



Citation for published version:

Bajoori, E & Vermeulen, D 2019, 'Equilibrium selection in interdependent-value auctions', *Mathematical Social Sciences*, vol. 98, pp. 47-56. <https://doi.org/10.1016/j.mathsocsci.2019.01.003>

DOI:

[10.1016/j.mathsocsci.2019.01.003](https://doi.org/10.1016/j.mathsocsci.2019.01.003)

Publication date:

2019

Document Version

Peer reviewed version

[Link to publication](#)

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Equilibrium selection in interdependent value auctions ^{*}

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January 23, 2019

Abstract

In second-price auctions with interdependent values, bidders do not necessarily have dominant strategies. Moreover, such auctions may have many equilibria. In order to rule out the less intuitive equilibria, we define the notion of distributional strictly perfect equilibrium (DSPE) for Bayesian games with infinite type and action spaces. This equilibrium is robust against arbitrary small perturbations of strategies. We apply DSPE to a class of symmetric second-price auctions with interdependent values. We show that the efficient equilibrium defined by Milgrom [28] is a DSPE, while a class of less intuitive, inefficient, equilibria introduced by Birulin [13] is not.

JEL Codes. D44, C72.

Keywords. Interdependent value auctions, Equilibrium selection, Strictly perfect equilibrium.

^{*}The authors would like to thank Philip J. Reny and János Flesch for their very helpful and constructive comments.

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1 Introduction

In private value second-price auctions each bidder has a dominant strategy in which he bids his own valuation of the object. However, in second-price auctions with interdependent values bidders may not have dominant strategies while there may exist multiple equilibria in undominated strategies. For example Birulin (2003) [13] shows that any auction with an efficient ex-post equilibrium has a continuum of inefficient undominated ex-post equilibria. Therefore, a selection tool is needed in such Bayesian games to rule out the less intuitive equilibria. We develop a concept of trembling hand perfect equilibrium for Bayesian games with infinite type and action spaces, called distributional strictly perfect equilibrium (DSPE). In this equilibrium concept, players' strategies are robust against any slight perturbations of their opponents' strategies.

In finite normal form games, DSPE is equivalent to strictly perfect equilibrium by Okada (1981) [30],¹ as both definitions require robustness against arbitrary slight perturbations of strategies. It is known that strictly perfect equilibrium may not exist (see example 1.5.5 in van Damme [35]), therefore DSPE may not exist even in finite games. However, we can prove existence of a weaker notion of DSPE, which is called distributional perfect equilibrium, in Bayesian games for which the type space of each player is a separable metric space, the action space of each player is a compact metric space, and player types are drawn from a prior probability measure on the product of the type spaces that is absolutely continuous with respect to the product measure of its marginal probabilities.^{2 3}

We apply this notion of perfection to second price auctions with interdependent values in order to select among equilibria. In our auction model, there is one continuous and efficient equilibrium introduced by Milgrom (1981) [28] and a set of inefficient ex-post equilibria introduced by Birulin [13]. All of these equilibria are undominated and there is no dominant strategy. Furthermore, all these equilibria all ex-post, so ex-post cannot be used as a selection criterion in this

¹The proof is available upon request.

²This property is called absolute continuity of information in Milgrom and Weber [29].

³For further studies on the existence of equilibrium in Bayesian games, we refer to Athey [3], Mcadams [24], Reny [32], Jackson et al. [18],[19], Mertens [27], Meirowitz [25], Milgrom and Weber [29], Mallozzi et al. [26], Kim and Yannelis [20], Zandt and Vives [38], and Balder [7].

model. We prove that the efficient equilibrium due to Milgrom [28] is DSPE, but the one's constructed in Birulin [13] are not. In other words, our results show that the efficient and continuous equilibrium is robust against arbitrary slight perturbation of strategies, while other discontinuous and inefficient equilibria are not. These discontinuous equilibria contain either overbidding or underbidding when compared with the efficient strategy profile by Milgrom [28]. We show that if each bidder believes that with a very small probability her opponent can bid according to the efficient bidding strategy, then overbidding or underbidding is not a best response.

Bajoori et al. (2016) [6] define the notion of Perfect Bayesian Nash Equilibrium and apply this notion to an example in interdependent value auctions. In a Perfect Bayesian Nash Equilibrium the strategies are robust against some slight perturbations of strategies. This definition is weaker than the notion of DSPE, but it is not strong enough to exclude all the discontinuous and inefficient equilibria in a more general model of an interdependent value auction.

To the best of our knowledge there is not much research done on the theory of equilibrium selection in interdependent value auctions. However, there are some relatively recent studies on the equilibrium selection in common value auctions such as Parreiras (2006) [31], Abraham et al. (2012) [2], Cheng and Tan (2008) [14], Larson (2009) [22], and Liu (2014) [23]. In the most relevant study by Liu [23], they use noisy bids rather than strategy perturbations. In their model, bidders need to believe that any bid is possible for their opponent even if this strategy does not have full support. They show that the continuous set of equilibria introduced by Milgrom [28] are robust against such noisy bids, while the set of discontinuous equilibria (similar to the ones introduced by Birulin [13]) are not. Our paper does not cover the case of common values, so that our results are disjoint from those in Liu [22].

Another relevant strand of literature is by Bergemann and Morris (2005) [9] on robust mechanism design. They show that, within the class of separable environments, interim implementability of a social choice correspondence for any given type space is equivalent to implementability in ex post equilibrium. Thus, in their approach, ex post equilibrium is used as the basic solution concept for

robust implementation. However, as we already mentioned, it is well known in the literature on auction design that ex post equilibrium may often be inefficient, and the refinement we propose often rules out the inefficient equilibria. Thus, our approach may in the future have consequences for mechanism design as well.

To illustrate our results, we discuss the following example.

Example 1.1 Consider a 2-bidder second-price auction with interdependent values $v_1 = t_1 + \frac{1}{2}t_2$ and $v_2 = \frac{1}{2}t_1 + t_2$. Types t_1, t_2 for bidder 1 and 2 are drawn independently from $[0, 1]$ according to the uniform distribution. Each bidder simultaneously submits a bid from the set $[0, \frac{3}{2}]$. The efficient equilibrium introduced by Milgrom [28] is the strategy pair (b_1, b_2) in which $b_i = \frac{3}{2}t_i$ for every bidder $i = 1, 2$. We can show that this equilibrium is robust against any slight perturbations of the strategies (Proposition 4.1). On the other hand, the class of inefficient ex post equilibria introduced by Birulin [13] are as follows. For fixed $s_1, s_2 \in [0, 1]$ with $s_1 < s_2$, define

$$\hat{b}_1(t_1) = \begin{cases} s_2 + \frac{1}{2}t_1 & \text{if } t_1 \in [s_1, s_2] \\ \frac{3}{2}t_1 & \text{otherwise,} \end{cases} \quad \hat{b}_2(t_2) = \begin{cases} s_1 + \frac{1}{2}t_2 & \text{if } t_2 \in [s_1, s_2] \\ \frac{3}{2}t_2 & \text{otherwise.} \end{cases}$$

An example of such an equilibrium is depicted in Figure 1. In each such an equilibrium, when the realizations of both types are within the set $[s_1, s_2]$, bidder 1 wins the auction and pays the amount $\hat{b}_2(t_2) = s_1 + \frac{1}{2}t_2$, which is his minimum ex post valuation. Thus, in this case bidder 1 overbids, while bidder 2 underbids. So, since such equilibria lead to inefficient outcomes, we would like to be able to rule them out. Proposition 4.3 shows that the equilibria (\hat{b}_1, \hat{b}_2) are not DSPEs. Intuitively, to exclude such equilibria it is enough to make the bidders believe that with a tiny probability their opponents bid according to the efficient bidding strategy. Then, the overbidding (underbidding) is not a best response against underbidding (overbidding).

The paper is structured as follows. First, we provide some preliminary notions in Section 2. In section 3 we introduce the concept of distributional strictly perfection. In section 4, first we present our interdependent value auction model. Then, we have two subsections; one on selected equilibrium and the other on discarding equilibria.

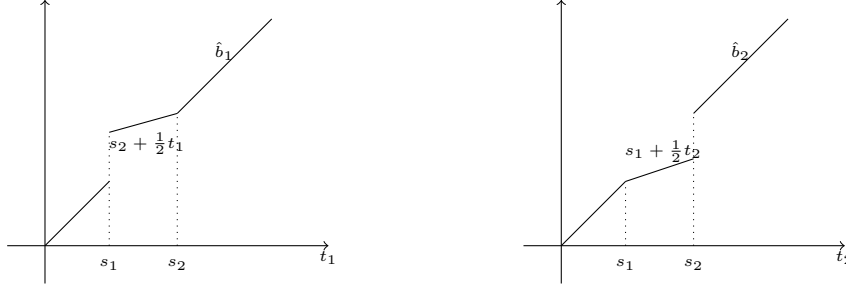


Figure 1

2 Preliminaries

A Bayesian game Γ is defined as follows. There are n players. For each player i , the set T_i of types is a complete separable metric space, and the set A_i of actions is a compact metric space. Let d_{T_i} and d_{A_i} denote the respective metrics, and let \mathcal{T}_i and \mathcal{A}_i denote the induced Borel σ -fields on T_i and A_i respectively. Moreover, let $T = \times_{i=1}^n T_i$ and $A = \times_{i=1}^n A_i$.

Let μ be a probability measure on the product σ -field $\mathcal{T} = \otimes_{i=1}^n \mathcal{T}_i$ with marginal probability μ_i on \mathcal{T}_i for every player i . We assume that each μ_i is a completely mixed probability measure.⁴

Player i 's payoff function $\pi_i : T \times A \rightarrow \mathbf{R}$ is assumed to be bounded and jointly measurable.

We define two classes of strategies: behavioral strategies and distributional strategies. We also define Bayesian Nash Equilibrium (BNE).

A pure strategy for player i in Γ is a measurable function $p_i : T_i \rightarrow A_i$. A behavioral strategy for player i is a function $\beta_i : T_i \times \mathcal{A}_i \rightarrow [0, 1]$ such that

[1] the section function $\beta_i(t_i, \cdot) : \mathcal{A}_i \rightarrow [0, 1]$ is a probability measure for every $t_i \in T_i$, and

[2] the section function $\beta_i(\cdot, B) : T_i \rightarrow [0, 1]$ is measurable for every $B \in \mathcal{A}_i$.

When player i plays according to the behavioral strategy β_i , for each type $t_i \in T_i$, he chooses his action according to the probability measure $\beta_i(t_i, \cdot)$.

⁴A probability measure is completely mixed if it assigns strictly positive weight to every nonempty open set.

A behavioral strategy β_i for player i is called pure if there is a pure strategy p_i for player i such that $\beta_i(t_i, \cdot) = D_{p_i(t_i)}(\cdot)$ for every type $t_i \in T_i$, where D denotes Dirac measure. For simplicity, we denote this pure behavioral strategy by p_i .

Following Milgrom and Weber [29], a distributional strategy for player i in a Bayesian game is a probability measure γ_i on $T_i \times A_i$ such that the marginal distribution of γ_i on T_i equals the probability measure μ_i . That is, $\gamma_i(U \times A_i) = \mu_i(U)$ for every $U \in \mathcal{T}_i$. Given a behavioral strategy β_i , the induced distributional strategy γ_i is uniquely determined by

$$\gamma_i(U \times B) = \int_U \beta_i(t_i, B) \mu_i(dt_i)$$

for every rectangle $U \times B \in \mathcal{T}_i \otimes \mathcal{A}_i$. This induces a many-to-one mapping from behavioral strategies to distributional strategies that preserves the players' ex ante expected payoffs.

The vector $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, where β_i is player i 's behavioral strategy, is called a behavioral strategy profile. For every player i we write $\beta_{-i} = (\beta_j)_{j \neq i}^n$ to denote the behavioral strategy profile for his opponents. The definitions are similar for distributional strategies.

Let Δ_i be the set of all distributional strategies for player i and $\Delta = \times_{i=1}^n \Delta_i$. Let $\Pi_i : \Delta \rightarrow \mathbf{R}$ be the expected payoff of player i playing distributional strategy γ_i against a distributional strategy profile γ_{-i} . So,

$$\Pi_i(\gamma_i, \gamma_{-i}) = \int \pi_i(t, a) \gamma_i(da_i | t_i) \gamma_{-i}(da_{-i} | t_{-i}) \mu(dt),$$

where $t = (t_1, \dots, t_n) \in T$, $a = (a_1, \dots, a_n) \in A$, and $\gamma_i(da_i | t_i)$ is a version of conditional probability on A_i given t_i . By integrals with respect to $\gamma_{-i}(da_{-i} | t_{-i})$ we mean the iterated integrals with respect to $\gamma_j(da_j | t_j)$ for all $j \neq i$.

As the measure μ is absolutely continuous with respect to $\hat{\mu}$, by the Radon-Nikodym theorem there is a measurable function $f : T \rightarrow \mathbf{R}$ such that for every measurable set $G \in \mathcal{T}$ we have $\mu(G) = \int_G f d\hat{\mu}$. Therefore, the expected payoff Π_i can be expressed in an easier form as

$$\Pi_i(\gamma_i, \gamma_{-i}) = \int \pi_i(t, a) f(t) d\gamma_i d\gamma_{-i}.$$

Let β_i be a behavioral strategy corresponding to the distributional strategy γ_i , for every player i . Then, by Theorem 10.2.1 in Dudley [15]⁵ we conclude that

$$\Pi_i(\gamma_i, \gamma_{-i}) = \int \pi_i(t, a) f(t) \beta_i(t_i, da_i) \mu_i(dt_i) \beta_{-i}(t_{-i}, da_{-i}) \mu_{-i}(dt_{-i}).$$

By $\Pi_i^b(\beta_i, \beta_{-i} \mid t_i)$ we denote player i 's expected payoff given his behavioral strategy β_i and his type t_i against a behavioral strategy profile β_{-i} . Thus,

$$\Pi_i^b(\beta_i, \beta_{-i} \mid t_i) = \int \pi_i(t, a) f(t) \beta_i(t_i, da_i) \beta_{-i}(t_{-i}, da_{-i}) \mu_{-i}(dt_{-i}).$$

Furthermore, notice that

$$\Pi_i(\gamma_1, \dots, \gamma_n) = \int \Pi_i^b(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

In the special case where player i chooses a pure behavioral strategy p_i , his expected payoff is

$$\Pi_i^b(p_i, \beta_{-i} \mid t_i) = \int \pi_i(t, (p_i(t_i), a_{-i})) f(t) \beta_{-i}(t_{-i}, da_{-i}) \mu_{-i}(dt_{-i}).$$

One can also define $\Pi_i^b(\sigma_i, \beta_{-i} \mid t_i)$ and $\Pi_i^b(a_i, \beta_{-i} \mid t_i)$ respectively for every probability measure σ_i on \mathcal{A}_i and every action $a_i \in A_i$.

A probability measure σ_i on \mathcal{A}_i is called a best response of player i for type $t_i \in T_i$ against a behavioral strategy profile τ_{-i} , if for every probability measure σ'_i on \mathcal{A}_i we have

$$\Pi_i^b(\sigma_i, \tau_{-i} \mid t_i) \geq \Pi_i^b(\sigma'_i, \tau_{-i} \mid t_i).$$

The set of such best responses is denoted by $BR_i(t_i, \tau)$. A behavioral strategy β_i is called a best response of player i against the behavioral strategy profile τ_{-i} , if $\beta_i(t_i, \cdot) \in BR_i(t_i, \tau)$ for every $t_i \in T_i$. A behavioral strategy profile

⁵Dudley's notation and terminology in [15] differs from ours. Part (II) of Theorem 10.2.1. in Dudley [15] with our terminology can be expressed as follows: For every measurable function $g : T_i \times A_i \rightarrow \mathbf{R}$ we have

$$\int_{T_i \times A_i} g d\gamma_i = \int_{T_i} \int_{A_i} g \beta_i(t_i, da_i) \mu_i(dt_i).$$

$\beta = (\beta_1, \dots, \beta_n)$ is a *Bayesian Nash Equilibrium (BNE)* if β_i is a best response to β_{-i} for every player i .

A distributional strategy γ_i of player i is a best response against a distributional strategy profile η_{-i} , if $\Pi_i(\gamma_i, \eta_{-i}) \geq \Pi_i(\tilde{\gamma}_i, \eta_{-i})$, for every distributional strategy $\tilde{\gamma}_i$ of player i . Let $BR_i : \Delta \rightarrow \Delta_i$ denote the best response correspondence for player i on the set of distributional strategies. Moreover, define $BR : \Delta \rightarrow \Delta$, for every $\gamma \in \Delta$, as $BR(\gamma) = \times_{i=1}^n BR_i(\gamma)$.

We say that a behavioral strategy β_i (weakly) dominates behavioral strategy $\hat{\beta}_i$ if, for their respective induced distributional strategies γ_i and $\hat{\gamma}_i$, it holds that

$$\Pi_i(\gamma_i, \gamma_{-i}) \geq \Pi_i(\hat{\gamma}_i, \gamma_{-i})$$

for all $\gamma_{-i} \in \Delta_{-i}$, and with a strict inequality for at least one $\gamma_{-i} \in \Delta_{-i}$. A behavioral strategy β_i is called (weakly) dominant if it dominates every other behavioral strategy of player i . A behavioral strategy β_i is undominated if there is no other behavioral strategy that dominates β_i .

3 Distributional strictly perfect equilibrium

In this section, we define perfect equilibrium for Bayesian games in terms of behavioral strategies. Since only the induced distribution on opponents' actions matter for computation of expected payoffs, we use distributional strategies as a tool to verify whether the behavioral strategy profile satisfies the conditions of trembling hand perfect equilibrium.

Let $\mathcal{C}_b(T_i \times A_i)$ be the set of all continuous and bounded functions with domain $T_i \times A_i$. The weak topology on Δ_i is the coarsest topology for which all the maps $\gamma_i \rightarrow \int h d\gamma_i$, $h \in \mathcal{C}_b(T_i \times A_i)$ are continuous. Since $T_i \times A_i$ is a separable metric space, the weak topology on Δ_i is separable and metrizable with the weak (Prohorov) metric ⁶, denoted by ρ^w .

⁶See Aliprantis and Border [1]. Let μ and ν be two probability measures on a σ -field Σ . Weak (Prohorov) metric is defined for μ and ν by $\rho^w(\mu, \nu) = \inf\{\varepsilon > 0 \mid \forall B \in \Sigma : \mu(B) < \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) < \mu(B^\varepsilon) + \varepsilon\}$.

Definition 3.1 A behavioral strategy profile $\beta = (\beta_1, \dots, \beta_n)$ is called *distributional strictly perfect equilibrium (DSPE)* if, given the induced profile $\gamma = (\gamma_1, \dots, \gamma_n)$ of distributional strategies, for every completely mixed strategy profile $\nu = (\nu_1, \dots, \nu_n)$ and every sequence $(\varepsilon_i^k)_{k=1}^\infty$ of real numbers converging to zero, the sequence $(\gamma_i^k)_{k=1}^\infty$ defined by $\gamma_i^k = (1 - \varepsilon_i^k) \cdot \gamma_i + \varepsilon_i^k \cdot \nu_i$ for every i is such that

$$\lim_{k \rightarrow \infty} \rho^w(\gamma_i^k, BR_i(\gamma^k)) = 0,$$

in which $\gamma^k = (\gamma_1^k, \dots, \gamma_n^k)$.

A behavioral strategy profile is a distributional strictly perfect BNE if it is both distributionally strictly perfect and a BNE.

We can show that DSPE is a refinement of BNE for almost all types of the players:

Proposition 3.2 *For every DSPE β , there is a BNE $\hat{\beta}$ such that for every player i we have $\beta_i(t_i, \cdot) = \hat{\beta}_i(t_i, \cdot)$, for μ_i -a.e. $t_i \in T_i$.*

Proof. : See appendix.

The above equilibrium concept may not exist even in finite games. DSPE is equivalent to strictly perfect equilibrium in finite normal form games⁷ and we see that in example 1.5.5 in van Damme [35] there is no strictly perfect equilibrium. However, we can define a weaker version of DSPE that is called distributional perfect equilibrium and prove that for every Bayesian game, the set of distributional BNEs is not empty under two additional assumptions: First, the measure μ to be absolutely continuous⁸ with respect to $\hat{\mu} = \times_{i=1}^n \mu_i$.⁹ Second, payoffs to be continuous in actions for every type, that is, $\pi_i(t, \cdot) : A \rightarrow \mathbf{R}$ is continuous for every $t \in T$.

A behavioral strategy profile $\beta = (\beta_1, \dots, \beta_n)$ is called distributional perfect if for the induced distributional strategy profile $\gamma = (\gamma_1, \dots, \gamma_n)$ there exists a

⁷The proof is available upon request

⁸Let μ and ν be two measures on a σ -field Σ . The measure μ is absolutely continuous with respect to ν if for every $G \in \Sigma$, $\nu(G) = 0$ implies $\mu(G) = 0$.

⁹In Milgrom and Weber [29] this property is called Absolutely Continuous Information. Absolutely continuous information is a rather weak assumption. It is satisfied when the type spaces are finite, or when each player's type is drawn independently from his type space.

sequence $(\gamma^k)_{k=1}^\infty = (\gamma_1^k, \dots, \gamma_n^k)_{k=1}^\infty$ of completely mixed distributional strategy profiles such that for every player i it holds that

- (i) $\lim_{k \rightarrow \infty} \rho^w(\gamma_i^k, \gamma_i) = 0,$
- (ii) $\lim_{k \rightarrow \infty} \rho^w(\gamma_i^k, BR_i(\gamma^k)) = 0.$

A behavioral strategy profile is a distributional perfect BNE if it is both distributional perfect and a BNE.

We can show that the set of distributional perfect BNEs is nonempty. The proof can be found in the appendix and it is based on the existence proof in Simon and Stinchcombe [34] and Milgrom and Weber [29].

4 Interdependent value auctions

In this section we study symmetric second-price auctions with interdependent values for two bidders and apply strict distributional perfection in this context. Let Λ be the following sealed-bid second-price auction for a single indivisible object. There are 2 bidders. Prior to bidding, each bidder i receives a private signal $t_i \in T_i = [\underline{t}, \bar{t}]$, which is called the type of bidder i . Signals are drawn independently according to a distribution μ_i . Bidder i 's valuation of the object may depend on both types and is denoted by $v_i(t_1, t_2)$. Each bidder i , after observing his own type, submits a bid from a set A_i independently of his opponent, where A_i is sufficiently large in the sense that it contains the range of v_i . Given the bids submitted by the bidders, the highest bid wins the auction for the price equal to the second highest bid.¹⁰ Suppose the following assumptions in this auction:

- [1] $v_1(t, t) = v_2(t, t)$, for every $t \in [\underline{t}, \bar{t}]$.
- [2] v_i is continuously differentiable.
- [3] v_i is strictly increasing in t_i and increasing in t_j for $j \neq i$.

¹⁰In the event of a tie for the highest bid, the winner is chosen according to a probability distribution which may depend on the identity of the highest bidder and his type as well. In our setting, a tie will only occur with probability 0, even given the type of one bidder, which makes the specification of this tie-breaking rule irrelevant for the expected profits.

[4] For every t_1, t_2 such that $v_1(t_1, t_2) = v_2(t_1, t_2)$, we have

$$\frac{\partial v_i}{\partial t_i}(t_1, t_2) > \frac{\partial v_j}{\partial t_i}(t_1, t_2).$$

Condition [4] is called the pairwise single crossing condition. Note that this condition in combination with condition [1] ensures that the bidder with the higher type has the higher ex post valuation.¹¹ Moreover, the pair of strategies (b_1, b_2) in which $b_i = v_i(t_i, t_i)$ for every bidder i , is an efficient equilibrium.¹² More information on pairwise single crossing condition can be found in for example Krishna [21].

In the auction Λ the set of available bids for bidder i is an interval $A_i = [\underline{a}, \bar{a}]$, where $v_i(\underline{t}, \underline{t}) = \underline{a}$ and $v_i(\bar{t}, \bar{t}) = \bar{a}$, for $i = 1, 2$. In this auction bidders do not have any dominant strategy (disregarding the classic case of private values for a moment), and there are many equilibria.

In the next two subsections we show how DSPE can be used to refine among these equilibria. We show that the efficient equilibrium introduced in Milgrom [28] is a DSPE. In contrast with this, we show that within the class of inefficient equilibria introduced in Birulin [13] none of them are DSPE.

4.1 Equilibrium refinement I: equilibrium selection

Define the strategy pair (b_1, b_2) by $b_1(t_1) = v_1(t_1, t_1)$ and $b_2(t_2) = v_2(t_2, t_2)$. In Proposition 4.1 we show that this strategy profile is a DSPE. We also argue that the BNE is not in dominant strategies (unless we are considering the special case of pure private values). Thus, dominance cannot be used as a means to refine among equilibria in this context.

Proposition 4.1 *The strategy pair (b_1, b_2) , where $b_i = v_i(t_i, t_i)$ for $i = 1, 2$ is a BNE and DSPE. Further, b_i is undominated for every bidder i .*

Proof. First we show that (b_1, b_2) is a BNE. Define $w(t) = v_1(t, t)$ for every $t \in [\underline{t}, \bar{t}]$. Clearly w is an increasing bijection from $[\underline{t}, \bar{t}]$ to $[\underline{a}, \bar{a}]$. Suppose that

¹¹Thus, common value auctions do not satisfy pairwise single crossing condition.

¹²An outcome in the auction is called efficient if the winner is the one with the highest ex post valuation.

bidder 1 uses bid function $b_1(t_1) = v_1(t_1, t_1)$. The expected payoff of bidder 2 having type t_2 and bidding p_2 is:

$$\Pi_2^b(b_1, p_2 \mid t_2) = \int_{\underline{t}}^{w^{-1}(p_2)} (v_2(t_1, t_2) - v_1(t_1, t_1)) \mu_1(dt_1).$$

As v_2 is strictly increasing in the second argument, and moreover

$$v_2(t_1, t_2) - v_1(t_1, t_1) = v_2(t_1, t_2) - v_2(t_1, t_1),$$

it is clear that $v_2(t_1, t_2) - v_1(t_1, t_1) \geq 0$ precisely when $t_1 \leq t_2$. So, the maximum of $\Pi_2^b(b_1, p_2 \mid t_2)$ is obtained by choosing p_2 such that $w^{-1}(p_2) = t_2$. Hence, the optimal bidding strategy is

$$b_2(t_2) = w(t_2) = v_1(t_2, t_2) = v_2(t_2, t_2).$$

Now, we prove that the BNE (b_1, b_2) is a DSPE. Let γ_i be the distributional strategy induced by b_i . For every $k \in \mathbf{N}$ define $\gamma_1^k = (1 - \varepsilon_k) \cdot \gamma_1 + \varepsilon_k \cdot \nu_1$, where $\varepsilon_k \in (0, 1)$, $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, and ν_1 is any completely mixed distributional strategy on the product space $T_1 \times A_1$.¹³

It suffices to show that for each k there is a strategy $\eta_2^k \in BR_2(\gamma_1^k)$ such that

$$\lim_{k \rightarrow \infty} \rho^w(\gamma_2, \eta_2^k) = 0.$$

The proof is in two steps. Let t_2 be a type of bidder 2.

A. We show that $p_2 = b_2$ is the unique maximizer of $F(p_2) = \Pi_2^b(b_1, p_2 \mid t_2)$ and $F(p_2)$ is continuous. Recall that

$$F(p_2) = \Pi_2^b(b_1, p_2 \mid t_2) = \int_{\underline{t}}^{w^{-1}(p_2)} (v_2(t_1, t_2) - v_1(t_1, t_1)) \mu_1(dt_1),$$

where w denotes the strictly increasing function $w(t) = v_1(t, t)$. Since the integrand

$$v_2(t_1, t_2) - v_1(t_1, t_1) = v_2(t_1, t_2) - v_2(t_1, t_1)$$

is continuous and strictly decreasing in t_1 , and $w^{-1}(p_2)$ is continuous and strictly increasing in p_2 , it is clear that $F(p_2)$ is continuous. Further, since the integrand

¹³That is, ν_1 is a completely mixed measure on $T_1 \times A_1$ and $\nu_1(S_1 \times A_1) = \mu_1(S_1)$ for every $S_1 \in \mathcal{T}_1$.

is zero at $t_1 = t_2$, it is also clear that $F(p_2)$ has a unique maximum location at value p_2 with $w^{-1}(p_2) = t_2$. Thus,

$$p_2 = w(t_2) = v_1(t_2, t_2) = v_2(t_2, t_2) = b_2(t_2)$$

is the unique maximizer of $F(p_2)$.

B. We prove that for each k there is a strategy $\eta_2^k \in BR_2(\gamma_1^k)$ such that

$$\lim_{k \rightarrow \infty} \rho^w(\gamma_2, \eta_2^k) = 0.$$

Let p_2 be a pure behavioral strategy for bidder 2 and σ_2 be the distributional strategy induced by p_2 . Then, the expected payoff of bidder 2 choosing σ_2 against γ_1^k is:

$$\begin{aligned} \Pi_2(\gamma_1^k, \sigma_2) &= \int \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) d\gamma_1^k d\sigma_2 \\ &= \int_{T_2} \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) d\gamma_1^k \mu_2(dt_2) \\ &= \int_{T_2} \Pi_2^b(\beta_1^k, p_2 | t_2) \mu_2(dt_2), \end{aligned}$$

where β_1^k is a behavioral strategy corresponding to γ_1^k for every $k \in \mathbf{N}$. Let b_2^k be any maximizer of $\Pi_2^b(\beta_1^k, p_2 | t_2)$ for every k , where

$$\begin{aligned} \Pi_2^b(\beta_1^k, p_2 | t_2) &= (1 - \varepsilon_k) \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) d\gamma_1 \\ &\quad + \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) d\nu_1 \\ &= (1 - \varepsilon_k) \int_{\underline{t}}^{w^{-1}(p_2)} (v_2(t_1, t_2) - v_1(t_1, t_1)) \mu_1(dt_1) \\ &\quad + \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) d\nu_1 \\ &= (1 - \varepsilon_k) \Pi_2^b(b_1, p_2 | t_2) + \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) d\nu_1. \end{aligned}$$

Note that the maximizer b_2^k exists, because $\Pi_2^b(\beta_1^k, p_2 | t_2)$ is a continuous function over the compact set T_2 . We know that $p_2 = b_2$ is the unique maximizer of $\Pi_2^b(b_1, p_2 | t_2)$. Thus, according to Lemma 6.1, we have that b_2^k converges to b_2 when $k \rightarrow \infty$. Now, let η_2^k be the induced distributional strategy by b_2^k . As

$b_2^k \in BR_2(t_2, \beta_1^k)$, then we have $\eta_2^k \in BR_2(\gamma_1^k)$. Furthermore, since $b_2^k(t_2)$ converges to $b_2(t_2)$ for every $t_2 \in T_2$, we conclude $\rho^w(\gamma_2, \eta_2^k) \rightarrow 0$ when $k \rightarrow \infty$. With a similar argument for bidder 2, we can conclude that (b_1, b_2) is a DSPE.

Finally, as bidding b_2 is the unique best response for bidder 2 against b_1 , we conclude that b_2 is undominated. A similar argument holds for b_1 . ■

In the next proposition we prove that this strategy pair is not in dominant strategies, unless we are in the pure private values case.

Proposition 4.2 *Suppose that the valuation function v_i is not private values. Then the strategy $b_i = v_i(t_i, t_i)$ is not dominant.*

Proof. Assume that v_1 is not private values. We prove that $b_1 = v_1(t_1, t_1)$ is not a dominant strategy. In particular, we construct a bid function d_2 for bidder 2 and an open set I_1 of types for bidder 1, such that there is a bid $a \in A_1$ for bidder 1 that yields a strictly higher payoff against d_2 than $b_1(t_1) = v_1(t_1, t_1)$ for any type $t_1 \in I_1$. This then is a sufficient argument.

Since v_1 is not private values, by condition [3] there is an s_1 in T_1 and an open set B_2 in T_2 such that the map $v_1(s_1, \cdot): T_2 \rightarrow A_2$ is strictly increasing on B_2 .

Suppose that $B_2 \cap [\underline{t}, s_1)$ is not empty. Then we may assume that $t_2 < s_1$ for all $t_2 \in B_2$. Then, for every $t_2 \in B_2$ we have $v_1(s_1, t_2) < v_1(s_1, s_1)$. Define

$$y(t_1) = \int_{\underline{t}}^{t_1} v_1(t_1, t_2) \mu_2(dt_2).$$

Since $v_1(s_1, t_2) \leq v_1(s_1, s_1)$ for every $t_2 < s_1$, and $v_1(s_1, t_2) < v_1(s_1, s_1)$ for every $t_2 \in B_2$, it follows that

$$y(s_1) = \int_{\underline{t}}^{s_1} v_1(s_1, t_2) \mu_2(dt_2) < v_1(s_1, s_1) = b_1(s_1).$$

So, $y(s_1) < b_1(s_1)$. Since $\underline{t} < s_1$ and b_1 is strictly increasing and continuous, there is $u_1 < s_1$ with $b_1(u_1) > y(s_1)$.

Take $I_1 = (u_1, s_1)$. Then, since y and b_1 are strictly increasing functions, for every $t_1 \in I_1$ we have $y(t_1) < y(s_1) < b_1(u_1) < b_1(t_1)$.

Take the (constant) bid function $d_2(t_2) = b_1(u_1)$ for every $t_2 \in T_2$ for bidder 2. Take $t_1 \in I_1$ and any $t_2 \in T_2$. Since bidder 1 bids according to $b_1(t_1)$, and $d_2(t_2) = b_1(u_1) < b_1(t_1)$, bidder 1 wins the auction. Consequently, his expected payoff against d_2 is

$$\int_{T_2} (v_1(t_1, t_2) - b_1(u_1)) \mu_2(dt_2) = y(t_1) - b_1(u_1) < 0.$$

A similar argument applies to the case where $B_2 \cap (s_1, \bar{t}]$ is not empty. This concludes the proof. ■

4.2 Equilibrium refinement II: discarding equilibria

Birulin in [13] on page 678 introduces the following class of discontinuous asymmetric strategy profiles:

For given $s_1, s_2 \in [t, \bar{t}]$ with $s_1 < s_2$, define the strategy pair (\hat{b}_1, \hat{b}_2) by

$$\hat{b}_1(t_1) = \begin{cases} v_2(t_1, s_2) & \text{if } t_1 \in [s_1, s_2] \\ v_1(t_1, t_1) & \text{otherwise,} \end{cases} \quad \hat{b}_2(t_2) = \begin{cases} v_1(s_1, t_2) & \text{if } t_2 \in [s_1, s_2] \\ v_2(t_2, t_2) & \text{otherwise.} \end{cases}$$

Birulin [13] shows that, under pairwise single crossing condition, the strategy pair (\hat{b}_1, \hat{b}_2) is an ex post equilibrium that allocates the object inefficiently. Moreover, this equilibrium may be in undominated strategies. In the equilibrium above, bidder 1 overbids and bidder 2 underbids for types between s_1 and s_2 . In the proposition below we prove that strict distributional perfection rules out these equilibria.¹⁴ The intuition behind this result is that if there is a small chance that one bidder bids $b_i = v_i(t_i, t_i)$ for $t_i \in [s_1, s_2]$ instead of \hat{b}_i , then \hat{b}_j will no longer be a best response for bidder j , where $j \neq i$. In other words, underbidding (overbidding) would not be a best response against overbidding (underbidding) if even with a tiny probability one bidder bids according to the efficient bidding strategy $b_i = v_i(t_i, t_i)$. As an extreme case, we see that the

¹⁴In finite games, an equilibrium is perfect if and only if it is in undominated strategies. This is in general not true in games with infinite action spaces, even if undominatedness is replaced by limit undominatedness. Bajoori et. al in [5] give a detailed analysis on this issue. They prove that, in normal form games under appropriate assumptions, every weak perfect equilibrium is in limit undominated strategies, but not vice versa. Example 6 in Bajoori et. al [5] provides the relevant counter-example.

ex post equilibrium ¹⁵ in which one bidder always bids \bar{a} and the other bidder always bids \underline{a} is not a DSPE.

Proposition 4.3 *The equilibrium (\hat{b}_1, \hat{b}_2) is not a DSPE.*

Proof. We prove the proposition in two parts. Before we need to define the followings:

Let $c(t_2) = v_2(s_1, t_2)$. For every $\lambda > 0$ define $\varepsilon = \varepsilon(\lambda)$ to be

$$\varepsilon = \frac{v_2(s_1, s_1 + \lambda) - v_1(s_1, s_1 + \lambda)}{2} = \frac{c(s_1 + \lambda) - \hat{b}_2(s_1 + \lambda)}{2}.$$

By pairwise single crossing we have $\varepsilon > 0$, also smaller λ generates smaller ε . Therefore, we can choose λ small enough such that

1. $s_1 + \lambda + \varepsilon < s_2 - \varepsilon$,
2. $\mu_2(s_1 + \varepsilon + \lambda, s_2 - \varepsilon) > \varepsilon$,

Moreover, for every $t_2 \in (s_1 + \lambda, s_2)$ define

$$B_{t_2} = \left(\hat{b}_2(t_2) - \varepsilon, \hat{b}_2(t_2) + \varepsilon \right).$$

Part A. We show that there is a completely mixed behavioral strategy $\hat{\eta}_1$ such that if bidder 1 bids according to

$$\beta_1^k(t_1) = (1 - 2\delta_k)\hat{b}_1(t_1) + \delta_k b_1(t_1) + \delta_k \hat{\eta}_1(t_1),$$

where $b_1(t_1) = v_1(t_1, t_1)$, and $\delta_k \rightarrow 0$ when $k \rightarrow \infty$, then for every $t_2 \in (s_1 + \lambda, s_2)$, bidding according to $\tilde{b}_2(t_2) = v_2(s_1, t_2)$ generates higher expected payoff than bidding $a_2 \in (B_{t_2})^\varepsilon$ against β_1^k for every k .

By pairwise single crossing, we have $\hat{b}_2(t_2) = v_1(s_1, t_2) < v_2(s_1, t_2) = c(t_2)$. Also, for every $a_2 \in (B_{t_2})^\varepsilon$, we have $a_2 < c(t_2)$. Let η_1 be any completely mixed behavioral strategy. Define

$$D_1(a_2) = \int_{\underline{t}}^{b_1^{-1}(c_2)} (v_2 - v_1(t_1, t_1)) \mu(dt_1) - \int_{\underline{t}}^{b_1^{-1}(a_2)} (v_2 - v_1(t_1, t_1)) \mu(dt_1),$$

¹⁵This kind of equilibrium is often called a wolf and sheep equilibrium.

and

$$D_2(a_2) = \int_{T_1} \int_{a_2 > a_1} (v_2 - a_1) \eta_1(t_1, da_1) \mu(dt_1) - \int_{T_1} \int_{c_2 > a_1} (v_2 - a_1) \eta_1(t_1, da_1) \mu(dt_1).$$

Since $a_2 < c(t_2)$ and the function $b_1(t) = v_1(t, t)$ is a strictly increasing bijection, it is clear that $b_1^{-1}(a_2) < b_1^{-1}(c_2)$. Also, in the first integral in $D_1(a_2)$ we have $v_2(t_1, t_2) - v_1(t_1, t_1) > 0$: having $t_1 < b_1^{-1}(c_2)$ implies $b_1(t_1) < c_2$. Also, we know

$$c_2 = v_2(s_1, t_2) \leq v_2(t_2, t_2) = v_1(t_2, t_2) = b_1(t_2).$$

Therefore, $b_1(t_1) < b_1(t_2)$. Since b_1 is strictly increasing, then we can conclude that $t_1 < t_2$. Consequently, we have $v_1(t_1, t_1) = v_2(t_1, t_1) < v_2(t_1, t_2)$. Hence, we have $D_1(a_2) > 0$, for every $a_2 \in (B_{t_2})^\varepsilon$. So we can choose $\xi > 0$ small enough such that

$$\inf_{a_2 \in (B_{t_2})^\varepsilon} D_1(a_2) > \xi \sup_{a_2 \in (B_{t_2})^\varepsilon} D_2(a_2).$$

Let $\hat{\eta}_1 = \xi \cdot \eta_1$. Therefore, for every $a_2 \in (B_{t_2})^\varepsilon$ we have

$$\begin{aligned} & \int_{\underline{t}}^{b_1^{-1}(c_2)} (v_2 - v_1(t_1, t_1)) \mu(dt_1) - \int_{\underline{t}}^{b_1^{-1}(a_2)} (v_2 - v_1(t_1, t_1)) \mu(dt_1) > \\ & \int_{T_1} \int_{a_2 > a_1} (v_2 - a_1) \hat{\eta}_1(t_1, da_1) \mu(dt_1) - \int_{T_1} \int_{c_2 > a_1} (v_2 - a_1) \hat{\eta}_1(t_1, da_1) \mu(dt_1) \quad (1) \end{aligned}$$

Assume bidder 1 bids according to

$$\beta_1^k(t_1) = (1 - 2\delta_k) \hat{b}_1(t_1) + \delta_k b_1(t_1) + \delta_k \hat{\eta}_1(t_1),$$

where $b_1(t_1) = v_1(t_1, t_1)$, and $\delta_k \rightarrow 0$ when $k \rightarrow \infty$. We prove that for every $t_2 \in (s_1 + \lambda, s_2)$, bidding according to $\tilde{b}_2(t_2) = c(t_2) = v_2(s_1, t_2)$ generates higher expected payoff compared with bidding $a_2 \in (B_{t_2})^\varepsilon$ against β_1^k for every k .

Take any $t_2 \in (s_1 + \lambda, s_2)$. According to strategy β_1^k , bidder 1 bids \hat{b}_1 with probability $1 - 2\delta_k$. It is easy to see that for $t_1 < s_1$, we have $\tilde{b}_2 > \hat{b}_1(t_1)$, therefore bidding \tilde{b}_2 wins against $\hat{b}_1(t_1)$ for $t_1 < s_1$. However, for $t_1 \in (s_1, s_2)$, we have

$$\tilde{b}_2 = v_2(s_1, t_2) < v_2(s_1, s_2) \leq v_2(t_1, s_2) = \hat{b}_1(t_1).$$

Also, for $t_1 > s_2$, we have $\tilde{b}_2 < \hat{b}_1(t_1)$. Hence, bidding \tilde{b}_2 loses against $\hat{b}_1(t_1)$ for any $t_1 > s_1$. Moreover, according to strategy β_1^k , bidder 1 bids $b_1(t_1) = v_1(t_1, t_1)$ with probability δ_k . Bidding \tilde{b}_2 wins against $b_1(t_1)$, if $\tilde{b}_2 > b_1(t_1)$, in other words if $t_1 < b_1^{-1}(\tilde{b}_2)$. So, bidder 2's expected payoff by bidding \tilde{b}_2 against β_1^k is

$$\begin{aligned} \Pi_2^b(\beta_1^k, \tilde{b}_2 \mid t_2) &= (1 - 2\delta_k) \int_{\underline{t}}^{s_1} (v_2 - v_1(t_1, t_1)) \mu(dt_1) \\ &+ \delta_k \int_{\underline{t}}^{b_1^{-1}(\tilde{b}_2)} (v_2 - v_1(t_1, t_1)) \mu(dt_1) \\ &+ \delta_k \int_{T_1} \int_{\tilde{b}_2 > a_1} (v_2 - a_1) \hat{\eta}_1(t_1, da_1) \mu(dt_1). \end{aligned}$$

Similarly, for any $a_2 \in (B_{t_2})^\varepsilon$, bidder 2's expected payoff by bidding a_2 against β_1^k is

$$\begin{aligned} \Pi_2^b(\beta_1^k, a_2 \mid t_2) &= (1 - 2\delta_k) \int_{\underline{t}}^{s_1} (v_2 - v_1(t_1, t_1)) \mu(dt_1) \\ &+ \delta_k \int_{\underline{t}}^{b_1^{-1}(a_2)} (v_2 - v_1(t_1, t_1)) \mu(dt_1) \\ &+ \delta_k \int_{T_1} \int_{a_2 > a_1} (v_2 - a_1) \hat{\eta}_1(t_1, da_1) \mu(dt_1). \end{aligned}$$

From inequality (1) we conclude that

$$\Pi_2^b(\beta_1^k, a_2 \mid t_2) < \Pi_2^b(\beta_1^k, \tilde{b}_2 \mid t_2),$$

for every $t_2 \in (s_1 + \lambda, s_2)$ and $a_2 \in (B_{t_2})^\varepsilon$.

Part B. By the Definition 3.1, to prove (\hat{b}_1, \hat{b}_2) is not a DSPE we show that for the induced distributional strategy profile $(\hat{\gamma}_1, \hat{\gamma}_2)$, there are completely mixed distributional strategies ν_1 and ν_2 and sequences of real numbers ε_1^k and ε_2^k converging to zero and an $\varepsilon > 0$, such that for $\gamma_1^k = (1 - \varepsilon_1^k) \cdot \gamma_1 + \varepsilon_1^k \cdot \nu_1$ and $\gamma_2^k = (1 - \varepsilon_2^k) \cdot \gamma_2 + \varepsilon_2^k \cdot \nu_2$ we have $\rho^w(\gamma_2^k, BR_2(\gamma_1^k)) > \varepsilon$.

Let $\hat{\eta}_1$ be the completely mixed behavioral strategy that we constructed in part A and assume bidder 1 bids according to

$$\beta_1^k(t_1) = (1 - 2\delta_k)\hat{b}_1(t_1) + \delta_k b_1(t_1) + \delta_k \hat{\eta}_1(t_1),$$

where $b_1(t_1) = v_1(t_1, t_1)$, and $\delta_k \rightarrow 0$ when $k \rightarrow \infty$.

Let $\tilde{\beta}_2^k(t_2, \cdot) \in BR_2(t_2, \beta_1^k)$ for every t_2 and every k . Moreover, let $\gamma_1^k, \tilde{\gamma}_2^k, \gamma_1$, and $\hat{\nu}_1$ be the distributional strategies induced by $\beta_1^k, \tilde{\beta}_2^k, b_1$, and $\hat{\eta}_1$ respectively. One can write

$$\gamma_1^k = (1 - 2\delta_k)\hat{\gamma}_1 + \delta_k\gamma_1 + \delta_k\hat{\nu}_1$$

So let $\nu_1 = \frac{\gamma_1 + \hat{\nu}_1}{2}$ and $\varepsilon_k^1 = 2\delta_k$. Also, let ν_2 be any completely mixed strategy and $\varepsilon_k^2 = 0$ for every k . Then we have $\gamma_2^k = \hat{\gamma}_2$ for every k .

To prove that (\hat{b}_1, \hat{b}_2) is not a DSPE, it is enough to show that for every k $\rho^w(\gamma_2^k, BR_2(\gamma_1^k)) > \varepsilon$, or equivalently, it is enough to show that for every k $\rho^w(\hat{\gamma}_2, BR_2(\gamma_1^k)) > \varepsilon$. Moreover, we have $\tilde{\gamma}_2^k \in BR_2(\gamma_1^k)$, therefore it is enough to show that for every k , $\rho^w(\hat{\gamma}_2, \tilde{\gamma}_2^k) > \varepsilon$.

Take any $t_2 \in (s_1 + \lambda, s_2)$. For every $a_2 \in (B_{t_2})^\varepsilon$, according to part A, a_2 cannot be a best response against β_1^k , for every k . Hence,

$$\tilde{\beta}_2^k(t_2, (B_{t_2})^\varepsilon) = 0, \text{ for every } t_2 \in (s_1 + \lambda, s_2).$$

Let $B = \{(t_2, a_2) \mid t_2 \in (s_1 + \lambda + \varepsilon, s_2 - \varepsilon), a_2 \in B_{t_2}\}$. Clearly, B is measurable. As $\hat{b}_2(t_2)(B_{t_2}) = 1$ ¹⁶ and $\tilde{\beta}_2^k(t_2, (B_{t_2})^\varepsilon) = 0$, we have

$$\hat{\gamma}_2(B) = \int_{s_1 + \lambda + \varepsilon}^{s_2 - \varepsilon} \hat{b}_2(t_2)(B_{t_2})\mu_2(dt_2) = \mu_2(s_1 + \lambda + \varepsilon, s_2 - \varepsilon)$$

and

$$\tilde{\gamma}_2^k(B^\varepsilon) = \int_{s_1 + \lambda}^{s_2} \tilde{\beta}_2^k(t_2, (B_{t_2})^\varepsilon)\mu_2(dt_2) = 0.$$

Consequently,

$$\hat{\gamma}_2(B) > \tilde{\gamma}_2^k(B^\varepsilon) + \varepsilon,$$

then $\rho^w(\hat{\gamma}_2, \tilde{\gamma}_2^k) > \varepsilon$ for every k . The proof is complete. \blacksquare

Uniqueness: We do not know whether the DSPE is unique in our model. In general, establishing uniqueness of equilibrium is difficult for auction games. For example, uniqueness of BNE for the first price sealed bid auction with independent types is only known within a specific category of bid functions.¹⁷

¹⁶Notice that $\hat{b}_2(t_2)$ is interpreted as a behavioral strategy that assigns probability 1 on $\hat{b}_2(t_2)$ for every $t_2 \in T_2$.

¹⁷See e.g. [37]

5 Discussion

We introduce the concept of distributional strictly perfect equilibrium (DSPE) as a refinement of Bayesian Nash equilibrium. We apply this concept in symmetric second-price auctions with interdependent values for two bidders to select the efficient BNE, and to rule out the less intuitive equilibria. In particular, we show that the efficient BNE discussed in Milgrom [28] is a DSPE. We also show that DSPE rules out all the discontinuous, inefficient, equilibria introduced by Birulin [13]. These equilibria feature overbidding and underbidding. Intuitively, we show that underbidding (overbidding) cannot be a best response against overbidding (underbidding), if one bidder bids according to the efficient equilibrium with a small probability.

In the sealed bid second price auction with more than two bidders with interdependent values, the only part that matters to a bidder is the amount added to his valuation by the realized types of opponents. The induced probability distribution, as well as the distribution on highest bids, become more complicated. But conceptually, adding more players does not seem to introduce any effects that aren't already present in the two-bidder case. For that reason, we decided to analyze the two-bidder case.

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6 Appendix

Lemma 6.1 *Let f and g be two continuous functions on an interval $[a, b]$. For every $k \in \mathbf{N}$, define*

$$f^k(x) = (1 - \varepsilon_k)f(x) + \varepsilon_k g(x)$$

in which $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Let y^k be a maximizer of f^k for every k . Suppose that function f has a unique maximizer $x = y$, then $y^k \rightarrow y$ when $k \rightarrow \infty$.

Proof. We have f^k converges to f uniformly. For every k , y^k is any maximizer of f^k , which exists since $[a, b]$ is compact. Consider any convergent subsequence $\{y^{k_\ell}\}_{k_\ell=1}^\infty$ of the sequence $\{y^k\}_{k=1}^\infty$. Assume y^{k_ℓ} converges to $y' \in [a, b]$ when $k_\ell \rightarrow \infty$. Then we have $f^{k_\ell}(y^{k_\ell}) \geq f^{k_\ell}(x)$ for every $x \in [a, b]$. Now, by uniform convergence, we have $f^{k_\ell}(y^{k_\ell}) \rightarrow f(y')$ and $f^{k_\ell}(x) \rightarrow f(x)$, when $k_\ell \rightarrow \infty$. Therefore, we have $f(y') \geq f(x)$ for every $x \in [a, b]$. Then, by uniqueness of maximizer we have $y' = y$. Thus, any convergent subsequence has the same limit y . Hence, we conclude that $\{y^k\}_{k=1}^\infty$ is convergent and converges to y when $k \rightarrow \infty$. ■

Lemma 6.2 *Consider a Bayesian game Γ . Let $\beta = (\beta_1, \dots, \beta_n)$ be a behavioral strategy profile and $\gamma = (\gamma_1, \dots, \gamma_n)$ be the induced distributional strategy profile. If $\gamma_i \in BR_i(\gamma)$, then $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for μ_i -a.e. $t_i \in T_i$.*

Proof. Suppose for every player i that $\gamma_i \in BR_i(\gamma)$. Then, we have

$$\Pi_i(\gamma_i, \gamma_{-i}) \geq \Pi_i(\tilde{\gamma}_i, \gamma_{-i}),$$

for every player i 's distributional strategy $\tilde{\gamma}_i$. Therefore, for every player i 's behavioral strategy $\tilde{\beta}_i$ we have

$$\int_{T_i} \Pi_i^b(\beta_i, \beta_{-i} | t_i) \mu_i(dt_i) \geq \int_{T_i} \Pi_i^b(\tilde{\beta}_i, \beta_{-i} | t_i) \mu_i(dt_i). \quad (2)$$

Theorem 9.1 in [36]¹⁸ implies that for every player i there is a behavioral strategy ζ_i such that $\zeta_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for every $t_i \in T_i$, which means having type $t_i \in T_i$

$$\Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) \geq \Pi_i^b(a_i, \beta_{-i} \mid t_i),$$

for every $a_i \in A_i$. To prove that $\beta_i(t_i, \cdot)$ for μ_i -a.e. $t_i \in T_i$ is a best response for every player i against β_{-i} , we show that $\Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) = \Pi_i^b(\beta_i, \beta_{-i} \mid t_i)$ for μ_i -a.e. $t_i \in T_i$. Suppose the opposite. First, let

$$S_i = \{t_i \in T_i \mid \Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) < \Pi_i^b(\beta_i, \beta_{-i} \mid t_i)\}.$$

Clearly, S_i is μ_i -measurable. Suppose $\mu_i(S_i) > 0$. Notice that given $t_i \in T_i$, for $\beta_i(t_i, \cdot)$ -a.e. $a_i \in A_i$ we have

$$\Pi_i^b(\beta_i, \beta_{-i} \mid t_i) = \Pi_i^b(a_i, \beta_{-i} \mid t_i).$$

Then, given $t_i \in S_i$, for $\beta_i(t_i, \cdot)$ -a.e. $a_i \in A_i$ we conclude that

$$\Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) < \Pi_i^b(a_i, \beta_{-i} \mid t_i),$$

which is a contradiction with optimality of ζ_i for every type. Second, let

$$S_i = \{t_i \in T_i \mid \Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) > \Pi_i^b(\beta_i, \beta_{-i} \mid t_i)\}$$

and suppose $\mu_i(S_i) > 0$. Therefore, we have

$$\int_{S_i} \Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) \mu_i(dt_i) > \int_{S_i} \Pi_i^b(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

Since $\zeta_i(t_i, \cdot)$ is a best response for every type $t_i \in T_i$ against β_{-i} , for every $a_i \in A_i$ we have

$$\Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) \geq \Pi_i^b(a_i, \beta_{-i} \mid t_i),$$

¹⁸Suppose (T, \mathcal{M}) is a measurable space, X is a topological space with the Borel σ -algebra \mathcal{B} , and F is a correspondence from T to X . Denote $GrF = \{(t, x) : x \in F(t)\}$. Suppose $u : GrF \rightarrow \mathbf{R}$ is $\mathcal{M} \otimes \mathcal{B}$ measurable and $u(t, \cdot)$ is upper semicontinuous on $F(t)$ for $t \in T$, and let $\nu(t) = \sup\{u(x, t) : x \in F(t)\}$ for $t \in T$. Theorem 9.1 in [36] states that if F is compact-valued and measurable, X is separable metric, and u is the limit of a decreasing sequence of Caratheodory maps, then ν is measurable and there exists $f : T \rightarrow X$ measurable such that $u(\cdot, f(\cdot)) = \nu$.

thus

$$\int_{T_i \setminus S_i} \Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) \mu_i(dt_i) \geq \int_{T_i \setminus S_i} \Pi_i^b(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

Consequently,

$$\int_{T_i} \Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) \mu_i(dt_i) > \int_{T_i} \Pi_i^b(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

This is a contradiction with (2). Overall, we showed that if there is a subset of T_i on which the expected payoff of player i , playing according to ζ_i is different from playing according to β_i , the measure of that subset is zero. In other words, for every i there exists a set $S_i \in \mathcal{T}_i$ with $\mu_i(S_i) = 0$ such that for every $t_i \in T_i \setminus S_i$ we have $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$. This completes the proof. ■

Lemma 6.3 *Consider a Bayesian game Γ . For every player i and every $\varepsilon > 0$, there is a set $E \in \mathcal{T}$ such that $\mu(E^c) < \varepsilon$ and the family $\{\pi_i(t, \cdot) \mid t \in E\}$ is equicontinuous.*¹⁹

Proof. Let $C(A)$ be the set of all continuous real functions on A . For every player i consider the mapping $t \rightarrow \pi_i(t, \cdot)$ which is a measurable mapping from T to $C(A)$. Therefore, the probability measure μ on T induces a probability measure on $C(A)$, say η . Since A is compact metric space, then $C(A)$ is a compact separable metric space that implies that η is tight. That means, for every $\varepsilon > 0$ there is a compact set $G \subset C(A)$ with $\eta(G^c) < \varepsilon$. Let E be the inverse image of G under the above mapping. Then we have $\mu(E^c) < \varepsilon$. Since G is compact, by Ascoli-Arzelà theorem one concludes that G is equicontinuous. This completes the proof. ■

In the following lemma, we use the following metrics: For every $t, s \in T$ and $a, b \in A$, define

$$d_T(t, s) = \left(\sum_{i=1}^n d_{T_i}^2(t_i, s_i) \right)^{\frac{1}{2}}, \quad d_A(a, b) = \left(\sum_{i=1}^n d_{A_i}^2(a_i, b_i) \right)^{\frac{1}{2}}.$$

Moreover, let $d_{T \times A}((t, a), (s, b)) = (d_T^2(t, s) + d_A^2(a, b))^{\frac{1}{2}}$.

¹⁹A family \mathfrak{F} of real functions on the metric space X is equicontinuous if to every $\varepsilon > 0$ corresponds a $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$ we have $|f(x) - f(y)| < \varepsilon$, for every $f \in \mathfrak{F}$.

Lemma 6.4 *Consider a Bayesian game Γ . For every player i and every $\varepsilon > 0$, there is a set $K \in \mathcal{T}$ with $\mu(K^c) < \varepsilon$ such that π_i is continuous on $K \times A$ for every i .*

Proof. Take $\varepsilon > 0$ and $a \in A$. First we prove that there is a set $K \in \mathcal{T}$ with $\mu(K^c) < \varepsilon$ such that the function $\pi_i(\cdot, a) : K \rightarrow \mathbf{R}$ is continuous. As A is separable, it has a countable dense subset, say B . Hence, there exists a sequence $(a_k)_{k=1}^\infty$ in B such that $a_k \neq a$, for every k , and $d_A(a_k, a) \rightarrow 0$ as $k \rightarrow \infty$. According to Lusin's Theorem for the collection of the functions $\{\pi_i(\cdot, a_k) \mid k = 1, 2, \dots\}$ there is a compact set $E \in \mathcal{T}$ with $\mu(E^c) < \frac{\varepsilon}{2}$ such that $\pi_i(\cdot, a_k)$ is continuous on E for every $k \in \mathbf{N}$. Moreover, by the lemma 6.3 we know that payoffs are equicontinuous, i.e. there is a set $F \in \mathcal{T}$ with $\mu(F^c) < \frac{\varepsilon}{2}$ such that the collection of the functions $\{\pi_i(t, \cdot) : A \rightarrow \mathbf{R} \mid t \in F\}$ is equicontinuous. Let $K = E \cap F$. It is clear that $\mu(K^c) < \varepsilon$. Moreover, because $\pi_i(t, \cdot)$ is continuous in actions for every $t \in K$, the sequence of the functions $(\pi_i(\cdot, a_k))_{k=1}^\infty$ converges pointwise to $\pi_i(\cdot, a)$ as $k \rightarrow \infty$. It is easy to check that equicontinuity of payoffs and the fact that $d_A(a_k, a) \rightarrow 0$ as $k \rightarrow \infty$, implies that the sequence of functions $(\pi_i(\cdot, a_k))_{k=1}^\infty$ uniformly converges to $\pi_i(\cdot, a)$ on K as $k \rightarrow \infty$. Hence, $\pi_i(\cdot, a) : K \rightarrow \mathbf{R}$ is continuous too.

Now, we prove that π_i is continuous at $(t, a) \in K \times A$. We know that $\{\pi_i(t, \cdot) \mid t \in K\}$ is equicontinuous, then there is a $\delta_1 > 0$ such that if $d_A(a, b) < \delta_1$, we have $|\pi_i(t, a) - \pi_i(t, b)| < \frac{\varepsilon}{2}$ for every $t \in K$. Also, we know $\pi_i(\cdot, a)$ is continuous at $t \in K$, so there is a $\delta_2 > 0$ such that if $d_T(t, s) < \delta_2$, we have $|\pi_i(t, a) - \pi_i(s, a)| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for every $(s, b) \in K \times A$ with $d_{T \times A}((t, a), (s, b)) < \delta$ we have $d_A(a, b) < \delta_1$ and $d_T(t, s) < \delta_2$. Consequently, we have

$$|\pi_i(t, a) - \pi_i(s, b)| < |\pi_i(t, a) - \pi_i(s, a)| + |\pi_i(s, a) - \pi_i(s, b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that π_i is continuous on $K \times A$. ■

Lemma 6.4 gives a slightly more general result than Lusin's theorem. Indeed, Lusin's theorem would guarantee the existence of a subset of $T \times A$ such that π_i is continuous on that subset, while Lemma 6.4 would do that for a set of the form $K \times A$, where $K \subseteq T$.

The following lemma is the basic continuity result in Milgrom and Weber [29]. Here we provide a detailed proof of that.

Lemma 6.5 *Consider a Bayesian game Γ . For every player i , the function $\Pi_i : \Delta \rightarrow \mathbf{R}$ is continuous with respect to the weak metric on Δ .*

Proof. To prove that Π_i is continuous at a distributional strategy profile γ , let $(\gamma^k)_{k=1}^\infty$ be a sequence of distributional strategy profiles converging to γ with respect to the weak metric as $k \rightarrow \infty$. Now, by Lemma 6.4, for every i and every $\delta > 0$, there exist a continuous and bounded function $v_\delta : T \times A \rightarrow \mathbf{R}$ and a set $K \in \mathcal{T}$ such that $\mu(K^c) < \delta$, and $v_\delta = \pi_i$ on $K \times A$.

As $\mu(K^c) < \delta$, we have $\gamma^k(K^c \times A) < \delta$, for every k . Similarly, $\gamma(K^c \times A) < \delta$. Moreover, because v_δ and π_i are bounded, there is an $M > 0$ such that $|\pi_i - v_\delta| \leq M$. Hence, for every k we have

$$\int_{T \times A} |\pi_i - v_\delta| f(t) d\gamma^k = \int_{K^c \times A} |\pi_i - v_\delta| f(t) d\gamma^k \leq M \cdot \delta.$$

Similarly,

$$\int_{T \times A} |\pi_i - v_\delta| f(t) d\gamma = \int_{K^c \times A} |\pi_i - v_\delta| f(t) d\gamma \leq M \cdot \delta.$$

Furthermore, as f is $\hat{\mu}$ -integrable, there is a sequence $(f_\ell)_{\ell=1}^\infty$ of bounded and continuous functions such that $\int_T |f(t) - f_\ell(t)| \hat{\mu}(dt) \rightarrow 0$, as $\ell \rightarrow \infty$. We have

$$\begin{aligned} |\Pi_i(\gamma^k) - \Pi_i(\gamma)| &= \left| \int \pi_i f(t) d\gamma^k - \int \pi_i f(t) d\gamma \right| \\ &\leq \int |\pi_i - v_\delta| f(t) d\gamma^k + \int |f(t) - f_\ell(t)| v_\delta d\gamma^k \\ &\quad + \left| \int v_\delta f_\ell(t) d\gamma^k - \int v_\delta f_\ell(t) d\gamma \right| \\ &\quad + \int |f(t) - f_\ell(t)| v_\delta d\gamma + \int |\pi_i - v_\delta| f(t) d\gamma \\ &\leq 2M\delta + \int |f(t) - f_\ell(t)| v_\delta d\gamma^k \\ &\quad + \left| \int v_\delta f_\ell d\gamma^k - \int v_\delta f_\ell d\gamma \right| + \int |f(t) - f_\ell(t)| v_\delta d\gamma. \end{aligned}$$

Hence, $|\Pi_i(\gamma^k) - \Pi_i(\gamma)|$ converges to zero when $k \rightarrow \infty$, $\ell \rightarrow \infty$, and $\delta \rightarrow 0$. This proves that Π_i is continuous at γ . ■

Proof of Proposition 3.2:

Let $\beta = (\beta_1, \dots, \beta_n)$ be a DSPE. By Definition 3.1, for every completely mixed strategy profile $\nu = (\nu_1, \dots, \nu_n)$ and every sequence $(\varepsilon_i^k)_{k=1}^\infty$ of real numbers converging to zero, the sequence $(\gamma_i^k)_{k=1}^\infty$ defined by $\gamma_i^k = (1 - \varepsilon_i^k) \cdot \gamma_i + \varepsilon_i^k \cdot \nu_i$, for every i , is such that

$$\lim_{k \rightarrow \infty} \rho^w(\gamma_i^k, BR_i(\gamma^k)) = 0.$$

Clearly $\lim_{k \rightarrow \infty} \rho^w(\gamma_i^k, \gamma_i) = 0$. Therefore, for every i , there exists a sequence $(\hat{\gamma}_i^k)_{k=1}^\infty$ such that $\hat{\gamma}_i^k \in BR_i(\gamma^k)$ for every $k \in \mathbf{N}$ and $\rho^w(\gamma_i^k, \hat{\gamma}_i^k) \rightarrow 0$ as $k \rightarrow \infty$. By the triangle inequality for ρ^w this implies for every i that $\rho^w(\gamma_i, \hat{\gamma}_i^k) \rightarrow 0$ as $k \rightarrow \infty$. Now, by Lemma 6.5 we conclude the upper hemicontinuity of the correspondence BR_i , for every player i . Hence, $\gamma_i \in BR_i(\gamma)$. Now, by Lemma 6.2 we have $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for μ_i -a.e. $t_i \in T_i$. In other words, for every i there exists a set $S_i \in \mathcal{T}_i$ with $\mu_i(S_i) = 0$ such that for every $t_i \in T_i \setminus S_i$ we have $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$. Now, define $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_n)$ by letting $\hat{\beta}_i(t_i, \cdot) = \beta_i(t_i, \cdot)$ for every $t_i \in T_i \setminus S_i$ and selecting any $\hat{\beta}_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for every $t_i \in S_i$. It is clear that $\hat{\beta}$ is a BNE and satisfies the requirement of the proposition. ■

Existence proof of distributional perfect BNE

We prove the existence of a distributional perfect BNE, with the help of the following fixed point theorem from Glicksberg [17].

Theorem 6.6 *Let S be a nonempty compact and convex subset of a locally convex Hausdorff space. Let $F : S \rightarrow S$ be an upper hemicontinuous correspondence with nonempty and convex values. Then, F has a fixed point.*

Theorem 6.7 *The set of distributional perfect BNEs is nonempty.*

First we prove that the set of distributional strategy profiles satisfying the conditions in the definition of distributional perfect equilibrium is not empty. For

every player i , let ν_i be a completely mixed distributional strategy. Note that such a ν_i exists, because $T_i \times A_i$ is separable. Define for every $k \in \mathbf{N}$

$$\Delta_i(k) = \{\gamma_i \in \Delta_i \mid \gamma_i(B_i) \geq \frac{1}{k} \cdot \nu_i(B_i), \forall B_i \in \mathcal{T}_i \otimes \mathcal{A}_i\}.$$

Moreover, let $\Delta(k) = \times_{i=1}^n \Delta_i(k)$ and $BR_i^k : \Delta(k) \rightarrow \Delta_i(k)$ be the best response correspondence for player i restricted to $\Delta_i(k)$ ²⁰ and define the correspondence $BR^k : \Delta(k) \rightarrow \Delta(k)$ to be

$$BR^k(\gamma) = \times_{i=1}^n BR_i^k(\gamma),$$

for every $\gamma \in \Delta(k)$. We verify the conditions of Theorem 6.6 for the correspondence BR^k .

Since T is a complete and separable metric space, by Theorem 1.4 in [12] μ is a tight measure.²¹ This fact together with the compactness of the set A implies that Δ_i is a tight set of probability measures.²² Now, by Prohorov's Theorem we conclude that Δ_i is a compact metric space with respect to the weak metric. It is easy to see that $\Delta_i(k)$ is a closed subset of Δ_i with respect to the strong metric. Consequently, by Theorem V.3.13 in [16], it is closed with respect to the weak metric. Hence, $\Delta_i(k)$ is compact with respect to the weak metric. Also, one can easily check that $\Delta_i(k)$ is convex. Moreover, upper hemicontinuity of BR^k follows by Lemma 6.5. Now, we apply Theorem 6.6, which leads us to the existence of an equilibrium point $\gamma^k \in \Delta(k)$, for every k .

Now, define²³

$$PBR_i(\gamma) = \{(t_i, a_i) \in T_i \times A_i \mid (t_i, a_i) \in \text{supp}(\gamma_i) \text{ and } \gamma_i \in BR_i(\gamma)\}.$$

It is clear that $\gamma_i^k(PBR_i(\gamma^k)) \geq 1 - \frac{1}{k} \nu_i(T_i \times A_i) = 1 - \frac{1}{k}$. This implies that $\rho^w(\gamma_i^k, BR_i(\gamma^k)) \leq \frac{1}{k}$. Furthermore, as Δ_i is compact with respect to the weak

²⁰Distributional strategy γ_i of player i is a best response restricted to $\Delta_i(k)$ against a distributional strategy profile $\eta_{-i} \in \Delta_{-i}(k)$, if $\Pi_i(\gamma_i, \eta_{-i}) \geq \Pi_i(\tilde{\gamma}_i, \eta_{-i})$, for every distributional strategy $\tilde{\gamma}_i \in \Delta_i(k)$ of player i .

²¹Let X be a metric space with the Borel σ -field Σ . A measure μ on Σ is tight if for every $G \in \Sigma$, $\mu(G) = \sup\{\mu(K) \mid K \subset G, \text{ and } K \text{ compact}\}$. Moreover, Theorem 1.4 in [12] states that if X is separable and complete, then each probability measure on Σ is tight.

²²A family of probability measures on a metric space is tight if for every $\varepsilon > 0$ there is a compact set K satisfying $\mu(K) > 1 - \varepsilon$, for every μ in the family.

²³Recall that $\text{supp}(\mu) = \{x \in X \mid \text{for every open set } G \text{ if } x \in G \text{ then } \mu(G) > 0\}$.

metric, without loss of generality we can assume that there is a distributional strategy profile γ such that $\rho^w(\gamma^k, \gamma) \rightarrow 0$ when $k \rightarrow \infty$. Let β be a behavioral strategy profile corresponding to γ . Obviously, β is distributionally perfect.

Now, we show that there is a BNE $\hat{\beta}$ such that for μ_i -a.e. $t_i \in T_i$ we have $\beta_i(t_i, \cdot) = \hat{\beta}_i(t_i, \cdot)$, for every player i . Then, since β is distributionally perfect, trivially the BNE $\hat{\beta}$ is also distributionally perfect. Hence, the set of distributional perfect BNEs is nonempty, as desired.

Since $\beta = (\beta_1, \dots, \beta_n)$ is distributionally perfect, by definition we know that for the induced distributional strategy profile $\gamma = (\gamma_1, \dots, \gamma_n)$ there exists a sequence $(\gamma^k)_{k=1}^\infty = (\gamma_1^k, \dots, \gamma_n^k)_{k=1}^\infty$ of completely mixed distributional strategy profiles such that for every player i we have $\lim_{k \rightarrow \infty} \rho^w(\gamma_i^k, \gamma_i) = 0$ and

$$\lim_{k \rightarrow \infty} \rho^w(\gamma_i^k, BR_i(\gamma^k)) = 0.$$

Therefore, for every i , there exists a sequence $(\hat{\gamma}_i^k)_{k=1}^\infty$ such that $\hat{\gamma}_i^k \in BR_i(\gamma^k)$ for every $k \in \mathbf{N}$ and $\rho^w(\gamma_i^k, \hat{\gamma}_i^k) \rightarrow 0$ as $k \rightarrow \infty$. By the triangle inequality for ρ^w this implies for every i that $\rho^w(\gamma_i, \hat{\gamma}_i^k) \rightarrow 0$ as $k \rightarrow \infty$. Now, by Lemma 6.5 we conclude the upper hemicontinuity of the correspondence BR_i , for every player i . Hence, $\gamma_i \in BR_i(\gamma)$.

Now, by Lemma 6.2 we have $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for μ_i -a.e. $t_i \in T_i$. In other words, for every i there exists a set $S_i \in \mathcal{T}_i$ with $\mu_i(S_i) = 0$ such that for every $t_i \in T_i \setminus S_i$ we have $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$. Define $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_n)$ by letting $\hat{\beta}_i(t_i, \cdot) = \beta_i(t_i, \cdot)$ for every $t_i \in T_i \setminus S_i$ and selecting any $\hat{\beta}_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for every $t_i \in S_i$. Note that the result of this selection is measurable. Clearly, $\hat{\beta}$ is a BNE and satisfies the requirement of the definition. ■