \in C$ *and* $Df \in L^1$, *then the solution,* u, *of (1.1) is in* C, $Du \in L^1$ *and*

$$(I - K)(Du - u(0)\kappa) = Df + u(0)K\kappa \ . \tag{3.10}$$

Proof

That $u \in C$ follows from the discussion following the assumptions on κ. Now write (1.1) as

$$u(x) - \int_0^\infty \frac{d}{dt}\left\{ - \int_{-\infty}^{x-t} \kappa(s)ds \right\} u(t)dt = f(x),$$

and integrate (formally) by parts to obtain

$$u(x) - \int_0^\infty \int_{-\infty}^{x-t} \kappa(s)ds\, Du(t)dt - u(0)\int_{-\infty}^{x} \kappa(s)ds = f(x). \tag{3.11}$$

Differentiating (3.11) with respect to x yields

$$Du - K\,Du - u(0)\kappa = Df, \tag{3.12}$$

and evaluating (3.11) at x=0 yields

$$u(0) - \int_0^\infty \int_{-\infty}^{-t} \kappa(s)ds\, Du(t)dt - u(0) \int_{-\infty}^{0} \kappa(s)ds = f(0). \tag{3.13}$$

Now for $\gamma \in \mathbf{C}$ and suitable functions v define

$$L\left[(v,\gamma)^{\mathrm{T}}\right] = (Kv,0)^{\mathrm{T}}$$

$$H\left[(v,\gamma)^{\mathrm{T}}\right] = (\gamma\kappa,\ h(v) + \Phi(0)\gamma)^{\mathrm{T}}$$

where

$$\Phi(x) = \int_{-\infty}^{x} \kappa(\sigma)d\sigma \ \text{ and } h(v) = \int_0^\infty \Phi(-\sigma)v(\sigma)d\sigma.$$

Then (3.12) and (3.13) may be written as

$$(I - L - H)\,(v,\gamma)^{\mathrm{T}} = (Df,f(0))^{\mathrm{T}} \ , \tag{3.14}$$

with v=Du, γ=$u(0)$. Now consider (3.14) in the space $\mathbf{B}:=L^1\times\mathbf{C}$, Note that $\mathbf{B}$ becomes a Banach space when equipped with the norm

$$\|(\nu,\gamma)^{\mathrm{T}}\|_B = \max\left\{\|\nu\|_1,|\gamma|\right\}.$$

If $(\nu,\gamma)^{\mathrm{T}} \in \mathbf{B}$ solves (3.14), then by reversing the process used to derive (3.14), it follows that

$$u(x) := \gamma + \int_0^x \nu \tag{3.15}$$

is the unique solution of (1.1). We shall now show that (3.14) has a unique solution $(\nu,\gamma)^{\mathrm{T}} \in C$.

First we note that if $(\nu,\gamma)^{\mathrm{T}} \in \mathbf{B}$ solves the homogeneous version of (3.14), then by reversing the process used to obtain (3.14), it follows that (3.15) is identically zero (since (1.1) has a unique solution $u \in C$), which shows $(\nu,\gamma)^{\mathrm{T}} = 0 \in \mathbf{B}$. Thus $(I - L - H)$ is one-one on $\mathbf{B}$. Also H is compact on $\mathbf{B}$. To see this, let $\mathbf{B}_1$ and L_1^1 be the unit balls in $\mathbf{B}$ and L^1 respectively. Now from the definition of H we have

$$H(\nu,\gamma)^{\mathrm{T}} = (0,h(\nu))^{\mathrm{T}} + \gamma(\kappa,\Phi(0))^{\mathrm{T}} ,$$

which implies

$$H(\mathbf{B}_1) \subseteq \{0\}\times h(L_1^1) + <(\kappa,\Phi(0))^{\mathrm{T}}> ,$$

where $<(\kappa,\Phi(0))^{\mathrm{T}}>$ denotes the span of $(\kappa,\Phi(0))^{\mathrm{T}}$ in $\mathbf{B}$. It then follows that

$$\overline{H(\mathbf{B})_1} \subseteq \{0\}\times \overline{h(L_1^1)} + <(\kappa,\Phi(0))> , \tag{3.16}$$

where the bar denotes closure. Now h is of finite rank on L^1 and hence h is compact on L^1, and so the set on the right hand side of (3.16) is compact. Thus $\overline{H\mathbf{B}_1}$ is compact which implies H is compact. Also, by A1, it follows that L is a contraction on $\mathbf{B}$. Hence, by the Fredholm Alternative, (3.14) has a unique solution $(\nu,\gamma)^{\mathrm{T}} \in \mathbf{B}$, and (3.15) describes the unique solution of (1.1) in terms of $(\nu,\gamma)^{\mathrm{T}} \in \mathbf{B}$.

To complete the proof note that by (3.14) and (3.15), $Du = \nu \in L^1$ and $u(0) = \gamma \in \mathbb{C}$ satisfies (3.12). Rearranging (3.12) slightly yields (3.10) and the result is proved. □

Now assume that the conditions of Lemma 3.2 hold, and consider (3.10), which is an equation of the form (1.1) for $Du-u(0)\kappa$. Let κ also satisfy A2 or A2′ and A3. If we know that $K\kappa \in C$, then by A1 it would follow that $(Du - u(0)\kappa) \in C$ provided $Df \in C$, and hence from A3, Du would have a logarithmic singularity at $x=0$. Further, if we know that $K\kappa(x)$ decays either exponentially or polynomially as $x \to \infty$, then, in view of Lemma 3.1, provided $Df(x)$ is suitably behaved as $x \to \infty$, we could determine the behaviour of $(Du(x) - u(0)\kappa(x))$ as $x \to \infty$, and hence by A2 or A2′ we would know the behaviour of $Du(x)$ as $x \to \infty$.

In Theorem 3.5 below, the process sketched above will be made rigorous and generalised to obtain information about the behaviour of higher derivatives of u. However, the proof of Theorem 3.5 requires certain very technical results about κ and its image under various operators. These technical results are given in the next Lemma and its corollary.

Notation

In the subsequent analysis we will need the following important notational devices.

1. If J is an interval of $\mathbf{R}^+$ and $l \in \mathrm{N}_0$, then $C^l(J)$ will denote the space of functions on $\mathbf{R}^+$ which have l continuous derivatives on J. Also, $C^\infty(J)$ will denote the space of functions on $\mathbf{R}^+$ which are infinitely continuously differentiable on J.
2. For any $l \in \mathrm{N}_0$ and $\delta > 0$, $\varphi_{l,\delta}$ will denote a generic $C^l[0,\delta]$ function. That is, in each case $\varphi_{l,\delta} \in C^l[0,\delta]$ but $\varphi_{l,\delta}$ may be allowed to vary from instance to instance.
3. $\{f_1+f_2+\ldots+f_n\}$ will denote a linear combination of the functions $f_1, f_2, \ldots$, and f_n. The coefficients of the linear combination will be unknown, but their values will be immaterial to the argument.
4. D will denote differentiation.

Lemma 3.3

Part A

If κ satisfies A1, A2, A3, *then for all $\delta>0$ and all $i,k,l\in N_0$,*

(a) $(DK)^i\kappa\in C^\infty(0,\infty)$

(b) $(DK)^i\kappa(x) = \{[(\log x)^{i+1}+(\log x)^i+\ldots+\log x][1+x+\ldots+x^l]\}+\varphi_{l,\delta}(x),\ \ x\in[0,\delta]$

(c) $|e^{\mu x}D^k(DK)^i\kappa(x)| \to 0$ *as* $x\to\infty$ *for all* $\mu\in[0,\rho)$.

Part B

If κ satisfies A1, A2′, A3, *then for all $\delta>0$ and all $i,k,l\in N_0$, (a) and (b) hold together with*

(c′) $|x^{\mu'}D^k(DK)^i\kappa(x)| \to 0$ *as* $x\to\infty$ *for all* $\mu'\in[0,\rho')$.

Proof

Part A. Let $i,k,l\in N_0$, $\delta>0$ and let κ satisfy A1, A2 and A3. Note that for $i=0$, (a), (b) and (c) follow trivially from the assumptions A2 and A3. The proof for $i\neq 0$ uses induction on i. We first show that $\psi(x)$ defined to be $\psi(x)=\kappa(x)$ for all $x\in R^+$ satisfies the assumptions Y1, Y2 and Y3 of Lemma A1, Appendix A. By A3, ψ satisfies Y1 and Y2 (with $p=1$), and by A2, ψ satisfies Y3. Then, Lemma A1(i) and (ii) show that (a) and (b) are true for $i=1$, and Lemma A1(iii) (with k replaced by $k+1$) shows that (c) is true for $i=1$. So let us assume that (a), (b) and (c) are true for $i=m\in N$. Then by (a), (b) and (c), ψ defined to be $\psi(x)=(DK)^m\kappa(x)$ for all $x\in R^+$ satisfies the assumptions Y1, Y2 (with $p=m+1$) and Y3 of Lemma A1. Then, by Lemma A1(i),

$$(DK)^{m+1}\kappa = (DK)\psi\in C^\infty(0,\infty), \tag{3.17}$$

and by Lemma A1(ii),

$$(DK)^{m+1}\kappa(x) = DK\psi(x)$$

$$=\{[(\log x)^{m+2}+(\log x)^{m+1}+\ldots+\log x][1+x+\ldots+x^l]\}+\varphi_{l,\delta}(x),\ \ x\in[0,\delta]\ . \tag{3.18}$$

Also, Lemma A1(iii) shows that for all $\mu\in[0,\rho)$

$$|e^{\mu x}D^k(DK)^{m+1}\kappa(x)| = |e^{\mu x}D^{k+1}K\psi(x)| \to 0 \text{ as } x\to\infty . \tag{3.19}$$

Then, by (3.17), (3.18) and (3.19), (a), (b) and (c) are true for $i=m+1$ and hence for all $i\in \mathbf{N}_0$, which proves the result.

Part B. Let $i,k,l\in \mathbf{N}_0$, $\delta>0$ and let κ satisfy A1, A2′ and A3. Note that for $i=0$, (a), (b) and (c′) follow trivially from the assumptions A2′ and A3. The proof for $i\neq 0$ follows analogously to the proof of Part A by first showing that $\psi(x)$ defined to be $\psi(x)=\kappa(x)$ for all $x\in \mathbf{R}^+$ satisfies the assumptions Y1, Y2 and Y3′ of Lemma A1, Appendix A. We omit the proof for the sake of brevity. □

Note that if κ satisfies A1, A2 or A2′ and A3, then from Lemma 3.3 it follows that, considering κ as a function on $\mathbf{R}^+$,

$$(DK)^i\kappa\in L^1, \tag{3.20}$$

for all $i\in \mathbf{N}_0$. Also from Lemma 3.3(a) (with i replaced by $i+1$), we see that for all $i\in \mathbf{N}_0$, $K(DK)^i\kappa$ is infinitely continuously differentiable on $(0,\infty)$ and so

$$K(DK)^i\kappa\in C(0,\infty) . \tag{3.21}$$

By integrating Lemma 3.3(b) (with i replaced by $i+1$) we obtain

$$K(DK)^i\kappa(x)=\{[(\log x)^{i+2}+(\log x)^{i+1}+\ldots+\log x][x+x^2+\ldots+x^l]\}+\varphi_{l,\delta}(x),\ x\in[0,\delta] , \tag{3.22}$$

for all $i\in \mathbf{N}_0$ and all $l\in \mathbf{N}$. Then (3.22) shows that $K(DK)^i\kappa(x)$ is continuous at $x=0$. Hence, if we write

$$K(DK)^i\kappa(x) = \int_0^x (DK)^{i+1}\kappa(t)dt + K(DK)^i\kappa(0) , \tag{3.23}$$

then (3.23), (3.22) and (3.20) (with i replaced with $i+1$) show that $K(DK)^i\kappa(x)$ has a limit at infinity. This fact together with (3.21) and (3.22) show that

$$K(DK)^i\kappa\in C \text{ for all } i\in \mathbf{N}_0 . \tag{3.24}$$

The following corollary generalises (3.24).

Corollary 3.4

(i) *If* κ *satisfies* A1, A2, A3, *then for all* $\mu \in [0,\rho)$ *and for all* $i \in \mathbf{N}_0$

$$e^{\mu x} K (DK)^i \kappa(x) \in C \ .$$

(ii) *If* κ *satisfies* A1, A2′, A3, *then for all* $\mu' \in [0,\rho')$ *and for all* $i \in \mathbf{N}_0$

$$(1+x)^{\mu'} K (DK)^i \kappa(x) \in C \ .$$

Proof

Let $i \in \mathbf{N}_0$. We define $\psi(x) = (DK)^i \kappa(x)$ for all $x \in \mathbf{R}^+$.

(i) Let κ satisfy A1, A2 and A3 and let $\mu \in [0,\rho)$. Then by Lemma 3.3, Part A, ψ satisfies the assumptions Y1, Y2(with $p=i+1$) and Y3 of Lemma A1. It then follows from Lemma A1(iii) that

$$|\ e^{\mu x} K \psi(x)\ | \to 0 \quad \text{as } x \to \infty \ . \tag{3.25}$$

Then (3.25) and (3.24) prove part (i).

(ii) Let κ satisfy A1, A2′ and A3 and let $\mu' \in [0,\rho')$. Then by Lemma 3.3, Part B, ψ satisfies the assumptions Y1, Y2(with $p=i+1$) and Y3′ of Lemma A1. It then follows from Lemma A1(iv) that

$$|\ x^{\mu'} K \psi(x)\ | \to 0 \quad \text{as } x \to \infty \ ,$$

which implies

$$|\ (1+x)^{\mu'} K \psi(x)\ | \to 0 \quad \text{as } x \to \infty. \tag{3.26}$$

Then (3.26) and (3.24) prove part (ii). □

This brings us to the main result of this chapter.

Theorem 3.5

(i) *Let* κ *satisfy* A1, A2, A3, *and let* μ *be the number given in Lemma 3.1(i). Then for all* $0<\beta<1$ *and all* $l \in \mathbf{N}$, *if* $f \in C_l^l[e^{-\mu x}]$, *the solution, u, of (1.1) satisfies* $u \in C_\beta^l[e^{-\mu x}]$.

(ii) *Let* κ *satisfy* A1, A2′, A3, *let* μ' *be the number given in Lemma 3.1(ii), and let* $\mu'' > \max\{1,\mu'\}$. *Then for all* $0<\beta<1$ *and all* $l \in \mathbf{N}$, *if* $f \in C_l^l[x^{-\mu''}]$, *the solution, u, of (1.1) satisfies* $u \in C_\beta^l[x^{-\mu'}]$.

Proof

We motivate the proof of (i) and (ii) by first determining the behaviour of $Du(x)$ and $D^2u(x)$ as $x \to 0$ under the conditions given in (i) and (ii) with $l \geq 2$. We then give the proof of (i) and (ii) simultaneously.

So let $0<\beta<1$, $l \geq 2$, and let μ, μ' and μ'' be determined as in the statement of the theorem. Let κ satisfy A1, A2 (respectively A2′), A3, and let $f \in C_l^l[e^{-\mu x}]$ (respectively $f \in C_l^l[x^{-\mu''}]$). Then the conditions of Lemma 3.2 are satisfied by (1.1) and so (3.10) holds. Now if we let

$$u_1 = Du - u(0)\kappa \tag{3.27}$$

and

$$f_1 = Df + u(0)K\kappa \ , \tag{3.28}$$

then we can write (3.10) as

$$u_1 - Ku_1 = f_1 \ . \tag{3.29}$$

Now $f_1 \in C$ since $Df \in C$ and (3.24) shows that $u(0)K\kappa \in C$. Hence from (3.29) and A1 we have $u_1 \in C$. Then, from (3.27) and A3, it follows that the behaviour of $Du(x)$ as $x \to 0$ is dominated by $\log x$.

To determine the behaviour of $D^2u(x)$ as $x \to 0$ consider (3.29). Now $f_1 \in C$ and $Df_1 \in L^1$ since $D^2 f \in L^1$ and (3.20) shows that $u(0)(DK)\kappa \in L^1$. Thus, the conditions of Lemma 3.2 are satisfied by (3.29) and so we have

$$(I - K)(Du_1 - u_1(0)\kappa) = Df_1 + u_1(0)K\kappa \ . \tag{3.30}$$

Now the right hand side of (3.30) is not, in general, continuous on $\mathbf{R}^+$, since Lemma 3.3(b) shows that $u(0)(DK)\kappa \notin C$, and so, by (3.28), it follows that $Df_1 \notin C$, in general. However we can regularize (3.30) by defining a new function u_2 where

$$u_2 = Du_1 - u_1(0)\kappa - u(0)(DK)\kappa. \tag{3.31}$$

Then we may write (3.30) as

$$u_2 - Ku_2 = f_2 \ , \tag{3.32}$$

where

$$f_2 = D^2 f + u_1(0)K\kappa + u(0)K(DK)\kappa \ . \tag{3.33}$$

Now $f_2 \in C$ since $D^2 f \in C$ and (3.24) shows that

$$u_1(0)K\kappa + u(0)K(DK)\kappa \in C \ .$$

Hence from (3.32) and A1 it follows that $u_2 \in C$. Now from (3.31) and (3.27) we have

$$u_2 = D^2 u - u(0)D\kappa - u_1(0)\kappa - u(0)(DK)\kappa, \tag{3.34}$$

and so, from Lemma 3.3(b) and A3, the behaviour of $D^2 u(x)$ as $x \to 0$ is dominated by $(1/x + \log x)$.

By applying the above model recursively, we can obtain information about higher derivatives of u. Using (3.20) and (3.24), it is straightforward to show that the sequences

$$\{u_k\}_{k=0}^{l} \ , \quad \{f_k\}_{k=0}^{l}$$

defined by

$$u_0 = u, \quad f_0 = f$$

$$u_k = Du_{k-1} - \sum_{m=0}^{k-1} u_m(0)(DK)^{k-1-m}\kappa, \qquad k=1,\ldots,l \tag{3.35}$$

$$f_k = D^k f + \sum_{m=0}^{k-1} u_m(0)K(DK)^{k-1-m}\kappa, \qquad k=1,\ldots,l \ , \tag{3.36}$$

are continuous and satisfy

$$u_k - Ku_k = f_k, \qquad k=0,\ldots,l. \tag{3.37}$$

From (3.35) it follows that

$$D^k u = \sum_{n=1}^{k} \sum_{m=0}^{k-n} u_m(0)D^{n-1}(DK)^{k-m-n}\kappa + u_k, \qquad k=1,\ldots,l, \tag{3.38}$$

where $u_k \in C$. This expansion is valid on R^+. Setting $k=l$, restricting the expansion to $[0,\delta]$ for arbitrary $\delta>0$ and integrating l times yields

$$u(x) = \sum_{n=1}^{l} \sum_{m=0}^{l-n} D^{n-1-l}(DK)^{l-m-n}\kappa(x) + D^{-l}u_l(x) + p_l(x), \quad x\in[0,\delta], \tag{3.39}$$

where D^{-l} denotes an l-fold indefinite integral and p_l is some polynomial of degree $l-1$. Now putting $q=l-n$ in (3.39) yields

$$u(x) = \sum_{q=0}^{l-1} \sum_{m=0}^{q} D^{-1-q}(DK)^{q-m}\kappa(x) + \varphi_{l,\delta}(x) , \quad x\in[0,\delta]. \tag{3.40}$$

Then, from (3.40), Lemma 3.3(b) and A3, it follows that

$$u(x) = \{[(\log x)^l+(\log x)^{l-1}+...+\log x][x+x^2+...+x^l]\} + \varphi_{l,\delta}(x) , \quad x\in[0,\delta]. \tag{3.41}$$

This determines the regularity of $u(x)$ as $x\to 0$.

To complete the proof of (i), let κ satisfy A1, A2 and A3, let $f\in C_l^l[e^{-\mu x}]$ and consider (3.37). Now $e^{\mu x}f_k(x)\in C$, $k=0,\ldots,l$, since $e^{\mu x}D^kf(x)\in C$, $k=0,\ldots,l$, and Corollary 3.4(i) shows that

$$e^{\mu x}\sum_{m=0}^{k-1} u_m(0)K(DK)^{k-1-m}\kappa(x) \in C, \quad k=1,\ldots,l.$$

Hence, from (3.37) and Lemma 3.1(i), it follows that $e^{\mu x}u_k(x)\in C$, $k=0,\ldots,l$. Then (3.38), Lemma 3.3(c) and A2 show that

$$D^ku(x) = O(e^{-\mu x}) \text{ as } x\to\infty, \quad k=0,\ldots,l. \tag{3.42}$$

Then (i) follows from (3.41) and (3.42).

Now for part (ii), let κ satisfy A1, A2′, A3, let $f\in C_l^l[x^{-\mu''}]$, and consider (3.37) again. Now $(1+x)^{\mu'}f_k(x)\in C$, $k=0,\ldots,l$, since $(1+x)^{\mu'}D^kf(x)\in C$, $k=0,\ldots,l$, and Corollary 3.4(ii) shows that

$$(1+x)^{\mu'}\sum_{m=0}^{k-1} u_m(0)K(DK)^{k-1-m}\kappa(x) \in C, \quad k=1,\ldots,l.$$

Hence, from (3.37) and Lemma 3.1(ii), it follows that $(1+x)^{\mu'}u_k(x)\in C$, $k=0,\ldots,l$.

Then (3.38), Lemma 3.3(c′) and A2′ show that

$$D^k u(x) = O((1+x)^{-\mu'}) = O(x^{-\mu'}) \text{ as } x\to\infty, \ k=0,\dots,l. \tag{3.43}$$

Then (ii) follows from (3.41) and (3.43). □

In the examples of (1.1) arising in radiative transfer it is often of interest to consider the equation

$$u - Ku = \mathrm{C} \tag{3.44}$$

where C>0 is a constant, (see Chapter 5, §5.2.2 and § 5.3.2). Recall the discussion in Remark 1 following the assumptions of this Chapter. Clearly $u(\infty)\neq 0$ in this case. A rearrangement of (3.44) results in an equation of the form (1.1) for $\tilde{u}=u-u(\infty)$, with

$$\tilde{u} - K\tilde{u} = F , \tag{3.45}$$

where in this case

$$\begin{aligned} F(x) &= \mathrm{C} - (1-K1(x))\frac{\mathrm{C}}{(1-\hat{\kappa}(0))} \\ &= \frac{\mathrm{C}}{(1-\hat{\kappa}(0))}(-\hat{\kappa}(0) + K1(x)) \\ &= -u(\infty)\int_x^\infty \kappa(t)dt . \end{aligned} \tag{3.46}$$

By A3, F does not satisfy either $D^lF\in C$ for $l\geq 1$, or $D^lF\in L^1$ for $l\geq 2$. Thus F does not satisfy the conditions of Theorem 3.5 for $l\geq 1$.

The following theorem describes the regularity of $\tilde{u}$ in (3.45) with F given by (3.46).

Theorem 3.6

(i) *Let* κ *satisfy* A1, A2 *and* A3, *and let* μ *be the number given in Lemma 3.1(i). Then for all* $0<\beta<1$ *and all* $l\in\mathrm{N}$, *the solution,* $\tilde{u}$, *of (3.45), with F given by (3.46), satisfies* $\tilde{u}\in C_\beta^l[e^{-\mu x}]$.

(ii) *Let* κ *satisfy* A1, A2′ *and* A3, *and let* μ' *be the number given in Lemma 3.1(ii). Then for all* $0<\beta<1$ *and all* $l\in\mathrm{N}$, *the solution,* $\tilde{u}$, *of (3.45), with F given by (3.46), satisfies* $\tilde{u}\in C_\beta^l[x^{-\mu'}]$.

Proof

We motivate the proof of (i) and (ii) by first determining the behaviour of $Du(x)$ and $D^2u(x)$ as $x\to 0$ under the conditions given in (i) and (ii). We then generate sequences corresponding to those given in the proof of Theorem 3.5, namely (3.35), (3.36) and (3.37). The proofs of (i) and (ii) will then follow analogously to the proofs of (i) and (ii) in Theorem 3.5.

So let $0<\beta<1$, $l\in\mathrm{N}$ and let μ and μ' be determined as in the statement of the theorem. Let κ satisfy A1, A2 (respectively A2′), A3, and consider (3.45) with F given by (3.46). Now from A3 we have $F\in C$. Also note that by A1 we have

$$DF = -u(\infty)D\int_x^\infty \kappa = u(\infty)\kappa \in L^1 \ .$$

Thus the conditions of Lemma 3.2 are satisfied by (3.45) and so (3.10) holds. Hence we have

$$\begin{aligned}(I-K)(D\tilde{u} - \tilde{u}(0)\kappa) &= DF + \tilde{u}(0)K\kappa \\ &= u(\infty)\kappa + \tilde{u}(0)K\kappa \ . \qquad (3.47)\end{aligned}$$

Now by A3, the right hand side of (3.47) is not continuous on R^+. However we can regularize (3.47) by defining a new function, $\tilde{u}_1$, where

$$\tilde{u}_1 = D\tilde{u} - \tilde{u}(0)\kappa - u(\infty)\kappa \ . \qquad (3.48)$$

Then we can write (3.47) as

$$\tilde{u}_1 - K\tilde{u}_1 = F_1 \ , \qquad (3.49)$$

where

$$F_1 = u(\infty)K\kappa + \tilde{u}(0)K\kappa \ . \tag{3.50}$$

Now (3.24) shows that $F_1 \in C$, and so, by (3.49) and A1, it follows that $\tilde{u}_1 \in C$. Then (3.48) and A3 show that the behaviour of $D\tilde{u}(x)$ as $x \to 0$ is dominated by $\log x$.

To determine the behaviour of $D^2\tilde{u}(x)$ as $x \to 0$ consider (3.49). We have $F_1 \in C$ and (3.20) shows that $DF_1 \in L^1$. Thus (3.49) satisfies the conditions of Lemma 3.2 and so we have

$$(I - K)(D\tilde{u}_1 - \tilde{u}_1(0)\kappa) = DF_1 + \tilde{u}_1(0)K\kappa.$$

Then, using (3.50), it follows that

$$(I - K)(D\tilde{u}_1 - \tilde{u}_1(0)\kappa) = u(\infty)(DK)\kappa + \tilde{u}(0)(DK)\kappa + \tilde{u}_1(0)K\kappa. \tag{3.51}$$

Now by Lemma 3.3(b), the right hand side of (3.51) is not continuous on $\mathbf{R}^+$. However we can regularize (3.51) by defining a new function, $\tilde{u}_2$, where

$$\tilde{u}_2 = D\tilde{u}_1 - \tilde{u}_1(0)\kappa - u(\infty)(DK)\kappa - \tilde{u}(0)(DK)\kappa. \tag{3.52}$$

Then we can write (3.51) as

$$\tilde{u}_2 - K\tilde{u}_2 = F_2 \ , \tag{3.53}$$

where

$$F_2 = u(\infty)K(DK)\kappa + \tilde{u}(0)K(DK)\kappa + \tilde{u}_1(0)K\kappa \ . \tag{3.54}$$

Now by (3.24) we have $F_2 \in C$, and so, from (3.53) and A1, it follows that $\tilde{u}_2 \in C$. From (3.52) and (3.48) we have

$$\tilde{u}_2 = D^2\tilde{u} - \tilde{u}(0)D\kappa - u(\infty)D\kappa - \tilde{u}_1(0)\kappa - u(\infty)(DK)\kappa - \tilde{u}(0)(DK)\kappa \in C. \tag{3.55}$$

Hence, from (3.55), Lemma 3.3(b) and A3, the behaviour of $D^2\tilde{u}(x)$ as $x \to 0$ is dominated by $(1/x + \log x)$.

By applying the above model recursively, we can obtain information about higher derivatives of $\tilde{u}$. Using (3.20) and (3.24), it is straightforward to show that the sequences

$$\{\tilde{u}_k\}_{k=0}^{l}, \quad \{F_k\}_{k=0}^{l}$$

defined by

$$\tilde{u}_0 = \tilde{u}, \quad F_0 = F$$

$$\tilde{u}_k = D\tilde{u}_{k-1} - \sum_{m=0}^{k-1} \tilde{u}_m(0)(DK)^{k-1-m}\kappa - u(\infty)(DK)^{k-1}\kappa, \qquad k=1,\ldots,l \quad (3.56)$$

$$F_k = \sum_{m=0}^{k-1} \tilde{u}_m(0)K(DK)^{k-1-m}\kappa + u(\infty)K(DK)^{k-1}\kappa, \quad k=1,\ldots,l,$$

are continuous and satisfy

$$\tilde{u}_k - K\tilde{u}_k = F_k, \qquad k=0,\ldots,l.$$

Now from (3.56) it follows that

$$D^k\tilde{u} = \sum_{n=1}^{k} \sum_{m=0}^{k-n} \tilde{u}_m(0)D^{n-1}(DK)^{k-m-n}\kappa + u(\infty)\sum_{m=0}^{k-1} D^m(DK)^{k-1-m}\kappa + \hat{u}_k, \quad k=1,\ldots,l,$$

where $\hat{u}_k \in C$.

The proofs of (i) and (ii) then follow analogously to the proofs of (i) and (ii) in Theorem 3.5. □

CHAPTER 4

THE NYSTROM-PRODUCT INTEGRATION METHOD

4.1 INTRODUCTION

In Chapter 1, § 1.4.2 we introduced the Nystrom-Product Integration approximation, u_n, for the solution, u, of (1.1), (see (1.29)). In this Chapter we shall carry out a convergence analysis for $u-u_n$ in the space C, where

$$u - u_n = (I - K_{i*,n})^{-1}(K - K_{i*,n})u \ , \tag{4.1}$$

(see (1.34)). Our analysis will consist of two parts. In §4.2 we prove a stability result which gives a bound on $\|(I - K_{i*,n})^{-1}\|_\infty$, under the assumptions, A1, A2 or A2′ and A3 of Chapter 3, and an additional assumption on the mesh $\Pi_n : 0=x_0<x_1<\ldots<x_n=\infty$. In §4.3 and §4.4 we prove consistency results under the assumptions A1, A2, A3 and A1, A2′, A3, respectively, for families of meshes which satisfy the stability requirements of §4.2. These consistency results bound the order of convergence of $\|(K - K_{i*,n})u\|_\infty$ as $n\to\infty$.

From now on C will denote a generic constant independent of n. Recall $I_i=(x_{i-1},x_i)$, $h_i=x_i-x_{i-1}$ and for any $v:\mathbf{R}^+\to\mathbf{C}$, $v_i=v$ on I_i, $v_i=0$ on $\mathbf{R}^+\setminus I_i$. Recall also that $m\le R\le 2m$ is the order of the quadrature rule (1.21), with points $\xi_1,\xi_2,\ldots,\xi_m$, and $x_{ij}=x_{i-1}+\xi_j h_i$.

We shall use the following proposition in our analysis, the proof of which is taken from [12, Lemma 1(ii)].

Proposition 4.1

Let $v\in C$ *and let* Π_n *be any mesh. If* $\|(Dv)_i\|_1<\infty$, *there exists a constant* C *independent of* n,i *and* v *such that for* $1\le i\le n-1$

$$h_i\sum_{j=1}^{m}|w_jv(x_{ij})| \le \mathrm{C}\,(\|v_i\|_1 + h_i\|(Dv)_i\|_1) \ .$$

Proof

For all $j=1,...,m$ and $x \in I_i$ we have

$$|v(x) - v(x_{ij})| = |\int_{x_{ij}}^{x} Dv(\sigma)d\sigma| \leq \|(Dv)_i\|_1 \ .$$

Then using the triangle inequality we obtain

$$|v(x_{ij})| \leq |v(x)| + \|(Dv)_i\|_1 \ ,$$

and integrating with respect to x yields

$$h_i |v(x_{ij})| \leq \|v_i\|_1 + h_i \|(Dv)_i\|_1 \ .$$

Then multiplying by $|w_j|$ and summing over $j=1,...,m$ gives the result, with

$$C = \sum_{j=1}^{m} |w_j| \ . \ \square$$

4.2 STABILITY

We shall prove stability under the following general assumption on Π_n.

M1. For each $\varepsilon>0$, there exists $i*=i*(\varepsilon,n)\geq 1$, with $1\leq i*<n$, such that $h_i<\varepsilon$ for all $1\leq i\leq n-i*$, and all n sufficiently large.

We will assume that for any n, $i*$ is chosen so that

$$h_i<\gamma_0 \ , \quad 1\leq i\leq n-i*, \tag{4.2}$$

for some constant γ_0 independent of n and i. Recall the definition of $\Omega^{[k]}(x)$, $k=1,2$, introduced in §1.4.2, and note that for all $x\in\mathbf{R}^+$,

$$\Omega^{[1]}(x)\subset[x-\gamma, \ x+\gamma], \text{ where } \gamma=\delta+\gamma_0. \tag{4.3}$$

In §4.3 and §4.4 we shall be concerned with specific meshes suitable for approximating u. These will satisfy M1.

Our proof of stability depends on the properties of K_{i*} and $K_{i*,n}$ as operators on a certain function space, Λ^β, defined as follows. For $0<\beta<1$ and $v:\mathbf{R}^+\rightarrow\mathbf{C}$, define

$$|v|_{\Lambda^\beta} = \sup\left\{ \|v(\cdot+\tau) - v(\cdot)\|_\infty / \tau^\beta \ : \tau > 0 \right\}.$$

Then,

$$\Lambda^\beta := \left\{ v \in C : |v|_{\Lambda^\beta} < \infty \right\} ,$$

and a norm on Λ^β may be defined by

$$\|v\|_{\Lambda^\beta} = \max\left\{ \|v\|_\infty , |v|_{\Lambda^\beta} \right\} .$$

We need the following two propositions in order to prove stability.

Proposition 4.2

Let $0<\beta<1$, *and let* $v \in C$. *If* $\|v(\cdot+\tau) - v(\cdot)\|_\infty / \tau^\beta$ *is bounded for sufficiently small* $\tau>0$, *then* $v \in \Lambda^\beta$, *with* $|v|_{\Lambda^\beta} \leq \mathrm{C}\|v\|_\infty$.

Proof

Suppose there exists $M, \tau_0 > 0$ such that

$$\frac{\|v(\cdot+\tau) - v(\cdot)\|_\infty}{\tau^\beta} < M , \quad 0<\tau<\tau_0 .$$

Then for $\tau \geq \tau_0$ we have

$$\frac{\|v(\cdot+\tau) - v(\cdot)\|_\infty}{\tau^\beta} \leq \frac{\|v(\cdot+\tau)\|_\infty + \|v\|_\infty}{\tau^\beta} \leq \frac{2}{\tau_0^\beta}\|v\|_\infty .$$

Hence,

$$\sup_{\tau>0} \frac{\|v(\cdot+\tau) - v(\cdot)\|_\infty}{\tau^\beta} \leq \max\{M , \frac{2}{\tau_0^\beta}\|v\|_\infty\} < \infty ,$$

and so $v \in \Lambda^\beta$ as required. Also, the last inequality shows

$$|v|_{\Lambda^\beta} \leq \mathrm{C} \max\{1 , \|v\|_\infty\} ,$$

where $\mathrm{C} = \max\{ M, 2/\tau_0^\beta \}$. Thus, if $\|v\|_\infty = 1$, then $|v|_{\Lambda^\beta} \leq \mathrm{C}$. Now for any $v \neq 0$

$$|v|_{\Lambda^\beta} = \left| \|v\|_\infty \frac{v}{\|v\|_\infty} \right|_{\Lambda^\beta}$$

$$\leq \|v\|_\infty \left| \frac{v}{\|v\|_\infty} \right|_{\Lambda^\beta} \leq C\|v\|_\infty ,$$

as required. □

Proposition 4.3

Let $0<\beta<1$, *let* Π_n *be any mesh and let* $1\leq i\leq n-1$. *If* $v\in\Lambda^\beta$ *and* φ *is continuously differentiable on* I_i, *then*

(i) $\|(v-v(x_i))_i\|_\infty \leq h_i^\beta \|v\|_{\Lambda^\beta}$.

(ii) $\|(I-P_n)(\varphi v)_i\|_\infty \leq C \max\left\{h_i^\beta, h_i\right\} \|v\|_{\Lambda^\beta}$.

Proof

(i) If $x\in I_i$ then

$$\begin{aligned} |v(x)-v(x_i)| &= |v(x+(x_i-x))-v(x)| \\ &\leq (x_i-x)^\beta \|v\|_{\Lambda^\beta} \leq h_i^\beta \|v\|_{\Lambda^\beta} , \end{aligned}$$

which proves part (i).

(ii) For $x\in I_i$ and any constant d, we have

$$|(I-P_n)(\varphi v)(x)| = |(I-P_n)(\varphi v-d)(x)| \leq C\|(\varphi v-d)_i\|_\infty .$$

Then, putting $d=\varphi(x_i)v(x_i)$, (i) above and the Mean Value Theorem give

$$\begin{aligned} |\varphi(x)v(x)-d| &\leq |\varphi(x)(v(x)-v(x_i))| + |(\varphi(x)-\varphi(x_i))v(x_i)| \\ &\leq h_i^\beta \|\varphi\|_\infty \|v\|_{\Lambda^\beta} + h_i \|(D\varphi)_i\|_\infty \|v\|_\infty , \end{aligned}$$

which yields the required estimate. □

The next two lemmas study K_{i*} and $K_{i*,n}$ in Λ^β. The first result shows $K_{i*,n}: C\rightarrow\Lambda^\beta$ is uniformly bounded.

Lemma 4.4

Let $v \in C$, *let* $0<\beta<1$ *and let* κ *satisfy* A1, A2(*or* A2′) *and* A3. *Then for any* n, *if* $i*$ *is chosen so that (4.2) is satisfied, we have* $K_{i*,n}v \in \Lambda^\beta$ *and*

$$\|K_{i*,n}v\|_{\Lambda^\beta} \le C\|v\|_\infty$$

with C *independent of* n, $i*$ *and* v.

Proof

By (1.26) and the triangle inequality,

$$|K_{i*,n}v(x)| \le |K_{i*,n}^{[1]}v(x)| + |K_{i*,n}^{[2]}v(x)|. \tag{4.4}$$

We shall bound each term on the right hand side of (4.4) separately. For the first term,

$$\begin{aligned} |K_{i*,n}^{[1]}v(x)| &\le \int_{\Omega^{[1]}(x)} |b(x-t)|\ |P_n(a(x-\cdot)v(\cdot))(t)|\,dt \\ &\le \|P_n\|_\infty\ \|b\|_{1,[-\gamma,\gamma]}\ \|a\|_{\infty,[-\gamma,\gamma]}\ \|v\|_\infty \\ &\le C\|v\|_\infty, \end{aligned} \tag{4.5}$$

with γ as in (4.3), where (4.5) follows by A3.

For the second term,

$$\begin{aligned} |K_{i*,n}^{[2]}v(x)| &= \Big|\sum_{i\in Q^{[2]}(x)} \sum_{j=1}^{m} w_j\ \kappa(x-x_{ij})v(x_{ij})h_i\Big| \\ &\le \sum_{i\in Q^{[2]}(x)} h_i \sum_{j=1}^{m} |w_j\ \kappa(x-x_{ij})|\ \|v\|_\infty \\ &\le C \sum_{i\in Q^{[2]}(x)} \Big[\|\kappa(x-\cdot)_i\|_1 + h_i\|D\kappa(x-\cdot)_i\|_1\Big]\ \|v\|_\infty \\ &\le C\|v\|_\infty. \end{aligned} \tag{4.6}$$

The second inequality follows from Proposition 4.1, and the final inequality follows from A2(or A2′) and (4.2). Hence, (4.4), (4.5) and (4.6) give

$$\|K_{i*,n}v\|_\infty \le C\|v\|_\infty. \tag{4.7}$$

To complete the proof we need to show $K_{i*,n}v \in C$ with

$$|K_{i*,n}v|_{\Lambda^\beta} \leq \mathrm{C}\|v\|_\infty . \tag{4.8}$$

First we note

$$|K_{i*,n}v(x+\tau) - K_{i*,n}v(x)| \leq |K^{[1]}_{i*,n}v(x+\tau) - K^{[1]}_{i*,n}v(x)| + |K^{[2]}_{i*,n}v(x+\tau) - K^{[2]}_{i*,n}v(x)| . \tag{4.9}$$

A careful examination of the definition of $\Omega^{[1]}(x)$ shows that when $\tau<1$ is sufficiently small, $\Omega^{[1]}(x+\tau) = \Omega^{[1]}(x)$. In this case, a simple manipulation shows

$$|K^{[1]}_{i*,n}v(x+\tau) - K^{[1]}_{i*,n}v(x)| \leq \int_{\Omega^{[1]}(x)} |b(x+\tau-t)-b(x-t)| \; |P_n(a(x-\cdot)v(\cdot))(t)|\,dt$$
$$+ \int_{\Omega^{[1]}(x)} |b(x+\tau-t)| \; |P_n \left[(a(x+\tau-\cdot) - a(x-\cdot))v(\cdot)\right](t)|\,dt.$$

Using the expression (1.11) for b and some elementary analysis (see, e.g. [21]), the first term in this inequality is bounded by C $\tau|\log\tau| \; \|v\|_\infty$. Since a is smooth and $|b|$ integrable on $[-\gamma,\gamma]$, the second term is bounded by C $\tau\|v\|_\infty$. Hence we have

$$|K^{[1]}_{i*,n}v(x+\tau) - K^{[1]}_{i*,n}v(x)| \leq \mathrm{C}\; \tau^\beta \|v\|_\infty . \tag{4.10}$$

Also observe that

$$|K^{[2]}_{i*,n}v(x+\tau) - K^{[2]}_{i*,n}v(x)| \leq \sum_{i\in Q^{[2]}(x)} h_i \left[\sum_j |w_j \; (\kappa(x+\tau-x_{ij}) - \kappa(x-x_{ij}))|\right] \|v\|_\infty$$
$$\leq \tau \sum_{i\in Q^{[2]}(x)} h_i \left[\sum_j |w_j \, D\kappa(\xi_x-x_{ij})|\right] \|v\|_\infty ,$$

with $x<\xi_x<x+\tau$, where we have used the Mean Value Theorem. Thus, Proposition 4.1, A2 (or A2′) and (4.2) yield

$$|K^{[2]}_{i*,n}v(x+\tau) - K^{[2]}_{i*,n}v(x)| \leq \mathrm{C}\; \tau\|v\|_\infty \int_\delta^\infty (|D\kappa| + |D^2\kappa|) \leq \mathrm{C}\; \tau^\beta \|v\|_\infty . \tag{4.11}$$

Hence, for $\tau<1$ sufficiently small, (4.9), (4.10) and (4.11) show

$$|K_{i*,n}v(x+\tau) - K_{i*,n}v(x)| \leq \mathrm{C}\; \tau^\beta \|v\|_\infty .$$

This shows $K_{i*,n}v$ is continuous on $\mathbf{R}^+$.

Also, when x is sufficiently large $\Omega^{[1]}(x) = \emptyset$, and since, by A1, $\kappa(x)\to 0$ as $x\to\infty$, we have $K_{i*,n}v(x) = K_{i*,n}^{[2]}v(x)\to 0$ as $x\to\infty$. So, $K_{i*,n}v\in C$. (4.8) now follows by applying Proposition 4.2. □

Lemma 4.5

Let $v\in\Lambda^\beta$ with $0<\beta<1$, and let κ satisfy A1, A2(or A2′) and A3. Then for any n and $i*$,

$$\|(K_{i*} - K_{i*,n})v\|_\infty \le C \max_{1\le i\le n-i*} \left\{h_i^\beta, h_i, h_i^R\right\} \|v\|_{\Lambda^\beta}.$$

Proof

First we write

$$(K_{i*} - K_{i*,n})v = (K_{i*}^{[1]} - K_{i*,n}^{[1]})v + (K_{i*}^{[2]} - K_{i*,n}^{[2]})v.$$

Using (1.10), (1.11) and Proposition 4.3(ii), we have

$$|(K_{i*}^{[1]} - K_{i*,n}^{[1]})v(x)| \le \sum_{i\in Q^{[1]}(x)} \int_{I_i} |b(x-t)|dt \; \|(I-P_n)\,(a(x-\cdot)v(\cdot))_i\|_\infty$$

$$\le C \max_{1\le i\le n-i*} \{h_i^\beta, h_i\} \; \|v\|_{\Lambda^\beta}. \qquad (4.12)$$

Also, some elementary manipulations show

$$|(K_{i*}^{[2]} - K_{i*,n}^{[2]})v(x)| \le \sum_{i\in Q^{[2]}(x)} \left|\int_{I_i} \kappa(x-t)(v(t)-v(x_i))dt\right|$$

$$+ \sum_{i\in Q^{[2]}(x)} \sum_j |w_j\kappa(x-x_{ij})(v(x_i)-v(x_{ij}))|\,h_i$$

$$+ \sum_{i\in Q^{[2]}(x)} |v(x_i)| \left[\int_{I_i} \kappa(x-t)dt - \sum_j w_j\kappa(x-x_{ij})h_i\right]| \,.$$

(4.13)

Now consider the right-hand side of (4.13). Using A2(or A2′) and Proposition 4.3(i), the first term may be bounded by

$$C \max_{1\le i\le n-i*} h_i^\beta \|v\|_{\Lambda^\beta}. \qquad (4.14)$$

Using, in addition, Proposition 4.1 and (4.2), the second term may also be bounded in the form (4.14). Similarly, (2.3) bounds the last term by

$$C \max_{1 \le i \le n-i*} h_i^R \|v\|_{\Lambda^\beta}.$$

Collecting these bounds together in (4.13) and combining with (4.12) proves the Lemma. □

The main result of this section then, is the following.

Theorem 4.6

Under the assumptions A1, A2*(or* A2′*)* A3 *and* M1, *there exists* $i*=i*(\varepsilon,n)$, *such that*

$$\|(I - K_{i*,n})^{-1}\|_\infty \le C,$$

for all n sufficiently large.

Proof

The key step will be to show that, for any $\eta>0$, there exists a sequence $i* = i*(\eta,n)$ such that

$$\|(K_{i*} - K_{i*,n})K_{i*,n}\|_\infty < \eta, \tag{4.15}$$

which will be sufficient to prove the theorem. To see this, note that

$$(I - K_{i*} + K_{i*,n})(I - K_{i*,n}) = I - K_{i*} + (K_{i*} - K_{i*,n})K_{i*,n}. \tag{4.16}$$

Since $\|K_{i*}\|_\infty < \|K\|_\infty < 1$, it follows that $\|(I - K_{i*})^{-1}\|_\infty \le C$. With η sufficiently small, (4.15) and the Banach Lemma imply that the right hand side of (4.16) is invertible with uniformly bounded inverse. This implies $I - K_{i*,n}$ is one-one, and, since $K_{i*,n}$ is finite rank, $I - K_{i*,n}$ is invertible. Hence by (4.16) and (4.7), $\|(I - K_{i*,n})^{-1}\|_\infty \le C$.

To obtain (4.15), we note that Lemma 4.4 and Lemma 4.5 show

$$\|(K_{i*} - K_{i*,n})K_{i*,n}v\|_\infty \le C \max_{1 \le i \le n-i*} \{h_i^\beta, h_i, h_i^R\} \|v\|_\infty.$$

Now for any $\eta>0$, choose $\varepsilon = \min\{1,(\eta/C)^{1/\beta}\}$, and let $i* = i*(\varepsilon,n)$ be as determined by M1. Then it is easy to see that

$$\|(K_{i*} - K_{i*,n})K_{i*,n}v\|_\infty < \eta\|v\|_\infty,$$

giving (4.15) as required. □

In a sense this proof is a generalisation of the proof in [12] of the stability of the Nystrom method for (1.1) when κ is smooth on $(-\infty,\infty)$. Then $K_{i*,n}$ is a bounded operator from C to C^R, and this fact makes the proof of (4.15) more straightforward than in this thesis. Here $K_{i*,n}$ has a much weaker smoothing property encapsulated in Lemma 4.4. However, this, together with Lemma 4.5, are still enough to prove stability as shown above.

4.3 CONSISTENCY - THE EXPONENTIAL CASE

Throughout this section we assume κ satisfies A1, A2 and A3. Recall Theorem 3.5(i), which says that, provided f behaves suitably, the solution, u, of (1.1) is in $C^l_\beta[e^{-\mu x}]$ for some $l\in$N, $\mu>0$ and all $0<\beta<1$. We will approximate such a u on the *exponential mesh*, (2.14) and (2.15), introduced in Chapter 2 §2.3.1. Now from (2.18), we have for $i=N+1,\ldots,(r+1)N-k_0-i*$ $(=n-i*)$ and any $1\le i*\le n-1$,

$$h_i < q_2\left[\frac{1}{(r+1)N-k_0-i}\right] \le q_2\left[\frac{1}{i*}\right]. \tag{4.17}$$

Then for arbitrary $\varepsilon>0$, choose $i*(\varepsilon)$ such that $q_2/i*<\varepsilon$. Thus (2.17) and (4.17) show that for all $1\le i\le n-i*$, we have $h_i<\varepsilon$ when n is sufficiently large. Hence, the *exponential mesh*, (2.14) and (2.15), satisfies M1, and in fact, in this case, $i*$ may be chosen independently of n.

We now prove the main theorem of this section, which shows that there exists $i*$ independent of n, such that, provided q_1 and q_2 are appropriately chosen, our method converges with at least $0(1/n^m)$, where m is the number of quadrature points in rule (1.21). If R (the order of (1.21)) is greater than m, then our method may converge even faster.

Theorem 4.7

Let Π_n be the exponential mesh, (2.14) and (2.15). Then, there exists $i*$ independent of n such that, for all n sufficiently large, stability holds. Let $\mu>0$ be determined as in Lemma 3.1(i), and let $0<\beta<1$.

(i) *If $R\geq m+1$ and $u\in C_\beta^{m+1}[e^{-\mu x}]$, then*

$$\|u-u_n\|_\infty \leq C\frac{1}{n^{m+\beta}}\|u:e^{-\mu x}\|_{m+1,\beta} ,$$

provided $q_1>(m+1)/\beta$ and $q_2\geq(m+1)/\mu$.

(ii) *If $R=m$ and $u\in C_\beta^{m}[e^{-\mu x}]$, then*

$$\|u-u_n\|_\infty \leq C\frac{1}{n^{m}}\|u:e^{-\mu x}\|_{m,\beta} ,$$

provided $q_1\geq m/\beta$ and $q_2\geq m/\mu$.

REMARK. Conditions sufficient to ensure the required regularity of u are given in Theorem 3.5(i).

Proof

Theorem 4.6 and the remarks following (4.17) prove stability for some $i*$ independent of n. Then (4.1) shows that we need only bound

$$(K - K_{i*,n})u(x) = (K - K_{i*})u(x) + (K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x) + (K_{i*}^{[2]} - K_{i*,n}^{[2]})u(x), \tag{4.18}$$

as $n\rightarrow\infty$ for any fixed $i*$. Note that for all sufficiently large n, x_{n-i*} will always lie in $(1,\infty)$.

Proof of (i)

For convenience, let $\|u\| = \|u:e^{-\mu x}\|_{m+1,\beta}$. Note that

$$|(K - K_{i*})u(x)| = |\int_{x_{n-i*}}^{\infty}\kappa(x-t)u(t)dt| \leq \|K\|_\infty \|u\|_{\infty,[x_{n-i*},\infty)} \tag{4.19}$$

$$\leq C \exp(-\mu x_{n-i*})\|u\| = C\left[\frac{i*}{rN}\right]^{\mu q_2}\|u\| \tag{4.20}$$

$$\leq C\frac{1}{n^{m+1}}\|u\|, \tag{4.21}$$

by (2.15), choice of q_2 and (2.16).

Also, using Proposition 2.2 (with $l=m+1\leq R$) we have, when $1\notin Q^{[2]}(x)$,

$$|(K_{i*}^{[2]} - K_{i*,n}^{[2]})u(x)| \leq C \sum_{i\in Q^{[2]}(x)} h_i^{m+1}\|D^{m+1}(\kappa(x-\cdot)u(\cdot))_i\|_1$$

$$\leq C \sum_{i\in Q^{[2]}(x)} h_i^{m+1} \sum_{v=0}^{m+1} \|D^{m+1-v}\kappa(x-\cdot)_i\|_1 \;\|(D^v u)_i\|_\infty \tag{4.22}$$

$$\leq C\frac{1}{n^{m+1}}\|u\|, \tag{4.23}$$

where we have used the Leibniz rule, Proposition 2.5 (with $t=m+1$), our choice of q_1 and q_2, (2.16) and A2. If $1\in Q^{[2]}(x)$, we use, in addition, the estimate

$$\left|\int_{I_1}(I-P_n)(\kappa(x-\cdot)u(\cdot))(t)dt\right| \leq C\, h_1\|\kappa(x-\cdot)_1\|_\infty\|u_1\|_\infty \tag{4.24}$$

$$\leq C\frac{1}{n^{m+1}}\|u\|,$$

by A2, (2.14), choice of q_1 and (2.16). Hence (4.23) remains true even when $1\in Q^{[2]}(x)$.

Now we deal with $(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x)$. Define, for each $x\in\mathbf{R}^+$, a piecewise constant function ψ_x by

$$\psi_x(t) = h_i^{-1}\int_{I_i} b(x-\xi)d\xi := \psi_{x,i} \text{ for } t\in I_i. \tag{4.25}$$

Then, since $R\geq m+1$, we have

$$(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x) = \sum_{i\in Q^{[1]}(x)}\int_{I_i}(b(x-t)-\psi_x(t))(I-P_n)(a(x-\cdot)u(\cdot)-v(\cdot))(t)dt$$

$$+ \sum_{i\in Q^{[1]}(x)}\psi_{x,i}\int_{I_i}(I-P_n)(a(x-\cdot)u(\cdot)-w(\cdot))(t)dt$$

$$:= T_1(x) + T_2(x), \text{ say,} \tag{4.26}$$

where, for each i, v_i, and w_i are chosen to be, respectively, the m^{th} and $(m+1)^{th}$ order Taylor polynomials for $a(x-\cdot)u(\cdot)$ about x_i.

Now suppose $1 \notin Q^{[1]}(x)$. Then Taylor's Theorem gives

$$\|(I-P_n)(a(x-\cdot)u(\cdot)-\zeta(\cdot))_i\|_\infty \le C\, h_i^l \;\|D^l(a(x-\cdot)u(\cdot))_i\|_\infty, \tag{4.27}$$

where $l=m$ (respectively $m+1$) when $\zeta=v$ (respectively w). Then the Leibniz rule, A3 and Proposition 2.5 (with $t=m+\beta$) bound $T_1(x)$ by

$$|T_1(x)| \le C \sum_{i\in Q^{[1]}(x)} \|(b(x-\cdot)-\psi_x)_i\|_1\, h_i^m \sum_{v=0}^{m} \|(D^v u)_i\|_\infty \tag{4.28}$$

$$\le C \frac{1}{n^{m+\beta}} \sum_{i\in Q^{[1]}(x)} \left[h_i^{-\beta}\|(b(x-\cdot)-\psi_x)_i\|_1\right]\|u\| ,$$

and Lemma B1(ii) of Appendix B shows

$$|T_1(x)| \le C \frac{1}{n^{m+\beta}} \|u\|. \tag{4.29}$$

Also, by Lemma B1(i), (4.27) and the Leibniz rule, we have

$$|T_2(x)| \le C \sum_{i\in Q^{[1]}(x)} |\log h_i|\, h_i^{m+2} \sum_{v=0}^{m+1} \|(D^v u)_i\|_\infty$$

$$\le C \sum_{i\in Q^{[1]}(x)} h_i h_i^{m+1-\varepsilon} \sum_{v=0}^{m+1} \|(D^v u)_i\|_\infty , \tag{4.30}$$

for any fixed $\beta>\varepsilon>0$. Now if $2\le i\le N$ we may argue as in the proof of Proposition 2.5 to show that

$$h_i^{m+1-\varepsilon}\|(D^{m+1}u)_i\|_\infty \le C\, \frac{1}{n^{m+1-\varepsilon}} \|u\| , \tag{4.31}$$

whenever

$$q_1 \ge (m+1-\varepsilon)/(\beta-\varepsilon). \tag{4.32}$$

But, since $(m+1-\varepsilon)/(\beta-\varepsilon)>(m+1)/\beta$, our choice of q_1 ensures that (4.32) will be satisfied for all ε sufficiently close to zero. Now fix ε in the range $0<\varepsilon<1-\beta$, so that (4.32) also holds. Then (4.31) yields

$$h_i^{m+1-\varepsilon} \| (D^{m+1} u)_i \|_\infty \le C \frac{1}{n^{m+\beta}} \| u \| .$$

The above remarks, together with Proposition 2.5, show

$$| T_2(x) | \le C \sum_{i \in Q^{[1]}(x)} h_i \frac{1}{n^{m+\beta}} \| u \| \le 2\gamma C \frac{1}{n^{m+\beta}} \| u \| . \tag{4.33}$$

Hence when $1 \notin Q^{[1]}(x)$ we have, from (4.26), (4.29) and (4.33),

$$\| (K_{i*}^{[1]} - K_{i*,n}^{[1]}) u \|_\infty \le C \frac{1}{n^{m+\beta}} \| u \| . \tag{4.34}$$

Now if $1 \in Q^{[1]}(x)$, the additional term

$$\int_{I_1} b(x-t)(I-P_n)(a(x-\cdot)u(\cdot))(t)dt \tag{4.35}$$

must be bounded. However,

$$\begin{aligned} | a(x-t)u(t) - a(x)u(0) | &= | \textstyle\int_0^t D(a(x-\cdot)u(\cdot))(\sigma) d\sigma | \\ &\le C \textstyle\int_0^t (| u(\sigma) | + | Du(\sigma) |) d\sigma \\ &\le C \, (h_1 + \textstyle\int_0^t \sigma^{\beta-1} d\sigma) \| u \| \\ &\le C \, h_1^\beta \| u \| \qquad (4.36) \\ &\le C \frac{1}{n^{m+1}} \| u \| , \end{aligned}$$

by (2.14), choice of q_1 and (2.16). So, replacing $a(x-\cdot)u(\cdot)$ by $a(x-\cdot)u(\cdot) - a(x)u(0)$ in (4.35), and recalling (1.11), shows that (4.34) remains true even when $1 \in Q^{[1]}(x)$.

Now substitution of (4.21), (4.23) and (4.34) in (4.18) yields the required estimate.

Proof of (ii)

For convenience let $\| u \| = \| u : e^{-\mu x} \|_{m,\beta}$. Note that (4.20), our choice of q_2 and (2.16) show

$$|(K - K_{i*})u(x)| = C\frac{1}{n^m}\|u\| \ . \tag{4.37}$$

Also, using Proposition 2.2 (with $l=m=R$) we have when $1 \notin Q^{[2]}(x)$,

$$|(K_{i*}^{[2]} - K_{i*,n}^{[2]})u(x)| \le C \sum_{i \in Q^{[2]}(x)} h_i^m \|D^m(\kappa(x-\cdot)u(\cdot))_i\|_1$$

$$\le C \sum_{i \in Q^{[2]}(x)} h_i^m \sum_{v=0}^{m} \|D^{m-v}\kappa(x-\cdot)_i\|_1 \ \|(D^v u)_i\|_\infty \tag{4.38}$$

$$\le C\frac{1}{n^m}\|u\| \ , \tag{4.39}$$

where we have used the Leibniz rule, Proposition 2.5 (with $t=m$), our choice of q_1 and q_2 and A2. If $1 \in Q^{[2]}(x)$, (4.24), A2, (2.14), choice of q_1 and (2.16) show that (4.39) remains true.

Now we deal with $(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x)$. When $1 \notin Q^{[1]}(x)$, we have

$$(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x) = \sum_{i \in Q^{[1]}(x)} \int_{I_i} b(x-t)(I-P_n)(a(x-\cdot)u(\cdot)-v(\cdot))(t)dt \ , \tag{4.40}$$

where for each i, v_i is chosen to be the *mth* order Taylor polynomial for $a(x-\cdot)u(\cdot)$ about x_i. Then using (4.27), the Leibniz rule, A3, Proposition 2.5 (with $t=m$), our choice of q_1 and q_2, and (2.16) we have

$$|(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x)| \le C\frac{1}{n^m}\|u\| \ . \tag{4.41}$$

When $1 \in Q^{[1]}(x)$, the additional term (4.35) must be bounded. However (4.36) holds. Thus, replacing $a(x-\cdot)u(\cdot)$ by $a(x-\cdot)u(\cdot)-a(x)u(0)$ in (4.35) shows that (4.41) remains true by A3, (2.14), our choice of q_1 and (2.16).

Then substituting (4.37), (4.39), and (4.41) in (4.18), yields the required estimate. □

Remarks.

(i) The proof of (4.34) in Theorem 4.7(i), generalises the techniques used by Schneider [37] and Chandler [10], for weakly singular integral equations on a finite interval.

(ii) Consider the case when $f = C =$ constant > 0 in (1.1) and recall the discussion following Theorem 3.5. Under the conditions of Theorem 3.6(i), the solution, $\tilde{u} = u - u(\infty)$, of (3.45) (an equation of the form (1.1) for $\tilde{u}$) with F, in (3.45), given by (3.46), satisfies $\tilde{u} \in C^l_\beta[e^{-\mu x}]$ for all $l \in \mathbb{N}$, $0<\beta<1$ and some $\mu>0$. Thus, Theorem 4.7 holds for (3.45).

4.4 CONSISTENCY - THE POLYNOMIAL CASE

Throughout this section we assume κ satisfies A1, A2$'$ and A3. Recall Theorem 3.5(ii) which says that, provided f behaves suitably, the solution, u, of (1.1) is in $C^l_\beta[x^{-\mu'}]$ for some $l \in \mathbb{N}$, $\mu'>0$ and all $0<\beta<1$. We will approximate such a u on the *polynomial mesh*, (2.33) and (2.34), introduced in Chapter 2 §2.3.2. Note that for any fixed $i* \geq 1$ and n sufficiently large, we have for $i=N+1,\ldots,(r+1)N-i*$ $(=n-i*)$

$$h_i \leq C\left[\frac{1}{i*}\right]\left[\frac{rN}{i*}\right]^{p_2}, \tag{4.42}$$

by (2.41). Recall that $N=O(n)$ and $n=O(N)$ (see (2.35)). Then from (4.42) it follows that, to ensure M1 (and therefore (4.2)), we must choose $i*=i*(n)$ to be an increasing function of n. Also, note that for large n, x_{n-i*} will always lie in $(1,\infty)$. Now (4.19) holds, and so if $u \in C^l_\beta[x^{-\mu'}]$ then (4.19) and (2.34) show that

$$\begin{aligned}|(K - K_{i*})u(x)| &\leq \|K\|_\infty \|u\|_{\infty,[x_{n-i*},\infty)} \\ &\leq C\, x_{n-i*}^{-\mu'} \|u:x^{-\mu'}\|_{l,\beta} \\ &\leq C\left[\frac{i*}{rN}\right]^{\mu' p_2} \|u:x^{-\mu'}\|_{l,\beta}\,,\end{aligned} \tag{4.43}$$

for all $x \in \mathbf{R}^+$.

Recall that $|(K - K_{i*})u(x)|$ is one of the terms in the consistency error (4.18). Suppose an increasing $i*(n)$ has been found so that M1 (and hence stability) holds, then the method cannot be shown to converge to 0 as $N \to \infty$ (or, equivalently as $n \to \infty$), unless the right hand side of (4.43) converges to 0 as $N \to \infty$. Now clearly $i*(n)$ must be chosen very carefully, since an $i*(n)$ which increases fast enough to ensure stability may prove to increase so fast so as to hinder the convergence of $\|(K - K_{i*})u\|_\infty$ (and hence the convergence of the Nystrom-Product Integration method) as $N \to \infty$. Thus we need to choose an $i*(n)$ which is finely tuned so that M1 holds but so that the convergence of (4.43) to 0 is as fast as possible. In this thesis we restrict to the choice of $i*(n) = (rN)^\alpha$ for some $0 < \alpha < 1$. Then, to satisfy M1, (4.42) shows that we need $\alpha(p_2+1) - p_2 > 0$ or

$$\alpha > \frac{p_2}{p_2+1} .$$

Now (2.35) shows that with this choice of $i*(n)$ we have

$$\left[\frac{i*}{rN}\right]^{\mu' p_2} = \frac{1}{(rN)^t} \left[(i*)^{\mu' p_2} \frac{1}{(rN)^{\mu' p_2 - t}}\right]$$

$$\leq C \frac{1}{n^t} , \tag{4.44}$$

provided $\mu' p_2 - t - \alpha\mu' p_2 \geq 0$, or

$$\alpha \leq 1 - \frac{t}{\mu' p_2} .$$

Thus, if $i* = (rN)^\alpha$ with

$$\frac{p_2}{p_2+1} < \alpha \leq 1 - \frac{t}{\mu' p_2} ,$$

for some $t > 0$, then M1 holds, and (4.43) and (4.44) show

$$\|(K - K_{i*})u\|_\infty \leq C \frac{1}{n^t} \|u : x^{-\mu'}\|_{l,\beta} . \tag{4.45}$$

Now such an α exists provided

$$\frac{p_2}{p_2+1}<1-\frac{t}{\mu' p_2} ,$$

or

$$p_2(\mu'-t)>t ,$$

which yields the choice $p_2>t/(\mu'-t)$, a valid choice of p_2 provided $t<\mu'$. Thus, we have the following lemma.

Lemma 4.8

Let $0<\beta<1$ *and let* $u\in C_\beta^l[x^{-\mu'}]$. *Let* Π_n *be the polynomial mesh, (2.33) and (2.34). Then for all* $0<t<\mu'$, *there exists* $i*=(rN)^\alpha$ *with*

$$\frac{p_2}{p_2+1}<\alpha\leq 1-\frac{t}{\mu' p_2}$$

such that M1 *holds and*

$$\|(K-K_{i*})u\|_\infty \leq C\frac{1}{n^t}\|u:x^{-\mu'}\|_{l,\beta} ,$$

provided $p_2>t/(\mu'-t)$.

The convergence analysis for the polynomial decay case is more complicated than in the exponential case. It turns out however that, under the conditions of Lemma 4.8 above, the whole consistency error (4.18) converges with $O(1/n^t)$. The main result of this section is Theorem 4.9 which shows that with $i*$ chosen as in Lemma 4.8, and with p_1 and p_2 appropriately chosen, our numerical method for (1.1) converges with at least $O(1/n^m)$ if $m<\mu'$. If $m\geq\mu'$ then our method converges with almost $O(1/n^{\mu'})$. If $\mu'>R\geq m+1$ then we can show $O(1/n^{m+\beta})$ convergence for all $0<\beta<1$.

Theorem 4.9

Let Π_n be the polynomial mesh, (2.33) and (2.34). Let $\mu'>0$ be determined as in Lemma 3.1(ii), let $0<\beta<1$ and let $u\in C_\beta^{m+1}[x^{-\mu'}]$.

(i) *If $R=m<\mu'$ then*

$$\|u-u_n\|_\infty \leq C\frac{1}{n^m}\|u:x^{-\mu'}\|_{m,\beta}\ , \qquad (4.46)$$

provided $p_1\geq m/\beta$ and $p_2>m/(\mu'-m)$.

(ii) *If $R\geq m+1>\mu'>m$ then (4.46) holds provided $p_1\geq m/\beta$ and $p_2>m/(\mu'-m)$.*

(iii) *If $R=m\geq\mu'$ then*

$$\|u-u_n\|_\infty \leq C\frac{1}{n^{\mu'-\varepsilon}}\|u:x^{-\mu'}\|_{m,\beta}\ , \qquad (4.47)$$

for all $0<\varepsilon<\mu'$, provided $p_1\geq m/\beta$ and $p_2>(\mu'-\varepsilon)/\varepsilon$.

(iv) *If $R\geq m+1$ and $m\geq\mu'$ then (4.47) holds for all $0<\varepsilon<\mu'$, provided $p_1\geq m/\beta$ and $p_2>(\mu'-\varepsilon)/\varepsilon$.*

(v) *If $\mu'>R\geq m+1$ then*

$$\|u-u_n\|_\infty \leq C\frac{1}{n^{m+\beta}}\|u:x^{-\mu'}\|_{m+1,\beta}\ ,$$

provided $p_1>(m+1)/\beta$ and $p_2>(m+1)/(\mu'-(m+1))$.

REMARK. Conditions sufficient to ensure the required regularity of u are described in Theorem 3.5(ii).

Proof

Proof of (i)

For convenience let $\|u\|=\|u:x^{-\mu'}\|_{m,\beta}$. Under the conditions of (i) we can apply Lemma 4.8 with $t=m<\mu'$, and our choice of p_2 ensures that there exists an

$i*=(rN)^\alpha$ with

$$\frac{p_2}{p_2+1}<\alpha\le 1-\frac{m}{\mu' p_2}$$

for which M1 holds and for which

$$\|(K-K_{i*})u\|_\infty \le C\frac{1}{n^m}\|u\|. \tag{4.48}$$

With this choice of $i*$, Theorem 4.6 ensures stability and then 4.1 shows that we need only bound

$$(K - K_{i*,n})u(x) =(K - K_{i*})u(x) + (K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x) + (K_{i*}^{[2]} - K_{i*,n}^{[2]})u(x), \tag{4.49}$$

as $n\to\infty$ for this choice of $i*$. Note that for all sufficiently large n, x_{n-i*} will always lie in $(1,\infty)$.

First consider $(K_{i*}^{[2]}-K_{i*,n}^{[2]})u(x)$. Note that (4.38) still holds and thus, when $1\notin Q^{[2]}(x)$, we have

$$\begin{aligned}|(K_{i*}^{[2]} - K_{i*,n}^{[2]})u(x)| &\le C \sum_{i\in Q^{[2]}(x)} h_i^m \sum_{v=0}^{m}\|D^{m-v}\kappa(x-\cdot)_i\|_1 \; \|(D^v u)_i\|_\infty \\ &\le C\frac{1}{n^m}\|u\| ,\end{aligned} \tag{4.50}$$

where the last inequality follows from A2′, Proposition 2.9 (with $t=m<\mu'$), our choice of p_1 and p_2, and (2.35). Now note that (4.24) still holds, and so if $1\in Q^{[2]}(x)$ then (4.50) remains true by (2.33), our choice of p_1, (2.35) and A2′.

Now we deal with $(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x)$. Observe that (4.40) holds and thus, when $1\notin Q^{[1]}(x)$, (4.27) (with $l=m$), the Leibniz rule and A3 show

$$\begin{aligned}|(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x)| &\le C \sum_{i\in Q^{[1]}(x)} \|b(x-\cdot)_i\|_1 h_i^m \sum_{v=0}^{m}\|(D^m u)_i\|_\infty \\ &\le C\frac{1}{n^m}\|u\| ,\end{aligned} \tag{4.51}$$

where (4.51) follows from Proposition 2.9 (with $t=m<\mu'$), our choice of p_1 and

p_2, (2.35) and A3. When $1\in Q^{[1]}(x)$, the additional term given by (4.35) must be bounded. However (4.36) holds. Thus, replacing $a(x-\cdot)u(\cdot)$ by $a(x-\cdot)u(\cdot)-a(x)u(0)$ in (4.35), shows that (4.51) remains true by A3, (2.33), our choice of p_1 and (2.35).

Now substitution of (4.48), (4.50) and (4.51) in (4.49) yields the required estimate.

Proof of (ii)

The proof of (ii) follows analogously to the proof of (i).

Proof of (iii)

For convenience set $\|u\|=\|u:x^{-\mu'}\|_{m,\beta}$. Also set $I[1]=\{i:\ 2\leq i\leq N\ \}$, $I[2]=\{i:\ N+1\leq i\leq n-i*\ \}$ and let $0<\varepsilon<\mu'$. Under the conditions of (iii) we can apply Lemma 4.8 with $t=(\mu'-\varepsilon)<\mu'$, and our choice of p_2 ensures that there exists an $i*=(rN)^{\alpha}$ with

$$\frac{p_2}{p_2+1}<\alpha\leq 1-\frac{(\mu'-\varepsilon)}{\mu' p_2}$$

for which M1 holds and for which

$$\|(K-K_{i*})u\|_{\infty}\leq C\frac{1}{n^{\mu'-\varepsilon}}\|u\|. \tag{4.52}$$

With this choice of $i*$, Theorem 4.6 ensures stability and then 4.1 shows that we need only bound (4.49) as $n\to\infty$ for this choice of $i*$.

Now consider $(K_{i*}^{[2]}-K_{i*,n}^{[2]})u(x)$. Proposition 2.2 (with $l=m=R$) and the Leibniz rule show that

$$|(K_{i*}^{[2]}-K_{i*,n}^{[2]})u(x)|\leq$$

$$C\left[\sum_{i\in Q^{[2]}(x)\cap I[1]}+\sum_{i\in Q^{[2]}(x)\cap I[2]}\right]h_i^m\sum_{v=0}^{m}\|D^{m-v}\kappa(x-\cdot)_i\|_1\|(D^v u)_i\|_{\infty}\ . \tag{4.53}$$

Thus, when $1\notin Q^{[2]}(x)$, the first term on the right hand side of (4.53) can be bounded by

$$C\frac{1}{n^m}\|u\| \ , \tag{4.54}$$

by (2.33), our choice of p_1, (2.35), (2.50) (with $t=m$) and A2$'$. Now we can write the second term on the right hand side of (4.53) as

$$C\sum_{i\in Q^{[2]}(x)\cap I[2]} h_i^{m-\mu'+\varepsilon}\left[h_i^{\mu'-\varepsilon}\sum_{v=0}^{m}\|D^{m-v}\kappa(x-\cdot)_i\|_1\|(D^v u)_i\|_\infty\right] . \tag{4.55}$$

Then, by (2.34), our choice of p_2, (2.35), Proposition 2.9 (with $t=\mu'-\varepsilon<\mu'$), A2$'$ and (4.2), we can bound (4.55) by

$$C\frac{1}{n^{\mu'-\varepsilon}}\|u\| \ , \tag{4.56}$$

and so collecting these bounds together in (4.53) we obtain when $1\notin Q^{[2]}(x)$

$$|(K_{i*}^{[2]} - K_{i*,n}^{[2]})u(x)| \le C\frac{1}{n^{\mu'-\varepsilon}}\|u\| \ . \tag{4.57}$$

Note that (4.24) holds. Thus, when $1\in Q^{[2]}(x)$, (2.33), our choice of p_1, (2.35) and A2$'$ show that (4.57) remains true.

Now we deal with $(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x)$. Observe that (4.40) holds, and so when $1\notin Q^{[1]}(x)$, (4.27) (with $l=m$), the Leibniz rule and A3 show that

$$|(K_{i*}^{[1]} - K_{i*,n}^{[1]})u(x)| \le C\left[\sum_{i\in Q^{[1]}(x)\cap I[1]} + \sum_{i\in Q^{[1]}(x)\cap I[2]}\right]\|b(x-\cdot)_i\|_1 h_i^m\sum_{v=0}^{m}\|(D^v u)_i\|_\infty$$

$$\le C\frac{1}{n^{\mu'-\varepsilon}}\|u\|, \tag{4.58}$$

where (4.58) follows using the same arguments as in the proof of (4.57) except that where that argument uses A2$'$, this argument uses A3. When $1\in Q^{[1]}(x)$, the additional term given by (4.35) must be bounded. However (4.36) holds. Thus, replacing $a(x-\cdot)u(\cdot)$ by $a(x-\cdot)u(\cdot)-a(x)u(0)$ in (4.35), shows that (4.58) remains true by A3, (2.33), our choice of p_1 and (2.35).

Then substituting (4.52), (4.57) and (4.58) in (4.49) yields the required estimate.

Proof of (iv)

The proof of (iv) follows analogously to the proof of (iii).

Proof of (v)

For convenience set $\|u\|=\|u:x^{-\mu'}\|_{m+1,\beta}$. Under the conditions of (v) we can apply Lemma 4.8 with $t=m+1<\mu'$, and our choice of p_2 ensures that there exists an $i*=(rN)^{\alpha}$ with

$$\frac{p_2}{p_2+1}<\alpha\leq 1-\frac{m+1}{\mu' p_2}$$

for which M1 holds and for which

$$\|(K-K_{i*})u\|_{\infty}\leq C\frac{1}{n^{m+1}}\|u\|. \tag{4.59}$$

With this choice of $i*$, Theorem 4.6 ensures stability and then 4.1 shows that we need only bound (4.49) as $n\to\infty$ for this choice of $i*$.

Consider $(K_{i*}^{[2]}-K_{i*,n}^{[2]})u(x)$ and observe that (4.22) holds. Then Proposition 2.9 (with $t=m+1<\mu'$), our choice of p_1 and p_2, (2.35) and A2$'$ show that when $1\notin Q^{[2]}(x)$,

$$\|(K_{i*}^{[2]}-K_{i*,n}^{[2]})u\|_{\infty}\leq C\frac{1}{n^{m+1}}\|u\| \ . \tag{4.60}$$

Now note that (4.24) holds. Thus, when $1\in Q^{[2]}(x)$, (2.33), our choice of p_1, (2.35) and A2$'$ show that (4.60) remains true.

Now we deal with

$$(K_{i*}^{[1]}-K_{i*,n}^{[1]})u(x) := T_1(x)+T_2(x) \ , \tag{4.61}$$

(see (4.26)). By arguments analogous to those used in the the proofs of (4.29) and (4.33) we have when $1\notin Q^{[1]}(x)$,

$$|T_1(x)| \leq C\frac{1}{n^{m+\beta}}\|u\| \ , \tag{4.62}$$

and

$$|T_2(x)| \leq 2\gamma\, C\frac{1}{n^{m+\beta}}\|u\| \ . \tag{4.63}$$

Hence, substituting (4.62) and (4.63) in (4.61) shows that when $1 \notin Q^{[1]}(x)$,

$$\|(K_{i*}^{[1]} - K_{i*,n}^{[1]})u\|_\infty \leq C\frac{1}{n^{m+\beta}}\|u\| \ . \tag{4.64}$$

When $1 \in Q^{[1]}(x)$ the additional term given by (4.35) must be bounded. However (4.36) holds. Thus, replacing $a(x-\cdot)u(\cdot)$ by $a(x-\cdot)u(\cdot)-a(x)u(0)$ in (4.35), shows that (4.64) remains true by A3, (2.33), our choice of p_1 and (2.35).

Then substituting (4.59), (4.60) and (4.64) in (4.49) yields the required estimate. □

Remark

Consider the case when f = C = constant > 0 in (1.1) and recall the discussion following Theorem 3.5. Under the conditions of Theorem 3.6(ii), the solution $\tilde{u} = u - u(\infty)$ of (3.45) (an equation of the form (1.1) for $\tilde{u}$) with F, in (3.45), given by (3.46), satisfies $\tilde{u} \in C_\beta^l[x^{-\mu'}]$ for all $l \in$ N, $0<\beta<1$ and some $\mu'>0$. Thus, Theorem 4.9 holds for (3.45).

CHAPTER 5
PRACTICAL APPLICATIONS

5.1 TEST EXAMPLE

We implemented the Nystrom-Product Integration method (see §1.4.2) on a particular test example for which the solution was known. We solved (1.1) with

$$\kappa(x) = \left[\frac{1-\varepsilon}{2}\right] E_1(|x|) = \left[\frac{1-\varepsilon}{2}\right] \int_0^1 e^{-|x|/\mu} \frac{1}{\mu} d\mu \;, \tag{5.1}$$

where $\varepsilon \in (0,1)$. This is the kernel which occurs in Example 1 §1.1. For this test, f was chosen so that $u(x) = x(\log x)e^{-x}$.

We first check the assumptions, A1, A2 and A3 of Chapter 3. Now it can be shown that

$$|\kappa|^{\wedge}(\omega) = \hat{\kappa}(\omega) = (1-\varepsilon)\arctan(\omega)/\omega,$$

and so we have $|\kappa|^{\wedge}(0) = \hat{\kappa}(0) = (1-\varepsilon)$, and thus A1 follows. By [1, (5.1.24), (5.1.26), (5.1.51)], A2 is satisfied for $0<\rho<1$. Also by [1, (5.1.11)],

$$E_1(x) = -\log x - \gamma - \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{kk!}, \qquad |\arg x| < \pi, \tag{5.2}$$

where γ is Euler's constant. By choosing

$$a(x) = -\frac{(1-\varepsilon)}{2}, \quad b(x) = -E_1(|x|), \quad c(x) = -E_1(x) - \log x, \tag{5.3}$$

A3 is satisfied.

We take $\varepsilon = \frac{1}{2}$ and $\delta = 1$ (where δ is the minimum radius of the region in which product integration is performed (see §1.4.2)), and for x in this region we compute $c(|x|)$ by the approximation

$$c(|x|) \approx \left[\gamma + \sum_{k=1}^{10} \frac{(-1)^k |x|^k}{kk!}\right], \tag{5.4}$$

with a maximum error of $|x|^{11}/(11)(11)!$.

The values of $f(x)$ for quadrature points x are not known analytically, but are approximated by setting

$$f(x) = u(x) - K_{i*,n_1}u(x),$$

for large n_1 (given below). We use the *exponential mesh*, (2.14) and (2.15), introduced in §2.3.1 with r=2 and various q_1 and q_2. By Theorem 4.7, there exists a cut-off point x_{n-i*} which ensures stability. In this experiment we chose $i*=1$ and observed no instability.

If we let e_N be $\max|u(x) - u_n(x)|$, taken over all quadrature points x, then an estimate of the order of uniform convergence is given by $EOC = \log(e_N/e_{2N})/\log 2$, (see discussion proceeding (2.5)).

Table 5.1 of § 5.1.1 gives results for the Trapezoidal rule (m=2, $\xi_1=0$, $\xi_2=1$, R=2) with $q_1=2=q_2$ and $n_1=284$. Table 5.2 of § 5.1.1 gives the results for Simpson's rule (m=3, $\xi_1=0$, $\xi_2=\frac{1}{2}$, $\xi_3=1$, R=4) with $q_1=4=q_2$ and $n_1=327$.

Slightly increasing q_2 accelerates the convergence of our numerical method. Table 5.3 of § 5.1.1 gives the results for the Trapezoidal rule with $q_1=2$ and $q_2=2.1$. Table 5.4 of §5.1.1 gives the results for Simpson's rule with $q_1=4$ and $q_2=4.1$. The results indicate that, using the Trapezium rule we have

$$\|u-u_n\|_\infty \approx O(1/n^2) ,$$

and for Simpson's rule we have

$$\|u-u_n\|_\infty \approx O(1/n^4) .$$

These results are as predicted by Theorem 4.7.

Figure 1 is a graph of the true solution (dark line) and the Nystrom-Product Integration solution (dashed line), using Simpson's rule with N=8, r=2 and with $q_1=4=q_2$. Figure 1 demonstrates the accuracy of our numerical method over a range of x. Figure 2 shows the graph in Figure 1 restricted to [0,1]. As can be seen, our numerical method seems to be working extremely well, even near the origin where the solution is not smooth. Figure 3 shows the graph in Figure 1 restricted to $x \geq 1$. Clearly our numerical method is modelling the decay of $u(x)$ as $x \rightarrow \infty$ to a high degree of accuracy as well.

5.1.1 NUMERICAL RESULTS

N	n	e_N	EOC
2	5	0.51×10^{-1}	
			1.21
4	9	0.22×10^{-1}	
			1.51
8	18	0.77×10^{-2}	
			1.62
16	36	0.25×10^{-2}	
			1.63
32	71	0.81×10^{-3}	

TABLE 5.1

N	n	e_N	EOC
2	6	0.83×10^{-2}	
			3.05
4	11	0.10×10^{-2}	
			3.32
8	21	0.10×10^{-3}	
			3.47
16	41	0.90×10^{-5}	
			3.62
32	82	0.73×10^{-6}	

TABLE 5.2

N	n	e_N	EOC
2	5	0.49×10^{-1}	
			1.37
4	9	0.19×10^{-1}	
			1.62
8	18	0.62×10^{-2}	
			1.78
16	36	0.18×10^{-2}	
			2.00
32	71	0.45×10^{-3}	

TABLE 5.3

N	n	e_N	EOC
2	6	0.70×10^{-2}	
			3.15
4	11	0.79×10^{-3}	
			3.52
8	21	0.69×10^{-4}	
			3.68
16	42	0.54×10^{-5}	
			3.95
32	83	0.35×10^{-6}	

TABLE 5.4

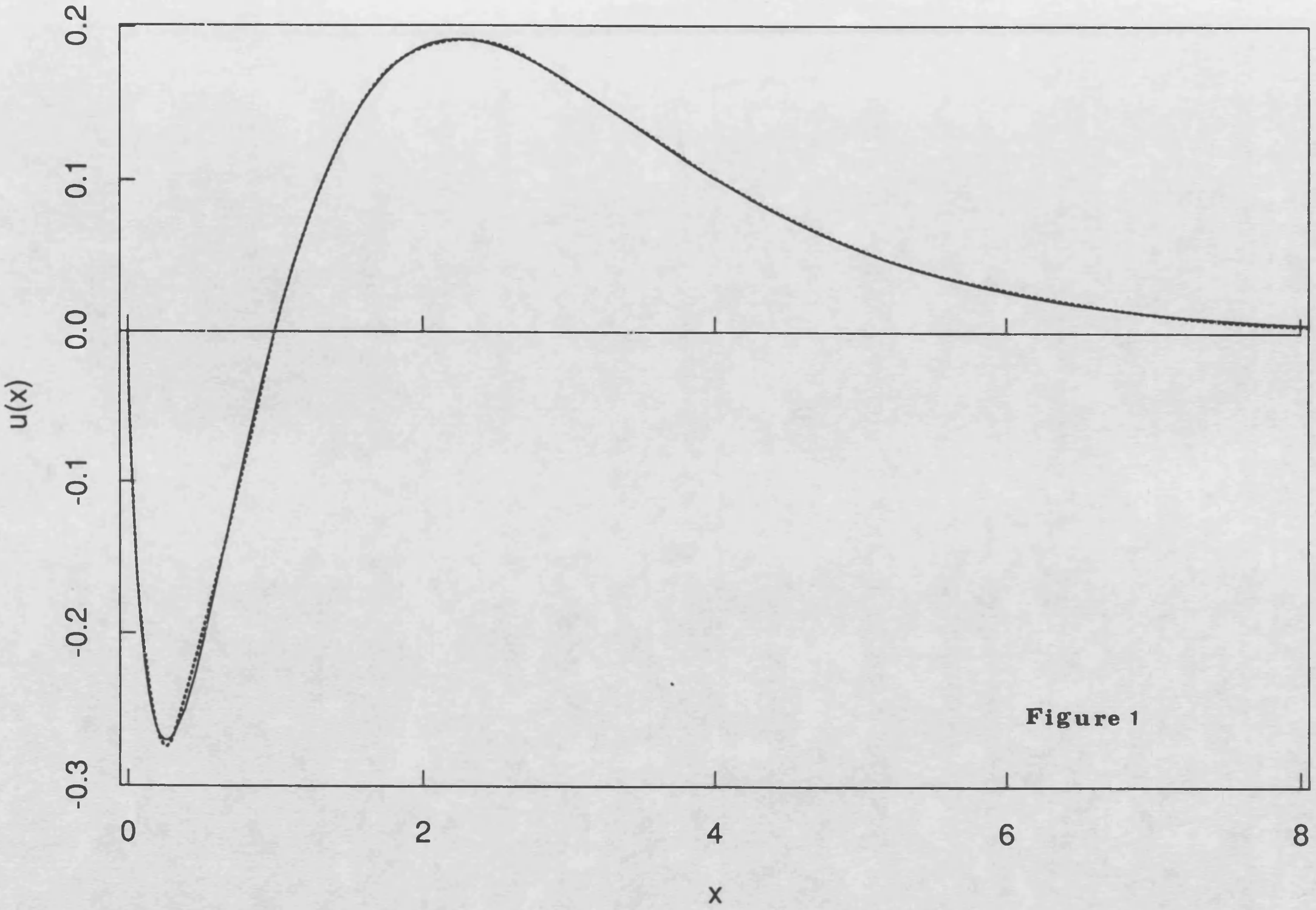

Figure 1

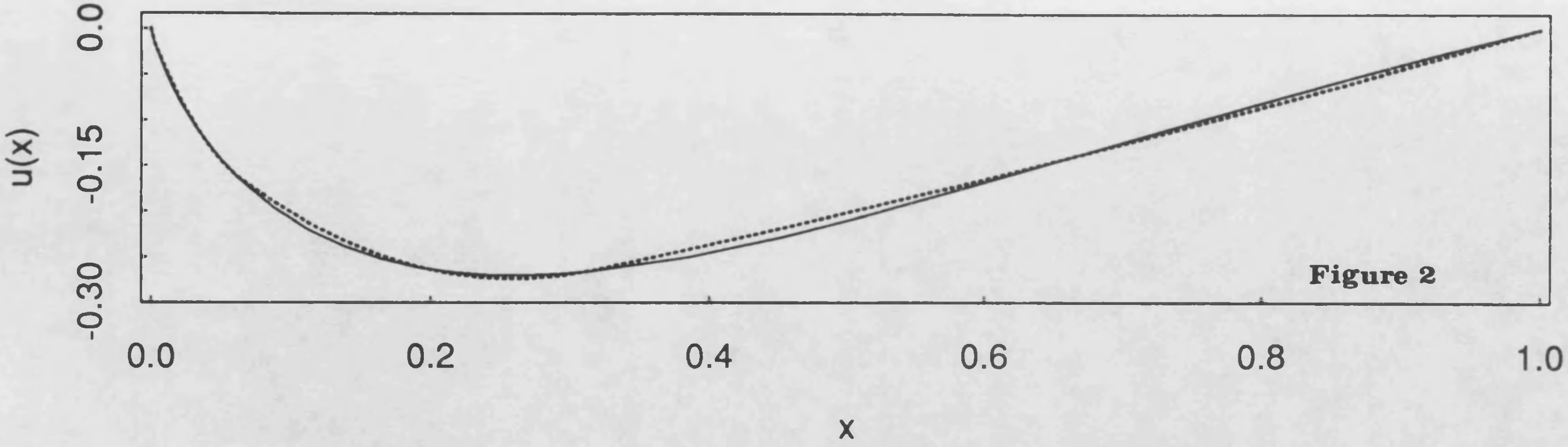

Figure 2

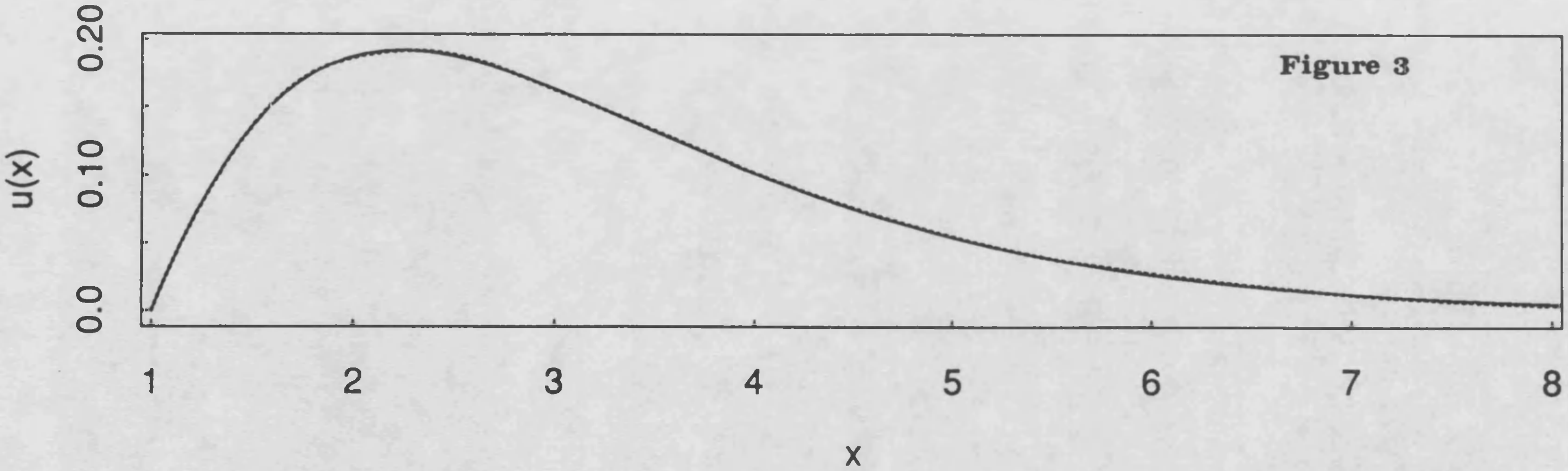

Figure 3

5.2 THE TRANSPORT EQUATION FOR COHERENT SCATTERING

5.2.1 REFORMULATION

Recall the linear Transport Equation of Chapter 1, Example 1,

$$\mu\frac{\partial}{\partial x}I(x,\mu) - I(x,\mu) = -S(x)\ , \qquad -1\le\mu\le 1,\quad x\in\mathbf{R}^+\ , \tag{5.5}$$

where the source function $S(x)$ is given by

$$S(x) = \left[\frac{1-\varepsilon}{2}\right]\int_{-1}^{1}I(x,\eta)d\eta + \varepsilon B(x)\ , \tag{5.6}$$

where $\varepsilon\in(0,1)$, $B\in C$ is the Planck function, and the boundary conditions are given by

$$I(0,\mu)=0 \text{ for } \mu<0$$

$$e^{-\alpha x}I(x,\mu)\to 0 \text{ as } x\to\infty, \text{ for all } \mu>0,\ \alpha>0\ .$$

We now reformulate (5.5) as an integral equation of the form (1.1) for $S(x)$. If we formally integrate (5.5) and build in the boundary conditions we obtain

$$I(x,\mu) = -\int_0^x e^{-(t-x)/\mu}\ \frac{S(t)}{\mu}dt, \quad \mu<0 \tag{5.7a}$$

$$I(x,\mu) = \int_x^\infty e^{-(t-x)/\mu}\ \frac{S(t)}{\mu}dt, \quad \mu>0. \tag{5.7b}$$

Now note that

$$\begin{aligned}\int_{-1}^{1}I(x,\eta)d\eta &= -\int_{-1}^{0}\int_0^x e^{-(t-x)/\eta}\,\frac{S(t)}{\eta}dtd\eta + \int_0^1\int_x^\infty e^{-(t-x)/\eta}\,\frac{S(t)}{\eta}dtd\eta \\ &= -\int_0^x\int_{-1}^{0} e^{-(t-x)/\eta}\,\frac{d\eta}{\eta}S(t)dt + \int_x^\infty\int_0^1 e^{-(t-x)/\eta}\,\frac{d\eta}{\eta}S(t)dt \\ &= \int_0^x\int_0^1 e^{(t-x)/\eta}\,\frac{d\eta}{\eta}S(t)dt + \int_x^\infty\int_0^1 e^{-(t-x)/\eta}\,\frac{d\eta}{\eta}S(t)dt \\ &= \int_0^\infty E_1(|x-t|)S(t)dt\ . \end{aligned} \tag{5.8}$$

Hence, by substituting (5.8) into (5.6), we obtain

$$S(x) - \int_0^\infty \kappa(x-t)S(t)dt = \varepsilon B(x) , \qquad x\in\mathbf{R}^+ , \tag{5.9}$$

an integral equation of the form (1.1), with

$$\kappa(x) = \left[\frac{1-\varepsilon}{2}\right]E_1(|x|) = \left[\frac{1-\varepsilon}{2}\right]\int_0^1 e^{-|x|/\mu}\frac{1}{\mu}d\mu . \tag{5.10}$$

Now recall that this is the same as the kernel (5.1) of the test example. Thus we have $|\kappa|^\wedge(0) = \hat{\kappa}(0) = (1-\varepsilon)$. Hence A1, is satisfied for all $\varepsilon\in(0,1)$, and so (5.9) has a unique solution $S\in C$.

Thus by reversing the process used to obtain (5.7), $I(x,\mu)$, given by (5.7), is well defined and gives a solution of the Transport Equation (5.5). We now show that it is the unique solution of (5.5). To this end, we define a class of functions, G, such that if $I(x,\mu)\in G$ then

(i) $I(0,\mu) = 0$ for $\mu<0$

(ii) $e^{-\alpha x}I(x,\mu)\to 0$ as $x\to\infty$, for all $\mu>0$ and $\alpha>0$

(iii) $\int_{-1}^1 I(x,\eta)d\eta \in C$.

Now note that $I(x,\mu)$, given by (5.7), is in this class, since (i) is satisfied trivially from (5.7a), (ii) is satisfied from (5.7b) (using the fact that $S\in C$), and (iii) is satisfied by observing that the right hand side of (5.8) is just $(2/(1-\varepsilon))KS(x)$ in (5.9), and then using the fact that K is a bounded operator from C to C, (see e.g. [4] and [6]).

Now suppose there exists two solutions, I_1, I_2, of (5.5), in this class, then $I:=I_1-I_2$ satisfies

$$\mu\frac{\partial I}{\partial x}(x,\mu) - I(x,\mu) = -\left[\frac{1-\varepsilon}{2}\right]\int_{-1}^1 I(x,\eta)d\eta \ (:=-h(x)), \qquad -1\le\mu\le 1, \ x\in\mathbf{R}^+$$

or

$$\frac{\partial}{\partial x}\left[e^{-x/\mu}I(x,\mu)\right] = -e^{-x/\mu}\frac{h(x)}{\mu}, \qquad -1\le\mu\le 1, \quad x\in\mathbf{R}^+. \tag{5.11}$$

Hence, by uniqueness of solution to ordinary differential equations [15], if we integrate (5.11) and build in the boundary conditions, which are given by (i) and (ii) above, we obtain the exact solution

$$I(x,\mu) = -\int_0^x e^{-(t-x)/\mu}\,\frac{h(t)}{\mu}\,dt\ ,\quad \mu<0 \tag{5.12a}$$

$$I(x,\mu) = \int_x^\infty e^{-(t-x)/\mu}\,\frac{h(t)}{\mu}\,dt\ ,\quad \mu>0. \tag{5.12b}$$

Then, substituting (5.12) into the defining equation for $h(x)$, and using the same techniques as in the derivation of (5.9), we obtain

$$h(x) - \int_0^\infty \kappa(x-t)h(t)dt = 0\ ,$$

where κ is given by (5.10). Since (5.9) has a unique solution, $S(x)$, it follows that $h(x)\equiv 0$ and hence $I\equiv 0$, which implies $I_1 = I_2$. Thus, $I(x,\mu)$ given by (5.7) is the unique solution of (5.5) in the class G.

5.2.2 IMPLEMENTATION

As a practical illustration of our numerical method we consider (5.9) with $\varepsilon=\frac{1}{2}$ and $B(x)=1$ (which is the case in certain isothermal atmospheres, see for example [8]). Then we have (1.1) with

$$\kappa(x) = \tfrac{1}{4}E_1(|x|), \tag{5.13}$$

and

$$f(x) = \tfrac{1}{2}\ ,$$

with $u(x)=S(x)$ the solution to be determined.

Now from the test example of §5.1, the kernel, κ, given by (5.13) satisfies the assumptions A1, A2 and A3 of Chapter 3.

Recall Remark 1 following the assumptions of Chapter 3 §3.1. Since $f(\infty)=\frac{1}{2}$, we have $S(\infty) = 1/(2(1-\hat{\kappa}(0)) = 1 \neq 0$ (since $\widehat{|\kappa|}(0) = (1-\varepsilon) = \frac{1}{2}$). Rearranging (5.9) gives

$$\tilde{S}(x) - \int_0^\infty \kappa(x-t)\tilde{S}(t)dt = F(x), \qquad x\in\mathbf{R}^+, \tag{5.14}$$

where

$$\tilde{S} = S - S(\infty) = S-1,$$

and

$$F(x) = f(x) - (1-K1(x))f(\infty)/(1-\hat{\kappa}(0)) = -\tfrac{1}{2} + K1(x).$$

Then we have

$$F(x) = -\int_{-\infty}^{\infty}\kappa(t)dt + \int_0^\infty \kappa(x-t)dt = -\int_x^\infty \kappa(t)dt \tag{5.15}$$

$$= -\tfrac{1}{4}\int_x^\infty E_1(t)dt \;(= \tfrac{1}{4}E_2(x)\,), \tag{5.16}$$

where E_2 is the second exponential integral (see Chapter 2, Example 5). We used NAG routine c05adf to show that the positive root (nearest to zero) of $1 - |\kappa|\hat{}(i\mu)$ is $\mu* = 0.957504$. Now by Theorem 3.6(i), $\tilde{S}\in C^l_\beta[e^{-\mu x}]$, for all $0<\beta<1$, $0<\mu<\mu*$, $l\in\mathbf{N}$.

We solve numerically (5.14), which in turn gives $S(x)=\tilde{S}(x)+1$ for all $x\in\mathbf{R}^+$. We use the *exponential mesh*, (2.14) and (2.15), with r=2, with various q_1 and q_2, and again we take $i*=1$. We use $a(x)$, $b(x)$, and $c(x)$ as in (5.3). We take δ=1, and for x in this region we compute $c(|x|)$ by the approximation (5.4). To compute $F(x)$ at the quadrature points x, we use the identity

$$F(x) = \tfrac{1}{4}E_2(x) = \tfrac{1}{4}\left[e^{-x} - xE_1(x)\right],$$

(see, [1, (5.1.26)]). Then for x=0, $F(x)=\frac{1}{4}$. For x>0, NAG routine S13aaf was used to compute $E_1(x)$, and hence $F(x)$ is known for all $x\in\mathbf{R}^+$. Alternatively, $F(x)$ may be computed by the method of Example 5 §2.4.1.

In our experiment, we solved (5.14) using the Trapezoidal rule and Simpson's rule, and then computed the solution $S_n(x) = \tilde{S}_n(x)+1$ to (5.9). For this particular problem it is known that $S(0) = \sqrt{\varepsilon} = 1/\sqrt{2}$ (see [17]). So as a measure of the error in our method we computed $e_N(0) = |S(0) - S_n(0)|$, where n+1 is the number of mesh points arising from discretisation parameter N (see (2.14) and

(2.15)). An approximation to the order of convergence at the origin is then given by $EOC(0) = \log(e_N(0)/e_{2N}(0))/\log 2$. Table 5.5 of §5.2.3 gives results for the Trapezoidal rule with $q_1=2=q_2$. Table 5.6 of §5.2.3 is for Simpson's rule with $q_1=4=q_2$. The results indicate that for the Trapezoidal rule we have

$$|S(0)-S_n(0)| \approx O(1/n^2) ,$$

and for Simpson's rule we have

$$|S(0)-S_n(0)| \approx O(1/n^4).$$

Of course, these results do not give any information about $\|S-S_n\|_\infty$, but they serve to indicate that the method is working as predicted by Theorem 4.7, at least at the origin.

Table 5.7 of §5.2.3 shows the values of $S_n(x)$ for various n and x, obtained by Simpson's rule as described above. The results indicate that refining the mesh yields excellent convergence throughout $\mathbf{R}^+$.

Figure 4 is a graph of $S_{21}(x)$ (obtained by using Simpson's rule as described above). Note how quickly the graph approaches 1. This is characteristic of the source function for coherent scattering.

5.2.3 NUMERICAL RESULTS

N	n	$e_N(0)$	$EOC(0)$
2	5	0.77×10^{-2}	
			0.94
4	9	0.40×10^{-2}	
			1.74
8	18	0.12×10^{-2}	
			1.78
16	36	0.35×10^{-3}	
			1.85
32	71	0.97×10^{-4}	
			1.90
64	142	0.26×10^{-4}	
			1.96
128	284	0.67×10^{-5}	

TABLE 5.5

N	n	$e_N(0)$	$EOC(0)$
2	6	0.14×10^{-2}	
			3.54
4	11	0.12×10^{-3}	
			3.77
8	21	0.88×10^{-5}	
			3.80
16	41	0.63×10^{-6}	
			3.98
32	82	0.40×10^{-7}	

TABLE 5.6

x	n=11	n=21	n=41	n=82
0.06	0.74809396	0.74822739	0.74823693	0.74823760
1.00	0.92739087	0.92742199	0.92742253	0.92742256
2.77	0.98910611	0.98912473	0.98912658	0.98912672
5.55	0.99932136	0.99932667	0.99932725	0.99932732
8.31	0.99995535	0.99995552	0.99995554	0.99995557

TABLE 5.7

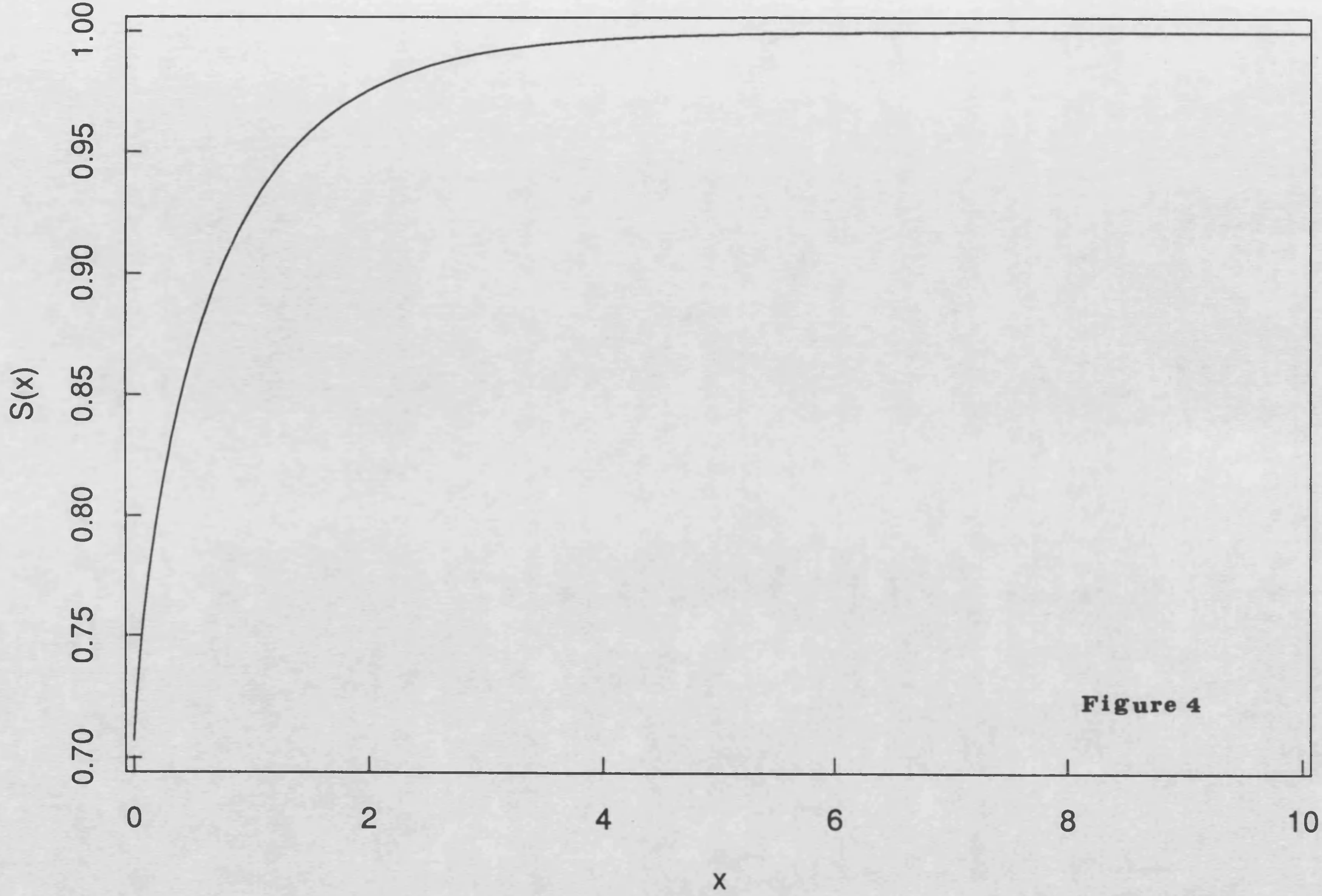

Figure 4

5.3 THE TRANSPORT EQUATION FOR NON-COHERENT SCATTERING

5.3.1 REFORMULATION

Recall the linear Transport Equation for non-coherent scattering introduced in Chapter 1, Example 2,

$$\mu\frac{\partial}{\partial x}I(x,\mu,s) - \varphi(s)I(x,\mu,s) = -\varphi(s)S(x)\ , \qquad -1\le\mu\le 1,\ x\in\mathbf{R}^+, \tag{5.17}$$

where s is a suitable measure of the frequency, the source function $S(x)$ is given by

$$S(x) = \left[\frac{1-\varepsilon}{2}\right]\int_{-\infty}^{\infty}\varphi(s')\int_{-1}^{1}I(x,\eta,s')d\eta ds' + \varepsilon B(x)\ , \tag{5.18}$$

and where φ is some positive profile (for examples see (1.7)) with

$$\int_{-\infty}^{\infty}\varphi = 1\ .$$

Again, $\varepsilon\in(0,1)$ and $B(x)$ is the Planck function. The boundary conditions are

$$I(0,\mu,s) = 0 \text{ for } \mu<0 \text{ and for all } s\ ,$$

$$\tag{5.19}$$

$$e^{-\alpha x}I(x,\mu,s) \to 0 \text{ as } x\to\infty, \text{ for all } \mu>0,\ \alpha>0 \text{ and for all } s.$$

We now reformulate (5.17) as an integral equation of the form (1.1) for the source function $S(x)$. If we formally integrate (5.17) and build in the boundary conditions we obtain

$$I(x,\mu,s) = -\frac{\varphi(s)}{\mu}\int_0^x \exp(-\varphi(s)(t-x)/\mu)\ S(t)dt, \quad \mu<0 \tag{5.20a}$$

$$I(x,\mu,s) = \frac{\varphi(s)}{\mu}\int_x^\infty \exp(-\varphi(s)(t-x)/\mu)\ S(t)dt, \quad \mu>0\ . \tag{5.20b}$$

Now note that

$$\int_{-1}^{1}I(x,\eta,s')d\eta = -\int_{-1}^{0}\frac{\varphi(s')}{\eta}\int_0^x \exp(-\varphi(s')(t-x)/\eta)\ S(t)dtd\eta$$

$$+ \int_0^1\frac{\varphi(s')}{\eta}\int_x^\infty \exp(-\varphi(s')(t-x)/\eta)\ S(t)dtd\eta$$

$$= -\int_0^x \varphi(s')\int_{-1}^0 \exp(-\varphi(s')(t-x)/\eta)\frac{d\eta}{\eta}S(t)dt$$

$$+ \int_x^\infty \varphi(s')\int_0^1 \exp(-\varphi(s')(t-x)/\eta)\frac{d\eta}{\eta}S(t)dt$$

$$= \int_0^\infty \left[\int_0^1 \exp(-\varphi(s')|x-t|/\eta)\frac{d\eta}{\eta}\right]\varphi(s')S(t)dt$$

$$= \int_0^\infty E_1(|x-t|\varphi(s'))\varphi(s')S(t)dt \ . \tag{5.21}$$

Hence by substituting (5.21) into (5.18) and changing the order of integration we obtain

$$S(x) - \int_0^\infty \kappa(x-t)S(t) = \varepsilon B(x) \ , \qquad x\in\mathbf{R}^+, \tag{5.22}$$

an integral equation of the form (1.1) for $u(x)=S(x)$, where

$$\kappa(x) = \left[\frac{1-\varepsilon}{2}\right]\int_{-\infty}^\infty \varphi^2(s)E_1(|x|\varphi(s))ds \tag{5.23}$$

or

$$\kappa(x) = (1-\varepsilon)K_1(|x|),$$

(see (1.6)).

For numerical illustration we will be interested in the case when Doppler broadening is dominant. In that case we have

$$\varphi(s) = \frac{1}{\sqrt{\pi}}\exp(-s^2). \tag{5.24}$$

Note that the integrand in (5.23) decays exponentially at ∞.

Now below we show that when Doppler broadening is dominant, the kernel given by (5.23) satisfies A1, and so (5.22) has a unique solution $S\in C$. Then using the same arguments as in §5.2.1, I, given by (5.20) is the unique solution of (5.17) within the class of functions satisfying the boundary conditions (5.19), together with

$$\int_{-1}^1 I(x,\eta,s)d\eta \in C \ .$$

5.3.2 IMPLEMENTATION

As a practical illustration of our method we consider (5.22) with κ given by (5.23) and φ given by (5.24). Again we take $\varepsilon = \frac{1}{2}$ and $B(x) = 1$. Then we have (1.1) with

$$\kappa(x) = \frac{1}{2} K_1(|x|)$$

and

$$f(x) = \tfrac{1}{2} ,$$

with $u(x) = S(x)$ the solution to be determined.

We first check the assumptions A1, A2′ and A3, of Chapter 3. Now it can be shown that

$$|\kappa|\hat{}(\omega) = \hat{\kappa}(\omega) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \int_0^1 \frac{\exp(-s^2)}{1+\exp(2s^2)\pi\omega^2\mu^2} d\mu ds .$$

Thus we have $|\kappa|\hat{}(0) = \hat{\kappa}(0) = \frac{1}{2}$, and so A1 follows. Now from [8], A2′ is satisfied at least for $l=0,1$, with $1<\rho'<2$. Also, by [8] we have

$$K_1(x) = \frac{1}{2\sqrt{2\pi}} \left[\frac{1}{4} - \gamma - \log\left[\frac{x}{\sqrt{\pi}}\right] - \sqrt{2}\sum_{k=1}^{\infty} \frac{(-x/\sqrt{\pi})^k}{kk!\sqrt{k+2}} \right] , \qquad x\geq 0 , \quad (5.25)$$

where γ is Euler's constant. Then by choosing

$$a(x) = -\frac{1}{4\sqrt{2\pi}},\ b(x) = -2\sqrt{2\pi}K_1(|x|),\ c(x) = -2\sqrt{2\pi}(K_1(x)) - \log x , \qquad (5.26)$$

A3 follows.

Now since $f(\infty) = \frac{1}{2}$, it follows that $S(\infty) = 1/(2(1-\hat{\kappa}(0)) = 1 \neq 0$. Rearranging (5.22) gives

$$\tilde{S}(x) - \int_0^{\infty} \kappa(x-t)\tilde{S}(t)dt = F(x), \qquad x \in \mathbf{R}^+, \qquad (5.27)$$

where again

$$\tilde{S} = S - S(\infty) = S - 1,$$

and

$$F(x) = -\int_x^\infty \kappa(t)dt \ ,$$

(see (5.15)). We solve numerically (5.27) which in turn gives $S(x)=\tilde{S}(x)+1$ for all $x \in \mathbf{R}^+$.

Now we may compute μ' in Lemma 3.1(ii) numerically in order to determine information about $\tilde{S}(x)$ as $x \to \infty$. We have not done this but asymptotic analysis shows (see for example [8]) that with $\varepsilon = \frac{1}{2}$ and $B(x)=1$ we have

$$\tilde{S}(x) \sim \frac{1}{\left[1 + (\ 4x\sqrt{\log(x/\sqrt{\pi})}\ )^{-1}\right]} - 1, \ \text{as } x \to \infty \ .$$

Thus, in implementing our numerical method we assume that $\tilde{S} \in C^l_\beta[x^{-(1+\eta)}]$ for all $0<\beta<1$, $l \in \mathrm{N}$ and some $\eta>0$, and we grade the mesh accordingly.

We use the *polynomial mesh*, (2.33) and (2.34), with $r=2$ and with various p_1 and p_2. We use $a(x)$, $b(x)$ and $c(x)$, as in (5.26). Again we take $\delta=1$ and for x in this region we compute $c(|x|)$ by using the approximation

$$c(|x|) \approx \left[-\frac{1}{4} + \gamma - \log\sqrt{\pi} + \sqrt{2}\sum_{k=1}^{l} \frac{(-|x|/\sqrt{\pi})^k}{kk!\sqrt{k+2}}\right] , \tag{5.28}$$

for suitable l (see below), with a maximum error of

$$\frac{\sqrt{2}(|x|/\sqrt{\pi})^{l+1}}{(l+1)(l+1)!\sqrt{l+3}} \ . \tag{5.29}$$

To compute $F(x)$ for quadrature points x, we use an approximation which is defined using our underlying quadrature rule. If $0 \le x \le 1$, we use

$$F(x) \approx -\int_x^1 \kappa(t)dt - \int_1^{x_{n-i*}} P_n \kappa(t)dt, \tag{5.30}$$

where P_n is the interpolatory projection introduced in §1.4.1. Writing $\kappa(t) = a(t)(\log|t| + c(|t|))$, and using the approximation (5.28) for $c(|t|)$ (with l given below), the first integral in (5.30) is calculated analytically. If $x_{n-i*} \ge x_i \ge x \ge 1$, we use

$$F(x) \approx -\int_x^{x_i} \tilde{P}^i_n \kappa(t)dt - \int_{x_i}^{x_{n-i*}} P_n \kappa(t)dt,$$

where $\tilde{P}^i_n\kappa$ is the polynomial of order m which interpolates κ at the points $x+\xi_j(x_i-x)$, for $j=1,...,m$. (For more computational details see Chapter 2, Example 8).

In order to evaluate $K^{[2]}_{i*,n}\tilde{S}(x)$ for $x\in\mathbf{R}^+$, a function routine DOPKER was written which approximates the integral defining $K_1(t)$ for $t>0$, using a high order Simpson's rule, on the mesh (2.6), with n=512 and q=4, (q.v. Example 5, § 2.4.1). The results were compared to 5-figure tables of $K_1(t)$ given in [8], (see Table 5.8 of §5.3.3). Excellent agreement was obtained uniformly over a range of $t\in\mathbf{R}^+$.

The next step was to determine a "suitable" value of l for the approximation (5.28) above. An experiment was carried out in which $K_1(|t|)$ $(=2\kappa(t))$ was approximated using

$$2\kappa(t) = 2a(t)(\log(|t| + c(|t|)) \ , \tag{5.31}$$

where a, b, c are given in (5.26), and $c(|t|)$ is approximated using (5.28) with l=5, 10, 15. Then for various values of $t>0$, these approximations were compared with $K_1(t)$ evaluated by DOPKER and the modulus of the error is given in Table 5.9 of §5.3.3. It was found that there was no advantage in taking $l>10$, and so in our numerical method we compute an approximation to $c(|t|)$ using (5.28) with $l=10$. Note that as t increases the error generally gets worse (see (5.29)).

Now consider the *polynomial mesh*, (2.33) and (2.34), with r=2, $p_1=2=p_2$ and $i*=1$. Then even for small values of N, the mesh diameter will be large and this will cause considerable approximation problems. To see this, for N=8 the mesh on $[0,\infty)$ is

$$x_i = \left[\frac{i}{8}\right]^2, \quad i=0,...,8$$

$$(5.32)$$

$$x_{8+j} = \left[\frac{16}{16-j}\right]^2, \quad j=1,...,15,$$

with $n=(r+1)N=3N=24$. Now recall $h_i=x_i-x_{i-1}$, $I_i=(x_{i-1},x_i)$, $i=1,\ldots,n$, and so from (5.32) we have

$$h_{23}=192, \quad h_{22}=35.56, \quad h_{21}=12.44, \quad h_{20}=5.76 \ .$$

Now from the definition of $\Omega^{[1]}(x)$ (see § 1.4.2), it is clear that $\Omega^{[1]}(x_{23})=[x_{22},x_{23}]$, and so evaluating $K^{[1]}_{i*,n}\tilde{S}_n(x_{23})$, say, will involve using the approximation (5.28) over a region of length ≥ 192. The error incurred in such an approximation could be extremely large (see (5.29)) and Table 5.9 of § 5.3.3.

In Table 5.9, we see that the error in using the approximation (5.28) at $x=4$ with $l=10$, is 0.12×10^{-5}. In order to overcome the approximation problem detailed above, an experiment was carried out which solves (5.27) using the Trapezoidal rule on a mesh consisting of the points defined in (5.32), and additional points which were added in each interval for which $h_i>4$, in order to make the mesh diameter no greater than 4. Thus, 47 points were added in I_{23}, 8 points were added in I_{22}, 3 points were added in I_{21} and 1 point was added in I_{20}. Thus the total number of mesh points used was 83. We call this approximation to $\tilde{S}$, $\tilde{S}^{(83)}_{24}$. We computed $S^{(83)}_{24} := \tilde{S}^{(83)}_{24} + 1$, to obtain an approximation to S.

Now again for this problem it is known that $S(0)=\sqrt{\varepsilon}=1/\sqrt{2}$, (see [17]). We obtained

$$S^{(83)}_{24}(0) = 0.70355658 \ ,$$

and so we have

$$|S(0) - S^{(83)}_{24}(0)| = 0.0036 \ .$$

Thus, as can be seen by the above discussion, the problem of solving the integral equation reformulation of Transport Equation for non-coherent scattering is much more difficult than the corresponding problem for coherent scattering. However, we have proposed a method which yields satisfactory results.

As a second experiment, (5.27) was solved using the Trapezoidal rule and Simpson's rule. We used the *polynomial mesh,* (2.33) and (2.34), with fixed N

(see below), $r=2$, and with $i*=3$, for various p_1 and p_2, together with additional points as described below. This choice of $i*$ will ensure that the mesh diameter is no greater than 4 for the particular value of N that we choose. Thus the mesh points used consisted of the points

$$x_i = \left[\frac{i}{N}\right]^{p_1}, \quad i=0,\ldots,N ,$$

$$x_{N+i} = \left[\frac{2N}{2N-i}\right]^{p_2}, \quad i=1,\ldots,2N-3,$$

together with the points

$$x_{3N-3+i} = x_{3N-3+i-1} + 4, \quad i=1,\ldots,k ,$$

for various k. Now let

$$S_N^{(s)}(0) = \tilde{S}_N^{(s)}(0) + 1$$

denote the approximation to $S(0)$, where s is the total number of mesh points used.

Table 5.10 of §5.3.3 gives the results for the Trapezoidal rule, with $N=4$, $p_1=2=p_2$. Table 5.11 of § 5.3.3 gives the results for Simpson's rule, with $N=2$, $p_1=4=p_2$.

The results indicate that our numerical method may be used to solve the integral equation for non-coherent scattering. However, the results are not as satisfactory as in the case for coherent scattering due to the unreasonably large size of the linear system needed to give results even for small values of the parameter N. Taking progressively larger values of k will enable us to determine an estimate of the rate of convergence at the origin. However, we present these results only for illustrative purposes.

Table 5.12 of §5.3.3 shows $S_2^{(s)}(x)$ for various x, obtained by using Simpson's rule with s=19, 34, 64. The results indicate that adding progressively more mesh points yields good convergence throughout $\mathbf{R}^+$.

Figure 5 is a graph of $S_2^{(34)}(x)$ (obtained by using Simpson's rule as described above). Note that $S_2^{(34)}(x)$ approaches 1 much more slowly than in the exponential case (see Figure 4).

5.3.3 NUMERICAL RESULTS

$\log_{10}t$	$K_1(t)*$	$K_1(t)$
−4	1.8861	–
−3	1.4269	–
−2	9.6842×10^{-1}	–
−1	5.1728×10^{-1}	–
0	1.3071×10^{-1}	–
1	2.1316×10^{-3}	–
2	1.2570×10^{-5}	–
3	9.9495×10^{-8}	–
4	8.5044×10^{-10}	8.5043×10^{-10}
5	7.5518×10^{-12}	–
6	6.8625×10^{-14}	6.8624×10^{-14}
7	6.3335×10^{-16}	6.3334×10^{-16}
8	5.9108×10^{-18}	–
9	5.5631×10^{-20}	5.5630×10^{-20}
10	5.2703×10^{-22}	–
11	5.0194×10^{-24}	–
12	4.8014×10^{-25}	4.8013×10^{-25}

TABLE 5.8

Note $K_1(t)*$ denotes the values of $K_1(t)$ obtained using Simpson's rule.

"-" denotes agreement with the corresponding figure in column 2.

t	l=5	l=10	l=15
0.5	0.11×10^{-7}	0.12×10^{-9}	0.12×10^{-9}
1	0.70×10^{-6}	0.89×10^{-10}	0.89×10^{-10}
2	0.42×10^{-4}	0.68×10^{-9}	0.38×10^{-11}
3	0.45×10^{-3}	0.52×10^{-7}	0.18×10^{-7}
4	0.24×10^{-2}	0.12×10^{-5}	0.10×10^{-5}
5	0.87×10^{-2}	0.13×10^{-4}	0.25×10^{-4}
6	0.25×10^{-1}	0.95×10^{-4}	0.35×10^{-3}
7	0.60×10^{-1}	0.50×10^{-3}	0.32×10^{-2}
8	0.13×10^{0}	0.21×10^{-2}	0.22×10^{-1}
9	0.25×10^{0}	0.75×10^{-2}	0.12×10^{0}
10	0.44×10^{0}	0.23×10^{-1}	0.60×10^{0}

TABLE 5.9

k	s	$\|S(0) - S_4^{(s)}(0)\|$
15	25	0.132×10^{-1}
30	40	0.782×10^{-2}
60	70	0.470×10^{-2}

TABLE 5.10

k	s	$\|S(0) - S_2^{(s)}(0)\|$
15	19	0.942×10^{-2}
30	34	0.473×10^{-2}
60	64	0.211×10^{-2}

TABLE 5.11

x	$S_2^{(19)}(x)$	$S_2^{(34)}(x)$	$S_2^{(64)}(x)$
1.00	0.83542651	0.84156143	0.84356154
5.16	0.95641860	0.96002345	0.96124545
11.16	0.98211213	0.98676942	0.98763152
21.16	0.99135412	0.99556173	0.99611297
31.16	0.99456215	0.99817341	0.99849497
41.16	0.99873051	0.99939409	0.99960532

TABLE 5.12

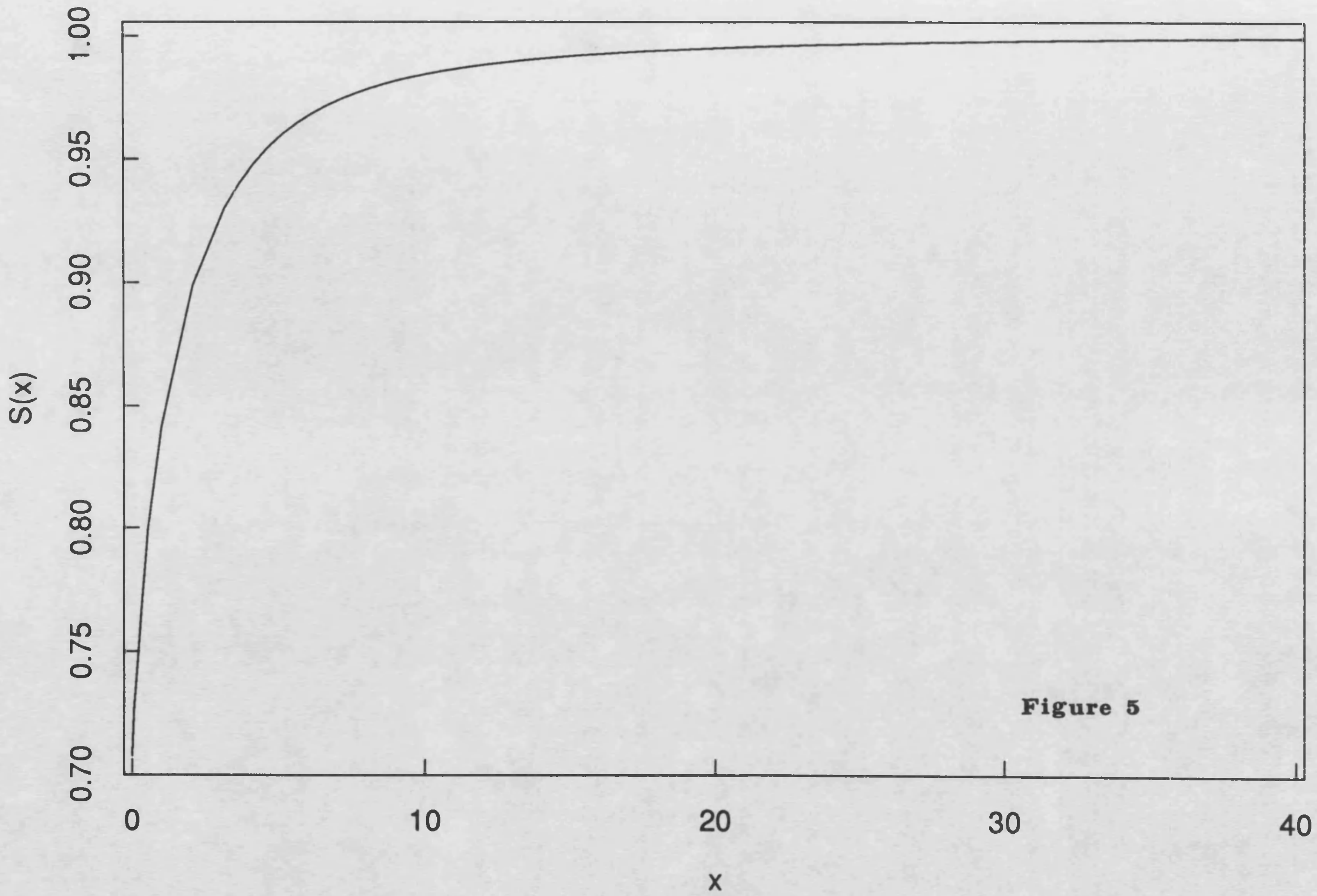

Figure 5

5.4 ELECTROMAGNETIC WAVE REFRACTION

Recall that an integral equation of the form (1.1) arises in electromagnetic wave refraction (see Chapter 1, Example 3) with

$$\kappa(x) = \frac{\lambda}{\pi}\int_0^\infty \frac{\cos(v|x|)}{(1+v^2)^{\frac{1}{2}}}dv \;, \qquad x>0,\;\; \lambda\notin(1,\infty) \;, \tag{5.33}$$

and

$$f(x) = e^{-x} \;. \tag{5.34}$$

We now provide some analysis for (1.1), with κ given by (5.33) and f given by (5.34).

We first check the assumptions A1, A2 and A3. We have

$$|\kappa\hat{}(\omega) = \frac{|\lambda|}{\pi}\left[\frac{\pi}{2}\right]^{\frac{1}{2}}\frac{1}{(1+\omega^2)^{\frac{1}{2}}} \;,$$

(see [26]). Thus it follows that

$$|\kappa\hat{}(0) = \frac{|\lambda|}{\pi}\left[\frac{\pi}{2}\right]^{\frac{1}{2}} \;,$$

and so A1 is satisfied for all λ with $|\lambda|<\sqrt{2\pi}$. Now in keeping with the notation in [1], we write

$$\kappa(x)=\frac{\lambda}{\pi}K_0(|x|) \;,$$

where K_0 is a modified Bessel function of integer order (see [1, (9.6.21)]). Then [1, (9.6.28), (9.7.2), (9.7.4)] show that A2 is satisfied with $0<\rho<1$. Also, note that

$$K_0(x) = -\{\log(\tfrac{1}{2}x)+\gamma\}I_0(x) + \frac{\frac{1}{4}x^2}{(1!)^2} + (1+\tfrac{1}{2})\frac{(\frac{1}{4}x^2)^2}{(2!)^2} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{4}x^2)^3}{(3!)^2} + \ldots, \tag{5.35}$$

where

$$I_0(x) = 1 + \frac{\frac{1}{4}x^2}{(1!)^2} + \frac{(\frac{1}{4}x^2)^2}{(2!)^2} + \frac{(\frac{1}{4}x^2)^3}{(3!)^2} + \ldots \;. \tag{5.36}$$

Now A3 does not follow from (5.35) and (5.36), since I_0 is an infinite series, and we provide no detailed analysis for this particular case. However the the regularity results of Chapter 3 and the convergence results of Chapter 4 hold for this example, with A3 suitably modified, (see Remark 3 following the assumptions of Chapter 3).

By Theorem 3.5(i), the solution u of (1.1) in this example is in $C^l_\beta[e^{-\mu x}]$ for all $l \in N$, $0<\beta<1$ and some $\mu>0$. Thus in implementing the Nystrom-Product Integration for this particular example one would use the *exponential mesh*, (2.14) and (2.15). Polynomial approximations to K_0 and I_0 are given in [1, (9.8.1), (9.8.4)], and it is these approximations that would be used in practice.

REFERENCES

1. ABRAMOVITZ, M., & STEGUN, I.A. 1965 *Handbook of Mathematical Functions.* Dover, New York.

2. APOSTOL, T.M. 1979 *Mathematical Analysis.* Addison-Wesley, Philippines.

3. ANSELONE, P.M. 1971 *Collectively Compact Operator and Approximation Theory.* Prentice Hall, Englewood Cliffs, New Jersey.

4. ANSELONE, P.M., & SLOAN, I.H. 1985 Integral equations on the half-line. *J. Integral Equations* [*suppl.*] **9**, 3-23.

5. ANSELONE, P.M., & SLOAN, I.H. Numerical solutions of integral equations on the half line. (ii) The Wiener-Hopf case. *Numer. Math.* (submitted for publication).

6. ATKINSON, K.E. 1969 The numerical solution of integral equations on the half line. *SIAM J. Numer. Analysis* **6**, 375-397.

7. ATKINSON, K.E. 1972 The numerical solution of Fredholm integral equations of the second kind with singular kernels. *Numer. Math.* **19**, 248-259.

8. AVERETT, E.H., & HUMMER, D.G. 1965 Non-coherent scattering. *Mon. Not. R. astr. Soc.* (4) **130**, 295-331.

9. BURDEN, R.L., & FAIRES, J.D. 1985 *Numerical Analysis.* Prindle, Weber & Schmidt, Boston.

10. CHANDLER, G.A. 1979 Superconvergence of numerical solutions to second kind integral equations. Ph.D. thesis, Australian National University, Canberra.

11. CHANDLER, G.A., & GRAHAM, I.G. 1988 Product integration-collocation methods for non-compact integral operator equations. *Math. Comp.* **50**, 125-138.

12. CHANDLER, G.A., & GRAHAM, I.G. 1988 The Convergence of Nystrom methods for Wiener-Hopf equations. *Numer. Math.* **52**, 345-364.

13. CHANDLER-WILDE, S.N., & HOTHERSALL, D.C. 1986. Integral equations in traffic noise simulation. Research Report No. 13. School of Civil and Structural Engineering, University of Bradford, U.K.

14. CHANDRASEKHAR, S. 1950 *Radiative Transfer.* Oxford University Press, London.

15. DERRICK, W.R., & GROSSMAN, S.I. 1978 *Elementary Differential Equations with Applications* . Addison-Wesley, Philippines.

16. FINN, G.D., & JEFFERIES, J.T. 1968 Studies in spectral line formation. I. Formulation and simple applications. *J. quant. Spectrosc. Radiat. Transfer* **8**, 1675-1703.

17. FRISCH, U., & FRISCH, H. 1975 Non-LTE transfer. $\sqrt{\varepsilon}$ revisited. *Mon. Not. R. Astr. Soc.* **173**, 167-182.

18. GAHLER, S., & GAHLER, W. 1987 Quadrature methods for the solution of integral equations of the second kind on the half line. *Math. Nach.* (to appear).

19. GOHBERG, I.C., & FELDMAN, I.A. 1974 *Convolution equations and projection methods for their solution.* American Mathematical Society, Translations of Mathematical Monographs No.41, Providence, Rhode Island.

20. GRAHAM, I.G. 1982 Singularity expansions for the solution of second kind Fredholm integral equations with weakly singular convolution kernels. *J. of Integral Equations* **4**, 1-30.

21. GRAHAM, I.G. 1982 Estimates for modulus of smoothness. *J. of Approx. Theory* **44**, 95-112.

22. GRAHAM, I.G., & MENDES, W.R. Nystrom-Product Integration for Wiener-Hopf equations with applications to radiative transfer. (Accepted for publication 1988) *IMA J. of Numer. Anal.*

23. GRINBERG, G.A. & FOK, V.A. 1948 On the theory of cliff-top refraction of electromagnetic waves. In: *Survey of Radio Wave Propagation. Collected papers, vol II.* (B.A.Vvedenskiy, Ed.) Izdat. Akad. Nauk SSR, Moscow-Leningrad.

24. HOPF, E. 1934 *Mathematical Problems of Radiative Equilibrium* . Cambridge University Press.

25. HUMMER, D.G., & RYBICKI, G.B. 1967 Computational methods for non-LTE line transfer problems. In: *Methods in Computational Physics* (53) **7**. Academic Press, New York.

26. IVANOV, V.V. 1976 *The Theory of Approximate Methods*, Noordhoff International Publishing, Leyden.

27. JOHNSON, C., & PITKARANTA, J. 1983 Convergence of a fully discrete scheme for two-dimensional neutron transport. *SIAM J. numer. Anal.* **20**, 951-966.

28. KAGIWADA, H.H., & KALABA, R. 1974 *Integral equations via imbedding methods*. Addison-Wesley.

29. KREIN, M.G. 1963 Integral equations on a half-line with kernel depending upon the difference of the arguments. *Amer. Math. Soc. Transl.* (2) **22**, 163-288.

30. MONEGATO, G., & COLUMBO, V. 1988 Product Integration for the linear transport equation in slab geometry. *Numerische Mathematik* **52**, 219-240.

31. PITKARANTA, J. 1985 Numerical approximation of the discrete-ordinate transport equation in slab geometry: error bounds for projection schemes. Report. Institue of Mathematics, Helsinki University of technology, Finland.

32. PITKARANTA, J., & SCOTT, L.R. 1983 Error estimates for the combined spatial and angular approximations of the transport equation for slab geometry. *SIAM J. numer. Anal.* **20**, 922-950.

33. RICE, J.R. 1969 On the degree of convergence of nonlinear spline approximation. In: *Aproximations with Special Emphasis on Spline functions.* (I.J.Schoenburg, Ed.). Academic Press, New York.

34. RICHARDS, F. 1972 On the saturation class for spline functions. *Proceedings of the American Mathematical Society*, **33**, 471-476.

35. RICHTER, G.R 1976 On weakly singular Fredholm integral equations with displacement kernels. *J. Math. Anal. Appl.* **55**, 32-42.

36. SCHNEIDER, C. 1979 Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind. *Integral Equations and Operator Theory* **2**, 62-68.

37. SCHNEIDER, C. 1981 Product integration for weakly singular integral equations. *Math. Comp.* **36**, 207-213.

38. SILBERMANN, B. 1986 Numerical analysis for Wiener-Hopf integral equations in spaces of measurable functions. In: *Seminar Analysis* (S. Prössdorf and B. Silbermann, Eds.). Akademie der Wissenschaften der DDR, Berlin.

39. SLOAN, I.H. 1981 Quadrature methods for integral equations of the second kind over infinite intervals. *Math Comput.* **36**, 511-523.

40. SLOAN, I.H., & SPENCE, A. 1986 Projection methods for integral equations on the half-line. *IMA J. Numer. Analysis* **6**, 153-172.

41. THOMAS, K.S. 1973 The numerical solution of integral equations. Ph.D. thesis, Oxford University, Oxford.

42. ZABREYKO, P.P. et al. 1975 *Integral Equations - a Reference Text.* Noordhoff International Publishing, Leydon.

APPENDIX A

Notation

In the subsequent analysis we will need the following important notational devices.

1. If J is an interval of $\mathbf{R}^+$ and $l \in \mathbf{N}_0$, then $C^l(J)$ will denote the space of functions on $\mathbf{R}^+$ which have l continuous derivatives on J. Also, $C^\infty(J)$ will denote the space of functions on $\mathbf{R}^+$ which are infinitely continuously differentiable on J.
2. For any $l \in \mathbf{N}_0$ and $\delta > 0$, $\varphi_{l,\delta}$ will denote a generic $C^l[0,\delta]$ function. That is, in each case $\varphi_{l,\delta} \in C^l[0,\delta]$ but $\varphi_{l,\delta}$ may be allowed to vary from instance to instance.
3. $\{f_1 + f_2 + \ldots + f_n\}$ will denote a linear combination of the functions $f_1, f_2, \ldots,$ and f_n. The coefficients of the linear combination will be unknown, but their values will be immaterial to the argument.
4. D will denote differentiation.

The following lemma is required in the proof of Lemma 3.3 of Chapter 3. Recall the assumptions A1, A2, A2′ and A3, of Chapter 3.

Lemma A1

Part A

Suppose for all $l,k \in \mathbf{N}_0$ and $\delta > 0$, $\psi : \mathbf{R}^+ \to \mathbf{C}$ satisfies

Y1. $\psi \in C^\infty(0,\infty)$

Y2. $\psi(x) = \{[(\log x)^p + (\log x)^{p-1} + \ldots + \log x][1 + x + \ldots + x^l]\} + \varphi_{l,\delta}(x),\ x \in [0,\delta]$, *for some $p \in \mathbf{N}$*

Y3. *For all $\mu \in [0,\rho)$, $|\ e^{\mu x} D^k \psi(x)\ | \to 0$ as $x \to \infty$ (where ρ is given in assumption* A2 *of Chapter 3).*

Then, if κ satisfies A1, A2, A3, *we have for all $l,k \in \mathbf{N}_0$ and $\delta > 0$,*

(i) $DK\psi \in C^\infty(0,\infty)$

(ii) $DK\psi(x) = \{[(\log x)^{p+1} + (\log x)^p + \ldots + \log x][1 + x + \ldots + x^l]\} + \varphi_{l,\delta}(x),\ x \in [0,\delta]$

(iii) $|\ e^{\mu x} D^k K\psi(x)\ | \to 0$ *as $x \to \infty$ for all $\mu \in [0,\rho)$.*

Part B

Suppose for all $l,k\in N_0$ *and* $\delta>0$, $\psi:R^+\to C$ *satisfies* Y1 *and* Y2 *together with*

Y3′. $|\ x^{\mu'}D^k\psi(x)\ | \to 0$ *as* $x\to\infty$ *for all* $\mu'\in[0,\rho')$ *(where* ρ' *is given in assumption* A2′ *of Chapter 3).*

Then, if κ *satisfies* A1, A2′ *and* A3, *we have for all* $l,k\in N_0$ *and* $\delta>0$, (i) *and* (ii) *above, together with*

(iv) $|\ x^{\mu'}D^kK\psi(x)\ | \to 0$ *as* $x\to\infty$ *for all* $\mu'\in[0,\rho')$.

Proof

Let $l,k\in N_0$ and let $\delta>0$. Before we give the proofs of (i), (ii), (iii) and (iv), we need the following technical preliminaries. So let ψ satisfy Y1, Y2, Y3 (respectively Y3′), and let κ satisfy A1, A2 (respectively A2′) and A3, unless otherwise stated. Then we write

$$K\psi(x) = \int_0^\delta \kappa(x-t)\ \psi(t)dt + \int_\delta^\infty \kappa(x-t)\ \psi(t)dt$$

$$:= J^{[1]}\psi(x) + J^{[2]}\psi(x)\ . \tag{a.1}$$

Observe that for all $x>\delta$ we have

$$D^kJ^{[1]}\psi(x) = \int_0^\delta \left[D^k\kappa\right](x-t)\ \psi(t)dt\ , \tag{a.2}$$

and so by A2 or A2′, and Y2, $D^kJ^{[1]}\psi(x)$ exists for all $x\in(\delta,\infty)$.

Also, we have

$$DJ^{[2]}\psi(x) = \int_\delta^\infty D_x\left[\kappa(x-t)\right]\ \psi(t)dt = -\int_\delta^\infty D_t\left[\kappa(x-t)\right]\ \psi(t)dt\ ,$$

where D_x and D_t denote differentiation with respect to x and t respectively. Then integrating by parts we obtain

$$DJ^{[2]}\psi(x) = -[\kappa(x-t)\ \psi(t)]_\delta^\infty + \int_\delta^\infty \kappa(x-t)\ D\psi(t)dt$$

$$= \kappa(x-\delta)\ \psi(\delta) + \int_\delta^\infty \kappa(x-t)\ D\psi(t)dt\ .$$

This may be generalised (using a proof by induction) to show that for all $k\in N_0$

$$D^kJ^{[2]}\psi(x) = \sum_{m=0}^{k-1} D^m\kappa(x-\delta)\ (D^{k-1-m}\psi)(\delta) + \int_\delta^\infty \kappa(x-t)\ D^k\psi(t)dt\ .$$

Now by Y1

$$(D^m\psi)(\delta) < \infty, \quad m=0,\dots,k-1,$$

and hence we can write

$$D^k J^{[2]}\psi(x) = \left\{\sum_{m=0}^{k-1} D^m \kappa(x-\delta)\right\} + \int_\delta^\infty \kappa(x-t) D^k \psi(t)\,dt$$

$$:= M_1(x) + M_2(x). \tag{a.3}$$

Observe that by A1, Y1 and Y3 (or Y3′), $M_2(x)$ exists for all $x \in \mathbf{R}^+$. Also, by Y2, it follows that

$$\int_0^\delta |\psi| < \infty\ , \tag{a.4}$$

and, by Y1 and Y3(or Y3′), it follows that

$$\int_\delta^\infty |D^k\psi| < \infty\ , \quad k \in \mathbf{N}_0\ . \tag{a.5}$$

Proof of (i)

In order to prove (i) we shall show

$$J^{[1]}\psi \in C^\infty(\delta,\infty) \tag{a.6}$$

$$J^{[2]}\psi \in C^\infty[0,\delta) \tag{a.7}$$

$$J^{[2]}\psi \in C^\infty(\delta,2\delta) \tag{a.8}$$

$$J^{[2]}\psi \in C^\infty(2\delta,\infty) \tag{a.9}$$

Then, since δ may be chosen arbitrarily, it will follow that

$$J^{[1]}\psi \in C^\infty(0,\infty) \tag{a.10}$$

$$J^{[2]}\psi \in C^\infty[0,\infty). \tag{a.11}$$

Hence, (a.1), (a.10) and (a.11) imply

$$K\psi \in C^\infty(0,\infty),$$

which implies

$$DK\psi \in C^{\infty}(0,\infty),$$

as required.

To prove (a.6), let $x \in (\delta,\infty)$ and let y be close to x. Without loss of generality assume $N \geq x \geq \varepsilon > \delta$ and $N \geq y \geq \varepsilon > \delta$, for some N, $\varepsilon \in \mathrm{R}^{+}$. Then, using (a.2), we have

$$\begin{aligned} | D^k J^{[1]}\psi(x) - D^k J^{[1]}\psi(y) | &= | \int_0^{\delta} \left[D^k \kappa\right](x-t)\psi(t)dt - \int_0^{\delta} \left[D^k \kappa\right](y-t)\psi(t)dt | \\ &\leq \int_0^{\delta} | \left[D^k \kappa\right](x-t) - \left[D^k \kappa\right](y-t) | \, |\psi(t)| dt \\ &\leq \sup_{t\in[0,\delta]} | \left[D^k \kappa\right](x-t) - \left[D^k \kappa\right](y-t) | \int_0^{\delta} |\psi(t)| dt . \end{aligned}$$

Now note that for all $t \in [0,\delta]$, $N \geq x-t \geq \varepsilon-\delta$ and $N \geq y-t \geq \varepsilon-\delta$, and $D^k\kappa$ is uniformly continuous on $[N,\varepsilon-\delta]$. Hence, using (a.4), it follows that

$$| D^k J^{[1]}\psi(x) - D^k J^{[1]}\psi(y) | \to 0 \ \text{ as } y \to x, \tag{a.12}$$

which shows that $D^k J^{[1]}\psi \in C(\delta,\infty)$ for all $k \in \mathrm{N}_0$, and thus (a.6) follows.

Now consider $D^k J^{[2]}\psi(x)$ for $x \in [0,\delta)$, given by (a.3). By A3 we have

$$M_1 \in C[0,\delta) \tag{a.13}$$

since $(\delta - x) > 0$. Also, by arguments analogous to those used in the proof of (a.12), it follows that

$$M_2 \in C[0,\delta). \tag{a.14}$$

Then (a.3), (a.13) and (a.14) show that $D^k J^{[2]}\psi \in C[0,\delta)$ for all $k \in \mathrm{N}_0$, which proves (a.7).

We now consider $D^k J^{[2]}\psi(x)$ for $x \in (\delta,2\delta)$ given by (a.3). By A3 we have

$$M_1 \in C(\delta,2\delta) \tag{a.15}$$

since $(x-\delta) > 0$. Now write $M_2(x)$ as

$$\begin{aligned} M_2(x) &= \left[\int_{\delta}^{x+\delta} + \int_{x+\delta}^{\infty}\right] \kappa(x-t)\, D^k\psi(t)dt \\ &:= M_3(x) + M_4(x) , \end{aligned} \tag{a.16}$$

and consider $M_4(x)$. We wish to show that

$$M_4 \in C(\delta,2\delta). \tag{a.17}$$

To prove (a.17), let $x,y\in(\delta,2\delta)$, let y be close to x, and without loss of generality assume $y>x$, $2\delta>\varepsilon\geq x>\delta$, $2\delta>\varepsilon\geq y>\delta$ for some $\varepsilon\in\mathbf{R}^+$. Then we have

$$| \; M_4(x) - M_4(y) \; | = | \int_{x+\delta}^{\infty}\kappa(x-t)D^k\psi(t)dt - \int_{y+\delta}^{\infty}\kappa(y-t)D^k\psi(t)dt \; |$$

$$\leq \int_{x+\delta}^{\infty}|\kappa(x-t) - \kappa(y-t)| \; |D^k\psi(t)|dt + \int_{x+\delta}^{y+\delta}|\kappa(y-t)| \; |D^k\psi(t)|dt$$

$$\leq \int_{2\delta}^{\infty}|\kappa(x-t) + \kappa(y-t)| \; |D^k\psi(t)|dt + \sup_{t\in[2\delta,3\delta]}|D^k\psi(t)| \int_x^y|\kappa(y-t+\delta)|dt \; .$$

Now note that

$$\int_{2\delta}^{\infty}|\kappa(x-t) - \kappa(y-t)||D^k\psi(t)|dt \leq \sup_{t\in[2\delta,\infty)}|\kappa(x-t) - \kappa(y-t)| \int_{2\delta}^{\infty}|D^k\psi(t)|dt$$

Observe that for all $t\in[2\delta,\infty)$, $-N\leq x-t\leq-(2\delta-\varepsilon)$, $-N\leq y-t\leq-(2\delta-\varepsilon)$ for some $N\in\mathbf{R}^+$, and $D^k\kappa$ is uniformly continuous on $[-N,-(2\delta-\varepsilon)]$. Hence, using (a.5), it follows that

$$\int_{2\delta}^{\infty}| \; \kappa(x-t) - \kappa(y-t) \; ||D^k\psi(t)|dt \to 0 \quad \text{as } y\to x \; .$$

Also, by Y1 and A1, we have

$$\sup_{[2\delta,3\delta]}|D^k\psi(t)| \int_x^y|\kappa(y-t+\delta)|dt \to 0 \quad \text{as } y\to x.$$

Thus it follows that

$$| \; M_4(x) - M_4(y) \; | \to 0 \quad \text{as } y\to x \; ,$$

which proves (a.17).

Also, by making the substitution $u=x-t$ in $M_3(x)$ we obtain

$$M_3(x) = \int_{-\delta}^{x-\delta}\kappa(u) \left[D^k\psi\right](x-u)du \; .$$

Then, by arguments analogous to those used in the proof of (a.17), we have

$$M_3 \in C(\delta,2\delta). \tag{a.18}$$

Now, (a.3), (a.15), (a.16), (a.17) and (a.18) show that $D^kJ^{[2]}\psi\in C(\delta,2\delta)$ for all

$k \in \mathbf{N}_0$, which proves (a.8).

Now consider $D^k J^{[2]} \psi(x)$ for $x \in (2\delta, \infty)$ given by (a.3). By A3 we have

$$M_1 \in C(2\delta,\infty), \tag{a.19}$$

since $(x-\delta)>0$.

Now write $M_2(x)$ as

$$M_2(x) = \left[\int_{\delta}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} \right] \kappa(x-t)\ D^k \psi(t) dt$$

$$:= M_5(x) + M_6(x) + M_7(x)\ . \tag{a.20}$$

Consider $M_5(x)$ and $M_7(x)$. By arguments analogous to those used in the proof of (a.17) we have

$$M_5 \in C(2\delta,\infty) \tag{a.21}$$

$$M_7 \in C(2\delta,\infty)\ . \tag{a.22}$$

Also, by making the substitution $u=x-t$ in $M_6(x)$ we obtain

$$M_6(x) = \int_{-\delta}^{\delta} \kappa(u) \left[D^k \psi \right](x-u) du\ .$$

Then, by arguments analogous to those used in the proof of (a.12), we have

$$M_6(x) \in C(2\delta,\infty). \tag{a.23}$$

Thus, (a.3), (a.19), (a.20), (a.21), (a.22) and (a.23) show that $D^k J^{[2]} \psi \in C(2\delta,\infty)$ for all $k \in \mathbf{N}_0$, which proves (a.9).

Proof of (ii)

By (a.1), (a.10) and (a.11), the singularities in $K\psi(x)$ as $x \to 0$ must be contained completely in $J^{[1]}\psi(x)$. Now by A3, for $x \in [0,\delta]$, we may write

$$J^{[1]}\psi(x) = \int_0^{\delta} a(x-t) \left[\log(|x-t|) + c(|x-t|) \right] \psi(t) dt$$

$$= \int_0^{\delta} a(x-t) \log(|x-t|)\ \psi(t) dt + \int_0^{\delta} a(x-t)\ c(|x-t|)\ \psi(t) dt$$

$$:= I_1(x) + I_2(x)\ . \tag{a.24}$$

We may write $I_2(x)$ as

$$I_2(x) = \left[\int_0^x + \int_x^\delta\right] a(x-t)\ c(|x-t|)\ \psi(t)dt$$

$$:= I_3(x) + I_4(x)\ . \tag{a.25}$$

So consider $I_3(x)$. We have

$$DI_3(x) = \left[ac\right](0)\ \psi(x) + \int_0^x \left[Dac\right](x-t)\ \psi(t)dt$$

$$D^2I_3(x) = \left[ac\right](0)\ D\psi(x) + \left[Dac\right](0)\ \psi(x) + \int_0^x \left[D^2ac\right](x-t)\ \psi(t)dt$$

$$\vdots$$

$$D^{l+1}I_3(x) = \sum_{m=0}^{l} \left[D^{l-m}ac\right](0)\ D^m\psi(x) + \int_0^x \left[D^{l+1}ac\right](x-t)\ \psi(t)dt$$

$$= \left\{\sum_{m=0}^{l} D^m\psi(x)\right\} + L_1(x),\ \text{say}\ . \tag{a.26}$$

Now by arguments analogous to those used in the proof of (a.17), we have $L_1 \in C[0,\delta]$. Then, by integrating (a.26) l times and using Y2, we obtain

$$DI_3(x) = \{[(\log x)^p + (\log x)^{p-1} + \ldots + \log x][1 + x + \ldots + x^l]\} + \varphi_{l,\delta}(x),\ \ x \in [0,\delta]\ . \tag{a.27}$$

Similarly, we can show

$$DI_4(x) = \{[(\log x)^p + (\log x)^{p-1} + \ldots + \log x][1 + x + \ldots + x^l]\} + \varphi_{l,\delta}(x),\ \ x \in [0,\delta]\ . \tag{a.28}$$

Then (a.25), (a.27) and (a.28) show that

$$DI_2(x) = \{[(\log x)^p + (\log x)^{p-1} + \ldots + \log x][1 + x + \ldots + x^l]\} + \varphi_{l,\delta}(x),\ \ x \in [0,\delta]. \tag{a.29}$$

Now consider $I_1(x)$ given in (a.24). Using Y2 with l replaced by $l+1$, we have

$$I_1(x) = \int_0^\delta a(x-t)\log(|x-t|)\{[(\log x)^p + (\log x)^{p-1} + \ldots + \log x][1 + x + \ldots + x^{l+1}]\}dt$$

$$+ \int_0^\delta a(x-t)\ \log(|x-t|)\ \varphi_{l+1,\delta}(t)dt$$

$$:= I_5(x) + I_6(x) \ . \tag{a.30}$$

We may write $I_6(x)$ as

$$I_6(x) = \left[\int_0^x + \int_x^\delta\right] a(x-t)\log(|x-t|)\ \varphi_{l+1,\delta}(t)dt$$

$$:= I_7(x) + I_8(x) \ . \tag{a.31}$$

Making the substitution $u=x-t$ in $I_7(x)$ we obtain

$$I_7(x) = \int_0^x a(u)\log u\ \varphi_{l+1,\delta}(x-u)du \ .$$

Then we have

$$DI_7(x) = a(x)\log x\ \varphi_{l+1,\delta}(0) + \int_0^x a(u)\log u\ D\varphi_{l+1,\delta}(x-u)du$$

$$D^2I_7(x) = D\left[a(x)\log x\right]\varphi_{l+1,\delta}(0) + a(x)\log x\ D\varphi_{l+1,\delta}(0) + \int_0^x a(u)\log u\ D^2\varphi_{l+1,\delta}(x-u)du$$

$$\vdots$$

$$D^{l+1}I_7(x) = \sum_{m=0}^{l} D^m\left[a(x)\log x\right]\left[D^{l-m}\varphi_{l+1,\delta}\right](0) + \int_0^x a(u)\log u\ D^{l+1}\varphi_{l+1,\delta}(x-u)du$$

$$= \left\{\sum_{m=0}^{l} D^m\left[a(x)\log x\right]\right\} + L_2(x), \text{ say.} \tag{a.32}$$

By arguments analogous to those used in the proof of (a.17) we have $L_2 \in C[0,\delta]$. Hence, by integrating (a.32) l times and using A3, we obtain

$$DI_7(x) = \{[\log x][1+x+\ldots+x^l]\} + \varphi_{l,\delta}(x), \ \ x\in[0,\delta] \ . \tag{a.33}$$

Similarly, by making the substitution $u=t-x$ in $I_8(x)$, we obtain

$$DI_8(x) = \{[\log(\delta-x)][1+(\delta-x)+\ldots+(\delta-x)^l]\} + \varphi_{l,\delta}(x), \ \ x\in[0,\delta]. \tag{a.34}$$

Since δ was chosen arbitrarily, we have from (a.31), (a.33) and (a.34),

$$DI_6(x) = \{[\log x][1+x+\ldots+x^l]\} + \varphi_{l,\delta}(x), \ \ x\in[0,\delta] \ . \tag{a.35}$$

Now recall that a is assumed to be a polynomial of degree M, say, and note that

$$I_5(x) = \sum_{m\le M, r\le l, j\le p} I_5^{m,r,j}$$

where

$$I_5^{m,r,j}(x) = \int_0^\delta (x-t)^m \log(|x-t|)\ t^r\ (\log t)^j dt$$

$$= \left[\int_0^x + \int_x^\delta \right](x-t)^m \log(|x-t|)\ t^r\ (\log t)^j\ dt$$

$$:= I_9(x) + I_{10}(x)\ . \tag{a.36}$$

Setting $t=ux$ in $I_9(x)$ yields

$$I_9(x) = x\int_0^1 \left[x^m\ (1-u)^m\ (\log x + \log(1-u))\ u^r\ x^r\ (\log u + \log x)^j\right] du$$

$$= x^{m+r+1}\int_0^1 \left[u^r\ (1-u)^m\ (\log x + \log(1-u))\ (\log x + \log u)^j\right] du$$

$$= \left\{ \sum_{k=0}^{j+1} x^{m+r+1} (\log x)^k \right\}, \qquad x \in [0,\delta]. \tag{a.37}$$

Now consider $I_{10}(x)$. Let $0<\varepsilon<\delta$ and let $x\in[\varepsilon,\delta]$. Then, expanding $t^r(\log t)^j$ in a Taylor series about δ, we have

$$t^r (\log t)^j = \left\{ \sum_{k=0}^{\infty} (\delta-t)^k \right\},$$

with uniform convergence for $t\in[x,\delta]$ for ε sufficiently close to δ, and thus we can integrate $I_{10}(x)$ term by term to obtain

$$I_{10}(x) = \left\{ \sum_{k=0}^{\infty} \int_x^\delta (t-x)^m \log(t-x)\ (\delta-t)^k\ dt \right\}. \tag{a.38}$$

Setting $(\delta-x)u=\delta-t$ in (a.38) yields

$$I_{10}(x) = \left\{ \sum_{k=0}^{\infty} \int_0^1 \left[(\delta-x)^m\ (1-u)^m\ [\log(\delta-x)+\log(1-u)]\ u^k\ (\delta-x)^{k+1}\right] du \right\}$$

$$= \left\{ \sum_{k=0}^{l+1} (\delta-x)^{k+1} \log(\delta-x) \right\} + \varphi_{l+1,\delta}^{[1]}(x), \qquad x\in[\varepsilon,\delta], \tag{a.39}$$

where $\varphi^{[1]}_{l+1,\delta} \in C^{l+1}[\varepsilon,\delta]$. Now return to $I_{10}(x)$ and let $x \in [0,\delta-\varepsilon]$. Then, for sufficiently small $\eta>0$, we write $I_{10}(x)$ as

$$I_{10}(x) = \left[\int_x^{(1+\eta)x} + \int_{(1+\eta)x}^{\delta}\right](x-t)^m \log(t-x)\ t^r\ (\log t)^j\ dt$$

$$:= I_{11}(x) + I_{12}(x)\ . \qquad \text{(a.40)}$$

By setting $t=ux$ in $I_{11}(x)$ we obtain

$$I_{11}(x) = \left\{\sum_{k=0}^{j+1} x^{m+r+1}(\log x)^k\right\}, \qquad x \in [0,\delta-\varepsilon], \qquad \text{(a.41)}$$

(see the derivation for the expression (a.37)). Now consider $I_{12}(x)$. We have

$$(x-t)^m \log(t-x) = (-1)^m (t-x)^m \log(t-x)$$

$$= (-1)^m t^m\ (1-x/t)^m \log t + (-1)^m t^m\ (1-x/t)^m \log(1-x/t)$$

$$= (-1)^m t^m \log t \sum_{k=0}^{m} a_k x^k t^{-k} + (-1)^m t^m \sum_{k=0}^{\infty} b_k x^k t^{-k}\ ,$$

for some scalars a_k and b_k (with uniform convergence for $t>x$). Hence, we may write

$$I_{12}(x) = (-1)^m \sum_{k=0}^{m} a_k x^k \int_{(1+\eta)x}^{\delta} t^{m+r-k}\ (\log t)^{j+1}\ dt$$

$$+ (-1)^m \sum_{k=0}^{\infty} b_k x^k \int_{(1+\eta)x}^{\delta} t^{m+r-k}\ (\log t)^j\ dt\ . \qquad \text{(a.42)}$$

Now note that for all $j \in \mathrm{N}$, if $k=m+r+1$ then

$$\int t^{m+r-k}\ (\log t)^j\ dt = \{(\log t)^{j+1}\},$$

and if $k \neq m+r+1$ we have

$$\int t^{m+r-k}\ (\log t)^j\ dt = \{t^{m+r-k+1}[(\log t)^j + (\log t)^{j-1} + \ldots + 1]\}\ ,$$

and so from (a.42) we have

$$I_{12}(x) = \left\{\sum_{k=0}^{j+1} x^{m+r+1}\ (\log x)^k\right\} + \varphi^{[2]}_{l+1,\delta}(x), \quad x \in [0,\delta-\varepsilon], \qquad \text{(a.43)}$$

where $\varphi^{[2]}_{l+1,\delta} \in C^{l+1}[0,\delta-\varepsilon]$. Then (a.40), (a.41) and (a.43) show that

$$I_{10}(x) = \left\{ \sum_{k=0}^{j+1} x^{m+r+1} (\log x)^k \right\} + \varphi^{[2]}_{l+1,\delta}(x), \quad x\in[0,\delta-\varepsilon] \ . \tag{a.44}$$

Now since ε and δ were chosen arbitrarily, we have from (a.39) and (a.44)

$$I_{10}(x) = \{[(\log x)^{j+1}+(\log x)^j+\ldots+\log x][x+x^2+\ldots+x^{l+1}]\} + \varphi_{l+1,\delta}(x), \quad x\in[0,\delta]. \tag{a.45}$$

Recall that

$$I_5(x) = \sum_{m\le M, r\le l, j\le p} I_5^{m,r,j}(x) \ ,$$

and so we have from (a.36), (a.37) and (a.45)

$$I_5(x) = \{[(\log x)^{p+1}+(\log x)^p+\ldots+\log x][x+x^2+\ldots+x^{l+1}]\} + \varphi_{l+1,\delta}(x), \quad x\in[0,\delta] \ , \tag{a.46}$$

and hence, from (a.30), (a.35) and differentiating (a.46), we obtain

$$DI_1(x) = \{[(\log x)^{p+1}+(\log x)^p+\ldots+\log x][1+x+\ldots+x^l]\} + \varphi_{l,\delta}(x), \quad x\in[0,\delta] \ . \tag{a.47}$$

Then (a.24), (a.29), (a.47), (a.1), (a.10) and (a.11) show that

$$DK\psi(x) = \{[(\log x)^{p+1}+(\log x)^p+\ldots+\log x][1+x+\ldots+x^l]\} + \varphi_{l,\delta}(x), \quad x\in[0,\delta] \ ,$$

which proves part (ii).

The last part of the proof of (ii) is based on the arguments of Richter [35].

Proof of (iii)

Now let the conditions of Part A hold. Let $\mu\in[0,\rho)$. First note that A2 implies that for each $k\in\mathbf{N}_0$, there exists $\varepsilon_1>0$ such that

$$D^k\kappa(x) = O(e^{-(\mu+\varepsilon_1)x}) \quad \text{as } x\to\infty \ , \tag{a.48}$$

and Y3 implies that for each $k\in\mathbf{N}_0$, there exists $\varepsilon_2>0$ such that

$$D^k\psi(x) = O(e^{-(\mu+\varepsilon_2)x}) \quad \text{as } x\to\infty \ . \tag{a.49}$$

From (a.1) it follows that

$$| \, e^{\mu x} D^k K\psi(x) \, | \le | \, e^{\mu x} D^k J^{[1]} \psi(x) \, | + | \, e^{\mu x} D^k J^{[2]} \psi(x) \, | \, . \qquad \text{(a.50)}$$

So first consider $D^k J^{[1]} \psi(x)$, given by (a.2), as $x \to \infty$. For all $t \in [0,\delta]$, $(x-t) \to \infty$ as $x \to \infty$, and so (a.48) and (a.4) show that for sufficiently large x,

$$| \, D^k J^{[1]} \psi(x) \, | \le \int_0^\delta | \left[D^k \kappa \right](x-t) | \, | \psi(t) | \, dt \le \mathrm{C} \, e^{-(\mu+\varepsilon_1)(x-\delta)} \int_0^\delta | \psi(t) | \, dt$$

$$\le \mathrm{C} \, e^{-(\mu+\varepsilon_1)x} \, , \qquad \text{(a.51)}$$

and so from (a.51) it follows that

$$| \, e^{\mu x} D^k J^{[1]} \psi(x) \, | \to 0 \ \text{ as } x \to \infty \, . \qquad \text{(a.52)}$$

Next consider $D^k J^{[2]} \psi(x)$, given by (a.3), as $x \to \infty$. From (a.48) it follows that

$$| \, e^{\mu x} M_1(x) \, | \to 0 \ \text{ as } x \to \infty \, . \qquad \text{(a.53)}$$

Now write $M_2(x)$ as given in (a.20) and consider $M_5(x)$, $M_6(x)$ and $M_7(x)$ separately.

For $M_5(x)$ we write

$$M_5(x) = \int_\delta^{x/2} \kappa(x-t) D^k \psi(t) \, dt + \int_{x/2}^{x-\delta} \kappa(x-t) D^k \psi(t) \, dt$$

$$:= M_{51}(x) + M_{52}(x) \, , \qquad \text{(a.54)}$$

(since $x/2 < (x-\delta)$ as $x \to \infty$). So consider $M_{51}(x)$. For all $t \in [\delta, x/2]$, $(x-t) \to \infty$ as $x \to \infty$, and so (a.48) and (a.5) show that for sufficiently large x,

$$| \, M_{51}(x) \, | \le \int_\delta^{x/2} | \kappa(x-t) | \, | D^k \psi(t) | \, dt \le \mathrm{C} \, e^{-(\mu+\varepsilon_1)\frac{x}{2}} \int_\delta^{x/2} | D^k \psi(t) | \, dt$$

$$\le \mathrm{C} \, e^{-(\mu+\varepsilon_1)x} \, , \qquad \text{(a.55)}$$

and hence from (a.55) it follows that

$$| \, e^{\mu x} M_{51}(x) \, | \to 0 \ \text{ as } x \to \infty \, . \qquad \text{(a.56)}$$

Now consider $M_{52}(x)$. (a.49) shows that for $t \in [x/2, x-\delta]$, $|D^k \psi(t)| \le \mathrm{C} e^{-(\mu+\varepsilon_2)\frac{x}{2}}$ as $x \to \infty$, and then A1 shows for sufficiently large x,

$$| M_{52}(x) | \le \int_{x/2}^{x-\delta} |\kappa(x-t)||D^k\psi(t)|dt \le C\, e^{-(\mu+\varepsilon_2)\frac{x}{2}} \int_{x/2}^{x-\delta} |\kappa(x-t)|dt$$

$$\le C\, e^{-(\mu+\varepsilon_2)x} \; . \tag{a.57}$$

Thus from (a.57) it follows that

$$| e^{\mu x} M_{52}(x) | \to 0 \;\; \text{as } x \to \infty \; . \tag{a.58}$$

From (a.54), (a.56) and (a.58) we have

$$| e^{\mu x} M_5(x) | \to 0 \;\; \text{as } x \to \infty \; . \tag{a.59}$$

Also, if we make the substitution $u=x-t$ in $M_6(x)$ we obtain

$$M_6(x) = \int_{-\delta}^{\delta} \kappa(u) \left[D^k\psi\right](x-u)du \; .$$

Now for $u \in [-\delta,\delta]$, $(x-u) \to \infty$ as $x \to \infty$, and so (a.49) and A1 show that for sufficiently large x,

$$| M_6(x) | \le \int_{-\delta}^{\delta} |\kappa(u)|| \left[D^k\psi\right](x-u)|du \le C\, e^{-(\mu+\varepsilon_2)(x-\delta)} \int_{-\delta}^{\delta} |\kappa(u)|du$$

$$\le C\, e^{-(\mu+\varepsilon_2)x} \; , \tag{a.60}$$

and hence from (a.60) it follows that

$$| e^{\mu x} M_6(x) | \to 0 \;\; \text{as } x \to \infty \; . \tag{a.61}$$

Now consider $M_7(x)$. For $t \in [x+\delta,\infty)$, (a.49) shows that $|D^k\psi(t)| \le Ce^{-(\mu+\varepsilon_2)(x+\delta)}$ as $x \to \infty$, and then A1 shows that for sufficiently large x,

$$| M_7(x) | \le \int_{x+\delta}^{\infty} |\kappa(x-t)||D^k\psi(t)|dt \le C\, e^{-(\mu+\varepsilon_2)(x+\delta)} \int_{x+\delta}^{\infty} |\kappa(x-t)|dt$$

$$\le C\, e^{-(\mu+\varepsilon_2)x} \; . \tag{a.62}$$

Thus from (a.62) it follows that

$$| e^{\mu x} M_7(x) | \to 0 \;\; \text{as } x \to \infty \; . \tag{a.63}$$

Then from (a.20), (a.59), (a.61) and (a.63) we have

$$| e^{\mu x} M_2(x) | \to 0 \;\; \text{as } x \to \infty \; . \tag{a.64}$$

Hence, from (a.50), (a.52), (a.3), (a.53) and (a.64), we have for all $k \in \mathbf{N}_0$

$$| \; e^{\mu x} D^k K\psi(x) \; | \to 0 \quad \text{as } x \to \infty \; ,$$

which proves part (iii).

Proof of (iv)

Now let the conditions of Part B hold. Let $\mu' \in [0,\rho')$. First note that A2$'$ implies that for each $k \in \mathbf{N}_0$, there exists $\varepsilon_1 > 0$ such that

$$D^k \kappa(x) = O(x^{-(\mu'+\varepsilon_1)}) \quad \text{as } x \to \infty \; , \tag{a.65}$$

and Y3$'$ implies that for each $k \in \mathbf{N}_0$, there exists $\varepsilon_2 > 0$ such that

$$D^k \psi(x) = O(x^{-(\mu'+\varepsilon_2)}) \quad \text{as } x \to \infty \; . \tag{a.66}$$

From (a.1) it follows that

$$| \; x^{\mu'} D^k K\psi(x) \; | \le | \; x^{\mu'} D^k J^{[1]} \psi(x) \; | + | \; x^{\mu'} D^k J^{[2]} \psi(x) \; | \; . \tag{a.67}$$

So first consider $D^k J^{[1]} \psi(x)$, given by (a.2), as $x \to \infty$. For all $t \in [0,\delta]$, $(x-t) \to \infty$ as $x \to \infty$, and so (a.65) and (a.4) show that for sufficiently large x,

$$| \; D^k J^{[1]} \psi(x) \; | \le \int_0^\delta \left| \left[D^k \kappa \right](x-t) \right| | \psi(t) | \, dt \le \mathrm{C} \, (x-\delta)^{-(\mu'+\varepsilon_1)} \int_0^\delta | \psi(t) | \, dt$$

$$\le \mathrm{C} \, x^{-(\mu'+\varepsilon_1)} \; , \tag{a.68}$$

and so from (a.68) it follows that

$$| \; x^{\mu'} D^k J^{[1]} \psi(x) \; | \to 0 \quad \text{as } x \to \infty \; . \tag{a.69}$$

Now consider $D^k J^{[2]} \psi(x)$, given by (a.3), as $x \to \infty$. From (a.65) it follows that

$$| \; x^{\mu'} M_1(x) \; | \to 0 \quad \text{as } x \to \infty \; . \tag{a.70}$$

Now write $M_2(x)$ as given in (a.20) and consider $M_5(x)$, $M_6(x)$ and $M_7(x)$ separately.

For $M_5(x)$ we write

$$M_5(x) = \int_\delta^{x/2} \kappa(x-t) D^k \psi(t) \, dt + \int_{x/2}^{x-\delta} \kappa(x-t) D^k \psi(t) \, dt$$

$$:= M_{51}(x) + M_{52}(x) \ , \tag{a.71}$$

(since $x/2 < (x-\delta)$ as $x \to \infty$). So consider $M_{51}(x)$. For all $t \in [\delta, x/2]$, $(x-t) \to \infty$ as $x \to \infty$, and so (a.65) and (a.5) show that for sufficiently large x,

$$| \ M_{51}(x) \ | \le \int_{\delta}^{x/2} |\kappa(x-t)||D^k \psi(t)| dt \le C \ (x/2)^{-(\mu'+\varepsilon_1)} \int_{\delta}^{x/2} |D^k \psi(t)| dt$$

$$\le C \, x^{-(\mu'+\varepsilon_1)} \ , \tag{a.72}$$

and hence from (a.72) it follows that

$$| \ x^{\mu'} M_{51}(x) \ | \to 0 \ \text{ as } x \to \infty \ . \tag{a.73}$$

Now consider $M_{52}(x)$. (a.66) shows that for $t \in [x/2, x-\delta]$, $|D^k \psi(t)| \le C(x/2)^{-(\mu'+\varepsilon_2)}$ as $x \to \infty$, and then A1 shows that for sufficiently large x,

$$| \ M_{52}(x) \ | \le \int_{x/2}^{x-\delta} |\kappa(x-t)||D^k \psi(t)| dt \le C \ (x/2)^{-(\mu'+\varepsilon_2)} \int_{x/2}^{x-\delta} |\kappa(x-t)| dt$$

$$\le C \, x^{-(\mu'+\varepsilon_2)} \ . \tag{a.74}$$

Thus from (a.74) it follows that

$$| \ x^{\mu'} M_{52}(x) \ | \to 0 \ \text{ as } x \to \infty \ . \tag{a.75}$$

Then from (a.71), (a.73) and (a.75) we have

$$| \ x^{\mu'} M_5(x) \ | \to 0 \ \text{ as } x \to \infty \ . \tag{a.76}$$

Also, if we make the substitution $u = x - t$ in $M_6(x)$ we obtain

$$M_6(x) = \int_{-\delta}^{\delta} \kappa(u) \left[D^k \psi \right] (x-u) du \ .$$

Now for $u \in [-\delta, \delta]$, $(x-u) \to \infty$ as $x \to \infty$, and so (a.66) and A1 show that for sufficiently large x,

$$| \ M_6(x) \ | \le \int_{-\delta}^{\delta} |\kappa(u)|| \left[D^k \psi \right] (x-u)| du \le C \ (x-\delta)^{-(\mu'+\varepsilon_2)} \int_{-\delta}^{\delta} |\kappa(u)| du$$

$$\le C \, x^{-(\mu'+\varepsilon_2)} \ , \tag{a.77}$$

and hence from (a.77) it follows that

$$| \ x^{\mu'} M_6(x) \ | \to 0 \ \text{ as } x \to \infty \ . \tag{a.78}$$

Now consider $M_7(x)$. For $t \in [x+\delta, \infty)$, (a.66) shows that $|D^k\psi(t)| \le C(x+\delta)^{-(\mu'+\varepsilon_2)}$ as $x \to \infty$, and then A1 shows that for sufficiently large x,

$$| M_7(x) | \le \int_{x+\delta}^{\infty} |\kappa(x-t)||D^k\psi(t)|dt \le C\,(x+\delta)^{-(\mu'+\varepsilon_2)} \int_{x+\delta}^{\infty} |\kappa(x-t)|dt$$

$$\le C\,x^{-(\mu'+\varepsilon_2)} \; . \tag{a.79}$$

Thus from (a.79) it follows that

$$| x^{\mu'} M_7(x) | \to 0 \text{ as } x \to \infty \; . \tag{a.80}$$

Then from (a.20), (a.76), (a.78) and (a.80) we have

$$| x^{\mu'} M_2(x) | \to 0 \text{ as } x \to \infty \; . \tag{a.81}$$

Hence, (a.67), (a.69), (a.3), (a.70) and (a.81) show that for all $k \in \mathbf{N}_0$

$$| x^{\mu'} D^k K\psi(x) | \to 0 \text{ as } x \to \infty \; ,$$

which proves part (iv). □

APPENDIX B

Notation

For any function v and any $\eta \in \mathrm{R}$ let $\Delta_\eta v(\cdot) = v(\cdot+\eta) - v(\cdot)$.

Suppose (1.11) holds, then for all $0<\beta<1$,

$$\sup_{\eta \neq 0} \frac{\|\Delta_\eta b\|_{1,[-\gamma,\gamma]}}{|\eta|^\beta} \leq \mathrm{C}. \tag{b.1}$$

This estimate may be obtained by direct calculation using, e.g., the methods of [21]. We need the following lemma in the proof of Theorems 4.7 and 4.9(v). Its proof is a simple modification of an argument first given in [10]. It is similar to a result used in [10] and [37] to analyse product integration for equations on finite intervals.

Lemma B1

Let ψ_x be given by (4.25), *where b in* (4.25) *is given by* (1.11). *Then we have*

(i) $\sup_{t \in I_i} |\psi_x(t)| \leq \mathrm{C}|\log h_i|$,

and

(ii) *For all* $0<\beta<1$,

$$\sum_{i \in Q^{[1]}(x)} h_i^{-\beta} \, \|(b(x-\cdot) - \psi_x(\cdot))_i\|_1 \leq \mathrm{C}.$$

In each case C *is independent of n,i and x.*

Proof

By (1.11), if $t \in I_i$ then

$$|\psi_x(t)| \leq h_i^{-1} \int_{I_i} |\log(|x-\xi|)| d\xi + \mathrm{C},$$

and (i) follows by straightforward calculus. For (ii), we note that

$$\int_{x_{i-1}-t}^{x_i-t} b(x-t-\eta) d\eta - h_i b(x-t) = \int_{x_{i-1}-t}^{x_i-t} (\Delta_\eta b(x-\cdot))(t) d\eta$$

which yields

$$\psi_x(t) - b(x-t) = \frac{1}{h_i} \int_{x_{i-1}-t}^{x_i-t} (\Delta_\eta b(x-\cdot))(t)\,d\eta.$$

Hence

$$\int_{I_i} |b(x-t) - \psi_x(t)|\,dt \le \frac{1}{h_i} \int_{I_i} \int_{x_{i-1}-t}^{x_i-t} |(\Delta_\eta b(x-\cdot))(t)|\,d\eta dt$$

$$\le \frac{1}{h_i} \int_0^{h_i} [\|\Delta_\eta b(x-\cdot)_i\|_1 + \|\Delta_{-\eta} b(x-\cdot)_i\|_1]\,d\eta$$

$$\le 2 \sup_{|\eta|\le h_i} \|\Delta_\eta b(x-\cdot)_i\|_1,$$

where the penultimate inequality follows from Fubini's theorem. Hence we have

$$\sum_{i\in Q^{[1]}(x)} h_i^{-\beta} \|(b(x-\cdot)-\psi_x)_i\|_1 \le 2 \sum_{i\in Q^{[1]}(x)} h_i^{-\beta} \sup_{|\eta|\le h_i} \|\Delta_\eta b(x-\cdot)_i\|_1$$

$$\le 2 \sum_{i\in Q^{[1]}(x)} \sup_{|\eta|\le h_i} \frac{\|\Delta_\eta b(x-\cdot)_i\|_1}{|\eta|^\beta}$$

$$\le 2 \sup_{|\eta|\le \gamma_0} \sum_{i\in Q^{[1]}(x)} \frac{\|\Delta_\eta b(x-\cdot)_i\|_1}{|\eta|^\beta}$$

$$\le 2 \sup_{|\eta|\le \gamma_0} \frac{\|\Delta_\eta b\|_{1,[-\gamma,\gamma]}}{|\eta|^\beta} \le C,$$

from (4.2), (4.3) and (b.1). □