



*Citation for published version:*

Cherednichenko, K, Ershova, Y & Kiselev, AV 2019, 'Time-dispersive behavior as a feature of critical-contrast media', *SIAM Journal on Applied Mathematics*, vol. 79, no. 2, pp. 690-715. <https://doi.org/10.1137/18M1187167>

*DOI:*

[10.1137/18M1187167](https://doi.org/10.1137/18M1187167)

*Publication date:*

2019

*Document Version*

Peer reviewed version

[Link to publication](#)

(C) Society for Industrial and Applied Mathematics, 2019.

## University of Bath

### Alternative formats

If you require this document in an alternative format, please contact:  
[openaccess@bath.ac.uk](mailto:openaccess@bath.ac.uk)

#### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

#### Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

1                   **TIME-DISPERSIVE BEHAVIOUR AS A FEATURE OF**  
2                   **CRITICAL-CONTRAST MEDIA\***

3                   KIRILL CHEREDNICHENKO<sup>†</sup>, YULIA ERSHOVA<sup>‡</sup>, AND ALEXANDER V. KISELEV<sup>§</sup>

4                   **Abstract.** Motivated by the urgent need to attribute a rigorous mathematical meaning to the  
5 term “metamaterial”, we propose a novel approach to the homogenisation of critical-contrast com-  
6 posites. This is based on the asymptotic analysis of the Dirichlet-to-Neumann map on the interface  
7 between different components (“stiff” and “soft”) of the medium, which leads to an asymptotic ap-  
8 proximation of eigenmodes. This allows us to see that the presence of the soft component makes  
9 the stiff one behave as a class of time-dispersive media. By an inversion of this argument, we also  
10 offer a recipe for the construction of such media with prescribed dispersive properties from periodic  
11 composites.

12                   **Key words.** Homogenisation, Effective properties, Operators, Time-dispersive media, Asymp-  
13 totics

14                   **AMS subject classifications.** 34E13, 34E05, 35P20, 47A20, 81Q35

15                   **1. Introduction.**

16                   **1.1. Physics context and motivation for quantitative analysis.** Under-  
17 standing the dependence of material properties of continuous media on frequency is a  
18 natural and practically relevant task, stemming from the theoretical and experimental  
19 studies of “metamaterials”, *e.g.* materials that exhibit negative refraction of propa-  
20 gating wave packets. Indeed, it was noted as early as in the pioneering work [37], that  
21 negative refraction is only possible under the assumption of frequency dispersion, *i.e.*  
22 when the material parameters (permittivity and permeability in electromagnetism,  
23 elastic moduli and mass density in acoustics) are not only frequency-dependent, but  
24 also become negative in certain frequency bands.

25                   Independently of the search for metamaterials, in the course of the development of  
26 the theory of electromagnetism, it has transpired in modern physics that the Maxwell  
27 equations need to be considered with time-nonlocal “memory” terms, see *e.g.* [24,  
28 Section 7.10] and also [7], [34]. The related generalised system (in the absence of  
29 charges and currents in the domain of interest) has the form

30 (1.1)            $\rho \partial_t u + \int_{-\infty}^t a(t - \tau) u(\tau) d\tau + iAu = 0, \quad A = \begin{pmatrix} 0 & i \operatorname{curl} \\ -i \operatorname{curl} & 0 \end{pmatrix},$

31 where  $u$  represents the (time-dependent) electromagnetic field  $(H, E)^\top$ , the matrix  $\rho$

---

\*Submitted to the editors 13 May 2018.

**Funding:** KDC and YE is grateful for the financial support of the Engineering and Physical Sciences Research Council: Grant EP/L018802/2 “Mathematical foundations of metamaterials: homogenisation, dissipation and operator theory”. AVK has been partially supported by the RFBR grant 16-01-00443-a.

<sup>†</sup>Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom ([K.Cherednichenko@bath.ac.uk](mailto:K.Cherednichenko@bath.ac.uk))

<sup>‡</sup>Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom AND Department of Mathematics, St.Petersburg State University of Architecture and Civil Engineering, 2-ya Krasnoarmeiskaya St. 4, 190005 St.Petersburg, Russia ([julija.ershova@gmail.com](mailto:julija.ershova@gmail.com))

<sup>§</sup>Department of Higher Mathematics and Mathematical Physics, St.Petersburg State University, Ulianovskaya 1, St.Peterhoff, 198504 St.Petersburg, Russia ([alexander.v.kiselev@gmail.com](mailto:alexander.v.kiselev@gmail.com))

32 depends on the electric permittivity and magnetic permeability, and  $a$  is a matrix-  
 33 valued “susceptibility” operator, set to zero in the more basic form of the system.<sup>1</sup>

34 Applying the Fourier transform in time  $t$  to (1.1), an equation in the frequency  
 35 domain is obtained:

$$36 \quad (1.2) \quad (i\omega\rho + \hat{a}(\omega))\hat{u}(\cdot, \omega) + iA\hat{u}(\cdot, \omega) = 0,$$

37 where  $\hat{u}$  is the Fourier transform of  $u$ , and  $\omega$  is the frequency. Equation (1.2) is  
 38 often interpreted as a “non-classical” version of Maxwell’s system of equations, where  
 39 the permittivity and/or permeability are frequency-dependent. The existence of such  
 40 media (commonly known as Lorentz materials) and the analysis of their properties go  
 41 back a few decades in time and has also attracted considerable interest quite recently,  
 42 *e.g.* in the study of plasma in tokamaks, see [15] and references therein.

43 Simultaneously with the above developments in the physics literature, recent  
 44 mathematical evidence, see [38], [6], suggests that such novel material behaviour,  
 45 which is incompatible (see [5, 10, 11]) with the mathematical assumption of uniform  
 46 ellipticity of the corresponding differential operators (such as  $A$  in (1.1)), may be ex-  
 47 plained by means of the asymptotic analysis (“homogenisation”) of operator families  
 48 with rapidly oscillating, and non-uniformly elliptic, coefficients.

49 It is therefore reasonable to ask the question of whether frequency dispersion  
 50 laws such as pertaining to (1.2), which in turn may provide one with metamaterial  
 51 behaviour in appropriate frequency intervals [37], can be derived by some process of  
 52 homogenisation of composite media with contrast (or, as we shall suggest below, any  
 53 other microscopic degeneracies resonating with the macroscopic wavefields).

54 **1.2. Basis for the mathematical framework.** If one were to look for an  
 55 asymptotic expansion of eigenmodes of a high-contrast composite, *restricted* to the  
 56 soft component of the medium, one would notice (see, *e.g.*, [9]) that their leading-  
 57 order terms can be understood as the eigenmodes of boundary-value problems with  
 58 impedance (*i.e.*, frequency-dependent) boundary conditions. Such problems have been  
 59 considered in the past (see, *e.g.*, [32]), motivated by the analysis of the wave equation.  
 60 On the other hand, by the celebrated analysis [29, 30] of the so-called generalised  
 61 resolvents, one knows that a problem of this type admits a conservative dilation,  
 62 which is constructed by adding the hidden degrees of freedom. In fact, this latter  
 63 observation has been used in [19, 20] in devising a conservative “extension” of a  
 64 time-dispersive system of the type (1.1). In the present paper we argue that the  
 65 aforementioned conservative dilation is precisely the asymptotic model of the original  
 66 high-contrast composite. Furthermore, the leading-order terms of its eigenmodes  
 67 restricted to the *stiff* component are solutions to a problem of the type (1.2) with  
 68 frequency dispersion. They can be easily expressed in terms of the above impedance  
 69 boundary value problems, thus yielding an explicit description of the link between the  
 70 resonant soft inclusions and the macroscopic time-dispersive properties. Therefore,  
 71 models of continuous media with frequency-dependent effective boundary conditions  
 72 can be seen as natural building blocks for media with frequency dispersion.

73 It is of a considerable value to relate these ideas to the earlier works [26, 27, 18],  
 74 where similar limiting impedance-type problems are obtained in the spectral analy-  
 75 sis of “thin” periodic structures, converging to metric graphs. Here, one obtains the

---

<sup>1</sup>From the rigorous operator-theoretic point of view,  $A$  in (1.1) is treated as a self-adjoint operator in a Hilbert space  $\mathbb{H}$  of functions of  $x \in \Omega$ , for example  $\mathbb{H} = L^2(\Omega; \mathbb{R}^6)$ , where  $\Omega$  is the part of the space occupied by the medium.

76 aforementioned impedance setup (see Fig. 1) on the limiting graph as the asymptotics  
 77 of the eigenmodes of a Neumann Laplacian, when the “thickness” of the structure vanishes  
 78 ishes for one particular (resonant) scaling between the “edge” and “vertex” volumes  
 of the structure.

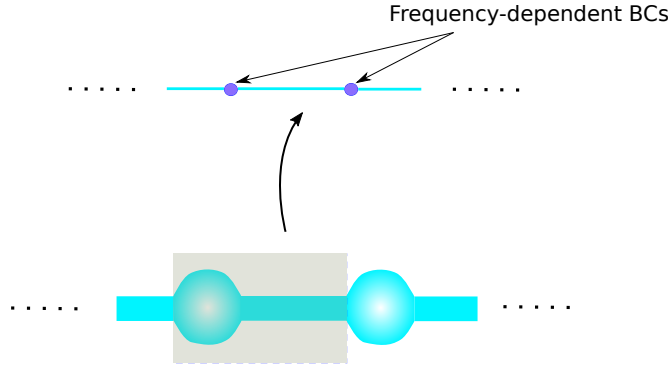


FIG. 1. AN EXAMPLE OF A RESONANT THIN NETWORK. *Edge volumes are asymptotically of the same order as vertex volumes. The stiffness of the material of the structure is of the order period-squared.*

79  
 80 It is instructive to point out that the results of [9] establish a thrilling relationship  
 81 between the analysis of thin structures and the homogenisation theory of high-contrast  
 composites. Namely, the paper [9] deals with the case of the so-called superlattices

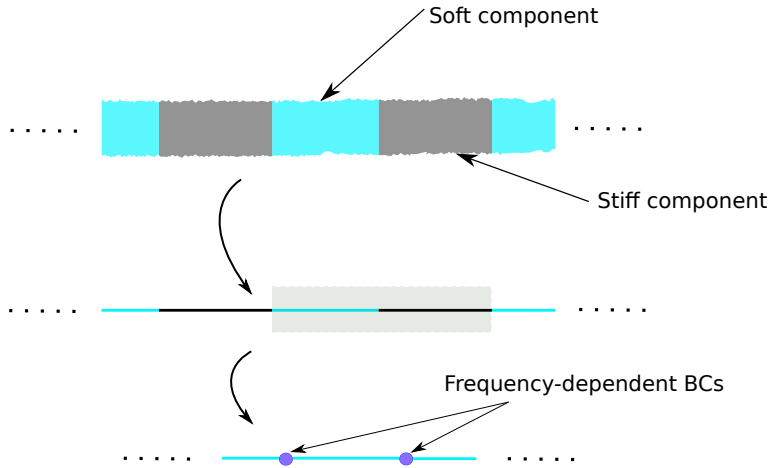


FIG. 2. HIGH-CONTRAST SUPERLATTICE. *The problem for a superlattice is reduced to a one-dimensional high-contrast problem. This is asymptotically equivalent to an impedance-type problem on the soft component.*

82  
 83 [36] with high contrast, see Fig. 2. While simple to set up, the related system of  
 84 ordinary differential equations (subject to the appropriate conditions of continuity

85 of fields and fluxes) is nontrivial from the point of view of quantitative analysis, see  
 86 also [8]. It is shown that the asymptotic model for this system is precisely the one  
 87 derived in [26, 27, 18] in the case of a resonant thin structure converging to a chain-  
 88 graph, see Fig. 1. As we shall argue in the present article, such superlattices (and  
 89 the corresponding chain-graphs) offer a simple prototype for a metamaterial, via the  
 90 mathematical approach outlined above.

91 The described result suggests that thin networks might acquire the same asymp-  
 92 totic properties as those of the corresponding high-contrast composites. It is therefore  
 93 a viable conjecture, that the metamaterial properties of a medium can be attained via  
 94 a version of geometric contrast instead of relying upon the contrast between material  
 95 components. This is especially promising when the required material contrast cannot  
 96 be guaranteed, as is commonly the case in elasticity and electromagnetism. The cor-  
 97 responding thin networks on the other hand have been made available in the study of  
 98 graphenes and related areas. This subject will be further pursued in a forthcoming  
 99 publication.

100 The above exposition vindicates the value of quantum graph models in the analysis  
 101 of high-contrast composites, where we follow the well-established convention, see [3],  
 102 to use the term *quantum graph* for an ordinary differential operator of second order  
 103 defined on a metric graph. These graph-based models are seen as natural limits of  
 104 composite thin networks consisting of a large number of channels (for, say, acoustic or  
 105 electromagnetic waves), where a combination of high-contrast and rapid oscillations  
 106 becomes increasingly taxing at small scales and often leads to impractical numerical  
 107 costs. For channels with low cross-section-to-length ratios, the material response of  
 108 such a system, see Fig. 3, is closely approximated by a quantum graph as described  
 above. Systems of this type are a particular example of high-contrast composites and

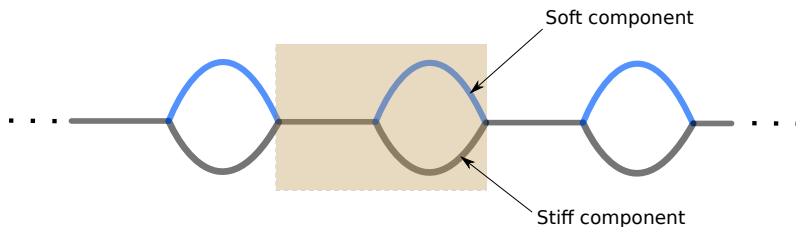


FIG. 3. THIN NETWORK. *An example of a high-contrast periodic network. Stiff channels are in grey, soft channels are in blue.*

109 thus, as explained above, they possess resonant properties at the microscale, which, in  
 110 turn, leads to macroscopic dispersion. At a very crude level, this is similar to the way  
 111 in which particle motion on the atomic scale leads to Lorentz-type electromagnetism,  
 112 see *e.g.* [31, Chapter 1] for the analysis of a related model of the damped harmonic  
 113 oscillator.  
 114

115 Furthermore, periodic quantum graphs with a vanishing period can serve as realis-  
 116 tic explicitly solvable ODE models for multidimensional continuous media, as demon-  
 117 strated<sup>2</sup>, *e.g.*, in [28], where an  $h$ -periodic cubic lattice, for small positive  $h$ , is shown  
 118 to be close (including the scattering properties) to the Laplacian in  $\mathbb{R}^d$ . More involved

<sup>2</sup>We remark, that it was Professor Pavlov who had pioneered the mathematical study of quantum graphs, see [21].

119 periodic graphs can be used to model non-trivial media, including anisotropic ones.

120 As a particular realistic example of a thin network with high contrast, consider  
 121 the problem of modelling acoustic wave propagation in a system of channels  $\Omega^{\varepsilon,\delta}$ ,  $\varepsilon$ -  
 122 periodic in one direction, of thickness  $\delta \ll \varepsilon$ , and with contrasting material properties  
 123 (cf. Fig. 3). To simplify the presentation, we assume the antiplane shear wave  
 124 polarisation (the so called S-waves), which leads to a scalar wave equation for the  
 125 only non-vanishing component  $W$ , of the form

126 
$$W_{tt} - \nabla_x \cdot (a^\varepsilon(x) \nabla_x W) = 0, \quad u = W(x, t), \quad x, t \in \mathbb{R},$$

127 where the coefficient  $a^\varepsilon$  takes values one and  $\varepsilon^2$  in different channels of the  $\varepsilon$ -periodic  
 128 structure. Looking for time-harmonic solutions  $W(x, t) = U(x) \exp(i\omega t)$ ,  $\omega > 0$ , one  
 129 arrives at the spectral problem

130 (1.3) 
$$-\nabla \cdot (a^\varepsilon \nabla U) = \omega^2 U.$$

131 As we argue below, the behaviour of (1.3) is close, in a quantitatively controlled way  
 132 as  $\varepsilon \rightarrow 0$ , to that of an “effective medium” on  $\mathbb{R}$  described by an equation of the form

133 (1.4) 
$$-U'' = \beta(\omega)U,$$

134 for an appropriate function  $\beta = \beta(\omega)$ , explicitly given in terms of the material pa-  
 135 rameters  $a^\varepsilon$  and the topology of the original system of channels.

136 The goal of the present paper is to derive an explicit general formula for the  
 137 function  $\beta$  in (1.4), in terms of the topology of the graph representing the original  
 138 domain of wave propagation, which is no longer restricted to the example shown in  
 139 Fig. 3. As noted above, the presence of both rapid oscillations and high contrast  
 140 make the task mathematically nontrivial. In our approach, which is new, we call  
 141 upon some recently developed machinery in the operator-theoretic analysis of abstract  
 142 boundary-value problems (which in our case take the form of boundary-value problems  
 143 for differential operators of interest). In the subsequent work [10] we develop the  
 144 corresponding analysis for the multidimensional case, which is neither included nor  
 145 an extension of the analysis for graphs presented in this article. However, it is based  
 146 on the same set of mathematical ideas, which makes us hope that the foundations for  
 147 (1.4) in the case of PDEs is clear from what follows.

148 Unlike the approach aimed at derivation of norm-resolvent convergence, which we  
 149 adopt in [11, 10], in the present paper, having the convenience of the more physically  
 150 inclined reader in mind, we systematically treat the subject from the point of view of  
 151 spectral problems and, in particular, of the asymptotic analysis of eigenmodes. We  
 152 refer the interested reader to the aforementioned papers, where further mathematical  
 153 details, which we think are out of scope here, are contained.

154 The present paper can be viewed as following in the footsteps of [9] in that it  
 155 relies upon the analysis of the fibre representations (obtained via the Floquet-Gelfand  
 156 transform) of the original periodic operator. This is carried out using the bound-  
 157 ary triples theory (see, *e.g.*, [22, 14]), which generalises the classical methods based  
 158 on the Weyl-Titchmarsh  $m$ -coefficient, applied to self-adjoint extensions of symmet-  
 159 ric operators. This allows us to develop a novel approach to the homogenisation of  
 160 a class of periodic high-contrast problems on “weighted quantum graphs”, *i.e.* one-  
 161 dimensional versions of thin composite media where the material parameters on one of  
 162 the components are much lower than on the others and scaled in a “critical” way with  
 163 respect to the period of the composite. We reiterate that the idea that such media

164 can be viewed as idealised models of thin periodic critical-contrast networks has been  
 165 explored in the mathematics literature, see [27], [18], [39] and elsewhere. The back-  
 166 bone of our approach is the study of eigenfunctions of the problem restricted to one  
 167 (“soft”) component of the composite. After the asymptotics for these is obtained, it  
 168 proves possible to reconstruct the “complete” eigenfunctions, where we implicitly rely  
 169 upon the classical results of operator theory, in particular dealing with out-of-space  
 170 self-adjoint extensions of symmetric operators and associated generalised resolvents.

171 **1.3. Physics interpretation and relevance to metamaterials.** Our argu-  
 172 ment leads to the understanding of the phenomenon of critical-contrast homogeni-  
 173 sation limit as a manifestation of a frequency-converting device: if one restricts the  
 174 eigenfunctions to the “stiff” component, they prove to be close to those of the medium  
 175 where the soft component has been replaced with voids *but* correspond to non-trivially  
 176 shifted eigenfrequencies. This is precisely what one would expect in the setting of  
 177 time-dispersive media after the passage to the frequency domain, *cf.* (1.2).

178 From the physics perspective, this link between homogenisation and frequency  
 179 conversion can be viewed as a justification of an “asymptotic equivalence” between  
 180 eigenvalue problems for periodic composites with high contrast and wave propagation  
 181 problems with nonlinear dependence on the spectral parameter, which in the frequency  
 182 domain characterise “time-dispersive media”, as in (1.1), see also [34, 35, 19, 20].

183 As we mention above, the phenomenon of frequency dispersion emerging as a  
 184 result of homogenisation has been observed in the two-scale formulation applied to  
 185 critical-contrast PDEs in, *e.g.*, [38, 6]. Our approach goes beyond the results of [38, 6]  
 186 in several ways. First, being based on an explicit asymptotic analysis of operators,  
 187 using the recent developments in the theory of abstract boundary-value problems (see  
 188 *e.g.* [33]), it provides an explicit procedure for recovering the dispersion relation and  
 189 does not draw upon the well-known two-scale asymptotic techniques. Second, the  
 190 convergence statements are obtained in the much stronger operator-norm topology.  
 191 Finally, our approach is not restricted to topologies where the stiff component forms  
 192 a connected set, see [11] for explicit dispersion formulae derived in such setups.

193 The approach we develop in the present paper offers a new perspective on frequen-  
 194 cy-dispersive (time non-local) continuous media, in the sense that it provides a recipe  
 195 for the construction of such media with prescribed dispersive properties from periodic  
 196 composites whose individual components are non-dispersive. It has been known that  
 197 time-dispersive media [19] in the frequency domain can be realised as a “restriction”  
 198 of a conservative Hamiltonian defined on a space which adds the “hidden” degrees of  
 199 freedom.<sup>3</sup>

200 In summary, the existing belief in the engineering and physics literature that time-  
 201 dispersive properties often arise as the result of complex microstructure of composites  
 202 suggests to look for a rather concrete class of such conservative Hamiltonian dilations,  
 203 namely, those pertaining to differential operators on composites with critical contrast.  
 204 Our results can be viewed as laying foundations for rigorously solving this problem.

**2. Infinite-graph setup.** Consider a graph  $\mathbb{G}_\infty$ , periodic in one direction, so  
 that  $\mathbb{G}_\infty + \ell = \mathbb{G}_\infty$ , where  $\ell$  is a fixed vector, which defines the graph axis. Let the  
 periodicity cell  $\mathbb{G}$  be a finite compact graph of total length  $\varepsilon \in (0, 1)$ , and denote by

---

<sup>3</sup> This is based on the observation that the equation (1.2) can be written in the form of an  
 eigenvalue problem  $\mathcal{A}U = \omega U$ ,  $U \in \mathcal{H}$ , for a suitable self-adjoint “dilation”  $\mathcal{A}$  of the operator  $A$ , so  
 that  $\mathcal{A}$  acts in a space  $\mathcal{H} \supset \mathbb{H}$ . The vector field  $U$  has a natural physical interpretation in terms of  
 additional electromagnetic field variables, the so-called polarisation  $P$  and magnetisation  $M$ , so that  
 the full (12-dimensional) field vector is  $(H, E, P, M)^\top$ .

$e_j$ ,  $j = 1, 2, \dots, n$ ,  $n \in \mathbb{N}$ , its edges. For each  $j = 1, 2, \dots, n$ , we identify  $e_j$  with the interval  $[0, \varepsilon l_j]$ , where  $\varepsilon l_j$  is the length of  $e_j$ . We associate with the graph  $\mathbb{G}_\infty$  the Hilbert space

$$L_2(\mathbb{G}_\infty) := \bigoplus_{\mathbb{Z}} \bigoplus_{j=1}^n L_2(0, \varepsilon l_j).$$

205 Consider a sequence of operators  $A^\varepsilon$ ,  $\varepsilon > 0$ , in  $L_2(\mathbb{G}_\infty)$ , generated by second-order  
 206 differential expressions

$$207 \quad (2.1) \quad - \frac{d}{dx} \left( (a^\varepsilon)^2 \frac{d}{dx} \right),$$

208 with positive  $\mathbb{G}$ -periodic coefficients  $(a^\varepsilon)^2$  defined on  $\mathbb{G}_\infty$ , with the domain  $\text{dom}(A^\varepsilon)$   
 209 that describes the coupling conditions at the vertices of  $\mathbb{G}_\infty$  :

$$210 \quad (2.2) \quad \text{dom}(A^\varepsilon) = \left\{ u \in \bigoplus_{e \in \mathbb{G}_\infty} W^{2,2}(e) \mid u \text{ continuous, } \sum_{e \ni V} \sigma_e (a^\varepsilon)^2 u'(V) = 0 \forall V \in \mathbb{G}_\infty \right\},$$

211 In the formula (2.2) the summation is carried out over the edges  $e$  sharing the vertex  
 212  $V$ , the coefficient  $(a^\varepsilon)^2$  in the vertex condition is calculated on the edge  $e$ , and  $\sigma_e = -1$   
 213 or  $\sigma_e = 1$  for  $e$  incoming or outgoing for  $V$ , respectively. The matching conditions (2.2)  
 214 represent the combined conditions of continuity of the function and of vanishing sums  
 215 of its co-normal derivatives at all vertices (*i.e.* the so-called Kirchhoff conditions).

216 **3. Gelfand transform.** We seek to apply the one-dimensional Gelfand trans-  
 217 form

$$218 \quad (3.1) \quad v(x) = \sqrt{\frac{\varepsilon}{2\pi}} \sum_{n \in \mathbb{Z}} u(x + \varepsilon n) e^{-it(x + \varepsilon n)}.$$

219 to the operator  $A^\varepsilon$  defined on  $\mathbb{G}_\infty$  in order to obtain the direct fibre integral for the  
 220 operator  $A^\varepsilon$  :

$$221 \quad (3.2) \quad A^\varepsilon = \int_{\oplus} A_t^\varepsilon dt.$$

222 In order to do achieve this goal, we first note that the geometry of  $\mathbb{G}_\infty$  is encoded in  
 223 the matching conditions (2.2) *only*. This opens up a possibility to embed the graph  
 224  $\mathbb{G}_\infty$  into  $\mathbb{R}^1$  by rearranging it edges as consecutive segments of the real line (leading  
 225 to a one-dimensional  $\varepsilon$ -periodic chain graph). In doing so we drop the customary  
 226 practice of drawing graphs in a way reflecting matching conditions (*i.e.*, so that these  
 227 are local relative to graph vertices). The above embedding leads to rather complex  
 228 non-local matching conditions, but, on the positive side, allows us to use the Gelfand  
 229 transform (3.1).

230 The Gelfand transform leads to periodic conditions on the boundary of the cell  
 231  $\mathbb{G}$  and thus in our case identifies the “left” boundary vertices of the graph  $\mathbb{G}$  with  
 232 their translations by  $\ell$ , which results in a modified graph  $\hat{\mathbb{G}}$ . Apart from this, the  
 233 matching conditions for the internal vertices of  $\mathbb{G}$  admit the same form as for  $A^\varepsilon$ ,  
 234 except for the fact that the Kirchhoff matching is replaced by a Datta-Das Sarma one  
 235 (the latter can be viewed as a weighted Kirchhoff), see below in (3.4). Unimodular  
 236 weights appearing in Datta-Das Sarma conditions are precisely due to the non-locality  
 237 of matching conditions mentioned above for the embedding of  $\mathbb{G}_\infty$  into  $\mathbb{R}^1$ .



238 The image of the Gelfand transform is described as follows. There exists a uni-  
 239 modular list  $\{w_V(e)\}_{e \ni V}$ , cf. [11], defined at each vertex  $V$  of  $\widehat{\mathbb{G}}$  as a finite collection  
 240 of values corresponding to the edges adjacent to  $V$ . For each  $t \in [-\pi/\varepsilon, \pi/\varepsilon)$ , the  
 241 fibre operator  $A_t^\varepsilon$  is generated by the differential expression

$$242 \quad (3.3) \quad \left( \frac{1}{i} \frac{d}{dx} + t \right) (a^\varepsilon)^2 \left( \frac{1}{i} \frac{d}{dx} + t \right)$$

243 on the domain  
 244

$$245 \quad (3.4) \quad \text{dom}(A_t^\varepsilon) = \left\{ v \in \bigoplus_{e \in \mathbb{G}} W^{2,2}(e) \mid \right.$$

$$246 \quad w_V(e)v|_e(V) = w_V(e')v|_{e'}(V) \text{ for all } e, e' \text{ adjacent to } V,$$

$$247 \quad \left. \sum_{e \ni V} \partial^{(t)}v(V) = 0 \text{ for each vertex } V \right\},$$

249 where  $\partial^{(t)}v(V)$  is the weighted ‘‘co-derivative’’  $\sigma_e w_V(e)(a^\varepsilon)^2(v' + itv)$  of the function  
 250  $v$  on the edge  $e$ , calculated at  $V$ .

251 **4. Boundary triples for extensions of symmetric operators.** In the analy-  
 252 sis of the asymptotic behaviour of the fibres  $A_t^\varepsilon$  of the original operator  $A^\varepsilon$  representing  
 253 the quantum graph, we employ the framework of boundary triples for a symmetric  
 254 operator with equal deficiency indices for the description of a class of its extensions.  
 255 Part of the toolbox of the theory of boundary triples is the generalisation of the clas-  
 256 sical Weyl-Titchmarsh  $m$ -function to the case of a matrix (finite deficiency indices)  
 257 and operators (infinite deficiency indices).

258 The boundary triples theory is a very convenient toolbox for dealing with exten-  
 259 sions of linear operators, originating in the works of M. G. Kreĭn. In essence, it is an  
 260 operator-theoretic interpretation of the second Green’s identity, see (4.1) below. As  
 261 such, it allows one to pass over from the consideration of functions in Hilbert spaces to  
 262 a formulation in which one deals with objects in the boundary spaces (such as traces  
 263 of functions and their normal derivatives), which in the context of quantum graphs  
 264 are finite-dimensional. Furthermore, it allows one to use explicit concise formulae for  
 265 the resolvents of operators under scrutiny and other related objects. Thus it facili-  
 266 tates the analysis by expressing the familiar, commonly used in this area, objects in  
 267 a concise way.

268 **DEFINITION 4.1** ([22, 25, 14]). *Suppose that  $A_{\max}$  is the adjoint to a densely de-*  
 269 *finied symmetric operator on a separable Hilbert space  $H$  and let  $\Gamma_0, \Gamma_1$  be linear*  
 270 *mappings of  $\text{dom}(A_{\max}) \subset H$  to a separable Hilbert space  $\mathcal{H}$ .*

271 *A. The triple  $(\mathcal{H}, \Gamma_0, \Gamma_1)$  is called a boundary triple for the operator  $A_{\max}$  if the*  
 272 *following two conditions hold:*

273 1. *For all  $u, v \in \text{dom}(A_{\max})$  one has the second Green’s identity*

$$274 \quad (4.1) \quad \langle A_{\max}u, v \rangle_H - \langle u, A_{\max}v \rangle_H = \langle \Gamma_1 u, \Gamma_0 v \rangle_{\mathcal{H}} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{\mathcal{H}}.$$

275 2. *The mapping  $\text{dom}(A_{\max}) \ni u \mapsto (\Gamma_0 u, \Gamma_1 u) \in \mathcal{H} \oplus \mathcal{H}$  is onto.*

276 *B. A restriction  $A_B$  of the operator  $A_{\max}$  such that  $A_{\max}^* =: A_{\min} \subset A_B \subset A_{\max}$*   
 277 *is called almost solvable if there exists a boundary triple  $(\mathcal{H}, \Gamma_0, \Gamma_1)$  for  $A_{\max}$  and a*  
 278 *bounded linear operator  $B$  defined on  $\mathcal{H}$  such that*

$$279 \quad \text{dom}(A_B) = \{u \in \text{dom}(A_{\max}) : \Gamma_1 u = B\Gamma_0 u\}.$$

280 C. The operator-valued Herglotz<sup>4</sup> function  $M = M(z)$ , defined by

$$281 \quad (4.2) \quad M(z)\Gamma_0 u_z = \Gamma_1 u_z, \quad u_z \in \ker(A_{\max} - z), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-,$$

282 is called the Weyl-Titchmarsh  $M$ -function of the operator  $A_{\max}$  with respect to the  
283 corresponding boundary triple.

284 Suppose  $A_B$  be a self-adjoint almost solvable restriction of  $A_{\max}$  with compact  
285 resolvent. Then  $M(z)$  is analytic on the real line away from the eigenvalues of  $A_\infty$ ,  
286 where  $A_\infty$  is the restriction of  $A_{\max}$  to domain  $\text{dom}(A_\infty) = \text{dom}(A_{\max}) \cap \ker(\Gamma_0)$ . It  
287 is a key observation for what follows that  $u \in \text{dom}(A_B)$  is an eigenvector of  $A_B$  with  
288 eigenvalue  $z_0 \in \mathbb{C} \setminus \text{spec}(A_\infty)$  if and only if

$$289 \quad (4.3) \quad (M(z_0) - B)\Gamma_0 u = 0.$$

290 In the next section we utilise a particular operator  $A_{\max}$  and a boundary triple  
291  $(\mathcal{H}, \Gamma_0, \Gamma_1)$ , which we use to analyse the resolvents of the operators on quantum graphs  
292 introduced in Sections 2, 3.

293 **5. Graph with high contrast: prototype for time-dispersive media.** In  
294 what follows we develop a general approach to the analysis of weighted quantum  
295 graphs with critical contrast. We demonstrate it on one particular example, which,  
296 as we show in Appendix A, exhibits all the properties of the generic case. We have  
297 thus chosen to present the analysis in the terms that are immediately applicable  
298 to the general case and, whenever advisable, we provide statements that carry over  
299 without modifications. Speaking of a “general” case, we imply an operator of the  
300 class introduced in Section 2, where some of the edges  $e_{\text{soft}}$  (“soft” edges) of the cell  
301 graph  $\mathbb{G}$  carry the weight  $a^\varepsilon = \varepsilon$ , with the remaining edges carrying weights of order  
302 1 uniformly in  $\varepsilon$ .

303 The rationale of the present section is in fact extendable to an even more general  
304 setup (including the one of periodic high-contrast PDEs), which we treat in the paper  
305 [10]. However, in the present work we consider a rather simplified model, in view  
306 of keeping technicalities to a bare minimum and thus hopefully making the matter  
307 transparent to the reader.

Consider the graph  $\mathbb{G}_\infty$  with the periodicity cell  $\mathbb{G}$  shown in Figure 4. The

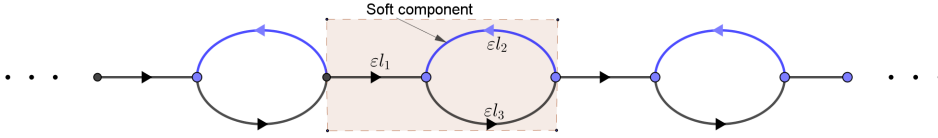


FIG. 4. PERIODICITY CELL  $\mathbb{G}$ . The intervals of lengths  $\varepsilon l_1$  and  $\varepsilon l_3$  are “stiff”, i.e. they carry the weights  $a_1^2$  and  $a_3^2$ , respectively, whereas the interval of length  $\varepsilon l_2$  is “soft”, with weight  $\varepsilon^2$ .

308 Gelfand transform, see Section 3, applied to this graph, yields the graph  $\widehat{\mathbb{G}}$  of Figure  
309 5. In the present section we show that there exists a boundary triple such that  $A_t^\varepsilon$   
310 is an almost solvable extension of the corresponding  $A_{\min}$ , and the  $M$ -function (which  
311 is in our case a matrix-valued function; for convenience, it is written as a function of  
312  $k := \sqrt{z}$ , with the branch chosen so that  $\Im k > 0$ ) of  $A_{\max}$  is given by

$$314 \quad (5.1) \quad M(k, \varepsilon, t) = k \widetilde{M}^{\text{stiff}}(\varkappa, \tau) + \varepsilon \widetilde{M}^{\text{soft}}(k, \tau), \quad \varkappa := \varepsilon k, \quad \tau := \varepsilon t,$$

<sup>4</sup>For a definition and properties of Herglotz functions, see e.g. [31].

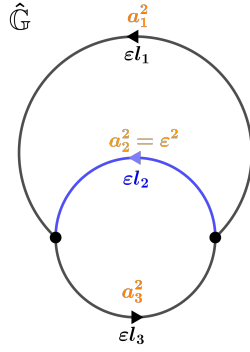


FIG. 5. THE GRAPH  $\widehat{\mathbb{G}}$ . *The left and right boundary vertices have been identified.*

315 where

$$316 \quad (5.2) \quad \widetilde{M}^{\text{stiff}}(\varkappa, \tau) := \begin{pmatrix} -a_1 \cot \frac{\varkappa l_1}{a_1} - a_3 \cot \frac{\varkappa l_3}{a_3} & a_1 \frac{e^{-i(l_1+l_3)\tau}}{\sin \frac{\varkappa l_1}{a_1}} + a_3 \frac{e^{il_2\tau}}{\sin \frac{\varkappa l_3}{a_3}} \\ a_1 \frac{e^{i(l_1+l_3)\tau}}{\sin \frac{\varkappa l_1}{a_1}} + a_3 \frac{e^{-il_2\tau}}{\sin \frac{\varkappa l_3}{a_3}} & -a_1 \cot \frac{\varkappa l_1}{a_1} - a_3 \cot \frac{\varkappa l_3}{a_3} \end{pmatrix},$$

317

$$318 \quad (5.3) \quad \widetilde{M}^{\text{soft}}(k, \tau) := k \begin{pmatrix} -\cot kl_2 & \frac{e^{il_2\tau}}{\sin kl_2} \\ \frac{e^{-il_2\tau}}{\sin kl_2} & -\cot kl_2 \end{pmatrix},$$

319 (Note that for all  $\tau \in [-\pi, \pi)$  the function  $\widetilde{M}^{\text{soft}}(\cdot, \tau)$  is meromorphic and regular at  
320 zero.)

321 Essentially, the claim made is a straightforward consequence of the double inte-  
322 gration by parts, followed by a simple rearrangement of terms. In the rest of this  
323 section we sketch the construction applicable in the general case, which in particular  
324 yields the result for the model graph considered. Under the definitions of Section  
325 4, the maximal operator  $A_{\max} = A_{\min}^*$  is defined by the same differential expression  
326 (3.3) on the domain  
327

$$328 \quad (5.4) \quad \text{dom}(A_{\max}) = \left\{ v \in \bigoplus_{e \in \widehat{\mathbb{G}}} W^{2,2}(e) \mid w_V(e)v|_e(V) = w_V(e')v|_{e'}(V) \right.$$

329

330

$$\left. \text{for all } e, e' \text{ adjacent to } V, \quad \forall V \in \widehat{\mathbb{G}} \right\}.$$

331 In what follows we use the triple  $(\mathbb{C}^m, \Gamma_0, \Gamma_1)$ , where  $m$  is the number of vertices in  
332 the graph  $\widehat{\mathbb{G}}$ , and

$$333 \quad (5.5) \quad \Gamma_0 v = \{v(V)\}_V, \quad \Gamma_1 v = \left\{ \sum_{e \ni V} \partial^{(t)} v(V) \right\}_V, \quad v \in \text{dom}(A_{\max}),$$

334 where  $v(V)$  is the common value of  $w_V(e)v|_e(V)$  for all edges  $e$  adjacent to  $V$ , and  
 335  $\partial^{(t)}v(V)$  is defined at the end of Section 3, see also (5.6) below.

By definition of the  $M$ -matrix one has  $\Gamma_1 v = M\Gamma_0 v$ , for functions  $v \in \ker(A_{\max} - z)$ , which have the form

$$v(x) = \exp(-ixt) \left\{ A_e \exp\left(-\frac{ikx}{a^\varepsilon}\right) + B_e \exp\left(\frac{ikx}{a^\varepsilon}\right) \right\}, \quad x \in e, \quad A_e, B_e \in \mathbb{C},$$

336 where  $k := \sqrt{z}$ , and the co-derivative is given by  
 (5.6)

$$337 \quad (a^\varepsilon)^2(v'(x) + itv(x)) = ika^\varepsilon \exp(-ixt) \left\{ -A_e \exp\left(-\frac{ikx}{a^\varepsilon}\right) + B_e \exp\left(\frac{ikx}{a^\varepsilon}\right) \right\}, \quad x \in e,$$

For the vertex  $V$  and for every ‘‘Dirichlet data’’ vector  $\Gamma_0 v$  one of whose entries is unity and the other entries vanish, the ‘‘Neumann data’’ vector  $\Gamma_1 v$  gives the column of the  $M$ -matrix corresponding to  $V$ . The elements of  $\Gamma_1 v$  corresponding to diagonal and off-diagonal entries of  $M(z)$  are, respectively,

$$-\sum_{e \in V} ka^\varepsilon \cot\left(\frac{k\varepsilon l_e}{a^\varepsilon}\right), \quad \sum_{e \in V} ka^\varepsilon \tilde{w}_V(e) \left(\sin \frac{k\varepsilon l_e}{a^\varepsilon}\right)^{-1},$$

338 where  $\{\tilde{w}_V(e)\}_{e \ni V}$  is a unimodular list uniquely determined by the list  $\{w_V(e)\}_{e \ni V}$ .  
 339 The resulting  $M$ -matrix is constructed from these columns over all vertices  $V$ .

340 In particular, for the example of Fig. 4–5, we have the following: the boundary  
 341 space  $\mathcal{H}$  pertaining to the graph  $\widehat{\mathbb{G}}$  is  $\mathcal{H} = \mathbb{C}^2$ . The unimodular list functions  $w_{V_1}$  and  
 342  $w_{V_2}$  are as follows, denoting by  $e^{(1)}$ ,  $e^{(3)}$  the stiff edges and by  $e^{(2)}$  the soft edge:

$$343 \quad \{w_{V_1}(e^{(j)})\}_{j=1}^3 = \{1, 1, e^{i\tau(l_2+l_3)}\}, \quad \{w_{V_2}(e^{(j)})\}_{j=1}^3 = \{e^{i\tau l_3}, 1, 1\},$$

344 and similarly

$$345 \quad \{\tilde{w}_{V_1}(e^{(j)})\}_{j=1}^3 = \{e^{-i\tau(l_1+l_3)}, e^{i\tau l_2}, e^{i\tau l_2}\},$$

$$\{\tilde{w}_{V_2}(e^{(j)})\}_{j=1}^3 = \{e^{i\tau(l_1+l_3)}, e^{-i\tau l_2}, e^{-i\tau l_2}\},$$

346 yielding the formulae (5.2), (5.3).

347 **6. Asymptotic diagonalisation of the  $M$ -matrix and the eigenvector**  
 348 **asymptotics.** The present section is the centrepiece of our approach. The major  
 349 difficulty to overcome is the fact that the operator  $A_t^\varepsilon$  entangles in a non-trivial way  
 350 the stiff and soft components of the medium. On the level of the analysis of the  
 351 operator itself this problem admits no obvious solution, unless one is prepared to in-  
 352 troduce a two-scale asymptotic ansatz. On the other hand, the  $M$ -matrix calculated  
 353 above will be shown to be additive with respect to the decomposition of the medium  
 354 (hence the notation  $M^{\text{soft}}$  and  $M^{\text{stiff}}$ ). Thus, via the representation (5.1), it proves  
 355 possible to use the asymptotic expansion of  $M^{\text{stiff}}$ , which is readily available, to re-  
 356 cover the asymptotics of eigenmodes, restricted to the soft component. This way, the  
 357 homogenisation task at hand can be viewed as a version of the perturbation analysis  
 358 in the boundary space pertaining to the problem.

359 In the example considered (and in the general case in view of Appendix A) it  
 360 follows from (4.3), (5.1) that  $u_\varepsilon$  is an eigenfunction of the operator  $A_t^\varepsilon$ , see (3.3)–  
 361 (3.4), if and only if

$$362 \quad (6.1) \quad M^{\text{soft}}\Gamma_0 u_\varepsilon = -M^{\text{stiff}}\Gamma_0 u_\varepsilon, \quad M^{\text{soft}} := \varepsilon \widetilde{M}^{\text{soft}}, \quad M^{\text{stiff}} := k \widetilde{M}^{\text{stiff}}.$$

363 In writing (6.1), we assume, without loss of generality, that the eigenvalue  $z_\varepsilon = k^2$   
 364 corresponding to the eigenfunction  $u_\varepsilon$  does not belong to the spectrum of the Dirichlet  
 365 decoupling  $A_\infty^t$ , defined according to the general theory of Section 4 for the operators  
 366 we introduce in Section 3. It follows from (5.2)–(5.3) that in any compact subset of  
 367  $\mathbb{C}$ , for small enough  $\varepsilon$ , this spectrum coincides with the  $\varepsilon$ -independent set of poles  
 368 of the matrix  $\widetilde{M}^{\text{soft}}$ . For this reason we can safely work under the assumption that  
 369 the eigenvalues  $z_\varepsilon$  do not belong to the spectrum of the Dirichlet operator on the  
 370 soft inclusion. This assumption ensures that the condition  $z_0 \in \mathbb{C} \setminus \text{spec}(A_\infty)$  for the  
 371 validity of (4.3) is satisfied in both cases: for the  $M$ -matrix of the operator  $A_t^\varepsilon$ , where  
 372  $B = 0$ , and for the  $M$ -matrix of the operator on the soft component represented by  
 373 (6.1), where the role of  $B$  is played by the matrix  $-M^{\text{stiff}}$ .

374 We proceed by observing that the matrices  $M^{\text{soft}}$  and  $M^{\text{stiff}}$  in (6.1) can be treated  
 375 as  $M$ -matrices of certain triples on their own. In particular, it will be instrumental in  
 376 what follows to attribute this meaning to  $M^{\text{soft}}$ . To this end, consider the decompo-  
 377 sition of the graph  $\widehat{\mathbb{G}}$  into its “soft”  $\mathbb{G}^{\text{soft}}$  and “stiff”  $\mathbb{G}^{\text{stiff}}$  components (each of these  
 378 is treated as a graph, so that  $\widehat{\mathbb{G}} = \mathbb{G}^{\text{soft}} \cup \mathbb{G}^{\text{stiff}}$ ) and the operator  $A_{\max}^{\text{soft}}$  defined by  
 379 (3.3), (5.4), with  $\widehat{\mathbb{G}}$  replaced by  $\mathbb{G}^{\text{soft}}$ . The boundary space for  $A_{\max}^{\text{soft}}$  can be defined  
 380 as  $\mathcal{H}$ , the same as the boundary space for the operator  $A_{\max}$  (again by Appendix A  
 381 in the general case). The boundary operators  $\Gamma_j^{\text{soft}}$ ,  $j = 0, 1$ , are defined as in (5.5) for  
 382 the graph  $\mathbb{G}^{\text{soft}}$ . Then, by inspection, the  $M$ -matrix for the operator  $A_{\max}^{\text{soft}}$  coincides  
 383 with  $M^{\text{soft}}$  (see [12] for further details).

384 For each  $v \in \text{dom}(A_{\max})$ , define  $\widetilde{v}$  to be the restriction of  $v$  to the soft component  
 385  $\mathbb{G}^{\text{soft}}$ , so that clearly  $\widetilde{v} \in \text{dom}(A_{\max}^{\text{soft}})$ . We notice that (6.1) implies, in particular, that

$$386 \quad (6.2) \quad M^{\text{soft}} \Gamma_0^{\text{soft}} \widetilde{u}_\varepsilon = B^\varepsilon \Gamma_0^{\text{soft}} \widetilde{u}_\varepsilon, \quad B^\varepsilon := -M^{\text{stiff}}.$$

387 Furthermore, since  $M^{\text{soft}}$  is the  $M$ -matrix for the pair  $(\Gamma_0^{\text{soft}}, \Gamma_1^{\text{soft}})$ , one has

$$388 \quad M^{\text{soft}} \Gamma_0^{\text{soft}} \widetilde{u}_\varepsilon = \Gamma_1^{\text{soft}} \widetilde{u}_\varepsilon,$$

389 so the condition (6.2) takes a form similar to (4.2):

$$390 \quad (6.3) \quad \Gamma_1^{\text{soft}} \widetilde{u}_\varepsilon = B^\varepsilon \Gamma_0^{\text{soft}} \widetilde{u}_\varepsilon.$$

391 This condition involves the Dirichlet data of the solution to the spectral equation  
 392 for  $A_{\max}^{\text{soft}}$  which is an ODE on the graph  $\mathbb{G}^{\text{soft}}$  with a constant coefficient. The Dirichlet  
 393 data  $\Gamma_0^{\text{soft}} \widetilde{u}_\varepsilon$  determine the vector  $\widetilde{u}_\varepsilon$  uniquely. The named vector is interpreted as a  
 394 solution to the spectral equation on the soft component of the graph  $\widehat{\mathbb{G}}$  subject to  $z$ -  
 395 dependent boundary conditions, encoded in (6.3). On the other hand, this vector can  
 396 also be used to reconstruct the vector  $u_\varepsilon$ : indeed, from  $\Gamma_0 u_\varepsilon = \Gamma_0^{\text{soft}} \widetilde{u}_\varepsilon$  it follows, that  
 397  $u_\varepsilon$ , which is by assumption an eigenvector to  $A_t^\varepsilon$  at the point  $z$ , is simply a continuation  
 398 of  $\widetilde{u}_\varepsilon$  to the rest of the graph  $\widehat{\mathbb{G}}$  based on its Dirichlet data at the boundary of the soft  
 399 component. It follows, *cf.* (6.3), that the asymptotic analysis can be reduced to the  
 400 soft component, with the information about the stiff component fed into the related  
 401 asymptotic procedure by means of the stiff-soft interface.

402 Before we proceed further, let us take another look at the equation  $M \Gamma_0 u_\varepsilon = 0$ ,  
 403 *cf.* (6.1), which is equivalent to  $u_\varepsilon$  being an eigenvector of  $A_t^\varepsilon$  at the value of spectral  
 404 parameter  $z$ . Using the fact that  $M = M^{\text{soft}} + M^{\text{stiff}}$  as well as the explicit expressions  
 405 for the matrices  $M^{\text{soft}}$ ,  $M^{\text{stiff}}$ , *cf.* (5.1), it is easily seen that the leading-order term of  
 406  $\Gamma_0 u_\varepsilon$ , and thus of  $u_\varepsilon$ , does not depend on the soft component of the medium, since the  
 407 elements of  $M^{\text{soft}}$  are  $\varepsilon$ -small. On the other hand, the situation is drastically different

408 from the viewpoint of the associated dispersion relation, which must be guaranteed  
 409 for the *solvability* of  $M\Gamma_0 u_\varepsilon = 0$ . The dispersion relation follows from the condition  
 410  $\det M = 0$ , and it is *here, and here only*, that the soft component of the medium  
 411 makes its presence felt in the problem. Due to the fact that  $M^{\text{stiff}}$  is rank one at  
 412  $\tau = 0$ , it transpires that the leading-order term of the equation  $\det M = 0$  *in the*  
 413 *case of critical contrast only* blends together in a non-trivial way the stiff and soft  
 414 components of the medium. Bearing this in mind, the phenomenon of critical-contrast  
 415 homogenisation can be seen as a manifestation of a frequency-converting device: if  
 416 one restricts the eigenfunctions to the stiff component, they are  $\varepsilon$ -close to those of the  
 417 medium where the soft component has been replaced with voids, *but* correspond to  
 418 non-trivially shifted eigenfrequencies. This is precisely what one would expect in the  
 419 setting of time-dispersive media after the passage to the frequency domain, *cf.* (1.1),  
 420 (1.2). We will come back to this discussion in Section 8.

421 Let us return to the analysis of (6.3), which, as explained above, contains all the  
 422 information on the asymptotic behaviour of  $A_\varepsilon^\xi$ . We notice that the named equation  
 423 corresponds to a homogeneous ODE; the non-trivial dependence on  $\varepsilon$  is concealed  
 424 in the right-hand side, which describes  $\varepsilon$ - and frequency-dependent boundary condi-  
 425 tions. The problem of asymptotic analysis of eigenfunctions of  $A_\varepsilon^\xi$  is thus effectively  
 426 reduced to the analysis of the asymptotic behaviour of these boundary conditions.  
 427 This analysis, however, is simplified by the fact that  $B^\varepsilon = -M^{\text{stiff}}$ , see (6.2), where  
 428  $M^{\text{stiff}}$  is shown to be the  $M$ -matrix of  $A_{\text{max}}^{\text{stiff}}$  (see Appendix A) by a similar argument  
 429 to that applied above to  $M^{\text{soft}}$ . Hence, the asymptotics sought for  $M^{\text{stiff}}$  is simply  
 430 the asymptotics of the Dirichlet-to-Neumann map of a uniformly elliptic problem at  
 431 zero frequency, which allows to use well-known elliptic techniques.

432 Firstly, we notice that the results of Section 5 combined with the asymptotic  
 433 formulae

$$434 \quad a_e \cot \frac{\varkappa l_e}{a_e} = \frac{a_e^2}{\varkappa l_e} - \frac{1}{3} \varkappa l_e + O(\varkappa^3), \quad a_e \left( \sin \frac{\varkappa l_e}{a_e} \right)^{-1} = \frac{a_e^2}{\varkappa l_e} + \frac{1}{6} \varkappa l_e + O(\varkappa^3),$$

435 yield the following statement.

436 LEMMA 6.1. *Suppose that  $K \subset \mathbb{C}$  is compact. One has*

$$437 \quad \widetilde{M}^{\text{stiff}}(\varkappa, \tau) = \varkappa^{-1} M_0(\tau) + \varkappa M_1(\tau) + O(\varkappa^3), \quad \tau \in [-\pi, \pi], \quad \varkappa = \varepsilon k, \quad \varepsilon \in (0, 1), \quad k \in K,$$

438 where  $M_0$  and  $M_1$  are analytic matrix functions of  $\tau$ .

439 It follows from Lemma 6.1 that, for all  $\tau \in [-\pi, \pi)$ ,

$$440 \quad (6.4) \quad B^\varepsilon(z) = \varepsilon^{-1} B_0 + \varepsilon z B_1 + O(\varepsilon^3 z^2), \quad \varepsilon \in (0, 1), \quad \sqrt{z} \in K,$$

441 where  $B_0, B_1$  are Hermitian matrices that depend on  $\tau$  only. The following two  
 442 lemmata, proved in Appendices B and C, carry over to the general case with only  
 443 minor modifications, since they pertain to the stiff component of the medium and  
 444 therefore rely upon the general uniformly elliptic properties of the latter.

445 LEMMA 6.2. *There exist  $\gamma \geq 0$  (where  $\gamma = 0$  if and only if the graph  $\mathbb{G}^{\text{stiff}}$  is a*  
 446 *tree<sup>5</sup>) and an eigenvalue branch  $\mu^{(\tau)}$  for the matrix  $B_0$ , such that  $\dim \text{Ker}(B_0 - \mu^{(\tau)}) =$   
 447  $1, \tau \in [-\pi, \pi)$ , and*

$$448 \quad (6.5) \quad \mu^{(\tau)} = \gamma \tau^2 + O(\tau^4).$$

---

<sup>5</sup>Recall that a tree is a connected forest [13].

449 We denote by  $\psi^{(\tau)}$  the normalised eigenvector for the eigenvalue  $\mu^{(\tau)}$ , so that  
 450  $\psi^{(0)} = (1/\sqrt{2})(1, 1)^\top$ , *i.e.* the trace of the first eigenvector of the Neumann problem  
 451 on the stiff component at zero quiasimomentum, which is clearly constant. Let  $\mathcal{P} :=$   
 452  $\langle \cdot, \psi^{(\tau)} \rangle_{\mathcal{H}} \psi^{(\tau)}$  and  $\mathcal{P}_\perp$  be the orthogonal projections in the boundary space onto  $\psi^{(\tau)}$   
 453 and its orthogonal complement, respectively.

454 LEMMA 6.3. *There exists  $C_\perp > 0$  such that*

$$455 \quad (6.6) \quad \mathcal{P}_\perp B_0 \mathcal{P}_\perp \geq C_\perp \mathcal{P}_\perp,$$

456 *in the sense that the operator  $\mathcal{P}_\perp (B_0 - C_\perp) \mathcal{P}_\perp$  is non-negative.*

457 We use Lemma 6.3 to solve (6.3) asymptotically. The overall idea is to diagonalise  
 458 the leading order term  $\varepsilon^{-1} B_0$  of the asymptotic expansion of  $B^\varepsilon$  in (6.3). From Lemma  
 459 6.2 we infer that  $B_0$  has precisely one eigenvalue quadratic in  $\tau$  (which thus gets  
 460 close to zero), while Lemma 6.3 provides us with a bound below on the remaining  
 461 eigenvalue. The fact that the eigenvalue  $\mu^{(\tau)}$  degenerates requires that the next  
 462 term in the asymptotics of  $B^\varepsilon$  be taken into account in the related eigenspace. This  
 463 additional term is easily seen to be  $z$ -dependent (in fact, linear in  $z$ ).

464 We start with an auxiliary rescaling of the soft component. Namely, we introduce  
 465 the unitary operator  $\Phi_\varepsilon$  mapping  $v \mapsto \hat{v}$  according to the formula  $\hat{v}(\cdot) = \sqrt{\varepsilon} v(\varepsilon \cdot)$ .  
 466 Under this mapping, the length of the soft component loses its dependence on  $\varepsilon$ . The  
 467 operator  $\hat{A}_{\max}^{\text{soft}}$  is defined as the unitary image of  $A_{\max}^{\text{soft}}$  under the mapping  $\Phi_\varepsilon$ , and  
 468  $\hat{\Gamma}_0^{\text{soft}}, \hat{\Gamma}_1^{\text{soft}}$  are the boundary operators for the rescaled soft component:

$$469 \quad \hat{\Gamma}_0^{\text{soft}} \hat{v} := \{\hat{v}(V)\}_V, \quad \hat{\Gamma}_1^{\text{soft}} \hat{v} := \left\{ \sum_{e \ni V} \hat{\partial}^{(\tau)} \hat{v}(V) \right\}_V, \quad \hat{v} \in \text{dom}(\hat{A}_{\max}^{\text{soft}}),$$

470 where we set  $\hat{v}(V)$  as the common value of  $w_V(e) \hat{v}|_e(V)$  for all  $e$  adjacent to  $V$ , and  
 471  $\hat{\partial}^{(\tau)} \hat{v}(V)$  is the expression  $\sigma_e w_V(e) (\hat{v}' + i\tau \hat{v})$  on the edge  $e$ , calculated at  $V$ . Note that  
 472  $\hat{\Gamma}_1^{\text{soft}}$  does not depend on  $\varepsilon$ .

473 Under the rescaling  $\Phi_\varepsilon$  the equation (6.3) becomes

$$474 \quad (6.7) \quad \hat{\Gamma}_1^{\text{soft}} \hat{u}_\varepsilon = \varepsilon^{-1} B^\varepsilon \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon,$$

475 where in accordance with the above convention  $\hat{u}_\varepsilon = \Phi_\varepsilon \tilde{u}_\varepsilon$ .

476 We start our diagonalisation procedure by considering the non-degenerate eigen-  
 477 space of  $B^\varepsilon$ . Applying  $\mathcal{P}_\perp$  to both sides of (6.7), replacing  $B^\varepsilon$  by its asymptotics (6.4)  
 478 and using (6.6) yields

$$479 \quad (6.8) \quad \mathcal{P}_\perp \hat{\Gamma}_1^{\text{soft}} \hat{u}_\varepsilon = \varepsilon^{-2} \mathcal{P}_\perp B_0 \mathcal{P}_\perp \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon + O(1) \geq \varepsilon^{-2} C_\perp \mathcal{P}_\perp \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon + O(1),$$

480 where we assume that  $u_\varepsilon$  is  $L^2$ -normalised. Multiplying by  $\varepsilon^2$  both sides of (6.8) and  
 481 applying the Sobolev embedding theorem to the left-hand side of (6.8), we infer

$$482 \quad (6.9) \quad \mathcal{P}_\perp \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon = O(\varepsilon^2).$$

483 Plugging this partial solution back into (6.7), to which  $\mathcal{P}$  is applied on both sides, we  
 484 obtain

$$485 \quad \begin{aligned} \mathcal{P} \hat{\Gamma}_1^{\text{soft}} \hat{u}_\varepsilon &= \varepsilon^{-2} \mathcal{P} B_0 \mathcal{P} \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon + z \mathcal{P} B_1 \mathcal{P} \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon + O(\varepsilon^2) \\ &= \varepsilon^{-2} \mu^{(\tau)} \mathcal{P} \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon + z \mathcal{P} B_1 \mathcal{P} \hat{\Gamma}_0^{\text{soft}} \hat{u}_\varepsilon + O(\varepsilon^2). \end{aligned}$$

486

488 We have proved that up to an error term admitting a uniform estimate  $O(\varepsilon^2)$  one  
 489 has the following asymptotically equivalent problem for the eigenvector  $\widehat{v}_\varepsilon$ :

$$490 \quad (6.10) \quad \mathcal{P}_\perp \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon = 0, \quad \mathcal{P} \widehat{\Gamma}_1^{\text{soft}} \widehat{u}_\varepsilon = \varepsilon^{-2} \mu^{(\tau)} \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon + z \mathcal{P} B_1 \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon.$$

491 We use Lemma 6.2 and expand  $\mathcal{P} B_1 \mathcal{P}$  in powers of  $\tau = \varepsilon t$  as follows<sup>6</sup>:  $\mathcal{P} B_1 \mathcal{P} =$   
 492  $\mathcal{P} B_1^{(0)} \mathcal{P} + O(\tau)$ . The second equation in (6.10) admits the form

$$493 \quad (6.11) \quad \mathcal{P} \widehat{\Gamma}_1^{\text{soft}} \widehat{u}_\varepsilon = \gamma t^2 \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon + z \mathcal{P} B_1^{(0)} \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon + (O(\tau) + O(\tau^4/\varepsilon^2)) \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon.$$

494 Expressing  $\mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon$  from the latter equation, it is easily seen based on embedding  
 495 theorems that (6.11) is asymptotically equivalent, up to an error uniformly estimated  
 496 as  $O(\varepsilon)$ , to the following equation:

$$497 \quad (6.12) \quad \mathcal{P} \widehat{\Gamma}_1^{\text{soft}} \widehat{u}_\varepsilon = \gamma t^2 \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon + z \mathcal{P} B_1^{(0)} \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}_\varepsilon.$$

498 We formulate the above result as the following theorem.

499 **THEOREM 6.4.** *Let  $\widehat{u}$  solve the following equation on the re-scaled soft component:*

$$\begin{aligned} & \widehat{A}_{\max}^{\text{soft}} \widehat{u} = z \widehat{u}, \\ 500 \quad (6.13) \quad & \mathcal{P}_\perp \widehat{\Gamma}_0^{\text{soft}} \widehat{u} = 0, \\ & \mathcal{P} \widehat{\Gamma}_1^{\text{soft}} \widehat{u} = \gamma t^2 \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u} + z \mathcal{P} B_1^{(0)} \mathcal{P} \widehat{\Gamma}_0^{\text{soft}} \widehat{u}. \end{aligned}$$

501 *Then the eigenvalues  $z_\varepsilon$  and their corresponding eigenfunctions  $u_\varepsilon$  of the operators*  
 502  *$A_t^\varepsilon$ , see (3.3), (3.4), are  $O(\varepsilon)$ -close uniformly in  $t \in [-\pi/\varepsilon, \pi/\varepsilon]$ , in the sense of  $\mathbb{C}$*   
 503 *and in the sense of the  $L^2$  norm, respectively, to the values  $z$  as above and functions*  
 504  *$u_{\text{eff}}$  defined as follows. On the soft component  $\mathbb{G}^{\text{soft}}$  we set  $u_{\text{eff}}(\cdot) := (1/\sqrt{\varepsilon}) \widehat{u}(\varepsilon^{-1} \cdot)$ ,*  
 505 *where  $\widehat{u}$  solves (6.13). On the stiff component  $\mathbb{G}^{\text{stiff}}$  the function  $u_{\text{eff}}$  is obtained as*  
 506 *the extension by  $(1/\sqrt{\varepsilon})v$ , where  $v$  is the solution of the operator equation*

$$507 \quad A_{\max}^{\text{stiff}} v = 0,$$

508 *determined by the Dirichlet data of  $\widehat{u}(\varepsilon^{-1} \cdot)$ , where  $A_{\max}^{\text{stiff}}$  is defined by (8.14), Appendix*  
 509 *A.*

*Remark 6.5.* It is straightforward to see that the eigenvalue  $\mu^{(\tau)}$  in Lemma 6.2 is  
 the least, by absolute value, Steklov eigenvalue of  $A_{\max}^{\text{stiff}}$ , i.e. the least  $\kappa$  such that the  
 problem

$$\begin{aligned} & A_{\max}^{\text{stiff}} \check{v} = 0, \quad \check{v} \in W^{2,2}(\mathbb{G}^{\text{stiff}}), \\ & \Gamma_1^{\text{stiff}} \check{v} = \kappa \Gamma_0^{\text{stiff}} \check{v}. \end{aligned}$$

510 admits a non-trivial solution  $\check{v}$ . Note that for this solution  $\check{v}$  one has  $\Gamma_0^{\text{stiff}} \check{v} = \psi^{(\tau)}$ ,  
 511 where  $\psi^{(\tau)}$  is defined in the text following Lemma 6.2. It follows that for the function  
 512  $v$  of Theorem 6.4 one has  $v = c \check{v}$ , where  $c$  is a constant determined by  $\widehat{u}$ .

<sup>6</sup>In the example considered in the present paper, as opposed to the general case, one can prove  
 that  $\mathcal{P} B_1 \mathcal{P} = \mathcal{P} B_1^{(0)} \mathcal{P} + O(\tau^2)$ , see the calculation in [11, Appendix B] for details. This yields the  
 error bound  $O(\varepsilon^2)$  in the statement of Theorem 6.4.



### 7. Eigenvalue and eigenvector asymptotics in the example of Section 5.

Here we provide the result of an explicit calculation applying the general procedure described in the previous section to the specific example of Section 5 (see [11] for details). We start by expanding the matrix  $B^\varepsilon$  as a series in powers of  $\varepsilon$ :

$$\widehat{B} := \varepsilon^{-1} B^\varepsilon = \widehat{B}_0 + z\widehat{B}_1 + O(\varepsilon^2 z^2), \quad \widehat{B}_0 := \frac{1}{\varepsilon^2} \begin{pmatrix} D & \bar{\xi} \\ \xi & D \end{pmatrix}, \quad \widehat{B}_1 := \begin{pmatrix} E & \bar{\eta} \\ \eta & E \end{pmatrix},$$

513 where

$$514 \quad (7.1) \quad \xi := -\frac{a_1^2}{l_1} \exp(i\tau(l_1 + l_3)) - \frac{a_3^2}{l_3} \exp(-i\tau l_2), \quad D := \frac{a_1^2}{l_1} + \frac{a_3^2}{l_3},$$

$$515 \quad \eta := \frac{1}{6} \left( l_1 \exp(i\tau(l_1 + l_3)) + l_3 \exp(-i\tau l_2) \right), \quad E := \frac{1}{3} (l_1 + l_3).$$

517 The matrix  $\varepsilon^2 \widehat{B}_0$  is Hermitian and has two distinct eigenvalues,  $\mu = D - |\xi|$  and  
 518  $\mu_\perp = D + |\xi|$ . The eigenvalue branch  $\mu$  is singled out by the condition  $\mu|_{\tau=0} = 0$ .  
 519 In order to diagonalise the matrix  $\widehat{B}_0$ , consider the normalised eigenvectors  $\psi^{(\tau)} =$   
 520  $(1/\sqrt{2})(1, -\xi/|\xi|)^\top$  and  $\psi_\perp^{(\tau)} = (1/\sqrt{2})(1, \xi/|\xi|)^\top$  corresponding to the eigenvalues  $\mu$   
 521 and  $\mu_\perp$ , respectively, as well as the matrix  $X := (\psi^{(\tau)}, \psi_\perp^{(\tau)})$ . The projections  $\mathcal{P}, \mathcal{P}_\perp$ ,  
 522 introduced in the previous section, are as follows:

$$523 \quad \mathcal{P} = \frac{1}{2} \begin{pmatrix} 1 & -\frac{\bar{\xi}}{|\xi|} \\ -\frac{\xi}{|\xi|} & 1 \end{pmatrix}, \quad \mathcal{P}_\perp = \frac{1}{2} \begin{pmatrix} 1 & \frac{\bar{\xi}}{|\xi|} \\ \frac{\xi}{|\xi|} & 1 \end{pmatrix}.$$

524 It follows by a straightforward calculation that the effective spectral problem is  
 525 given by

$$526 \quad (7.2) \quad -\left(\frac{d}{dx} + i\tau\right)^2 u = zu,$$

527

$$528 \quad (7.3) \quad u(0) = -\frac{\bar{\xi}}{|\xi|} u(l_2),$$

$$(u' + i\tau u)(0) + \frac{\bar{\xi}}{|\xi|} (u' + i\tau u)(l_2) = \left( \left( \frac{l_1}{a_1^2} + \frac{l_3}{a_3^2} \right)^{-1} \left( \frac{\tau}{\varepsilon} \right)^2 - (l_1 + l_3)z \right) u(0),$$

529 By invoking Theorem 6.4, the problem (7.2)–(7.3) on the scaled soft component  
 530 provides the asymptotics, as  $\varepsilon \rightarrow 0$ , of the eigenvalue problems for the family  $A_t^\varepsilon$ ,  
 531  $t = \tau/\varepsilon \in [-\pi/\varepsilon, \pi/\varepsilon)$ . Its spectrum, *i.e.* the set of values  $z$  for which (7.2)–(7.3)  
 532 has a non-trivial solution, as well as the corresponding eigenfunctions approximate,  
 533 up to terms of order  $O(\varepsilon^2)$ , the corresponding spectral information for the family  $A_t^\varepsilon$ ,  
 534 and consequently,  $A^\varepsilon$ . Notice that the stiff component of the original graph (where  
 535 the eigenfunctions converge to a constant, in a suitable scaled sense), appears in this  
 536 limit problem through the boundary datum  $u(0)$ . In the next section we show that an  
 537 appropriate extension of the function space for (7.2)–(7.3) by the (one-dimensional)  
 538 complementary space of constants leads to an eigenvalue problem for a self-adjoint  
 539 operator, describing a conservative system. Solving this latter eigenvalue problem for  
 540 the element in the complementary space yields a frequency-dispersive formulation we  
 541 announced in the introduction.

542 **8. Frequency dispersion in a “complementary” medium.**

 543 **8.1. Self-adjoint out-of-space extension.** Following the strategy outlined at  
 544 the end of the last section, we treat  $u(0)$  in (7.3) as an additional field variable, and  
 545 reformulate (7.2)–(7.3) as an eigenvalue problem in a space of pairs  $(u, u(0))$ , see (8.4).

 546 More precisely, for all values  $\tau \in [-\pi, \pi)$ , consider an operator  $A_\tau^{\text{hom}}$  in the space  
 547  $L^2(0, l_2) \oplus \mathbb{C}$  defined as follows. The domain  $\text{dom}(A_\tau^{\text{hom}})$  consist of all pairs  $(u, \beta)$   
 548 such that  $u \in W^{2,2}(0, l_2)$  and the quasiperiodicity condition

549 (8.1) 
$$u(0) = \overline{w_\tau} u(l_2) =: \frac{\beta}{\sqrt{l_1 + l_3}}, \quad w_\tau \in \mathbb{C},$$

 550 is satisfied. On  $\text{dom}(A_\tau^{\text{hom}})$  the action of the operator is set by

551 (8.2) 
$$A_\tau^{\text{hom}} \begin{pmatrix} u \\ \beta \end{pmatrix} = \begin{pmatrix} -\left(\frac{d}{dx} + i\tau\right)^2 u \\ \frac{1}{\sqrt{l_1 + l_3}} \Gamma_\tau \begin{pmatrix} u \\ \beta \end{pmatrix} \end{pmatrix},$$

 552 where  $\Gamma_\tau : W^{2,2}(0, l_2) \oplus \mathbb{C} \rightarrow \mathbb{C}$  is bounded. We set

553 (8.3) 
$$\Gamma_\tau \begin{pmatrix} u \\ \beta \end{pmatrix} = -(u' + i\tau u)(0) + \overline{w_\tau}(u' + i\tau u)(l_2) + \frac{(\sigma t)^2}{\sqrt{l_1 + l_3}} \beta, \quad \sigma^2 := \left(\frac{l_1}{a_1^2} + \frac{l_3}{a_3^2}\right)^{-1},$$

 554 where  $w_\tau = -\xi/|\xi|$  (see (7.1) for the definition of  $\xi$ ), in which case  $A_\tau^{\text{hom}}$  is a self-  
 555 adjoint operator on the domain described by (8.1). Moreover, (7.2)–(7.3) is the prob-  
 556 lem on the first component of spectral problem for the operator  $A_\tau^{\text{hom}}$  :

557 (8.4) 
$$A_\tau^{\text{hom}} \begin{pmatrix} u \\ \beta \end{pmatrix} = z \begin{pmatrix} u \\ \beta \end{pmatrix}.$$

 558 We now re-write this spectral problem in terms of the complementary component  
 559  $\beta \in \mathbb{C}$ . In order to do this, we represent the function  $u$  in (8.4) as a sum of two: one  
 560 of them is a solution to the related inhomogeneous Dirichlet problem, while the other  
 561 takes care of the boundary condition. More precisely, consider the solution  $v$  to the  
 562 problem

563 
$$-\left(\frac{d}{dx} + i\tau\right)^2 v = 0, \quad v(0) = 1, \quad v(l_2) = w_\tau,$$

 564 *i.e.*

565 (8.5) 
$$v(x) = \left\{1 + l_2^{-1} \left(w_\tau \exp(i\tau l_2) - 1\right) x\right\} \exp(-i\tau x), \quad x \in (0, l_2).$$

566 The function

567 
$$\tilde{u} := u - \frac{\beta}{\sqrt{l_1 + l_3}} v$$

568 satisfies

569 
$$-\left(\frac{d}{dx} + i\tau\right)^2 \tilde{u} - z\tilde{u} = \frac{z\beta}{\sqrt{l_1 + l_3}} v, \quad \tilde{u}(0) = \tilde{u}(l_2) = 0.$$

570 In other words, one has

$$571 \quad \tilde{u} = \frac{z\beta}{\sqrt{l_1 + l_3}} (A_D^{(\tau)} - zI)^{-1} v,$$

572 where  $A_D^{(\tau)}$  is the Dirichlet operator in  $L^2(0, l_2)$  associated with the differential ex-  
573 pression

$$574 \quad -\left(\frac{d}{dx} + i\tau\right)^2.$$

575 We now write the ‘‘boundary’’ part of the spectral equation (8.4) as  
(8.6)

$$576 \quad K(\tau, z)\beta = z\beta, \quad K(\tau, z) := \frac{1}{l_1 + l_3} \left\{ z\Gamma_\tau \begin{pmatrix} (A_D^{(\tau)} - zI)^{-1} v \\ 0 \end{pmatrix} + \Gamma_\tau \begin{pmatrix} v \\ \sqrt{l_1 + l_3} \end{pmatrix} \right\}.$$

577 In accordance with the rationale for introducing the component  $\beta$ , the effective dis-  
578 persion relation for the operator  $A_{\tau/\varepsilon}^\varepsilon$ ,  $\tau \in [-\pi, \pi)$ , is given by

$$579 \quad K(\tau, z) = z.$$

580 The explicit expression for this relation that we have obtained, see (8.6), is new, and it  
581 quantifies explicitly the rôle of the soft component of the composite in the macroscopic  
582 frequency-dispersive properties. In particular, the expression (8.6) shows that the soft  
583 inclusions enter the macroscopic equations via a Dirichlet-to-Neumann map on the  
584 boundary of the inclusions.

585 **8.2. Explicit formula for the time-dispersion kernel.** Here we compute  
586 explicitly the kernel  $K(\tau, z)$  entering the effective dispersion relation for  $A_\tau^\varepsilon$ . In view  
587 of possible generalisations, and recalling the pioneering formula in [38, Section 8] for  
588 effective dispersion in double-porosity media, we represent the action of the resolvent  
589  $(A_D^{(\tau)} - zI)^{-1}$  as a series in terms of the normalised eigenfunctions

$$590 \quad (8.7) \quad \phi_j(x) = \sqrt{\frac{2}{l_2}} \exp(-i\tau x) \sin \frac{\pi j x}{l_2}, \quad x \in (0, l_2), \quad j = 1, 2, 3, \dots,$$

591 of the operator  $A_D^{(\tau)}$ . This yields

$$592 \quad (8.8) \quad K(\tau, z) := \frac{1}{l_1 + l_3} \left\{ z \sum_{j=1}^{\infty} \frac{\langle v, \phi_j \rangle_{L^2(0, l_2)}}{\mu_j - z} \Gamma_\tau \begin{pmatrix} \phi_j \\ 0 \end{pmatrix} + \Gamma_\tau \begin{pmatrix} v \\ \sqrt{l_1 + l_3} \end{pmatrix} \right\}.$$

593 where  $\mu_j = (\pi j / l_2)^2$ ,  $j = 1, 2, 3, \dots$ , are the eigenvalues corresponding to (8.7). For  
594 the choice (8.3) of  $\Gamma_\tau$  we obtain (see (8.5), (8.7))

$$595 \quad \Gamma_\tau \begin{pmatrix} v \\ \sqrt{l_1 + l_3} \end{pmatrix} = \frac{2}{l_2} (1 - \Re\theta(\tau)) + \left(\frac{\sigma\tau}{\varepsilon}\right)^2, \quad \theta(\tau) := \frac{\frac{a_1^2}{l_1} e^{-i\tau} + \frac{a_3^2}{l_3}}{\left| \frac{a_1^2}{l_1} e^{-i\tau} + \frac{a_3^2}{l_3} \right|},$$

$$596 \quad \Gamma_\tau \begin{pmatrix} \phi_j \\ 0 \end{pmatrix} = -\sqrt{\frac{2}{l_2}} \frac{\pi j}{l_2} ((-1)^{j+1} \overline{\theta(\tau)} + 1),$$

$$597 \quad \langle v, \phi_j \rangle_{L^2(0, l_2)} = \frac{\sqrt{2l_2}}{\pi j} ((-1)^{j+1} \theta(\tau) + 1), \quad j = 1, 2, \dots$$

598 Substituting the above expressions into (8.8) and making use of the formulae, see *e.g.*  
 599 [23, p. 48],

$$600 \sum_{j=1}^{\infty} \frac{1}{(\pi j)^2 - x^2} = \frac{1}{2} \left( \frac{1}{x^2} - \frac{\cos x}{x \sin x} \right), \quad \sum_{j=1}^{\infty} \frac{(-1)^j}{(\pi j)^2 - x^2} = \frac{1}{2} \left( \frac{1}{x^2} - \frac{1}{x \sin x} \right), \quad x \notin \pi\mathbb{Z},$$

601 we obtain

$$602 (8.9) \quad K(\tau, z) = \frac{1}{l_1 + l_3} \left\{ \frac{2\sqrt{z} \cos(l_2\sqrt{z})}{\sin(l_2\sqrt{z})} - \frac{2\sqrt{z}}{\sin(l_2\sqrt{z})} \Re\theta(\tau) + \left( \frac{\sigma\tau}{\varepsilon} \right)^2 \right\}.$$

603 **8.3. Asymptotically equivalent model on the real line.** In this section we  
 604 are going to treat (8.6), (8.9) as a nonlinear eigenvalue problem in the space of second  
 605 components of pairs  $(u, \beta) \in L^2(0, l_2) \oplus \mathbb{C}$ . As is evident from above, this problem is  
 606 closely related to (7.2)–(7.3), via the construction presented in Section 8.1. We show  
 607 next that the aforementioned macroscopic field is governed by a certain frequency-  
 608 dispersive formulation. In order to obtain the latter, we will use a suitable inverse  
 609 Gelfand transform.

610 Our strategy can be seen as motivated by the following elementary observation,  
 611 closely linked with the Birman-Suslina study [5] of homogenisation in the moderate  
 612 contrast case, albeit understood in terms of spectral equations. Starting with the  
 613 spectral problem

$$614 (8.10) \quad -\frac{d^2u}{dx^2} = zu \quad \text{on } L_2(\mathbb{R}),$$

one applies the Gelfand transform<sup>7</sup> (well defined on generalised eigenvectors due to  
 the rigging procedure, see, *e.g.*, [2, 4]) to obtain for  $\tilde{u} := \mathcal{G}u$

$$-\left( \frac{d}{dx} + it \right)^2 \tilde{u}(x, t) = z\tilde{u}(x, t), \quad x \in (0, \varepsilon), \quad t \in [-\pi/\varepsilon, \pi/\varepsilon].$$

We compute the inner products of both sides in  $L_2(0, \varepsilon)$  with the normalised constant  
 function  $(1/\sqrt{\varepsilon})\mathbb{1}$ , which yields the dispersion relation of the original problem via the  
 equation

$$t^2 \hat{u}(t) = z \hat{u}(t),$$

615 where  $\hat{u}$  is the Fourier transform of the function  $u \in L_2(\mathbb{R})$ . The latter equation is  
 616 then solved in the distributional sense,

$$617 (8.11) \quad \beta(t) = \sum_m c_m \delta(t - t_m),$$

618 where  $\beta(t) := \hat{u}(t)$  and the sum in (8.11) is taken over  $m = 1, 2$ , so that  $t_1, t_2$  are  
 619 the solutions of the equation  $t^2 = z$ , and  $c_m$  are arbitrary constants. Ultimately, one

<sup>7</sup>Recall, *cf.* Section 3, that the Gelfand transform is a map  $L^2(\mathbb{R}) \rightarrow L^2((0, \varepsilon) \times (-\pi/\varepsilon, \pi/\varepsilon))$   
 given by

$$\mathcal{G}u(y, t) = \sqrt{\frac{\varepsilon}{2\pi}} \sum_{n \in \mathbb{Z}} u(x + \varepsilon n) \exp(-it(x + \varepsilon n)), \quad t \in [-\pi/\varepsilon, \pi/\varepsilon], \quad x \in (0, \varepsilon).$$

620 applies the inverse Gelfand transform

$$621 \quad (\mathcal{G}^* f)(x) = \sqrt{\frac{\varepsilon}{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} f(t) \exp(itx) dt, \quad f \in L^2\left(-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right), \quad x \in \mathbb{R},$$

to the function  $\mathfrak{B}(x, t) := (1/\sqrt{\varepsilon})\beta(t)\mathbb{1}(x)$ , *i.e.*

$$v(x) := \sqrt{\frac{\varepsilon}{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \mathfrak{B}(x, t) \exp(itx) dt, \quad x \in \mathbb{R}.$$

622 It is easily seen that this function is precisely the solution to (8.10).

623 We emulate the above argument for the case of interest to us, starting from  
624 the eigenvalue problem  $K(\tau, z)\beta = z\beta$ , which we now treat as an equation in the  
625 distributional sense with  $K$  given by (8.9). It admits the form

$$626 \quad (8.12) \quad (\sigma t)^2 \beta = \left\{ (l_1 + l_3)z - \frac{2\sqrt{z} \cos(l_2\sqrt{z})}{\sin(l_2\sqrt{z})} + \frac{2\sqrt{z}}{\sin(l_2\sqrt{z})} \Re\theta(\varepsilon t) \right\} \beta, \quad t = \frac{\tau}{\varepsilon},$$

627 The solution is defined by (8.11), where  $\{t_m\}$  is the set of zeroes of the equation  
628  $K(\varepsilon t, z) = z$ .

629 Second, we argue that the function  $\mathfrak{B}(x, t)$  as defined above is the  $\varepsilon$ -periodic  
630 Gelfand transform of the solution to a spectral equation on  $\mathbb{R}$  for a differential operator  
631 with constant coefficients, where the conventional spectral parameter  $z$  is replaced by  
632 a nonlinear in  $z$  expression, as on the right-hand side of (8.12).

633 Indeed, expand the function  $\Re\theta(\tau)$  into Fourier series

$$634 \quad \Re\theta(\tau) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n \exp(in\tau), \quad c_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \Re\theta(\tau) \exp(-in\tau) d\tau, \quad n \in \mathbb{Z}.$$

635 and apply to  $\mathfrak{B}(x, t)$  the inverse Gelfand transform  $\mathcal{G}^*$  :

$$636 \quad (\mathcal{G}^* f)(x) = \sqrt{\frac{\varepsilon}{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} f(t) \exp(itx) dt, \quad f \in L^2\left(-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right), \quad x \in \mathbb{R}.$$

637 We denote  $U := \mathcal{G}^*\mathfrak{B}$  and notice that

$$638 \quad \sqrt{\frac{\varepsilon}{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} t^2 \mathfrak{B}(x, t) \exp(itx) dt = -\frac{d^2}{dx^2} \left( \sqrt{\frac{\varepsilon}{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \mathfrak{B}(x, t) \exp(itx) dt \right) = -U''(x)$$

639 and

$$640 \quad \sqrt{\frac{\varepsilon}{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \Re\theta(\varepsilon t) \mathfrak{B}(x, t) \exp(itx) dt = \sum_{n=-\infty}^{\infty} c_n \frac{\sqrt{\varepsilon}}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \mathfrak{B}(x, t) \exp(it(x + \varepsilon n)) dt$$

641

$$642 \quad = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n U(x + \varepsilon n) \sim \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n U(x) = \Re\theta(0)U(x) = U(x), \quad \varepsilon \rightarrow 0.$$

643

644 The above asymptotics as  $\varepsilon \rightarrow 0$  is understood in the sense of  $W^{-2,2}(\mathbb{R})$ . It can  
 645 be demonstrated, see [11], that the order of convergence is  $O(\varepsilon^2)$  (and  $O(\varepsilon)$  in the  
 646 general case), however we do not dwell on the complete proof here. The idea of the  
 647 proof, which is standard, can be, for example, the following. Instead of the function  
 648  $\beta$ , define  $\beta^0$  by the expression (8.11), where the sequence  $\{t_m\}$  is replaced by the  
 649 sequence  $\{t_m^0\}$  of zeros of the equation  $K^0(\tau, z) = z$ . Here  $K^0$  is defined by (8.9)  
 650 with  $\Re\theta(\tau)$  replaced by  $\Re\theta(0) = 1$ . It is then shown that  $\beta$  is  $O(\varepsilon^2)$ -close, in the  
 651 sense of distributions, to  $\beta^0$ , and one obtains the claim by taking the inverse Gelfand  
 652 transform of the function  $\mathfrak{B}^0(x, t) = (1/\sqrt{\varepsilon})\beta^0(t)\mathbb{1}(x)$ .

653 It follows that the limit equation on the function  $U$  takes the form

$$654 \quad (8.13) \quad -\sigma^2 U''(x) = \left\{ (l_1 + l_3)z + 2\sqrt{z} \tan\left(\frac{l_2\sqrt{z}}{2}\right) \right\} U(x), \quad x \in \mathbb{R}.$$

655 In particular, the limit spectrum is given by the set of  $z \in \mathbb{R}$  for which the expression  
 656 in brackets on the right-hand side of (8.13) is non-negative, see Fig. 6.

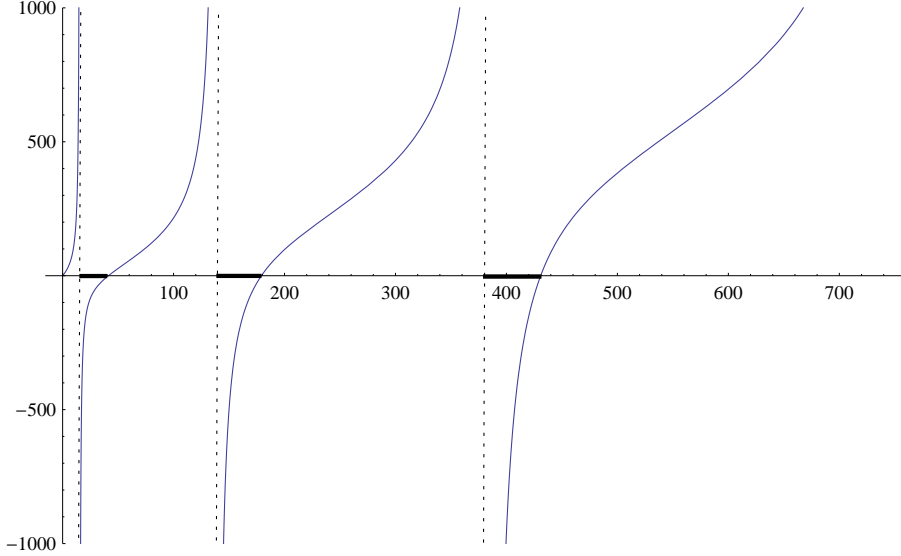


FIG. 6. DISPERSION FUNCTION. *The plot of the dispersion function on the right-hand side of (8.13), for  $l_1 + l_3 = 1 - l_2 = 0.2$ . The spectral gaps are highlighted in bold.*

657 **Appendix A: The reduction of the general case to the one treated in**  
 658 **Section 6.** We proceed as follows. First, we decompose the graph  $\widehat{\mathbb{G}}$  into the union  
 659 of its stiff and soft components,  $\widehat{\mathbb{G}} = \mathbb{G}^{\text{soft}} \cup \mathbb{G}^{\text{stiff}}$ , each of these being a graph on  
 660 its own. The common boundary of them is  $\partial\mathbb{G} := \mathbb{G}^{\text{soft}} \cap \mathbb{G}^{\text{stiff}}$ , and it is treated  
 661 as a set of vertices. Second, we consider two maximal operators  $\check{A}_{\text{max}}^{\text{soft}}$  and  $\check{A}_{\text{max}}^{\text{stiff}}$ ,  
 662 which are densely defined in  $L_2(\mathbb{G}^{\text{soft}})$  and  $L_2(\mathbb{G}^{\text{stiff}})$ , respectively, by (3.3), (5.4)  
 663 applied to  $\mathbb{G}^{\text{soft}}$  and  $\mathbb{G}^{\text{stiff}}$ . Furthermore, we introduce the orthogonal projections  
 664  $P^{\text{soft}}, P^{\text{stiff}}$  in the boundary space  $\mathcal{H}$  onto the subspaces pertaining to vertices of  $\mathbb{G}^{\text{soft}}$   
 665 and  $\mathbb{G}^{\text{stiff}}$ , respectively. Finally, we construct boundary triples for  $\check{A}_{\text{max}}^{\text{soft(stiff)}}$  with  
 666 boundary spaces  $P^{\text{soft(stiff)}}\mathcal{H}$  and boundary operators  $\check{\Gamma}_j^{\text{soft(stiff)}}$ ,  $j = 0, 1$  (cf. (5.5)),  
 667 respectively.

668 Now consider the restrictions

$$669 \quad (8.14) \quad \begin{aligned} A_{\max}^{\text{soft (stiff)}} &= \check{A}_{\max}^{\text{soft (stiff)}} \Big|_{\text{dom}(A_{\max}^{\text{soft (stiff)})}}, \\ \text{dom}(A_{\max}^{\text{soft (stiff)}}) &:= \left\{ u \in \text{dom}(\check{A}_{\max}^{\text{soft (stiff)}}) \Big| (1 - P_{\partial\mathbb{G}})\check{\Gamma}_1^{\text{soft (stiff)}} u = 0 \right\}, \end{aligned}$$

where  $P_{\partial\mathbb{G}}$  is defined as an orthogonal projection in  $\mathcal{H}$  onto the subspace pertaining to the vertices belonging to  $\partial\mathbb{G}$ . For these two maximal operators, one has the common boundary space  $P_{\partial\mathbb{G}}\mathcal{H}$  and boundary operators defined by

$$\Gamma_j^{\text{soft (stiff)}} := P_{\partial\mathbb{G}}\check{\Gamma}_j^{\text{soft (stiff)}}, \quad j = 0, 1.$$

670 The corresponding  $M$ -matrices  $M^{\text{soft (stiff)}}$  are computed as inverses of the matrices  
671  $P_{\partial\mathbb{G}}(\check{M}^{\text{soft (stiff)}})^{-1}P_{\partial\mathbb{G}}$ , where the latter are considered in the reduced space  
672  $P_{\partial\mathbb{G}}\mathcal{H}$  and  $\check{M}^{\text{soft (stiff)}}$  are  $M$ -matrices of  $\check{A}_{\max}^{\text{soft (stiff)}}$  relative to the boundary triples  
673  $(P^{\text{soft (stiff)}}, \check{\Gamma}_0^{\text{soft (stiff)}}, \check{\Gamma}_1^{\text{soft (stiff)}})$ .

674 It is easily shown that the operator  $A_t^\varepsilon$  is expressed as an almost solvable extension  
675 parameterised by the matrix  $B = 0$  relative to a triple which has the  $M$ -matrix  
676  $M = M^{\text{soft}} + M^{\text{stiff}}$ . It follows that all the prerequisites of the analysis carried out in  
677 Section 6 are met.

678 **Appendix B: Proof of Lemma 6.2.** The proof could be carried out on the  
679 basis of [16], [17] and is rather elementary. Nevertheless, in the present paper we have  
680 elected to follow an alternative approach to this proof, which has an advantage of  
681 carrying over to the PDE case with minor modifications.

682 For simplicity we set  $w_V(e) = 1$  for all  $e, V$  in (3.4), as the argument below is  
683 unaffected by the concrete choice of the list  $\{w_V(e)\}_{e \ni V}$ ,  $V \in \widehat{\mathbb{G}}$ , in the construction  
684 of Section 3. For convenience, we also imply that the unitary rescaling to a graph of  
685 length one has been applied to the operator family  $A_t^\varepsilon$ . For brevity, we keep the same  
686 notation for the unitary images of graphs  $\widehat{\mathbb{G}}$ ,  $\mathbb{G}^{\text{stiff}}$  and  $\partial\mathbb{G}$  under this transform.

687 For each  $\tau \in [-\pi, \pi)$ , the eigenvalues of  $B_0(\tau)$  are those  $\mu \in \mathbb{C}$  for which there  
688 exists  $u \neq 0$  satisfying

$$689 \quad (8.15) \quad \begin{cases} \left( \frac{d}{dx} + i\tau \right)^2 u = 0 & \text{in } \mathbb{G}^{\text{stiff}}, \\ -\sum_{e \ni V} \sigma_e (u'_e(V) + i\tau u(V)) = \mu u(V), & V \in \partial\mathbb{G}, \\ u \text{ continuous on } \mathbb{G}^{\text{stiff}}, \end{cases}$$

690 where  $u'_e(V)$  is the derivative of  $u$  along the edge  $e$  of  $\mathbb{G}^{\text{stiff}}$  evaluated at  $V \in \partial\mathbb{G}$ ,  
691 and, as before,  $\sigma_e = -1$  or  $\sigma_e = 1$ , depending on whether  $e$  is incoming or outgoing  
692 for  $V$ , respectively. It is known that the spectrum of (8.15) is discrete and the least  
693 eigenvalue, which clearly coincides with  $\mu^{(\tau)}$ , is simple.

694 *Formal series.* In order to show (6.5), we first consider series in powers of  $i\tau$  :

$$695 \quad (8.16) \quad \mu = \sum_{k=1}^{\infty} \alpha_j (i\tau)^{2k}, \quad u = \sum_{j=0}^{\infty} u_j (i\tau)^j,$$

696 where  $u_j$ ,  $j = 1, 2, \dots$  are continuous on  $\mathbb{G}^{\text{stiff}}$ .

697 Note that the expansion for  $\mu$  contains only even powers of the parameter  $\tau$ , as  
 698 it is an even function of  $\tau$ . Indeed, the function obtained from the eigenfunction  $u$  in  
 699 (8.15) by changing the directions of all edges of the graph is clearly an eigenfunction  
 700 for (8.15) with  $\tau$  replaced by  $-\tau$ . (On such a change of edge direction, the weights  
 701  $w_e(V)$ ,  $e \ni V$ ,  $V \in \widehat{\mathbb{G}}$ , are replaced by their complex conjugates.) In view of the fact  
 702 that for all  $\tau \in (-\pi, \pi]$  the eigenvalue  $\mu^{(\tau)}$  is simple, we obtain  $\mu^{(-\tau)} = \mu^{(\tau)}$ .

703 Substituting the expansion (8.16) into (8.15) and equating the coefficients on  
 704 different powers of  $\tau$ , we obtain a sequence of recurrence relations for  $u_j$ ,  $j = 0, 1, \dots$   
 705 In particular, the problem for  $u_0$  is obtained by comparing the coefficients on  $\tau^0$ :

$$706 \quad \begin{cases} u_0'' = 0 & \text{on } \mathbb{G}^{\text{stiff}}, \\ \sum_{e \ni V} \sigma_e (u_0)'_e(V) = 0, & V \in \partial \mathbb{G}, \\ u_0 \text{ continuous on } \mathbb{G}^{\text{stiff}}. \end{cases}$$

707 Assuming that  $\mathbb{G}^{\text{stiff}}$  contains a loop, it follows that  $u_0$  is a constant, which we set to  
 708 be unity. In the case opposite, i.e., when  $\mathbb{G}^{\text{stiff}}$  is a tree,  $\mu^{(\tau)} \equiv 0$  for all  $\tau$ , and the  
 709 claim of Lemma follows trivially.

710 We impose the condition of vanishing mean of  $u_j$ ,  $j = 1, 2, \dots$  over  $\mathbb{G}^{\text{stiff}}$ . This is  
 711 justified by the convergence estimates below as well as the fact that the eigenvalue  $\mu$   
 712 is simple. The choice  $u_0 = 1$  thus corresponds to the “normalisation” condition that  
 713 the mean over  $\mathbb{G}^{\text{stiff}}$  of the eigenfunction  $u$  for (8.15) is close to unity<sup>8</sup> for small values  
 714 of  $\tau$ .

715 Proceeding with the asymptotic procedure, the problem for  $u_1$  is obtained by  
 716 comparing the coefficients on  $\tau^1$ :

$$717 \quad \begin{cases} u_1'' = 0 & \text{on } \mathbb{G}^{\text{stiff}}, \\ \sum_{e \ni V} \sigma_e ((u_1)'_e(V) + 1) = 0, & V \in \partial \mathbb{G}, \\ u_1 \text{ continuous on } \mathbb{G}^{\text{stiff}}, \\ \int_{\mathbb{G}^{\text{stiff}}} u_1 = 0. \end{cases}$$

718 Further, the equation for  $u_2$  is obtained by comparing the coefficients on  $\tau^2$ :

$$719 \quad (8.17) \quad \begin{cases} u_2'' = -2u_1' - 1 & \text{on } \mathbb{G}^{\text{stiff}}, \\ -\sum_{e \ni V} \sigma_e ((u_2)'_e(V) + u_1(V)) = \alpha_2, & V \in \partial \mathbb{G}, \\ u_2 \text{ continuous on } \mathbb{G}^{\text{stiff}}, \\ \int_{\mathbb{G}^{\text{stiff}}} u_2 = 0. \end{cases}$$

720 The condition for solvability of the problem (8.17) yields the expression for  $\alpha_2$ , as  
 721 follows:

$$722 \quad \int_{\mathbb{G}^{\text{stiff}}} (-2u_1' - 1) = \int_{\mathbb{G}^{\text{stiff}}} u_2'' = - \sum_{V \in \partial \mathbb{G}} \sum_{e \ni V} \sigma_e (u_2)'_e(V) = \sum_{V \in \partial \mathbb{G}} \left( \sum_{e \ni V} \sigma_e u_1(V) + \alpha_2 \right).$$

723 Re-arranging the terms in the last equation, we obtain

$$724 \quad \alpha_2 = -|\partial \mathbb{G}|^{-1} \int_{\mathbb{G}^{\text{stiff}}} (u_1' + 1).$$

<sup>8</sup>The eigenfunction  $u$  clearly does not vanish identically, at least for small values of  $\tau$ .



725 The above asymptotic procedure is continued, to obtain the terms of all orders in  
726 (8.16). In particular, for the term  $u_3$  in the expansion for  $u$  we obtain

$$727 \quad \begin{cases} u_3'' = -2u_2' - u_1 & \text{on } \mathbb{G}^{\text{stiff}}, \\ -\sum_{e \ni V} \sigma_e((u_3)'_e(V) + u_2(V)) = \alpha_2 u_1, & V \in \partial \mathbb{G}, \\ u_3 \text{ continuous on } \mathbb{G}^{\text{stiff}}, \\ \int_{\mathbb{G}^{\text{stiff}}} u_3 = 0. \end{cases}$$

728 *Error estimates.* We write

$$729 \quad u = 1 + i\tau u_1 + (i\tau)^2 u_2 + (i\tau)^3 u_3 + R, \quad \mu^{(\tau)} = \alpha_2 (i\tau)^2 + r,$$

730 so that  $R, r$  satisfy

$$731 \quad \left. \begin{aligned} (8.18) \quad & \left( \frac{d}{dx} + i\tau \right)^2 R = -(i\tau)^4 (2u_3' + u_2) - (i\tau)^5 u_3 && \text{on } \mathbb{G}^{\text{stiff}}, \\ (8.19) \quad & -\sum_{e \ni V} \sigma_e(R'_e(V) + i\tau R(V)) = \\ & = (r + \alpha_2 (i\tau)^2) (1 + i\tau u_1 + (i\tau)^2 u_2 + (i\tau)^3 u_3 + R) \\ & - \alpha_2 (i\tau)^2 (1 + i\tau u_1), \quad V \in \partial \mathbb{G} \\ & R \text{ continuous on } \mathbb{G}^{\text{stiff}}, \\ & \int_{\mathbb{G}^{\text{stiff}}} R = 0. \end{aligned} \right\}$$

732 Notice first that

$$733 \quad (8.20) \quad r + \alpha_2 (i\tau)^2 = \mu^{(\tau)} = \min_{u \in W^{2,2}(\mathbb{G}^{\text{stiff}})} \left( \sum_{\partial \mathbb{G}} |u|^2 \right)^{-1} \int_{\mathbb{G}^{\text{stiff}}} \left| \left( \frac{d}{dx} + i\tau \right) u \right|^2$$

$$734 \quad \leq |\partial \mathbb{G}|^{-1} |\mathbb{G}^{\text{stiff}}| \tau^2.$$

735 Multiplying (8.18) by  $R$ , integrating by parts, and using (8.19), we obtain the estimate

$$736 \quad (8.21) \quad \|R\|_{L^2(\mathbb{G}^{\text{stiff}})}^2 \leq C(|\tau| |r| \|R\|_{L^2(\mathbb{G}^{\text{stiff}})} + |\tau|^4 \|R\|_{L^2(\mathbb{G}^{\text{stiff}})} + |r|^2), \quad C > 0,$$

737 and hence, by virtue of (8.20), we obtain

$$738 \quad (8.22) \quad \|R\|_{L^2(\mathbb{G}^{\text{stiff}})} \leq C\tau^2.$$

739 Next, we re-arrange the right-hand side of (8.19):

$$740 \quad \begin{aligned} (r + \alpha_2 (i\tau)^2) (1 + i\tau u_1 + (i\tau)^2 u_2 + (i\tau)^3 u_3 + R) - \alpha_2 (i\tau)^2 (1 + i\tau u_1) \\ 741 \quad = r(1 + i\tau u_1 + (i\tau)^2 u_2 + (i\tau)^3 u_3 + R) + \alpha_2 (i\tau)^2 ((i\tau)^2 u_2 + (i\tau)^3 u_3 + R). \end{aligned}$$

742 Multiplying (8.18) by 1, integrating by parts, and using (8.19) once again yields the  
743 existence of  $C > 0$  such that

$$744 \quad (8.23) \quad |r| \leq C(|\tau| \|R\|_{L^2(\mathbb{G}^{\text{stiff}})} + |\tau|^4).$$

749 Combining this with (8.22) yields  $|r| \leq C\tau^3$ , which, by virtue of (8.21) again, implies

750 (8.24) 
$$\|R\|_{L^2(\mathbb{G}^{\text{stiff}})} \leq C|\tau|^3.$$

751 Finally, the inequalities (8.23) and (8.24) together yield

752 (8.25) 
$$|r| \leq C|\tau|^4,$$

753 as claimed.<sup>9</sup>

754 **Appendix C: Proof of Lemma 6.3.** For all  $\tau \in [-\pi, \pi)$ , using the formula for  
 755 the second eigenvalue  $\mu_2^{(\tau)}$  of the problem (8.15) via the Rayleigh quotient, we obtain

756 
$$\mu_2^{(\tau)} = \min \left\{ \left( \sum_{\partial\mathbb{G}} |u|^2 \right)^{-1} \int_{\mathbb{G}^{\text{stiff}}} \left| \left( \frac{d}{dx} + i\tau \right) u \right|^2 : u \in W^{2,2}(\mathbb{G}^{\text{stiff}}), \int_{\mathbb{G}^{\text{stiff}}} u = 0 \right\}$$
  
 757 
$$\geq \min \left\{ \left( \sum_{\partial\mathbb{G}} |u|^2 \right)^{-1} \int_{\mathbb{G}^{\text{stiff}}} |u'|^2 : u \in W^{2,2}(\mathbb{G}^{\text{stiff}}), \int_{\mathbb{G}^{\text{stiff}}} u = 0 \right\} = \mu_2^{(0)} > 0,$$
  
 758

759 from which the claim follows by setting  $C_{\perp} = \mu_2^{(0)}$ .

760 **Acknowledgements.** We are grateful to Professor S. Naboko for suggesting a  
 761 calculation in Section 8.

762 REFERENCES

763 [1] Arendt, W., Mazzeo, R., 2012. Friedlander’s eigenvalue inequalities and the Dirichlet-to- Neu-  
 764 mann semigroup. *Commun. Pure Appl. Anal.* **11**(6), 2201–2212  
 765 [2] Berezansky, Yu., 1968. *Expansions in eigenfunctions of selfadjoint operators*. Translated from  
 766 the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman. AMS Translations  
 767 of Mathematical Monographs, Vol. 17.  
 768 [3] Berkolaiko, G., Kuchment, P., 2012. *Introduction to Quantum Graphs*, Mathematical Surveys  
 769 and Monographs **186**, American Mathematical Society.  
 770 [4] Birman, M. Sh., Solomyak, M. Z., 1987. *Spectral theory of selfadjoint operators in Hilbert space*.  
 771 Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht.  
 772 [5] Birman, M. Sh., Suslina, T. A., 2004. Second order periodic differential operators. Threshold  
 773 properties and homogenisation. *St. Petersburg. Math. J.* **15**(5), 639–714.  
 774 [6] Bouchitté, G., Felbacq, D., 2004. Homogenisation near resonances and artificial magnetism  
 775 from dielectrics. *C. R. Math. Acad. Sci. Paris* **339**(5), 377–382.  
 776 [7] Cessenat, M., 1996. *Mathematical Methods in Electromagnetism: Linear Theory and Applica-*  
 777 *tions*, World Scientific.  
 778 [8] Cherednichenko, K., Cooper, S., Guenneau, S., 2015. Spectral analysis of one-dimensional high-  
 779 contrast elliptic problems with periodic coefficients. *Multiscale Modeling and Simulation*  
 780 (*SIAM*) **13**(1), 72–98 (2015)  
 781 [9] Cherednichenko, K. D., Kiselev, A. V., 2017. Norm-resolvent convergence of one-dimensional  
 782 high-contrast periodic problems to a Kronig-Penney dipole-type model. *Comm. Math.*  
 783 *Phys.* **349**(2), 441–480.  
 784 [10] Cherednichenko, K. D., Ershova, Yu. Yu., Kiselev, A. V., 2018. Unified approach to critical-  
 785 contrast homogenisation of PDEs with explicit links to time-dispersive media. I. *arXiv:*  
 786 *1808.03961*.  
 787 [11] Cherednichenko, K. D., Ershova, Yu. Yu., Kiselev, A. V., Naboko, S. N., 2018. Unified approach  
 788 to critical-contrast homogenisation with explicit links to time-dispersive media. *arXiv:*  
 789 *1805.00884*.

---

<sup>9</sup>Combining (8.25) with (8.20), we also obtain the estimate  $\|R\|_{L^2(\mathbb{G}^{\text{stiff}})} \leq C\tau^4$ .

- 790 [12] Cherednichenko, K. D., Kiselev, A. V., Silva, L. O., 2018. Functional model for extensions  
791 of symmetric operators and applications to scattering theory. *Netw. and Heterog. Media*  
792 **13**(2).
- 793 [13] Cvetković D. M., Doob M., Sachs H. 1995. *Spectra of graphs. Theory and applications*. Johann  
794 Ambrosius Barth, Heidelberg.
- 795 [14] Derkach, V. A., Malamud M. M., 1991. Generalised resolvents and the boundary value problems  
796 for Hermitian operators with gaps, *J. Funct. Anal.* **95**, 1–95.
- 797 [15] Després B., Imbert-Gérard L.-M., Weder R., 2014. Hybrid resonance of Maxwell’s equations in  
798 slab geometry. *Journal de Mathématiques Pures et Appliquées* **101**, no. 5, 623–659
- 799 [16] Ershova, Yu., Karpenko, I. I., Kiselev, A.V., 2016. Isospectrality for graph Laplacians under  
800 the change of coupling at graph vertices, *J. Spectral Th.* **6** (2016), no. 1, 43–66.
- 801 [17] Ershova, Yu., Karpenko, I. I., Kiselev, A.V., 2016. Isospectrality for graph Laplacians under  
802 the change of coupling at graph vertices: necessary and sufficient conditions. *Mathematika*  
803 *(UCL)* **62** (2016), no. 1, 210–242.
- 804 [18] Exner, P., Post, O., 2005. Convergence of spectra of graph-like thin manifolds. *J. Geom. Phys.*  
805 **54**(1), 77–115.
- 806 [19] Figotin, A., Schenker, J. H., 2005. Spectral analysis of time dispersive and dissipative systems,  
807 *Journal of Statistical Physics*, **118**(1–2), 199–263.
- 808 [20] Figotin, A., Schenker, J. H., 2007. Hamiltonian structure for dispersive and dissipative dynam-  
809 ical systems. *Journal of Statistical Physics* **128** (4), 969–1056.
- 810 [21] Gerasimenko, N.I., Pavlov, B.S., 1988. A scattering problem on non-compact graphs. *Theoret.*  
811 *and Math. Phys.* **74** (3), 230–240.
- 812 [22] Gorbachuk, V. I., Gorbachuk, M. L., 1991. *Boundary Value Problems for Operator Differential*  
813 *Equations*, Springer.
- 814 [23] Gradshteyn, I. S., Ryzhik, I. M., 2007. *Table of Integrals, Series, and Products*, Academic  
815 Press.
- 816 [24] Jackson, J. D., 1998. *Classical Electrodynamics*, John Wiley & Sons.
- 817 [25] Kočubei, A. N., 1975. On extension of symmetric operators and symmetric binary relations,  
818 *Math. Notes* **17**, 41–48.
- 819 [26] Kuchment, P., Zeng, H., 2001. Convergence of spectra of mesoscopic systems collapsing onto a  
820 graph. *J. Math. Anal. Appl.* **258**(2), 671–700.
- 821 [27] Kuchment, P., Zeng, H., 2004. Asymptotics of spectra of Neumann Laplacians in thin domains.  
822 *Contemporary Mathematics* **327**, Amer. Math. Soc., Providence, Rhode Island, 199–213.
- 823 [28] Melnikov, Yu., Pavlov, B., 2001. Scattering on graphs and one-dimensional approximations to  
824  $N$ -dimensional Schrödinger operators. *Journal of Mathematical Physics* **42**, 1202.
- 825 [29] Neumark, M. A., 1940. Spectral functions of a symmetric operator. *Izv. Acad. Nauk SSSR,*  
826 *Mathematics* **4:3**, 277–318.
- 827 [30] Neumark, M., 1943. Positive definite operator functions on a commutative group. (Russian)  
828 *Bull. Acad. Sci. URSS Ser. Math. [Izvestia Akad. Nauk SSSR]* **7**, 237–244.
- 829 [31] Nussenzweig, H. M., 1972. *Causality and Dispersion Relations*. Mathematics in Science and  
830 Engineering **95**, Elsevier.
- 831 [32] Pavlov, B. S.; Faddeev, M. D., 1978. Construction of selfadjoint dilation for a problem with an  
832 impedance boundary condition. (Russian) *Investigations on linear operators and the theory*  
833 *of functions, VIII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **73**,  
834 217–223, 236.
- 835 [33] Ryzhov, V., 2018. Spectral boundary value problems and their linear operators, 38 pp.,  
836 *arXiv:0904.0276; To appear in Oper. Theory: Adv. Appl.*
- 837 [34] Tip, A., 1998. Linear absorptive dielectrics. *Phys. Rev. A* **57**: 4818–4841.
- 838 [35] Tip, A., 2006. Some mathematical properties of Maxwell’s equations for macroscopic dielectrics.  
839 *J. Math. Phys.* **47** 012902–012902-22.
- 840 [36] Tsu, R., 2010. *Superlattice to Nanoelectronics*, Elsevier.
- 841 [37] Veselago, V. G., 1964. The electrodynamics of substances with simultaneously negative values  
842 of  $\varepsilon$  and  $\mu$ . *Soviet Phys. Uspekhi* **10**(4), 509–514.
- 843 [38] Zhikov, V. V., 2000. On an extension of the method of two-scale convergence and its applica-  
844 tions. *Sbornik: Mathematics* **191**(7), 973–1014.
- 845 [39] Zhikov, V. V., 2002. Averaging of problems in the theory of elasticity on singular structures.  
846 *Izv. Math.* **66** (2), 299–365.